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Modular Forms Associated to Real Quadratic Fields

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The purpose of this paper is to construct modular forms, both for $SL_2\mathbb{Z}$ (and certain of its congruence subgroups) and for the Hilbert modular group of a real quadratic field.

In §1 we fix a real quadratic field K and even integer $k>2$ and construct a series of functions $\omega_m(z_1, z_2)$ $(m=0, 1, 2, \ldots)$ which are modular forms of weight k for the Hilbert modular group $SL_2\mathcal{O}$ ($\mathcal{O}$ = ring of integers in K). The form ω_0 is a multiple of the Hecke-Eisenstein series for K, while all of the other ω_m are cusp forms.

The Fourier expansion of $\omega_m(z_1, z_2)$ is calculated in §2; each Fourier coefficient is expressed as an infinite sum whose typical term is the product of a finite exponential sum (analogous to a Kloosterman sum) and a Bessel function of order $k-1$.

The main result is that, for any points z_1 and z_2 in the upper half-plane $\mathfrak{H}$, the numbers $m^{k-1}\omega_m(z_1, z_2)$ $(m=1, 2, \ldots)$ are the Fourier coefficients of a modular form (in one variable) of weight k. More precisely, let D be the discriminant of K, $\varepsilon=(D/\ )$ the character of K, and $S(D, k, \varepsilon)$ the space of cusp forms of weight k for $\Gamma_0(D)$ with character ε; then for fixed $z_1, z_2 \in \mathfrak{H}$, the function

$$\Omega(z_1, z_2; \tau) = \sum_{m=1}^{\infty} m^{k-1}\, \omega_m(z_1, z_2)\, e^{2\pi i m\tau} \qquad (\tau \in \mathfrak{H})$$

(considered as a function of τ) belongs to $S(D, k, \varepsilon)$. What we in fact prove is an identity expressing Ω as a linear combination of Poincaré series for the group $\Gamma_0(D)$ and character ε. The necessary facts about such Poincaré series are collected in §3; the proof that Ω is a modular form is given in §4 (in these sections we assume for simplicity that $D \equiv 1 \pmod 4$).

In [2] and [8], K. Doi and H. Naganuma prove the following: assume D is a prime ($\equiv 1 \pmod 4$) having class number one, and let $f(\tau)=\sum_{n=1}^{\infty} a_n e^{2\pi i n\tau} \in S(D, k, \varepsilon)$ be a normalized eigenfunction of all Hecke operators $T(n)$. Then

$$\left(\sum_{n=1}^{\infty} a_n n^{-s}\right)\left(\sum_{n=1}^{\infty} \bar{a}_n n^{-s}\right)$$

is the Mellin transform (in a suitable sense) of a function $\hat{f}$ of two variables which satisfies

$$\hat{f}\left(\frac{-1}{z_1}, \frac{-1}{z_2}\right) = z_1^k z_2^k \hat{f}(z_1, z_2) \qquad (z_1, z_2 \in \mathfrak{H})$$

as well as the trivial invariance property

$$f(\varepsilon^2 z_1+\mu, \varepsilon'^2 z_2+\mu')=f(z_1, z_2) \qquad (\mu \text{ an integer of } K, \varepsilon \text{ a unit of } K).$$

In particular, if K is Euclidean, so that the matrices $\begin{pmatrix}\varepsilon & \varepsilon^{-1}\mu\\ 0 & \varepsilon^{-1}\end{pmatrix}$ and $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ generate $SL_2\mathcal{O}$, then $\hat{f}$ is a Hilbert modular form; to be sure, this is the case only for $D=5, 13, 17, 29, 37, 41, 73$ (cf. [3], Theorem 247). We show (in §5) that $\hat{f}(z_1, z_2)$ is, up to a scalar factor, the Petersson product (with respect to the variable τ) of $f(\tau)$ with $\Omega(z_1, z_2; \tau)$. It follows that $\hat{f}(z_1, z_2)$ is always a cusp form for $SL_2\mathcal{O}$.[1] Thus $f\mapsto\hat{f}$ extends to a linear map from $S(D, k, \varepsilon)$ to the vector space of cusp forms of weight k for $SL_2\mathcal{O}$; the image of this map has dimension $\frac{1}{2}\dim S(D, k, \varepsilon)$ and is spanned by the forms $\omega_m(z_1, z_2)$ $(m=1, 2, \ldots)$.

In Appendix 1 we define the forms $\omega_m(z_1, z_2)$ for the previously excluded case $k=2$ and reprove all the results of the paper in this case; in particular, the restriction $k>2$ can be lifted in the work of Doi-Naganuma. In Appendix 2 we investigate briefly the modular forms

$$f_\Delta(z)=\sum_{\substack{a, b, c\in\mathbb{Z}\\ b^2-4ac=\Delta}} \frac{1}{(az^2+bz+c)^k} \qquad (z\in\mathfrak{H}, \Delta\in\mathbb{N})$$

of weight $2k$ for the full modular group $SL_2\mathbb{Z}$; these forms arise naturally when one looks at the restriction of $\omega_m(z_1, z_2)$ to the diagonal $z_1=z_2$ in $\mathfrak{H}\times\mathfrak{H}$.

The main results of this paper (in the case when the discriminant D is a prime) were announced in a Comptes Rendus note [12].

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§1. Definition and Properties of ω_m

We will use the following notation:

K a real quadratic number field;

D the discriminant of K;

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[1] Lenstra has pointed out to the author that, by a recent theorem of Vaserstein, the matrices $\begin{pmatrix}1 & \mu\\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ always generate $SL_2\mathcal{O}$, so that this does in fact follow from the work of Doi-Naganuma.

$\mathcal{O}$ the ring of integers of K;
$\mathcal{O}^*$ the group of units of $\mathcal{O}$;
$\mathfrak{d}$ the different of K (i.e. the principal ideal $(\sqrt{D})$);
x' the conjugate over $\mathbb{Q}$ of an element $x \in K$;
$N(x)$ the norm of x, $N(x) = xx'$;
$Tr(x)$ the trace of x, $Tr(x) = x + x'$;
$\mathfrak{H}$ the upper half-plane $\{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$;
k a fixed even integer >2.

For each integer $m \geqq 0$, we define

$$\omega_m(z_1, z_2) = \sum_{\substack{a,\, b \in \mathbb{Z} \\ \lambda \in \mathfrak{d}^{-1} \\ N(\lambda) - ab = m/D}}' \frac{1}{(a z_1 z_2 + \lambda z_1 + \lambda' z_2 + b)^k} \qquad (z_1, z_2 \in \mathfrak{H}), \tag{1}$$

where the summation is over all triples (a, b, λ) satisfying the given conditions, and the notation $\sum'$ indicates that, for $m=0$, the triple $(0,0,0)$ is to be omitted. One easily checks that, for $z_1, z_2 \in \mathfrak{H}$, the expression $a z_1 z_2 + \lambda z_1 + \lambda' z_2 + b$ never vanishes $\left(\text{indeed, } a z_1 z_2 + \lambda z_1 + \lambda' z_2 + b = 0 \text{ implies } z_1 = \dfrac{-\lambda' z_2 - b}{a z_2 + \lambda}, \text{ and this is impossible since the determinant of } \begin{pmatrix} -\lambda' & -b \\ a & \lambda \end{pmatrix} \text{ is } \leqq 0\right)$ and that the series converges absolutely. Therefore ω_m is a holomorphic function in $\mathfrak{H} \times \mathfrak{H}$. Its main properties are summarized in the following theorem.

Theorem 1. (i) *For each $m \geqq 0$, $\omega_m(z_1, z_2)$ is a modular form of weight k with respect to the Hilbert modular group $SL_2\mathcal{O}$.*

(ii) *ω_0 is a multiple of the Hecke-Eisenstein series of weight k of the field K.*

(iii) *ω_m is a cusp form for $m>0$.*

(iv) *$\omega_m = 0$ if $-4m$ is not a quadratic residue of D.*

Proof. (i) We recall that a Hilbert modular form of weight k for $SL_2\mathcal{O}$ is a holomorphic function F in $\mathfrak{H} \times \mathfrak{H}$ satisfying

$$F\left(\frac{\alpha z_1 + \beta}{\gamma z_1 + \delta}, \frac{\alpha' z_2 + \beta'}{\gamma' z_2 + \delta'}\right) = (\gamma z_1 + \delta)^k (\gamma' z_2 + \delta')^k F(z_1, z_2) \qquad (z_1, z_2 \in \mathfrak{H}) \tag{2}$$

for any matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2\mathcal{O}$ and also satisfying certain regularity conditions (explained below) at the "cusps". We verify only Eq. (2), since the conditions on the behaviour of ω_m at the cusps are contained in the statements (ii) and (iii) of the theorem.

If $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2\mathcal{O}$, then, for any $a, b \in \mathbb{Z}$, $\lambda \in \mathfrak{d}^{-1}$,

$$a\left(\frac{\alpha z_1 + \beta}{\gamma z_1 + \delta}\right)\left(\frac{\alpha' z_2 + \beta'}{\gamma' z_2 + \delta'}\right) + \lambda\left(\frac{\alpha z_1 + \beta}{\gamma z_1 + \delta}\right) + \lambda'\left(\frac{\alpha' z_2 + \beta'}{\gamma' z_2 + \delta'}\right) + b$$
$$= \frac{a^* z_1 z_2 + \lambda^* z_1 + \lambda^{*\prime} z_2 + b^*}{(\gamma z_1 + \delta)(\gamma' z_2 + \delta')}$$

with

$$a^* = a\alpha\alpha' + \lambda\alpha\gamma' + \lambda'\alpha'\gamma + b\gamma\gamma',$$
$$\lambda^* = a\alpha\beta' + \lambda\alpha\delta' + \lambda'\beta'\gamma + b\gamma\delta',$$
$$b^* = a\beta\beta' + \lambda\beta\delta' + \lambda'\beta'\delta + b\delta\delta'.$$

Thus $a^* = aN(\alpha) + Tr(\lambda(\alpha\gamma')) + bN(\gamma) \in \mathbb{Z}$ since (by definition) the trace of the product of an element of $\mathcal{O}$ and an element of $\mathfrak{d}^{-1}$ is an integer; similarly, $b^* \in \mathbb{Z}$ and $\lambda^* \in \mathfrak{d}^{-1}$. Furthermore $N(\lambda^*) - a^* b^* = N(\lambda) - ab$. Therefore

$$\begin{aligned}\omega_m&\left(\frac{\alpha z_1+\beta}{\gamma z_1+\delta}, \frac{\alpha' z_2+\beta'}{\gamma' z_2+\delta'}\right)\\ &= (\gamma z_1+\delta)^k(\gamma' z_2+\delta')^k \sum_{\substack{a,b,\lambda\\ N(\lambda)-ab=m/D}}' (a^* z_1 z_2 + \lambda^* z_1 + \lambda^{*\prime} z_2 + b^*)^{-k}\\ &= (\gamma z_1+\delta)^k(\gamma' z_2+\delta')^k\, \omega_m(z_1, z_2),\end{aligned}$$

the latter equality because (a^*, b^*, λ^*) runs over the same set of triples as does (a, b, λ) and because the sum converges absolutely.

Remark 1. From the general congruence $x' \equiv x \pmod{\mathfrak{d}}$ $(x \in \mathcal{O})$ and the equation $\alpha\delta - \beta\gamma = 1$, one deduces that in the above formula

$$\lambda^* = \lambda \pmod{\mathcal{O}}.$$

Thus, for each $\nu \in \mathfrak{d}^{-1}$ such that $N(\nu) \equiv m/D \pmod 1$ we could define a Hilbert modular form $\omega_m(z_1, z_2; \nu)$ by restricting the summation in (1) to those λ with $\lambda \equiv \nu \pmod{\mathcal{O}}$ (notice that, for $\lambda \in \mathfrak{d}^{-1}$, $N(\lambda) \pmod 1$ depends only on $\lambda \pmod{\mathcal{O}}$). The function $\omega_m(z_1, z_2)$ would then be the sum of the (finitely many) functions $\omega_m(z_1, z_2; \nu)$ with ν running over the residue classes of $\mathfrak{d}^{-1} \pmod{\mathcal{O}}$ for which $N(\nu) \equiv m/D \pmod 1$.

Remark 2. A more invariant way of writing ω_m and seeing that it is a Hilbert modular form is as follows. For any matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2\mathbb{R}$, let

$$\begin{aligned}\phi_M(z_1, z_2) &= \frac{1}{\det M}\frac{d}{dz_1}\left(\frac{1}{z_2 - Mz_1}\right)\\ &= \frac{1}{(cz_1z_2 - az_1 + dz_2 - b)^2} \qquad (z_1, z_2 \in \mathfrak{H})\end{aligned} \tag{3}$$

$\left(\text{here } Mz_1 = \frac{az_1+b}{cz_1+d}\right)$; the second formula serves to define $\phi_M(z_1, z_2)$ even if $\det M = 0$, while the first shows that $\phi_M(z_1, z_2)$ has no poles in $\mathfrak{H} \times \mathfrak{H}$ if $\det M \leqq 0$. One easily checks that, for $A_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}$, $A_2 = \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix} \in GL_2\mathbb{R}$,

$$\phi_M(A_1 z_1, A_2 z_2) = (\gamma_1 z_1 + \delta_1)^2(\gamma_2 z_2 + \delta_2)^2\, \phi_{A_2^* M A_1}(z_1, z_2), \tag{4}$$

where $A_2^* = \begin{pmatrix} \delta_2 & -\beta_2 \\ -\gamma_2 & \alpha_2 \end{pmatrix} = (\det A_2)A_2^{-1}$ is the adjoint of A_2. Let

$$\mathscr{A} = \{M \in \mathfrak{M}_2(\mathcal{O}) \mid M^* = M'\} \tag{5}$$

be the set of matrices whose adjoints equal their conjugates over $\mathbb{Q}$; a typical matrix of $\mathscr{A}$ has the form $M=\begin{pmatrix}\theta & b\sqrt{D}\\ -a\sqrt{D} & \theta'\end{pmatrix}$ with $a, b\in\mathbb{Z}$, $\theta\in\mathcal{O}$. Write $\theta=-\lambda\sqrt{D}$ with $\lambda\in\mathfrak{d}^{-1}$; then $\phi_M(z_1, z_2)=D^{-1}(a z_1 z_2+\lambda z_1+\lambda' z_2+b)^{-2}$. Hence

$$\omega_m(z_1, z_2)=D^{k/2}\sum_{\substack{M\in\mathscr{A}\\ \det M=-m}}{}' \phi_M(z_1, z_2)^{k/2}, \tag{6}$$

where the prime on the sigma indicates that, for $m=0$, the zero matrix is to be omitted from the summation. That ω_m satisfies (2) now follows immediately from Eq. (4).

(ii) We recall the definition of the Hecke-Eisenstein series. Let (temporarily) K be a totally real number field of arbitrary degree n over $\mathbb{Q}$, $\mathcal{O}$ its ring of integers, $\mathcal{O}^*$ the group of units of $\mathcal{O}$, C an ideal class. Set

$$F_k(z_1, \dots, z_n; C)=2N(\mathfrak{a})^k \sum_{(\mu, \nu)\in(\mathfrak{a}\times\mathfrak{a}-\{(0,0)\})/\mathcal{O}^*} \frac{1}{(\mu^{(1)} z_1+\nu^{(1)})^k\dots(\mu^{(n)} z_n+\nu^{(n)})^k}, \tag{7}$$

where $z_1, \dots, z_n$ are in $\mathfrak{H}$ and $\mathfrak{a}$ is any ideal in C; here $\mu^{(i)}$, $\nu^{(i)}$ are the conjugates of μ, ν and the summation is over non-associated non-zero pairs of numbers in $\mathfrak{a}$. (The factor 2 is inserted so that for $K=\mathbb{Q}$ F_k agrees with the usual Eisenstein series $\sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}-\{(0,0)\}}(mz+n)^{-k}$ where one does not divide by the action of $\mathcal{O}^*=\{\pm 1\}$.) Clearly, replacing $\mathfrak{a}$ by $(\alpha)\,\mathfrak{a}$ ($\alpha\in K^*$) does not change the right-hand side of (7), so that the expression really does depend only on the ideal class C. The Hecke-Eisenstein series of the field K is then defined as the (finite) sum

$$F_k(z_1, \dots, z_n)=\sum_C F_k(z_1, \dots, z_n; C).$$

We wish to prove (in the case that K is a quadratic field) the formula

$$\omega_0(z_1, z_2)=\frac{\zeta(k)}{\zeta_K(k)} F_k(z_1, z_2). \tag{8}$$

Here $\zeta(s)$ is the Riemann zeta-function, $\zeta_K(s)$ the Dedekind zeta-function of K.

First of all, one of the summation conditions in the definition of ω_0 is $\lambda\lambda'-ab=0$. This implies $\lambda\in\mathcal{O}$ (since $N(\lambda)$, $Tr(\lambda)\in\mathbb{Z}$). It is also homogeneous in (a, b, λ), i.e. if the triple (a, b, λ) appears in the sum, then so does $(ra, rb, r\lambda)$ for each positive integer r. Hence

$$\omega_0(z_1, z_2)=\omega_0^*(z_1, z_2)\sum_{r=1}^{\infty} r^{-k}=\zeta(k)\,\omega_0^*(z_1, z_2),$$

with

$$\omega_0^*(z_1, z_2)=\sum_{\substack{(a, b, \lambda)\text{ primitive}\\ \lambda\lambda'=ab}} (a z_1 z_2+\lambda z_1+\lambda' z_2+b)^{-k} \tag{9}$$

(where "primitive" means that $(a, b, \lambda)\in\mathbb{Z}\times\mathbb{Z}\times\mathcal{O}$ is not divisible by any integer >1).

Similarly, define

$$F_k^*(z_1, z_2; C)=2N(\mathfrak{a})^k \sum_{\delta(\mu,\nu)=\mathfrak{a}}{}^{*} \frac{1}{(\mu z_1+\nu)^k(\mu' z_2+\nu')^k}, \tag{10}$$

where the summation is over non-associated pairs of numbers $\mu, \nu \in K$ whose greatest common divisor, $\delta(\mu,\nu)$, is the ideal $\mathfrak{a}$ (again, this depends only on C, not on $\mathfrak{a}$). Then

$$F_k(z_1,z_2;C)=2N(\mathfrak{a})^k \sum_{\text{ideals }\mathfrak{b}} \sum^*_{\delta(\mu,\nu)=\mathfrak{a}\mathfrak{b}} \frac{1}{(\mu z_1+\nu)^k(\mu' z_2+\nu')^k}$$

(since for any (μ,ν) in the summation of (7), $\delta(\mu,\nu)$ is divisible by $\mathfrak{a}$)

$$=2N(\mathfrak{a})^k \sum_{\mathfrak{b}} \frac{1}{2N(\mathfrak{a}\mathfrak{b})^k} F_k^*(z_1,z_2;[\mathfrak{a}\mathfrak{b}]),$$

(where $[\mathfrak{a}\mathfrak{b}]$ is the ideal class containing $\mathfrak{a}\mathfrak{b}$)

$$=\sum_{\mathfrak{b}} N(\mathfrak{b})^{-k} F_k^*(z_1,z_2;C\cdot[\mathfrak{b}]).$$

Hence

$$F_k(z_1,z_2)=\sum_C F_k(z_1,z_2;C)$$

$$=\sum_{\mathfrak{b}}\sum_C F_k^*(z_1,z_2;C\cdot[\mathfrak{b}])\,N(\mathfrak{b})^{-k}$$

(the reversal being permitted since $\sum_C$ is finite)

$$=\sum_{\mathfrak{b}} N(\mathfrak{b})^{-k}\sum_C F_k^*(z_1,z_2;C)$$

(since, for each $\mathfrak{b}$, $C[\mathfrak{b}]$ runs over all ideal classes as C does)

$$=\zeta_K(k)\sum_C F_k^*(z_1,z_2;C).$$

Hence to prove Eq. (8) we must show that

$$\omega_0^*(z_1,z_2)=\sum_C F_k^*(z_1,z_2;C). \tag{11}$$

For each ideal class C, let

$$\omega_0^*(z_1,z_2;C)=\sum_{\substack{(a,b,\lambda)\text{ primitive}\\ \lambda\lambda'=ab\\ \delta(a,\lambda')\in C}} (a z_1 z_2+\lambda z_1+\lambda' z_2+b)^{-k}. \tag{12}$$

Clearly $\omega_0^*(z_1,z_2)=\sum_C \omega_0^*(z_1,z_2;C)$, so that (11) will follow if we show that, for each C,

$$\omega_0^*(z_1,z_2;C)=F_k^*(z_1,z_2;C). \tag{13}$$

Fix the class C and an ideal $\mathfrak{a}\in C$, and let

$$S=\{(a,b,\lambda)\in\mathbb{Z}\times\mathbb{Z}\times\mathcal{O}\,|\,(a,b,\lambda)\text{ primitive},\ \lambda\lambda'=ab,\ \delta(a,\lambda')\in C\},$$

$$T=\{(\mu,\nu)\in(\mathfrak{a}\times\mathfrak{a}-\{(0,0)\})/\mathcal{O}^*\,|\,\delta(\mu,\nu)=\mathfrak{a}\}$$

(recall that $\delta(\,,)$ denotes g.c.d.). We define a map

$$m\colon S\to T$$

as follows. Given $(a,b,\lambda)\in S$, one can write the quotient $a/\lambda'=\lambda/b$ as x/y for some $x,y\in\mathfrak{a}$. Then the ideals $\delta(x,y)$ and $\delta(a,\lambda')$ belong to the same class, i.e. $\delta(x,y)\in C$. Also $\delta(x,y)$ is divisible by $\mathfrak{a}$, and $\mathfrak{a}\in C$. Therefore $\delta(x,y)=(\sigma)\,\mathfrak{a}$ for some principal ideal (σ), and it follows that x and y are divisible by σ and that $\mu=x/\sigma$, $\nu=y/\sigma$ are in $\mathfrak{a}$ and have greatest common divisor $\mathfrak{a}$. On the other hand, it is clear that the two conditions $\mu/\nu=\lambda/b$ and $\delta(\mu,\nu)=\mathfrak{a}$ determine the pair (μ,ν) up to possible multiplication by a unit. We set $m(a,b,\lambda)=(\mu,\nu)$.

The map m is onto, for given μ,ν we choose an integer $b_1\in\mathbb{Z}$, $b_1\neq 0$ such that $\lambda_1=b_1\frac{\mu}{\nu}$ is in $\mathcal{O}$; then $a_1=N(\lambda_1)/b_1$ is in $\mathbb{Q}$. Multiplying a_1,b_1,λ_1 by the denominator of a_1, we obtain $(a_2,b_2,\lambda_2)\in\mathbb{Z}\times\mathbb{Z}\times\mathcal{O}$ with $a_2/\lambda_2'=\lambda_2/b_2=\mu/\nu$. Dividing (a_2,b_2,λ_2) by the largest rational integer r for which $(a,b,\lambda)=(a_2/r,b_2/r,\lambda_2/r)$ satisfies the same conditions, we obtain a primitive triple $(a,b,\lambda)\in S$ whose image under m is clearly (μ,ν).

Finally, the mapping m is two-to-one: $m(a_1,b_1,\lambda_1)=m(a_2,b_2,\lambda_2)$ if and only if $(a_1,b_1,\lambda_1)=\pm(a_2,b_2,\lambda_2)$. Indeed, $m(a_1,b_1,\lambda_1)=m(a_2,b_2,\lambda_2)$ implies $\lambda_1/b_1=\lambda_2/b_2$. Let b_0 be the greatest common divisor of b_1 and b_2 and write $\lambda_0=\frac{b_0}{b_1}\lambda_1$, $a_0=\lambda_0\lambda_0'/b_0$. Then $\lambda_0\in\mathcal{O}$ and $a_0\in\mathbb{Z}$, for we can choose $r,s\in\mathbb{Z}$ such that $b_0=rb_1+sb_2$, and then $\lambda_0=r\lambda_1+s\lambda_2$, $a_0=ra_1+sa_2$. Then $(a_1,b_1,\lambda_1)=\frac{b_1}{b_0}(a_0,b_0,\lambda_0)$, so (since (a_1,b_1,λ_1) is primitive) $b_1/b_0=\pm 1$. Similarly $b_2/b_0=\pm 1$ and so $(a_2,b_2,\lambda_2)=\pm(a_1,b_1,\lambda_1)$.

If $m(a,b,\lambda)=(\mu,\nu)$, then

$$a z_1 z_2+\lambda z_1+\lambda' z_2+b=\pm N(\mathfrak{a})^{-1}(\mu z_1+\nu)(\mu' z_2+\nu'). \tag{14}$$

Indeed, from $\mu/\nu=a/\lambda'=\lambda/b$ we deduce that

$$(\mu z_1+\nu)(\mu' z_2+\nu')=n(a z_1 z_2+\lambda z_1+\lambda' z_2+b)$$

for some $n\in\mathbb{Q}$, $n\neq 0$, and the primitivity of (a,b,λ) implies that n is an integer, and in fact that $|n|$ is the largest integer dividing $\delta(\mu\mu',\mu\nu',\nu\nu')$. But the greatest common divisor (in the sense of ideals) of $\mu\mu'$, $\mu\nu'$, $\mu'\nu$, and $\nu\nu'$ is

$$\begin{aligned}\delta(\mu\mu',\mu\nu',\mu'\nu,\nu\nu')&=\delta\big((\mu)\,\delta(\mu',\nu'),(\nu)\,\delta(\mu',\nu')\big)\\&=\delta(\mu,\nu)\,\delta(\mu',\nu')=\mathfrak{a}\,\mathfrak{a}'=\big(N(\mathfrak{a})\big),\end{aligned}$$

so $n=\pm N(\mathfrak{a})$. It follows immediately from (14) and the fact that m is two-to-one and surjective that

$$\begin{aligned}\omega_0^*(z_1,z_2;C)&=\sum_{(a,b,\lambda)\in S}(a z_1 z_2+\lambda z_1+\lambda' z_2+b)^{-k}\\&=2\sum_{(\mu,\nu)\in T}N(\mathfrak{a})^k(\mu z_1+\nu)^{-k}(\mu' z_2+\nu')^{-k}\\&=F_k^*(z_1,z_2;C).\end{aligned}$$

(iii) This property will be proved in § 2; here we only recall what it means for ω_m to be a cusp form for $SL_2\mathcal{O}$. A function F holomorphic on $\mathfrak{H}\times\mathfrak{H}$ and satisfying (2), satisfies in particular

$$F(z_1+\theta,z_2+\theta')=F(z_1,z_2)\qquad(\theta\in\mathcal{O}),$$

and therefore has a Fourier development of the form

$$F(z_1, z_2) = \sum_{\nu \in \mathfrak{d}^{-1}} a_\nu \, e^{2\pi i(\nu z_1 + \nu' z_2)}.$$

The regularity condition which a modular form $F(z_1, z_2)$ must satisfy is that $a_\nu = 0$ if $\nu < 0$ or $\nu' < 0$, i.e. that F has a Fourier development

$$F(z_1, z_2) = a_0 + \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} a_\nu \, e^{2\pi i(\nu z_1 + \nu' z_2)} \tag{15}$$

($\nu \gg 0$ means ν totally positive, i.e. $\nu > 0$ and $\nu' > 0$).

If F is a cusp form, then also $a_0 = 0$; an equivalent condition is that

$$F(z_1, z_2) = O(e^{-c\sqrt{y_1 y_2}}) \quad \text{as} \quad y_1 y_2 \to \infty \qquad (y_1 = \operatorname{Im} z_1,\ y_2 = \operatorname{Im} z_2)$$

$\Big($because, for $\nu \in \mathfrak{d}^{-1}$ totally positive, $\nu \nu' y_1 y_2 \geqq \frac{y_1 y_2}{D}$ so $\nu y_1 + \nu' y_2 \geqq 2\sqrt{\frac{y_1 y_2}{D}}$, $|e^{2\pi i(\nu z_1 + \nu' z_2)}| \leqq e^{-4\pi\sqrt{y_1 y_2/D}}\Big)$.

If the class number of K is 1, then this is the only condition that F must satisfy. In general, we require that, for each $W = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2 K$, the function

$$(F|W)(z_1, z_2) = (c z_1 + d)^{-k} (c' z_2 + d')^{-k} \, F\left(\frac{a z_1 + b}{c z_1 + d}, \frac{a' z_2 + b'}{c' z_2 + d'}\right)$$

satisfy a similar condition as $y_1 y_2 \to \infty$. By (2), $F|W$ satisfies

$$(F|W)(z_1 + \theta, z_2 + \theta') = (F|W)(z_1, z_2) \tag{16}$$

whenever $W \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix} W^{-1} \in SL_2 \mathcal{O}$. But $W \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix} W^{-1} = \begin{pmatrix} 1 - ac\theta & a^2\theta \\ -c^2\theta & 1 + ac\theta \end{pmatrix}$, so this condition is equivalent to the three conditions

$$\theta a^2 \in \mathcal{O}, \qquad \theta ac \in \mathcal{O}, \qquad \theta c^2 \in \mathcal{O},$$

i.e. the fractional ideal (θ^{-1}) divides (a^{-2}), $(a^{-1}c^{-1})$ and (c^{-2}). Hence (16) holds whenever $(\theta^{-1}) | \delta(a^{-2}, a^{-1}c^{-1}, c^{-2})$, where (as in the proof of ii)) $\delta(x_1, \ldots, x_r)$ denotes the greatest common divisor of the fractional ideals $(x_1), \ldots, (x_r)$. But $\delta(a^{-2}, a^{-1}c^{-1}, c^{-2}) = \delta(a^{-1}, c^{-1})^2$, so the condition is that $\theta \in \delta(a^{-1}, c^{-1})^{-2}$. Denote $\delta(a^{-1}, c^{-1})^{-2}$ by M. Then we have shown that (16) holds whenever $\theta \in M$, and it follows that $F|W$ has a Fourier expansion of the form

$$(F|W)(z_1, z_2) = \sum_{\nu \in M^*} a_\nu \, e^{2\pi i(\nu z_1 + \nu' z_2)}, \tag{17}$$

where $M^* = \{\nu \in K \mid Tr(\nu\theta) \in \mathbb{Z} \text{ for all } \theta \in M\}$ is the complementary module of M (here $M^* = \mathfrak{d}^{-1}\delta(a^{-1}, c^{-1})^2$). The condition that F be a modular form is that, for each $W \in SL_2 K$, the coefficients a_ν in (17) with $\nu < 0$ or $\nu' < 0$ vanish; if also $a_0 = 0$ in (17), i.e. if $F|W$ has an expansion

$$(F|W)(z_1, z_2) = \sum_{\substack{\nu \in M^* \\ \nu \gg 0}} a_\nu \, e^{2\pi i(\nu z_1 + \nu' z_2)} \tag{18}$$

for each $W \in SL_2 K$, then F is a cusp form. There are in fact only finitely many such conditions, since, as one easily checks, the condition on $W = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ that $F|W$ have a development of the form (18) depends only on the ideal class of the ideal $\delta(a, c)$.

Now take $F = \omega_m$, and expand

$$\omega_m(z_1, z_2) = \sum_{v \in \mathfrak{d}^{-1}} c_{mv}\, e^{2\pi i (v z_1 + v' z_2)}. \tag{19}$$

We will calculate the Fourier coefficients c_{mv} in § 2, and will see then that $c_{mv} = 0$ if $v < 0$ or $v' < 0$ and also that $c_{m0} = 0$ for $m > 0$, i.e. that ω_m for $m > 0$ does have a Fourier expansion in which only totally positive v occur. If $W = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2 K$, then, by the same argument used to prove i),

$$\begin{aligned}(\omega_m | W)(z_1, z_2) &= D^{k/2} (\gamma z_1 + \delta)^{-k} (\gamma' z_2 + \delta')^{-k} \sum_{\substack{M \in \mathscr{A} \\ \det M = -m}}' \phi_M(W z_1, W' z_2) \\ &= D^{k/2} \sum_{\substack{M \in \mathscr{A} \\ \det M = -m}} \phi_{W'^* M W}(z_1, z_2)^{k/2} \\ &= D^{k/2} \sum_{\substack{M \in \mathscr{A}_1 \\ \det M = -m}} \phi_M(z_1, z_2)^{k/2},\end{aligned}$$

where

$$\begin{aligned}\mathscr{A}_1 &= W'^* \mathscr{A} W \\ &= W'^{-1} \mathscr{A} W \\ &= \{M \in W'^{-1} \mathfrak{M}_2(\mathcal{O})\, W | M' = M^*\}.\end{aligned}$$

A typical matrix $M \in \mathscr{A}_1$ has the form $M = \begin{pmatrix} \theta & b\sqrt{D} \\ -a\sqrt{D} & \theta' \end{pmatrix}$ with $\theta \in K$, $a, b \in \mathbb{Q}$, and, writing $\theta = \lambda \sqrt{D}$, we obtain

$$(\omega_m | W)(z_1, z_2) = \sum_{\substack{(a, b, \lambda) \in L \\ \lambda\lambda' - ab = m/D}} (a z_1 z_2 + \lambda z_1 + \lambda' z_2 + b)^{-k}, \tag{20}$$

where $L \subset \mathbb{Q} \times \mathbb{Q} \times K$ is the lattice (i.e. free $\mathbb{Z}$-module of rank 4) of triples (a, b, λ) for which $W' \begin{pmatrix} \lambda\sqrt{D} & b\sqrt{D} \\ -a\sqrt{D} & -\lambda'\sqrt{D} \end{pmatrix} W^{-1} \in \mathfrak{M}_2\, \mathcal{O}$. We will not in fact calculate the Fourier expansion of $\omega_m | W$, but in view of the similarity between Eqs. (20) and (1) it will be clear that the method used in § 2 to find the Fourier development of ω_m applies equally to prove that $\omega_m | W$ has a Fourier series of the type (18).

(iv) This property is clear, since the summation in (1) is empty unless $-4m \equiv x^2 \pmod{D}$ for some x $\left(\text{write } \lambda = \dfrac{x + y\sqrt{D}}{2\sqrt{D}} \text{ in (1) and multiply by } -4D\right)$.

§ 2. The Fourier Coefficients of ω_m

We wish to evaluate the Fourier coefficients c_{mv} ($m \geqq 0$, $v \in \mathfrak{d}^{-1}$) defined by Eq. (19). We first break up the sum (1) into subsums corresponding to various values of a:

$$\omega_m(z_1, z_2) = \sum_{a\in\mathbb{Z}} \omega_m^a(z_1, z_2) = \omega_m^0(z_1, z_2) + 2\sum_{a=1}^{\infty} \omega_m^a(z_1, z_2),$$

where

$$\omega_m^a(z_1, z_2) = \sum_{\substack{b\in\mathbb{Z}\\ \lambda\in\mathfrak{d}^{-1}\\ \lambda\lambda'-ab=m/D}}' (a z_1 z_2 + \lambda z_1 + \lambda' z_2 + b)^{-k}. \tag{21}$$

The individual pieces ω_m^a are no longer modular forms of weight k, but they do satisfy the periodicity property $\omega_m^a(z_1+\theta, z_2+\theta') = \omega_m^a(z_1, z_2)$ $(\theta\in\mathcal{O})$, and hence each ω_m^a has a Fourier expansion

$$\omega_m^a(z_1, z_2) = \sum_{\nu\in\mathfrak{d}^{-1}} c_{m\nu}^a e^{2\pi i(\nu z_1 + \nu' z_2)}. \tag{22}$$

The Fourier coefficients of ω_m are then given by

$$c_{m\nu} = c_{m\nu}^0 + 2\sum_{a=1}^{\infty} c_{m\nu}^a. \tag{23}$$

We wish to compute the $c_{m\nu}^a$ and, in particular, to see that $c_{m\nu}^a = 0$ unless $\nu \gg 0$ or $\nu = m = 0$.

The computation of $c_{m\nu}^a$ is different according as $a=0$ or $a>0$.

Case 1. $a=0$.

The condition $\lambda\lambda' - ab = m/D$ now becomes just $\lambda\lambda' = m/D$, so $\omega_m^0(z_1, z_2) \equiv 0$ if $-m$ is not the norm of an element of $\mathcal{O}$. The summation on $b\in\mathbb{Z}$ is unrestricted. Thus (for $m\neq 0$)

$$\begin{aligned}\omega_m^0(z_1, z_2) &= \sum_{\substack{\lambda\in\mathfrak{d}^{-1}\\ \lambda\lambda'=m/D}} \sum_{b\in\mathbb{Z}} (\lambda z_1 + \lambda' z_2 + b)^{-k}\\ &= \sum_{\substack{\lambda\in\mathfrak{d}^{-1}\\ \lambda\lambda'=m/D}} h_k(\lambda z_1 + \lambda' z_2),\end{aligned} \tag{24}$$

where

$$h_k(t) = \sum_{b=-\infty}^{\infty} \frac{1}{(b+t)^k}. \tag{25}$$

The functions h_k can be expressed in trigonometric terms: $h_2(t) = \dfrac{\pi^2}{\sin^2 \pi t}$ and $h_k(t) = \dfrac{-1}{k-1}\dfrac{d}{dt} h_{k-1}(t)$ for $k \geqq 3$. Now

$$\pi^2 \csc^2 \pi t = -4\pi^2 \sum_{r=1}^{\infty} r e^{2\pi i r t} \qquad \text{for } t\in\mathfrak{H},$$

and by successive differentiation we obtain

$$h_k(t) = \frac{(2\pi i)^k}{(k-1)!} \sum_{r=1}^{\infty} r^{k-1} e^{2\pi i r t} \qquad (t\in\mathfrak{H}). \tag{26}$$

For convenience, we abbreviate

$$c_k = \frac{(2\pi i)^k}{(k-1)!}. \tag{27}$$

In Eq. (24), it follows from $\lambda\lambda' = m/D > 0$ that $\lambda \gg 0$ or $-\lambda \gg 0$. Since h_k is an even function (because k is even), we can write

$$\omega_m^0(z_1, z_2) = 2 \sum_{\substack{\lambda \in \mathfrak{d}^{-1} \\ \lambda \gg 0 \\ \lambda\lambda' = m/D}} h_k(\lambda z_1 + \lambda' z_2)$$

or, using (26),

$$\omega_m^0(z_1, z_2) = 2 c_k \sum_{\substack{\lambda \in \mathfrak{d}^{-1} \\ \lambda \gg 0 \\ \lambda\lambda' = m/D}} \sum_{r=1}^{\infty} r^{k-1} e^{2\pi i r(\lambda z_1 + \lambda' z_2)}.$$

Thus we have proved:

Proposition 1. *For $m > 0$, $\nu \in \mathfrak{d}^{-1}$, the Fourier coefficient $c_{m\nu}^0$ defined by (22) is zero unless $\nu \gg 0$ and $\nu = r\lambda$ with $r \in \mathbb{N}$, $\lambda \in \mathfrak{d}^{-1}$, $\lambda\lambda' = m/D$, in which case*

$$c_{m\nu}^0 = 2 c_k r^{k-1}. \tag{28}$$

$\left(\text{Notice that } \nu \text{ can be written as } r\lambda \text{ in at most one way, since necessarily } r = \sqrt{\nu\nu' \frac{D}{m}} \text{ and } \lambda = \nu/r.\right)$

The excluded case $m = 0$ is even simpler, since from $\lambda\lambda' - ab = m/D$, $a = 0$, $m = 0$ we deduce $\lambda = 0$, and hence

$$\omega_0^0(z_1, z_2) = \sum_{b \in \mathbb{Z}}' b^{-k} = 2\zeta(k).$$

Thus we can complete the above proposition by

$$c_{0\nu}^0 = \begin{cases} 2\zeta(k) & \text{if } \nu = 0, \\ 0 & \text{if } \nu \neq 0. \end{cases} \tag{29}$$

Case 2. $a > 0$.

We have

$$\omega_m^a(z_1, z_2) = \sum_{M \in S} \phi_M(z_1, z_2)^{k/2}, \tag{30}$$

where ϕ_M is defined by (3) and S is the set of matrices

$$S = \left\{ \begin{pmatrix} -\lambda & -b \\ a & \lambda' \end{pmatrix} \middle| \lambda \in \mathfrak{d}^{-1}, b \in \mathbb{Z}, \lambda\lambda' - ab = \frac{m}{D} \right\}.$$

For $M = \begin{pmatrix} -\lambda & -b \\ a & \lambda' \end{pmatrix} \in S$ and $\theta \in \mathcal{O}$, define $M_\theta \in S$ by

$$M_\theta = \begin{pmatrix} 1 & \theta' \\ 0 & 1 \end{pmatrix}^{-1} M \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\lambda - a\theta' & -b - \theta\lambda - \theta'\lambda' - a\theta\theta' \\ a & \lambda' + a\theta \end{pmatrix}.$$

Conversely, if $M_1=\begin{pmatrix}-\lambda_1 & -b_1\\ a & \lambda_1'\end{pmatrix}\in S$ with $\lambda_1\equiv\lambda \pmod{a\mathcal{O}}$, say $\lambda_1=\lambda+a\theta'$, then $b_1=(\lambda_1\lambda_1'-m/D)/a=b+\theta\lambda+\theta'\lambda'+a\theta\theta'$ and $M_1=M_\theta$. Therefore we can break up the sum (30) according to the values of $\lambda \pmod{a\mathcal{O}}$:

$$\omega_m^a(z_1,z_2)=\sum_{\lambda\in R}\sum_{\theta\in\mathcal{O}}\phi_{M(\lambda)_\theta}(z_1,z_2)^{k/2}, \tag{31}$$

where R is a set of representatives for the set of $\lambda\in\mathfrak{d}^{-1} \pmod{a\mathcal{O}}$ for which

$$N(\lambda\sqrt{D})\equiv -m \pmod{aD} \tag{32}$$

and, for each $\lambda\in R$,

$$M(\lambda)=\begin{pmatrix}-\lambda & -b\\ a & \lambda'\end{pmatrix},\qquad b=\frac{\lambda\lambda'-m/D}{a}. \tag{33}$$

By virtue of (4), we have $\phi_{M_\theta}(z_1,z_2)=\phi_M(z_1+\theta, z_2+\theta')$, so each inner sum in (31) has a Fourier development. Since R is finite, the determination of the Fourier coefficients of ω_m will be known as soon as we know those of the sums $\sum_\theta \phi_{M(\lambda)_\theta}(z_1,z_2)^{k/2}$. They are given by the following lemma, which will be proved at the end of the section.

Lemma 1. *Let* $M=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\mathfrak{M}_2\mathbb{R}$, $\alpha\delta-\beta\gamma=-\Delta\leqq 0$. *Then*

$$\sum_{\theta\in\mathcal{O}}\phi_M(z_1+\theta, z_2+\theta')^{k/2}=\sum_{\substack{\nu\in\mathfrak{d}^{-1}\\ \nu\gg 0}} c_k(\nu, M)\, e^{2\pi i(\nu z_1+\nu' z_2)} \tag{34}$$

with

$$c_k(\nu,M)=\begin{cases}\dfrac{1}{\gamma\sqrt{D}}\dfrac{(2\pi)^{k+1}}{(k-1)!}\left(\dfrac{N(\nu)}{\Delta}\right)^{\frac{k-1}{2}} e^{\frac{2\pi i}{\gamma}(\nu\delta-\nu'\alpha)} J_{k-1}\left(\dfrac{4\pi}{\gamma}\sqrt{N(\nu)\Delta}\right) & \text{if } \Delta>0,\\[2ex] \dfrac{1}{\gamma^k\sqrt{D}}\left(\dfrac{(2\pi)^k}{(k-1)!}\right)^2 N(\nu)^{k-1} e^{\frac{2\pi i}{\gamma}(\nu\delta-\nu'\alpha)} & \text{if } \Delta=0.\end{cases} \tag{35}$$

Here

$$J_{k-1}(t)=\sum_{r=0}^{\infty}\frac{(-1)^r(t/2)^{2r+k-1}}{r!(r+k-1)!} \tag{36}$$

is the Bessel function of order $k-1$.

Remark. The second formula in (35) is the limiting case of the first, since

$$\lim_{x\to 0}\left(J_{k-1}(x)/x^{k-1}\right)=\frac{2^{1-k}}{(k-1)!}.$$

Substituting (35) into Eq. (31), we obtain for the ν^{th} Fourier coefficient the formula:

Proposition 2. *For $m>0$, $\nu\in\mathfrak{d}^{-1}$, $a>0$, the Fourier coefficient $c^a_{m\nu}$ defined by (22) is zero unless $\nu \gg 0$ and is then given by*

$$c^a_{m\nu}=\frac{(2\pi)^{k+1}}{(k-1)!}\,\frac{D^{\frac{k}{2}-1}}{a}\left(\frac{N(\nu)}{m}\right)^{\frac{k-1}{2}} G_a(m,\nu)\,J_{k-1}\left(\frac{4\pi}{a}\sqrt{\frac{mN(\nu)}{D}}\right), \tag{37}$$

where J_{k-1} is the Bessel function of order $k-1$ and $G_a(m,\nu)$ the finite exponential sum

$$G_a(m,\nu)=\sum_{\substack{\lambda\in\mathfrak{d}^{-1}/a\mathcal{O}\\ \lambda\lambda'\equiv m/D\ (\mathrm{mod}\ a\mathbb{Z})}} e^{2\pi i\, Tr(\nu\lambda)/a}. \tag{38}$$

For $m=0$, we have

$$c^a_{0\nu}=\frac{(2\pi)^{2k}}{(k-1)!^2}\,\frac{D^{-1/2}}{a^k}\,N(\nu)^{k-1}\,G_a(0,\nu). \tag{39}$$

Putting the results of Propositions 1 and 2 into Eq. (23), we obtain the following theorem, which is the main result of this section.

Theorem 2. *For $m>0$, the Fourier coefficient $c_{m\nu}$ of $\omega_m(z_1,z_2)$ defined by Eq. (19) is given by*

$$c_{m\nu}=\frac{2(2\pi)^k}{(k-1)!}\left\{(-1)^{k/2}\sum_{\substack{r\in\mathbb{N}\\ r\mid\nu\sqrt{D}\\ N(\nu\sqrt{D}/r)=-m}} r^{k-1} + 2\pi D^{\frac{k}{2}-1}\left(\frac{N(\nu)}{m}\right)^{\frac{k-1}{2}}\sum_{a=1}^{\infty}\frac{1}{a}J_{k-1}\left(\frac{4\pi}{a}\sqrt{\frac{mN(\nu)}{D}}\right)G_a(m,\nu)\right\} \tag{40}$$

if $\nu\gg 0$ and is zero otherwise. If $m=0$, then

$$c_{0\nu}=\frac{2^{2k+1}\pi^{2k}}{(k-1)!^2\sqrt{D}}\,N(\nu)^{k-1}\sum_{a=1}^{\infty}\frac{G_a(0,\nu)}{a^k} \tag{41}$$

if $\nu\gg 0$,

$$c_{00}=2\zeta(k),$$

and $c_{m\nu}=0$ unless ν is zero or totally positive.

Notice the resemblance between Theorem 2 and the Hardy-Ramanujan-Rademacher partition formula, in which also the Fourier coefficients of a modular form are expressed by infinite series whose terms are products of a finite exponential sum and a Bessel function.

Before giving the proof of Lemma 1, we will look more carefully at the case $m=0$, where the Fourier coefficients are given by (41). We write χ for the Dirichlet character associated to the field K, i.e. $\chi(n)=(D/n)$.

Proposition 3. *For fixed $\nu\in\mathfrak{d}^{-1}$ the sum $G_a(0,\nu)$ defined by Eq. (38) with $m=0$ is a multiplicative function of a, i.e. $G_a(0,\nu)=\prod_i G_{q_i^{r_i}}(0,\nu)$ if $a=\prod q_i^{r_i}$ is the prime decomposition of a. For prime powers, $G_a(0,\nu)$ is given as follows:*

(i) *If $\chi(q)=-1$, so that $(q)=\mathfrak{q}$ is a prime ideal in $\mathcal{O}$, and if $\mathfrak{q}^\alpha$ is the largest power of $\mathfrak{q}$ dividing the integral ideal $(\nu)\mathfrak{d}$, then*

$$G_{q^r}(0,\nu)=\begin{cases} q^{2[r/2]} & \text{if } r\leqq 2\alpha+1,\\ 0 & \text{if } r\geqq 2\alpha+2.\end{cases} \tag{43a}$$

(ii) *If* $\chi(q)=0$, *so that* $(q)=\mathfrak{q}^2$, *and* $\mathfrak{q}^\alpha$ *is the largest power of* $\mathfrak{q}$ *dividing* $(v)\mathfrak{d}$, *then*

$$G_{q^r}(0, v)=\begin{cases} q^r & \text{if } r\leqq\alpha, \\ 0 & \text{if } r>\alpha. \end{cases} \tag{43b}$$

(iii) *If* $\chi(q)=+1$, *so that* $(q)=\mathfrak{q}\mathfrak{q}'$, *and* $\mathfrak{q}^\alpha, \mathfrak{q}'^\beta$ *are the largest powers of* $\mathfrak{q}$, $\mathfrak{q}'$ *dividing* $(v)\mathfrak{d}$, *then*

$$G_{q^r}(0, v)=\begin{cases} (r+1)\,q^{r-1}(q-1)+q^{r-1} & \text{if } r\leqq\min(\alpha,\beta) \\ (\min(\alpha,\beta)+1)\,q^{r-1}(q-1) & \text{if } \min(\alpha,\beta)<r\leqq\max(\alpha,\beta) \\ (\alpha+\beta+1-r)\,q^{r-1}(q-1)-q^{r-1} & \text{if } \max(\alpha,\beta)<r\leqq\alpha+\beta+1 \\ 0 & \text{if } r>\alpha+\beta+1. \end{cases} \tag{43c}$$

Proof. In (38), the equation $\lambda\lambda'\equiv 0 \pmod{a\mathbb{Z}}$ implies in particular $N(\lambda)\in\mathbb{Z}$, and, since $Tr(\lambda)\in\mathbb{Z}$ (by the definition of $\mathfrak{d}^{-1}$), $\lambda\in\mathcal{O}$. Thus

$$G_a(0, v)=\sum_{\substack{\lambda\in\mathcal{O}/a\mathcal{O} \\ \lambda\lambda'\equiv 0 \ (\mathrm{mod}\ a)}} e_a\big(Tr(v\lambda)\big), \tag{44}$$

where we use the standard abbreviation $e_a(n)=e^{2\pi i n/a}$. That this is multiplicative is almost obvious: if $a=a_1a_2$ with $(a_1,a_2)=1$, choose integers x_1, x_2 with $a_1x_1+a_2x_2=1$. Then

$$\begin{aligned} G_a(0, v)&=\sum_{\substack{\lambda_1 (\mathrm{mod}\ a_1) \\ \lambda_2 (\mathrm{mod}\ a_2) \\ \lambda_1\lambda_1'\equiv 0\ (\mathrm{mod}\ a_1) \\ \lambda_2\lambda_2'\equiv 0\ (\mathrm{mod}\ a_2)}} e_{a_1}\big(Tr(v\lambda_1)\big)\,e_{a_2}\big(Tr(v\lambda_2)\big) \\ &=G_{a_1}(0, v)\cdot G_{a_2}(0, v), \end{aligned}$$

since there is a one-to-one correspondence between the pairs (λ_1,λ_2) in this summation and the integers λ of (44) given by $\lambda\mapsto(x_2\lambda, x_1\lambda)$, $(\lambda_1,\lambda_2)\mapsto a_2\lambda_1+a_1\lambda_2$.

Now let $a=q^r$ with $\chi(q)=-1$. Then $\lambda\lambda'\equiv 0 \pmod{q^r}$ implies $q^{r-[r/2]}\mid\lambda$; writing $\lambda=q^{r-[r/2]}\mu$, we find

$$G_a(0, v)=\sum_{\mu\in\mathcal{O}/q^{[r/2]}\mathcal{O}} e_{q^{[r/2]}}\big(Tr(v\mu)\big).$$

If $q^{[r/2]}\mid v$, then each term of this sum is 1 and so $G_a(0, v)=|\mathcal{O}/q^{[r/2]}\mathcal{O}|=q^{2[r/2]}$; if not, then the permutation $\mu\mapsto\mu+x$ of the summation set (where $x\in\mathcal{O}$ is such that $Tr(xv)\not\equiv 0 \pmod{q^{[r/2]}}$) gives $G_a(0, v)=e_{q^{[r/2]}}\big(Tr(xv)\big)\,G_a(0, v)$, so $G_a(0, v)=0$.

Cases (ii) and (iii) are similar, although the details in the case of decomposable primes are more complicated.

Corollary 1. *Let* $v\in\mathfrak{d}^{-1}$, $\mathfrak{a}=(v)\mathfrak{d}$. *Then*

$$\sum_{a=1}^{\infty}\frac{G_a(0, v)}{a^s}=\frac{1}{L(s,\chi)}\sum_{\mathfrak{b}\mid\mathfrak{a}}\frac{1}{N(\mathfrak{b})^{s-1}}$$

for $s\in\mathbb{C}$ *with* $\mathrm{Re}(s)>1$; *here* $L(s,\chi)=\sum_{n=1}\frac{\chi(n)}{n^s}$ *is the L-series of the character* χ *and the summation is over all integral ideals dividing* $\mathfrak{a}$.

Proof. This just summarizes the contents of Proposition 3, for that proposition implies that $\sum G_a(0, v)a^{-s}$ ($\mathrm{Re}(s)$ large) admits an Euler product whose factor corresponding to a prime q is

$$(1+q^{-s})(1+q^{2(1-s)}+\cdots+q^{2\alpha(1-s)}) \qquad \text{if } (q)=\mathfrak{q},\ \mathfrak{q}^{\alpha}\,\|\,\mathfrak{a},$$

$$(1+q^{1-s}+\cdots+q^{\alpha(1-s)}) \qquad \text{if } (q)=\mathfrak{q}^2,\ \mathfrak{q}^{\alpha}\,\|\,\mathfrak{a},$$

$$(1-q^{-s})(1+q^{1-s}+\cdots+q^{\alpha(1-s)})(1+q^{1-s}+\cdots+q^{\beta(1-s)}) \qquad \text{if } (q)=\mathfrak{q}\mathfrak{q}',\ \mathfrak{q}^{\alpha}\mathfrak{q}'^{\beta}\,\|\,\mathfrak{a}.$$

Corollary 2. *The form $\omega_0(z_1, z_2)$ has the Fourier expansion*

$$\omega_0(z_1,z_2)=2\zeta(k)+\frac{1}{L(k,\chi)}\frac{2(2\pi)^{2k}D^{1-k}}{(k-1)!^2\sqrt{D}}\sum_{\substack{\nu\in\mathfrak{d}^{-1}\\ \nu\gg 0}}\Big(\sum_{\mathfrak{b}\mid(\nu)\mathfrak{d}}N(\mathfrak{b})^{k-1}\Big)e^{2\pi i(\nu z_1+\nu' z_2)}.$$

Proof. Immediate from Theorem 2 and Corollary 1.

Comparing this with the known Fourier expansion

$$F_k(z_1,z_2)=2\zeta_K(k)+\frac{2(2\pi)^{2k}}{(k-1)!^2\sqrt{D^{k-\frac{1}{2}}}}\sum_{\substack{\nu\in\mathfrak{d}^{-1}\\ \nu\gg 0}}\Big(\sum_{\mathfrak{b}\mid(\nu)\mathfrak{d}}N(\mathfrak{b})^{k-1}\Big)e^{2\pi i(\nu z_1+\nu' z_2)}$$

of the Hecke-Eisenstein series of weight k ([4], p. 385) and observing that $\zeta_K(k)=\zeta(k)L(k,\chi)$, we obtain a second proof of the identity

$$\omega_0(z_1,z_2)=\frac{\zeta(k)}{\zeta_K(k)}F_k(z_1,z_2)$$

which was proved by a more direct method in § 1.

Finally, we must prove Lemma 1.

Lemma 2. *Let* $\alpha\in\mathbb{R}$, $\alpha\geq 0$. *Then*

$$\sum_{\theta\in\mathcal{O}}\frac{1}{((z_1+\theta)(z_2+\theta')-\alpha)^k}=\sum_{\nu\in\mathfrak{d}^{-1}}j_{k-1}(\alpha,\nu)e^{2\pi i(\nu z_1+\nu' z_2)} \tag{45}$$

with Fourier coefficients $j_{k-1}(\alpha,\nu)$ $(\nu\gg 0)$ *given by*

$$j_{k-1}(\alpha,\nu)=\begin{cases}\dfrac{(2\pi)^{k+1}}{(k-1)!\sqrt{D}}\left(\dfrac{N(\nu)}{\alpha}\right)^{\frac{k-1}{2}}J_{k-1}(4\pi\sqrt{\alpha N(\nu)}) & \textit{if } \alpha>0,\\[2ex] \dfrac{(2\pi)^{2k}}{(k-1)!^2\sqrt{D}}N(\nu)^{k-1} & \textit{if } \alpha=0.\end{cases} \tag{46}$$

We observe that Lemma 1 follows immediately, since (in the notation of that lemma)

$$\phi_M(z_1,z_2)=\frac{1}{(\gamma z_1 z_2-\alpha z_1+\delta z_2-\beta)^2}$$

$$=\frac{1}{\gamma^2((z_1+\delta/\gamma)(z_2-\alpha/\gamma)-\Delta/\gamma^2)^2}$$

and hence

$$c_k(\nu,M)=\gamma^{-k}e^{2\pi i(\frac{\nu\delta}{\gamma}-\frac{\nu'\alpha}{\gamma})}j_{k-1}\left(\frac{\Delta}{\gamma^2},\nu\right).$$

Proof. The sum on the left-hand side of (45) is absolutely convergent since $k>2$, and is clearly invariant under translations T_θ: $(z_1, z_2)\mapsto(z_1+\theta, z_2+\theta')$ with $\theta\in\mathcal{O}$. It therefore possesses a Fourier expansion as in (45) with

$$j_{k-1}(\alpha, \nu)=\frac{1}{\sqrt{D}}\iint_A \sum_{\theta\in\mathcal{O}} \frac{1}{((z_1+\theta)(z_2+\theta')-\alpha)^k}\, e^{-2\pi i(\nu z_1+\nu' z_2)}\,dz_1\,dz_2,$$

where A is a subset of the plane $\{\operatorname{Im} z_1=C_1>0, \operatorname{Im} z_2=C_2>0\}$ which is a fundamental domain for the group of translations $\{T_\theta\}$ (the factor $1/\sqrt{D}$ enters because A has area $\sqrt{D}$). By virtue of the compactness of A and the absolute convergence of the sum, we may interchange $\iint_A$ and $\sum_\theta$ to get

$$j_{k-1}(\alpha, \nu)=\frac{1}{\sqrt{D}}\sum_{\theta\in\mathcal{O}} \iint_{(z_1, z_2)\in T_\theta A} \frac{1}{(z_1 z_2-\alpha)^k}\, e^{-2\pi i(\nu z_1+\nu' z_2)}\,dz_1\,dz_2.$$

But the domains $T_\theta A$ $(\theta\in\mathcal{O})$ exactly cover the plane $\{\operatorname{Im} z_1=C_1, \operatorname{Im} z_2=C_2\}$, so

$$j_{k-1}(\alpha, \nu)=\frac{1}{\sqrt{D}}\int_{\operatorname{Im} z_2=C_2} z_2^{-k} e^{-2\pi i\nu' z_2}\left(\int_{\operatorname{Im} z_1=C_1}\left(z_1-\frac{\alpha}{z_2}\right)^{-k} e^{-2\pi i\nu z_1}\,dz_1\right)dz_2. \quad (47)$$

In the inner integral, the integrand has its unique pole at $z_1=\alpha/z_2$; since $z_2\in\mathfrak{H}$ and $\alpha\geqq 0$, this pole is on $\mathbb{R}$ or in the lower half-plane, i.e. below the line of integration $\operatorname{Im} z_1=C_1$. Hence if $\nu\leqq 0$, we can deform the path of integration up to $+i\infty$ without crossing any poles, so the inner integral is 0 for each z_2. This proves that $j_{k-1}(\alpha, \nu)=0$ if $\nu\leqq 0$ or (by symmetry) if $\nu'\leqq 0$, i.e. $j_{k-1}(\alpha, \nu)=0$ unless $\nu\gg 0$. We therefore suppose that $\nu\gg 0$; then the inner integral in (47) equals

$$\begin{aligned}
&-2\pi i\operatorname{res}_{z_1=\alpha/z_2}\left(\left(z_1-\frac{\alpha}{z_2}\right)^{-k} e^{-2\pi i\nu z_1}\right)\\
&=-2\pi i\operatorname{res}_{t=0}\left(t^{-k} e^{-2\pi i\nu\alpha/z_2} e^{-2\pi i\nu t}\right)\\
&=\frac{(-2\pi i)^k \nu^{k-1}}{(k-1)!}\, e^{-2\pi i\nu\alpha/z_2},
\end{aligned}$$

and so

$$j_{k-1}(\alpha, \nu)=\frac{(-2\pi i)^k \nu^{k-1}}{(k-1)!\sqrt{D}}\int_{\operatorname{Im} z_2=C_2} z_2^{-k} e^{-2\pi i\nu' z_2-2\pi i\nu\alpha/z_2}\,dz_2.$$

If $\alpha=0$, then the same calculation as we just did for the z_1-integral gives $\frac{(-2\pi i)^k \nu'^{k-1}}{(k-1)!}$ for the integral and we obtain the second formula of (46); if $\alpha>0$, then the substitution $z_2=i\sqrt{\frac{\alpha\nu}{\nu'}}\,t$ gives

$$j_{k-1}(\alpha, \nu)=-i\frac{(2\pi)^k}{(k-1)!\sqrt{D}}\left(\frac{\nu\nu'}{\alpha}\right)^{\frac{k-1}{2}}\int_{C_2'-i\infty}^{C_2'+i\infty} t^{-k} e^{2\pi\sqrt{\alpha\nu\nu'}\left(t-\frac{1}{t}\right)}\,dt,$$

and the integral equals $2\pi i J_{k-1}(4\pi\sqrt{\alpha\nu\nu'})$ (standard integral representation of J_{k-1}).

§ 3. Poincaré Series for $\Gamma_0(D)$

In this section we recall the basic facts about Poincaré series and their Fourier expansions (a more detailed account can be found in [7], Chapter VIII) and introduce the linear combinations of Poincaré series which will be needed in § 4 for the formulation of the main result.

We use the following notation:

k (as previously) is a fixed even integer >2 (for a discussion of the case $k=2$ see Appendix 1).

For $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{R}$, $z \in \mathfrak{H}$,

$$\mu_A(z)=\left(\frac{dAz}{dz}\right)^{k/2}=(cz+d)^{-k}.$$

If furthermore f is any map $\mathfrak{H}\to\mathbb{C}$, then $f|A$ is the function defined by

$$\begin{aligned}(f|A)(z)&=\mu_A(z)\,f(Az)\\ &=(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).\end{aligned}$$

(Observe that $\mu_{AB}(z)=\mu_A(Bz)\,\mu_B(z)$, $f|AB=(f|A)|B$, $f|-A=f|A$.) For $c\in\mathbb{N}$, $h\in\mathbb{Z}/c\mathbb{Z}$, $e_c(h)$ denotes $e^{2\pi i h/c}$ (notice that h is taken modulo c; thus, if $(b,c)=1$, the symbol $e_c(a/b)$ denotes not $e^{2\pi i a/bc}$ but rather $e^{2\pi i x/c}$ where $bx\equiv a \pmod c$).

Let $\Gamma\subset SL_2\mathbb{Z}$ be a subgroup of finite index containing $-I$, and $\chi\colon\Gamma\to\{\pm1\}$ a character such that $\chi(-I)=1$. We denote by $S_k(\Gamma,\chi)$ the space of cusp forms for Γ of weight k and "reellem Nebentypus" (Hecke's terminology) χ. A function $f\in S_k(\Gamma,\chi)$ is a holomorphic function in $\mathfrak{H}$ satisfying

(i) $f|A=\chi(A)f$ for all $A\in\Gamma$.

(ii) f is holomorphic and vanishes at the cusps of Γ.

The second condition means the following. A *cusp* P of Γ is an equivalence class of points of $\mathbb{Q}\cup\{\infty\}$ under the action of Γ. For each cusp P we fix a matrix A_P transforming the cusp P to ∞ (i.e. such that $A_P^{-1}(\infty)\in P$). The *width* w_P of the cusp P is defined by

$$w_P=\left[\begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix}:\Gamma_P\right],\qquad \Gamma_P=A_P\Gamma A_P^{-1}\cap\begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \tag{48}$$

(this index is finite since $[SL_2\mathbb{Z}\colon\Gamma]<\infty$); thus $\Gamma_P=\begin{pmatrix} 1 & w_P\mathbb{Z} \\ 0 & 1 \end{pmatrix}=\left\{\begin{pmatrix} 1 & nw_P \\ 0 & 1 \end{pmatrix}\middle|\, n\in\mathbb{Z}\right\}$.

The width w_P is independent of the choice of A_P. Now for any $f\in S_k(\Gamma,\chi)$, the function $f|A_P^{-1}$ is periodic of period w_P; we require that it have a Fourier expansion of the form

$$(f|A_P^{-1})(z)=\sum_{n=1}^{\infty} a_n^P(f)\,e^{2\pi i n z/w_P}. \tag{49}$$

The numbers $a_n^P(f)$ are called the *Fourier coefficients of f at P*, and they do depend on the choice of A_P, but in a very trivial way – a different choice of A_P replaces $a_n^P(f)$ by $\zeta^n a_n^P(f)$, where ζ is some w_Pth root of unity.

If $f, g \in S_k(\Gamma, \chi)$, the *Petersson product* of f and g is defined as

$$(f, g) = \iint_{\mathscr{F}} f(z)\,\overline{g(z)}\, y^{k-2}\, dx\, dy, \tag{50}$$

where $z = x + iy \in \mathfrak{H}$ and $\mathscr{F}$ is some fundamental domain for the action of Γ on $\mathfrak{H}$; it is easily checked that the integral converges (for $k > 2$) and is independent of the choice of $\mathscr{F}$. The Petersson product makes $S_k(\Gamma, \chi)$ into a finite-dimensional Hilbert space; thus any linear map $S_k(\Gamma, \chi) \to \mathbb{C}$ can be represented as $f \mapsto (f, g)$ for some (unique) $g \in S_k(\Gamma, \chi)$. In particular, the map $f \mapsto a_n^P(f)$ sending a cusp form to its n^{th} Fourier coefficient at P can be represented in this way; the function g which achieves this is (up to a factor) the *Poincaré series*

$$G_n^P(z) = \tfrac{1}{2} \sum_{A \in \Gamma_P \setminus A_P \Gamma} \chi(A_P^{-1} A)\, \mu_A(z)\, e^{2\pi i n A z / w_P} \tag{51}$$

(summation over the orbits of the left action of Γ_P on $A_P \Gamma$; the series is easily checked to be convergent for $k > 2$ and independent of the choices of representatives A); indeed, it is an easy calculation to check that $G_n^P \in S_k(\Gamma, \chi)$ and satisfies

$$(f, G_n^P) = \frac{(k-2)!}{(4\pi n)^{k-1}}\, w_P^k\, a_n^P(f) \tag{52}$$

for all $f \in S_k(\Gamma, \chi)$.

Now $G_n^P \in S_k(\Gamma, \chi)$ and so has itself a Fourier expansion of the form (49) at each cusp Q of Γ. For simplicity we take $Q = (\infty)$ and suppose that the width w_∞ is 1 (this will be the case for the group we need); we can then choose $A_Q = 1$ and have $\Gamma_\infty = \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix}$ in (48). Thus G_n^P has a Fourier expansion

$$G_n^P(z) = \sum_{m=1}^{\infty} g_{nm}^P\, e^{2\pi i m z} \qquad (z \in \mathfrak{H}), \tag{53}$$

and we wish to calculate the coefficients g_{nm}^P.

Lemma. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{R}$, $\gamma > 0$. *Then we have the following Fourier expansion:*

$$\sum_{r=-\infty}^{\infty} \frac{e^{2\pi i \gamma [a(z+r)+b]/[c(z+r)+d]}}{[c(z+r)+d]^k} = \frac{2\pi(-1)^{k/2}}{c} \sum_{m=1}^{\infty} \left(\frac{m}{\gamma}\right)^{\frac{k-1}{2}} J_{k-1}\left(\frac{4\pi}{c}\sqrt{m\gamma}\right) e^{\frac{2\pi i}{c}(\gamma a + m d)}\, e^{2\pi i m z}, \qquad (z \in \mathfrak{H}) \tag{54}$$

where $J_{k-1}(t)$ is the Bessel function of order $k-1$ (Eq. (36)).

Proof. It suffices to treat the case $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, i.e.

$$\sum_{r=-\infty}^{\infty} \frac{e^{-2\pi i \gamma/(z+r)}}{(z+r)^k} = 2\pi(-1)^{k/2} \sum_{m=1}^{\infty} \left(\frac{m}{\gamma}\right)^{\frac{k-1}{2}} J_{k-1}(4\pi\sqrt{m\gamma})\, e^{2\pi i m z}, \tag{55}$$

since Eq. (54) then follows on replacing z by $z+\frac{d}{c}$ and γ by γ/c^2 and multiplying both sides of the resulting equation by $c^{-k}e^{2\pi i\gamma a/c}$. To prove (55), we observe that the series on the left converges absolutely and uniformly in $\mathfrak{H}$ (for $k>2$) and is periodic with period 1, so equals $\sum c_m e^{2\pi imz}$ for some c_m; then

$$c_m = \int_{iC}^{iC+1} e^{-2\pi imz}\Big(\sum_{r\in\mathbb{Z}}(z+r)^{-k}e^{-2\pi i\gamma/(z+r)}\Big)dz$$

$$= \int_{iC-\infty}^{iC+\infty} e^{-2\pi imz} z^{-k} e^{-2\pi i\gamma/z}\,dz$$

(applying the usual Poisson summation trick); the integral can be evaluated as in proof of Lemma 2 of §2 and equals

$$2\pi i^k (m/\gamma)^{\frac{k-1}{2}} J_{k-1}(4\pi\sqrt{m\gamma}).$$

Now consider (51). Two matrices $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ and $A'=\begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix}$ in $A_P\Gamma$ are left equivalent under Γ_P iff $(c'\,d')=(c\,d)$; thus the sum is over all rows $(c\,d)$ which occur as the bottom rows of matrices of $A_P\Gamma$, the whole matrix $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ then being determined (mod Γ_P) by the conditions $ad-bc=1$, $A_P^{-1}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma$. Also $(c\,d)$ occurs iff $(d,c)=1$ and $-\frac{d}{c}\in P$. Thus

$$G_n^P(z)=\tfrac{1}{2}\sum_{(c,d)=1,\ -\frac{d}{c}\in P}\chi\left(A_P^{-1}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right)(cz+d)^{-k}e^{2\pi inw_P^{-1}(az+b)/(cz+d)}.$$

We break up the sum into the terms with $c=0$ and twice the sum of terms with $c>0$. Clearly $c=0$ can only occur if $P=\infty$, and then d must be ±1, $\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\pm I$, $w_P=w_\infty=1$; thus the terms with $c=0$ contribute $\delta_{P\infty}e^{2\pi inz}$ ($\delta_{P\infty}$ = Kronecker delta). To study the terms with $c>0$, we observe that $\Gamma\supset\Gamma_\infty=\begin{pmatrix}1 & \mathbb{Z}\\ 0 & 1\end{pmatrix}$ and hence $\begin{pmatrix}* & *\\ c & d\end{pmatrix}\in A_P\Gamma\Rightarrow\begin{pmatrix}* & *\\ c & d+rc\end{pmatrix}=\begin{pmatrix}* & *\\ c & d\end{pmatrix}\begin{pmatrix}1 & r\\ 0 & 1\end{pmatrix}\in A_P\Gamma$ for all $r\in\mathbb{Z}$, i.e. the condition $\frac{-d}{c}\in P$ only depends on $d(\bmod c)$. Therefore the terms with $c>0$ yield

$$\sum_{c=1}^{\infty}\ \sideset{}{^*}\sum_{d(\bmod c)}\chi\left(A_P^{-1}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right)\sum_{r\in\mathbb{Z}}\frac{e^{2\pi inw_P^{-1}[a(z+r)+b]/[c(z+r)+d]}}{[c(z+r)+d]^k},$$

where the inner summation is over all $d\,(\bmod c)$ satisfying $(d,c)=1$, $\frac{-d}{c}\in P$ and we have made some choice of matrix $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in A_P\Gamma$ with lower row $(c\,d)$. The Fourier expansion of the inner sum is now given by the lemma, and we obtain

Proposition. *The Poincaré series* $G_n^P(z)$ *has a Fourier expansion of the form* (53) *with*

$$g_{nm}^P = \delta_{P\infty}\,\delta_{nm} + 2\pi(-1)^{k/2}\left(\frac{m w_P}{n}\right)^{\frac{k-1}{2}} \sum_{c=1}^{\infty} H_c^P(n,m)\, J_{k-1}\left(\frac{4\pi}{c}\sqrt{\frac{mn}{w_P}}\right) \tag{56}$$

with

$$H_c^P(n,m) = \frac{1}{c} \sum_{d \,(\mathrm{mod}\, c)}^{*} \chi\left(A_P^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) e^{2\pi i c^{-1}(n a w_P^{-1} + md)}, \tag{57}$$

where the summation is over all $d \,(\mathrm{mod}\, c)$ *such that* $(d,c)=1$, $\frac{-d}{c} \in P$, *and where* a, b *satisfy* $A_P^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

For example, if $\Gamma = SL_2\mathbb{Z}$, $\chi = \chi_0$ the trivial character, then

$$H_c^\infty(n,m) = \frac{1}{c} \sum_{\substack{d \,(\mathrm{mod}\, c) \\ (d,c)=1}} e_c(nd^{-1} + md)$$

is a Kloosterman sum.

We now return to our quadratic field K of discriminant D and take

$$\Gamma = \Gamma_0(D) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{Z} \mid c \equiv 0 \,(\mathrm{mod}\, D) \right\}$$

and

$$\chi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \varepsilon(a) = \varepsilon(d) \qquad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(D)\right),$$

where $\varepsilon = \varepsilon_D$ is the fundamental character of K with $\varepsilon(p) = (D/p)$ for $p \nmid 2D$. The space $S_k(\Gamma, \chi)$ is usually denoted $S(D, k, \varepsilon)$.

For the rest of the paper we suppose

$$D \equiv 1 \,(\mathrm{mod}\, 4),$$

or (equivalently) D square-free; this simplifies the formalism and proofs. In particular, it is easy to check that, for x/y, $x'/y' \in \mathbb{Q} \cup \{\infty\}$ with $(x', y') = (x, y) = 1$, the equation

$$\frac{x'}{y'} = \frac{ax+by}{cx+dy}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(D)$$

can be solved if and only if $(y', D) = (y, D)$. The equivalence classes of $\mathbb{Q} \cup \{\infty\}$ modulo $\Gamma_0(D)$ are thus described by the positive divisors D_1 of D. Let the cusp P be given by D_1 and write $D_2 = D/D_1$; then $(D_1, D_2) = 1$ since D is square-free, and we can find $p, q \in \mathbb{Z}$ such that $pD_1 + qD_2 = 1$; we choose

$$A_P = \begin{pmatrix} D_2 & -p \\ D_1 & q \end{pmatrix} \in SL_2\mathbb{Z}. \tag{58}$$

The cusp P is easily checked to have width $w_P = D_2$. We will denote the cusp P simply by D_1; thus for $f \in S(D, k, \varepsilon)$ and $D_1 \mid D$ we have the Fourier expansion

$$(f \mid A_{D_1}^{-1})(z) = \sum_{n=1}^{\infty} a_n^{D_1}(f)\, e^{2\pi i n z / D_2}, \tag{59}$$

the coefficients $a_n^{D_1}(f)$ being independent of the choice of p, q in (58) and given by

$$a_n^{D_1}(f)=\frac{(4\pi n)^{k-1}}{(k-2)!}D_2^{-k}(f, G_n^{D_1}) \tag{60}$$

with $G_n^{D_1}$ defined by (51). By the proposition, we have

$$G_n^{D_1}(z)=\sum_{m=1}^{\infty} g_{nm}^{D_1} e^{2\pi imz},$$

$$g_{nm}^{D_1}=\delta_{D_1 D}\delta_{nm}+2\pi(-1)^{k/2}\left(\frac{mD_2}{n}\right)^{\frac{k-1}{2}}\sum_{\substack{c=1\\(c,D)=D_1}}^{\infty} H_c^{D_1}(n,m)J_{k-1}\left(\frac{4\pi}{c}\sqrt{\frac{mn}{D_2}}\right),$$

$$H_c^{D_1}(n,m)=\frac{1}{c}\sum_{\substack{d \,(\mathrm{mod}\, c)\\(d,c)=1}}\chi\left(A_P^{-1}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right)e^{2\pi ic^{-1}(na/D_2+md)}.$$

Now $A_P^{-1}\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}aq+pc & bq+dp\\ -aD_1+cD_2 & -bD_1+dD_2\end{pmatrix}$ will be in $\Gamma_0(D)$ only if $D_2|a$ (since $D_1|c$), so a is determined (mod cD_2) by

$$ad\equiv 1 \,(\mathrm{mod}\, c), \qquad D_2|a.$$

Then

$$\begin{aligned}\chi\left(A_P^{-1}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right)=\varepsilon(aq+pc)&=\left(\frac{aq+pc}{D}\right)\\ &=\left(\frac{aq+pc}{D_1}\right)\left(\frac{aq+pc}{D_2}\right)\\ &=\left(\frac{aq}{D_1}\right)\left(\frac{pc}{D_2}\right)\\ &=\left(\frac{a/D_2}{D_1}\right)\left(\frac{c/D_1}{D_2}\right)\\ &=\left(\frac{dD_2}{D_1}\right)\left(\frac{c/D_1}{D_2}\right)\\ &=\left(\frac{-d}{D_1}\right)\left(\frac{c}{D_2}\right)\end{aligned}$$

where in the last line we have used quadratic reciprocity and $D_1D_2\equiv 1 \,(\mathrm{mod}\, 4)$ to set $\left(\frac{D_2}{D_1}\right)\left(\frac{D_1}{D_2}\right)=\left(\frac{-1}{D_1}\right)$. Therefore for $(c, D)=D_1$,

$$H_c^{D_1}(n,m)=\frac{1}{c}\left(\frac{c}{D_2}\right)\sum_{\substack{d \,(\mathrm{mod}\, c)\\(d,c)=1}}\left(\frac{-d}{D_1}\right)e_c(nD_2^{-1}d^{-1}+md) \tag{61}$$

(note that D_2 and d are prime to c, so $e_c(nD_2^{-1}d^{-1})$ makes sense).

Finally, we introduce certain linear combinations of the Poincaré series:

Definition. Let $D\equiv 1 \pmod 4$ be square-free and positive and n a positive integer. We set

$$G_n(z)=\sum_{\substack{D=D_1D_2\\ D_2\mid n}} \psi(D_2)D_2^{-k}\, G^{D_1}_{n/D_2}(z) \qquad (z\in\mathfrak{H}), \tag{62}$$

where

$$\psi(D_2)=\begin{cases} \left(\dfrac{D_1}{D_2}\right)\sqrt{D_2} & \text{if } D_1\equiv D_2\equiv 1 \pmod 4\\[2mm] -i\left(\dfrac{D_1}{D_2}\right)\sqrt{D_2} & \text{if } D_1\equiv D_2\equiv 3 \pmod 4. \end{cases} \tag{63}$$

Thus $G_n(z)$ is a linear combination of Poincaré series at certain of the cusps of $\Gamma_0(D)$. Notice that the coefficient $\psi(D_2)$ is just the Gauss sum

$$\psi(D_2)=\sum_{x \,(\mathrm{mod}\, D_2)} \left(\frac{x}{D_2}\right) e_{D_2}(-D_1x);$$

it is also easy to check that, if $D_2=D_2'D_2''$ divides D (in which case $(D_2', D_2'')=1$),

$$\psi(D_2)=\psi(D_2')\psi(D_2''). \tag{64}$$

Finally, from the formula above for the Fourier coefficients of $g^{D_1}_{nm}$ we obtain the Fourier expansion of $G_n(z)$:

$$G_n(z)=\sum_{m=1}^{\infty} g_{nm}\, e^{2\pi i m z} \tag{65}$$

with

$$\begin{aligned} g_{nm}&=\sum_{\substack{D=D_1D_2\\ D_2\mid n}} \psi(D_2)D_2^{-k}\, g^{D_1}_{n/D_2\, m}\\ &=\delta_{nm}+2\pi(-1)^{k/2}\left(\frac{m}{n}\right)^{\frac{k-1}{2}} \sum_{\substack{D=D_1D_2\\ D_2\mid n}} \frac{\psi(D_2)}{D_2} \sum_{\substack{c=1\\ (c,D)=D_1}}^{\infty} H^{D_1}_c\left(\frac{n}{D_2}, m\right) J_{k-1}\left(\frac{4\pi}{cD_2}\sqrt{nm}\right)\\ &=\delta_{nm}+2\pi(-1)^{k/2}\left(\frac{m}{n}\right)^{\frac{k-1}{2}} \sum_{b=1}^{\infty} H_b(n,m)\, J_{k-1}\left(\frac{4\pi}{bD}\sqrt{nm}\right), \end{aligned} \tag{66}$$

where

$$H_b(n,m)=\sum_{\substack{D=D_1D_2\\ D_2\mid n\\ (b,D_2)=1}} \frac{\psi(D_2)}{D_2} H^{D_1}_{bD_1}\left(\frac{n}{D_2}, m\right), \tag{67}$$

$H^{D_1}_{bD_1}$ being given by (61).

§ 4. The Form $\Omega(z_1, z_2; \tau)$

As before we fix a real quadratic field $K=\mathbb{Q}(\sqrt{D})$ ($D\equiv 1 \pmod 4$ squarefree) and even integer $k>2$. We define a function of three variables

$$\Omega(z_1, z_2; \tau)=\sum_{m=1}^{\infty} m^{k-1}\, \omega_m(z_1, z_2)\, e^{2\pi i m\tau} \qquad (z_1, z_2, \tau\in\mathfrak{H}), \tag{68}$$

where the $\omega_m(z_1, z_2)$ are the forms defined by Eq. (1). The series converges absolutely. It follows from the results of § 1 that, for fixed $\tau \in \mathfrak{H}$, $\Omega(z_1, z_2; \tau)$ is a Hilbert cusp form for $SL_2\mathcal{O}$ of weight k with respect to the variables z_1, z_2. Our goal is to show that, for fixed $z_1, z_2 \in \mathfrak{H}$, $\Omega(z_1, z_2; \tau)$ is a cusp form for $\Gamma_0(D)$ of weight k and Nebentypus $(\ /D)$ with respect to the variable τ. We will do this by proving an identity which expresses Ω as a linear combination of the functions $G_n(\tau) \in S(D, k, \varepsilon)$ constructed in the preceding section:

Theorem 3. *For all* $z_1, z_2, \tau \in \mathfrak{H}$, *the identity*

$$\Omega(z_1, z_2; \tau) = \sum_{n=1}^{\infty} n^{k-1}\, \omega_n^0(z_1, z_2)\, G_n(\tau) \tag{69}$$

holds.

Proof. We will expand both sides as triple Fourier series. Of course, the definition of $\Omega(z_1, z_2; \tau)$ already gives the Fourier series of the left-hand side of (69) with respect to the variable τ; its Fourier development with respect to z_1, z_2 is given by Theorem 2 of § 2, which tells us that

$$\Omega(z_1, z_2; \tau) = \sum_{\substack{m \in \mathbb{Z} \\ m > 0}} \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} m^{k-1} c_{m\nu}\, e^{2\pi i m\tau} e^{2\pi i(\nu z_1 + \nu' z_2)} \tag{70}$$

with $c_{m\nu}$ as in (40).

As to the right-hand side of (69), we recall that the function $\omega_n^0(z_1, z_2)$ is defined by (24) and has the Fourier expansion

$$\begin{aligned}
\omega_n^0(z_1, z_2) &= 2c_k \sum_{\substack{\lambda \in \mathfrak{d}^{-1} \\ \lambda \gg 0 \\ D\lambda\lambda' = n}} \sum_{r=1}^{\infty} r^{k-1} e^{2\pi i(r\lambda z_1 + r\lambda' z_2)} \\
&= 2c_k \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} \Big(\sum_{\substack{r \mid \nu \\ D\nu\nu' = nr^2}} r^{k-1} \Big) e^{2\pi i(\nu z_1 + \nu' z_2)}
\end{aligned}$$

with c_k the constant (27); here the inner sum is over all natural numbers r such that $\frac{1}{r}\nu \in \mathfrak{d}^{-1}$ and $N\left(\frac{1}{r}\nu\right) = \frac{n}{D}$, and contains at most one summand. On the other hand, $G_n(\tau)$ has the Fourier development (65), so the right-hand side of Eq. (69) equals

$$\begin{aligned}
&\sum_{m=1}^{\infty} \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} \Big(2c_k \sum_{n=1}^{\infty} n^{k-1} g_{nm} \sum_{\substack{r \mid \nu \\ D\nu\nu' = r^2 n}} r^{k-1} \Big) e^{2\pi i(\nu z_1 + \nu' z_2)} e^{2\pi i m\tau} \\
&= 2c_k \sum_{m=1}^{\infty} \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} \left(\sum_{r \mid \nu} \left(\frac{D\nu\nu'}{r} \right)^{k-1} g_{\frac{D\nu\nu'}{r^2}, m} \right) e^{2\pi i(\nu z_1 + \nu' z_2)} e^{2\pi i m\tau}.
\end{aligned}$$

Comparing this with (70), we see that we must prove

$$m^{k-1} c_{m\nu} = 2c_k \sum_{r \mid \nu} \left(\frac{D\nu\nu'}{r} \right)^{k-1} g_{\frac{D\nu\nu'}{r^2}, m} \tag{71}$$

for $m\in\mathbb{Z}$, $m>0$, $\nu\in\mathfrak{d}^{-1}$, $\nu\gg 0$. Substituting for $c_{m\nu}$ and g_{nm} from Eqs. (40) and (66), respectively, we see that the identity to be proved is

$$m^{k-1}\sum_{\substack{r|\nu\\ \nu\nu' D=mr^2}} r^{k-1}+2\pi(-1)^{\frac{k}{2}} D^{\frac{k}{2}-1}(m\nu\nu')^{\frac{k-1}{2}}\sum_{a=1}^{\infty}\frac{1}{a}G_a(m,\nu)\,J_{k-1}\left(\frac{4\pi}{a}\sqrt{\frac{m\nu\nu'}{D}}\right)$$
$$=\sum_{\substack{r|\nu\\ D\nu\nu'=r^2 m}}\left(\frac{D\nu\nu'}{r}\right)^{k-1}$$
$$+2\pi(-1)^{\frac{k}{2}}(m\nu\nu' D)^{\frac{k-1}{2}}\sum_{r|\nu}\sum_{b=1}^{\infty}H_b\left(\frac{D\nu\nu'}{r^2},m\right)J_{k-1}\left(\frac{4\pi}{br}\sqrt{\frac{m\nu\nu'}{D}}\right).$$

The first terms on the two sides of this identity are plainly equal, and comparing the coefficients of $J_{k-1}\left(\frac{4\pi}{a}\sqrt{\frac{m\nu\nu'}{D}}\right)$ on the two sides of the equation, we find that the theorem will follow once we have proved the following identity between finite exponential sums:

Proposition. *For $a, m\in\mathbb{Z}$, $\nu\in\mathfrak{d}^{-1}$, $a>0$,*

$$\frac{1}{a\sqrt{D}}G_a(m,\nu)=\sum_{\substack{r|\nu\\ r|a}}H_{a/r}\left(\frac{D\nu\nu'}{r^2},m\right). \tag{72}$$

Here $G_a(m,\nu)$ is the sum defined by (38) *and $H_b(n,m)$ the sum defined by* (67), (61) *and* (63).

It is convenient to write μ for $\nu\sqrt{D}$, so that $\mu\in\mathcal{O}$; then

$$G_a(m,\nu)=G_a\left(m,\frac{\mu}{\sqrt{D}}\right)=\sum_{\substack{\lambda\,(\mathrm{mod}\,a\mathfrak{d})\\ N(\lambda)\equiv -m\,(\mathrm{mod}\,aD)}} e_{aD}(Tr\,\lambda\mu) \tag{73}$$

(sum over $\lambda\in\mathcal{O}/a\mathfrak{d}$) and we wish to show that this equals

$$\begin{aligned} &a\sqrt{D}\sum_{\substack{r|\mu\\ r|a}}H_{a/r}\left(-N\left(\frac{\mu}{r}\right),m\right)\\ &=a\sqrt{D}\sum_{\substack{r|\mu\\ r|a}}\ \sum_{\substack{D=D_1D_2\\ D_2|N(\mu/r)\\ (a/r,D_2)=1}}\frac{\psi(D_2)}{D_2}H^{D_1}_{\frac{a}{r}D_1}\left(-N(\mu)/r^2D_2,m\right). \end{aligned} \tag{74}$$

Now both expressions (73) and (74) are clearly periodic in m with period aD, so to prove them equal, it suffices to show the equality of their finite Fourier transforms. Thus we must multiply both expressions by $e_{aD}(-hm)$ and sum over $m\,(\mathrm{mod}\,aD)$, i.e. we must show that, for every $h\in\mathbb{Z}/aD\mathbb{Z}$,

$$\begin{aligned} &\sum_{m=1}^{aD}e_{aD}(-hm)\sum_{\substack{\lambda\,(\mathrm{mod}\,a\mathfrak{d})\\ N(\lambda)\equiv -m\,(\mathrm{mod}\,aD)}}e_{aD}(Tr\,\lambda\mu)\\ &=a\sqrt{D}\sum_r\sum_{D_2}\frac{\psi(D_2)}{D_2}\sum_{m=1}^{aD}e_{aD}(-hm)\,H^{D_1}_{\frac{a}{r}D_1}\left(-\frac{\mu\mu'}{r^2D_2},m\right), \end{aligned} \tag{75}$$

where the conditions of summation on r and D_2 are the same as in (74). The left-hand side of (75) clearly equals

$$\sum_{\lambda \,(\mathrm{mod}\, a\mathfrak{d})} e_{aD}(hN(\lambda)+Tr(\mu\lambda)).$$

As to the right-hand side, we observe that $H_c^{D_1}(n,m)$, considered as a function of m, is a linear combination of terms ζ^m with $\zeta=e_c(d)$ a *primitive* c^{th} root of unity; therefore for $c\,|\,aD$, the sum $\sum_{m\,(\mathrm{mod}\, aD)} e_{aD}(-hm)\,H_c^{D_1}(n,m)$ is 0 unless $e_{aD}(h)$ is a primitive c^{th} root of unity, i.e. unless (h,aD) equals aD/c, in which case it equals $aD/c(c/D_2)(-d/D_1)e_c(nD_2^{-1}d^{-1})$ with d defined by $h/aD=d/c$ (cf. (61)). Hence the inner sum in (75) can only be different from zero if

(i) $(h,aD)=rD_2$.

On the other hand the conditions on r, D_2 are that $D_1D_2=D$, $(a/r,D_2)=1$, so $(a,rD_2)=r$ and hence

(ii) $(a,h)=(a,h,aD)=(a,rD_2)=r$.

From (i) and (ii) we see that r and D_2 are determined by h: given h, we must set

$$r=(a,h),\qquad D_2=\frac{(h,aD)}{(h,a)}. \tag{76}$$

These values of r and D_2 automatically satisfy $r\,|\,a$, $D_2\,|\,D$, $(a/r,D_2)=1$, but must still satisfy $r\,|\,\mu$, $D_2\,|\,N(\mu/r)$ in order for the right-hand side of (75) to be non-zero. Thus the identity we have to prove now reads:

Lemma 1. *Let* $a>0$, $\mu\in\mathcal{O}$, $h\in\mathbb{Z}$, *and define integers* r, D_2, D_1 *by* (76) *and* $D_1=D/D_2$. *Then*

$$\sum_{\lambda\in\mathcal{O}/a\mathfrak{d}} e_{aD}(h\lambda\lambda'+\mu\lambda+\mu'\lambda')$$
$$=\begin{cases} a\sqrt{D}\,\dfrac{\psi(D_2)}{D_2}\,rD_2\left(\dfrac{aD_1/r}{D_2}\right)\left(\dfrac{-h/rD_2}{D_1}\right)e_{aD_1/r}\left(-\dfrac{\mu\mu'}{r^2D_2}(h/r)^{-1}\right) & \text{if } r\,|\,\mu,\ D_2\,|\,N\left(\dfrac{\mu}{r}\right),\\ 0 & \text{otherwise.}\end{cases} \tag{77}$$

Proof. We first show that the left-hand side is zero unless $r\,|\,\mu$, $D_2r^2\,|\,N(\mu)$.

If we replace λ by $\lambda+\frac{a}{r}\tau\sqrt{D}$ $(\tau\in\mathcal{O})$ in the summation, then

$$hN(\lambda)\mapsto hN\left(\lambda+\frac{a}{r}\tau\sqrt{D}\right)=hN(\lambda)+aD\frac{h}{r}\,Tr\left(\frac{\lambda'\tau}{\sqrt{D}}\right)-aD\frac{a}{r}\frac{h}{r}N(\tau)$$
$$\equiv hN(\lambda)\ (\mathrm{mod}\ aD)$$

and

$$Tr(\mu\lambda)\mapsto Tr(\mu\lambda)+\frac{aD}{r}\,Tr\left(\frac{\mu\tau}{\sqrt{D}}\right),$$

so $e_{aD}(h\lambda\lambda'+\mu\lambda+\mu'\lambda')$ is multiplied by the factor $e_r\left(Tr\frac{\mu\tau}{\sqrt{D}}\right)$ independent of λ. Therefore the sum on the left-hand side of (77) equals $e_r\left(Tr\frac{\mu\tau}{\sqrt{D}}\right)$ times itself, and can only be different from 0 if $Tr\left(\frac{\mu}{r\sqrt{D}}\tau\right)\in\mathbb{Z}$ for all $\tau\in\mathcal{O}$, i.e. if $\frac{\mu}{r\sqrt{D}}\in\mathfrak{d}^{-1}$, $r|\mu$.

If we repeat the argument with $\tau\in\mathfrak{a}^{-1}$ instead of $\tau\in\mathcal{O}$, $\mathfrak{a}$ being the ideal with $\mathfrak{a}^2=(D_2)$ ($D_2|D$ is a product of ramified primes), then from $D_2|h/r$ we again see that $\lambda\mapsto\lambda+(a/r)\tau\sqrt{D}$ does not change $hN(\lambda)$ (mod aD) and that therefore we must have

$$e_r\left(Tr\frac{\mu\tau}{\sqrt{D}}\right)=1 \qquad (\tau\in\mathfrak{a}^{-1})$$

if the left-hand side of (77) is different from zero; this implies that

$$\frac{\mu}{r\sqrt{D}}\in\mathfrak{a}\mathfrak{d}^{-1}, \qquad \frac{\mu}{r}\in\mathfrak{a}, \qquad D_2|N\left(\frac{\mu}{r}\right).$$

We can therefore suppose that the conditions $r|\mu$, $D_2|N(\mu/r)$ hold. Then a, h and μ are now all divisible by r, and we see easily that both sides of (77) are r^2 times their value when a, h, μ and r are replaced by a/r, h/r, μ/r and 1. It therefore suffices to prove Lemma 1 in the case $r=1$. We thus assume that $(h,a)=1$ and that $D_2=(h,D)$ divides $\mu\mu'$, and have to prove the identity

$$\sum_{\lambda\in\mathcal{O}/a\mathfrak{d}} e_{aD}(h\lambda\lambda'+\mu\lambda+\mu'\lambda') = a\sqrt{D}\,\psi(D_2)\left(\frac{aD_1}{D_2}\right)\left(\frac{-h/D_2}{D_1}\right)e_{aD_1}\left(-\frac{N(\mu)}{D_2}h^{-1}\right) \tag{78}$$

(here h^{-1} makes sense because $(h,aD_1)=1$). Because $D_2|h$ and $D_2|\mu\mu'$, we have

$$\begin{aligned} e_{aD}(hN(\lambda)+Tr(\mu\lambda)) &= e_{aD_1}\left(\frac{h}{D_2}N(\lambda)+Tr\frac{\mu}{D_2}\lambda\right)\\ &= e_{aD_1}\left(\frac{h}{D_2}N(\lambda+h^{-1}\mu')\right)e_{aD_1}\left(-h^{-1}\frac{N(\mu)}{D_2}\right)\end{aligned}$$

with $\frac{h}{D_2}\in\mathbb{Z}$, $\frac{\mu}{D_2}\in\mathfrak{d}^{-1}$. Therefore, multiplying both sides of (78) by $e_{aD_1}\left(\frac{N(\mu)}{D_2}h^{-1}\right)$ and replacing λ by $\lambda-h^{-1}\mu'$ (mod aD_1), h by h/D_2 and aD_1 by b, we find as the identity to be proved

$$\sum_{\lambda\in\mathcal{O}/a\mathfrak{d}} e_b(h\lambda\lambda')=a\sqrt{D}\,\psi(D_2)\left(\frac{b}{D_2}\right)\left(\frac{-h}{D_1}\right) \tag{79}$$

where $(h,b)=1$, $(b,D)=D_1$, $D/D_1=D_2$, $b/D_1=a$. It is convenient to replace the summation in (79) by one over $\mathcal{O}/aD_1$; it is clear that this multiplies its value by D_1/D_2 (both sums are multiples of a sum over $\mathcal{O}/a\mathfrak{a}$ with $\mathfrak{a}^2=(D_1)$). We have thus reduced the complicated identity (77) to the simpler one given in the following lemma:

Lemma 2. *Let b be a natural number, h an integer prime to b. Set $D_1=(b,D)$, $D_2=D/D_1$. Then*

$$\frac{1}{b}\sum_{\lambda\,(\mathrm{mod}\,b)} e_b(hN(\lambda))=\begin{cases}\left(\frac{b/D_1}{D_2}\right)\left(\frac{h}{D_1}\right)\sqrt{D_1} & \text{if } D_1\equiv 1\ (\mathrm{mod}\ 4),\\ i\left(\frac{b/D_1}{D_2}\right)\left(\frac{h}{D_1}\right)\sqrt{D_1} & \text{if } D_1\equiv 3\ (\mathrm{mod}\ 4),\end{cases} \tag{80}$$

where the summation runs over integers $\lambda\in\mathcal{O}$ modulo the principal ideal (b).

Proof. Denote the left-hand side of (80) by $C(h/b)$. Thus $C(h/b)$ depends on the class of h in $(\mathbb{Z}/b\mathbb{Z})^*$ modulo squares. It is easily checked that $C(h/b)$ has the multiplicative property

$$C\left(\frac{h}{b}\right)=C\left(\frac{hb''}{b'}\right)C\left(\frac{hb'}{b''}\right),\qquad b=b'b'',\ (b',b'')=1.$$

The right-hand side of (80) has the same property, since (setting $D_1'=(b',D)$, $D_2'=D/D_1'$ and similarly for b'')

$$\left(\frac{b'/D_1'}{D_2'}\right)\left(\frac{hb''}{D_1'}\right)\sqrt{\pm D_1'}\left(\frac{b''/D_1''}{D_2''}\right)\left(\frac{hb'}{D_1''}\right)\sqrt{\pm D_1''}$$

$$=\left(\frac{b'}{D_2'/D_1''}\right)\left(\frac{b''}{D_2''/D_1'}\right)\left(\frac{h}{D_1'D_1''}\right)\left(\frac{D_1'}{D_2'}\right)\left(\frac{D_1''}{D_2''}\right)\sqrt{\pm D_1'}\sqrt{\pm D_1''}$$

(where $\sqrt{\pm n}$ is $+\sqrt{|n|}$ or $i\sqrt{|n|}$ depending whether $n=1$ or $3\ (\mathrm{mod}\ 4)$)

$$=\left(\frac{b'}{D_2}\right)\left(\frac{b''}{D_2}\right)\left(\frac{h}{D_1}\right)\left(\frac{D_1}{D_2}\right)\sqrt{\pm D_1}=\left(\frac{h}{D_1}\right)\left(\frac{b/D_1}{D_2}\right)\sqrt{\pm D_1}$$

as we see after a short calculation. Because both sides of (80) behave multiplicatively, it suffices to prove Lemma 2 for $b=q^\beta$ a prime power. Clearly

$$C\left(\frac{h}{b}\right)=\frac{1}{b}\sum_{n=1}^{b} N_b(n)\,e_b(hn), \tag{81}$$

where

$$N_b(n)=\#\{\lambda\in\mathcal{O}/b\mid N(\lambda)\equiv n\ (\mathrm{mod}\ b)\}, \tag{82}$$

and the above multiplicativity property can be stated simply $N_{b'b''}(n)=N_{b'}(n)N_{b''}(n)$ for $(b',b'')=1$. To evaluate $C(h/b)$, therefore, we have to find a closed formula for $N_b(n)$ when $b=q^\lambda$. There are three cases, according to the value of (D/q); we summarize the results in a lemma.

Lemma 3. *Let $b=q^\lambda$ (q prime), $n=n_0q^\nu$ with $\nu\geqq 0$, $q\nmid n_0$. Then*

(i) *If $(D/q)=+1$,*

$$N_b(n)=\begin{cases}(\nu+1)q^{\lambda-1}(q-1) & \text{if } \nu<\lambda,\\ (\lambda+1)q^\lambda-\lambda q^{\lambda-1} & \text{if } \nu\geqq\lambda.\end{cases} \tag{83a}$$

(ii) *If $(D/q)=-1$,*

$$N_b(n)=\begin{cases}q^{\lambda-1}(q+1) & \text{if } \nu<\lambda,\ \nu \text{ even},\\ 0 & \text{if } \nu<\lambda,\ \nu \text{ odd},\\ q^\lambda & \text{if } \nu\geqq\lambda,\ \lambda \text{ even},\\ q^{\lambda-1} & \text{if } \nu\geqq\lambda,\ \lambda \text{ odd}.\end{cases} \tag{83b}$$

(iii) *If* $q\,|\,D$, $D_2=D/q$,

$$N_b(n)=\begin{cases}\left(1+\left(\frac{n_0}{q}\right)\right)q^\lambda & \text{if } \nu<\lambda,\ \nu \text{ even},\\ \left(1+\left(\frac{-n_0D_2}{q}\right)\right)q^\lambda & \text{if } \nu<\lambda,\ \nu \text{ odd},\\ q^\lambda & \text{if } \nu\geqq\lambda.\end{cases} \tag{83c}$$

The proof of the lemma is straightforward and tedious and will be omitted.

We now substitute (83) into (81). If $b=q^\lambda$, $q\nmid D$, then $N_n(b)$ only depends on the largest power q^ν of q dividing n; hence

$$C\left(\frac{h}{b}\right)=\frac{1}{b}\sum_{\nu=0}^{\lambda} N_b(q^\nu)\sum_{\substack{n=1\\ q^\nu\|n}}^{b} e_b(hn),$$

and the inner sum equals 1 if $\nu=\lambda$, -1 if $\nu=\lambda-1$, 0 if $\nu<\lambda-1$. Hence

$$C\left(\frac{h}{b}\right)=\frac{1}{b}\,[N_b(q^\lambda)-N_b(q^{\lambda-1})].$$

If $(D/q)=+1$, then we see by (83a) that this equals 1, in agreement with (80) (here $D_1=1$, $D_2=D$, $(b/D_2)=+1$). If $(D/q)=-1$, then using (83b) we find $C(h/b)=(-1)^\lambda=(b/D)$, again in accordance with Lemma 2.

If $q|D$, $D_2=D/q$, then (81) and (83c) give

$$C\left(\frac{h}{q^\lambda}\right)=\sum_{\substack{0\leqq\nu<\lambda\\ \nu\text{ even}}}\ \sum_{\substack{n_0\,(\mathrm{mod}\ q^{\lambda-\nu})\\ (n_0,q)=1}}\left(\frac{n_0}{q}\right)e_{q^{\lambda-\nu}}(hn_0)$$
$$+\left(\frac{-D_2}{q}\right)\sum_{\substack{0\leqq\nu<\lambda\\ \nu\text{ odd}}}\ \sum_{\substack{n_0\,(\mathrm{mod}\ q^{\lambda-\nu})\\ (n_0,q)=1}}\left(\frac{n_0}{q}\right)e_{q^{\lambda-\nu}}(hn_0)\qquad (q\nmid h).$$

The inner sum in both terms is zero if $\nu<\lambda-1$, as we see on replacing n_0 by n_0+q. Thus only the terms $\nu=\lambda-1$ contribute, and we find

$$C\left(\frac{h}{q^\lambda}\right)=\begin{cases}\left(\frac{h}{q}\right)\sqrt{\pm q} & \text{if } \lambda \text{ is odd},\\ \left(\frac{-D_2}{q}\right)\left(\frac{h}{q}\right)\sqrt{\pm q} & \text{if } \lambda \text{ is even},\end{cases} \tag{84}$$

where again $\sqrt{\pm q}$ is $\sqrt{q}$ or $i\sqrt{q}$ depending whether $q\equiv 1$ or 3 modulo 4. In the notation of Lemma 2, $D_1=q$, $(-D_2/q)=(q/D_2)$, so (84) can be stated as

$$C\left(\frac{h}{b}\right)=\left(\frac{q}{D_2}\right)^{\lambda-1}\left(\frac{h}{q}\right)\sqrt{\pm q}$$
$$=\left(\frac{b/D_1}{D_2}\right)\left(\frac{h}{D_1}\right)\sqrt{\pm D_1}\qquad (b=q^\lambda,\ q\nmid h),$$

in agreement with Eq. (80). This completes the proof of Lemma 2 and hence of Theorem 3.

§ 5. The Doi-Naganuma Map

In the last section we proved the identity

$$\sum_{m=1}^{\infty} m^{k-1}\,\omega_m(z_1,z_2)\,e^{2\pi i m\tau} = \sum_{m=1}^{\infty} m^{k-1}\,\omega_m^0(z_1,z_2)\,G_m(\tau)$$

relating the Hilbert modular forms of weight k constructed in § 1 to the Poincaré series of weight k and "Nebentypus" constructed in § 3. Depending whether we read this identity from left to right or from right to left, we can deduce two statements asserting that some infinite series defines a cusp form: on the one hand, since $G_m(\tau)$ is a cusp form of Nebentypus, we have

1. *For each point* $(z_1,z_2)\in\mathfrak{H}\times\mathfrak{H}$, *the series*

$$\sum_{m=1}^{\infty} m^{k-1}\,\omega_m(z_1,z_2)\,e^{2\pi i m\tau},$$

considered as a function of τ, *defines a cusp form for* $\Gamma_0(D)$ *of weight* k *and "Nebentypus";*

on the other hand, since we know that the ω_m are Hilbert cusp forms, we have

2. *For each point* $\tau\in\mathfrak{H}$, *the series*

$$\sum_{m=1}^{\infty} m^{k-1}\,G_m(\tau)\,\omega_m^0(z_1,z_2),$$

considered as a function of (z_1,z_2), *defines a cusp form of weight* k *for the Hilbert modular group* $SL_2(\mathcal{O})$.

The first of these assertions has several interesting consequences; for example, using Hecke's well-known estimate for the Fourier coefficients of cusp forms, we have the corollary,

For fixed $(z_1,z_2)\in\mathfrak{H}\times\mathfrak{H}$, *the absolute values of* $\omega_m(z_1,z_2)$ $(m=1,2,\ldots)$ *satisfy*

$$|\omega_m(z_1,z_2)| = O(m^{k/2})$$

as $m\to\infty$ $\Big($indeed, using the recently proved Petersson conjecture we can improve $\frac{k}{2}$ to $\frac{k-1}{2}+\varepsilon\Big)$. A considerably more interesting corollary of 1. is obtained by integrating the forms ω_m along certain curves. If the field K has a unit of negative norm, say $\varepsilon>0>\varepsilon'$, then the function $y^k\,\omega_m(\varepsilon z,\varepsilon'\bar{z})$ $(z=x+iy\in\mathfrak{H})$ is invariant with respect to $SL_2\mathbb{Z}$, and so we can consider the numbers

$$a_m = m^{k-1}\int_{\mathscr{F}} \omega_m(\varepsilon z,\varepsilon'\bar{z})\,y^{k-2}\,dx\,dy,$$

where $\mathscr{F}=\{z\,|\,|z|\geqq 1, |\mathrm{Re}(z)|<\frac{1}{2}\}$ is a fundamental domain for the action of $SL_2\mathbb{Z}$ on $\mathfrak{H}$. It follows from 1. above that the series $\sum_{m=1}^{\infty} a_m e^{2\pi i m\tau}$ is a cusp form of weight k and Nebentypus for the group $\Gamma_0(D)$ and character $(\ /D)$. On the other hand, the a_m can be evaluated explicitly and turn out to be (apart from a trivial factor) integers expressible as a finite sum of class numbers of imaginary quadratic fields. In this way we are able to construct a large number of cusp forms of Nebentypus whose Fourier coefficients are explicitly given by certain expressions involving class numbers. That these class number expressions really are Fourier coefficients of cusp forms had been conjectured (in the case $k=2$) by Hirzebruch and the author on the basis of another interpretation of them as the intersection numbers of certain curves on the Hilbert modular surface $\mathfrak{H}\times\mathfrak{H}/SL_2\mathcal{O}$. (This conjecture was the original motivation for studying the forms ω_m.) The proofs of these relations will be given in a subsequent paper.

In this section, however, we will investigate the significance of the second assertion above. We fix, as usual, an even weight $k>2$ and discriminant $D\equiv 1 \pmod 4$. We denote by S_1 the vector space of cusp forms of weight k and Nebentypus for $\Gamma_0(D)$ and the character $\varepsilon=(D/\ )$ (so $S_1=S_k(\Gamma_0(D),\varepsilon)=S(D,k,\varepsilon)$ in the notation of § 3) and by S_2 the space of cusp forms of weight k for the Hilbert modular group $SL_2(\mathcal{O})$. (The subscripts 1 and 2 refer to the number of variables of the functions involved.) The assertion of the theorem in Section 3 is then that the function $\Omega(z_1,z_2;\tau)$ defined by (68) is in $S_1\otimes S_2$. On the other hand, as discussed in § 3, the Petersson product on S_1 is a non-degenerate scalar product and provides a canonical identification of S_1 with its dual $S_1^*=\operatorname{Hom}(S_1,\mathbb{C})$. Using this, we can identify $S_1\otimes S_2$ with $\operatorname{Hom}(S_1,S_2)$ and thus think of Ω as a map from S_1 to S_2, namely the map sending a cusp form $f=f(\tau)\in S_1$ to

$$(f,\Omega)_\tau=\int_{\mathcal{F}} f(\tau)\,\overline{\Omega(z_1,z_2;\tau)}\,y^{k-2}\,dx\,dy$$

($\tau=x+iy$, $\mathcal{F}=$fundamental domain for $\mathfrak{H}/\Gamma_0(D)$).

On the other hand, we have expressed Ω as a linear combination of Poincaré series and therefore can easily evaluate its Petersson product with any cusp form. Let $f\in S_1$ and let $a_n^{D_1}(f)$ $(n=1,2,\dots;D_1|D)$ be its Fourier coefficients at the various cusps as defined by Eq. (59). Recalling the definition (62) of G_n as a linear combination of Poincaré series and the basic property (60) of Poincaré series, we find

$$\begin{aligned} n^{k-1}(f,G_n)&=n^{k-1}\sum_{\substack{D=D_1D_2\\ D_2|n}}\psi(D_2)\,D_2^{-k}(G^{D_1}_{n/D_2},f)\\ &=\frac{(k-2)!}{(4\pi)^{k-1}}\sum_{\substack{D=D_1D_2\\ D_2|n}}\psi(D_2)\,D_2^{k-1}\,a^{D_1}_{n/D_2}(f)\end{aligned}$$

and hence

$$\begin{aligned}(f,\Omega)_\tau&=\sum_{n=1}^{\infty}n^{k-1}(f,G_n)\,\omega_n^0(z_1,z_2)\\ &=\frac{(k-2)!}{(4\pi)^{k-1}}\sum_{D=D_1D_2}\psi(D_2)D_2^{k-1}\sum_{m=1}^{\infty}a_m^{D_1}(f)\,\omega^0_{mD_2}(z_1,z_2).\end{aligned}$$

Into this expression we substitute the Fourier expansion

$$\omega_m^0(z_1,z_2)=\frac{2(2\pi)^k}{(k-1)!}(-1)^{k/2}\sum_{\substack{\lambda\in\mathfrak{d}^{-1}\\ \lambda\gg 0\\ N(\lambda)=m/D}}\sum_{r=1}^{\infty}r^{k-1}e^{2\pi i(r\lambda z_1+r\lambda' z_2)}$$

obtained in § 2, and obtain the following theorem.

Theorem 4. *Let $f \in S_k(\Gamma_0(D), \varepsilon)$ be a cusp form of weight k and "Nebentypus" and $a_n^{D_1}(f)$ $(n \in \mathbb{N}, D_1 \mid D)$ its Fourier coefficients as defined by* (59). *For each integral ideal $\mathfrak{a}$ of the field K, define*

$$c(\mathfrak{a}) = \sum_{r \mid \mathfrak{a}} r^{k-1} \sum_{D_2 \mid (D, N(\mathfrak{a})/r^2)} \psi(D_2) D_2^{k-1} a_{N(\mathfrak{a})/r^2 D_2}^{D_1}(f), \tag{85}$$

where the first sum is over all natural numbers r dividing $\mathfrak{a}$, the second sum over all positive integers dividing D and $N(\mathfrak{a})/r^2$, $D_1 = D/D_2$ and $\psi(D_2)$ is defined by (63). *Then the series*

$$F(z_1, z_2) = \sum_{\substack{v \in \mathfrak{d}^{-1} \\ v \gg 0}} c((v)\mathfrak{d}) e^{2\pi i(v z_1 + v' z_2)} \tag{86}$$

is a cusp form of weight k for the Hilbert modular group. The map $f \mapsto F$ defines a linear map

$$\iota: S_1 \to S_2$$

which is, up to a factor, the map sending a cusp form $f(\tau)$ to its Petersson product (with respect to τ) with $\Omega(z_1, z_2; \tau)$.

We have written the Fourier expansion of F in the form (86), with coefficients $c((v)\mathfrak{d})$ depending on the integral ideal $(v)\mathfrak{d}$ rather than simply c_v as previously, first of all because this form puts into evidence the invariance of F under

$$(z_1, z) \mapsto (\varepsilon z_1, \varepsilon' z_2) \qquad (\varepsilon \text{ a totally positive unit}) \tag{87}$$

as well as under the translations

$$(z_1, z_2) \mapsto (z_1 + \theta, z_2 + \theta'), \tag{88}$$

and, secondly, because this is the appropriate form for writing down the Mellin transform of F. Namely, to a cusp form with Fourier expansion (86) we associate the Dirichlet series

$$\Phi(s) = \sum_{\mathfrak{a}} c(\mathfrak{a}) N(\mathfrak{a})^{-s} \tag{89}$$

(at least if the class number of K is 1, in which case every integral ideal $\mathfrak{a}$ can be written as $(v)\mathfrak{d}$ with $v \in \mathfrak{d}^{-1}$, $v \gg 0$); then Φ and F are related in the same way as are Dirichlet series and modular forms in one variable, namely that the invariance of F under the modular group is reflected by a functional equation of the function $\Phi(s)$.

We now describe the relationship of Theorem 4 to a construction of K. Doi and H. Naganuma, in which these Dirichlet series play a basic role. The original paper of Doi and Naganuma [2] treats the case of modular forms of "Haupt-typus" (i.e. trivial character); we will in fact describe the modification for forms of "Nebentypus" given by Naganuma [8]. Let

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z} \tag{90}$$

be a cusp form of weight k for $\Gamma_0(D)$, where $D = p$ is now assumed to be a prime of class number 1. We assume that f is an eigenfunction of all the Hecke operators T_n, normalized with $a_1 = 1$ (so that a_n is just the eigenvalue of f under T_n); then the associated Dirichlet series

$$\varphi(s)=\sum_{n=1}^{\infty} a_n n^{-s} \qquad (\operatorname{Re} s \gg 1) \tag{91}$$

has an Euler product expansion of the form

$$\varphi(s)=\prod_q \left(1-a_q q^{-s}+\left(\frac{q}{p}\right) q^{k-1-2s}\right)^{-1} \tag{92}$$

(product over all rational primes q). The series

$$\varphi^\rho(s)=\sum_{n=1}^{\infty} a_n^\rho n^{-s}, \qquad a_n^\rho=\bar{a}_n,$$

whose coefficients are the complex conjugates of those of $\varphi(s)$, will then be the Mellin transform of the modular form $f^\rho(z)=\sum \bar{a}_n e^{2\pi i n z}$. Thanks to Hecke [5], one has the following information about the eigenvalues a_q of $\varphi(s)$ in (92):

$$\textit{for } (q/p)=1, \qquad a_q \textit{ is real;} \tag{93a}$$

$$\textit{for } (q/p)=-1, \qquad a_q \textit{ is pure imaginary;} \tag{93b}$$

$$\textit{for } \quad q=p, \qquad |a_q|=p^{(k-1)/2} \tag{93c}$$

This gives us the possibility of writing

$$\Phi(s)=\varphi(s)\,\varphi^\rho(s)=\prod_{\mathfrak{q}} \left(1-b(\mathfrak{q})\,N(\mathfrak{q})^{-s}+N(\mathfrak{q})^{k-1-2s}\right)^{-1}, \tag{94}$$

where the product is extended over all prime ideals $\mathfrak{q}$ of $\mathbb{Q}(\sqrt{p})$ and the coefficients $b(\mathfrak{q})$ are defined by

$$b(\mathfrak{q})=a_q \qquad \textit{if } \mathfrak{q}\mathfrak{q}'=(q),\ (q/p)=1, \tag{95a}$$

$$b(\mathfrak{q})=a_q^2+2q^{k-1} \qquad \textit{if } \mathfrak{q}=(q),\ (q/p)=-1, \tag{95b}$$

$$b(\mathfrak{q})=a_p+\bar{a}_p \qquad \textit{if } \mathfrak{q}^2=(p). \tag{95c}$$

Indeed, for decomposable primes q we know by (93a) that $a_q=\bar{a}_q$, so the factor $(1-a_q q^{-s}+q^{k-1-2s})^{-1}$ occurs twice in $\varphi(s)\varphi^\rho(s)$, and since there are two prime ideals with norm q, it also occurs twice in the product in (94). For inert primes q, (93b) tells us that $\bar{a}_q=-a_q$, so the corresponding local factor in $\varphi(s)\,\varphi^\rho(s)$ is

$$\begin{aligned}
&(1-a_q q^{-s}-q^{k-1-2s})^{-1}(1+a_q q^{-s}-q^{k-1-2s})^{-1}\\
&\quad=(1-a_q^2 q^{-2s}-2q^{k-1-2s}+q^{2k-2-4s})^{-1}\\
&\quad=\left(1-b(\mathfrak{q})\,N(\mathfrak{q})^{-s}+N(\mathfrak{q})^{k-1-2s}\right)^{-1}
\end{aligned}$$

with $\mathfrak{q}=(q)$, $b(\mathfrak{q})$ as in (95b), $N(\mathfrak{q})=q^2$. Finally, for the ramified prime p, $(p)=\mathfrak{q}^2$, we deduce from (93c) that the local factor in $\varphi(s)\,\varphi^\rho(s)$ is

$$\begin{aligned}
(1-a_p p^{-s})^{-1}(1-\bar{a}_p p^{-s})^{-1}&=\left(1-(a_p+\bar{a}_p)p^{-s}+p^{k-1-2s}\right)^{-1}\\
&=\left(1-b(\mathfrak{q})\,N(\mathfrak{q})^{-s}+N(\mathfrak{q})^{k-1-2s}\right)^{-1}.
\end{aligned}$$

We now extend the definition of $b(\cdot)$ to all integral ideals, defining first $b(\mathfrak{q}^r)$ ($\mathfrak{q}$ prime, $r\in\mathbb{N}$) by

$$1+\sum_{r=1}^{\infty} b(\mathfrak{q}^r)\,t^r=\left(1-b(\mathfrak{q})\,t+N(\mathfrak{q})^{k-1}\,t^2\right)^{-1} \tag{96}$$

(as formal power series in $\mathbb{R}[[t]]$) and then requiring b to be multiplicative, i.e.

$$b(\mathfrak{q}_1^{r_1}\dots\mathfrak{q}_n^{r_n})=b(\mathfrak{q}_1^{r_1})\dots b(\mathfrak{q}_n^{r_n}). \tag{97}$$

Then clearly

$$\Phi(s)=\prod_{\mathfrak{q}}(1+b(\mathfrak{q})N(\mathfrak{q})^{-s}+b(\mathfrak{q}^2)N(\mathfrak{q})^{-2s}+\cdots)=\sum_{\mathfrak{a}} b(\mathfrak{a})N(\mathfrak{a})^{-s}$$

(sum over all integral ideals). We can now state the result of Doi and Naganuma:

Theorem (Naganuma [8]). *Let* $p\equiv 1 \pmod 4$ *be a prime with* $h(p)=1$, *and* $f\in S_k(\Gamma_0(p),(\ /p))$ *a normalized eigenfunction of the Hecke operators with the Fourier expansion as in* (90). *Define numbers* $b(\mathfrak{a})\in\mathbb{R}$ *for all integral ideals* $\mathfrak{a}$ *of* $\mathbb{Q}(\sqrt{p})$ *by* (95)–(97). *Then the function*

$$F(z_1,z_2)=\sum_{\substack{\nu\in\mathfrak{d}^{-1}\\ \nu\gg 0}} b((\nu)\mathfrak{d})\,e^{2\pi i(\nu z_1+\nu' z_2)} \qquad (z_1,z_2\in\mathfrak{H}) \tag{98}$$

satisfies the functional equation

$$F\left(\frac{-1}{z_1},\frac{-1}{z_2}\right)=z_1^k z_2^k F(z_1,z_2). \tag{99}$$

As the author remarks, if the ring of integers $\mathcal{O}$ is Euclidean, then (99) (together with the obvious invariance of (98) with respect to translations and multiplication by totally positive units) is sufficient to ensure that F is in fact a Hilbert modular form, for in this case the transformation (87) and (88) together with

$$(z_1,z_2)\mapsto\left(\frac{-1}{z_1},\frac{-1}{z_2}\right) \tag{100}$$

certainly generate $SL_2\mathcal{O}$. However, it is known that the only primes $p\equiv 1 \pmod 4$ for which $\mathbb{Q}(\sqrt{p})$ is Euclidean (at least with respect to the norm map) are 5, 13, 17, 29, 37, 41 and 73 (cf. [3], Theorem 247).

In fact, a recent and very difficult theorem of Vaserstein (Mat. Sbornik **131** (89) 1972) tells us that the transformations (88) and (100) always generate the full group $SL_2\mathcal{O}$, and combining this with the theorem of Naganuma just enunciated, we deduce that (98) is in fact always a Hilbert modular form. We now give a different proof of this fact by showing that the function constructed by Naganuma is precisely the function we constructed in Theorem 4.

Theorem 5. *Let the assumptions be as in the theorem of Naganuma above:* $p\equiv 1 \pmod 4$ *a prime with class number one,* $f\in S_k(\Gamma_0(p),(\ /p))$ *a normalized Hecke eigenfunction. Then the function defined by* (98) *is identical with that defined by* (86), *and in particular is a cusp form of weight k for the Hilbert modular group.*

Thus the modular form $\Omega(z_1,z_2;\tau)$ in three variables constructed in this paper has an interpretation as the "kernel" (in the sense of integral operators) of the Doi-Naganuma mapping.

Proof. We must show that, for each integral ideal $\mathfrak{a}$, the number $b(\mathfrak{a})$ defined by (95)–(97) equals the number $c(\mathfrak{a})$ defined by (85). Since $D=p$ is prime, the number D_2 in (85) can only have the values 1 or p, and so (85) simplifies to

$$c(\mathfrak{a})=\sum_{r|\mathfrak{a}} r^{k-1}a^{p}_{N(\mathfrak{a})/r^2}(f)+\sum_{r\mathfrak{d}|\mathfrak{a}} r^{k-1}p^{k-\frac{1}{2}}a^{1}_{N(\mathfrak{a})/r^2p}(f), \tag{101}$$

where $a_n^p(f)$ and $a_n^1(f)$ are the Fourier coefficients of f at the two cusps of $\Gamma_0(p)$ and the second sum vanishes if $\mathfrak{d}\nmid\mathfrak{a}$ (to obtain (101) we have substituted the values $\psi(1)=1$, $\psi(p)=\sqrt{p}$). Now the coefficients $a_n^p(f)$ are the Fourier coefficients of $f\Big|\begin{pmatrix}1&0\\p&1\end{pmatrix}$ (cf. (58), (59)), and since $\begin{pmatrix}1&0\\p&1\end{pmatrix}\in\Gamma_0(p)$, these are precisely the a_n defined by (90). Similarly, the $a_n^1(f)$ are the coefficients of $f\Big|\begin{pmatrix}p&-1\\1&0\end{pmatrix}$, i.e. are given by

$$(z-p)^{-k}f\left(\frac{-1}{z-p}\right)=\sum_{n=1}^{\infty}a_n^1(f)\,e^{2\pi i n z/p}\qquad (z\in\mathfrak{H})$$

or (replacing $z-p$ by pz)

$$p^{-k}z^{-k}f\left(\frac{-1}{pz}\right)=\sum_{n=1}^{\infty}a_n^1(f)\,e^{2\pi i n z}\qquad (z\in\mathfrak{H}).$$

But, by Lemma 2 of [8],

$$p^{-k}z^{-k}f\left(\frac{-1}{pz}\right)=p^{-k+1/2}\bar{a}_p f^\rho(z)$$

($f^\rho(z)=\sum\bar{a}_n e^{2\pi i n z}$ as above), and therefore

$$a_n^1(f)=p^{-k+1/2}\bar{a}_p\bar{a}_n.$$

Also, because the local factor corresponding to p in $\varphi^\rho(s)$ is simply

$$(1-\bar{a}_p p^{-s})^{-1}=\sum\bar{a}_p^r p^{-rs},$$

we see that $\bar{a}_n\bar{a}_p=\bar{a}_{np}$ for all n. Therefore (101) can be rewritten as

$$c(\mathfrak{a})=\sum_{r\mid\mathfrak{a}}r^{k-1}a'_{N(\mathfrak{a})/r^2},\tag{102}$$

where

$$a'_n=\begin{cases}a_n & \text{if } p\nmid n,\\ a_n+\bar{a}_n & \text{if } p\mid n.\end{cases}\tag{103}$$

Since $c(\mathfrak{a})$ is then clearly multiplicative for ideals with relatively prime norms (i.e. $c(\mathfrak{a}_1\mathfrak{a}_2)=c(\mathfrak{a}_1)c(\mathfrak{a}_2)$ for $(N(\mathfrak{a}_1),N(\mathfrak{a}_2))=1$), we only have to prove the equality $c(\mathfrak{a})=b(\mathfrak{a})$ for prime powers $\mathfrak{q}^m$ with $\mathfrak{q}$ inert or ramified or products $\mathfrak{q}^r\mathfrak{q}'^r$ if $\mathfrak{q}\mathfrak{q}'=(q)$. We consider the three cases separately.

(i) $\mathfrak{a}=\mathfrak{q}^m$, $\mathfrak{q}=(q)$, $(q/p)=-1$.

Then $r\mid\mathfrak{a}$ for $r=1,q,\ldots,q^m$, so

$$c(\mathfrak{q}^m)=\sum_{i=0}^{m}q^{i(k-1)}a_{q^{2m-2i}}.$$

To evaluate this, we introduce the generating series:

$$\begin{aligned}\sum_{m=0}^{\infty}c(\mathfrak{q}^m)t^{2m}&=\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}q^{i(k-1)}a_{q^{2l}}t^{2l+2i}\\&=\left(\sum_{i=0}^{\infty}q^{i(k-1)}t^{2i}\right)\left(\sum_{l=0}^{\infty}a_{q^{2l}}t^{2l}\right)\end{aligned}$$

$$=(1-q^{k-1}t^2)^{-1}\left(\tfrac{1}{2}\sum_{l=0}^{\infty} a_{q^l}t^l+\tfrac{1}{2}\sum_{l=0}^{\infty} a_{q^l}(-t)^l\right)$$

$$=\tfrac{1}{2}(1-q^{k-1}t^2)^{-1}\left(\frac{1}{1-a_q t-q^{k-1}t^2}+\frac{1}{1+a_q t-q^{k-1}t^2}\right)$$

$$=\frac{1}{(1-a_q t-q^{k-1}t^2)(1+a_q t-q^{k-1}t^2)}$$

$$=\frac{1}{1-b(\mathfrak{q})t^2+q^{2k-2}t^4}$$

$$=\sum_{m=0}^{\infty} b(\mathfrak{q}^m)t^{2m},$$

where the various steps use the precise form of the Euler factor for q in (92) as well as the properties (95 b), (96) of $b(\mathfrak{q}^m)$. Thus $c(\mathfrak{q}^m)=b(\mathfrak{q}^m)$ for all m.

(ii) $\mathfrak{a}=\mathfrak{q}^m$, $\mathfrak{q}^2=(p)$.

Now $r|\mathfrak{a}$ for $r=1, p, \ldots, p^{[m/2]}$, so (102), (103) give

$$c(\mathfrak{q}^m)=\sum_{i=0}^{[m/2]} p^{i(k-1)}a'_{p^{m-2i}}$$

$$=\sum_{0\leqq i\leqq \frac{m}{2}} p^{i(k-1)}a_{p^{m-2i}}+\sum_{0\leqq i\leqq \frac{m-1}{2}} p^{i(k-1)}\bar{a}_{p^{m-2i}}.$$

Also, by the above noted total multiplicativity, $a_{p^m}=(a_p)^m$, $\bar{a}_{p^m}=(\bar{a}_p)^m$. We again use a generating series:

$$\sum_{m=0}^{\infty} c(\mathfrak{q}^m)t^m=\sum_{i=0}^{\infty}\sum_{l=0}^{\infty} p^{i(k-1)}a_{p^l}t^{l+2i}+\sum_{i=0}^{\infty}\sum_{l=1}^{\infty} p^{i(k-1)}\bar{a}_{p^l}t^{l+2i}$$

$$=\left(\sum_{i=0}^{\infty} p^{i(k-1)}t^{2i}\right)\left(\sum_{l=0}^{\infty} a_p^l t^l\right)+\left(\sum_{i=0}^{\infty} p^{i(k-1)}t^{2i}\right)\left(\sum_{l=1}^{\infty} \bar{a}_p^l t^l\right)$$

$$=\frac{1}{1-p^{k-1}t^2}\left(\frac{1}{1-a_p t}+\frac{\bar{a}_p t}{1-\bar{a}_p t}\right)$$

$$=\frac{1}{1-p^{k-1}t^2}\cdot\frac{1-a_p\bar{a}_p t^2}{(1-a_p t)(1-\bar{a}_p t)}$$

$$=\frac{1}{1-(a_p+\bar{a}_p)t+p^{k-1}t^2}$$

(since $a_p\bar{a}_p=p^{k-1}$ by (93 c))

$$=\frac{1}{1-b(\mathfrak{q})t+p^{k-1}t^2}$$

$$=\sum_{m=0}^{\infty} b(\mathfrak{q}^m)t^m,$$

and again we deduce $c(\mathfrak{q}^m)=b(\mathfrak{q}^m)$ for all m.

(iii) $\mathfrak{a}=\mathfrak{q}^m\mathfrak{q}'^n$, $\mathfrak{q}\mathfrak{q}'=(q)$, $(q/p)=1$.

Then $r\,|\,\mathfrak{a}$ for $r=1, q, \dots, q^{\min(m,n)}$, so

$$c(\mathfrak{q}^m\mathfrak{q}'^n)=\sum_{i=0}^{\min(m,n)} q^{i(k-1)} a_{q^{m+n-2i}}.$$

Now we have to introduce a double generating series with two formal variables t, u and find

$$\begin{aligned}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c(\mathfrak{q}^m\mathfrak{q}'^n)t^m u^n &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} q^{i(k-1)} a_{q^{j+l}} t^{i+j} u^{i+l}\\
&=\left(\sum_{i=0}^{\infty} q^{i(k-1)} t^i u^i\right)\left(\sum_{n=0}^{\infty} a_{q^n}(t^n+t^{n-1}u+\dots+tu^{n-1}+u^n)\right)\\
&=\frac{1}{1-q^{k-1}tu}\sum_{n=0}^{\infty} a_{q^n}\frac{t^{n+1}-u^{n+1}}{t-u}\\
&=\frac{1}{1-q^{k-1}tu}\cdot\frac{1}{t-u}\left(t\sum_{n=0}^{\infty} a_{q^n}t^n-u\sum_{n=0}^{\infty} a_{q^n}u^n\right)\\
&=\frac{1}{1-q^{k-1}tu}\cdot\frac{1}{t-u}\left(\frac{t}{1-a_q t+q^{k-1}t^2}-\frac{u}{1-a_q u+q^{k-1}u^2}\right)\\
&=\frac{1}{(1-a_q t+q^{k-1}t^2)(1-a_q u+q^{k-1}u^2)}\\
&=\frac{1}{1-b(\mathfrak{q})t+N(\mathfrak{q})^{k-1}t^2}\;\frac{1}{1-b(\mathfrak{q}')u+N(\mathfrak{q}')^{k-1}u^2}\\
&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} b(\mathfrak{q}^m)b(\mathfrak{q}'^n)t^m u^n\\
&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} b(\mathfrak{q}^m\mathfrak{q}'^n)t^m u^n\end{aligned}$$

(the last equation because $\mathfrak{q}^m$ and $\mathfrak{q}'^n$ are relatively prime and $b(\mathfrak{a})$ multiplicative). Thus $c(\mathfrak{q}^m\mathfrak{q}'^n)=b(\mathfrak{q}^m\mathfrak{q}'^n)$ for all m, n and the proof of Theorem 5 is complete.

As a consequence of Theorem 5, we find that we have two descriptions of the map ι of Theorem 4: as the Petersson product with Ω and as the map of Doi-Naganuma. Using these two descriptions, we obtain a better understanding of this mapping.

First of all, there is a natural involution ρ on $S_1=S_k(\Gamma_0(p),(\ /p))$ which sends a normalized Hecke eigenfunction $f=\sum a_n e^{2\pi i n z}$ to $f^\rho=\sum a_n^\rho e^{2\pi i n z}$ ($a_n^\rho=\bar{a}_n$) and is then defined on all S_1 by linearity (the normalized Hecke eigenfunctions form a basis of S_1). If we denote by a_n the Fourier coefficients of f^ρ ($f\in S_1$ arbitrary), then it follows from (93a) and (93b) that

$$a_n^\rho=\left(\frac{n}{p}\right)a_n \quad \text{if } p\nmid n; \tag{104}$$

i.e. $\rho\colon S_1 \to S_1$ "twists" the Fourier coefficients of a cusp form by the character $(\ /p)$. Let

$$S_1^+ = \left\{ f\in S_1 \mid f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z},\ a_n = 0 \text{ for } \left(\frac{n}{p}\right) = -1 \right\},$$
$$S_1^- = \left\{ f\in S_1 \mid f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z},\ a_n = 0 \text{ for } \left(\frac{n}{p}\right) = +1 \right\}.$$

Then $S_1^+ \cap S_1^- = \{0\}$ since a well-known lemma of Hecke (see e.g. Ogg [9], p. 32) states that a non-zero cusp form cannot have a development of the form $\sum_n a_n e^{2\pi i n p z}$. It is easy to see that

$$S_1 = S_1^+ \oplus S_1^- \tag{105}$$

and that $S_1^\pm$ are just the (± 1)-eigenspaces of the involution ρ. It follows easily from either description of the map ι that ι is zero on S_1^-. One can in fact show

Proposition 1. *For the splitting* (105) *of* S_1 *according to the eigenvalues of* ρ,

(i) $\dim S_1^+ = \dim S_1^- = \frac{1}{2}\dim S_1$;

(ii) *the map* $\iota\colon S_1 \to S_2$ *is zero on* S_1^- *and injective on* S_1^+.

We omit the proof.

This describes the kernel of ι; we now describe the image.

Proposition 2. *The image of* $\iota\colon S_1 \to S_2$ *is precisely the subspace of* S_2 *spanned by the cusp forms* $\omega_1, \omega_2, \ldots$ *defined in* § 1.

Proof. Consider the Poincaré series $G_n^p(z)$ for the cusp at infinity of $\Gamma_0(p)$. By the basic property of Poincaré series, (G_n^p, f) is a constant $(\neq 0)$ times a_n for an arbitrary $f(z) = \sum a_n e^{2\pi i n z}$ in S_1. In particular, the G_n^p $(n = 1, 2, \ldots)$ generate S_1, since a cusp form orthogonal to all the G_n^p would have all its Fourier coefficients zero and hence vanish. But by the same property of G_n^p,

$$\iota(G_n^p) = (G_n^p, \Omega) = (\text{const}) \cdot \omega_n$$

since

$$\Omega = \sum_{n=1}^{\infty} \omega_n e^{2\pi i n \tau},$$

and therefore the ω_n generate $\operatorname{Im}(\iota)$.

Propositions 1 and 2 give some insight into the nature of ι. Since the map ι, relating as it does modular forms in one variable with Hilbert modular forms, seems to play quite a significant role in understanding Hilbert modular forms (cf. for example the rather theoretical discussion in § 20 of Jacquet [6]), it would be of considerable interest to acquire more information about its properties. The following questions suggest themselves.

I. Elucidate the relationship of ι to the Hecke operators in S_1 and S_2. Just as there are Hecke operators $T_m\colon S_1 \to S_1$ sending a form Fourier coefficients a_n to one whose n^{th} coefficient is

$$\sum_{t \mid (n,m)} \left(\frac{t}{p}\right) t^{k-1} a_{nm/t^2},$$

there are Hecke operators $T_{\mathfrak{b}}: S_2 \to S_2$ sending a form $F(z_1, z_2)$ with Fourier coefficients $c(\mathfrak{a})$ to one whose $\mathfrak{a}^{\text{th}}$ coefficient is

$$\sum_{\mathfrak{t} \mid (\mathfrak{a}, \mathfrak{b})} N(\mathfrak{t})^{k-1} c(\mathfrak{a}\mathfrak{b}\mathfrak{t}^{-2}).$$

It is, of course, clear from the Doi-Naganuma description that $\imath$ maps Hecke eigenfunctions in S_1 to eigenfunctions of the Hecke operators in S_2, and the assertion of Proposition 1 is essentially that this map from the (finite) set of Hecke eigenfunctions of S_1 to the similar set for S_2 is precisely two-to-one, sending the two eigenfunctions f and f^ρ (which are always distinct, by i) of the proposition) to the same eigenfunction. One also has the Doi-Naganuma map for forms of "Haupttypus," i.e. a map $\imath_0: S_k(SL_2\mathbb{Z}) \to S_2$. This map also takes Hecke eigenfunctions to Hecke eigenfunctions and seems to be one-to-one and have an image disjoint from that of $\imath$. The image of $\imath \oplus \imath_0$ seems to be precisely the set of eigenfunctions in S_2 whose eigenvalues $c(\mathfrak{a})$ satisfy

$$c(\mathfrak{a}) = c(\mathfrak{a}')$$

for all $\mathfrak{a}$. Otherwise stated, one has an involution on the set of Hecke eigenfunctions which sends a function with Fourier coefficients $c(\mathfrak{a})$ to the function whose $\mathfrak{a}^{\text{th}}$ coefficient is $c(\mathfrak{a}')$, and the image of $\imath \oplus \imath_0$ is the space spanned by those eigenfunctions fixed under this involution. This would imply, in particular, that

$$\begin{aligned}\tfrac{1}{2}\dim S_1 + \dim S_k(SL_2\mathbb{Z}) &= \dim(\operatorname{Im} \imath \oplus \imath_0)\\ &\equiv \dim S_2 \pmod 2.\end{aligned}$$

That this is in fact the situation seems to follow from work of Saito, at least if the discriminant D is a prime with class number $h(D)=1$ ("Algebraic extensions of number fields and automorphic forms," to appear). Saito also finds a similar map for Hilbert modular forms associated to certain cyclic number fields of prime degree.

II. Since Hecke eigenfunctions are mapped to Hecke eigenfunctions under $\imath$ and since these eigenfunctions are orthogonal under the Petersson product, it is natural to ask whether the map $\imath: S_1 \to S_2$ is not perhaps an *isometry* (up to a constant factor) with respect to this product, i.e. whether the relation

$$(\imath(f), \imath(g)) = \kappa(f, g)$$

holds for all $f, g \in S_1$ for some constant $\kappa \neq 0$. Because the Hecke eigenfunctions form an orthogonal basis, it would suffice to show that

$$(\imath(f), \imath(f)) = \kappa(f, f)$$

for all normalized eigenfunctions f, with κ independent of f. This can be attacked using the Doi-Naganuma description of the associated Dirichlet series and Rankin's description [10] of the Petersson product. As far as the author can check, however, the answer seems to be negative: $\imath$ is not an isometry.

III. Finally, one can ask for a geometrical description of $\imath$; namely, if we think of modular forms as functions of pairs (E, ω) with E as elliptic curve and ω an abelian differential on E, and of Hilbert modular forms similarly as functions of 2-dimensional abelian varieties, then $\imath$ should have some interpretation in terms of a relationship between abelian varieties of dimension two and one. However, it is not yet evident what such an interpretation might be.

Appendix 1: The Case $k=2$

In this paper we have constructed and studied a series of Hilbert modular forms $\omega_0, \omega_1, \omega_2, \ldots$ of arbitrary even weight $k>2$. The case $k=2$ was excluded because the series defining ω_m is not absolutely convergent. However, Hecke [4] has shown that the Hecke-Eisenstein series of weight 2 for a real quadratic field can be defined by an appropriate limiting procedure and is a modular form having the same properties as the series of higher weights. Since – as we saw in § 1 – the Hecke-Eisenstein series is a multiple of our ω_0, it is reasonable to expect that all of the ω_m can be defined also for $k=2$ by applying Hecke's method. This will be carried out here.

Hecke's idea is to replace a conditionally convergent series $\sum f(z)^{-2}$ by the absolutely convergent series $\sum f(z)^{-2}|f(z)|^{-2s}$, where $s\in\mathbb{C}$ has positive real part. The latter function is no longer holomorphic in z but is an entire function of s. It also is periodic in z with real period, so has a Fourier expansion $\sum c_\alpha(y,s)e^{2\pi i\alpha z}$ whose Fourier coefficients depend on s and also on the imaginary part of z. The coefficients $c_\alpha(y,s)$ are then holomorphic in the entire s-plane, and in favourable cases $c_\alpha(y,0)$ is independent of y; then $\lim_{s\to 0}\left(\sum f(z)^{-2}|f(z)|^{-2s}\right) = \sum c_\alpha(y,0)e^{2\pi i\alpha z}$ is holomorphic.

We thus define (for $m\in\mathbb{N}$, $z_1, z_2\in\mathfrak{H}$, $s\in\mathbb{C}$, $\mathrm{Re}(s)>0$)

$$\omega_{m,s}(z_1,z_2)=\sum_{\substack{a,b\in\mathbb{Z}\\ \lambda\in\mathfrak{d}^{-1}\\ \lambda\lambda'-ab=m/D}} \varphi_s(az_1z_2+\lambda z_1+\lambda' z_2+b), \tag{1}$$

where for convenience we have used the abbreviation

$$\varphi_s(z)=\frac{1}{z^2|z|^{2s}}=z^{-2}e^{-2s\log|z|}. \tag{2}$$

The function $\omega_{m,s}$ satisfies

$$\omega_{m,s}\left(\frac{\alpha z_1+\beta}{\gamma z_1+\delta}, \frac{\alpha' z_2+\beta'}{\gamma' z_2+\delta'}\right)=\frac{1}{\varphi_s(\gamma z_1+\delta)\varphi_s(\gamma' z_2+\delta')}\,\omega_{m,s}(z_1,z_2)$$

for all $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in SL_2\mathcal{O}$; in particular, the function ω_m defined by

$$\omega_m(z_1,z_2)=\lim_{s\to 0}\omega_{m,s}(z_1,z_2) \tag{3}$$

(if the limit exists) will satisfy

$$\omega_m\left(\frac{\alpha z_1+\beta}{\gamma z_1+\delta}, \frac{\alpha' z_2+\beta'}{\gamma' z_2+\delta'}\right)=(\gamma z_1+\delta)^2(\gamma' z_2+\delta')^2\,\omega_m(z_1,z_2). \tag{4}$$

We want to prove that ω_m exists, is holomorphic, and has a Fourier expansion given by Theorem 2, § 2, with $k=2$. Since the Hecke-Eisenstein series was treated by Hecke, we assume $m>0$.

As in § 2, we split up $\omega_{m,s}$ as $\omega_{m,s}+2\sum_{a=1}^{\infty}\omega_{m,s}^{a}$, where $\omega_{m,s}^{a}(z_1,z_2)$ is defined as the subsum of (1) with a fixed value of a. The series

$$\omega_m^0(z_1,z_2)=\sum_{\substack{\lambda\lambda'=m/D\\ b\in\mathbb{Z}}}\frac{1}{(\lambda z_1+\lambda' z_2+b)^2}$$

is absolutely convergent and $\lim_{s\to 0}\omega_{m,s}^0=\omega_m^0$. Therefore we only have to worry about $a\neq 0$. Let $a>0$ and split up the sum defining $\omega_{m,s}^a$ as in Eq. (31), § 2:

$$\omega_{m,s}^a(z_1,z_2)=\sum_{\lambda\in R}\sum_{\theta\in\mathcal{O}}\varphi_s\left(a(z_1+\theta)(z_2+\theta')+\lambda(z_1+\theta)+\lambda'(z_2+\theta')+\frac{\lambda\lambda'-m/D}{a}\right)\tag{5}$$

with R a finite set. The inner sum equals

$$a^{-2-2s}\sum_{\theta\in\mathcal{O}}\varphi_s\left(\left(z_1+\frac{\lambda'}{a}+\theta\right)\left(z_2+\frac{\lambda}{a}+\theta'\right)-\frac{m}{a^2D}\right).\tag{6}$$

Denote by $b_{\nu,s}(\alpha,y_1,y_2)$ the Fourier coefficients in the expansion

$$\sum_{\theta\in\mathcal{O}}\varphi_s((z_1+\theta)(z_2+\theta')-\alpha)=\sum_{\nu\in\mathfrak{d}^{-1}}b_{\nu,s}(\alpha,y_1,y_2)\,e^{2\pi i(\nu z_1+\nu' z_2)}\tag{7}$$

($\alpha>0$, $z_1=x_1+iy_1$, $z_2=x_2+iy_2$). Substituting this into (5), we see that $\omega_{m,s}^a$ has a Fourier expansion

$$\omega_{m,a}^s(z_1,z_2)=\sum_{\nu\in\mathfrak{d}^{-1}}e^{2\pi i(\nu z_1+\nu' z_2)}\left(a^{-2-2s}\sum_{\lambda\in R}e^{2\pi i\left(\nu\frac{\lambda'}{a}+\frac{\nu'\lambda}{a}\right)}b_{\nu,s}\left(\frac{m}{a^2D},y_1,y_2\right)\right)$$

and therefore

$$\begin{aligned}&\omega_{m,s}(z_1,z_2)\\&\quad=\omega_{m,s}^0(z_1,z_2)+\sum_{\nu\in\mathfrak{d}^{-1}}\left(2\sum_{a=1}^{\infty}\frac{G_a(m,\nu)}{a^{2+2s}}b_{\nu,s}\left(\frac{m}{a^2D},y_1,y_2\right)\right)e^{2\pi i(\nu z_1+\nu' z_2)}\end{aligned}\tag{8}$$

with $G_a(m,\nu)$ as in Eq. (38), § 2.

As usual, $b_{\nu,s}$ in (7) can be evaluated by Poisson summation:

$$\begin{aligned}b_{\nu,s}(\alpha,y_1,y_2)&=\frac{1}{\sqrt{D}}\int_{\operatorname{Im}z_2=y_2}\int_{\operatorname{Im}z_1=y_1}e^{-2\pi i(\nu z_1+\nu' z_2)}\varphi_s(z_1z_2-\alpha)\,dz_1\,dz_2\\&=\frac{1}{\sqrt{D}}\int_{\operatorname{Im}z_2=y_2}\varphi_s(z_2)\,e^{-2\pi i(\nu' z_2+\nu\alpha/z_2)}\\&\quad\cdot\int_{\operatorname{Im}z_1=y_1}e^{-2\pi i\nu(z_1-\alpha/z_2)}\varphi_s\left(z_1-\frac{\alpha}{z_2}\right)dz_1\,dz_2\\&=\frac{1}{\sqrt{D}}\int_{\operatorname{Im}z_2=y_2}\varphi_s(z_2)\,e^{-2\pi i(\nu' z_2+\nu\alpha/z_2)}\\&\quad\cdot\int_{\operatorname{Im}t=y_1-\operatorname{Im}(\alpha/z_2)}e^{-2\pi i\nu t}\varphi_s(t)\,dt\,dz_2.\end{aligned}\tag{9}$$

Consider the inner integral. Write y for $y_1 - \mathrm{Im}(\alpha/z_2)$.

$$\begin{aligned}\int\limits_{\mathrm{Im}\,t=y} e^{-2\pi i v t}\varphi_s(t)\,dt &= \int\limits_{-\infty}^{\infty} e^{-2\pi i v(x+iy)}\varphi_s(x+iy)\,dx \\ &= e^{2\pi v y}\int\limits_{-\infty}^{\infty}\frac{e^{-2\pi i v x}\,dx}{(x+iy)^2(x^2+y^2)^s} \\ &= y^{-1-2s}e^{2\pi v y}\int\limits_{-\infty}^{\infty}\frac{e^{2\pi i v y u}\,du}{(u+i)^2(u^2+1)^s},\end{aligned} \tag{10}$$

where in the last line we have set $u=x/y$. The integrand is now holomorphic and one-valued in the cut region $\mathbb{C}-[i,i\infty]-[-i\infty,-i]$. If v is negative, we can deform the path of integration upwards to a path Γ starting at $-\varepsilon+i\infty$, circling counterclockwise about the point i and ending at $+\varepsilon+i\infty$; if v is positive, we deform the path of integration to the mirror image of Γ. The resulting integral in both cases is then holomorphic in s for all s and satisfies a uniform estimate (Eq. (10) of [4]) which suffices to make the series

$$\sum_{\substack{v\in\mathfrak{d}^{-1}\\ v\neq 0}}\left(\sum_{a=1}^{\infty} a^{-2-2s}G_a(m,v)\,b_{v,s}\left(\frac{m}{a^2D},y_1,y_2\right)e^{2\pi i(vz_1+v'z_2)}\right) \tag{11}$$

absolutely convergent for all s; therefore this series has a limit at $s=0$ which is obtained simply by setting $s=0$ in each term. By Lemma 2 of § 2 (with $k=2$),

$$b_{v,0}(\alpha,y_1,y_2)=4\pi^2\sqrt{\frac{vv'}{\alpha D}}\,J_1(4\pi\sqrt{\alpha v v'})$$

(this is independent of y_1, y_2, so the limit of (11) as $s\to 0$ is holomorphic). Therefore in (8) the sum of $\omega_{m,s}^0$ and the terms with $v\neq 0$ tend as $s\to 0$ to the holomorphic function $\sum_{v\gg 0} c_{mv}e^{2\pi i(vz_1+v'z_2)}$ with c_{mv} as in Theorem 2, § 2.

It remains to treat the term $v=0$ in (8). Eq. (10) for $v=0$ becomes

$$\int\limits_{\mathrm{Im}\,t=y}\varphi_s(t)\,dt=\frac{c(s)}{y^{1+2s}},$$

where

$$c(s)=\int\limits_{-\infty}^{\infty}\frac{du}{(u+i)^2(u^2+1)^s}=-s\frac{\Gamma(s+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(s+2)}. \tag{12}$$

Substituting this into (9), we find

$$\begin{aligned}b_{0,s}(\alpha,y_1,y_2) &= \frac{c(s)}{\sqrt{D}}\int\limits_{\mathrm{Im}\,z_2=y_2}\frac{\varphi_s(z_2)\,dz_2}{\left(y_1-\mathrm{Im}\dfrac{\alpha}{z_2}\right)^{1+2s}} \\ &= \frac{c(s)}{\sqrt{D}}\int\limits_{-\infty}^{\infty}\frac{dx}{(x+iy_2)^2(x^2+y_2^2)^s\left(y_1+\dfrac{\alpha y_2}{x^2+y_2^2}\right)^{1+2s}} \\ &= \frac{c(s)}{\sqrt{D}}\frac{1}{(y_1y_2)^{2s+1}}\int\limits_{-\infty}^{\infty}\frac{du}{(u+i)^2(u^2+1)^s\left(1+\dfrac{\alpha/y_1y_2}{u^2+1}\right)^{2s+1}}.\end{aligned}$$

The integral equals $c(s)+O(\alpha/y_1 y_2)$, with the constant implied by O uniform in s. Therefore

$$2\sum_{a=1}^{\infty}\frac{G_a(m,0)}{a^{2+2s}}\,b_{0,s}\left(\frac{m}{a^2 D},y_1,y_2\right)$$
$$=\frac{2c(s)^2}{\sqrt{D}}(y_1 y_2)^{-1-2s}\sum_{a=1}^{\infty}\frac{G_a(m,0)}{a^{2+2s}}+c(s)\sum_{a=1}^{\infty}O\left(\frac{G_a(m,0)}{a^{4+2s}}\right). \tag{13}$$

Since $|G_a(m,0)|\leqq a^2$, the last sum is bounded for $s\to 0$ and, since $c(s)\to 0$ for $s\to 0$, the second term tends to zero with s. Also, $G_a(m,0)=\frac{1}{D}N_{aD}(m)$ in the notation of Eq. (82), § 4, and – by Lemma 3 of that section – the function $\sum N_{aD}(m)/a^s$ has an Euler product with q-factor $\frac{1-(D/q)q^{-s}}{1-q^{1-s}}$ for $q\nmid D_m$ and therefore has a simple pole at $s=2$. Hence $\sum G_a(m,0)/a^{2+2s}$ has a simple pole at $s=0$ and, since $c(s)^2$ has a zero of second order, we see that also the first term of (13) tends to zero with s. Therefore the terms in (8) with $\nu=0$ contribute nothing in the limit as $s\to 0$, and we deduce for the limit (3) the Fourier expansion

$$\omega_m(z_1,z_2)=\sum_{\substack{\nu\in\mathfrak{d}^{-1}\\ \nu\gg 0}} c_{m\nu}\,e^{2\pi i(\nu z_1+\nu' z_2)} \tag{14}$$

with $c_{m\nu}$ given by the same formula as in § 2. In particular, $\omega_m(z_1,z_2)$ is holomorphic and (by virtue of (4) and the absence of a term with $\nu=0$ in (14)) is in fact a cusp form of weight 2 for $SL_2\mathcal{O}$.

A similar calculation shows that if, in the context of § 3, we define a Poincaré series of weight 2 by

$$G_n^P(z)=\lim_{s\to 0}\Big(\tfrac{1}{2}\sum_{A=\binom{*\,*}{c\,d}\in\Gamma_P\setminus A_P\Gamma}\chi(A_P^{-1}A)\,e^{2\pi i n Az/w_P}\,\varphi_s(cz+d)\Big)$$

(cf. Eq. (51), § 3), then for non-trivial characters χ $G_n^P(z)$ is a holomorphic cusp form of weight 2 for the group Γ and character χ and has the Fourier expansion $\sum_{m=1}^{\infty} g_{nm}^P e^{2\pi i m z}$ with g_{nm}^P as in (56) of § 3. In particular, Eq. (65) and (66) of § 3 define a function $G_n\in S_2(\Gamma_0(D),(\ /D))$. Since the calculation of § 4 used only the Fourier coefficients of ω_m and G_n, it remains true without any change. We have proved:

Theorem. *Let $k=2$ and, for $m>0$, define*

$$\omega_m(z_1,z_2)=\lim_{s\to 0}\sum_{\substack{a,b\in\mathbb{Z}\\ \lambda\in\mathfrak{d}^{-1}\\ \lambda\lambda'-ab=m/D}}\frac{1}{(az_1z_2+\lambda z_1+\lambda' z_2+b)^2\,|az_1z_2+\lambda z_1+\lambda' z_2+b|^{2s}}.$$

Then the functions ω_m have all the properties which are asserted in Theorems 1, 2 *and* 3 (§§ 1, 2 *and* 4 *respectively*) *for the corresponding forms of higher weight.*

Appendix 2: Restriction to the Diagonal

Given a Hilbert modular form $F(z_1, z_2)$ of weight k, the restriction of F to the diagonal in $\mathfrak{H} \times \mathfrak{H}$, i.e. the function $F(z, z)$ $(z \in \mathfrak{H})$, is an ordinary modular form of weight $2k$ for the full modular group $SL_2\mathbb{Z}$. This process can lead to interesting modular forms: for instance, Siegel [11] has studied the restriction to the diagonal of the Hecke-Eisenstein series of weight k. Since the Hecke-Eisenstein series is (up to a factor) just our form ω_0, this suggests that it might be of interest to study the forms $\omega_m(z, z)$.

We have

$$\omega_m(z,z) = \sum_{\substack{a,c\in\mathbb{Z} \\ \lambda \in \mathfrak{d}^{-1} \\ \lambda\lambda' - ac = m/D}}' (az^2 + (Tr\,\lambda)z + c)^{-k}. \tag{1}$$

Let $b = Tr\,\lambda \in \mathbb{Z}$ and write $\lambda = \dfrac{d + b\sqrt{D}}{2\sqrt{D}}$ with $d^2 \equiv b^2 D \pmod 4$. Then (1) becomes

$$\omega_m(z,z) = \sum_{\substack{a,b,c,d\in\mathbb{Z} \\ d^2 - D(b^2-4ac) = -4m}}' (az^2 + bz + c)^{-k},$$

where $\sum'$ means that the 4-tuple $a = b = c = d = 0$ is to be omitted if $m = 0$. (The condition $d^2 \equiv b^2 D \pmod 4$ can be left out since it follows from the equation $d^2 - (b^2 - 4ac)D = -4m$.)

Definition. For $k > 2$ an even integer and for any nonnegative integer Δ, set

$$f_k(\Delta, z) = \sum_{\substack{a,b,c\in\mathbb{Z} \\ b^2-4ac=\Delta}}' \frac{1}{(az^2+bz+c)^k}, \tag{2}$$

where $\sum'$ means that the term $a = b = c = 0$ is to be omitted from the summation in case $\Delta = 0$.

Then the above equation can be written

$$\omega_m(z,z) = \sum_{\substack{d\in\mathbb{Z} \\ d^2 \equiv -4m \pmod D}} f_k\left(\frac{d^2+4m}{D}, z\right). \tag{3}$$

Thus the modular form $\omega_m(z, z)$ breaks up into an infinite sum of functions $f_k(\Delta, z)$. We now show that these functions are modular forms having similar properties to the properties of the ω_m considered in § 1.

Theorem. (i) *For each $\Delta \geqq 0$, $f_k(\Delta, z)$ is a modular form of weight $2k$ with respect to $SL_2\mathbb{Z}$.*

(ii) *$f_k(0, z)$ is a multiple of the Eisenstein series of weight $2k$.*

(iii) *$f_k(\Delta, z)$ is a cusp form for $\Delta > 0$.*

(iv) *$f_k(\Delta, z) \equiv 0$ unless $\Delta \equiv 0$ or $1 \pmod 4$.*

Proof. (i) This is clear, since

$$a\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)^2 + b\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right) + c = (a^* z^2 + b^* z + c^*)/(\gamma z+\delta)^2,$$

with a^*, b^*, c^* given by

$$\begin{pmatrix} a^* & b^*/2 \\ b^*/2 & c^* \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^t \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Then $b^{*2} - 4a^* c^* = \Delta$ and (a^*, b^*, c^*) runs over the same set as does (a, b, c).

(ii) If $\Delta = 0$, the equation $b^2 - 4ac = \Delta$ becomes homogeneous in a, b, c, so we can remove the greatest common divisor of a, b, c to obtain

$$f_k(0, z) = \zeta(k) \sum_{\substack{a,b,c \in \mathbb{Z} \\ (a,b,c)=1 \\ b^2 = 4ac}} \frac{1}{(az^2 + bz + c)^k}.$$

Now $(a, b, c) = 1$, $b^2 = 4ac$ implies $(a, c) = 1$, $a = \pm m^2$, $c = \pm n^2$, $b = 2mn$ (with the signs of a and c agreeing) for some relatively prime integers m, n. Hence

$$f_k(0, z) = \zeta(k) \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n)=1}} \frac{1}{(mz+n)^{2k}}.$$

By the same argument, the Eisenstein series

$$G_{2k}(z) = \sum_{m,n \in \mathbb{Z}}' \frac{1}{(mz+n)^{2k}}$$

equals

$$\zeta(2k) \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n)=1}} \frac{1}{(mz+n)^{2k}}.$$

Therefore

$$f_k(0, z) = \frac{\zeta(k)}{\zeta(2k)} G_{2k}(z). \tag{4}$$

(iii) The whole Fourier expansion of $f_k(\Delta, z)$ can be found by the method of § 2: one breaks up the sum

$$f_k(\Delta, z) = f_k^0(\Delta, z) + 2 \sum_{a=1}^{\infty} f_k^a(\Delta, z)$$

with $f_k^a(\Delta, z)$ defined by Eq. (2) with the summation restricted to a fixed value of a; thus

$$f_k^0(\Delta, z) = \sum_{b^2 = \Delta} h_k(bz)$$

(in the notation of (25), § 2) and

$$f_k^a(\Delta, z) = \sum_{\substack{b \in \mathbb{Z} \\ b^2 \equiv \Delta \pmod{4a}}} \left(az^2 + bz + \frac{b^2 - \Delta}{4a}\right)^{-k}$$

for $a = 0$. The Fourier expansion of f_k^0 is then given by (26) and has no constant term; that of f_k^a is found by breaking up the sum as

$$f_k^a(\Delta, z) = \sum_{\substack{b \pmod{2a} \\ b^2 \equiv \Delta \pmod{4a}}} \sum_{n \in \mathbb{Z}} \left(a(z+n)^2 + b(z+n) + \frac{b^2 - \Delta}{4a}\right)^{-k}.$$

The first summation is finite, so it suffices to show that the inner sum has a Fourier expansion with no constant term. But the r^{th} Fourier coefficient of the inner sum is

$$\int_{-\infty+iC}^{\infty+iC} \left(az^2+bz+\frac{b^2-\Delta}{4a}\right)^{-k} e^{-2\pi irz}\,dz \qquad (C>0) \tag{5}$$

by the usual argument, and this vanishes for $r\leqq 0$ because the poles of the integrand are on the real axis, below the line of integration.

(iv) This assertion is clear since the summation is empty otherwise.

We can evaluate the integral (5) by substituting $t=-i\left(z+\dfrac{b}{2a}\right)$:

$$\int_{-\infty+iC}^{\infty+iC} \frac{e^{-2\pi irz}\,dz}{\left(az^2+bz+\frac{b^2-\Delta}{4a}\right)^k} = -i\,e^{\pi irb/a}\int_{C-i\infty}^{C+i\infty} \frac{e^{2\pi rt}\,dt}{\left(at^2+\frac{\Delta}{4A}\right)^k}$$

$$=\begin{cases} \dfrac{2^{k+\frac12}\pi^{k+1}r^{k-\frac12}}{\Delta^{\frac{k}{2}-\frac14}\sqrt{a}(k-1)!}\,J_{k-\frac12}\left(\dfrac{\pi r\sqrt{\Delta}}{a}\right)e^{\pi irb/a} & \text{if } \Delta>0 \qquad (6\text{a})\\[2ex] \dfrac{2^{2k}\pi^{2k}}{(2k-1)!\,a^k}\,r^{2k-1}\,e^{\pi irb/a} & \text{if } \Delta=0 \qquad (6\text{b})\end{cases}$$

(the integral is essentially the inverse Laplace transform of $(s^2+1)^{-k}$ and is evaluated in [1], (29. 3. 57). Recall that $J_{k-\frac12}$ is an elementary function: the expression (6a) equals $4\pi\dfrac{(-4a)^{k-1}}{(k-1)!}\dfrac{d^{k-1}}{d\Delta^{k-1}}\left(\dfrac{\sin\frac{\pi}{a}\sqrt{r\Delta}}{\sqrt{\Delta}}\right)$.) Therefore we have

Proposition. *For $\Delta>0$, $f_k(\Delta,z)$ has the Fourier expansion*

$$f_k(\Delta,z)=\sum_{r=1}^{\infty} c_r^{\Delta}\,e^{2\pi irz},$$

$$c_r^{\Delta}=\frac{(2\pi i)^k}{(k-1)!}\sum_{r^2=d^2\Delta} d^{k-1}+\frac{2^{k+\frac32}\pi^{k+2}r^{k-\frac12}}{\Delta^{\frac{k}{2}-\frac14}(k-1)!}\sum_{a=1}^{\infty} a^{-\frac12}S_a(r,\Delta)J_{k-\frac12}\left(\frac{\pi r\sqrt{\Delta}}{a}\right), \tag{7}$$

where

$$S_a(r,\Delta)=\sum_{\substack{b \,(\mathrm{mod}\, 2a)\\ b^2\equiv\Delta \,(\mathrm{mod}\, 4a)}} e_{2a}(rb). \tag{8}$$

For $\Delta=0$, we have

$$f_k(0,z)=2\zeta(k)+\sum_{r=1}^{\infty} c_r^0\,e^{2\pi irz},$$

$$c_r^0=\frac{2^{2k+1}\pi^{2k}}{(2k-1)!}\,r^{2k-1}\sum_{a=1}^{\infty}\frac{1}{a^k}\,S_a(r,0). \tag{9}$$

As in § 2, we can evaluate $S_a(r,0)$, for clearly

$$S_a(r,0)=\sum_{\substack{c\,(\mathrm{mod}\,a)\\ c^2\equiv 0\,(\mathrm{mod}\,a)}} e_a(rc)$$

is multiplicative as a function of a and, for $a=p^m$, is given by

$$S_{p^m}(r,0)=\begin{cases} p^{[m/2]} & \text{if } p^{[m/2]}\mid r,\\ 0 & \text{if } p^{[m/2]}\nmid r.\end{cases}$$

This implies

$$\sum_{a=1}^{\infty} S_a(r,0)\,a^{-s}=\prod_p (1+p^{-s})\cdot\sum_{d\mid r} d^{1-2s}=\frac{\zeta(s)}{\zeta(2s)}\sum_{d\mid r}\frac{1}{d^{2s-1}}$$

and thus (9) becomes simply

$$c_r^0=\frac{\zeta(k)}{\zeta(2k)}\,\frac{2^{2k+1}\pi^{2k}}{(2k-1)!}\,\sigma_{2k-1}(r).$$

This gives a second proof of (4).

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