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The Hilbert Modular Group for the Field $\mathbb{Q}(\sqrt{13})$*

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Dedicated to F. Hirzebruch

Introduction

The action of the classical modular group $SL_2(\mathbb{Z})$ on the complex upper half-plane $\mathfrak{H}$ has a natural generalization to an action of the group $SL_2(\mathcal{O}_K)$ on the n-fold cartesian product $\mathfrak{H}^n$, where $\mathcal{O}_K$ is the ring of integers of a totally real number field K of degree n. This is the Hilbert modular group, introduced by Hilbert towards the end of the last century and studied by his student Blumenthal and later by Hecke, Maass, Gundlach and others. These authors showed how to compactify the quotient $\mathfrak{H}^n/SL_2(\mathcal{O}_K)$ to a (singular) projective variety $\overline{\mathfrak{H}^n/SL_2(\mathcal{O}_K)}$ by the addition of finitely many points ("cusps"), determined the function field in a few cases, constructed Eisenstein series and other modular forms for the group $SL_2(\mathcal{O}_K)$ and gave various arithmetic applications.

The theory was given new impetus in 1970 when Hirzebruch [4] showed how to resolve the singularities of $\overline{\mathfrak{H}^n/SL_2(\mathcal{O}_K)}$ in the case $n=2$. He also calculated the Chern numbers of the non-singular models thus obtained and

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studied the properties of modular curves on these surfaces, thus making it possible to apply the techniques of algebraic geometry and in particular to determine completely how the Hilbert modular surfaces fit into Kodaira's "rough classification scheme" (i.e. whether they are rational, K3, elliptic or of general type) [4, 8, 9]. However, there remains the problem of determining the isomorphism class of the Hilbert modular surfaces, rather than just their birational equivalence class or Kodaira type. This problem seems to be very difficult and has been solved only in a few cases. In 1976, Hirzebruch gave the answer for the field $\mathbb{Q}(\sqrt{5})$ [6] and several other fields of small discriminant [7]. For example, the modular surface for $\mathbb{Q}(\sqrt{5})$ is related to a famous cubic surface studied by Klein.

In this article we study various modular surfaces associated to the field $K=\mathbb{Q}(\sqrt{13})$. In particular, let Y be the minimal desingularization of the surface obtained by compactifying $\mathfrak{H}^2/\Gamma$, where

$$\Gamma=\left\{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in SL_2(\mathcal{O}_K)\,|\,\alpha\equiv\delta\equiv 1,\ \beta\equiv\gamma\equiv 0 (\mathrm{mod}\ 2)\right\} \tag{1}$$

is the principal congruence subgroup of $SL_2(\mathcal{O}_K)$ for the prime ideal generated by 2. In §§ 3–4 we show that Y contains 10 exceptional curves and that the surface Y^0 obtained by blowing down these curves is minimal and is isomorphic to the minimal desingularization of the quintic surface

$$S=\left\{(x_0:\ldots:x_4)\in\mathbb{P}_4(\mathbb{C})\,\middle|\,\sum_{i=0}^{4}x_i=0,\ \sum_{i=0}^{4}x_i^5=\frac{5}{12}\sum_{i=0}^{4}x_i^2\sum_{j=0}^{4}x_j^3\right\}. \tag{2}$$

In §§ 5–7 we study modular curves on Y and their images on S, showing that these images are always complete intersections and illustrating how their equations can be determined. In the last three sections we use the birational equivalence $Y\to S$ to study the modular forms on Γ and on $SL_2(\mathcal{O}_K)$ and show how to express the coordinates of the map $\mathfrak{H}\times\mathfrak{H}\to\mathfrak{H}\times\mathfrak{H}/\Gamma\to S$ in terms of Eisenstein series of weight one and weight two. Our thirteenth and last theorem (§ 10) gives the structure of the ring of modular forms of arbitrary weight for the Hilbert modular group $SL_2(\mathcal{O}_K)$.

§ 1. The Hilbert Modular Surface Y

We denote by K the real quadratic field $\mathbb{Q}(\sqrt{13})$ and by $\mathcal{O}=\mathcal{O}_K$ its ring of integers. The group $SL_2(\mathcal{O})$ acts on $\mathfrak{H}^2=\mathfrak{H}\times\mathfrak{H}$ (where $\mathfrak{H}=\{z\in\mathbb{C}\,|\,\mathrm{Im}\,z>0\}$ denotes the upper half-plane) by

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\circ(z_1,z_2)=\left(\frac{\alpha z_1+\beta}{\gamma z_1+\delta},\frac{\alpha' z_2+\beta'}{\gamma' z_2+\delta'}\right),$$

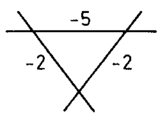
where $x\mapsto x'$ denotes conjugation over $\mathbb{Q}$ in K. The group $SL_2(\mathcal{O})/\{\pm 1\}$ acts effectively on $\mathfrak{H}\times\mathfrak{H}$. The subgroup $\Gamma/\{\pm 1\}$ (with Γ defined by (1)) acts freely. In this section we collect some basic facts about the surfaces $\mathfrak{H}^2/SL_2(\mathcal{O})$ and $\mathfrak{H}^2/\Gamma$, a reference for everything being the article [4] of Hirzebruch.

The surface $\overline{\mathfrak{H}^2/SL_2(\mathcal{O})}$ is the quotient of $\overline{\mathfrak{H}^2/\Gamma}$ by the group

$$SL_2(\mathcal{O})/\Gamma \cong SL_2(\mathcal{O}/2\mathcal{O}) \cong SL_2(\mathbb{F}_4)$$

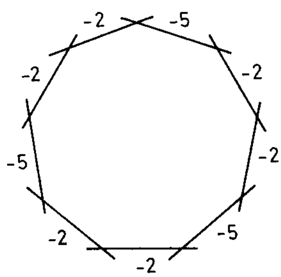
of order 60. Both surfaces are non-compact and must be compactified by adding cusps; these cusps are in 1:1 correspondence with the orbits of $\mathbb{P}_1(K)=K\cup\{\infty\}$ under the corresponding group. Since the class number of K is 1, there is only one orbit of $\mathbb{P}_1(K)$ under $SL_2(\mathcal{O})$, represented (say) by ∞, so that $\mathfrak{H}^2/SL_2(\mathcal{O})$ is compactified by adding one point: $\overline{\mathfrak{H}^2/SL_2(\mathcal{O})}=\mathfrak{H}^2/SL_2(\mathcal{O})\cup\{\infty\}$. The action of Γ on $\mathbb{P}_1(K)$ has five orbits, corresponding to the points of $\mathbb{P}_1(\mathbb{F}_4)$, so $\mathfrak{H}^2/\Gamma$ must be compactified by adding 5 cusps, represented (say) by ∞, 0, 1, ε_0 and ε_0^2, where $\varepsilon_0=\frac{3+\sqrt{13}}{2}$ is the fundamental unit of K. We number these cusps 0, 1, 2, 3 and 4, respectively. The group $SL_2(\mathcal{O})/\Gamma\cong SL_2(\mathbb{F}_4)$ acts on these 5 cusps as $\mathfrak{A}_5$, the alternating group on 5 elements. In future we identify $SL_2(\mathcal{O})/\Gamma$ with $\mathfrak{A}_5$.

Let $Y(13)$ denote the minimal desingularization of $\overline{\mathfrak{H}^2/SL_2(\mathcal{O})}$ and Y that of $\overline{\mathfrak{H}^2/\Gamma}$. The recipe for resolving cusp singularities was given by Hirzebruch in [4] and involves the periods of certain continued fractions. In our case, we find that the resolution of the cusp singularity of $\overline{\mathfrak{H}^2/SL_2(\mathcal{O})}$ consists of 3 non-singular rational curves with the intersection diagram



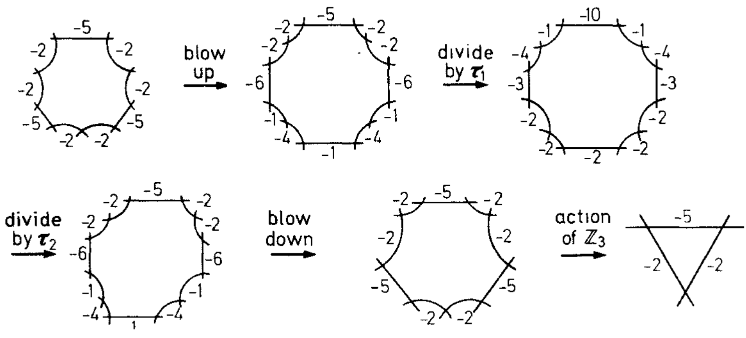
(3)

(the negative integers denote self-intersection numbers), whereas that of each cusp singularity of $\overline{\mathfrak{H}^2/\Gamma}$ consists of 9 non-singular rational curves with the intersection diagram



(4)

(the 5 cusps have isomorphic resolutions, since they are permuted by the group $\mathfrak{A}_5$). Under the action of $\mathfrak{A}_5$, each cusp has an isotropy group isomorphic to $\mathfrak{A}_4$. This group contains two involutions τ_1 and τ_2 such that $\{1, \tau_1, \tau_2, \tau_1\tau_2\}$ is a normal subgroup of index 3. The involution τ_1 has three isolated fixed points in a neighbourhood of the configuration (4). After blowing them up, we can visualize the action of the isotropy group which produces (3) as follows:



(Cf. [6], where a similar process for the cusps of the Hilbert modular surface for $\mathbb{Q}(\sqrt{5})$ is described in more detail.)

Finally, we give the values of the numerical invariants of the surface Y. Since $\Gamma/\{\pm 1\}$ acts freely on $\mathfrak{H}^2$, the Euler characteristic of the quotient is given by

$$e(\mathfrak{H}^2/\Gamma)=\int_{\mathfrak{H}^2/\Gamma}\omega_1\wedge\omega_2=[SL_2(\mathcal{O}):\Gamma]\int_{\mathfrak{H}^2/SL_2(\mathcal{O})}\omega_1\wedge\omega_2,$$

where $\omega_j=-\frac{1}{2\pi}y_j^{-2}\,dx_j\wedge dy_j$ $(z_j=x_j+iy_j,\ j=1,2)$ is the invariant volume form in $\mathfrak{H}$. Furthermore,

$$\int_{\mathfrak{H}^2/SL_2(\mathcal{O})}\omega_1\wedge\omega_2=2\zeta_K(-1),$$

where $\zeta_K(s)$ is the Dedekind zeta-function of K. Hence

$$e(\mathfrak{H}^2/\Gamma)=60\cdot 2\zeta_K(-1)=20,$$

and, since Y is the union of $\mathfrak{H}^2/\Gamma$ and 5 configurations (4), the Euler number of Y equals 65. On the other hand, the signature of $\mathfrak{H}^2/\Gamma$ is zero and the 45 curves of the cusp resolutions (4) have a negative definite intersection matrix, so the signature of Y is -45. Hence by the signature theorem of Hirzebruch and the theorem of Noether we find the values

$$c_1^2(Y)=-5,\qquad c_2(Y)=65,\qquad \chi(Y)=\frac{c_1^2+c_2}{12}=5$$

for the Chern numbers and arithmetic genus of Y. We recall that

$$\chi(Y)=\sum_{i=0}^{2}(-1)^i\dim H^i(Y,\mathcal{O}_Y),$$

where $\mathcal{O}_Y$ is the structure sheaf of Y. Since $\dim H^0(Y,\mathcal{O}_Y)=1$ and the irregularity $q=\dim H^1(Y,\mathcal{O}_Y)$ vanishes for all Hilbert modular surfaces, the geometric genus $p_g(Y)=\dim H^2(Y,\mathcal{O}_Y)$ is equal to 4.

We denote by K_Y the divisor class of a canonical divisor (i.e. the divisor of a meromorphic 2-form), or an element representing this class if no confusion can arise. Then $K_Y^2 = c_1^2 = -5$.

§ 2. The Curves F_N

In this section we give the definition and main properties of the curves F_N and T_N which were studied in [4, 5, 9] and [10].

Let N be a natural number and consider in $\mathfrak{H} \times \mathfrak{H}$ the graphs of all linear fractional transformations

$$z \mapsto \frac{\lambda' z - a_2\sqrt{13}}{a_1\sqrt{13}\, z + \lambda}$$

of determinant $13a_1a_2 + \lambda\lambda' = N$, where $a_1, a_2 \in \mathbb{Z}$, $\lambda \in \mathcal{O}$ and the triple (a_1, a_2, λ) is primitive (i.e. no natural number >1 divides a_1, a_2 and λ). The union of these graphs is invariant under the action of $SL_2(\mathcal{O})$; its image in $\mathfrak{H}^2/SL_2(\mathcal{O})$, or in $\mathfrak{H}^2/\Gamma$, will be denoted by F_N. We also consider the curves T_N, which are defined similarly but without the condition that (a_1, a_2, λ) be primitive, since many formulas are simpler in terms of the T_N. Clearly $T_N = \bigcup_{d>0,\, d^2/N} F_{N/d^2}$. The curve F_N in $\mathfrak{H}^2/\Gamma$ is mapped into itself by the action of the group $\mathfrak{A}_5$, the quotient being the curve F_N in $\mathfrak{H}^2/SL_2(\mathcal{O})$. The curve F_N is non-empty if N is a quadratic residue of 13.

The curves F_N have no singularities except ordinary multiple points and intesect one another transversally. Their intersection numbers were determined in [10] and can be given in terms of class numbers. These numbers are in general not integral, but only rational, since $\mathfrak{H}^2/SL_2(\mathcal{O})$ is a rational homology manifold. For example, if N is not a square,

$$(T_1 \cdot T_N)_{\mathfrak{H}^2/SL_2(\mathcal{O})} = H_{13}^0(N) \underset{\text{DEF}}{=} \sum_{\substack{x \in \mathbb{Z} \\ x^2 < 4N \\ x^2 \equiv 4N \pmod{13}}} H\left(\frac{4N - x^2}{13}\right), \tag{5}$$

where $H(n) \in \frac{1}{6}\mathbb{Z}$ is the number of classes of binary quadratic forms φ of discriminant $-n$, each counted with multiplicity $\dfrac{1}{|\mathrm{Aut}(\varphi)|}$, and more generally

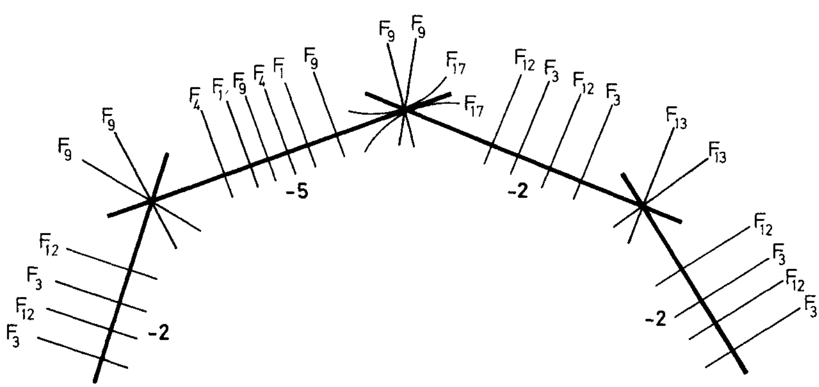
$$(T_M \cdot T_N)_{\mathfrak{H}^2/SL_2(\mathcal{O})} = \sum_{d/(m,n)} d\left(\frac{d}{13}\right) H_{13}^0\left(\frac{MN}{d^2}\right) \tag{6}$$

if $13 \nmid (M, N)$ and MN is not a square. The intersection numbers on $\mathfrak{H}^2/\Gamma$ are 60 times those given by (6) and are of course integers.

The curves F_N (resp. T_N) on $\mathfrak{H}^2/SL_2(\mathcal{O})$ and $\mathfrak{H}^2/\Gamma$ determine curves on the non-singular models $Y(13)$ and Y; these curves will also be denoted by F_N (resp. T_N). The curve F_N intersects the cusp resolution(s) if there exists an element $x \in \mathbb{P}_1(K)$ satisfying

$$a_1\sqrt{13}xx'-\lambda' x+\lambda x'+a_2\sqrt{13}=0,$$

i.e. if N is a norm in K. The way F_N intersects the cusp resolution on $Y(13)$ was described in [9]. Using our knowledge of the action of $\mathfrak{A}_5$, we can determine how F_N and T_N meet the cusp resolutions of Y. We find, for instance, that T_{n^2} meets each (-5)-curve in the cycle (4) transversally in $2n$ points and that T_{3n^2} meets each (-2)-curve in (4) transversally in $2n$ points, the intersection points being given in both cases by $u^{2n}=1$, where u is a coordinate on the (-5)- or (-2)-curve which takes on the values 0 and ∞ at the intersection points with the adjacent curves in the cycle. All other intersections of the T_N with the cusp resolutions occur at intersection points of adjacent curves of a cycle. At a common point of a (-2)-curve and a (-5)-curve, the curve T_N is given by the equations $u^{2q}=v^{2p}$, where (u,v) are local coordinates in Y such that $u=0$ is the local equation of the (-5)-curve and $v=0$ that of the (-2)-curve, and where p and q are positive integers such that $p^2+5pq+3q^2=N$. Similarly, at the meeting point of two (-2)-curves, T_N is given by $u^{2q}=v^{2p}$ with $p,q>0$ and $3p^2+7pq+3q^2=N$, where $u=0$ and $v=0$ define the two (-2)-curves. Thus we get the following intersection picture:



We denote the (-5)-curves of the cusp resolutions (4) by D_β $(\beta=1,\dots,15)$ and the (-2)-curves by E_α $(\alpha=1,\dots,30)$. Then the above description of the intersection of T_N with the cusps leads to the formulas

$$\begin{aligned}(T_N\cdot D_\beta)_Y&=a_N=2\sum_{\substack{p,q>0\\ p^2+5pq+3q^2=N}}2p+\begin{cases}2n & (N=n^2)\\ 0 & (N\neq\text{square})\end{cases}\\ (T_N\cdot E_\alpha)_Y&=b_N=\sum_{\substack{p,q>0\\ p^2+5pq+3q^2=N}}2q+\sum_{\substack{p,q>0\\ 3p^2+7pq+3q^2=N}}2p\\ &\quad+\begin{cases}2n & (N=3n^2)\\ 0 & (N/3\neq\text{square}).\end{cases}\end{aligned}\tag{7}$$

On the other hand, the Chern class c_1 of Y can be represented by a differential form $\gamma_1+\gamma_2$, where γ_2 represents in $H^2(Y)$ the Poincaré dual of the homology class of the cusp resolutions, and where γ_1 has support disjoint from the cusp resolutions and satisfies

$$\int_{F_N} \gamma_1 = \int_{F_N} (\omega_1+\omega_2)$$

with ω_1, ω_2 as in § 1 (cf. [4], 4.3). Also

$$\int_{F_N} \omega_1 = \int_{F_N} \omega_2 = \mathrm{vol}(F_N),$$

where $\mathrm{vol}(F_N)$ is the Euler volume of F_N. Hence

$$(c_1 \cdot F_N)_Y = 2\,\mathrm{vol}(F_N) + F_N \cdot (\text{cusps}).$$

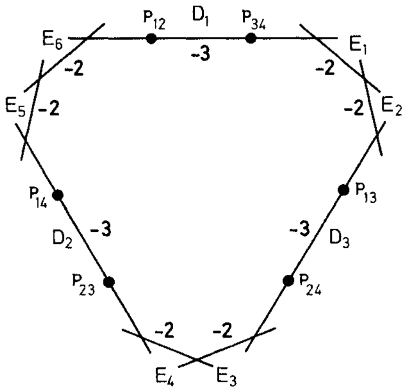
Of course a similar formula holds for T_N. Using the formula for $\mathrm{vol}(T_N)$ given in [10], we find

$$(c_1 \cdot T_N)_Y = -10 \sum_{d|n} \left(\frac{d}{13}\right)\left(d+\frac{N}{d}\right) + 15a_N + 30b_N \tag{8}$$

with a_N, b_N given by (7). Note that $(c_1 \cdot F_N)_Y = -K_Y \cdot F_N$.

We consider three examples of curves F_N in detail.

N = 1. The curve F_1 on $Y(13)$ consists of one component, so the components of F_1 on Y are permuted by $\mathfrak{A}_5$. It therefore suffices to consider the component given by $z_1 = z_2$. The subgroup of Γ that maps the diagonal $z_1 = z_2$ in $\mathfrak{H}^2$ into itself equals $\Gamma(2)$, the usual congruence subgroup of level 2 in $SL_2(\mathbb{Z})$. Since $\Gamma(2)/\{\pm 1\}$ has index 6 in $SL_2(\mathbb{Z})/\{\pm 1\}$ (which is the subgroup of $SL_2(\mathcal{O})/\{\pm 1\}$ preserving the diagonal), we find that F_1 has $60/6 = 10$ components in $\mathfrak{H}^2/\Gamma$. Each is isomorphic to $\mathfrak{H}/\Gamma(2)$ and hence is compactified by adding three cusps, the result being a non-singular rational curve. The component defined by $z_1 = z_2$ intersects each of the cusps 0, 1 and 2 transversally in a point of a (-5)-curve. By the action of $\mathfrak{A}_5$ we see that each component of F_1 meets exactly three of the five cusps, so that the components can be conveniently numbered F_1^{ij} $(0 \leqq i < j \leqq 4)$, where i and j are the indices of the two cusps which the component does not meet. From (7) and (8) we find $(c_1 \cdot F_1)_Y = 10$, so the value of $K_Y \cdot F_1^{ij}$ (which must be the same for all i, j) equals -1. Hence *each component of F_1 is an exceptional curve* (i.e. a non-singular rational curve with self-intersection number -1). Blowing down these 10 curves we obtain a new surface Y^0. This surface is non-singular and contains 10 points p_{ij} which are the images of the F_1^{ij} under the blowing down map $Y \to Y^0$. We draw a picture of the image of one of the configurations (4) (say the resolution of the 0-th cusp) in Y^0:



(9)

N = 4. Consider the component of F_4 defined by

$$z_1 = t + \frac{\sqrt{13}}{4}, \quad z_2 = t - \frac{\sqrt{13}}{4} \quad (t \in \mathfrak{H}).$$

This component intersects only cusp 0; in fact, $\frac{\sqrt{13}}{2}$ represents ∞ in $\mathbb{P}_1(\mathbb{F}_4)$ and $x - x' = \frac{\sqrt{13}}{2}$ holds only if x is $\Gamma(2)$-equivalent to ∞ in $\mathbb{P}_1(K)$. Comparing the subgroups of $SL_2(\mathcal{O})$ and of Γ which map this curve in $\mathfrak{H} \times \mathfrak{H}$ to itself, we see that the curve F_4 on Y has five components. Each component intersects one of the configurations (4) transversally in six points, two on each (-5)-curve. We denote by F_4^i $(i = 0, 1, 2, 3, 4)$ the component of F_4 which meets the resolution of the i-th cusp. The subgroup of $\mathfrak{A}_5$ that keeps a component F_4^i invariant is isomorphic to $\mathfrak{A}_4$; by the Hurwitz formula for branched coverings we find $e(F_4^i) = 2$, so F_4^i is rational. Applying formula (8) we find $(c_1 \cdot F_4)_Y = -10$, so $K_Y \cdot F_4^i = 2$ for each component. By the adjunction formula

$$K_Y \cdot F_4^i + (F_4^i)^2 = -2,$$

the self-intersection of F_4^i is -4. Since F_4^i is disjoint from F_1, the image curve in Y^0 (for which we use the same symbol F_4^i) also has self-intersection number -4 and hence $K_{Y^0} \cdot F_4^i = 2$.

N = 13. The curve in $\mathfrak{H} \times \mathfrak{H}$ defined by

$$\varepsilon_0^{-3} \sqrt{13} z_1 - \varepsilon_0^3 \sqrt{13} z_2 = 0$$

is transformed into itself by a subgroup of Γ that has index 6 in the corresponding subgroup of $SL_2(\mathcal{O})$. We find 10 components and $(c_1 \cdot F_{13})_Y = -80$. Since $(F_1 \cdot F_{13})_Y = 30$ by formula (6), $K_{Y^0} \cdot F_{13} = 50$. Just as in the case of F_1, each component of F_{13} meets exactly three cusps and we can write $F_{13} = \bigcup F_{13}^{ij}$ where F_{13}^{ij} does not meet cusp i and j.

§ 3. A Canonical Model for Y

As we just saw, the curve F_1 on Y consists of 10 exceptional curves which can be blown down to give a non-singular surface Y^0. This surface has the numerical invariants $\chi=5$, $c_1^2=5$. This implies that Y^0 (and hence Y) is of general type, i.e. that the sections in a sufficiently high tensor power of the canonical bundle determine a birational map of Y^0 onto a projective algebraic surface (see [8]). In this section we show that Y^0 is a minimal model of Y (i.e. does not contain any exceptional curves) and determine its image under the 1-canonical mapping.

Proposition 1. *The surface Y^0 is minimal.*

Proof. We use the method introduced in [2], which is based on the following assertion: *Any non-singular rational curve C on Y has at least three points in common with the curves of the cusp resolutions.* Indeed, if not, then by deleting from C its intersection points with the cusp resolutions we would obtain a curve in $\mathfrak{H}\times\mathfrak{H}/\Gamma$ isomorphic to $\mathbb{P}_1$, $\mathbb{C}$ or $\mathbb{C}^*$ and hence a curve in the universal covering $\mathfrak{H}^2$ of $\mathfrak{H}^2/\Gamma$ isomorphic to $\mathbb{P}_1$, $\mathbb{C}$ or $\mathbb{C}^*$. But this is impossible since, according to the theorem of Picard, none of these curves admit a non-constant holomorphic mapping to $\mathfrak{H}$. Of course, the same principle applies to Y^0.

Now suppose that Y^0 contains an exceptional curve E. Then E has at least three points in common with the curves coming from the cusp resolutions. These curves are either (-2)-curves or (-3)-curves (i.e. non-singular rational curves with self-intersection -2 or -3). There are now four possibilities:

i) $E\cdot C\geqq 2$ for some (-2)-curve or (-3)-curve C;

ii) E intersects at least two (-2)-curves, each transversally in one point;

iii) E intersects two (-3)-curves and one (-2)-curve transversally in one point;

iv) E intersects at least three (-3)-curves transversally in one point.

By blowing down E in case i) we obtain a non-singular rational curve with self-intersection $\geqq 0$. In case ii) we get two intersecting exceptional curves. By blowing down E and the exceptional curve arising from the (-2)-curve in case iii) we obtain two (-2)-curves with intersection number $\geqq 2$. Finally, in case iv) we obtain three (-2)-curves with a common intersection point. However, as explained in [8], such configurations cannot occur on a regular surface of general type.

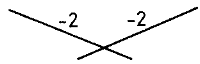
Theorem 1. *There exist 5 sections $s_i\in H^0(Y^0, K_{Y^0})$ such that the canonical mapping*

$$\Phi\colon\ y\mapsto (s_0(y):s_1(y):s_2(y):s_3(y):s_4(y)) \tag{10}$$

is a holomorphic mapping of degree 1 onto a quintic surface S in $\mathbb{P}_4$ determined by the equations

$$\sigma_1=0,\qquad \lambda\sigma_2\sigma_3+\mu\sigma_5=0 \tag{11}$$

for some $\lambda,\mu\in\mathbb{C}$, where σ_i is the i-th elementary symmetric function in the coordinates $x_0,\dots,x_4$ of $\mathbb{P}_4$. Moreover, S has 15 singular points, each resolved by *a configuration*



in Y^0.

Proof. The geometric genus $p_g(Y^0)$ of Y^0 equals 4. This means that for any three points $q_1, q_2, q_3 \in Y^0$ there is at least one effective canonical divisor passing through q_1, q_2 and q_3.

Now consider the configuration (9) on Y^0 arising from the resolution of the cusp 0 on Y, and let D be an effective canonical divisor passing through p_{12}, p_{34} and p_{13}. Since $K_{Y^0} \cdot D_1 = 1$, the divisor D must contain D_1. But $K_{Y^0} \cdot E_\beta = 0$, so D must also contain E_1, E_6 and hence also E_2 and E_5. Then it also contains D_3, E_3, E_4 and D_2. Thus D contains all nine curves in (9). Also, the component F_4^0 of F_4 passes through each of D_1, D_2 and D_3 twice (see above) and $K_{Y^0} F_4^0 = 2$, so D must contain F_4^0. Hence

$$D = \sum_{\alpha=1}^{6} a_\alpha E_\alpha + \sum_{\beta=1}^{3} b_\beta D_\beta + c F_4^0 + R$$

with $a_\alpha, b_\beta, c \geqq 1$ and $RE_\alpha \geqq 0$ $(1 \leqq \alpha \leqq 6)$, $RD_\beta \geqq 0$ $(1 \leqq \beta \leqq 3)$, $RF_4^0 \geqq 0$ and $K_{Y^0} R \geqq 0$. The relation

$$5 = K_{Y^0}^2 = K_{Y^0} D = \sum_{\beta=1}^{3} b_\beta + 2c + K_{Y^0} R$$

implies $b_1 = b_2 = b_3 = c = 1$ and $K_{Y^0} R = 0$. By intersecting D with D_1 we find

$$1 = K_{Y^0} D_1 = DD_1 = -3b_1 + a_1 + a_6 + 2c + RD_1 = a_1 + a_6 + RD_1 - 1,$$

so $a_1 = a_6 = 1$ and $RD_1 = 0$. Thus $a_\alpha = 1$ for all α and $RD_\beta = 0$ for all β. Similarly $RE_\alpha = 0$, $RF_4^0 = 0$. Since the canonical divisor D must be connected ([1], § 4), R is empty. It also follows that $E_1, \ldots, E_{15}$ *are the only* (-2)-*curves on* Y^0. Indeed, by the assertion used in the proof of the preceding proposition, any other (-2)-curve would have to intersect some cusp, say the 0-th, and would then be contained in D, a contradiction.

In this way, we have constructed a canonical divisor D consisting of the curves of the 0-th cusp resolution together with F_4^0, all with multiplicity 1. Let s_0 be a section of K_{Y^0} with zero-divisor $(s_0) = D$. By the action of a subgroup $\mathbb{Z}_5$ of $\mathfrak{A}_5$ which permutes the cusps cyclically we get 5 sections $s_i \in H^0(Y^0, K_{Y^0})$, $i = 0, 1, 2, 3, 4$, where the zero-divisor of s_i consists of F_4^i and the curves of the i-th cusp resolution. We claim that

(i) the sections $s_0, s_1, \ldots, s_4$ satisfy the linear relation $\sum_{i=0}^{4} s_i = 0$ and no other linear relation,

(ii) any element $\pi \in \mathfrak{A}_5$ acts on the s_i by $s_i \mapsto c\, s_{\pi(i)}$, where $c \neq 0$ depends on π but not on i,

(iii) the sections s_i do not have a common zero.

Indeed, since $p_g(Y^0)=4$, there must be some relation $\sum_{i=0}^{4} A_i s_i=0$ among the s_i, and this relation is unique (s_1, s_2, s_3, s_4 are linearly independent, since exactly one of them is non-zero at each of the four points p_{0i}, $1\leqq i\leqq 4$). Applying the action of the group $\mathbb{Z}_5$ to the relation, we see that all A_i are equal. This proves (i) but also (ii), since clearly $\pi^* s_i=c_i s_{\pi(i)}$ for some number $c_i\neq 0$ (the two sections have the same divisor), and the only way for the relation $\sum_{i=0}^{4} s_i=0$ to be preserved is that c_i should be independent of i. Finally, since the curve F_4^1 does not intersect the 0-th cusp, we have $F_4^0 F_4^1=DF_4^1=K_{Y^0} F_4^1=2$, i.e. any two F_4^i meet in two points (which are outside the cusp resolution). On the other hand, on F_4^i we have 8 points corresponding to the quotient singularity of order three on $F_4\subset\mathfrak{H}^2/SL_2(\mathcal{O})$, and these must be the intersection points of F_4^i with the other F_4^j. Hence there is no point lying on all of the divisors (s_i).

We have proved that the mapping (10) is a well-defined mapping from Y^0 into the hypersurface $\mathbb{P}_3\subset\mathbb{P}_4$ defined by $\sigma_1=\sum_{i=0}^{4} x_i=0$ and that it is equivariant with respect to the action of $\mathfrak{A}_5$ on $\mathbb{P}_3$ defined by the permutation of the coordinates. Since $K_{Y^0}^2=5$, it is either a holomorphic mapping of degree 5 onto a surface of degree 1 in $\mathbb{P}_3$ or a holomorphic mapping of degree 1 onto a quintic surface. But the first case would contradict (i) above, so the image of Y^0 is a quintic surface. The defining equation of this surface is invariant under $\mathfrak{A}_5$ and hence must have the form (11) for some constants λ, μ not both zero. Finally, the mapping (10) sends (-2)-configurations to rational double points and is otherwise biholomorphic. Since we have already shown that the only (-2)-curves on Y^0 are the 30 curves E_α, this proves the last assertion of the theorem.

§ 4. Quintic Surfaces in $\mathbb{P}_3$ which are Invariant under $\mathfrak{A}_5$

In this section we determine the coefficients λ and μ in the equation (11). We will show that the equation of S is

$$\sigma_1=0, \qquad 2\sigma_5=\sigma_2\sigma_3 \tag{12}$$

(this is equivalent to the equations in (2)). This follows immediately from Theorem 1 and the following result

Theorem 2. *In the family of quintic surfaces $S_{(\lambda:\mu)}$ in $\mathbb{P}_4$ given by*

$$S_{(\lambda:\mu)}=\{(x_0:\ldots:x_4)\in\mathbb{P}_4 \mid \sigma_1=0,\ \lambda\sigma_2\sigma_3+\mu\sigma_5=0\}$$

(where σ_k denotes the k-th elementary symmetric polynomial in $x_0,\ldots,x_4$), all but the following 6 are non-singular:

(i) $\sigma_5=0$ *(reducible, consisting of 5 planes, meeting along 10 lines which in turn meet 3 at a time in 10 points),*

(ii) $\sigma_2\sigma_3=0$ *(reducible; consisting of a quadric and a cubic surface meeting along a non-singular sextic curve),*

(iii) $2\sigma_5+\sigma_2\sigma_3=0$ *(20 singularities, namely the $\mathfrak{S}_5$-orbit of* $(-2:-2:-2:3+\sqrt{-7}:3-\sqrt{-7})$),

(iv) $25\sigma_5-12\sigma_2\sigma_3=0$ (10 *singularities, namely the $\mathfrak{S}_5$-orbit of* $(-2:-2:-2:3:3)$),

(v) $50\sigma_5+\sigma_2\sigma_3=0$ (5 *singularities, namely the $\mathfrak{S}_5$-orbit of* $(1:1:1:1:-4)$),

(vi) $2\sigma_5-\sigma_2\sigma_3=0$ (15 *singularities, namely the $\mathfrak{S}_5$-orbit of* $(0:1:-1:1:-1)$).

Proof. It will be more convenient to work with the power sums $S_k=\sum_{i=0}^{4} x_i^k$, which are related to the σ_k by the well-known formulas of Newton (here $S_1=\sigma_1=0$, $S_2=-2\sigma_2$, $S_3=3\sigma_3$, $S_4=-4\sigma_4+2\sigma_2^2$, $S_5=5\sigma_5-5\sigma_2\sigma_3$). Then the equation of $S_{(\lambda:\mu)}$ becomes

$$S_1=0, \quad tS_2S_3+\tfrac{1}{5}S_5=0 \tag{13}$$

with $t=-\dfrac{\lambda+\mu}{6\mu}$. Since the cases $\lambda=0$ and $\mu=0$ correspond to the obvious reducible cases (i), (ii), we assume $t\neq-\frac{1}{6}$.

Let $p=(x_0:\ldots:x_4)$ be a singular point of $S_{(\lambda:\mu)}$. Its coordinates must satisfy

$$\frac{\partial}{\partial x_i}(tS_2S_3+\tfrac{1}{5}S_5)=u\frac{\partial}{\partial x_i}(S_1) \qquad (i=0,\ldots,4)$$

for some u, i.e.

$$x_i^4+(3tS_2)x_i^2+(2tS_3)x_i-u=0 \qquad (i=0,\ldots,4). \tag{14}$$

Summing over i, we find that u is given by

$$S_4+3tS_2^2-5u=0. \tag{15}$$

Since the five coordinates x_i satisfy the quartic equation (14), at least two x_i must be equal (but not all x_i, since $S_1=0$). We can then distinguish the following five cases

Case	p is equivalent to	Number of points in $\mathfrak{S}_5$-orbit
I	$(\alpha,\alpha,\alpha,\alpha,\beta)$, $4\alpha=-\beta\neq 0$	5
II	$(\alpha,\alpha,\alpha,\beta,\beta)$, $3\alpha=-2\beta\neq 0$	10
III	$(\alpha,\alpha,\alpha,\beta,\gamma)$, $3\alpha+\beta+\gamma=0$	20
IV	$(\alpha,\alpha,\beta,\beta,\gamma)$, $2\alpha+2\beta+\gamma=0$	15
V	$(\alpha,\alpha,\beta,\gamma,\delta)$, $2\alpha+\beta+\gamma+\delta=0$	30

where α,β,γ and δ are distinct. In cases I and II, (13) is satisfied only for $t=-\frac{17}{100}$ and $t=-\frac{13}{150}$, respectively, giving cases (v) and (iv) of the theorem. As to the other three cases, we observe that the coefficient of x_i^3 in (14) is 0, so the sum of the roots of the quartic equation is 0. This implies that the roots of the quartic are α, β, γ and 2α in Case III and α, β, γ and $\alpha+\beta$ in Case IV. Consequently, in Case III the quartic (14) is identically equal to $(x-\alpha)(x-\beta)(x-\gamma)(x-2\alpha)$; comparing coefficients (using Eq. (15)) we get

$$(16t+2)\,\alpha^3-(6t+1)\,\alpha\beta\gamma=0,$$

$$(36t+7)\,\alpha^2-(6t+1)\,\beta\gamma=0$$

and hence (since $t \neq -\frac{1}{6}$ implies $\alpha \neq 0$) $t=-\frac{1}{4}$ and $\beta, \gamma=\dfrac{-3\pm\sqrt{-7}}{2}$, giving Case (iii) of the theorem. Similarly, in Case IV the quartic (14) is identical with $(x-\alpha)(x-\beta)(x-\gamma)(x-\alpha-\beta)$ and comparing coefficients we find $t=-\frac{1}{12}$ and $\gamma=-2(\alpha+\beta)=0$, giving Case (vi) of the theorem. Finally, in Case V, the four roots of (14) are $\alpha,\beta,\gamma,\delta$ so we must have $-\alpha=\alpha+\beta+\gamma+\delta=0$. Then $S_2=-2\sigma_2(\beta,\gamma,\delta)$ and $S_3=3\sigma_3(\beta,\gamma,\delta)$, so the quartic equation having the roots $\alpha=0$, β, γ and δ is $x_i^4-\frac{1}{2}S_2x_i^2-\frac{1}{3}S_3x_i=0$, which can agree with (14) only in the excluded case $t=-\frac{1}{6}$. This completes the proof.

Observe that the surfaces $S_{(\lambda:\mu)}$ are the fibres of the projection map $V \to \mathbb{P}_1$, where $V \subset \mathbb{P}_4 \times \mathbb{P}_1$ is the threefold defined by

$$V=\{((x_0:\ldots:x_4),(\lambda:\mu)) \in \mathbb{P}_4 \times \mathbb{P}_1 \mid \sigma_1=0,\ \lambda\sigma_2\sigma_3+\mu\sigma_5=0\}.$$

Each of these fibres contains the 15 lines

$$x_i+x_j=0, \quad x_k+x_l=0, \quad x_m=0 \qquad (\{i,j,k,l,m\}=\{0,1,2,3,4\}) \tag{16}$$

and the 5 conics

$$x_i=0, \quad \sum_{j=0}^{4} x_j=0, \quad \sum_{j=0}^{4} x_j^2=0 \qquad (i\in\{0,1,2,3,4\}). \tag{17}$$

In order to determine the singularities of V, note that, if $\lambda\mu \neq 0$, a point $q=((x_0:\ldots:x_4),(\lambda:\mu)) \in V$ is singular if and only if $\sigma_2\sigma_3=0$, $\sigma_5=0$ and the x_i satisfy (14) $\left(\text{with } t=-\dfrac{\lambda+\mu}{6\mu}\right)$. If $\lambda=0$, then $\sigma_2\sigma_3=0$, $\sigma_5=0$ and the x_i satisfy (14) with $t=-\frac{1}{6}$ and $u=0$. If $\mu=0$, then $\sigma_2\sigma_3=0$, $\sigma_5=0$ and the x_i satisfy $3S_2x_i^2+2S_3x_i=0$ $(0\leq i\leq 4)$, so $S_2=0$ and $S_3=0$. Therefore V has 75 singular points:

15 in the $\mathfrak{S}_5$-orbit of $((0:1:1:-1:-1),(1:-2))$,

20 in the $\mathfrak{S}_5$-orbit of $((0:0:1:\rho:\rho^2),(0:1))$, where $\rho=e^{2\pi i/3}$,

30 in the $\mathfrak{S}_5$-orbit of $((0:1:-1:i:-i),(1:0))$, and

10 in the $\mathfrak{S}_5$-orbit of $((0:0:0:1:-1),(0:1))$.

Digression: The extended Hilbert modular group for $\mathbb{Q}(\sqrt{21})$

The methods introduced so far apply to another case. Until the end of this section, let $K=\mathbb{Q}(\sqrt{21})$. Again there is a subgroup $\Gamma \subset SL_2(\mathcal{O}_K)$ of index 60 defined by (1) and we can define a modular surface Y by compactifying $\mathfrak{H}^2/\Gamma$

and resolving the cusp singularities; here the numerical invariants of Y are $c_1^2 = -10$, $\chi = 5$ and the resolutions of the five cusps are cycles consisting of six (-5)-curves. There is an involution α on $\mathfrak{H}^2/\Gamma$ defined by $(z_1, z_2) \mapsto (\varepsilon z_1, \varepsilon' z_2)$, where $\varepsilon = 55 + 12\sqrt{21}$ generates the group U_1 of units $\equiv 1 \pmod 2$. The quotient of $\mathfrak{H}^2/\Gamma$ by α is $\mathfrak{H}^2/\Gamma_e$ where

$$\Gamma_e = \Gamma \cup \left\{ \begin{pmatrix} \alpha/\sqrt{\varepsilon} & \beta/\sqrt{\varepsilon} \\ \gamma/\sqrt{\varepsilon} & \delta/\sqrt{\varepsilon} \end{pmatrix} \middle| \, \alpha, \delta \in 1 + 2\mathcal{O}_K, \beta, \gamma \in 2\mathcal{O}_K, \alpha\delta - \beta\gamma = \varepsilon \right\}$$

is the extended Hilbert modular group. Let Y_e be the minimal desingularization of $\mathfrak{H}^2/\Gamma_e \cup \{\text{cusps}\}$; then Y_e is isomorphic to $\tilde{Y}/\tilde{\alpha}$, where $\tilde{Y}$ denotes the surface obtained by blowing up the fixed points of α on Y and $\tilde{\alpha}$ the extension of the involution α to $\tilde{Y}$. By the Hurwitz formula, therefore,

$$c_1^2(Y_e) = -5, \quad \chi(Y_e) = \frac{5}{2} + \frac{r}{8},$$

where r is the number of fixed points of α on Y. In particular, $r \neq 0$. But then $r \geqq 20$ since the set of fixed points of α must be invariant under the operation of $G = SL_2(\mathcal{O}_K)/\Gamma$ and the surface $\mathfrak{H}^2/SL_2(\mathcal{O}_K) = (\mathfrak{H}^2/\Gamma)/G$ has quotient singularities only of order 2 and 3. Hence $\chi(Y_e) \geqq 5$. By blowing down F_1 (which again consists of 10 exceptional curves) on Y_e we get a surface Y_e^0 with $c_1^2(Y_e^0) = 5$ and $p_g(Y_e^0) = p_g(Y_e) = \chi(Y_e) - 1$. By an argument like that in § 3 one shows that Y_e^0 is minimal; now the inequality $c_1^2 \geqq 2p_g - 4$, valid for minimal surfaces of general type ([1], Theorem 9) implies $\chi(Y_e) = 5$. Using the cusp resolutions and F_4 as in the proof of Theorem 1, we conclude that the canonical map is a holomorphic map of Y_e^0 onto a quintic surface in $\mathbb{P}_3 \subset \mathbb{P}_4$ having 20 singularities, each with a resolution consisting of one (-2)-curve. Theorem 2 then implies the following result:

Theorem 3. *The minimal model Y_e^0 of the Hilbert modular surface corresponding to the congruence subgroup of level 2 in the extended Hilbert modular group for $\mathbb{Q}(\sqrt{21})$ is isomorphic to the minimal desingularization of the quintic*

$$\left\{ x = (x_0 : \ldots : x_4) \in \mathbb{P}_4(\mathbb{C}) \,\middle|\, \sum_{i=0}^{4} x_i = 0, \ \sum_{i=0}^{4} x_i^5 = \tfrac{5}{4} \sum_{i=0}^{4} x_i^2 \sum_{j=0}^{4} x_j^3 \right\}.$$

§ 5. The Involution τ and a $K3$ Surface

The involution $(z_1, z_2) \mapsto (z_2, z_1)$ of $\mathfrak{H} \times \mathfrak{H}$ induces an involution τ on Y. Since the curve F_1 is transformed into itself under τ we obtain an involution τ^0 of Y^0. From the definition of the involution it follows that the cusps 0, 1, and 2 (corresponding to ∞, 0 and 1 resp.) are fixed and that the cusps 3 and 4 are interchanged. Therefore $(\tau^0)^* s_i = \pm s_i$ for $i = 0, 1$ and 2, and $(\tau^0)^* s_3 = \pm s_4$, where $(\tau^0)^*$ is the involution on $H^0(Y^0, K_{Y^0})$ induced by τ^0 and the sections s_i are as in Theorem 1. Since the relation $\sum_{i=0}^{4} s_i = 0$ has to be preserved the signs are equal.

Hence the involution of the quintic surface S which is induced by τ^0 is given by

$$\tau_S\colon (x_0:x_1:x_2:x_3:x_4)\mapsto(x_0:x_1:x_2:x_4:x_3).$$

The fixpoint locus of τ_S can easily be determined. It equals:

$$\{(x_0:\ldots:x_4)\in\mathbb{P}_3\subset\mathbb{P}_4 \mid x_3-x_4=0\}\cup\{(0:0:0:1:-1)\}. \tag{18}$$

It is known ([4]) that the 1-dimensional part of the fixed-point set of the involution of $\mathfrak{H}^2/SL_2(\mathcal{O})$ induced by $(z_1,z_2)\mapsto(z_2,z_1)$ consists of the two curves F_1 and F_{13}. Hence the 1-dimensional part of the fixed-point set of the involution on $\mathfrak{H}\times\mathfrak{H}/\Gamma$ is contained in $F_1\cup F_{13}$ on $\mathfrak{H}\times\mathfrak{H}/\Gamma$. One easily verifies that none of the curves of the cusp resolutions in Y is pointwise fixed under τ. Also the inverse image of the point $(0:0:0:1:-1)\in S$ in Y^0 is the point p_{34} obtained by blowing down the curve $F_1^{34}\subset Y$. It now follows from (18) that the image of F_{13}^{34} in S is the curve given by $x_3-x_4=0$ (recall that F_{13}^{34} denotes the component of F_{13} which intersects the resolutions of cusps 0, 1 and 2) and that the fixed point locus of τ in Y consists of the two curves F_1^{34} and F_{13}^{34}. By virtue of the $\mathfrak{A}_5$-symmetry, it is clear that the image of F_{13}^{ij} in S is given by $x_i-x_j=0$.

Now consider the surface Y' obtained by blowing down the nine components F_1^{ij} $(i,j)\neq(3,4)$ of F_1 in Y (or by blowing up the point p_{34} in Y^0).The involution τ induces an involution τ' of Y'.

Theorem 4. *The quotient Y'/τ' is non-singular. Its minimal model, obtained by blowing down 3 exceptional curves, is a $K3$ surface, namely the non-singular model of the double covering of $\mathbb{P}_2$ branched along the sextic curve*

$$(\Sigma_1^3+\Sigma_3)^2-4\Sigma_1(\Sigma_2-\Sigma_1^2)(\Sigma_3-\Sigma_1\Sigma_2)=0, \tag{19}$$

where $\Sigma_1,\Sigma_2,\Sigma_3$ denote the elementary symmetric functions in the coordinates of $\mathbb{P}_2$.

Proof. The fixed-point locus of τ' is 1-dimensional (it consists of the images of F_1^{34} and F_{13}^{34}), so Y'/τ' is non-singular. We use the Hurwitz-formula for branched coverings. The canonical classes of Y' and Y'/τ' are related by

$$K_{Y'}=\alpha^*K_{Y'/\tau'}+F_1^{34}+F_{13}^{34}, \tag{20}$$

where $\alpha\colon Y'\to Y'/\tau'$ is the natural map. The divisor $F_1^{34}+F_{13}^{34}+\sum_{i=1}^{6}E_{\alpha_i}$, where $E_{\alpha_1},\ldots,E_{\alpha_6}$ are the images of the (-2)-curves in the cusp resolutions which intersect F_{13}^{34}, is a canonical divisor on Y' (namely the pull back of (x_3-x_4) on S plus the exceptional curve). These six (-2)-curves E project down to three exceptional curves on Y'/τ' and by (20) these three curves together form a canonical divisor. Blowing them down, we obtain a $K3$ surface.

Now consider the projection of S from $(0:0:0:1:-1)$ onto the (x_0,x_1,x_2)-plane. This projection factors through the involution τ_S and exhibits S/τ_S as a twofold covering of $\mathbb{P}_2$ branched along a curve R of degree 6. If we write $\Sigma_1,\Sigma_2,\Sigma_3$ for the elementary symmetric functions in x_0,x_1 and x_2, then the equations of S take the form

$$x_3+x_4+\Sigma_1=0,$$
$$\Sigma_1(x_3x_4)^2-(\Sigma_1^3+\Sigma_3)x_3x_4+(\Sigma_2-\Sigma_1^2)(\Sigma_3-\Sigma_1\Sigma_2)=0.$$

The equation of the branch curve R is obtained by setting the discriminant of the quadratic equation equal to zero.

Remark. Each component F_{13}^{ij} of F_{13} is isomorphic to $\overline{\mathfrak{H}/\Gamma_0^*(52)}$, where

$$\Gamma_0^*(52)=\Gamma_0(52)\cup\Gamma_0(52)\begin{pmatrix}0 & 1/\sqrt{52}\\ -\sqrt{52} & 0\end{pmatrix},$$

and is a non-singular curve of genus 3. The equation for F_{13}^{ij} on S enables us to give a (singular) model for this curve in $\mathbb{P}_2(x_0,x_1,x_2)$. In fact, F_{13}^{34} is given by the equations

$$x_3-x_4=0$$
$$\sigma_1=0$$
$$2\sigma_5-\sigma_2\sigma_3=0.$$

Using the symmetric functions Σ_i in x_0,x_1,x_2 as defined above, we find by elimination

$$3\Sigma_1^5-16\Sigma_1^3\Sigma_2+20\Sigma_1^2\Sigma_3+16\Sigma_1\Sigma_2^2-16\Sigma_2\Sigma_3=0.$$

This curve in $\mathbb{P}_2$ has 3 singular points: $(0:1:1)$, $(1:0:1)$ and $(1:1:0)$.

§ 6. Defining Equations for Modular Curves

In §§ 3 and 4 we defined a map from the Hilbert modular surface Y onto a quintic surface $S\subset\mathbb{P}_4$. In this section we study the images of the modular curves $F_N\subset Y$ in S and in particular prove that the image of any F_N ($N\neq 1$) is a complete intersection in $\mathbb{P}_4$.

We have already determined, at least implicitly, the equation in S of the images of several curves on Y. In particular, the ten components F_1^{ij} of F_1 are mapped to the ten non-singular points $\bar{p}_{ij}$ of S lying in the $\mathfrak{A}_5$-orbit of $\bar{p}_{01}=(1:-1:0:0:0)$ (since the image of F_1^{ij} in Y^0 is the point p_{ij} where the three sections s_k $(k\neq i,j)$ vanish) and the thirty (-2)-curves of the cusp resolutions are mapped to the fifteen singular points of S, i.e. to the points in the $\mathfrak{A}_5$-orbit of $(0:1:-1:1:-1)$. We claim that the fifteen (-5)-curves of the cusp resolutions and the five components of the curve F_4 are mapped to the fifteen lines and the five conics on S given by Equations (16) and (17), respectively. Indeed, from the construction of the s_i in the proof of Theorem 1 it is clear that the divisor of $\sigma_5=\prod_{i=0}^{4}x_i$ consists of the image of F_4 together with all of the curves coming from the cusp resolutions. Since $\sigma_5=\frac{1}{2}\sigma_2\sigma_3$ on S, this divisor is the union of the divisors of σ_3 and σ_2, which are given by Equations (16) and (17), respectively. The intersection points of the various components of F_4 are the twenty points in the $\mathfrak{A}_5$-orbit of $(0:0:1:\rho:\rho^2)$, $\rho=e^{2\pi i/3}$. Finally, as we saw in § 5, the image of F_{13} in S is the zero-divisor of the polynomial

$$f_{13}(x_0, \dots, x_4) = \prod_{0\leq i<j\leq 4} (x_i - x_j). \tag{21}$$

We now show that all of the modular curves F_N $(N\neq 1)$ are complete intersections in S, i.e. that each such curve can be given in $\mathbb{P}_4$ by the defining equations of S and one further equation. We will determine this equation for several values of N in § 7.

Theorem 5. *The image in S of each modular curve F_N, $N\neq 1$, can be given by a single equation of the form*

$$f_N(x_0, \dots, x_4) = 0, \tag{22}$$

where f_N is a homogeneous polynomial whose degree α_N is given by

$$\sum_{\substack{d\geq 1\\ d^2|N}} \alpha_{N/d^2} = 2\sum_{d|N} \left(\frac{d}{13}\right)\left(d+\frac{N}{d}\right) - 12H_{13}^0(N) - 3a_N - 6b_N, \tag{23}$$

with $H_{13}^0(N)$, a_N, b_N as in Equations (5) *and* (7). *The polynomial f_N is given by* (21) *for $N=13$ and can be written as a polynomial in the elementary symmetric functions $\sigma_2, \sigma_3, \sigma_4$ if $N\neq 13$.*

Proof. We will prove that for each F_N $(N\neq 1)$ there exists an effective divisor S_N on Y consisting only of (-2)-curves from the cusp resolutions such that

$$F_N + (\alpha_N + \tfrac{1}{10} F_N F_1) F_1 + S_N \tag{24}$$

is an α_N-canonical divisor. The pull-back of the hyperplane bundle of S under the 1-canonical map (10) is the canonical bundle of Y^0. Since S is a surface whose only singularities are rational double points, each section of the α_N-th tensor power of the hyperplane bundle of Y can be extended to a section of the α_N-th tensor power of the hyperplane bundle of $\mathbb{P}_3$. Hence, if the divisor (24) is α_N-canonical, its image in S is the divisor of some homogeneous polynomial f_N of degree α_N. Since (24) is invariant under $\mathfrak{S}_5$, this polynomial is either invariant under $\mathfrak{S}_5$ or else changes sign under odd permutations of the coordinates. In the former case f_N can be expressed in terms of the σ_i (and hence of $\sigma_2, \sigma_3, \sigma_4$); in the latter case, f_N is divisible by the polynomial (21) and hence its divisor F_N contains F_{13}, so $N=13$.

In order to show that (24) is an α_N-canonical divisor, note that $H_2(Y;\mathbb{Q})$ splits as the direct sum of $\operatorname{Im}(H_2(\mathfrak{H}^2/\Gamma;\mathbb{Q})\to H_2(Y;\mathbb{Q}))$ and W, where W is the subspace of $H_2(Y;\mathbb{Q})$ generated by the homology classes of the 45 curves E_α and D_β. We define F_N^c as the component of the homology class $[F_N]$ of F_N in $\operatorname{Im}(H_2(\mathfrak{H}^2/\Gamma;\mathbb{Q})\to H_2(Y;\mathbb{Q}))$. The Poincaré dual of F_N^c is invariant under the action of $\mathfrak{A}_5$ and hence comes from a class in

$$\operatorname{Im}(H^2(\mathfrak{H}^2/SL_2(\mathcal{O});\mathbb{Q})\to H^2(Y(13);\mathbb{Q}))$$

which is invariant under the involution τ^* and of type $(1,1)$ in the Hodge decomposition of $H^2(Y(13);\mathbb{C})$. But the dimension of the space of such classes was computed in [5] and equals 1. Hence there exists a $\lambda\in\mathbb{Q}$ such that

$$F_N^c+(\alpha_N+\tfrac{1}{10}F_N F_1)F_1^c=\lambda(F_4^c+2F_1^c),$$

i.e.

$$[F_N]+(\alpha_N+\tfrac{1}{10}F_N F_1)[F_1]+S_N=\lambda([F_4]+2[F_1]) \tag{25}$$

with $S_N\in W$. On the other hand, the number α_N defined by (23) is an integer satisfying $5\alpha_N=K_{Y^0}F_N$ on Y^0 (this follows from (5) and (8) and a comparison of K_Y and K_{Y^0}; recall that $T_N=\bigcup_{d^2|N}F_{N/d^2}$). Also, F_4 is a 2-canonical divisor on Y^0, because its image in S is the divisor of σ_2. Hence

$$F_4 F_N=10\alpha_N \tag{26}$$

(the intersection number is the same on Y and on Y^0, since $F_4 F_1=0$). Therefore, intersecting (25) with F_1 and F_4 and using $F_1^2=-10$, $F_4^2=20$, we obtain

$$-10\alpha_N+S_N F_1=-20\lambda,\qquad 10\alpha_N+S_N F_4=20\lambda$$

and hence $S_N(F_4+F_1)=0$. On the other hand, S_N lies in the space W generated by the classes of the E_α und D_β and is $\mathfrak{A}_5$-invariant, so

$$S_N=d_N\sum_{\beta=1}^{15}[D_\beta]+e_N\sum_{\alpha=1}^{30}[E_\alpha]$$

for some d_N, e_N. Since $(F_4+F_1)D_\beta\neq0$ and $(F_4+F_1)E_\alpha=0$, we must have $d_N=0$. Intersecting both sides of (25) with E_α now gives $e_N=F_N E_\alpha$. In particular, $e_N\in\mathbb{Z}$. Using the well-known sequence

$$\cdots\to H^1(Y,\mathcal{O}_Y)\to H^1(Y,\mathcal{O}_Y^*)\to H^2(Y;\mathbb{Z})\to\cdots$$

and the fact that $q(Y)=\dim H^1(Y,\mathcal{O}_Y)=0$ we see that homology implies linear equivalence. Therefore (25) and the fact that F_4+2F_1 is a 2-canonical divisor on Y imply that (24) is an α_N-canonical divisor. Q.E.D.

We observe that Theorem 5 and its proof can be used to obtain class-number relations in the style of [5]. For example, from (6) and (26) we find

$$10\sum_{d^2|N}\alpha_{N/d^2}=F_4 T_N=T_4 T_N-T_1 T_N=60(H_{13}^0(4N)-H_{13}^0(N)) \tag{27}$$

if N is odd and not a square; in combination with (23) this gives

$$2\sum_{d|N}\left(\frac{d}{13}\right)\left(d+\frac{N}{d}\right)=6(H_{13}^0(4N)+H_{13}^0(N))+3a_N+6b_N.$$

As another example we take two curves T_M, T_N with T_N compact, MN prime to 26 and not a square and $M\neq1$. The intersection number of T_M and T_N on Y is given by (6):

$$(T_M \cdot T_N)_Y = 60 \sum_{d|(M,N)} \left(\frac{d}{13}\right) d H_{13}^0(MN/d^2).$$

After blowing down we get for the intersection number on Y^0

$$(T_M \cdot T_N)_{Y^0} = 60 \sum_{d|(M,N)} \left(\frac{d}{13}\right) d H_{13}^0(MN/d^2) + 6 H_{13}^0(M)\, H_{13}^0(N).$$

By Theorem 5, this equals

$$5\Big(\sum_{d^2|M} \alpha_{M/d^2}\Big)\Big(\sum_{d^2|N} \alpha_{N/d^2}\Big),$$

since $K_{Y^0}^2 = 5$ and T_N is disjoint from the (-2)-curves of the cusp resolutions. Using (27) we infer the following class number relation:

$$3(H_{13}^0(4M) - H_{13}^0(M))(H_{13}^0(4N) - H_{13}^0(N))$$
$$= \sum_{d|(M,N)} \left(\frac{d}{13}\right) d H_{13}^0(MN/d^2) + \tfrac{1}{10} H_{13}^0(M)\, H_{13}^0(N).$$

§ 7. Explicit Equations for Modular Curves

In this section we illustrate how the polynomials f_N of Theorem 5 can be determined. Each such determination gives in principle an explicit (in general singular) projective model for a modular curve like the model for $\overline{\mathfrak{H}/\Gamma_0^*}$ (52) given at the end of § 5.

The degree of f_N for $N \leqq 26$ is given by the following table

N	3	4	9	10	12	13	14	16	17	22	23	25	26
α_N	4	2	12	8	12	10	12	12	24	20	36	32	12

(28)

(Equation (27) implies that $\alpha_N \equiv 0 \pmod 4$ for $N \neq 4, 13$). We will determine f_N for the 9 values of N with $\alpha_N \leqq 12$.

The equation of F_4 was determined in § 6: we have (up to a constant)

$$f_4 = \sigma_2. \tag{29}$$

The polynomial f_{13} is given by Equation (21). The polynomial f_3 has degree 4 and must vanish at the point $(0:1:-1:1:-1) \in S$, since F_3 passes through the curves E_α. Hence it is given by

$$f_3 = 4\sigma_4 - \sigma_2^2. \tag{30}$$

However, for other values of N it is not so easy to determine f_N. For example, f_{10} has degree 8 and must vanish quadruply at the 10 points $\bar{p}_{ij}$ where $\sigma_3 = \sigma_4 = 0$ (since $F_1 \cdot F_{10} = 40$), so it has the form

$$f_{10} = \sigma_4^2 + A\sigma_3^2\sigma_2 \tag{31}$$

for some A, but there is no simple way to determine A. For other values of N, where $\alpha_N \geqq 12$, there are even more unknown coefficients. To find them, we will use the following two methods.

1. In [10], the intersection behaviour of curves F_N on Hilbert modular surfaces was described completely: these intersections occur in "special points", and at each special one can say exactly which F_N pass through the point and determine their tangent directions. Even though we do not know how to identify the tangent space of the Hilbert modular surface at a point with the tangent space of S at the corresponding point, we know that the cross-ratio of the tangent directions of any four curves must be the same in both spaces, and this gives conditions on the coefficients of the defining equations.

2. The components of the modular curves F_3 and F_4 are rational curves which can be represented as branched coverings of $\overline{\mathfrak{H}/SL_2(\mathbb{Z})}$ and hence parametrised by an algebraic function of $j(z)$, $z \in \mathfrak{H}$. The intersections of F_3 and F_4 with other F_N at special points correspond to numbers z which are quadratic over $\mathbb{Q}$, for which j can be calculated by well-known methods. On the other hand, we can give explicit parametrisations of F_3 and F_4 on S, so by comparing cross-ratios we can determine the coordinates on S of the intersection points of any F_N with F_3 or F_4. A similar argument works with F_1 instead of F_3 or F_4: now the points on $F_1^{ij} \subset Y$ correspond to tangent directions at $\bar{p}_{ij} \in S$, so by computing j-invariants one can determine the tangent directions of the branches of F_N which pass through that point.

We now explain both methods in more detail.

Every branch of $F_N \subset \mathfrak{H}^2/\Gamma$ is the image of a curve in $\mathfrak{H}^2$ defined by

$$a_1 \sqrt{13} z_1 z_2 - \lambda' z_1 + \lambda z_2 + a_2 \sqrt{13} = 0, \tag{32}$$

where $\begin{pmatrix} a_1\sqrt{13} & \lambda \\ -\lambda' & a_2\sqrt{13} \end{pmatrix}$ $(a_1, a_2 \in \mathbb{Z},\ \lambda \in \mathcal{O})$ is a skew-hermitian matrix of determinant N. Conversely, for each $z=(z_1, z_2) \in \mathfrak{H}^2$, the skew-hermitian matrices satisfying (32) form a $\mathbb{Z}$-module $\mathfrak{M}_z$ of rank $\leqq 2$, and the map assigning to such a matrix its determinant $13 a_1 a_2 + \lambda\lambda'$ is a positive definite quadratic form φ_z on $\mathfrak{M}_z$ whose values are the integers N for which $z \in F_N$. The map assigning to a non-zero element of $\mathfrak{M}_z$ the tangent direction at z of the corresponding curve on $\mathfrak{H}^2$ defines a projective map $\mathbb{P}(\mathfrak{M}_z \otimes \mathbb{C}) \to \mathbb{P}(T_z(\mathfrak{H}^2))$. Also, since Γ acts freely, $\mathbb{P}(T_z(\mathfrak{H}^2)) \xrightarrow{\sim} \mathbb{P}(T_\mathfrak{z}(\mathfrak{H}^2/\Gamma))$, where $\mathfrak{z} \in \mathfrak{H}^2/\Gamma$ is the image of z. Thus, if $\mathfrak{z}$ is "special", i.e. if $\operatorname{rk}(\mathfrak{M}_z)=2$, we obtain a map $\mathbb{P}(\mathfrak{M}_z \otimes \mathbb{C}) \to \mathbb{P}(T_\mathfrak{z}(\mathfrak{H}^2/\Gamma))$ which is an isomorphism of projective lines. In this case, φ_z is a binary quadratic form which can be written with respect to a $\mathbb{Z}$-basis of $\mathfrak{M}_z$ as $a u^2 + b u v + c v^2$, and any parameter on $\mathbb{P}(T_\mathfrak{z}(\mathfrak{H}^2/\Gamma))$ is related to u/v by a fractional linear transformation.

For example, the curves F_3 and F_4 meet in a point $\mathfrak{z} \in \mathfrak{H}^2/\Gamma$ with $\varphi_z(u,v) = 3u^2 + 3uv + 4v^2$. This point corresponds to the point $p=(0:0:1:\rho:\rho^2) \in S$ $(\rho = e^{2\pi i/3})$ which is a common root of the polynomials (29) and (30). (Note that φ_z represents 4 twice, in agreement with the statement in § 6 that two components of F_4 meet in p.) The form $3u^2+3uv+4v^2$ represents 3 and 13 once and 4 and 10 twice, while the polynomials (30), (21) (resp. (29), (31)) vanish simply (resp.

doubly) at p, and we have the following table:

N	f_N	$\left(\frac{\partial f_N}{\partial x_0}\Big/\frac{\partial f_N}{\partial x_1}\right)\Big\vert_p$	$(u,v), 3u^2+3uv+4v^2=N$
3	$4\sigma_4-\sigma_2^2$	1	$\pm(1,0)$
4	σ_2	$0, \infty$	$\pm(0,1),\ \pm(1,-1)$
13	$\prod(x_i-x_j)$	-1	$\pm(1,-2)$
10	$\sigma_4^2+A\sigma_3^2\sigma_2$	$1+A\pm\sqrt{2A+A^2}$	$\pm(1,1),\ \pm(2,-1)$

Comparing the entries for $N=3,4,13$ we obtain

$$\left(\frac{\partial f_N}{\partial x_0}\Big/\frac{\partial f_N}{\partial x_1}\right)\Big\vert_p=\frac{u}{u+v}$$

and hence $1+A\pm\sqrt{2A+A^2}=\frac{1}{2}$, 2 and $A=\frac{1}{4}$. Thus

$$f_{10}=\sigma_4^2+\tfrac{1}{4}\sigma_3^2\sigma_2. \tag{33}$$

For the second method we must first describe the components of F_3. Arguing as in § 2, we find that F_3 has 10 rational components, each meeting 3 cusps in the two (-2)-curves adjacent to some (-5)-curve D_β. Let F_3^{ij} be the component of F_3 not meeting the i-th and j-th cusps. For example, F_3^{34} is the image of the curve

$$(4-\sqrt{13})\,z_1-(4+\sqrt{13})\,z_2=0 \tag{34}$$

in $\mathfrak{H}^2$ and is isomorphic to $\mathfrak{H}/\Gamma_0(12)$. We claim that the image of F_3^{ij} in S is contained in the divisor of x_i+x_j. Indeed, the canonical divisor on Y^0 corresponding to (x_i+x_j) has the form

$$\sum_{\mu=1}^{3} D_{\beta_\mu}+\sum_{\nu=1}^{12} E_{\alpha_\nu}+R \qquad (R\geqq 0),$$

where the D_{β_μ} are the three D_β passing through p_{ij} and the E_{α_ν} are the curves of the (-2)-configurations adjacent to these D_β. But F_3^{ij} meets $\sum E_{\alpha_\nu}$ in 6 points and $K_{Y^0}\cdot F_3^{ij}=2$, so F_3^{ij} must be contained in R. From the equations of S we deduce that $\sigma_2=-2x_i^2=2x_ix_j$ on F_3^{ij} and then obtain easily a parametrisation, e.g.

$$\begin{aligned}F_3^{34}=\{(&4tu:3t^2-2tu-u^2:-3t^2-2tu\\ &+u^2:3t^2+u^2:-3t^2-u^2)\,|\,(t:u)\in\mathbb{P}_1(\mathbb{C})\}.\end{aligned} \tag{35}$$

On the other hand, the curve $\mathfrak{H}/\Gamma_0(12)$ can be parametrised by a certain function $k(z)$ $(z\in\mathfrak{H})$ related algebraically to $j(z)$; the functions $k(z)$ and t/u must then be related by a fractional linear transformation. It is, however, more convenient to divide by the action of $\mathfrak{S}_5$ (i.e. to work on $\overline{(\mathfrak{H}^2/SL_2(\mathcal{O}))/\tau}$ instead of $\mathfrak{H}^2/\Gamma$), since $F_3/\mathfrak{S}_5$ is the simpler modular curve $\overline{\mathfrak{H}/\Gamma_0(3)}$ and the set $F_3\cap F_N$ is $\mathfrak{S}_5$-invariant. We represent $F_3/\mathfrak{S}_5$ as $F_3^{34}/\mathfrak{S}_3$, where an element of $\mathfrak{S}_3$ acts by permuting

x_2, x_3, x_4 in the obvious way and, if it is odd, also interchanges x_0 and x_1. Then we can take

$$\Delta = 4\frac{(x_2-x_3)(x_3-x_4)(x_4-x_2)}{(x_0-x_1)^3} \qquad ((x_0:\ldots:x_4)\in F_3^{34}) \tag{36}$$

as a parameter on $F_3^{34}/\mathfrak{S}_3$; it is related to $\sigma_2, \sigma_3, \sigma_4$ by

$$\Delta^2 = \frac{\sigma_2^3 + 54\sigma_3^2}{\sigma_2^3}. \tag{37}$$

In particular, the two cusps of $\overline{\mathfrak{H}/\Gamma_0(3)}$ correspond to $\Delta = \pm 1$ (since $\sigma_3 = 0$) and the fixed point of order 3 to $\Delta = \infty$ (namely to the intersection of F_3 and F_4). On the other hand, the map $\overline{\mathfrak{H}/\Gamma_0(3)} \to \overline{\mathfrak{H}/SL_2(\mathbb{Z})}$ is a 4-fold branched covering with 2 points over ∞ (one unramified, the other triply ramified), two over ρ (again one unramified and the other triply ramified) and two points over i (both doubly ramified). The values of j at ∞, ρ and i are ∞, 0 and 1728, respectively. Together, this implies that j and Δ are related by

$$j = 432\frac{(4\Delta-5)^3}{(\Delta-1)(\Delta+1)^3}. \tag{38}$$

A special point $\mathfrak{z}$ on F_3 with associated quadratic form $a u^2 + b u v + c v^2$ corresponds to a point $z \in \mathfrak{H}$ satisfying a quadratic equation of discriminant $d = \frac{b^2 - 4ac}{13} < 0$.

As a check of (38), note that $\Delta = 0$ corresponds to the intersection points of F_3 and F_{13} not lying on F_4 (compare (21) and (36)), which have the quadratic form $3u^2 + 13v^2$ with $d = -12$, and the value of $j(z)$ for z satisfying a quadratic equation of discriminant -12 is 54000, in accordance with (38). Proceeding in this way for other special points on F_3 we find the following table:

Point	N	d	j	Δ	$\sigma_2, \sigma_3, \sigma_4$	
cusp	$3, 9, 12, 13, \ldots$	–	∞	± 1	$-2, 0, 1$	
$[3,3,4]$	$3, 4, 10, 13, \ldots$	-3	0	∞	$0, 1, 0$	
$[3,0,13]$	$3, 13, 16, \ldots$	-12	54000	0	$-6, 2, 9$	(39)
$[3,2,9]$	$3, 9, 10, 14, \ldots$	-8	8000	$\pm 5\sqrt{-2}/2$	$-1, \frac{1}{2}, \frac{1}{4}$	
$[3,1,12]$	$3, 12, 14, 16, 26, \ldots$	-11	-32768	$\pm\sqrt{-11}/4$	$-2, \frac{1}{2}, 1$	
$[3,0,26]$	$3, 26, \ldots$	-24	$1399 \pm 988\sqrt{2}$	$\pm\sqrt{2}/2$	$-1, \frac{1}{6}, \frac{1}{4}$	

In the first column we have indicated a special point $z \in \mathfrak{H}$ with quadratic form $\varphi_z = a u^2 + b u v + c v^2$ by the symbol $[a, b, c]$. The second column gives the values of N for which F_N passes through z. The values $\sigma_2, \sigma_3, \sigma_4$ in the last column are obtained from (37) and are defined up to $(\sigma_2, \sigma_3, \sigma_4) \sim (\alpha^2\sigma_2, \alpha^3\sigma_3, \alpha^4\sigma_4)$. For example, the fact that F_{10} passes through the point with $\sigma_2 = -1$, $\sigma_3 = \frac{1}{3}$, $\sigma_4 = \frac{1}{4}$ gives another proof that the number A in (31) equals $\frac{1}{4}$.

Arguing similarly for F_4 we find that $F_4/\mathfrak{S}_5 \cong \overline{\mathfrak{H}/SL_2(\mathbb{Z})}$, the parameter σ_4^3/σ_3^4 on F_4 being related to j by

$$j(z)=2^{14}\sigma_4^3/\sigma_3^4,$$

and obtain the table

Point	N	d	j	$\sigma_2,\sigma_3,\sigma_4$
cusp	4	–	∞	0, 0, 1
[4, 3, 3]	3, 4, 10, 13, ...	−3	0	0, 1, 0
[4, 0, 13]	4, 13, ...	−4	$1728=12^3$	0, 4, 3
[4, 1, 9]	4, 9, 12, 14, ...	−11	$-32768=-32^3$	0, 1, −2
[4, 3, 16]	4, 16, 26, ...	−19	$-884736=-96^3$	0, 1, −6

(40)

Finally, the tangent directions at the point $\bar{p}_{01}=(1:-1:0:0:0)\in S$ correspond (after blowing up $\bar{p}_{01}$) to points on $F_1^{01}=\overline{\mathfrak{H}/\Gamma_0(4)}$. Again dividing by the action of the isotropy group, we can take $\sigma_4^3/\sigma_3^2\sigma_2^3$ as a parameter for the tangent directions and $j(z)$ as a parameter for the modular curve. They are related by $j=-2^{10}\sigma_4^3/\sigma_3^2\sigma_2^3$, because j equals ∞, 0, and 1728 at the cusp and the two fixed points on $\overline{\mathfrak{H}/SL_2(\mathbb{Z})}$ and these points correspond to the intersections of F_1 with the cusp resolution and the curves F_{10} and F_{13}, where $\sigma_4^3/\sigma_3^2\sigma_2^3$ equals ∞, 0 and $-27/16$, respectively. The curves F_{14} and F_{26} intersect F_1 at the special points [1, 0, 13] and [1, 0, 26] where $d=-4$, $j=1728$ and $d=-8$, $j=8000$, respectively. Hence

$$\lim_{\substack{z\to\bar{p}_{01}\\ z\in F_{14}}}\sigma_4^3/\sigma_3^2\sigma_2^3=-\tfrac{27}{16},\qquad \lim_{\substack{z\to\bar{p}_{01}\\ z\in F_{26}}}\sigma_4^3/\sigma_3^2\sigma_2^3=-\tfrac{125}{16}. \tag{41}$$

We now use the information we have obtained to determine the equation of F_N for the five values of N in (28) with $\alpha_N=12$, namely $N=9$, 12, 14, 16 and 26. Each such equation has the form

$$f_N=A\sigma_4^3+B\sigma_4^2\sigma_2^2+C\sigma_4\sigma_2^4+D\sigma_2^6+E\sigma_4\sigma_3^2\sigma_2+F\sigma_3^2\sigma_2^3+G\sigma_3^4 \tag{42}$$

where the coefficients $A,\dots,G$ are determined up to a constant. To determine them, we first use our knowledge of the way F_N passes through the cusp resolutions. The line D_1 of the cusp resolution (cf. (9)) can be parametrised as

$$(0:1:-1:t:-t)\qquad (t\in\mathbb{P}_1(\mathbb{C})=\mathbb{C}\cup\{\infty\}) \tag{43}$$

with $\sigma_2=-1-t^2$, $\sigma_3=0$, $\sigma_4=t^2$. The points $t=\pm 1$ correspond to the images of curves E_α, the points $t=0,\infty$ to the images of F_1^{12} and F_1^{34}, and the points $t=\dfrac{\zeta+1}{\zeta-1}$ (ζ^2 a primitive n-th root of unity, $n>1$) to the intersection of F_{n^2} with D_1 (cf. § 2). We deduce that the restrictions of $f_9, f_{12}, f_{14}, f_{16}$ and f_{26} to the line (43) are given (up to a constant) by

$$(t^2-1)^4(t^4+\tfrac{10}{3}t^2+1),\ t^4(t^2-1)^2,\ t^6,\ t^4(t^4+6t^2+1)$$

and t^6, respectively. This gives the coefficients A, B, C, D in (42). Then $E+4F$ and G can be determined by using the values of $\sigma_2,\sigma_3,\sigma_4$ at the intersection points of F_N with F_3, F_4 as given by (39) and (40), and the coefficient F can be determined for $N=14$ and $N=26$ by using (41). This gives

$$\begin{aligned} f_9 &= (4\sigma_4-\sigma_2^2)^2(4\sigma_4+3\sigma_2^2)+32\sigma_3^2(\sigma_2^3+4\sigma_3^2)+E_9\,\sigma_2\,\sigma_3^2(\sigma_4-\tfrac{1}{4}\sigma_2^2)\\ f_{12} &= \sigma_4^2(4\sigma_4-\sigma_2^2)+\tfrac{1}{4}\sigma_3^2(\sigma_2^3+32\sigma_3^2)+E_{12}\,\sigma_2\,\sigma_3^2(\sigma_4-\tfrac{1}{4}\sigma_2^2)\\ f_{14} &= \sigma_4^3-\tfrac{9}{2}\sigma_2\,\sigma_3^2\,\sigma_4+\tfrac{27}{16}\sigma_2^3\,\sigma_3^2+2\sigma_3^4\\ f_{16} &= \sigma_4^2(4\sigma_4+\sigma_2^2)+\tfrac{43}{4}\sigma_2\,\sigma_3^2\,\sigma_4+216\sigma_3^4+E_{16}\,\sigma_2\,\sigma_3^2(\sigma_4-\tfrac{1}{4}\sigma_2^2)\\ f_{26} &= \sigma_4^3-\tfrac{45}{2}\sigma_2\,\sigma_3^2\,\sigma_4+\tfrac{125}{16}\sigma_2^3\,\sigma_3^2+54\sigma_3^4. \end{aligned} \tag{44}$$

To determine the remaining coefficients E_9, E_{12} and E_{16}, we use the cross-ratios of the tangent directions of the curves F_N passing through the special points $\mathfrak{z}$ given in tables (39) and (40). For example, since F_3, F_9, F_{10} and F_{14} pass through the point $[3,2,9]$ and three of them are known, we can determine the tangent direction of the remaining one. The values of (u,v) with $3u^2+2uv+9v^2=N$ for $N=3$, 10 and 14 are $(1,0)$, $(1,-1)$ and $(1,1)$, respectively, and the limiting value T of $\sigma_2(\sigma_2^2-4\sigma_4)/(\sigma_2^3+4\sigma_3^2)$ as $\mathfrak{z}$ tends to $[3,2,9]$ along these curves equals 0, 1/2 and $-1/3$, respectively. Hence $T=\dfrac{-2v}{u+v}$. Since F_9 corresponds to $(u,v)=(0,1)$, we obtain $\dfrac{128}{E_9}=T=-2$, $E_9=-64$. Similarly, using the special point $[3,1,12]$ and the equations of F_3, F_{14} and F_{26} we find that the limiting value of $\sigma_2(\sigma_2^2-4\sigma_4)/(\sigma_2^3+32\sigma_3^2)$ along the curve $F_{3u^2+uv+12v^2}$ equals $\dfrac{-v}{4u-2v}$; applying this to $(u,v)=(0,1)$ and $(1,1)$ yields the values $E_{12}=10$, $E_{16}=18$ for the two remaining unknown coefficients in (44). We can now use the point $[4,1,9]$ in (40), where the limiting value of $(\sigma_4^3+2\sigma_3^4)/\sigma_2\,\sigma_3^2\,\sigma_4$ along $F_{4u^2+uv+9v^2}$ equals $\dfrac{7u+2v}{12v}$, to check the values of E_9 and E_{12}. A further check is provided by blowing up the point $\bar{p}_{01}\in S$ and using the cross-ratios of tangent directions of curves at the special point $[1,1,10]$ (the fixed point of order 3 on F_1), where the limiting value of $\sigma_4^2/\sigma_2\,\sigma_3^2$ along $F_{u^2+uv+10v^2}$ equals $-\left(\dfrac{2u+v}{2v}\right)^2$. We can also use the special point $[9,3,10]$, through which F_9, F_{10} and F_{16} pass, or the special point $[9,10,10]$, through which F_9, F_{10} and F_{26} pass (and in fact F_9 doubly, which gives an extra condition), or the cross-ratios of the intersection points of F_N with the (-2)-curve E obtained by blowing up the singular points on S (and on which we can take $(4\sigma_4-\sigma_2^2)\,\sigma_2^2/\sigma_3^2$ as a parameter). Thus we obtain the coefficients $A,\ldots,G$ for our five curves F_N with considerable "overkill". Nevertheless, it is not clear whether this type of argument would suffice to determine f_N in all or even in infinitely many cases. In § 9 we will give a method with which one could in principle calculate the defining equations of all non-compact curves F_N.

§ 8. The Ring of Hilbert Modular Forms of Even Weight

In this section we will determine the structure of the ring of modular forms of even weight for Γ and for the full Hilbert modular group $SL_2(\mathcal{O})$ by making use of the canonical map $\Phi\colon Y^0\to S$ constructed in § 3. Let

$$R=\mathbb{C}[x_0,\ldots,x_4]/(\sigma_1,2\sigma_5-\sigma_2\,\sigma_3)\qquad (\sigma_i=\sigma_i(x_0,\ldots,x_4))$$

be the coordinate ring of S and

$$\psi\colon \mathbb{C}[x_0, \ldots, x_4] \to R$$

the natural projection map.

Theorem 6. *Let $M_{ev}(\Gamma) = \bigoplus_k M_{2k}(\Gamma)$ be the ring of Hilbert modular forms of even weight for Γ and let L_k be the subspace of $\mathbb{C}[x_0, \ldots, x_4]$ consisting of all homogeneous polynomials of degree $4k$ which vanish with multiplicity $\geqq 2k$ at the points in the $\mathfrak{A}_5$-orbit of $(0:0:0:1:-1)$. There exist five cusp forms ξ_i $(i=0, \ldots, 4)$ of weight 2 on Γ such that the map*

$$\alpha\colon L_k \to M_{2k}(\Gamma)$$

given by

$$F(x_0, \ldots, x_4) \mapsto \frac{F(\xi_0, \ldots, \xi_4)}{\sigma_3(\xi_0, \ldots, \xi_4)^k}$$

induces an isomorphism between $\psi(L_k)$ and $M_{2k}(\Gamma)$.

Proof of Theorem 6. Each Hilbert modular form $\omega = g(z_1, z_2)(dz_1 \wedge dz_2)^k$ of weight $2k$ for Γ defines a holomorphic section s of the k-fold tensor power of the canonical bundle $K_{\mathfrak{H}^2/\Gamma}$ of $\mathfrak{H}^2/\Gamma$. This section can be extended to a meromorphic section $\bar{s}$ of the k-fold canonical bundle of Y whose divisor $(\bar{s})$ contains the curves of the cusp resolutions with multiplicity $\geqq -k$ (cf. lemma in [4], 3.6) and is effective outside the cusp resolutions. Let K_S denote the hyperplane bundle of S. Then we have the isomorphisms

$$H^0(Y, K_Y^k) \xleftarrow{\pi^*} H^0(Y^0, K_{Y^0}^k) \xleftarrow{\phi^*} H^0(S, K_S^k),$$

where π is the map $Y \to Y^0$ which blows down F_1 and ϕ is the 1-canonical map defined by (10) in § 3. Also, the curves of the cusp resolutions in Y correspond to the divisor of σ_3 in S. Hence $\bar{s}$ defines a meromorphic section s' of K_S^k such that

$$\sigma_3^k\, s' \in H^0(S, K_S^k).$$

An element of $H^0(S, K_S^k)$ can be given by a homogeneous polynomial of degree $4k$ in $x_0, \ldots, x_4$. We claim: $\sigma_3^k\, s'$ vanishes with multiplicity $\geqq 2k$ in the points $\bar{p}_{ij}$ on S. Indeed, the divisor of the corresponding element in $H^0(Y, K_Y^{4k})$ has the form

$$\pi^*\, \phi^*(\sigma_3^k\, s') + 4k\, F_1,$$

while the divisor of the section of $H^0(Y, K_Y^3)$ corresponding to σ_3 equals

$$\sum_{\beta=1}^{15} D_\beta + \sum_{\alpha=1}^{30} E_\alpha + 3F_1 + 3F_1$$

(the first three terms form the total transform of (σ_3)). Hence the divisor $(\bar{s})$ on Y takes the form

$$\pi^*\, \phi^*(\sigma_3^k\, s') - k\sum D_\beta - k\sum E_\alpha - 2k\, F_1. \tag{45}$$

Since (45) is effective outside the cusp resolutions, $\pi^* \phi^*(\sigma_3^k s')$ contains F_1 with multiplicity $\geqq 2k$.

Conversely, if $f \in \mathbb{C}[x_0, \ldots, x_4]$ is a homogeneous polynomial of degree $4k$ such that f vanishes with multiplicity $\geqq 2k$ at the points $\bar{p}_{ij}$ on S, then f/σ_3^k defines a meromorphic section of the bundle K_Y^k on Y whose divisor is effective outside the cusp resolutions and hence a Hilbert modular form of weight $2k$. Clearly the five sections $x_i \in H^0(S, K_S)$ define (up to a constant) five cusp forms $\xi_i \in M_2(\Gamma)$ with the properties stated.

As the proof shows, the Hilbert modular forms of weight $2k$ which can be extended holomorphically over the resolutions of the cusps correspond to elements of the form $\psi(g\,\sigma_3^k)$ where g is a homogeneous polynomial of degree k, i.e. they correspond to sections in the k-fold tensor product of the hyperplane bundle of Y. Hence the dimension of the space of Hilbert modular forms that can be extended holomorphically over the cusp resolutions equals the k-th plurigenus $P_k(S)$ of S. By a well-known formula, we have:

$$P_k(S) = \begin{cases} \frac{1}{2}k(k-1)K^2 + \chi = \frac{5}{2}k(k-1) + 5 & \text{if } k \geqq 2, \\ \chi - 1 = 4 & \text{if } k = 1. \end{cases}$$

On the other hand, the Shimizu dimension formula [12] gives

$$\dim M_{2k}(\Gamma) = 20k(k-1) + 10 \qquad \text{if } k \geqq 2.$$

For $k=1$ we have $\dim M_2(\Gamma) = \dim S_2(\Gamma) + 5 = P_1(S) + 4 = 9$, since all cusp forms of weight 2 can be extended holomorphically over the cusps. The space of cusp forms of weight 2 is generated by the forms ξ_i and has dimension 4. The meromorphic sections

$$\eta_i = \sigma_2(\xi_0, \ldots, \xi_4)/\xi_i = 2\xi_0 \ldots \hat{\xi}_i \ldots \xi_4/\sigma_3(\xi_0, \ldots, \xi_4)$$

of K_S determine 5 non-cusp forms. Since η_i is holomorphic at all cusps but the i-th, these 5 forms are linearly independent and are also independent of the ξ_i. We normalize the ξ_i, which up to now have been defined only up to a multiplicative constant, by the requirement that the constant term of the Fourier expansion of $\eta_i = \sigma_2(\xi_0, \ldots, \xi_4)/\xi_i$ at the i-th cusp should be equal to 1. Thus Theorem 6 permits us to give a basis of $M_2(\Gamma)$. However, it seems to be quite difficult to write down explicitly a set of generators of the ring $M_{\mathrm{ev}}(\Gamma)$ or even to give additive generators of $M_{2k}(\Gamma)$, $k > 1$.

By contrast, we can derive from Theorem 6 both additive and ring generators for the modular forms on the full Hilbert modular group. Let $M_{\mathrm{ev}}^s(SL_2(\mathcal{O}))$ be the ring of *symmetric* Hilbert modular forms of even weight, i.e. modular forms $f(z_1, z_2)$ satisfying $f(z_1, z_2) = f(z_2, z_1)$. It is clear that $M_{\mathrm{ev}}^s(SL_2(\mathcal{O}))$ is isomorphic to the $\mathfrak{S}_5$-invariant part of $M_{\mathrm{ev}}(\Gamma)$ and hence by Theorem 6, the space $M_{2k}^s(SL_2(\mathcal{O}))$ is isomorphic to the image under ψ of all symmetric polynomials of degree $4k$ in $x_0, \ldots, x_4$ which vanish with multiplicity $\geqq 2k$ in $(0:0:0:-1:1)$. An additive basis of $M_{\mathrm{ev}}^s(SL_2(\mathcal{O}))$ is given by the elements

$$\left\{ \frac{\sigma_2^a \sigma_3^{2b} \sigma_4^c}{\sigma_3^k} \,\middle|\, a + 3b + 2c = 2k,\ 3b + c \geqq k \right\}, \tag{46}$$

where $\sigma_i=\sigma_i(\xi_0,\ldots,\xi_4)$. Indeed, since σ_3 and σ_4 vanish at $\bar{p}_{ij}$ with the multiplicities 3 and 2, respectively, the monomials (46) certainly lie in $M^s_{2k}(SL_2(\mathcal{O}))$. Conversely, suppose that a polynomial

$$f=\sum_{\substack{a_i,b_i,c_i\geq 0\\ a_i+3b_i+2c_i=2k\\ 3b_i+c_i<k}} n_{a_i,b_i,c_i}\,\sigma_2(x_0,\ldots,x_4)^{a_i}\,\sigma_3(x_0,\ldots,x_4)^{2b_i}\,\sigma_4(x_0,\ldots,x_4)^{c_i}$$

vanishes with multiplicity $\geq 2k$ in the points $\bar{p}_{ij}$ on S. Let σ_3^{2d} be the largest power of σ_3 dividing f. Then for every monomial $\sigma_2^{a_i}\sigma_3^{2b_i}\sigma_4^{c_i}$ of f we have $b_i\geq d$ and hence

$$a_i=(a_i+3b_i+2c_i)-2(3b_i+c_i)+3b_i>3d,$$

so f is divisible by σ_2^{3d+1} and we can write $f=\sigma_2^{3d+1}\sigma_3^{2d}g$, where $\sigma_3\nmid g$. The intersection number of the divisor (σ_3) and (g) (which have no common components) must on the one hand equal $30k-180d-30$ (since g and σ_3 have degree $2k-12d-2$ and 3, respectively, and $K_S^2=5$) and on the other hand be at least $30k-180d$ (since g and σ_3 vanish at each of the 10 points $\bar{p}_{ij}$ with multiplicity $\geq 2k-6d$ and 3, respectively). This contradiction proves our claim.

By counting the elements in (46) we deduce

$$\dim M^s_{2k}(SL_2(\mathcal{O}))=\left[\frac{k^2+3k}{6}\right]\qquad(k>0). \tag{47}$$

Also, any element of (46) can be written in the form

$$\left(\frac{\sigma_4}{\sigma_3}\right)^c\sigma_2^{a-3[a/3]}\,\sigma_3^{[(3b-a)/6]}\left(\frac{\sigma_2^3}{\sigma_3}\right)^{[a/3]},$$

where all the exponents are positive. This proves:

Theorem 7. *The graded ring of symmetric Hilbert modular forms of even weight on $SL_2(\mathcal{O})$ is isomorphic to $\mathbb{C}[A,B,C,C']/(B^3-CC')$ where the generators A, B, C, C' have degree 2, 4, 6, and 6, respectively. The isomorphism is given by*

$$A\mapsto\frac{\sigma_4}{\sigma_3},\qquad B\mapsto\sigma_2,\qquad C\mapsto\sigma_3,\qquad C'\mapsto\frac{\sigma_2^3}{\sigma_3},$$

where σ_i is the i-th elementary symmetric polynomial in the modular forms $\xi_0,\ldots,\xi_4$ of Theorem 6.

We now determine the structure of the ring $M_{\mathrm{ev}}(SL_2(\mathcal{O}))\cong M_{\mathrm{ev}}(\Gamma)^{\mathfrak{A}_5}$. The elements of $M_{2k}(SL_2(\mathcal{O}))$ have the form f/σ_3^k, where f is a polynomial in $\xi_0,\ldots,\xi_4$ which is invariant under $\mathfrak{A}_5$ and hence can be expressed in terms of $\sigma_2,\sigma_3,\sigma_4$ and $\prod_{i<j}(\xi_i-\xi_j)$. The latter polynomial has weight 10 and vanishes with multiplicity 3 at each point $\bar{p}_{ij}$, so Theorem 6 tells us that

$$\Delta_8=\Big(\prod_{i<j}(\xi_i-\xi_j)\Big)/\sigma_3(\xi_0,\ldots,\xi_4)^2 \tag{48}$$

is a holomorphic modular form (in fact a cusp form) of weight 8 on $SL_2(\mathcal{O})$. We claim that

$$M_{2k}(SL_2(\mathcal{O}))=M^s_{2k}(SL_2(\mathcal{O}))\oplus M^s_{2k-8}(SL_2(\mathcal{O}))\cdot\Delta_8.$$

Indeed, any modular form is the sum of a symmetric and an anti-symmetric form, and an anti-symmetric form must vanish on the fixed point set $F_1\cup F_{13}$ of the involution τ (cf. § 4). But the divisor of Δ_8 in $\mathfrak{H}\times\mathfrak{H}$ is precisely $F_1\cup F_{13}$ (compare Eq. (21)), so any anti-symmetric form is divisible by Δ_8. By computing the discriminant of the quintic polynomial $x^5+\sigma_2x^3-\sigma_3x^2+\sigma_4x-\frac{1}{2}\sigma_2\sigma_3$, we find the relation

$$\Delta_8^2=P\left(\frac{\sigma_4}{\sigma_3},\sigma_2,\sigma_3,\frac{\sigma_2^3}{\sigma_3}\right), \tag{49}$$

where

$$P(A,B,C,D)=256A^5C-128A^4B^2+16A^3BD-656A^3BC+776A^2B^3 \\ -261AB^2D+27BD^2-27A^2C^2+\tfrac{495}{2}AB^2C-\tfrac{947}{16}B^4+54BC^2. \tag{50}$$

This proves

Theorem 8. *The graded ring of Hilbert modular forms of even weight on $SL_2(\mathcal{O})$ is isomorphic to*

$$\mathbb{C}[A,B,C,C',D]/(B^3-CC',D^2-P(A,B,C,C')),$$

where A, B, C, C', D have degree 2, 4, 6, 6 *and* 8, *respectively, and P is the polynomial* (50).

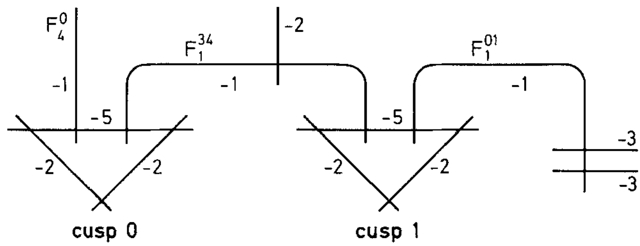
As a further application of Theorem 6 we determine the modular forms of even weight on Γ_0, where

$$\Gamma_0=\left\{\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}\in SL_2(\mathcal{O})\,\middle|\,\gamma\equiv0\ (\mathrm{mod}\ 2)\right\}.$$

It contains Γ as a normal subgroup of index 12. If we compactify the quotient $\mathfrak{H}^2/\Gamma_0$ in the usual way and resolve the singularities we obtain a non-singular algebraic surface $Y(\Gamma_0)$. It is simply-connected. The isotropy group at ∞ of Γ has index 12 in the isotropy group at ∞ of Γ_0. But the subgroup of $\mathfrak{A}_5$ which preserves the resolution of cusp 0 has order 12 and is isomorphic to $\mathfrak{A}_4$. Hence if we divide Y by the action of this group and resolve the singularities created we obtain $Y(\Gamma_0)$. From Theorem 6 we deduce a descripion for the ring $M_{\mathrm{ev}}(\Gamma_0)$ of Hilbert modular forms of even weight on Γ_0: $M_{\mathrm{ev}}(\Gamma_0)=M_{\mathrm{ev}}(\Gamma)^{\mathfrak{A}_4}$, where $\mathfrak{A}_4$ acts on $\xi_1,\ldots,\xi_4$. In particular, we see that $M_2(\Gamma_0)$ is generated by ξ_0, η_0 and $\sum_{i=1}^{4}\eta_i$. Since $S_2(\Gamma_0)\subset M_2(\Gamma_0)$ is isomorphic to $H^0(Y(\Gamma_0),K_{Y(\Gamma_0)})$ we find $p_g(Y(\Gamma_0))=1$.

Theorem 9. *The Hilbert modular surface $Y(\Gamma_0)$ associated to the congruence subgroup Γ_0 of $SL_2(\mathcal{O})$ is a blown-up $K3$ surface. The ring of symmetric Hilbert modular forms of even weight on Γ_0 is generated over $M_{\mathrm{ev}}(SL_2(\mathcal{O}))$ by the forms ξ_0, $\xi_0^2\sigma_2^2/\sigma_3$, $\xi_0^4\sigma_2/\sigma_3$, ξ_0^6/σ_3 (of weight* 2, 6, 6, 6, *respectively) with the obvious relations.*

Proof. On $Y(\Gamma_0)$ there are two cusps (the images of cusp 0 and 1) both with a resolution as in (3). The curve F_1 has two components, both exceptional curves, namely the images of F_1^{34} and F_1^{01}; the former meets the (-5)-curves of both cusp resolutions and a (-2)-curve coming from a quotient singularity, while the latter meets the (-5)-curve of the resolution of cusp 1. Also the image of F_4^0 on $Y(\Gamma_0)$ is exceptional and meets the (-5)-curve of cusp 0. Blowing down F_1^{34}, F_1^{01}, F_4^0 and the (-2)-curve meeting F_1^{34} we obtain an elliptic $K3$ configuration (namely two intersecting cycles of (-2)-curves). The resulting surface is minimal because $K^2_{Y(\Gamma_0)} = -4$ and we have blown down 4 times.



To prove the second statement, remark that in the coordinate ring R of S the elementary symmetric polynomials in $x_1, \dots, x_4$ can be expressed in x_0 and σ_2, σ_3 and σ_4. As in the proof of Theorem 7, one checks that

$$\left\{\frac{\xi_0^a\,\sigma_2^b\,\sigma_3^c\,\sigma_4^d}{\sigma_3^k}\,\middle|\, a+2b+3c+4d=4k,\ 3c+2d\geqq 2k\right\}$$

constitute an additive basis of $M^s_{2k}(\Gamma_0)$ and that all of these elements can be expressed as monomials in σ_4/σ_3, σ_2, σ_3, σ_2^3/σ_3, ξ_0, $\xi_0^2\,\sigma_2^2/\sigma_3$, $\xi_0^4\,\sigma_2/\sigma_3$ and ξ_0^6/σ_3.

As a final application of Theorems 5 and 6 we now show that every curve F_N is the zero-set of a modular form on $SL_2(\mathcal{O})$. For $N=1$, for instance, we can take $\sigma_3(\xi_0, \dots, \xi_4)$, whose divisor in $\mathfrak{H}\times\mathfrak{H}$ is $6F_1$. Since $\mathfrak{H}\times\mathfrak{H}$ is simply connected, this function has a sixth root Ω_1 which lies in $S_1(SL_2(\mathcal{O}), \varepsilon)$ for a certain character ε on $SL_2(\mathcal{O})$ with $\varepsilon^6=1$. More generally:

Theorem 10. *For every curve F_N there exists a Hilbert modular form of Nebentypus on $SL_2(\mathcal{O})$, unique up to a constant, whose zero-divisor in $\mathfrak{H}\times\mathfrak{H}$ is the curve F_N. This form has weight $\alpha_N - \frac{1}{10}F_N\cdot F_1$ and character $\varepsilon^{-\alpha_N-\frac{1}{10}F_N\cdot F_1}$, where α_N and $F_N\cdot F_1$ are given by* (23) *and* (5) *respectively.*

Proof. In the proof of Theorem 5 we showed the existence of a polynomial f_N such that the zero-divisor of the pull-back of $f_N(s_0, \dots, s_4)\in H^0(Y^0, K_{Y^0})$ to Y is an α_N-canonical divisor of the form (24), where S_N is a divisor consisting only of (-2)-curves coming from the cusp resolutions. Hence, by the identification given in the proof of Theorem 6, $f_N(\xi_0, \dots, \xi_4)$ is a modular form of weight $2\alpha_N$ whose divisor in $\mathfrak{H}\times\mathfrak{H}$ equals

$$F_N + (\alpha_N + \tfrac{1}{10}F_N\cdot F_1)\,F_1.$$

Consider the modular form of Nebentypus defined by

$$\Omega_N = f_N(\xi_0, \dots, \xi_4)/\Omega_1^{\alpha_N + \frac{1}{10} F_N \cdot F_1}.$$

This modular form is of weight $\alpha_N - \frac{1}{10} F_N \cdot F_1$, its character is $\varepsilon^{-\alpha_N - \frac{1}{10} F_N \cdot F_1}$ and its zero-divisor in $\mathfrak{H} \times \mathfrak{H}$ is F_N. Suppose that Ω'_N is another Hilbert modular form on $SL_2(\mathcal{O})$ such that the divisor of Ω'_N equals precisely F_N. Then $(\Omega'_N/\Omega_N)^6$ defines on S a meromorphic section of some tensor power of K_s whose divisor is a combination $\sum m_\beta D_\beta$ of the cusp curves D_β. Moreoever, since Ω_N and Ω'_N come from $SL_2(\mathcal{O})$, this section is invariant under $\mathfrak{A}_5$, so all m_β are equal and the section in question is a constant times σ_3^m. Since the divisors of Ω_N and Ω'_N do not contain F_1, we must have $m=0$, so Ω'_N/Ω_N is constant.

Remark. Both α_N and $\frac{1}{10} F_N F_1$ are even for all $N \neq 1, 13$. (*Proof*: By (5), $\frac{1}{10} \sum F_{N/d^2} F_1 = 6H_{13}^0(n) = 6\sum H((4N-x^2)/13)$. The number $6H((4N-x^2)/13)$ is an integer and is odd only when $(4N-x^2)/13$ is a square, so $6H_{13}^0(N)$ is odd if and only if $13N$ is a square.) Therefore α_N has even weight and a character of order at most 3 for all such N. (In fact, we will show in § 10 that $\varepsilon^3 = 1$, so that Ω_N has a character of order $\leqq 3$ in any case.) If $N \equiv 2 \pmod 3$, then α_N and $\frac{1}{10} F_N F_1$ are also divisible by 3. (*Proof*: The sum $\sum \left(\frac{d}{13}\right)\left(d + \frac{N}{d}\right)$ is divisible by 3, since $d \equiv -N/d \pmod 3$. Also, $6H(n)$ is congruent to $2 \pmod 3$ if n is three times a square and is divisible by 3 otherwise, so $\frac{1}{10} \sum F_{N/d^2} F_1 = 6\sum H((4N-x^2)/13)$ is congruent to twice the number of representations of N as $(x^2 + 39y^2)/4$, i.e. as a norm in $\mathbb{Q}(\sqrt{-39})$. For $N \equiv 2 \pmod 3$ there are no such representations.) Therefore Ω_N is of Haupttypus in this case. In general, we can obtain a modular form of Haupttypus which vanishes on F_N either by replacing Ω_N by its third power or by multiplying it by Ω_1^a for some appropriate integer $a \leqq 5$.

§ 9. The Canonical Mapping in Terms of Eisenstein Series

The aim of this section is to show how the modular forms ξ_i $(i=0, \dots, 4)$ can be expressed in terms of Eisenstein series of weight 1 (with a character) and Eisenstein series of weight 2. Thus we obtain a description of the modular forms ξ_i as functions on $\mathfrak{H} \times \mathfrak{H}$ and hence a more explicit description of the rings $M_{\text{ev}}(\Gamma)$ and $M_{\text{ev}}(SL_2(\mathcal{O}))$. The fact that the mapping

$$(z_1, z_2) \mapsto (\xi_0(z_1, z_2) : \xi_1(z_1, z_2) : \xi_2(z_1, z_2) : \xi_3(z_1, z_2) : \xi_4(z_1, z_2))$$

is a birational map from $\mathfrak{H}^2/\Gamma$ to its image on the quintic surface S can then be seen as the analogue of the classical parametrisation

$$z \mapsto (E_4(z)^3 : E_6(z)^2) \in \mathbb{P}_1(\mathbb{C})$$

of $\mathfrak{H}/SL_2\mathbb{Z}$ by Eisenstein series.

Consider the five normalised Eisenstein series E_2^i $(i=0, \dots, 4)$ of weight 2 on Γ defined by

$$E_2^i(z_1,z_2)=\sum_{\kappa,\mu}{}' \frac{1}{\mathrm{N}(\kappa z+\mu)^2} \qquad (\mathrm{N}(\kappa z+\mu) \overset{\text{def}}{=} (\kappa z_1+\mu)(\kappa' z_2+\mu')) \tag{51}$$

where (κ,μ) runs over all non-associated relatively prime pairs of integers in $\mathcal{O}$ such that μ/κ represents the i-th cusp and the summation has to be performed according to Hecke's well-known procedure for obtaining convergence. Let $x_i \in \mathcal{O}$ $(i \neq 0)$ represent the i-th cusp. Then we have the Fourier developments

$$E_2^0(z_1,z_2)=1+\tfrac{8}{5}\sum_{\substack{\nu\in\mathfrak{d}^{-1}\\ \nu\gg 0}} (16\sigma_1(\tfrac{1}{2}\nu\mathfrak{d})-\sigma_1(\nu\mathfrak{d}))\, e^{2\pi i(\nu z_1+\nu' z_2)} \tag{52}$$

$$E_2^i(z_1,z_2)=\tfrac{8}{5}\sum_{\substack{\nu\in\mathfrak{d}^{-1}\\ \nu\gg 0}} (-1)^{\mathrm{Tr}(\nu x_i)}(\sigma_1(\nu\mathfrak{d})-4\sigma_1(\tfrac{1}{2}\nu\mathfrak{d}))\, e^{\pi i(\nu z_1+\nu' z_2)} \quad (0<i\leqq 4), \tag{53}$$

where $\mathfrak{d}=(\sqrt{13})$ is the different of K and

$$\sigma_1(\mathfrak{a})=\begin{cases}\sum_{\mathfrak{b}|\mathfrak{a}} N(\mathfrak{b}) & \text{if } \mathfrak{a} \text{ is integral}\\ 0 & \text{otherwise.}\end{cases}$$

If F is a Hilbert modular form of weight k on Γ and $A=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in SL_2(\mathcal{O})$ we denote by $F|A$ the modular form of weight k defined by

$$(F|A)(z_1,z_2)=(\gamma z_1+\delta)^{-k}(\gamma' z_2+\delta')^{-k}\, F\left(\frac{\alpha z_1+\beta}{\gamma z_1+\delta}, \frac{\alpha' z_2+\beta'}{\gamma' z_2+\delta'}\right).$$

Since the form $F|A$ depends only on F and on the equivalence class of A mod Γ, the group $\mathfrak{A}_5=SL_2(\mathcal{O})/\Gamma$ acts on $M_{\mathrm{ev}}(\Gamma)$. This group acts on the E_2^i $(i=0,\dots,4)$ in the same way as on the five cusps; in particular, E_2^i vanishes at all cusps except the i-th (by (52) and (53)) and any modular form of weight 2 on Γ is the sum of a cusp form and a linear combination of the E_2^i. The non-cusp forms $\eta_i=\sigma_2(\xi_0,\dots,\xi_4)/\xi_i\in M_2(\Gamma)$ also have the value 1 at the i-th cusp and vanish at the other ones. Hence E_2^i is related to ξ_i and η_i by

$$E_2^i=\eta_i+\sum c_{ij}\,\xi_j.$$

Since E_2^i and η_i are invariant under the subgroup $\mathfrak{A}_4\subset\mathfrak{A}_5$ fixing i, the difference $E_2^i-\eta_i$ is a linear combination of ξ_i and $\sum_{j\neq i}\xi_j$ and hence, since $\sum\xi_j=0$, a multiple $c_i\,\xi_i$ of ξ_i. Moreover, the invariance also implies that the coefficient c_i is independent of i:

$$E_2^i=\eta_i+c\,\xi_i. \tag{54}$$

In order to determine the constant c we will use Eisenstein series of weight 1. An Eisenstein series of odd weight k (for a field whose fundamental unit has norm -1) must be associated to some character. For example, the Eisenstein series of odd weight $k\geqq 3$ with the ray class character $\chi(\mathfrak{a})=\left(\frac{-4}{\mathrm{N}\,\mathfrak{a}}\right)$ of conductor 4 is given by

$$E_k^\chi(z_1,z_2)=\sum_{\kappa,\mu}{}' \frac{\chi((\mu))}{\mathrm{N}(\kappa z+\mu)^k}$$
$$=1+\frac{4}{L(1-k,\chi)}\sum_{\substack{\nu\in\mathfrak{d}^{-1}\\ \nu\gg 0}}\Big(\sum_{\mathfrak{a}\mid\nu\mathfrak{d}}\chi(\mathfrak{a})\,\mathrm{N}(\mathfrak{a})^{k-1}\Big)\,e^{\pi i(\nu z_1+\nu' z_2)} \tag{55}$$

where the summation runs over non-associated relatively prime pairs κ, μ with μ/κ representing the 0-th cusp (i.e. $2|\kappa$). The functions $E_k^\chi|A$ ($A\in SL_2(\mathcal{O})$) depend (up to sign) only on the class of A in $SL_2(\mathcal{O})/\Gamma=\mathfrak{A}_5$. We obtain in this way 20 functions (since E_k is invariant under the matrices $\begin{pmatrix}\varepsilon^{-1} & 0\\ 0 & \varepsilon\end{pmatrix}$, which represent three elements of $\mathfrak{A}_5$). For example, the function $E_k^\chi\Big|\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$, which has the same character as E_k^χ, is given by

$$E_k^\chi\Big|\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}=\sum_{\kappa,\mu}{}' \frac{\chi((\mu))}{\mathrm{N}(\mu z+\kappa)^k}$$
$$=-\frac{4^k}{L(1-k,\chi)}\sum_{\substack{\nu\in\mathfrak{d}^{-1}\\ \nu\gg 0}}\Big(\sum_{\mathfrak{a}\mid\nu\mathfrak{d}}\chi(\mathfrak{a})\,\mathrm{N}(\nu\mathfrak{d}/\mathfrak{a})^{k-1}\Big)\,e^{\pi i(\nu z_1+\nu' z_2)}. \tag{56}$$

Each function $E_k^\chi|A$ is non-zero at precisely one cusp.

For weight one, Hecke [3] showed how to define Eisenstein series by replacing the factor $\mathrm{N}(\kappa z+\mu)^k$ by $\mathrm{N}(\kappa z+\mu)|\mathrm{N}(\kappa z+\mu)|^{2s}$ with $\mathrm{Re}(s)>\frac{1}{2}$, computing the Fourier coefficients as a function of s, and then extending analytically in s to obtain for $s=0$ a function transforming like a modular form of weight 1. The result of this computation is that the higher Fourier coefficients are the same as they would be if the Eisenstein series were convergent, whereas for the constant term there is an extra contribution coming from the analytic continuation procedure. Hecke claimed (loc. cit., p. 394) that this extra piece vanishes; if this were true, the Eisenstein series of weight 1 would behave exactly like those of higher weight. However, this is incorrect (cf. correction by Schoeneberg, loc. cit.). Indeed, there are even situations where an Eisenstein series which is non-zero for $k>1$ vanishes for $k=1$, the contribution from the Hecke procedure in this case exactly cancels out the constant term of the series. In our situation, if we define E_1^χ by applying Hecke's procedure to the Eisenstein series (55), then the extra contribution to the constant term vanishes and we obtain the same Fourier development

$$E_1^\chi(z_1,z_2)=1+4\sum_{\substack{\nu\in\mathfrak{d}^{-1}\\ \nu\gg 0}}\rho(\nu\mathfrak{d})\,e^{\pi i(\nu z_1+\nu' z_2)},\qquad \rho(\mathfrak{a})=\sum_{\mathfrak{b}\mid\mathfrak{a}}\chi(\mathfrak{b}), \tag{57}$$

as for $k>1$, but when we calculate the Fourier series of $E_1^\chi\Big|\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$, we find that this function equals $-E_1^\chi$, the constant term now arising solely from the analytic continuation procedure (it is clear that this must be so, since the higher Fourier coefficients in (55) and (56) coincide up to sign for $k=1$). As a result we find that the

Eisenstein series E_1^χ is non-zero at *two* cusps rather than only at one as in higher weight, and that (up to sign) only *ten* of the modular forms $E_1^\chi | A$ ($A \in SL_2(\mathcal{O})$) are distinct. More precisely:

Lemma. *For each pair indices i, j between 0 and 4 there is a modular form $E_2^{ij} = E_2^{ji}$ on Γ of weight 2 and with trivial character which takes on the value 1 at cusps i and j and 0 at the other three cusps and which has even divisor in $\mathfrak{H} \times \mathfrak{H}$, namely $E_2^{ij} = (E_1^\chi)^2 | A_{ij}$, where A_{ij} is any matrix in $SL_2(\mathcal{O})$ mapping cusps i and j to cusps 0 and 1. The Fourier developments of these forms are given by*

$$E_2^{0j} = [1 + 4 \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} (-1)^{\mathrm{Tr}(\nu x_j)} \rho(\nu \mathfrak{d})\, e^{\pi i (\nu z_1 + \nu' z_2)}]^2 \qquad (0 \leqq j \leqq 4),$$

$$E_2^{ij} = [-4 \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} i^{\mathrm{Tr}(\nu x_j)} \rho(\nu \mathfrak{d})\, e^{\frac{1}{2}\pi i (\nu z_1 + \nu' z_2)}]^2 \qquad (0 \leqq i, j \leqq 4,\ i \neq j)$$

with $\rho(\mathfrak{a})$ as in (57) and $x_j \in \mathcal{O}$ representing the j-th cusp.

We can now proceed with the determination of c. The modular form $E_2^{ij} - E_2^i - E_2^j$ is a cusp form of weight 2 and is invariant under the subgroup $\mathfrak{S}_3 \subset \mathfrak{A}_5$ of permutations which fix or permute i and j. It is therefore a multiple of the cuspform $\xi_i + \xi_j$:

$$E_2^{ij} - E_2^i - E_2^j = \lambda(\xi_i + \xi_j) \tag{58}$$

(λ is independent of i, j because of the $\mathfrak{A}_5$-symmetry). Substituting (54), we find that the modular form

$$\eta_i + \eta_j + (c + \lambda)(\xi_i + \xi_j) = (\xi_i + \xi_j)\left(\frac{\sigma_2}{\xi_i \xi_j} + c + \lambda\right) \tag{59}$$

equals $(E_1^\chi | A_{ij})^2$ and hence has an even divisor in $\mathfrak{H} \times \mathfrak{H}$, so the corresponding section of the hyperplane bundle of S has an even divisor outside the curve $\sigma_3 = 0$ corresponding to the cusps. Hence $\sigma_2 + (c + \lambda)\, x_i x_j$ must vanish on the component of the curve $x_i + x_j = 0$ which is not contained in the divisor of σ_3 (and which, as we saw in § 7, corresponds to $F_3^{ij} \subset Y$). In particular, $\sigma_2 + (c + \lambda)\, x_0 x_1$ must vanish at the point $(\sqrt{3}: -\sqrt{3}: 1: 1: -2) \in S$, so

$$c + \lambda = -2 \tag{60}$$

(cf. (35)). The component F_4^0 of F_4 given by $(z_1, z_2) = \left(z + \frac{\sqrt{13}}{4}, z - \frac{\sqrt{13}}{4}\right)$ corresponds to the curve $\sigma_2 = 0$, $x_0 = 0$ on S, so the restrictions of ξ_0 and η_1 to F_4^0 vanish. Hence (52), (58) and (60) imply

$$E_2^{01}\left(z + \frac{\sqrt{13}}{4}, z - \frac{\sqrt{13}}{4}\right) - E_2^0\left(z + \frac{\sqrt{13}}{4}, z - \frac{\sqrt{13}}{4}\right) = -2\xi_1\left(z + \frac{\sqrt{13}}{4}, z - \frac{\sqrt{13}}{4}\right)$$

$$E_2^1\left(z + \frac{\sqrt{13}}{4}, z - \frac{\sqrt{13}}{4}\right) = c\,\xi_1\left(z + \frac{\sqrt{13}}{4}, z - \frac{\sqrt{13}}{4}\right). \tag{61}$$

Using the Fourier developments given by (52), (53) and the lemma we find the Fourier series

$$E_2^0\left(z+\frac{\sqrt{13}}{4}, z-\frac{\sqrt{13}}{4}\right)=1+0q+0q^2+0q^3-240q^4+\cdots$$

$$E_2^1\left(z+\frac{\sqrt{13}}{4}, z-\frac{\sqrt{13}}{4}\right)=0+\frac{24\sqrt{2}}{5}q-\frac{96}{5}q^2-\frac{96\sqrt{2}}{5}q^3+0q^4+\cdots$$

$$E_2^{01}\left(z+\frac{\sqrt{13}}{4}, z-\frac{\sqrt{13}}{4}\right)=1-8\sqrt{2}q+32q^2+32\sqrt{2}q^3-240q^4+\cdots. \tag{62}$$

Together with (61) this gives $c=\frac{6}{5}$, $\lambda=\frac{-16}{5}$. We have proved:

Theorem 11. *The modular forms ξ_i, η_i are related to the Eisenstein series of weight* 1 *and* 2 *by*

$$E_2^i=\eta_i+\tfrac{6}{5}\xi_i,$$
$$E_2^{ij}=(E_1^\chi)^2|A_{ij}=\eta_i+\eta_j-2(\xi_i+\xi_j).$$

Inverting these formulas, we can express the forms ξ_i, η_i in terms of the Eisenstein series and hence obtain their Fourier expansions. We find, for example,

$$\begin{aligned}\xi_0(z_1,z_2)&=-\xi_1-\xi_2-\xi_3-\xi_4\\&=\tfrac{5}{16}(E_2^{12}+E_2^{34}-E_2^1-E_2^2-E_2^3-E_2^4)\\&=8(x-x^{-1})^2(x+x^{-1})\{-q+(x^3+3x+3x^{-1}+x^{-3})q^2\\&\quad-(x^6+4x^4-x^2+4-x^{-2}+4x^{-4}+x^{-6})q^3\\&\quad+(x^{11}-4x^7-7x^5-10x^3-12x-12x^{-1}-10x^{-3}\\&\quad-7x^{-5}-4x^{-7}+x^{-11})q^4+\cdots\}\end{aligned} \tag{63}$$

where we have written

$$q=e^{\pi i(z_1+z_2)},\qquad x=e^{\pi i(z_1-z_2)/\sqrt{13}}\qquad (z_1,z_2\in\mathfrak{H}),$$

(i.e. $q=e^{2\pi i u}$, $x=e^{2\pi i v}$ with $z_1,z_2=u\pm v\sqrt{13}$); this way of writing the Fourier series of Hilbert modular forms is convenient if one wants to see the effect of interchanging z_1 and z_2 (corresponding to $x\mapsto x^{-1}$) or of restricting to the diagonal F_1 (corresponding to $x=1$). The Fourier developments of the other ξ_i can be deduced from (63) using the relations (whose proof we leave as an exercise)

$$\xi_j(z_1,z_2)=\tfrac{1}{4}\xi_0\left(\frac{z_1}{2}+x_j, \frac{z_2}{2}+x_j'\right)\qquad (j=1,2,3,4)$$

where x_j as before represents the j-th cusp. The forms ξ_0, ξ_1 and ξ_2 vanish doubly on $F_1^{34}=\{z_1=z_2\}=\{x=1\}$, while ξ_3 and ξ_4 vanish simply there. Computing the elementary symmetric functions of the ξ_i, we find

$$\begin{aligned}\sigma_2 = & -8(x-x^{-1})^2(x+x^{-1})\{q+(7x^3-19x-19x^{-1}+7x^{-3})\,q^2 \\ & +(x^6-44x^4+119x^2+100+119x^{-2}-44x^{-4}+x^{-6})\,q^3+\cdots\},\end{aligned}$$

$$\begin{aligned}\sigma_3 = & \, 32(x-x^{-1})^6\{q^2-6(x+x^{-1})^3\,q^3 \\ & +(27x^6+114x^4+237x^2+324+237x^{-2}+114x^{-4}+27x^{-6})\,q^4+\cdots\},\end{aligned}$$

$$\begin{aligned}\sigma_4 = & \, 16(x-x^{-1})^6\{q^2+(18x^3+78x+78x^{-1}+18x^{-3})\,q^3 \\ & +(3x^6-582x^4-2019x^2-3084-2019x^{-2}-582x^{-4}+3x^{-6})\,q^4+\cdots\}. \quad (64)\end{aligned}$$

The function $\frac{\sigma_4}{\sigma_3}=\frac{1}{2}\sum_{i=0}^{4}\eta_i$ is invariant under $SL_2(\mathcal{O})$ and takes on the value $\frac{1}{2}$ at all cusps, so it must be $\frac{1}{2}$ the normalized Eisenstein series of weight 2 for the full modular group:

$$\begin{aligned}2\frac{\sigma_4}{\sigma_3} &= E_2(z_1,z_2)=1+24\sum_{\substack{v\in\mathfrak{d}^{-1}\\ v\gg 0}}\sigma_1(v\,\mathfrak{d})\,e^{2\pi i(vz_1+v'z_2)} \\ &= 1+24(x^3+4x+4x^{-1}+x^{-3})\,q \\ &\quad +24(5x^6+13x^4+20x^2+14+20x^{-2}+13x^{-4}+5x^{-6})\,q^2+\cdots.\end{aligned}$$

The form σ_2 is -8 times the (unique) normalized cusp form of weight 4 on $SL_2(\mathcal{O})$ and is also equal to $\frac{29}{144}(E_4-E_2^2)$, where E_4 is the normalized Eisenstein series of weight 4 for $SL_2(\mathcal{O})$. Similarly we find

$$\begin{aligned}\frac{\sigma_2^3}{\sigma_3} = & -16(x+x^{-1})^3\{q+(27x^3-39x-39x^{-1}+27x^{-3})\,q^2 \\ & +(285x^6-792x^4-45x^2+1356-45x^{-2}-792x^{-4}+285x^{-6})\,q^3+\cdots\};\end{aligned}$$

this is a Hilbert cusp form of weight 6 on $SL_2(\mathcal{O})$ whose restriction to $F_1=\mathfrak{H}/SL_2(\mathbb{Z})$ is $-128\Delta(z)$. In this way we get an entirely explicit description of the modular forms occurring in Theorem 7.

Finally, observe that the Fourier developments of the ξ_i permit us to determine the defining polynomials f_N of § 6 for any non-compact curve F_N: since we know the degree of f_N as a function of the ξ_i, we can by computing sufficiently many Fourier coefficients of $\xi_i(\lambda z_1, \lambda' z_2)$ (where $\lambda\lambda'=N$) determine the polynomial $f_N(\xi_0,\ldots,\xi_4)$ which vanishes along F_N. This is the algorithm we referred to at the end of § 7. However, even for $N=9$ the amount of computation involved seems to be prohibitive.

§ 10. Hilbert Modular Forms of Odd Weight. Applications

In this section we will determine the structure of the ring of all modular forms on $SL_2(\mathcal{O})$ and give some applications. We start with the following lemma.

Lemma. *The modular form* $\omega_3=\sqrt{\sigma_3(\xi_0,\ldots,\xi_4)}$ *is a modular form of weight* 3 *of Haupttypus on* $SL_2(\mathcal{O})$.

Proof. It is clear that ω_3 is well-defined (up to sign), since the divisor of σ_3 on $\mathfrak{H}\times\mathfrak{H}$ is even, and that $\omega_3|A=\pm\omega_3$ for all $A\in SL_2(\mathcal{O})$. We have to prove that the sign is always $+$, i.e. that $\varepsilon^3=1$, where ε is the character of $\Omega_1=\sqrt[6]{\sigma_3}$ (cf. Theorem 10, § 8). By (64) we have

$$\omega_3=4\sqrt{2}(x-x^{-1})^3\{q-3(x+x^{-1})^3q^2 \\ +(9x^6+30x^4+51x^2+72+51x^{-2}+30x^{-4}+9x^{-6})q^3+\cdots\}. \tag{65}$$

From this it is obvious that ω_3 is invariant under the translations

$$(z_1,z_2)\to(z_1+1,z_2+1),\quad (z_1,z_2)\to(z_1+\sqrt{13},z_2-\sqrt{13}),$$
$$(z_1,z_2)\to\left(z_1+\frac{1+\sqrt{13}}{2},z_2+\frac{1+\sqrt{13}}{2}\right),$$

so $\omega_3\left|\begin{pmatrix}1&\lambda\\0&1\end{pmatrix}\right.=\omega_3$ for all $\lambda\in\mathcal{O}$. We claim that ω_3 is invariant under $B=\begin{pmatrix}0&\varepsilon_0\\\varepsilon_0'&0\end{pmatrix}$, where $\varepsilon_0=\dfrac{3+\sqrt{13}}{2}$. Indeed, the functions ω_3 and $\omega_3|B$ are equal up to sign and agree at the fixed point $(\varepsilon_0 i,\varepsilon_0^{-1}i)$ of B. Since ω_3 has no zeroes in $\mathfrak{H}\times\mathfrak{H}$ outside F_1, their common value at this point is different from 0 and our claim follows. Finally, $\omega_3\left|\begin{pmatrix}\varepsilon_0&0\\0&\varepsilon_0^{-1}\end{pmatrix}\right.$ and ω_3 are equal, since they are equal up to sign and (by (65)) have the same coefficient of x^3q. Since $\mathcal{O}_K$ is Euclidian, the ring $SL_2(\mathcal{O})$ is generated by $\begin{pmatrix}\varepsilon_0&0\\0&\varepsilon_0^{-1}\end{pmatrix}$, $\begin{pmatrix}0&\varepsilon_0\\\varepsilon_0'&0\end{pmatrix}$ and $\begin{pmatrix}1&\lambda\\0&1\end{pmatrix}$. The lemma follows.

In § 9 we used the fact that E_2^{ij} has an even divisor on $\mathfrak{H}\times\mathfrak{H}$ to find the relation between the ξ_i $(i=0,\dots,4)$ and the Eisenstein series E_2^{ij} $(0\leqq i<j\leqq 4)$ and E_2^i $(i=0,\dots,4)$. We now show that the divisor of E_2^{ij} is in fact divisible by 4. To show that

$$E_2^{ij}=\frac{(\xi_i+\xi_j)}{\xi_i\xi_j}(\sigma_2(\xi_0,\dots,\xi_4)-2\xi_i\xi_j)$$

has even divisor on $\mathfrak{H}\times\mathfrak{H}$, we proved that the divisor of $\sigma_2(x_0,\dots,x_4)-2x_ix_j$ contains the component of the divisor (x_i+x_j) lying outside the image of the cusp resolutions. Hence $\sigma_3(x_0,\dots,x_4)(\sigma_2(x_0,\dots,x_4)-2x_ix_j)$ must be divisible by x_i+x_j in the ring $R=\mathbb{C}[x_0,\dots,x_4]/(\sigma_1,2\sigma_5-\sigma_2\sigma_3)$. Calculating we find

$$\begin{aligned}&-\tfrac{1}{2}\sigma_3(\sigma_2-2x_0x_1)\\&\quad=-\sigma_5+x_0x_1[x_2x_3x_4+(x_0+x_4)(x_2x_3+x_2x_4+x_3x_4-x_0x_1)]\\&\quad=x_0x_1(x_0+x_1)[\sigma_2-2x_0x_1+(x_0+x_1)^2]\end{aligned}$$

or

$$(\sigma_2-2x_0x_1)(\tfrac{1}{2}\sigma_3+x_0x_1(x_0+x_1))=-x_0x_1(x_0+x_1)^3.$$

Consequently, the divisor of the meromorphic section of K_S given by

$$\frac{x_0+x_1}{x_0 x_1}(\sigma_2-2x_0 x_1)$$

consists of 4 times the divisor of (x_0+x_1) plus a linear combination of curves in the image of the cusp resolutions. This section vanishes triply at $\bar{p}_{01}=(1:-1:0:0:0)\in S$ and has a simple pole at the nine other points $\bar{p}_{ij}$. Therefore its pullback to Y is a section of K_Y whose divisor has the form

$$4F_3^{01}+4F_1^{01}+(\text{curves of the cusp resolutions})$$

(recall that the divisor of x_i+x_j on S consists of the image of F_3^{ij} plus curves coming from the cusp resolutions). This proves:

Theorem 12. *The modular form $E_2^{ij}(z_1,z_2)$ is the fourth power of a modular form of weight $\frac{1}{2}$ on Γ which vanishes with multiplicity one along the curves F_1^{ij} and F_3^{ij} and has no other zeroes.*

Now consider the θ-series

$$\theta(z_1,z_2)=\sum_{\lambda\in\mathcal{O}} e^{2\pi i(\varepsilon_0\lambda^2 z_1-\varepsilon_0'\lambda'^2 z_2)/\sqrt{13}}. \tag{66}$$

This is a modular form on Γ of weight $\frac{1}{2}$ satisfying

$$\theta(\varepsilon_0^2 z_1,\varepsilon_0'^2 z_2)=\theta(z_1,z_2),$$

$$\theta\left(-\frac{1}{z_1},-\frac{1}{z_2}\right)=\frac{\sqrt{z_1 z_2}}{i}\,\theta(z_1,z_2)$$

(the latter by the Poisson summation formula). Hence θ^4 is a modular form in $M_2(\Gamma)$ which (a) is invariant under the action of the six elements of $\mathfrak{A}_5$ that fix or interchange the cusps 0 and 1, (b) has the value 1 at cusps 0 and 1 and 0 at the other three cusps, and (c) has an even divisor in $\mathfrak{H}\times\mathfrak{H}$. It follows that $\theta^4=E_2^{01}$. In particular, θ vanishes on F_1^{01} and F_3^{01}. Thus E_1^{χ} has been identified (up to sign) as the square of the theta-series θ and similarly E_2^{ij} as the fourth power of the theta-series $\theta_{ij}=\theta|A_{ij}$ where A_{ij} is some element of $SL_2(\mathcal{O})$ mapping cusps i and j to cusps 0 and 1.

Now consider $\Theta_5=\prod_{0\leq i<j\leq 4}\theta_{ij}$, which is a modular form of weight 5. By (59) and (60),

$$\Theta_5^4=\prod_{i<j}E_2^{ij}=\prod_{i<j}\frac{(\xi_i+\xi_j)(\sigma_2-2\xi_i\xi_j)}{\xi_i\xi_j}.$$

The divisor of the corresponding section consists of F_1+F_3-2 (image of the cusp resolutions), so from (30) we obtain $\Theta_5^4=c^4(\sigma_4-\frac{1}{4}\sigma_2^2)^4/\sigma_3^2$ for some constant c (in fact $c^4=-2^{10}$). In the notation of the lemma, therefore, we have

$$\Theta_5\,\omega_3=c(\sigma_4-\tfrac{1}{4}\sigma_2^2), \tag{67}$$

where $\sigma_i=\sigma_i(\xi_0,\ldots,\xi_4)$. The lemma now implies that Θ_5 is a modular form of Haupttypus (i.e. without a character) on $SL_2(\mathcal{O})$.

We now show that every symmetric modular form of odd weight can be written as $a\omega_3 + b\Theta_5$ with a, b symmetric modular forms of even weight. Together with Theorem 8 this will permit a complete description of the ring of all modular forms on $SL_2(\mathcal{O})$. Before giving our main result, we introduce the following notation. By $M_{2k}^{(i)}$ we mean the subspace of $M_{2k}(SL_2(\mathcal{O}))$ of symmetric modular forms of weight $2k$ which vanish with multiplicity $\geq 2i$ on F_1 in $\mathfrak{H}\times\mathfrak{H}$. A basis of $M_{2k}^{(i)}$ is given by the elements of (46) with $3b+c \geq k+i$. By counting the elements of such a basis we find

$$\dim M_{2k}^{(0)} = \left[\frac{k^2+3k+6}{6}\right], \quad \dim M_{2k}^{(1)} = \left[\frac{k^2+k}{6}\right] \quad \text{and} \quad \dim M_{2k}^{(2)} = \left[\frac{k^2-k}{6}\right],$$

while $M_{2k}^{(i)} = M_{2k-6}^{(i-3)} \cdot \sigma_3$ for $i \geq 3$.

Theorem 13. *The ring $M_*(SL_2(\mathcal{O}))$ of modular forms on the full modular group for the field $\mathbb{Q}(\sqrt{13})$ is generated by the modular forms*

$$\sigma_4/\sigma_3,\ \sqrt{\sigma_3},\ \sigma_2,\ \Theta_5,\ \sigma_2^3/\sigma_3,\ \Delta_8 \ \textit{and}\ \sigma_2\Delta_8/\sqrt{\sigma_3},$$

of weight 2, 3, 4, 5, 6, 8 *and* 9, *respectively, with the relations* (49) *and* (67) *together with the obvious relations implied by these relations and by the notation* (e.g.

$$(\sigma_2^3/\sigma_3)\,\sigma_3 = (\sigma_2)^3,$$

$$\Theta_5^2 = \frac{1}{2i}[(\sigma_4/\sigma_3)^2\,\sigma_3 - \tfrac{1}{2}(\sigma_4/\sigma_3)\,\sigma_2^2 + \tfrac{1}{16}\sigma_2(\sigma_2^3/\sigma_3)]).$$

Proof. Let $M_k^s = M_k^s(SL_2(\mathcal{O}))$ be the space of Hilbert modular forms f of weight k on $SL_2(\mathcal{O})$ such that $f(z_1,z_2) = (-1)^k f(z_2,z_1)$ ("symmetric" forms). For example $\omega_3 \in M_3^s$ (cf. (65)). If $f \in M_{2k+1}^s$ then $f\cdot\omega_3$ lies in M_{2k+4}^s and vanishes with multiplicity ≥ 3 on $F_1 \subset \mathfrak{H}\times\mathfrak{H}$. But, as one sees from (46) and the proof of Theorem 7, every element of $M_{ev}^s(SL_2(\mathcal{O}))$ vanishes with even multiplicity on F_1. Hence $f\cdot\omega_3 \in M_{2k+4}^{(2)}$. Conversely, if $g \in M_{2k+4}^{(2)}$ then g is divisible by ω_3 (as a modular form) and $g/\omega_3 \in M_{2k+1}^s$. Thus $M_{2k+1}^s \xrightarrow{\sim} M_{2k+4}^{(2)}$, where the isomorphism is given by multiplication with ω_3. In particular, $\dim M_{2k+1}^s = \left[\frac{k^2+3k+2}{6}\right]$. The space M_{2k+1}^s contains the subspace $M_{2k-2}^{(0)}\cdot\omega_3 + M_{2k-4}^{(0)}\cdot\Theta_5$, with $M_{2k-2}^{(0)}\cdot\omega_3 \cap M_{2k-4}^{(0)}\cdot\Theta_5 = M_{2k-4}^{(1)}\cdot\Theta_5$, and by comparing dimensions we find that these spaces coincide. Hence $M_*^s(SL_2(\mathcal{O}))$ is generated over $M_{ev}^s(SL_2(\mathcal{O}))$ by ω_3 and Θ_5 with the relations $\omega_3\Theta_5 = c(\sigma_4 - \frac{1}{4}\sigma_2^2)$, $\omega_3^2 = \sigma_3$ and $\Theta_5^2 = c^2(\sigma_4 - \frac{1}{4}\sigma_2^2)^2/\sigma_3$.

Next, let M_k^a be the subspace of $M_k(SL_2(\mathcal{O}))$ consisting of elements f satisfying $f(z_2,z_1) = -(-1)^k f(z_1,z_2)$ ("anti-symmetric"). If $f \in M_{2k+1}^a$ then $f\cdot\omega_3 \in M_{2k+4}^a = M_{2k-4}^s\cdot\Delta_8$ (cf. Theorem 8). Since $f\cdot\omega_3$ vanishes triply and Δ_8 simply on F_1

$$M_{2k+1}^a = M_{2k-4}^{(1)}\cdot\frac{\Delta_8}{\omega_3}.$$

But $M_{2k-4}^{(1)} = M_{2k-8}^{(0)}\cdot\sigma_2 + M_{2k-10}^{(0)}\cdot\sigma_3$ (this follows by looking at the basis (46) or by computing dimensions, using $M_{2k-8}^{(0)}\cdot\sigma_2 \cap M_{2k-10}^{(0)}\cdot\sigma_3 = M_{2k-8}^{(2)}\cdot\sigma_2$). This proves the theorem.

Incidentally, we have obtained the dimension formulas

$$\begin{aligned}\dim M_{2k} &= \dim M_{2k}^s + \dim M_{2k}^a \\ &= \left[\frac{k^2+3k+6}{6}\right] + \left[\frac{k^2-5k+10}{6}\right] = \left[\frac{k^2-k+6}{3}\right] \qquad (k>1),\end{aligned}$$

$$\begin{aligned}\dim M_{2k+1} &= \dim M_{2k+1}^s + \dim M_{2k+1}^a \\ &= \left[\frac{k^2+3k+2}{6}\right] + \left[\frac{k^2-3k+2}{6}\right] = \left[\frac{k^2+2}{3}\right].\end{aligned}$$

This is in accordance with the formula

$$\begin{aligned}\dim S_k &= \frac{(k-1)^2}{12} + \begin{cases}\frac{1}{4} & (k \text{ even}) \\ 0 & (k \text{ odd})\end{cases} + \begin{cases}\frac{2}{3} & (k \not\equiv 1 \bmod 3) \\ 0 & (k \equiv 1 \bmod 3)\end{cases} \\ &= \left[\frac{k^2-2k}{12}\right] + 1 \qquad (k>2)\end{aligned}$$

obtained from Shimizu's dimension formula, since there is an Eisenstein series of weight k for k even but not for k odd. It also agrees with the formula

$$\begin{aligned}&\dim S_k^s - \dim S_k^a \\ &\quad = 2\left[\frac{k}{3}\right] - 1 = \tfrac{1}{2}\dim S_k\left(\Gamma_0(13), \left(\frac{\cdot}{13}\right)\right) + \dim S_k(SL_2(\mathbb{Z})) \quad (k \text{ even})\end{aligned}$$

expressing the fact that all symmetric Hecke eigenforms of even weight for $SL_2(\mathcal{O})$ are the Doi-Naganuma lifts of eigenforms on $SL_2(\mathbb{Z})$ or of eigenforms of Nebentypus on $\Gamma_0(13)$ (see e.g. [11]). The nature of the Fourier coefficients of $\omega_3 \in S_3(SL_2(\mathcal{O}))$ seems to indicate that the symmetric Hilbert eigenforms of odd weight are also liftings of modular forms of one variable.

We end with two number-theoretical applications. The identity $\theta(z_1, z_2)^4 = E_2^\chi(z_1, z_2)$ for the function (66), together with the Fourier development (57), gives the following.

Proposition 2. *Let $\mu \in \mathcal{O}$ be a totally positive integer in $\mathbb{Q}(\sqrt{13})$. The number of representations of μ as a sum of two squares in $\mathcal{O}$ equals* $4\sum_{\mathfrak{a}|\mu}\left(\frac{-4}{\mathrm{N}\mathfrak{a}}\right)$.

This is the analogue of the well-known identity

$$r_2(n) = \begin{cases}1 & (n=0) \\ 4\sum\limits_{\substack{d|n \\ d>0}}\left(\frac{-4}{d}\right) & (n>0)\end{cases} \tag{68}$$

for the number of representations of a natural number as the sum of two squares, which is equivalent to the fact that the square of the ordinary θ-series

$$\theta(z) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 z} \qquad (z \in \mathfrak{H})$$

is the Eisenstein series of weight 1 with character $\left(\frac{-4}{\cdot}\right)$. By comparing the first few Fourier coefficients, one shows that $\theta(z,z)$ and $\theta(z)^2$ (which are both modular forms on $\Gamma(2)\subset SL_2\mathbb{Z}$) are equal, so

$$E_1^{\chi}(z,z)=\theta(z,z)^2=\theta(z)^4.$$

Comparing Fourier coefficients and evaluating the coefficients $\rho(\mathfrak{a})$ of E_1^{χ} by the method given in [13] (Lemma, § 3) for $\sigma_k(\mathfrak{a})$, one gets as a further application

Proposition 3. *The number of representations of a natural number as a sum of four squares is given by*

$$r_4(n)=4\sum_{\substack{d\mid n\\ d>0}}\left(\frac{-52}{d}\right)c\left(13\frac{n^2}{d^2}\right),$$

where

$$c(m)=\sum_{\substack{x\in\mathbb{Z}\\ x^2\leq m\\ x^2\equiv m\,(\mathrm{mod}\,4)}} r_2\left(\frac{m-x^2}{4}\right).$$

with r_2 as in (68).

This proposition can also be obtained by applying Shimura's theorem on lifting modular forms of half-integral weight to $\sum_{m=0}^{\infty} c(m)\,e^{2\pi i m z}$, which is a modular form of weight $\frac{3}{2}$.

By restricting various modular forms on Y to other curves F_N, one can obtain many further identities of this type.

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