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Values of L-series of Modular Forms at the Center of the Critical Strip

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Let $f(z)$ be a cusp form of weight $2k$ which is an eigenfunction of all Hecke operators, and denote by $L(f,s)$ the associated L-series. It is known that $L(f,s)$ satisfies a functional equation under $s \mapsto 2k-s$ and that the values of $L(f,s)$ at integral arguments within the critical strip $0<\mathrm{Re}(s)<2k$ can be expressed as algebraic multiples of certain "periods" associated to f. A remarkable fact, first discovered empirically, is then the following: The algebraic number appearing at the particular argument $s=k$ (the point of symmetry of the functional equation) is a perfect square. This is of course a somewhat vague statement whose precise formulation would depend on the particular situation. One well-known example is provided by the Birch-Swinnerton-Dyer conjecture, which says that for a cusp form f attached to an elliptic curve $E/\mathbb{Q}$, the quotient of $L(f,1)$ by the natural period is either 0 (if $E(\mathbb{Q})$ is infinite), or else, up to a "fudge factor", the quotient of the order of the Tate-Shafarevich group (known, if finite, to be a square) by $|E(\mathbb{Q})|^2$.

Another example, for forms of higher weight associated to Grossencharacters of imaginary quadratic fields, was described in [3]; here as in the case of the Birch-Swinnerton-Dyer conjecture the numerical evidence is very convincing. A natural question is then to try to explain the appearance of these squares.

Another, seemingly unrelated but equally important question in the theory of modular forms arises from Shimura's theory of forms of half-integral weight [14]. This theory provides a correspondence between certain modular forms of even weight $2k$ and modular forms of half-integral weight $k+\frac{1}{2}$; the correspondence is such that if f and g are corresponding forms, both assumed to be eigenforms of all Hecke operators, and if $c(n)$ denotes the n^{th} Fourier coefficient of g, then the ratio of $c(n)$ to $c(m)$ can be expressed in terms of the Fourier coefficients of f if n/m is a square. Shimura's theory, however, gives no information about this ratio if n/m is not a square. A formula for the coefficients $c(n)$ as integrals of certain differential forms attached to f over geodesic cycles in $\mathfrak{H}/SL_2(\mathbb{Z})$ was given by Shintani [17], who observed that this formula allows one to express these coefficients in terms of period integrals of

f and hence in terms of $L(f,k)$ (see pp. 117–118 of [17], where the example $k=6$, $f=\Delta$ is discussed). However, the precise nature of this relation remained unclear. The question of describing the $c(n)$ (n square-free) and in particular of relating them to the values of the L-series of f at integral points in the critical strip was also mentioned by Shimura in the "miscellaneous remarks" at the end of his paper [16].

Both of the questions just described were answered brilliantly in two recent papers of Waldspurger ([20, 21]; this work is reported on in [19]). The answer is unexpected and very satisfying: Given an eigenform f of even weight $2k$ and a form $g=\sum c(n)e^{2\pi i n z}$ of weight $k+\frac{1}{2}$ which corresponds to it in the sense of Shimura, the *square* of $c(n)$ (n square-free) is essentially proportional to the special value at $s=k$ of the *twist* of $L(f,s)$ by the character of the quadratic field $\mathbb{Q}(\sqrt{(-1)^k n})$. Thus one obtains simultaneously an explanation for the appearance of squares in the values of $L(f,k)$ and a formula for the Fourier coefficients $c(n)$ (up to sign; the sign of $c(n)$ is still utterly mysterious).

Waldspurger's proof, which was presented originally in the language of adeles and in terms of the representation theory of the metaplectic group as developed by Gelbart, Piatetskii-Shapiro, Flicker and others, is rather difficult, as is the translation back into the classical language of modular forms [21]; due to the variety of possible local behavior (at primes dividing the level, even primes, or primes at which the character ramifies), the statement of the final result is also extremely complicated. Furthermore, Waldspurger's proof does not seem to give the value of the constant of proportionality between the twists of $L(f,k)$ and the numbers $c(n)^2$; thus one gets information about the ratios of the various $c(n)$ but none about the individual coefficients. In this paper we would like to present an elementary proof of a version of Waldspurger's theorem for the special case of modular forms on the full modular group whose statement is very simple and in which the constant of proportionality between the squares of the $c(n)$ and the special values of the twists of $L(f,s)$ is given explicitly. This will permit us to deduce as corollaries several results about the arithmetic nature of the $c(n)$ and about the distribution of the values of the twists.

One reason that we can obtain such a good result for forms of level 1 is that in this case a completely satisfactory version of the Shimura correspondence between forms of integral and half-integral weight is available. This theory, which was worked out by one of the authors [4], gives an isomorphism as modules over the Hecke algebra between the space S_{2k} of cusp forms of even weight $2k$ on the full modular group $SL_2(\mathbb{Z})$ and the space $S^+_{k+\frac{1}{2}}$ of cusp forms of weight $k+\frac{1}{2}$ on $\Gamma_0(4)$ having a Fourier development of the form

$$g(z)=\sum c(n)q^n, \qquad c(n)=0 \text{ unless } (-1)^k n\equiv 0 \text{ or } 1 \pmod 4 \tag{1}$$

($q=e^{2\pi i z}$). If $f(z)=\sum a(n)q^n\in S_{2k}$ is a normalized eigenform (i.e. $a(1)=1$) and g as in (1) the corresponding form of half-integral weight, then the Fourier coefficients of f and g are related by

$$c(n^2|D|)=c(|D|)\cdot\sum_{d|n}\mu(d)\left(\frac{D}{d}\right)d^{k-1}a\left(\frac{n}{d}\right), \tag{2}$$

where D is an arbitrary fundamental discriminant (i.e. 1 or the discriminant of a quadratic field) with $(-1)^k D>0$, $\left(\frac{D}{d}\right)$ the corresponding character (Kronecker symbol), $\mu(d)$ the Möbius function, and $\sum_{d\mid n}$ a sum over the positive divisors of n. The form g, which is determined by (2) only up to a scalar multiple, can (and will) be chosen in such a way that its Fourier coefficients $c(n)$ lie in the field generated over $\mathbb{Q}$ by the $a(n)$ (in particular, they are real and algebraic). We can now state the main result of the paper.

Theorem 1. *Let $f\in S_{2k}$ be a normalized Hecke eigenform, $g\in S^+_{k+\frac{1}{2}}$ a form corresponding to f as above, D a fundamental discriminant with $(-1)^k D>0$, and $L(f,D,s)$ the "twisted" L-series of f, defined by analytic continuation of the Dirichlet L-series $\sum_{n=1}^{\infty}\left(\frac{D}{n}\right)a(n)n^{-s}$. Then*

$$\frac{c(|D|)^2}{\langle g,g\rangle}=\frac{(k-1)!}{\pi^k}|D|^{k-\frac{1}{2}}\frac{L(f,D,k)}{\langle f,f\rangle}. \tag{3}$$

Here $\langle g,g\rangle$ and $\langle f,f\rangle$ denote the Petersson scalar products

$$\langle g,g\rangle=\tfrac{1}{6}\int_{\mathfrak{H}/\Gamma_0(4)}|g(z)|^2\,y^{k-\frac{3}{2}}\,dx\,dy,$$

$$\langle f,f\rangle=\int_{\mathfrak{H}/SL_2(\mathbb{Z})}|f(z)|^2\,y^{2k-2}\,dx\,dy$$

(the "6" enters as the index of $\Gamma_0(4)$ in $SL_2(\mathbb{Z})$). Notice that g is defined by our normalization up to a real scalar multiple, so the left-hand side of (3) is well-defined; in fact, (3) holds for any $g\in S_{k+\frac{1}{2}}$ corresponding to f, normalized or not, if we replace $c(|D|)^2$ by $|c(|D|)|^2$.

Before going on to state the corollaries and give the proof of the theorem, we give a numerical example for $k=6$, the first non-trivial case. The one-dimensional spaces S_{12} and $S^+_{13/2}$ are generated by the functions

$$\Delta(z)=8\,000\,G_4(z)^3-147\,G_6(z)^2=\sum_{n=1}^{\infty}\tau(n)\,q^n$$

and

$$\delta(z)=\frac{60}{2\pi i}(2G_4(4z)\,\theta'(z)-G_4'(4z)\,\theta(z))=\sum_{\substack{n=1\\ n\equiv 0,1\,(\mathrm{mod}\,4)}}^{\infty}\alpha(n)\,q^n,$$

where

$$G_k(z)=\tfrac{1}{2}\zeta(1-k)+\sum_{n=1}^{\infty}\sigma_{k-1}(n)\,q^n,\qquad \theta(z)=1+2\sum_{n=1}^{\infty}q^{n^2}$$

($\sigma_\nu(n)=\sum_{d\mid n}d^\nu$). The first few Fourier coefficients $\tau(n)$ and $\alpha(n)$ are given in the accompanying table; we have given a fairly large number of the coefficients $c(n)$ for the benefit of the reader who wants to try to guess the law of

Fourier coefficients of Δ and δ

n	$\tau(n)$
1	1
2	-24
3	252
4	-1,472
5	4,830
6	-6,048
7	-16,744
8	84,480
9	-113,643
10	-115,920
11	534,612
12	-370,944

n	$\alpha(n)$
1	1
4	-56
5	120
8	-240
9	9
12	1,440
13	-1,320
16	-704
17	-240
20	960
21	5,040
24	-12,960
25	1,705
28	13,440
29	-3,960
32	5,760
33	-6,480
36	-504
37	-23,880
40	23,520
41	16,320
44	-43,680
45	59,400
48	-34,560
49	-33,551
52	-10,560
53	4,200
56	87,360
57	65,520
60	-51,840
61	-141,240
64	131,584
65	-111,360
68	13,440
69	64,800
72	-118,800
73	145,200
76	58,080
77	110,880
80	-268,800
81	-174,879
84	40,320
85	137,520
88	-153,120
89	267,600
92	312,960
93	-231,120
96	311,040
97	-357,360
100	-95,480
101	-460,920
104	-92,640
105	272,160
108	362,880
109	505,800
112	-322,560
113	-188,640
116	-31,680
117	-11,880
120	-123,840
121	373,561
124	-1,340,160
125	579,600
128	353,280
129	-422,640
132	362,880
133	300,720
136	1,629,120
137	-46,080
140	-651,840
141	-1,982,880
144	-6,336
145	428,160
148	-191,040
149	-59,640

formation of their signs. The series $L(\Delta,s)=\sum \tau(n)\,n^{-s}$ diverges at $s=6$, but using the integral representation

$$(2\pi)^{-s}\Gamma(s)\,L(\Delta,s)=\int_0^\infty \Delta(iy)\,y^{s-1}\,dy=\int_1^\infty \Delta(iy)(y^s+y^{12-s})\frac{dy}{y}$$

(convergent for all s) we find

$$(2\pi)^{-6}\Gamma(6)\,L(\Delta,6)=2\int_1^\infty \Delta(iy)\,y^5\,dy=2\sum_{n=1}^\infty \tau(n)\,\phi_5(2\pi n),$$

where

$$\phi_5(x)=\int_1^\infty y^5\,e^{-xy}\,dy=\frac{5!}{x^6}\,e^{-x}\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}\right).$$

Similarly, the easily checked functional equation $\Delta^{(D)}(-1/z)=z^{12}\Delta^{(D)}(z)$ for the function

$$\Delta^{(D)}(z)=\frac{1}{\sqrt{D}}\sum_{r(\mathrm{mod}\,D)}\left(\frac{D}{r}\right)\Delta\left(\frac{z+r}{D}\right)=\sum_{n=1}^{\infty}\left(\frac{D}{n}\right)\tau(n)\,q^{n/D}$$

($D>0$ a fundamental discriminant) gives

$$(2\pi/D)^{-6}\Gamma(6)\,L(\Delta,D,6)=2\sum_{n=1}^{\infty}\left(\frac{D}{n}\right)\tau(n)\,\phi_5(2\pi n/D).$$

Since the new series converge exponentially fast, these formulas can be used to compute the L-values in question numerically; using only a few values of $\tau(n)$ and a few minutes on a desk computer one finds the values

D	$L(\Delta,D,6)$	$D^{\frac{11}{2}}L(\Delta,D,6)/L(\Delta,6)$
1	0.7921228386449	1
5	1.632375257462	14,399.99999998
8	0.4922889527919	57,599.9999994
12	1.905551392180	2,073,600.000002
13	1.030984081679	1,742,399.999994
17	0.007793740670	57,600.000029
21	1.075096596506	25,401,600.00005
24	3.410677913003	167,961,600.0000
28	1.57116537691	180,633,600.0001

where within the accuracy of the computation the last column is $\alpha(D)^2$.

We now turn to the corollaries. Since $c(D)$ is real, we deduce immediately from (3):

Corollary 1. $L(f,D,k)\geqq 0$ *for all* D *with* $(-1)^k D>0$.

(Here the condition $(-1)^k D>0$ could be omitted since $L(f,D,k)$ vanishes in the contrary case because of the sign in the functional equation.) Notice that Waldspurger's result on the proportionality of $L(f,D,k)$ and $c(|D|)^2$ would imply only that the values $L(f,D,k)$ for a given f are either all $\geqq 0$ or all $\leqq 0$; of course, one would expect them to be $\geqq 0$ since the contrary would contradict the Riemann hypothesis for $L(f,D,s)$. Even using the Ramanujan-Petersson conjecture there does not seem to be any direct way to prove Corollary 1, since the point $s=k$ lies half a unit to the left of the abscissa of absolute convergence of the L-series or its Euler product.

Corollary 1 does not say when $L(f,D,k)$ is strictly positive; of course, it follows from the theorem that this is equivalent to $c(|D|)\neq 0$ (this was proved in [4] for the case k even, $D=1$), but we do not have any general criterion for this. However, we can prove a partial result. In the table of coefficients for the form in $S^{+}_{13/2}$ given above, one sees by inspection that, for D not a perfect square, $\alpha(D)/120$ is always integral and is odd if (and only if) D is a prime congruent to 5 (mod 8); in particular it is non-zero for such D. A similar statement holds for certain forms of higher weight. Using it, we will prove:

Corollary 2. *For $k \equiv 2 \pmod 4$ and D a prime congruent to $5 \pmod 8$, there is at least one Hecke eigenform $f \in S_{2k}$ for which both $L(f, k)$ and $L(f, D, k)$ are different from zero.*

The next result concerns the arithmetic nature of the Petersson scalar product $\langle g, g \rangle$. The corresponding question for forms of integral weight has a well-known answer: As was shown by Shimura [13] and Manin [6], one can attach to f two real numbers ω_+ and ω_- such that the values of $\pi^{-s} L(f, s)$ for integral s between 0 and $2k$ are algebraic multiples (in fact, multiples in the number field generated by the coefficients of f) of either ω_+ or ω_- depending on the parity of f. On the other hand, a much earlier result of Rankin [11] implies that the product of $\pi^{-2k+1} L_f(2k-1)$ and $\pi^{-m} L_f(m)$ (m even, $k+2 \leqq m \leqq 2k-4$) is an algebraic multiple (in the same field) of $\langle f, f \rangle$; hence $\langle f, f \rangle$ is an algebraic multiple of $\omega_+ \omega_-$. On the other hand, the special values of the twisted L-series $L(f, D, s)$ at integral arguments between 0 and $2k$ are of the form $\pi^s \cdot$(algebraic number)$\cdot \omega_\pm$, where the sign $\pm$ is the same as for $L(f, s)$ if $D > 0$ and the opposite if $D < 0$ ([15, 12]). Combining this with (3) gives

Corollary 3. *Let $g \in S^+_{k+\frac{1}{2}}$ be a Hecke eigenform with algebraic coefficients and f the corresponding normalized eigenform in S_{2k}. Then $\langle g, g \rangle$ is an algebraic multiple of one of the periods $\omega_\pm$ attached to f.*

As an example we take $g = \delta \in S^+_{13/2}$, $f = \Delta$. The values ω_+, ω_- can be chosen such that the values of $L(\Delta, s)$ in the critical strip are given by

s	1, 11	2, 10	3, 9	4, 8	5, 7	6
$(2\pi)^{-s}\Gamma(s)L(\Delta, s)$	$\frac{192}{691}\omega_+$	$\frac{384}{5}\omega_-$	$\frac{16}{135}\omega_+$	$40\omega_-$	$\frac{8}{105}\omega_+$	$32\omega_-$

(numerically $\omega_+ \cong 0.0214460667068$, $\omega_- \cong 0.00004827748001$); then $\langle \Delta, \Delta \rangle = \omega_+ \omega_-$, and the Eq. (3) with $D = 1$ gives

$$\langle \delta, \delta \rangle = \frac{\pi^6}{120} \frac{\langle \Delta, \Delta \rangle}{L(\Delta, 6)} = 2^{-11} \omega_+.$$

Observe that the statement $L(\Delta, D, 6)/\pi^6 \sqrt{D}\, \omega_- \in \mathbb{Q}$ also follows from (3).

Remark. Corollary 3 is also contained in some recent results of Shimura ("The critical values of certain zeta functions associated with modular forms of half-integral weight", to appear in J. Math. Soc. Japan).

Our next corollary is in fact a strengthening of the theorem. In [23], an L-series $L_D(s)$ was defined for all integers D in such a way that $L_D(s)$ equals $\sum \left(\frac{D}{n}\right) n^{-s}$ if D is a fundamental discriminant and that $L_D(s)$ equals $L_{D_0}(s)$ times a finite Euler product over the prime divisors of n if D is n^2 times a fundamental discriminant D_0 ($L_D(s)$ is defined to be identically 0 if D is not congruent to 0 or 1 (mod 4), so this covers all cases $D \neq 0$). The exact definition will be recalled below. We can now define $L(f, D, s)$ for arbitrary D as the convolution of $L(f,s)$ with $L_D(s)$. Then $L(f,Dn^2,s)$ and $L(f,D,s)$ (D a fundamental discriminant) differ only in finitely many factors of their Euler products, and by an elementary computation we will show that their ratio at $s=k$ is the square of the sum appearing in (2). Combining this with formula (3) for fundamental discriminants gives

Corollary 4. *Formula* (3) *holds for arbitrary* $D\in\mathbb{Z}$, $(-1)^k D>0$.

This extension of the theorem is useful for analytic statements, since now we can use all the coefficients $c(n)$. In particular, by applying Rankin's method to the form g we will prove

Corollary 5. *Let* $f\in S_{2k}$ *be a normalized Hecke eigenform. Then the Dirichlet series* $\mathscr{L}_f(s)=\sum_{n=1}^{\infty} L(f,(-1)^k n,k)\,n^{-s}$, *absolutely convergent for* $\mathrm{Re}(s)>1$, *has a meromorphic continuation to the entire complex plane, the only singularity being a simple pole at* $s=1$, *and satisfies the functional equation*

$$\pi^{-2s}\Gamma(s)\Gamma(s+k-\tfrac{1}{2})\zeta(2s)\mathscr{L}_f(s)=(\text{same with } s\mapsto 1-s).$$

The residue of $\mathscr{L}_f(s)$ *at* $s=1$ *equals* $\dfrac{3}{\pi}\dfrac{(4\pi)^{2k}}{(2k-1)!}\langle f,f\rangle$.

By applying Rankin's method to f, we find that the residue of $\sum a(n)^2 n^{-s}$ at the (unique) pole $s=2k$ is also given by $\dfrac{3}{\pi}\dfrac{(4\pi)^{2k}}{(2k-1)!}\langle f,f\rangle$, so we can express the last statement of Corollary 5 more picturesquely as

Corollary 6. *The mean value of* $L(f,D,k)$ $((-1)^k D>0)$ *equals the mean value of* $\dfrac{a(n)^2}{n^{2k-1}}$ $(n\in\mathbb{N})$.

Thus in our example with $k=6$ we have

$$\begin{aligned}\lim_{x\to\infty}\frac{1}{x}\sum_{n=1}^{x} L(\Delta,n,6)&=\frac{2^{16}\pi^{11}}{3^3\cdot 5^2\cdot 7\cdot 11}\langle\Delta,\Delta\rangle\\ &=\lim_{x\to\infty}\frac{1}{x}\sum_{1}^{x}\frac{\tau(n)^2}{n^{11}}=\lim_{x\to\infty}\frac{12}{x^{12}}\sum_{1}^{x}\tau(n)^2.\end{aligned}$$

The numerical value 0.3840840544358 of this limit can be compared with the following experimental data:

x	240	480	720	960
$\frac{1}{x}\sum_{1}^{x} L(\Delta,n,6)$	0.387141325	0.390298507	0.380172606	0.383229191

Corollary 6 is to be compared with the recent paper [2] of Goldfeld and Viola, in which the authors give a conjectural formula for $L(f,D,k)$ for certain cusp forms. Their formula is more complicated than the statement in Corollary 6 because they consider only the twists by fundamental discriminants, and this introduces extra factors. Indeed, using the above-mentioned elementary relation between $L(f,D,s)$ and $L(f,n^2D,s)$ one finds

$$\mathscr{L}_f(s)=\sum_D \frac{L(f,D,k)}{|D|^s}\prod_{p\nmid D}\left(1+p^{-2s-1}-\frac{2p^{-k}\left(\frac{D}{p}\right)a(p)}{p^{2s}+1}\right)$$

where the sum is now over fundamental discriminants D with $(-1)^k D>0$, so Corollary 6 gives us the mean value of the numbers

$$L(f,D,k)\prod_{p\nmid D}\left(1+\frac{1}{p^3}-\frac{2p^{-k}\left(\frac{D}{p}\right)a(p)}{p^2+1}\right).$$

With this observation our result is precisely that conjectured by Goldfeld and Viola (expect that we can treat only the case of level 1).

We observe that Corollaries 1, 2, 5 and 6 all involve only forms of integral weight, even though their proofs need the theory of half-integral weight.

To formulate the next corollary, we define for $g\in S^+_{k+\frac{1}{2}}$ and D a fundamental discriminant with $(-1)^k D>0$

$$\mathscr{S}_D^+ g(z)=\sum_{n=1}^{\infty}\left(\sum_{d\mid n}\left(\frac{D}{d}\right)d^{k-1}c\left(\frac{n^2}{d^2}|D|\right)\right)q^n. \tag{4}$$

Then $\mathscr{S}_D^+ g\in S_{2k}$ (this was proved in [4] and will also follow from our proof of Theorem 1; the map $\mathscr{S}_D^+$ is a modification of the lifting map introduced by Shimura in [14]), and the map $\mathscr{S}_D^+\colon S^+_{k+\frac{1}{2}}\to S_{2k}$ commutes with Hecke operators. In particular, if $f\in S_{2k}$ is a normalized eigenform and g the corresponding form as in (2), then $\mathscr{S}_D^+ g$ is a multiple of f, and comparing the coefficients of q^1 we find $\mathscr{S}_D^+ g=c(|D|)f$. Now let $P^+_{k+\frac{1}{2},D}$ be the $|D|$-th Poincaré series in $S^+_{k+\frac{1}{2}}$. It is characterized by the formula

$$\langle g, P^+_{k+\frac{1}{2},D}\rangle=\frac{\Gamma(k-\frac{1}{2})}{(4\pi|D|)^{k-\frac{1}{2}}}c(|D|)\qquad (\forall g\in S^+_{k+\frac{1}{2}}).$$

The following statement is then easily seen to be equivalent to the main theorem.

Corollary 7. *The image of the Poincaré series $P^+_{k+\frac{1}{2},D}$ under the map $\mathscr{S}_D^+$ defined by* (4) *is the cusp form in S_{2k} characterized by the formula*

$$\langle f,\mathscr{S}_D^+ P^+_{k+\frac{1}{2}}\rangle=\frac{(2k-1)!}{(4\pi)^{k-1}}L(f,D,k)\qquad (\forall f\in S_{2k}).$$

Next, we can compare the main theorem in the case k even, $D=1$, with Shintani's description of the Fourier coefficients of the function in $S^+_{k+\frac{1}{2}}$ corresponding to $f\in S_{2k}$ as integrals of f over certain geodesic cycles in $\mathfrak{H}/SL_2(\mathbb{Z})$.

More precisely, Shintani ([17]) defines for $f \in S_{2k}$, and $n \in \mathbb{N}$, $n \equiv 0$ or $1 \pmod 4$ numbers

$$C(n) = \sum_{i=1}^{h(n)} \int_{\Omega_i} f(z)(a_i z^2 + b_i z + c_i)^{k-1}\, dz$$

where $a_i x^2 + b_i x y + c_i y^2$ $(i = 1, \ldots, h(n))$ are representatives for the equivalence classes of binary quadratic forms of discriminant n and Ω_i is the image in $\mathfrak{H}/SL_2(\mathbb{Z})$ of the semicircle $a_i |z|^2 + b_i x + c_i = 0$. For $n \equiv 2$ or $3 \pmod 4$ set $C(n) = 0$. Then Shintani proves that $\sum_{n=1}^{\infty} C(n) q^n$ is a cusp form of weight $k + \frac{1}{2}$ on $\Gamma_0(4)$ (clearly in the space $S^+_{k+\frac{1}{2}}$) and that the map Φ which sends f to $(-1)^{k/2} 2^k \sum_{n>0} C(n) q^n$ commutes with Hecke operators. Thus if $f \in S_{2k}$ and $g \in S^+_{k+\frac{1}{2}}$ are corresponding eigenforms, we must have $\Phi(f) = \lambda g$ for some $\lambda \in \mathbb{C}$. Comparing the coefficients of q we find

$$\lambda c(1) = (-1)^{k/2} 2^k C(1) = (-1)^{k/2} 2^k \int_{x=0} f(z) z^{k-1}\, dz = \pi^{-k} \Gamma(k) L(f, k).$$

Substituting for $L(f, k)$ from Eq. (3) (we need only the easiest case $D = 1$) we find that $\lambda c(1) = \frac{\langle f, f \rangle}{\langle g, g \rangle} c(1)^2$ and hence $c(1) \langle \Phi f, g \rangle = c(1) \langle f, \mathcal{S}_1^+ g \rangle$. If $c(1) \neq 0$ (or equivalently $L(f, k) \neq 0$; this condition is fulfilled for all eigenforms if and only if $\mathcal{S}_1^+$ is an isomorphism), we can cancel it on both sides of this equation. We thus obtain:

Corollary 8. *If $\mathcal{S}_1^+$ is an isomorphism, then it is the adjoint with respect to the Petersson scalar product of Shintani's map Φ.*

The last corollary to Theorem 1 we wish to state involves the functions

$$f_k(D, z) = \sum_{\substack{a, b, c \in \mathbb{Z} \\ b^2 - 4ac = D}} \frac{1}{(a z^2 + b z + c)^k} \qquad (k > 2 \text{ even}, D > 0).$$

These functions were introduced in [22], Appendix 2, where they were shown to be cusp forms in S_{2k} and their Fourier developments computed. We now have

Theorem 2. *The function*

$$\Omega_k(z, \tau) = \sum_{D>0} D^{k-\frac{1}{2}} f_k(D, z)\, e^{2\pi i D \tau} \qquad (z, \tau \in \mathfrak{H})$$

is for each fixed z a cusp form in $S^+_{k+\frac{1}{2}}$ with respect to τ.

This is an immediate consequence of a result of Vignéras [18] on theta series associated to indefinite quadratic forms. For $z \in \mathfrak{H}$ and $x = (a, b, c) \in \mathbb{R}^3$ define

$$q(x) = b^2 - 4ac, \qquad p_z(x) = \frac{q(x)^{k-\frac{1}{2}}}{(a z^2 + b z + c)^k}.$$

Then $\Omega_k(z,\tau)$ can be written as

$$\Omega_k(z,\tau)=\sum_{\substack{x\in\mathbb{Z}^3\\ q(x)>0}} p_z(x)\,e^{2\pi i q(x)\tau},$$

and the theorem of [18] then implies that Ω_k transforms like a modular form on $\Gamma_0(4)$ of weight $k+\frac{1}{2}$; since it is holomorphic in τ, has no constant term, and has D^{th} Fourier coefficients equal to 0 for $D\not\equiv 0,\ 1 \pmod 4$, we have $\Omega_k(z,\cdot)\in S^+_{k+\frac{1}{2}}$ as claimed.

Remark. Theorem 2 (for forms of arbitrary level, though not specifically for $S^+_{k+\frac{1}{2}}$) was also proved by Michèle Vergne. See §2.7 (particularly Th. 2.7.17, p. 277) of Lion and M. Vergne, The Weil Representation, Maslov Index and Theta Series, Progress in Math. No. 6, Birkhäuser, Boston-Basel-Stuttgart 1980.

Our last corollary to Theorem 1 now states that, under the same restriction as in Corollary 8, the function $\Omega_k(z,\tau)$ is (up to a constant) the holomorphic kernel function for the lifting $\mathscr{S}_1^+\colon S^+_{k+\frac{1}{2}}\to S_{2k}$:

Corollary 9. *Let* $C_k=\dfrac{(-1)^{k/2}\pi}{2^{3k-2}}\dbinom{2k-2}{k-1}$. *Then the scalar product of* $\Omega_k(z,\cdot)$ *with* g *is given by*

$$\tfrac{1}{6}\iint_{\mathfrak{H}/\Gamma_0(4)} g(\tau)\,\Omega_k(z,-\bar\tau)\,y^{k+\frac{1}{2}}\frac{dx\,dy}{y^2}=C_k\mathscr{S}_1^+g(z)$$

for all forms $g\in S^+_{k+\frac{1}{2}}$ *which are orthogonal to* $\operatorname{Ker}(\mathscr{S}_1^+)$. *Thus, if the map* $\mathscr{S}_1^+$ *is an isomorphism, it is given by the kernel function* $C_k^{-1}\Omega_k(z,\tau)$.

Remarks. 1. The functions $f_k(D,z)$ were introduced in [22] for no particular reason other than that they exist. Their significance is made clear by Corollary 9. It is amusing to compare the remark on p. 43 of [22] that "these functions are modular forms having similar properties to the properties of the ω_m"; indeed, something much more formal was meant there, but in fact the formula in Corollary 9 is the exact analogue of the main result of [22], which said that the functions ω_m are the Fourier coefficients of the kernel function for the Doi-Naganuma correspondence between elliptic and Hilbert modular forms.

2. The fact that the same assumption on $\mathscr{S}_1^+$ was needed in Corollaries 8 and 9 is no coincidence, for one can show without any hypothesis that Ω_k is the kernel function for Shintani's map Φ (i.e. that the scalar product of $f\in S_{2k}$ and $f_k(n,\cdot)$ is proportional to the integral $C(n)$ defined above). In fact, both corollaries can be proved without any assumption by comparing the Fourier expansions of the $f_k(D,z)$ as given in [22] and the Fourier expansions of Poincaré series in $S^+_{k+\frac{1}{2}}$ (i.e. by the method used in [22] to prove the corresponding result for the ω_m). The proof is given in [5]. In any case, the assumption that $\mathscr{S}_1^+$ is an ismorphism is probably always fulfilled (it certainly is whenever S_{2k} is irreducible as a module over the rational Hecke algebra, which has been checked up to quite large values of k).

3. Kernel functions for $\mathscr{S}_1^+$ and Φ were previously given by Niwa [7] and Shintani [17], respectively; however, both are non-holomorphic (Niwa's with

respect to z, Shintani's with respect to τ). All three constructions of kernel functions fit into the theory of theta series and the Weil representation as developed, among other places, in Oda [9] and Rallis-Schiffmann [10].

We now turn to the proofs of Theorem 1 and its corollaries. We must first recall some of the results proved in [4] about the spaces $S_{k+\frac{1}{2}}^{+}$ and $M_{k+\frac{1}{2}}^{+}$ (which is defined similarly but with "cusp forms" replaced by "modular forms").

The space $S_{k+\frac{1}{2}}^{+}$ has a basis $\{g_\nu\}$ corresponding by (2) to the canonical basis $\{f_\nu\}$ of normalized eigenforms in S_{2k}. The spaces $M_{k+\frac{1}{2}}^{+}$ is generated by the g_ν and an Eisenstein series $G_{k+\frac{1}{2}}^{+}$ (found by Cohen [1]) whose constant term is $\zeta(1-2k)$ and whose $|D|^{\text{th}}$ Fourier coefficient for $(-1)^k D>0$ is $L_D(1-k)$. The map $\mathscr{S}_D^{+}$ defined by (4) extends to a map $M_{k+\frac{1}{2}}^{+}\to M_{2k}$ if we add the constant term $\frac{1}{2}L_D(1-k)\,c(0)$ to the right-hand side of (4).

There is a canonical projection operator pr^{+} from the space of all modular forms of weight $k+\frac{1}{2}$ on $\Gamma_0(4)$ to $M_{k+\frac{1}{2}}^{+}$, given by

$$pr^{+}=(-1)^{\left[\frac{k+1}{2}\right]}2^{-k}W_4\,U_4+\tfrac{1}{3},$$

where U_4, W_4 are the standard operators

$$U_4\left(\sum c(n)q^n\right)=\sum c(4n)q^n,\qquad (W_4 g)(z)=\left(\frac{2z}{i}\right)^{-k-\frac{1}{2}} g\left(\frac{-1}{4z}\right).$$

We now introduce two special functions

$$\mathscr{F}_D(z)=Tr_1^D(G_{k,D}(z)^2)\in M_{2k},$$

$$\mathscr{G}_D(z)=\tfrac{3}{2}\left(1-\left(\frac{D}{2}\right)2^{-k}\right)^{-1} pr^{+}\,Tr_4^{4D}(G_{k,4D}(z)\,\theta(|D|\,z))\in M_{k+\frac{1}{2}}^{+},$$

where $G_{k,D}$ and $G_{k,4D}$ are the Eisenstein series

$$G_{k,D}(z)=\tfrac{1}{2}L_D(1-k)+\sum_{n=1}^{\infty}\left(\sum_{d\mid n}\left(\frac{D}{d}\right)d^{k-1}\right)q^n\in M_k\left(\Gamma_0(D),\left(\frac{D}{\cdot}\right)\right),$$

$$G_{k,4D}(z)=G_{k,D}(4z)-2^{-k}\left(\frac{D}{2}\right)G_{k,D}(2z)\in M_k\left(\Gamma_0(4D),\left(\frac{D}{\cdot}\right)\right)$$

and Tr_N^M ($M,N\in\mathbb{Z}\setminus\{0\}, N\mid M$) denotes the trace operator (adjoint to the inclusion map) from modular forms on $\Gamma_0(M)$ to forms on $\Gamma_0(N)$. Then we have

Proposition 1. *Let $f=\sum a(n)q^n$ be a normalized eigenform in S_{2k}. Then*

$$\langle f,\mathscr{F}_D\rangle=\frac{1}{2}\frac{(2k-2)!}{(4\pi)^{2k-1}}\frac{L_D(1-k)}{L_D(k)}L(f,2k-1)\,L(f,D,k).$$

Proposition 2. *Let f be as above and $g=\sum c(n)q^n\in S_{k+\frac{1}{2}}^{+}$ an eigenform corresponding to f as in* (2). *Then*

$$\langle g,\mathscr{G}_D\rangle=\frac{1}{4}\frac{\Gamma(k-\frac{1}{2})}{(4\pi)^{k-\frac{1}{2}}}\frac{L_D(1-k)}{L_D(k)}|D|^{k-\frac{1}{2}}L(f,2k-1)\,c(|D|).$$

Proposition 3. *$\mathscr{F}_D$ is the image of $\mathscr{G}_D$ under the mapping $\mathscr{S}_D^+$.*

All three propositions are proved in the case k even, $D=1$ in [4], 2.4. The proofs of Propositions 1 and 2 in the general case are very similar and will therefore only be sketched here. By a standard property of the trace operator we have

$$\langle f, \mathscr{F}_D\rangle = \int_{\mathfrak{H}/\Gamma_0(D)} f(z)\overline{G_{k,D}(z)^2}\, y^{2k-2}\,dx\,dy.$$

By a result of Rankin [11], as reformulated and generalized in [23], the scalar product of any cusp form f with the product of an Eisenstein series of smaller weight and a modular form h of complementary weight is given (up to an elementary factor) by the value at $s=(\mathrm{wt}.f)-1$ of the convolution of the L-series of f and of h. Applying this here and observing that the convolution of the L-series of f and $G_{k,D}$ equals $L(f,s)L(f,D,s-k+1)/L_D(2s-3k+1)$ we obtain Proposition 1. Similarly, since the operator pr^+ is hermitian on cusp forms (this was proved for the operator W_4U_4 by Niwa [8]), we obtain for $g\in S^+_{k+\frac{1}{2}}$

$$4\left(1-\left(\frac{D}{2}\right)2^{-k}\right)\langle g, \mathscr{G}_D\rangle$$
$$= \int_{\mathfrak{H}/\Gamma_0(4D)} g(z)\overline{G_{k,4D}(z)\theta(|D|z)}\, y^{k-\frac{3}{2}}\,dx\,dy,$$

and by Rankin's identity this equals a certain constant times the value at $s=k-\frac{1}{2}$ of the convolution of the L-series of $g(z)$ and $\theta(|D|z)$. But this convolution equals $\dfrac{2c(|D|)}{|D|^s}\,\dfrac{L(f,2s)}{L_D(2s-k+1)}$ by virtue of Eq. (2). This proves Proposition 2.

The proof of Proposition 3 for $D=1$ is a very simple and beautiful calculation, due to H. Cohen and apparently found earlier by Selberg, which we reproduce here. The proof for $D>1$ is based on the same idea, but requires explicit calculations of Fourier coefficients of Eisenstein and theta series at the various cusps of $\Gamma_0(D)$; since these are tedious and of a routine nature, we have relegated them to an appendix. We have

$$\mathscr{G}_1(z)=G_k(4z)\theta(z)=\left(\sum_{n=0}^{\infty}\sigma_{k-1}(n)q^n\right)\left(\sum_{r=-\infty}^{\infty}q^{r^2}\right)$$

(where we have set $\sigma_{k-1}(0)=\frac{1}{2}\zeta(1-k)$), i.e.

$$\mathscr{G}_1(z)=\sum_{n=0}^{\infty}c(n)q^n,\qquad c(n)=\sum_{\substack{r\in\mathbb{Z}\\ r^2\leqq n\\ r^2\equiv n \pmod 4}}\sigma_{k-1}\left(\frac{n-r^2}{4}\right).$$

Hence

$$\mathscr{S}_1^+\mathscr{G}_1(z)=\sigma_{k-1}(0)c(0)+\sum_{n=1}^{\infty}\left(\sum_{d|n}d^{k-1}c\left(\frac{n^2}{d^2}\right)\right)q^n$$
$$=\sigma_{k-1}(0)^2+\sum_{n=1}\left(\sum_{d|n}d^{k-1}\sum_{\substack{|r|\leqq\sqrt{n/d}\\ r\equiv\frac{n}{d}\pmod 2}}\sigma_{k-1}\left(\frac{n^2-r^2d^2}{4d^2}\right)\right)q^n.$$

Writing $n_1=\frac{n-rd}{2}$, $n_2=\frac{n+rd}{2}$, we see that the coefficient of q^n $(n>0)$ equals

$$\sum_{\substack{n_1,n_2\geqq 0\\ n_1+n_2=n}} \sum_{d|(n_1,n_2)} d^{k-1}\sigma_{k-1}\left(\frac{n_1 n_2}{d^2}\right).$$

By the multiplicative properties of σ_{k-1}, the inner sum here equals $\sigma_{k-1}(n_1)\sigma_{k-1}(n_2)$, so

$$\begin{aligned}\mathcal{S}_1^+\mathcal{G}_1(z) &= \sum_{n=0}^{\infty}\Big(\sum_{\substack{n_1,n_2>0\\ n_1+n_2=n}} \sigma_{k-1}(n_1)\sigma_{k-1}(n_2)\Big)q^n\\ &= G_k(z)^2=\mathcal{F}_1(z),\end{aligned}$$

as was to be shown. (Note that the same proof shows that $\mathcal{S}_1^+(f(4z)\theta(z)) = f(z)^2$ for any normalized Hecke eigenform $f\in M_k$, a fact also noticed by both Cohen and Selberg.)

Proof of Theorem 1. Let

$$\mathcal{G}_D(z)=\lambda G_{k+\frac{1}{2}}^+(z)+\sum_{\nu}\lambda_\nu\, g_\nu(z)$$

be the expansion of $\mathcal{G}_D$ with respect to the basis of $M_{k+\frac{1}{2}}^+$ described above. By the remarks preceding Corollary 7 we have $\mathcal{S}_D^+(g_\nu)=c_\nu(|D|)f_\nu$, so Proposition 3 implies

$$\mathcal{F}_D(z)=\lambda L_D(1-k)G_{2k}(z)+\sum_{\nu}\lambda_\nu c_\nu(|D|)f_\nu(z).$$

Therefore we have

$$\langle f_\nu,\mathcal{F}_D\rangle=c_\nu(|D|)\langle f_\nu,f_\nu\rangle\lambda_\nu=c_\nu(|D|)\langle f_\nu,f_\nu\rangle\frac{\langle\mathcal{G}_D,g_\nu\rangle}{\langle g_\nu,g_\nu\rangle},$$

and (3) now follows immediately from Propositions 1 and 2 and the fact that $L(f,2k-1)$ and $L_D(1-k)$ are non-zero.

We now give the proof of Corollaries 2, 4, 5 and 9 (the other corollaries of Theorem 1 are either obvious or have already been proved).

Proof of Corollary 2. For k even we define

$$\delta_k=\frac{1}{4\pi i}\left(\left(\frac{k}{2}-1\right)G_{k-2}(4z)\theta'(z)-G'_{k-2}(4z)\theta(z)\right)=\sum_{n=1}^{\infty}\alpha_k(n)q^n.$$

(Thus δ_6 is $\frac{1}{120}$ times the function $\delta(z)$ used earlier.) Then $\delta_k\in S_{k+\frac{1}{2}}^+$ and for g as in (2) the Petersson product $\langle\delta_k,g\rangle$ is up to a non-zero constant factor equal to $c(1)$ (this follows from the generalization of Rankin's method given in [23], Prop. 6, p. 147). The function δ_k is therefore a linear combination of those eigenforms $g_\nu\in S_{k+\frac{1}{2}}^+$ for which $c_\nu(1)\neq 0$. Hence we have

$$\begin{aligned}&L(f_\nu,k)\neq 0 \quad\text{and}\quad L(f_\nu,D,k)\neq 0 \quad \text{for some } \nu\\ &\Leftrightarrow c_\nu(1)\neq 0 \quad\text{and}\quad c_\nu(D)\neq 0 \quad \text{for some } \nu\\ &\Leftarrow \alpha_k(D)\neq 0.\end{aligned}$$

The Fourier coefficients $\alpha_k(n)$ of δ_k are given by

$$\alpha_k(n) = -\frac{n}{8}\,\sigma_{k-3}\left(\frac{n}{4}\right) + \sum_{0<x\leq\sqrt{n}} \left(\left(\frac{k}{2}-1\right)x^2 + \frac{x^2-n}{4}\right)\sigma_{k-3}\left(\frac{n-x^2}{4}\right)$$

(with the conventions $\sigma_{k-3}(0)=\frac{1}{2}\zeta(3-k)$ and $\sigma_{k-3}(n)=0$ for $n\notin\mathbb{Z}$). We need only consider the case that $n=D$ is a fundamental discriminant >1. Using the fact that $\sigma_{k-3}(n)$ is odd if and only if n or $\frac{n}{2}$ is a square, we find after a little calculation involving the number of representations of an integer as a norm in $\mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-2})$ that the first term is always even except for $D=8$ and that the second is odd if and only if $D=8$ or $D=p\equiv 5 \pmod 8$ and $\frac{k}{2}\equiv 1 \pmod 2$. Thus $\alpha_k(D)$ is odd and hence nonzero for D a prime $\equiv 5 \pmod 8$ and $k\equiv 2 \pmod 4$.

Proof of Corollary 4. Write D as $n^2 D_0$, where $n\in\mathbb{N}$ and D_0 is the discriminant of $\mathbb{Q}(\sqrt{D})$. In view of Eq. (2), it suffices to prove the identity

$$n^{2k-1} L(f,D,k) = L(f,D_0,k)\left(\sum_{d\mid n}\mu(d)\left(\frac{D_0}{d}\right)d^{k-1}a\left(\frac{n}{d}\right)\right)^2.$$

According to Prop. 3, iii) of [23], the L-series $L_D(s)$ is given by

$$L_{D_0}(s) = \sum_{r=1}^{\infty}\left(\frac{D_0}{r}\right)r^{-s}, \qquad L_D(s) = L_{D_0}(s)\sum_{d\mid n}\mu(d)\left(\frac{D_0}{d}\right)d^{-s}\sigma_{1-2s}\left(\frac{n}{d}\right)$$

(this can be taken as the definition of $L_D(s)$). Denote the coefficient of m^{-s} in $L_D(s)$ by $\varepsilon_D(m)$. Then from the above equation we find

$$\sum_{m=1}^{\infty}\varepsilon_D(m)m^{-s} = \sum_{r=1}^{\infty}\left(\frac{D_0}{r}\right)r^{-s}\cdot\sum_{\substack{d,g\\ dg\mid n}}\mu(d)\left(\frac{D_0}{d}\right)d^{-s}g^{1-2s}$$

or, using a well-known property of the Möbius function,

$$\varepsilon_D(m) = \sum_{\substack{r,d,g>0\\ gd\mid n,\, g^2dr=m}}\mu(d)\left(\frac{D_0}{rd}\right)g = \sum_{\substack{g>0\\ g\mid n,\, g^2\mid m\\ \left(\frac{n}{g},\frac{m}{g^2}\right)=1}}\left(\frac{D_0}{g^{-2}m}\right)g$$

(the final sum is either empty or contains exactly one term). Hence the convolution $L(f,D,s) = \sum_{m=1}^{\infty}\varepsilon_D(m)a(m)m^{-s}$ is given by

$$L(f,D,s) = \sum_{g\mid n} g^{1-2s}\sum_{\substack{r=1\\ \left(r,\frac{n}{g}\right)=1}}^{\infty}\left(\frac{D_0}{r}\right)\frac{a(rg^2)}{r^s}.$$

Since f is a Hecke eigenform, we can compute this as an Euler product $\prod_p L_{f,D,p}(p^{-s})$, where

$$L_{f,D,p}(t)=\sum_{\mu=0}^{\nu-1} p^{\mu} t^{2\mu} a(p^{2\mu})+p^{\nu} t^{2\nu} \sum_{\lambda=0}^{\infty} \left(\frac{D_0}{p^{\nu}}\right) a(p^{2\nu+\lambda}) t^{\lambda}$$

(here p^{ν} is the exact power of p dividing n and we have set $g=p^{\mu}$, $r=p^{\lambda}$). Let α and β be the roots of $x^2-a(p)x+p^{2k-1}=0$. Then $a(p^i)$ can be written as $\dfrac{\alpha^{i+1}-\beta^{i+1}}{\alpha-\beta}$, and summing the geometric series we find

$$\begin{aligned} L_{f,D,p}(t)=&\frac{1}{\alpha-\beta}\left(\alpha\frac{1-p^{\nu}\alpha^{2\nu}t^{2\nu}}{1-p\alpha^2t^2}-\beta\frac{1-p^{\nu}\beta^{2\nu}t^{2\nu}}{1-p\beta^2t^2}\right)\\ &+\frac{p^{\nu}}{\alpha-\beta}\left(\frac{\alpha^{2\nu+1}t^{2\nu}}{1-\varepsilon\alpha t}-\frac{\beta^{2\nu+1}t^{2\nu}}{1-\varepsilon\beta t}\right), \end{aligned}$$

where $\varepsilon=\left(\frac{D_0}{p}\right)$. For $p\nmid n$ we clearly have $L_{f,D,p}(t)=L_{f,D_0,p}(t)$. Therefore $L(f,D,s)$ and $L(f,D_0,s)$ differ in only finitely many Euler factors, and continuing analytically to $s=k$ we find

$$L(f,D,k)=L(f,D_0,k)\prod_{p|n}\frac{L_{f,D,p}(p^{-k})}{L_{f,D_0,p}(p^{-k})}.$$

But an elementary computation with the above formula at $t=p^{-k}=(p\alpha\beta)^{-1/2}$ gives

$$\begin{aligned} \frac{L_{f,D,p}(p^{-k})}{L_{f,D_0,p}(p^{-k})}&=(\alpha\beta)^{-\nu}\left(\frac{\beta^{\nu+1}(1-\varepsilon\alpha p^{-k})-\alpha^{\nu+1}(1-\varepsilon\beta p^{-k})}{\alpha-\beta}\right)^2\\ &=p^{-\nu(2k-1)}(a(p^{\nu})-\varepsilon p^{k-1}a(p^{\nu-1}))^2. \end{aligned}$$

Multiplying these expressions together for all $p|n$ gives the desired formula.

Proof of Corollary 5. Corollary 4 gives $\mathscr{L}_f(s)=\dfrac{\pi^k}{(k-1)!}\dfrac{\langle f,f\rangle}{\langle g,g\rangle}\sum_n c(n)^2/n^{s+k-\frac{1}{2}}$. By Rankin's method we find for sufficiently large s the identity

$$\begin{aligned} (4\pi)^{-s-k+\frac{1}{2}}\Gamma(s+k-\tfrac{1}{2})\sum_{n>0}\frac{c(n)^2}{n^{s+k-\frac{1}{2}}}&=\iint_{\mathfrak{H}/\Gamma_\infty}|g(z)|^2 y^{s+k-\frac{3}{2}}dx\,dy\\ &=\iint_{\mathfrak{H}/\Gamma_0(4)} y^{k+\frac{1}{2}}|g(z)|^2 E_s^{(4)}(z)\frac{dx\,dx}{y^2}. \end{aligned}$$

Here $z=x+iy$, Γ_∞ is the group of matrices $\pm\begin{pmatrix}1&n\\0&1\end{pmatrix}$ $(n\in\mathbb{Z})$, and $E_s^{(4)}(z)=\sum_{\gamma\in\Gamma_\infty\setminus\Gamma_0(4)}\operatorname{Im}(\gamma z)^s$ denotes the non-holomorphic Eisenstein series for $\Gamma_0(4)$.

Since this Eisenstein series is known to have a meromorphic continuation to all s, holomorphic except for a simple pole at $s=1$ with residue

$$\operatorname{res}_{s=1} E_s^{(4)}(z)=\frac{1}{[SL_2(\mathbb{Z})\colon \Gamma_0(4)]}\operatorname{res}_{s=1} E_s(z)=\frac{1}{6}\frac{3}{\pi}=\frac{1}{2\pi}$$

(independent of z), where $E_s(z)=\sum_{\gamma\in\Gamma_\infty\backslash SL_2(\mathbb{Z})}\operatorname{Im}(\gamma z)^s$ is the Eisenstein series for $SL_2(\mathbb{Z})$, all of the statements of the Corollary except for the functional equation of $\mathscr{L}_f(s)$ follow immediately from this identity. These properties would hold for $\sum c(n)^2/n^{s+k-\frac{1}{2}}$ for any function $g=\sum c(n)q^n$ in $S_{k+\frac{1}{2}}(\Gamma_0(4))$. To prove the functional equation, however, we must use the fact that $g\in S_{k+\frac{1}{2}}^+$, because this implies relationships among the Fourier expansions of g at the three cusps of $\Gamma_0(4)$, and the function $E_s^{(4)}(z)$ is mapped by $s\mapsto 1-s$ to a linear combination of the Eisenstein series of these cusps. More specifically, one has

$$\left(\frac{2z}{i}\right)^{-k-\frac{1}{2}} g\left(\frac{-1}{4z}\right)=\alpha_1^{-1}\, g_0(z),$$

$$\left(\frac{2z}{i}\right)^{-k-\frac{1}{2}} g\left(\frac{-1}{4z}+\frac{1}{2}\right)=\alpha_1^{-1}\, g_1(z),$$

where $\alpha_1=(-1)^{\left[\frac{k+1}{2}\right]}\,2^k$ and g_0 and g_1 are defined by

$$g_j(z)=\sum_{n\equiv j\,(\mathrm{mod}\ 2)} c(n)\,q^{n/4}\qquad (j=0,1).$$

(The first of these identities is Prop. 2 of [4], and the second is an easy consequence of the first, namely:

$$\left(\frac{2z}{i}\right)^{-k-\frac{1}{2}} g\left(\frac{1}{2}-\frac{1}{4z}\right)=\left(\frac{2z}{i}\right)^{-k-\frac{1}{2}}\left[2g_0\left(-\frac{1}{z}\right)-g\left(\frac{-1}{4z}\right)\right]$$
$$=\frac{\alpha_1}{2^{2k}}\, g\left(\frac{z}{4}\right)-\alpha_1^{-1}\, g_0(z)=\alpha_1^{-1}\, g_1(z).)$$

Applying Rankin's method to the functions g_0 and g_1 (which are modular forms with multiplier systems of weight $k+\frac{1}{2}$ on $\Gamma_0(4)$), we deduce the two identities

$$\pi^{-s-k+\frac{1}{2}}\Gamma(s+k-\tfrac{1}{2})\sum_{n\equiv j\,(\mathrm{mod}\ 2)}\frac{c(n)^2}{n^{s+k-\frac{1}{2}}}$$
$$=\iint_{\mathfrak{H}/\Gamma_0(4)} y^{k+\frac{1}{2}}|g_j(z)|^2\, E_s^{(4)}(z)\,\frac{dx\,dy}{y^2}$$
$$=2^{2k}\iint_{\mathfrak{H}/\Gamma_0(4)} y^{k+\frac{1}{2}}|g(z)|^2\, E_s^{(4)}\left(\frac{-1}{4z+2j}\right)\frac{dx\,dy}{y^2}$$

$(j=0,1)$. Adding them, comparing with the identity given at the beginning of the proof, and using the two easy formulas

$$E_s^{(4)}(z)=\frac{1}{4^s-1}\,(E_s(4z)-2^{-s}E_s(2z)),$$

$$E_s^{(4)}\left(\frac{-1}{4z}\right)+E_s^{(4)}\left(\frac{-1}{4z+2}\right)=\frac{1}{4^s-1}\,(2^s E_s(2z)-E_s(4z)),$$

we obtain the two new integral representations

$$\pi^{-s-k+\frac{1}{2}}\Gamma(s+k-\tfrac{1}{2})\sum_n \frac{c(n)^2}{n^{s+k-\frac{1}{2}}}=\frac{2^{2k}}{3}\iint_{\mathfrak{H}/\Gamma_0(4)} y^{k+\frac{1}{2}}|g(z)|^2 E_s(4z)\frac{dx\,dy}{y^2}$$
$$=\frac{2^{2k}}{2^s+2^{1-s}}\iint_{\mathfrak{H}/\Gamma_0(4)} y^{k+\frac{1}{2}}|g(z)|^2 E_s(2z)\frac{dx\,dy}{y^2}.$$

Since $\pi^{-s}\Gamma(s)\zeta(2s)E_s(z)$ is invariant under $s\mapsto 1-s$, either of them implies the desired functional equation.

Proof of Corollary 9. By Theorem 2, $\Omega_k(z,\tau)$ is a cusp form in S_{2k} with respect to z and a cusp form in $S_{k+\frac{1}{2}}^{+}$ with respect to τ. Therefore we can write Ω_k as

$$\Omega_k(z,\tau)=\sum_{\nu,\mu}\gamma_{\nu\mu}f_\nu(z)\,g_\mu(\tau)$$

for some coefficients $\gamma_{\nu\mu}\in\mathbb{C}$. We first claim that $\gamma_{\nu\mu}=0$ for $\nu\neq\mu$, i.e.

$$\Omega_k(z,\tau)=\sum_{\nu}\gamma_\nu f_\nu(z)\,g_\nu(\tau).$$

Indeed, in view of the "strong multiplicity 1" theorem proved in [4], this assertion is equivalent to the statement that the map $S_{k+\frac{1}{2}}^{+}\to S_{2k}$ defined by scalar product with Ω_k commutes with Hecke operators, i.e.

$$\Omega_k|_{2k}T(p)=\Omega_k|_{k+\frac{1}{2}}T^{+}(p^2),$$

where $T(p)$ on the left is the p^{th} Hecke operator in S_{2k}, acting on the variable z and $T^{+}(p^2)$ on the right the p^{th} Hecke operator in $S_{k+\frac{1}{2}}^{+}$ (as defined in [4]), acting on the variable τ. By the definition of $T^{+}(p^2)$, this is equivalent to the identity

$$f_k(D,\cdot)|_{2k}T(p)=p^{2k-1}f_k(Dp^2,\cdot)+\left(\frac{D}{p}\right)p^{k-1}f_k(D,\cdot)+f_k\left(\frac{D}{p^2},\cdot\right)$$

$\left(\text{with the usual convention } f_k\left(\frac{D}{p^2},z\right)=0 \text{ if } p^2\nmid D\right)$. The proof of this relation, which is quite elementary, is exactly the same as the proof of the analogous statement for non-holomorphic modular forms of weight zero given in [24] (proof of Eq. (36)) and will not be repeated here.

Finally, we must compute the coefficients γ_ν. The statement that $C_k^{-1}\Omega_k(z,\tau)$ is the kernel function for $\mathcal{S}_1^{+}$ is equivalent to the identity

$$\gamma_\nu=C_k\frac{c_\nu(1)}{\langle g_\nu,g_\nu\rangle}.$$

We will prove this identity with both sides multiplied by $c_\nu(1)$, which is equivalent to the statement of Corollary 9. By comparing the coefficients of $e^{2\pi i D\tau}$ in both sides of the identity defining the γ_ν, we can write that identity in the form

$$f_k(D,z)=\sum_\nu \gamma_\nu \frac{c_\nu(D)}{D^{k-\frac{1}{2}}} f_\nu(z).$$

We apply this identity with $D=n^2$ and sum over all $n>0$, obtaining

$$\sum_{n=1}^{\infty} f_k(n^2,z)=\frac{1}{\zeta(k)}\sum_\nu \gamma_\nu c_\nu(1) L(f_\nu, 2k-1) f_\nu(z).$$

(The identity $\sum_{n\geq 1} c_\nu(n^2) n^{-s}=c_\nu(1) L(f_\nu,s)/\zeta(s-k+1)$ was already used in the proof of Proposition 2.) On the other hand, a quadratic form az^2+bz+c whose discriminant is a non-zero square factors as $r(mz+n)(m'z+n')$ with $r\in\mathbb{N}$, $(m,n)=(m',n')=1$, $mn'\neq m'n$; this factorization is unique up to the possibility of interchanging the two factors $mz+n$ and $m'z+n'$ or changing both their signs. Hence

$$\begin{aligned}\sum_{n=1}^{\infty} f_k(n^2,z)&=\tfrac{1}{4}\zeta(k)\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)=1}}\ \sum_{\substack{m',n'\in\mathbb{Z}\\(m',n')=1\\(m,n)\neq\pm(m',n')}} \frac{1}{(mz+n)^k(m'z+n')^k}\\ &=\zeta(k)(E_k(z)^2-E_{2k}(z)),\end{aligned}$$

where $E_k(z)=\frac{1}{2}\sum_{(m,n)=1}(mz+n)^{-k}=\frac{2}{\zeta(1-k)}G_k(z)$ denotes the normalized Eisenstein series in M_k. Taking the Petersson scalar product of both sides with an eigenform $f_\nu\in S_{2k}$ and using the case $D=1$ of Proposition 1, we find

$$\begin{aligned}&\frac{1}{\zeta(k)^2}\gamma_\nu c_\nu(1)\langle f_\nu, f_\nu\rangle L(f_\nu, 2k-1)\\ &\quad=\frac{4}{\zeta(1-k)^2}\langle f_\nu, \mathscr{F}_1\rangle=\frac{2(2k-2)!}{(4\pi)^{2k-1}}\,\frac{1}{\zeta(k)\zeta(1-k)}L(f_\nu 2k-1)L(f_\nu,k)\end{aligned}$$

and hence, by the case $D=1$ of Theorem 1,

$$\gamma_\nu c_\nu(1)=\frac{2(2k-2)!}{(4\pi)^{2k-1}}\,\frac{\zeta(k)}{\zeta(1-k)}\,\frac{\pi^k}{(k-1)!}\,\frac{c_\nu(1)^2}{\langle g_\nu, g_\nu\rangle}=C_k\frac{c_\nu(1)^2}{\langle g_\nu, g_\nu\rangle}.$$

This completes the proof of Corollary 9.

Appendix. Computations of Fourier Coefficients

We introduce the following notations. For D_1, D_2 relatively prime fundamental discriminants with $(-1)^k D_1 D_2>0$ set

$$G_{k,D_1,D_2}(z)=\gamma_{k,D_1}^{-1}\cdot\frac{1}{2}\sum_{m,n}{}' \left(\frac{D_1}{n}\right)\left(\frac{D_2}{m}\right)(mD_1z+n)^{-k}$$

where $\gamma_{k,D_1}=\left(\frac{D_1}{-1}\right)^{\frac{1}{2}}|D_1|^{-k+\frac{1}{2}}\frac{(-2\pi i)^k}{(k-1)!}$ and $\sum'$ has the usual meaning. The function G_{k,D_1,D_2} is an Eisenstein series in $M_k\left(\Gamma_0(D),\left(\frac{D}{\cdot}\right)\right)$ $(D=D_1D_2)$ for the cusp $\frac{1}{D_1}$. Its Fourier expansion is given by

$$G_{k,D_1,D_2}(z)=\sum_{n\geq 0}\sigma_{k-1,D_1,D_2}(n)\,q^n$$

$$\sigma_{k-1,D_1,D_2}(n)=\begin{cases}-L_{D_1}(1-k)L_{D_2}(0) & (n=0)\\ \sum\limits_{\substack{d_1,d_2>0\\ d_1d_2=n}}\left(\frac{D_1}{d_1}\right)\left(\frac{D_2}{d_2}\right)d_1^{k-1} & (n>0)\end{cases}$$

(the constant term is zero unless $D_2=1$).

For $m\in\mathbb{N}$ and $f(z)=\sum_{n\geq 0}a(n)q^n$ we define U_mf by

$$U_mf(z)=\frac{1}{m}\sum_{v\bmod m}f\left(\frac{z+v}{m}\right)=\sum_{n\geq 0}a(mn)q^n.$$

Finally, for D a fundamental discriminant we define three functions u, v and w by $u(D)=(D,8)$, $v(D)=\frac{(D,8)}{(D,2)}$ and $w(D)=\frac{(D,8)^2}{(D,4)^2}$, i.e. by the table

	$D\equiv 1 \pmod 4$	$D\equiv 4 \pmod 8$	$D\equiv 0 \pmod 8$
$u(D)$	1	4	8
$v(D)$	1	2	4
$w(D)$	1	1	4

The following lemma describes the Fourier developments of $\mathcal{F}_D$ and $U_{u(D)}\mathcal{G}_D$, where $\mathcal{F}_D$ and $\mathcal{G}_D$ are the functions defined before Proposition 1.

Lemma. *Let D be a fundamental discriminant with $(-1)^kD>0$. Then we have*

$$\mathcal{F}_D(z)=\sum_{D=D_1D_2}\left(\frac{D_2}{-1}\right)U_{v(D_1)|D_2|}(G_{k,D_1,D_2}(z)^2),$$

$$U_{u(D)}\mathcal{G}_D(z)=\sum_{D=D_1D_2}\left(\frac{D_2}{-|D_1|}\right)U_{u(D)w(D_1)|D_2|}(G_{k,D_1,D_2}(4z)\,\theta(|D_1|z))$$

where $\sum_{D=D_1D_2}$ denotes summation over all decompositions of D as a product of two fundamental discriminants.

Let us first show how the Lemma implies Proposition 3. Suppose D is odd (the case of even D is similar). The 2^{nd} equation of the Lemma says that the m^{th} coefficient of $\mathcal{G}_D$ equals

$$c(m)=\sum_{D=D_1D_2}\left(\frac{D_2}{-|D_1|}\right)\sum_{x\in\mathbb{Z}}\sigma_{k-1,D_1,D_2}\left(\frac{m|D_2|-|D_1|x^2}{4}\right)$$

where we have put $\sigma_{k-1,D_1,D_2}(n)=0$ for $n\notin\mathbb{N}\cup\{0\}$. For $m=n^2|D|$ the expression $\frac{m|D_2|-|D_1|x^2}{4}$ factors as $|D_1|\frac{n|D_2|-x}{2}\cdot\frac{n|D_2|+x}{2}$, so the inner sum equals

$$\sum_{\substack{a_1,a_2\geqq 0\\ a_1+a_2=n|D_2|}}\sigma_{k-1,D_1,D_2}(|D_1|a_1a_2)=\left(\frac{D_2}{|D_1|}\right)\sum_{\substack{a_1,a_2\geqq 0\\ a_1+a_2=n|D_2|}}\sigma_{k-1,D_1,D_2}(a_1a_2).$$

Hence we find for $n>0$

$$\begin{aligned}&\sum_{d|n}\left(\frac{D}{d}\right)d^{k-1}c\left(\frac{n^2}{d^2}|D|\right)\\ &=\sum_{D=D_1D_2}\left(\frac{D_2}{-1}\right)\sum_{d|n}\left(\frac{D}{d}\right)d^{k-1}\sum_{\substack{a_1,a_2\geqq 0\\ a_1+a_2=\frac{n}{d}|D_2|}}\sigma_{k-1,D_1,D_2}(a_1a_2)\\ &=\sum_{D=D_1D_2}\left(\frac{D_2}{-1}\right)\sum_{\substack{a_1,a_2\geqq 0\\ a_1+a_2=n|D_2|}}\sum_{d|(a_1,a_2)}\left(\frac{D}{d}\right)d^{k-1}\sigma_{k-1,D_1,D_2}\left(\frac{a_1a_2}{d^2}\right)\end{aligned}$$

which by the first equation of the Lemma is the n-th coefficient of $\mathscr{F}_D$, if one takes into account the identity

$$\sum_{d|(a_1,a_2)}\left(\frac{D}{d}\right)d^{k-1}\sigma_{k-1,D_1,D_2}\left(\frac{a_1a_2}{d^2}\right)=\sigma_{k-1,D_1,D_2}(a_1)\,\sigma_{k-1,D_1,D_2}(a_2).$$

Also $\frac{1}{2}L_D(1-k)\sigma_{k-1,D_1,D_2}(0)$ is the constant term of $\mathscr{F}_D$. Thus we have shown $\mathscr{S}_D^+(\mathscr{G}_D)=\mathscr{F}_D$.

It remains to prove the Lemma. Since the Fourier expansions of the theta series θ and the Eisenstein series G_{k,D_1,D_2} in every cusp of $\Gamma_0(4D)$ are known, it is clear the Fourier coefficients of $\mathscr{F}_D$ and $\mathscr{G}_D$ can be computed. However, the calculations are rather tricky, so we will sketch the main steps.

The trace operator from $\Gamma_0(D)$ to $SL_2(\mathbb{Z})$ is given by

$$(Tr_1^D f)(z)=\sum_{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_0(D)\setminus SL_2(\mathbb{Z})}(cz+d)^{-k}f\left(\frac{az+b}{cz+d}\right).$$

As a system of representations for $\Gamma_0(D)\backslash SL_2(\mathbb{Z})$ we can take the matrices

$$\begin{pmatrix}1&0\\ d^{-1}|D_1|&1\end{pmatrix}\begin{pmatrix}1&\mu\\0&1\end{pmatrix}\quad(D=D_1D_2,\ d\,|\,v(D_1),\ \mu \bmod \alpha_d D_2)$$

where D_1, D_2 and $v(D_1)$ are as in the Lemma and $\alpha_d=\frac{d}{(d,2)}$ $(=1$ or $2)$. Now one shows as an easy consequence of the definition of G_{k,D_1,D_2} that

$$(d^{-1}|D_1|z+1)^{-k}G_{k,D}\left(\frac{z}{d^{-1}|D_1|z+1}\right)$$
$$=\left(\frac{D_2}{-1}\right)^{-\frac{1}{2}}\left(\frac{D_2}{|D_1|}\right)|D_2|^{-\frac{1}{2}}G_{k,D_1,D_2,d}\left(\frac{z+d|D_1|^*}{d|D_2|}\right)$$

where $|D_1|^*$ is an integer with $|D_1||D_1|^*\equiv 1 \pmod{D_2}$ and

$$G_{k,D_1,D_2,d}(z)=\frac{1}{d}\sum_{\nu \bmod d}\left(\frac{-2d}{2\nu+1}\right)G_{k,D_1,D_2}\left(z+\frac{\nu}{d}\right).$$

From identity (5) we deduce

$$\sum_{d|v(D_1)}\sum_{\mu \bmod \alpha_d D_2}(d^{-1}|D_1|(z+\mu)+1)^{-2k}G_{k,D}\left(\frac{z}{d^{-1}|D_1|(z+\mu)+1}\right)^2$$
$$=\left(\frac{D_2}{-1}\right)U_{v(D_1)|D_2|}(G_{k,D_1,D_2}(z)^2),$$

and this immediately implies the first equality of the Lemma.

Now let us prove the second one. The operator pr^+ from $M_{k+\frac{1}{2}}(\Gamma_0(4))$ to $M^+_{k+\frac{1}{2}}(\Gamma_0(4))$ is given by

$$(pr^+g)(z)=\frac{1-(-1)^k i}{6}(Tr_4^{16}Vg)(z)+\tfrac{1}{3}g(z)\qquad (g\in M_{k+\frac{1}{2}}(\Gamma_0(4)))$$

where V is the map sending $g(z)$ to $g(z+\frac{1}{4})$. This definition of pr^+ is more convenient for us than the one given previously. That they are equivalent is implicit in [8] (proof of Lemma, p. 200–201) or [4] (§2.3., p. 260). Now we distinguish the cases of odd and even D.

i) D odd.

We have

$$V\,Tr_4^{4D}g=Tr_{16}^{16D}Vg\qquad (g\in M_{k+\frac{1}{2}}(\Gamma_0(4D))).$$

Substituting this into the definition of $\mathscr{G}_D$ we find

$$\mathscr{G}_D=\tfrac{3}{2}\left(1-\left(\frac{D}{2}\right)2^{-k}\right)^{-1}Tr_4^{4D}g_D \qquad (6)$$

with

$$g_D(z)=\frac{1-(-1)^k i}{6}Tr_{4D}^{16D}V(G_{k,4D}(z)\theta(|D|z))+\tfrac{1}{3}G_{k,4D}(z)\theta(|D|z).$$

The matrices $\begin{pmatrix}1 & 0\\ 4|D|\nu & 1\end{pmatrix}$, where ν runs over integers mod 4, form a set of representatives for $\Gamma_0(16D)\backslash\Gamma_0(4D)$. Using this one obtains after some calculation

$$g_D(z)=\tfrac{2}{3}\left(1-\left(\frac{D}{2}\right)2^{-k}\right)G_{k,D}(4z)\theta(|D|z). \qquad (7)$$

As a set of representatives for $\Gamma_0(4D)\backslash\Gamma_0(4)$ we choose the matrices

$$\begin{pmatrix} 1 & 0 \\ 4|D_1| & 1 \end{pmatrix} \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \quad (D=D_1 D_2,\ \mu \bmod D_2),$$

where D_1 and D_2 have the same meaning as in the Lemma. Using (5) and well-known transformation formulae of θ one can check that

$$(4|D_1|z+1)^{-k-\frac{1}{2}} G_{k,D}\left(\frac{4z}{4|D_1|z+1}\right)\theta\left(\frac{|D|z}{4|D_1|z+1}\right)$$
$$=\left(\frac{D_2}{-|D_1|}\right)\frac{1}{|D_2|}G_{k,D_1,D_2}\left(\frac{4z+|D_1|^*}{|D_2|}\right)\theta\left(\frac{|D_1|z+4^*}{|D_2|}\right)$$

(where $a^*\in\mathbb{Z}$ with $aa^*\equiv 1 \pmod{D_2}$). From this and (6) and (7) the first equality of the Lemma for odd D follows immediately.

ii) D even.

We shall sketch only the proof for $D\equiv 4 \pmod 8$, since the case $D\equiv 0 \pmod 8$ is similar. From

$$V\,Tr_4^D = Tr_{16}^{4D}\,Vg \qquad (g\in M_{k+\frac{1}{2}}(\Gamma_0(D)))$$

we obtain

$$\mathscr{G}_D=\tfrac{3}{2}\,Tr_4^D\,g_D$$

with

$$g_D(z)=\frac{1-(-1)^k i}{6}\,Tr_D^{4D}\,V\,Tr_D^{4D}(G_{k,4D}(z)\,\theta(|D|z))$$
$$+\tfrac{1}{3}\,Tr_D^{4D}(G_{k,4D}(z)\,\theta(|D|z)).$$

As a set of representatives for $\Gamma_0(4D)\backslash\Gamma_0(D)$ we choose the matrices $\begin{pmatrix} 1 & 0 \\ |D|\nu & 1 \end{pmatrix}$, $\nu \bmod 4$. Using this we obtain

$$g_D(z)=\tfrac{2}{3}\,Tr_D^{4D}(G_{k,4D}(z)\,\theta(|D|z))+\frac{(-1)^k}{3}\,G_{k,-\frac{D}{4},-4}(z)\,\theta\left(\frac{|D|}{4}z\right).$$

Hence

$$\mathscr{G}_D(z)=\tfrac{2}{3}\,Tr_4^{4D}(G_{k,4D}(z)\,\theta(|D|z))$$
$$+\frac{(-1)^k}{3}\,Tr_4^D\left(G_{k,-\frac{D}{4},-4}(z)\,\theta\left(\frac{|D|}{4}z\right)\right). \tag{8}$$

To calculate Tr_4^{4D} we use as a system of representatives of $\Gamma_0(4D)\backslash\Gamma_0(4)$ the matrices

$$\begin{pmatrix} 1 & 0 \\ 4|d_1|(\nu d_2^2+1) & 1 \end{pmatrix}\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix},$$

where ν runs over integers mod 4, d_1 runs over all decompositions $-\frac{D}{4}=d_1 d_2$

(d_1, d_2 fundamental discriminants) and for d_1 fixed μ runs modulo d_2. One now checks that for each v, d_1, d_2 there is an identity of the form

$$\begin{pmatrix} 1 & 0 \\ |d_1|(vd_2^2+1) & 1 \end{pmatrix} = X \begin{pmatrix} 1 & 0 \\ s_v|d_1| & 1 \end{pmatrix} \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

with $s_v = (4, v+1)$ and some $X \in \Gamma_0(D)$, $\lambda \in \mathbb{Z}$. Therefore multiplying the above representatives by $\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$ on the left gives the matrices

$$X \begin{pmatrix} 1 & 0 \\ s_v|d_1| & 1 \end{pmatrix} \begin{pmatrix} 4 & \lambda+4\mu \\ 0 & 1 \end{pmatrix}.$$

But $G_{k,4D}(z)\,\theta(|D|\,z) = G_{k,D}(4z)\,\theta\left(\frac{|D|}{4}\cdot 4z\right)$ and $G_{k,D}(z)$, $\theta\left(\frac{|D|}{4}z\right)$ are forms on $\Gamma_0(D)$, so we can absorb the X into these functions and then calculate the effect of $\begin{pmatrix} 1 & 0 \\ s_v|d_1| & 1 \end{pmatrix}$ on $G_{k,D}(z)$ and $\theta\left(\frac{|D|}{4}z\right)$ by using Eq. (5) (with $D_1 = d_1$ or $-4d_1$ and $d = 1$ or 2 depending on s_v) and the standard transformation equations of θ. The effect of the remaining matrix $\begin{pmatrix} 4 & \lambda+4\mu \\ 0 & 1 \end{pmatrix}$ on the Fourier developments is, of course, trivial. Carrying out the computation, one finds

$$\begin{aligned} &U_4\, Tr_4^{4D}(G_{k,4D}(z)\,\theta(|D|\,z)) \\ &= U_4 \left(\tfrac{1}{2} \sum_{\substack{D=D_1D_2 \\ D_1 \text{ odd}}} \left(\frac{D_2}{-|D_1|}\right) U_{|D_2|}(G_{k,D_1,D_2}(4z)\,\theta(|D_1|\,z)) \right. \\ &\quad \left. + \sum_{\substack{D=D_1D_2 \\ D_1 \text{ odd}}} \left(\frac{D_2}{-|D_1|}\right) U_{|D_2|}(G_{k,D_1,D_2}(4z)\,\theta(|D_1|\,z)) \right). \end{aligned}$$

Finally one shows that

$$\begin{aligned} &Tr_4^D \left(G_{k,-\frac{D}{4},-4}(z)\,\theta\left(\frac{|D|}{4}z\right) \right) \\ &= (-1)^k \sum_{\substack{D=D_1D_2 \\ D_1 \text{ odd}}} \left(\frac{D_2}{-|D_1|}\right) U_{|D_2|}(G_{k,D_1,D_2}(4z)\,\theta(|D_1|\,z)). \end{aligned}$$

Together with (8) these formulas imply the desired result.

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