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Hyperbolic manifolds and special values of Dedekind zeta-functions

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§ 1. Introduction

A famous theorem, proved by Euler in 1734, is that the sum $\sum_{n=1}^{\infty} \frac{1}{n^{2m}}$ is a rational multiple of π^{2m} for all natural numbers m:

$$\sum_1^\infty \frac{1}{n^2}=\frac{\pi^2}{6}, \quad \sum_1^\infty \frac{1}{n^4}=\frac{\pi^4}{90}, \ldots, \quad \sum_1^\infty \frac{1}{n^{12}}=\frac{691\pi^{12}}{638512875}, \ldots.$$

This result was generalized some years ago by Klingen [3] and Siegel [5], who showed that for an arbitrary totally real number field K the value of the Dedekind zeta function

$$\zeta_K(s)=\sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s} \qquad \text{(sum over non-zero integral ideals } \mathfrak{a} \text{ of } K)$$

at a positive even integral argument $s=2m$ can be expressed by a formula of the form

$$\zeta_K(2m)=\text{rational number} \times \frac{\pi^{2nm}}{\sqrt{D}},$$

where n and D denote the degree and discriminant of K, respectively. However, little is known about the numbers $\zeta_K(2m)$ for K not totally real. We will prove the following theorem which describes the nature of these numbers for $m=1$.

Theorem 1. *Let $A(x)$ be the real-valued function*

$$A(x)=\int_0^x \frac{1}{1+t^2}\log\frac{4}{1+t^2}\,dt \qquad (x\in\mathbb{R}) \tag{1}$$

(see Fig. 1). Then the value of $\zeta_K(2)$ for an arbitrary number field K can be expressed by a formula of the form

$$\zeta_K(2)=\frac{\pi^{2r+2s}}{\sqrt{|D|}}\times\sum_v c_v A(x_{v,1})\ldots A(x_{v,s}) \qquad \textit{(finite sum)}, \tag{2}$$

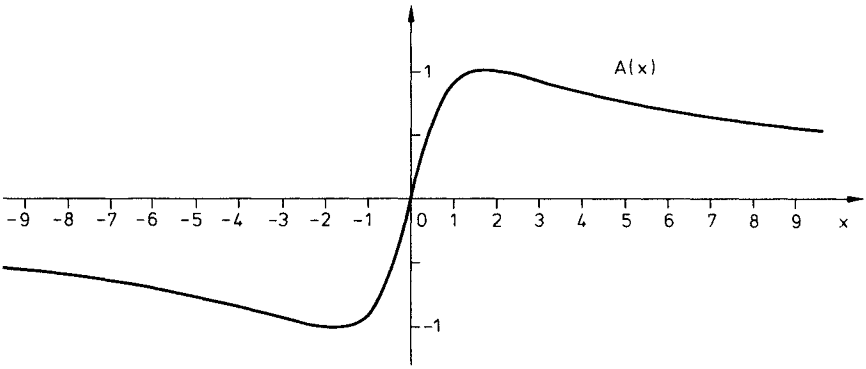

Fig. 1

where D, r and s denote the discriminant and numbers of real and complex places of K, respectively, the c_ν are rational, and the $x_{\nu,j}$ are real algebraic numbers.

The proof will show that the $x_{\nu,j}$ can be chosen of degree at most 8 over K, and will in fact yield the following stronger statement: Let $\sigma_1, \bar{\sigma}_1, \ldots, \sigma_s, \bar{\sigma}_s$ denote the distinct complex embeddings of K; then for any totally imaginary quadratic extension K_1/K and embeddings $\tilde{\sigma}_j\colon K_1 \to \mathbb{C}$ extending σ_j $(1 \leqq j \leqq s)$ there is a formula of the form (2) with $x_{\nu,j}\sqrt{-1}$ of degree $\leqq 2$ over $\tilde{\sigma}_j(K_1)$.

More picturesquely stated, the Klingen-Siegel theorem says that a single transcendental *number*, π^2, suffices to give the contribution of each real place of a field to the value of its zeta-function at $s=2$, and our result says that a single transcendental *function*, $\pi^2 A(x)$, evaluated at algebraic arguments, suffices to give the contribution of each complex place.

The proof of Theorem 1 will be geometric, involving the interpretation of $\zeta_K(2)$ as the volume of a hyperbolic manifold (the function $A(x)$ is equivalent to the dilogarithm and Lobachevsky functions occurring in the formulas for the volumes of 3-dimensional hyperbolic tetrahedra). Since it is only $\zeta_K(2)$ which can be interpreted geometrically in this way, we did not get a formula for $\zeta_K(2m)$, $m>1$. However, we conjecture that an analogous result holds here, namely:

Conjecture 1. *For each natural number m let $A_m(x)$ be the real-valued function*

$$A_m(x) = \frac{2^{2m-1}}{(2m-1)!} \int_0^\infty \frac{t^{2m-1}\,dt}{x \sinh^2 t + x^{-1} \cosh^2 t}. \tag{3}$$

Then the value of $\zeta_K(2m)$ for an arbitrary number field K equals $\pi^{2m(r+s)}/\sqrt{|D|}$ times a rational linear combination of products of s values of $A_m(x)$ at algebraic arguments.

The formulation of this conjecture, and the choice of A_m, are motivated by:

Theorem 2. *Conjecture 1 holds if K is abelian over $\mathbb{Q}$; in fact, in this case the arguments x can be chosen of the form $x = \cot\dfrac{\pi n}{N}$, where N is the conductor of K*

(the smallest natural number such that $K \subset \mathbb{Q}(e^{2\pi i/N})$*). For* $m=1$*, the function defined by* (3) *agrees with the function* $A(x)$ *in Theorem* 1.

Theorems 1 and 2 and the Siegel-Klingen Theorem show that Conjecture 1 is true if K is totally real (i.e. $s=0$), if $m=1$, or if K is abelian, special cases of a sufficiently varied nature to make its truth in general very plausible. The proof of Theorem 2, given in § 4, uses routine number-theoretical tools, and it is worth noting that, even for abelian fields, the geometrically proved Theorem 1 gives a stronger statement (for $m=1$), namely that the arguments of $A(x)$ can be chosen to be of bounded degree over K. Thus, in the simplest case of imaginary quadratic fields ($r=0$, $s=1$), the proof of Theorem 2 gives

$$\zeta_K(2)=\frac{\pi^2}{6\sqrt{|D|}}\sum_{0<n<|D|}\left(\frac{D}{n}\right)A\left(\cot\frac{\pi n}{|D|}\right), \tag{4}$$

where the arguments of $A(x)$ for $(n,D)=1$ are of degree $\phi(|D|)$ or $\phi(|D|)/2$ over $\mathbb{Q}$. For example, when $D=-7$ it gives

$$\zeta_{\mathbb{Q}(\sqrt{-7})}(2)=\frac{\pi^2}{3\sqrt{7}}\left(A\left(\cot\frac{\pi}{7}\right)+A\left(\cot\frac{2\pi}{7}\right)+A\left(\cot\frac{4\pi}{7}\right)\right), \tag{5}$$

whereas the proof of Theorem 1 will lead to the formula

$$\zeta_{\mathbb{Q}(\sqrt{-7})}(2)=\frac{2\pi^2}{7\sqrt{7}}(2A(\sqrt{7})+A(\sqrt{7}+2\sqrt{3})+A(\sqrt{7}-2\sqrt{3})), \tag{6}$$

where now the arguments of $A(x)$, multiplied by $\sqrt{-1}$, are quadratic rather than cubic over K. In this connection we observe that the values of $A(x)$ at algebraic arguments satisfy many non-trivial linear relations over the rational numbers; I know of no direct proof, for instance, of the equality of the right-hand sides of Eq. (5) and (6).

We will discuss (6) and other examples of Theorem 1 later, after giving its proof.

§ 2. Proof of Theorem 1

Assume first that $s=1$, i.e. K is a field of degree $r+2$ with r real places and one complex place. Let B be a quaternion algebra over K which is ramified at all real places (i.e. $B\otimes_K\mathbb{R}\cong$ Hamiltonian quaternions for each real completion $\mathbb{R}$ of K), $\mathcal{O}$ an order in B, and Γ a torsion-free subgroup of finite index in the group $\mathcal{O}^1$ of units of $\mathcal{O}$ of reduced norm 1. Then choosing one of the two complex embeddings of K into $\mathbb{C}$ and an identification of $B\otimes_K\mathbb{C}$ with $M_2(\mathbb{C})$ gives an embedding of Γ into $SL_2(\mathbb{C})$ as a discrete subgroup and hence, identifying $SL_2(\mathbb{C})/\{\pm 1\}$ with the group of isometries of hyperbolic 3-space $\mathfrak{H}_3$, a free and properly discontinuous action of Γ on $\mathfrak{H}_3$. The quotient $\mathfrak{H}_3/\Gamma$ is smooth and is compact if $B\neq M_2(K)$ (which is automatic if $r>0$ and can be assumed in any case) and its volume is well-known to be a rational multiple of

$\zeta_K(2)/\pi^{2r+2}\sqrt{|D|}$ (see e.g. [8], IV, § 1 or [1], 9.1(1)). We therefore have to show that this volume can be expressed as a rational linear combination of values of $A(x)$ at algebraic arguments x.

[The choice of B, $\mathcal{O}$ and Γ plays no role; the reader not familiar with quaternion algebras can take

$$\Gamma \underset{\substack{\text{finite}\\ \text{index}}}{\subset} \left\{ \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} \middle| \, a,b,c,d \in R,\ a^2+b^2+c^2+d^2=1 \right\} \subset SL_2(\mathbb{C}), \tag{7}$$

where $R \subset K \subset \mathbb{C}$ is the ring of integers of K or a subring of finite index (e.g. the ring $\mathbb{Z}[\alpha]$, where α is one of the two non-real roots of a polynomial $f(x)=x^n+\ldots \in \mathbb{Z}[x]$ defining K) and $i=\sqrt{-1}$, corresponding to

$$\mathcal{O}=R+Ri+Rj+Rij \subset B=K+Ki+Kj+Kij \qquad (i^2=j^2=-1,\ ij=-ji).$$

With this choice of B, the field K_1 occurring below can be taken to be $K(i)$.]

Choose a quadratic extension K_1 of K which is a splitting field for B, i.e. such that $B \otimes_K K_1 \cong M_2(K_1)$, and choose an embedding $K_1 \subset \mathbb{C}$ extending the chosen complex place of K and an identification of $B \otimes \mathbb{C}$ with $M_2(\mathbb{C})$ extending the isomorphism $B \otimes K_1 \cong M_2(K_1)$. Then $SL_2(K_1)$ is embedded into $SL_2(\mathbb{C})$ as a countable dense subgroup containing the discrete group Γ, and Γ acts on $\mathfrak{H}_3$ preserving the dense set of points whose coordinates z, r in the standard representation of $\mathfrak{H}_3$ as $\mathbb{C} \times \mathbb{R}_+$ belong to K_1. Hence if we choose a geodesic triangulation of $\mathfrak{H}_3/\Gamma$ with sufficiently small simplices, then by moving the vertices slightly to lie on this dense set we can get a new geodesic triangulation whose vertices have coordinates which are algebraic and in fact lie in the chosen splitting field K_1. To prove the theorem (still for $s=1$), it therefore suffices to show that the volume of a hyperbolic tetrahedron whose four vertices have coordinates belonging to a field $K_1 \subset \mathbb{C}$ can be expressed as a rational linear combination of values of $A(x)$ at arguments x of degree $\leqq 4$ over K_1. In fact, we will show that it is a combination of at most 36 such values, with coefficients $\pm\frac{1}{8}$ or $\pm\frac{1}{4}$.

Let, then, $\Delta \subset \mathfrak{H}_3$ be a tetrahedron with vertices $P_i=(z_i, r_i) \in K_1 \times (K_1 \cap \mathbb{R})_+^\times \subset \mathbb{C} \times \mathbb{R}_+^\times$ $(i=0,1,2,3)$. The geodesic through P_0 and P_1, continued in the direction from P_0 to P_1, meets the ideal boundary $\mathbb{P}_1(\mathbb{C})=\mathbb{C} \cup \{\infty\}$ of $\mathfrak{H}_3$ in a point of $\mathbb{P}_1(K_1)$, and by applying an element of $SL_2(K_1)$ (which does not change the volume of Δ) we may assume that this point is ∞, i.e. that P_0 is vertically below P_1. Then Δ is the difference of two tetrahedra with three vertices $P_i \in \mathfrak{H}_3$ and one vertex at ∞ (Fig. 2). Such a tetrahedron is bounded by (parts of) three vertical planes and one hemisphere with base on $\mathbb{C} \times 0 \subset \partial(\overline{\mathfrak{H}}_3)$. Let P be the top point of this hemisphere. Looking down from infinity, we see a triangle and a point P; drawing the straight lines from P to the vertices and the perpendiculars from P to the sides of this triangle decomposes the triangle into six right triangles and the tetrahedron into six tetrahedra of the kind shown in Fig. 3 (Fig. 4). The volume of the tetrahedron of Fig. 3 is given by the formula

$$\mathrm{Vol}(\mathfrak{S}_{\alpha,\gamma})=\frac{1}{4}\left(\text{Л}(\alpha+\gamma)+\text{Л}(\alpha-\gamma)+2\,\text{Л}\left(\frac{\pi}{2}-\alpha\right)\right) \tag{8}$$

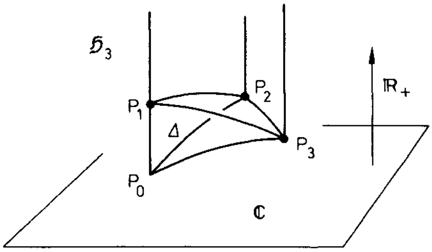

Fig. 2.

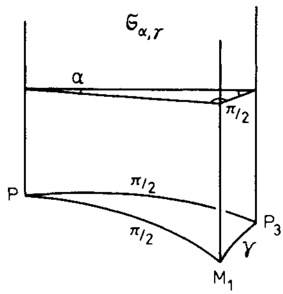

Fig. 3

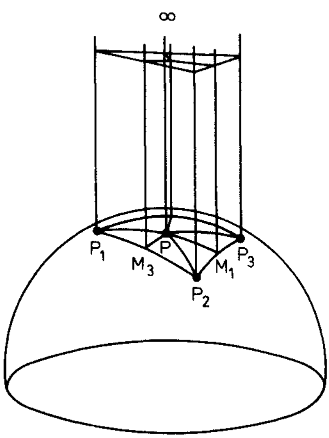

Fig. 4

(cf. Chap. 7 of [6], by Milnor, Lemma 7.2.2), where $Л(\theta)$ is the "Lobachevsky function" (actually introduced by Clausen in 1832, and discussed extensively in Chap. 4 of [4]), defined by

$$Л(\theta)=\frac{1}{2}\sum_{n=1}^{\infty}\frac{\sin 2n\theta}{n^2}=-\int_0^{\theta}\log|2\sin t|\,dt. \tag{9}$$

From

$$\frac{d}{dx}\,Л(\operatorname{arc}\cot x)=-\frac{1}{1+x^2}\,Л'(\operatorname{arc}\cot x)=\frac{1}{2}\,\frac{1}{1+x^2}\log\frac{4}{1+x^2}$$

we deduce that

$$A(x)=2\,Л(\operatorname{arc}\cot x). \tag{10}$$

Hence (8) is equivalent to

$$\operatorname{Vol}(\mathfrak{S}_{\alpha,\gamma})=\frac{1}{8}\left(A\left(\frac{1-ac}{a+c}\right)+A\left(\frac{1+ac}{a-c}\right)+2A(a)\right)\quad (a=\tan\alpha,\ c=\tan\gamma), \tag{11}$$

so to complete the proof we need only check that the tangents of α and γ for the particular tetrahedra $\mathfrak{S}_{\alpha,\gamma}$ occurring in the decomposition of Fig. 4 are algebraic and satisfy $a\sqrt{-1}\in K_1$, $c^2\in K_1$ (so that the three arguments of $A(x)$ in (11), multiplied by $\sqrt{-1}$, are at most quadratic over K_1). This is a question of elementary analytic geometry. Let (Z,R) be the coordinates of the point P in Fig. 4. Then the point $(Z,0)$ is at a distance R from each $P_i=(z_i,r_i)$, so

$$|z_i-Z|^2+r_i^2=R^2 \qquad (i=1,2,3).$$

This leads to the linear system of equations

$$\begin{pmatrix}\bar z_1 & z_1 & 1\\ \bar z_2 & z_2 & 1\\ \bar z_3 & z_3 & 1\end{pmatrix}\begin{pmatrix}Z\\ \bar Z\\ R^2-|Z|^2\end{pmatrix}=\begin{pmatrix}r_1^2+|z_1|^2\\ r_2^2+|z_2|^2\\ r_3^2+|z_3|^2\end{pmatrix}.$$

Since the numbers r_i and z_i belong to K_1, these imply that Z and R^2 belong to K_1. Referring to the picture, we see that the angle $\frac{\pi}{2}-\alpha$ is the argument of $\lambda=(z_j-z_i)/(Z-z_i)\in K_1$ for some i, j, from which $\sqrt{-1}\tan\alpha=\frac{\bar\lambda+\lambda}{\bar\lambda-\lambda}\in K_1$. We also find $\cos\gamma=\frac{D}{R}$ and hence $\tan^2\gamma=(R^2-D^2)/D^2$, where D is the distance from Z to the line joining z_i and z_j, and a simple calculation shows that

$$D^2=-\tfrac{1}{4}(-Z\bar z_i+Z\bar z_j+\bar Z z_i-\bar Z z_j+\bar z_i z_j-z_i\bar z_j)^2/|z_i-z_j|^2\in K_1,$$

as claimed. This completes the proof of the theorem for $s=1$.

Now let s be arbitrary. We choose B, $\mathcal{O}$ and Γ as before (i.e. $B\neq M_2(K)$ a totally definite quaternion algebra over K, $\mathcal{O}$ an order in B, and $\Gamma\subset\mathcal{O}^1$ torsion-free and of finite index). The embeddings $\sigma_1,\dots,\sigma_s\colon K\hookrightarrow\mathbb{C}$ give a map $\sigma\colon B\to M_2(\mathbb{C})^s$ such that $\sigma(\Gamma)$ is a discrete subgroup of $SL_2(\mathbb{C})^s$, and this gives a properly discontinuous, free action of Γ on $\mathfrak{H}_3^s$. Let $M=\mathfrak{H}_3^s/\Gamma$ denote the quotient; then M is a smooth, compact $3s$-dimensional hyperbolic manifold whose volume is a rational multiple of $\zeta_K(2)/\pi^{2r+2s}\sqrt{|D|}$ (loc. cit.). We will show that M can be decomposed as the union (with multiplicities) of sets of the form $\pi(\Delta^{(1)}\times\dots\times\Delta^{(s)})$, where $\pi\colon\mathfrak{H}_3^s\to M$ is the projection and $\Delta^{(j)}\subset\mathfrak{H}_3$ is a hyperbolic tetrahedron each of whose four vertices has both coordinates in $\tilde{\sigma}_j(K_1)$ (K_1 a splitting field of B over K, $\tilde{\sigma}_j$ as in the remark following Theorem 1). Then by the calculation just given, $\mathrm{Vol}(\Delta^{(j)})$ is a rational linear combination of values $A(x)$ with $x\sqrt{-1}$ quadratic over $\tilde{\sigma}_j(K_1)$, and the desired result will follow.

Since M is compact, we can choose compact sets $F_1, \ldots, F_s \subset \mathfrak{H}_3$ so large that $F_1 \times \ldots \times F_s$ contains a fundamental domain for the action of Γ on $\mathfrak{H}_3^s$. We can clearly assume that F_j is triangulated by finitely many small tetrahedra $\Delta_a^{(j)}$ whose coordinates lie in the dense subset $\tilde{\sigma}_j(K_1) \times (\tilde{\sigma}_j(K_1) \cap \mathbb{R}_+)$ of $\mathfrak{H}_3$; here "small" means that each product $\Delta_{\mathbf{a}} = \Delta_{a_1}^{(1)} \times \ldots \times \Delta_{a_s}^{(s)}$ is mapped isomorphically onto its image in M by π. Hence M is covered by finitely many such products $\pi(\Delta_{\mathbf{a}})$, and by the principle of inclusion-exclusion

$$\mathrm{Vol}(M) = \sum_{\mathbf{a}} \mathrm{Vol}(\Delta_{\mathbf{a}}) - \sum_{\mathbf{a}<\mathbf{b}} \mathrm{Vol}(\Delta_{\mathbf{a}} \cap \Delta_{\mathbf{b}}) + \sum_{\mathbf{a}<\mathbf{b}<\mathbf{c}} \mathrm{Vol}(\Delta_{\mathbf{a}} \cap \Delta_{\mathbf{b}} \cap \Delta_{\mathbf{c}}) - \ldots,$$

where we have ordered the multi-indices $\mathbf{a}$ in some way. But each intersection $\Delta_{\mathbf{a}} \cap \Delta_{\mathbf{b}} \cap \ldots$ is itself a product $(\Delta_{a_1}^{(1)} \cap \Delta_{b_1}^{(1)} \cap \ldots) \times \ldots \times (\Delta_{a_s}^{(s)} \cap \Delta_{b_s}^{(s)} \cap \ldots)$, and each factor $\Delta_{a_j}^{(j)} \cap \Delta_{b_j}^{(j)} \cap \ldots$ can be further subdivided into small simplices with coordinates in $\tilde{\sigma}_v(K_1)$, giving a decomposition of the type claimed. This completes the proof of Theorem 1.

§ 3. Numerical examples

Various examples of arithmetic hyperbolic 3-manifolds with explicit triangulations are given in Thurston's notes [6]. Consider, for instance, the knot shown in Fig. 5(a). It was shown by Gieseking in 1912 that the complement M of this knot in S^3 can be triangulated by two 3-simplices (minus their vertices), the triangulation being such that six tetrahedron edges meet along each of the two 1-simplices of the triangulation. Hence, if the two 3-simplices are given the structure of ideal hyperbolic tetrahedra (=tetrahedra with vertices in $\partial \mathfrak{H}_3$) with all dihedral angles equal to 60°, then M acquires a smooth hyperbolic structure with volume $2 \times 3 Л\left(\frac{\pi}{3}\right) = 3A\left(\frac{1}{\sqrt{3}}\right)$ (cf. (10); we have used the fact, proved in [6], that the volume of an ideal hyperbolic tetraheron with dihedral angles α, β, γ is $Л(\alpha) + Л(\beta) + Л(\gamma)$). On the other hand, Riley showed in 1975 that the same knot complement M has a fundamental group isomorphic to a subgroup Γ of $PSL_2(R)$ of index 12, where $R = \mathbb{Z} + \mathbb{Z}\frac{1+i\sqrt{3}}{2}$ is the ring of integers of $\mathbb{Q}(\sqrt{-3})$, so

$$\mathrm{Vol}(M) = \mathrm{Vol}(\mathfrak{H}_3/\Gamma) = 12\,\mathrm{Vol}(\mathfrak{H}_3/SL_2(R)) = 12 \times \frac{3\sqrt{3}}{4\pi^2} \zeta_{\mathbb{Q}(\sqrt{-3})}(2).$$

Comparing these formulas, we find

$$\zeta_{\mathbb{Q}(\sqrt{-3})}(2) = \frac{2\pi^2}{3\sqrt{3}} Л\left(\frac{\pi}{3}\right) = \frac{\pi^2}{3\sqrt{3}} A\left(\frac{1}{\sqrt{3}}\right).$$

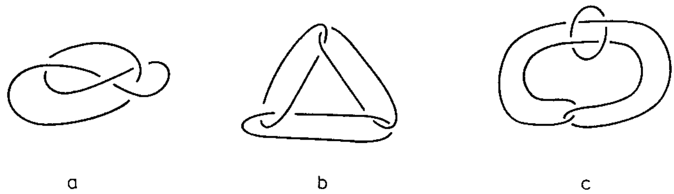

Fig. 5

This formula is not too interesting since it agrees with the formula (4) obtained by straight number-theoretical means $\Bigl($indeed, $\zeta_{\mathbb{Q}(\sqrt{-3})}(s)/\zeta(s) = 1 - 1/2^s + 1/4^s - \ldots$, which at $s=2$ reduces to the series defining $\frac{4}{\sqrt{3}}\,\text{Л}\left(\frac{\pi}{3}\right)\Bigr)$. However, if we take M instead to be the complement of one of the links is 5(b) or 5(c), then Thurston [6, pp. 6.38, 6.40] shows $\mathrm{Vol}(M) = 6\,\mathrm{Vol}(\mathfrak{H}_3/SL_2(R))$, where now R is the ring of integers of $\mathbb{Q}(\sqrt{-7})$. On the other hand, for the manifold of 5(b) he gives a decomposition into two pieces of the form

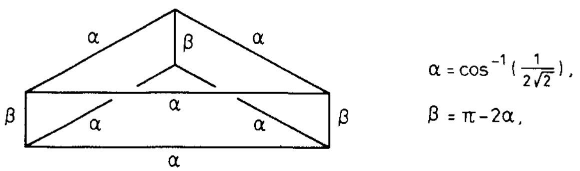

Fig. 6

and applying the volume formula on p. 7.16 of [6] we find that each of these pieces has volume

$$2A(\sqrt{7}) + A(\sqrt{7}+\sqrt{12}) + A(\sqrt{7}-\sqrt{12}).$$

Comparing these two formulas (and using the formula for $\mathrm{Vol}(\mathfrak{H}_3/SL_2(R))$), we obtain Eq. (6) of the introduction. This time, as we remarked at that point, the result is quite different from the formula (5) obtained number-theoretically; as a numerical check, we have the values

$$A(\sqrt{7}) \cong 0.962673014617 \qquad A\left(\cot\frac{\pi}{7}\right) \cong 1.004653150540$$

$$A(\sqrt{7}+\sqrt{12}) \cong 0.690148299958 \qquad A\left(\cot\frac{2\pi}{7}\right) \cong 0.826499033472$$

$$A(\sqrt{7}-\sqrt{12}) \cong -0.837664473558 \qquad A\left(\cot\frac{4\pi}{7}\right) \cong -0.307298022053$$

so that both (5) and (6) give the value $\zeta_{\mathbb{Q}(\sqrt{-7})}(2) \cong 1.89484144897$ to twelve places. (We explain in Appendix 1 how to calculate $A(x)$ numerically.) We can also compute $\zeta_{\mathbb{Q}(\sqrt{-7})}(2)$ directly by the method explained in Appendix 2 and check that this is the right value.

Finally, we consider the field $K=\mathbb{Q}(\sqrt{3+2\sqrt{5}})$ of degree 4 with $r=2$, $s=1$, $|D|=275$ (this is the smallest discriminant for this r and s). Taking an appropriate Γ here gives a quotient $\mathfrak{H}_3/\Gamma$ which can be triangulated by a single tetrahedron Δ with angles as shown in Fig. 7, while the arithmetic description of Γ leads to

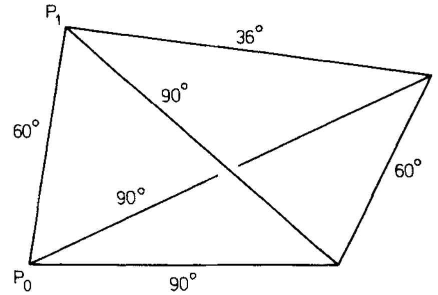

Fig. 7

$$\mathrm{Vol}(\mathfrak{H}_3/\Gamma)=\frac{275^{3/2}}{2^7\pi^6}\,\zeta_K(2).$$

This example, due to Thurston, is discussed in Borel [1], p. 30. The group Γ has torsion, so $\mathfrak{H}_3/\Gamma$ is only an "orbifold" rather than a smooth hyperbolic manifold; it is of special interest because it has the smallest known volume of any orientable hyperbolic orbifold, arithmetic or otherwise. We can compute this volume either number-theoretically or geometrically. The number-theoretical method is described in Appendix 2. The geometrical method is the one used in the proof of Theorem 1. If we choose P_0, P_1 as in Fig. 7 and extend P_0P_1 to ∞ as in Fig. 2, then because of the many right angles in Δ we can subdivide Δ into four simplices $\mathfrak{S}_{\alpha,\gamma}$ of the sort shown in Fig. 3 rather than the usual twelve. Their angles can be computed in a straightforward way, and we find

$$\Delta=\mathfrak{S}_{\frac{\pi}{3},\theta}-\mathfrak{S}_{\beta,\theta-\frac{\pi}{3}}-\mathfrak{S}_{\frac{\pi}{6},\frac{\pi}{5}}+\mathfrak{S}_{\frac{\pi}{6}+\beta,\frac{\pi}{5}}$$

with

$$\theta=\operatorname{arc\,cot}\left(\frac{\sqrt{-3+2\sqrt{5}}}{3}\right),\qquad \beta=\operatorname{arc\,cot}\left(\frac{3+\sqrt{5}}{2}\sqrt{3}+\frac{7+3\sqrt{5}}{2\sqrt{2\sqrt{5}-3}}\right).$$

Now Eq. (8) gives a formula for $\mathrm{Vol}(\Delta)$ as a sum of 12 values of $A(x)$ at (complicated!) algebraic arguments. Computing these values by the method given in Appendix 1, we find $\mathrm{Vol}(\Delta)\cong 0.039050286$, in agreement with the number-theoretical calculation.

We have discussed this last example in some detail because it shows how complicated the formula promised by Theorem 1 can be, even when the geometry of the hyperbolic manifold is very simple (in this case triangulated by a single, and very special, hyperbolic tetrahedron). In general, it is very hard to find examples of arithmetic hyperbolic manifolds for which one has both a good arithmetic and geometric description. Thus it is clear that getting actual formulas for $\zeta_K(2)$ by this method is usually impractical, so that, unless an arithmetical proof giving an explicit formula of the form (2) is found, Theorem 1 must be considered as of mostly theoretical interest.

Appendices to § 3

1. Computation of $A(x)$

By (10), calculating $A(x)$ is equivalent to calculating the Lobachevsky function $Л(\theta)$. Neither the sum nor the integral in (9) are very convenient for numerical work, but there is a very rapidly convergent method. By periodicity, we can assume $|\theta|<\frac{\pi}{2}$. Then $Л(\theta)$ is given by

$$\frac{1}{\pi}\,Л(\pi t)=t(2N+1-\log|2\sin\pi t|)-\sum_{n=1}^{N} n\log\frac{n+t}{n-t}$$
$$-\sum_{k=1}^{\infty}\left(\zeta(2k)-\sum_{n=1}^{N}\frac{1}{n^{2k}}\right)\frac{t^{2k+1}}{k+\frac{1}{2}}$$

for any $N\geqq 0$. This formula, which is easily proved by differentiation, is a special case of the results of [2]. The series converges for $|t|\leqq N+1$ and therefore converges very rapidly for $|t|\leqq\frac{1}{2}$ and quite modest N. Taking $N=4$ and breaking off the series at $k=4$, for example, we get

$$\frac{1}{\pi}\,Л(\pi t)=t(9-\log|2\sin\pi t|)-\sum_{n=1}^{4}\left(c_n t^{2n+1}+n\log\frac{n+t}{n-t}\right)+\varepsilon$$

with

$$c_1=0.147548637158,\qquad c_2=0.00142852188,$$
$$c_3=0.00002919407,\qquad c_4=0.00000076258$$

and $|\varepsilon|<1.2\times10^{-11}$ for $|t|\leqq\frac{1}{2}$.

2. Epstein zeta functions

Let $Q(x,y)=ax^2+bxy+cy^2$ be a positive definite binary quadratic form with integer coefficients and for $n\in\mathbb{N}$ let

$$r(Q,n)=\#\{(x,y)\in\mathbb{Z}^2/\{\pm1\}\mid Q(x,y)=n\}$$

denote one-half the number of representations of n by Q. The series $\sum_{n=1}^{\infty} r(Q,n)\,n^{-s}$ is called an Epstein zeta-function, and the zeta-functions of imaginary quadratic fields are finite sums of such series. For instance, since

$\mathbb{Q}(\sqrt{-7})$ has class number 1, the norms of ideals are just the values of the norm form $x^2+xy+2y^2$, so

$$\zeta_{\mathbb{Q}(\sqrt{-7})}(s)=\sum_{n=1}^{\infty}\frac{r(x^2+xy+2y^2,n)}{n^s}.$$

The Epstein zeta-functions have a well-known Fourier expansion, which at $s=2$ becomes

$$\sum_{n=1}^{\infty}\frac{r(ax^2+bxy+cy^2,n)}{n^2}$$
$$=\frac{\pi^4}{90a^2}+4\pi\zeta(3)\frac{a}{\delta^3}+\frac{8\pi}{\delta^2}\sum_{n=1}^{\infty}\left(\pi n+\frac{a}{\delta}\right)\sigma_{-3}(n)\,e^{\frac{-\pi n\delta}{a}}\cos\frac{\pi nb}{a} \qquad (12)$$

with $\delta=\sqrt{4ac-b^2}$, $\sigma_{-3}(n)=\sum_{\substack{d\mid n\\ d\geqq 1}} d^{-3}$, $\zeta(3)=\sum_{d\geqq 1} d^{-3}=1.202056903\ldots$. The series converges exponentially, and four terms of (12) suffice to compute $\zeta_{\mathbb{Q}(\sqrt{-7})}(2)$ to twelve places.

Epstein zeta-functions can also be used to compute zeta-functions for number fields other than quadratic fields. For example, the field $K=\mathbb{Q}(\sqrt{3+2\sqrt{5}})$ is related to the genus field of the imaginary quadratic field $\mathbb{Q}(\sqrt{-55})$, and using this relationship one can prove the formula

$$\zeta_K(s)=\zeta_{\mathbb{Q}(\sqrt{5})}(s)\times\sum_{n=1}^{\infty}\frac{r(x^2+xy+14y^2,n)-r(4x^2+3xy+4y^2,n)}{n^s}; \qquad (13)$$

since $\zeta_{\mathbb{Q}(\sqrt{5})}(2)=\frac{2\pi^4}{75\sqrt{5}}$, this permits us to calculate $\zeta_K(2)$ easily using Eq. (12) (in fact, *very* easily, since $e^{-\pi\sqrt{55}}<10^{-10}$, so the series in (12) is negligible for $x^2+xy+14y^2$ and extremely rapidly convergent for $4x^2+3xy+4y^2$). We find

$$\zeta_K(2)\cong\frac{2\pi^4}{75\sqrt{5}}(1.1193564009-0.2122647724)\cong 1.053742217$$

and hence $\mathrm{Vol}(\mathfrak{H}_3/\Gamma)\cong 0.0390502856$ for the group Γ discussed at the end of §3.

§4. Proof of Theorem 2

We begin by proving the special case (4), even though this is well-known (see e.g., Milnor [6], p. 7.19), since it illustrates the general case. Let $K=\mathbb{Q}(\sqrt{D})$ be an imaginary quadratic field with discriminant $D<0$ and $\chi(n)=\left(\frac{D}{n}\right)$ the associated character. Then $\zeta_K(s)$ factors as $\zeta(s)L(s,\chi)$, where $L(s,\chi)=\sum_{n=1}^{\infty}\chi(n)\,n^{-s}$, so $\zeta_K(2)=\frac{\pi^2}{6}L(2,\chi)$. The function $\chi(n)$ is odd and periodic with period $|D|$, so it has a Fourier sine expansion, well known to be

$$\chi(n)=\frac{1}{\sqrt{|D|}}\sum_{0<k<|D|}\chi(k)\sin\frac{2\pi k n}{|D|}.$$

Hence, by (9) and (10),

$$L(2,\chi)=\frac{2}{\sqrt{|D|}}\sum_{0<k<|D|}\chi(k)\,\text{Л}\left(\frac{\pi k}{|D|}\right)=\frac{1}{\sqrt{|D|}}\sum_{0<k<|D|}\chi(k)\,A\left(\cot\frac{\pi k}{|D|}\right).$$

Now let K be an arbitrary abelian field. Then $\zeta_K(s)$ is the product of $[K:\mathbb{Q}]$ L-series $L(s,\chi)$, where the χ are primitive Dirichlet characters whose conductors f divide the conductor N of K. If χ is an even character, then $\chi(n)$ has a Fourier expansion

$$\chi(n)=\frac{1}{G_\chi}\sum_{k=1}^{f}\chi(k)\cos\frac{2\pi k n}{f}$$

where G_χ (defined by setting $n=1$ in this formula) is a certain algebraic integer, the Gauss sum attached to χ. Therefore

$$L(2m,\chi)=\frac{\pi^{2m}}{G_\chi}\sum_{k=1}^{f}\chi(k)\,b_{m,k,f}\qquad(\chi\text{ even})$$

where $b_{m,k,f}=\pi^{-2m}\sum_{n=1}^{\infty}\frac{1}{n^{2m}}\cos\frac{2\pi k n}{f}$, which is known to be a rational number $\left(b_{m,k,f}=\frac{2^{2m-1}}{(2m)!}B_{2m}\left(\frac{k}{f}\right)\right.$, where B_r denotes the r-th Bernoulli polynomial$\left.\right)$. If χ is an odd character, then instead

$$\chi(n)=\frac{i}{G_\chi}\sum_{k=1}^{f-1}\chi(k)\sin\frac{2\pi k n}{f},$$

(where again G_χ is defined by setting $n=1$). But

$$\begin{aligned}\frac{(2m-1)!}{2^{2m-1}}\sum_{n=1}^{\infty}\frac{\sin 2n\theta}{n^{2m}}&=2\sum_{n=1}^{\infty}\sin 2n\theta\int_0^{\infty}e^{-2nt}\,t^{2m-1}\,dt\\&=2\int_0^{\infty}\operatorname{Im}\left(\sum_{n=1}^{\infty}e^{2in\theta-2nt}\right)t^{2m-1}\,dt\\&=\sin 2\theta\int_0^{\infty}\frac{t^{2m-1}\,dt}{\cosh 2t-\cos 2\theta}\\&=\int_0^{\infty}\frac{t^{2m-1}\,dt}{\cosh^2 t\tan\theta+\sinh^2 t\cot\theta},\end{aligned}$$

and comparing this with the definition of $A_m(x)$ (Eq. (3)) we find

$$A_m(\cot\theta)=\sum_{n=1}^{\infty}\frac{\sin 2n\theta}{n^{2m}}$$

(which in view of (9) and (10) proves that $A_1=A$) and

$$L(2m,\chi)=\frac{i}{G_\chi}\sum_{k=1}^{f-1}\chi(k)\,A_m\left(\cot\frac{\pi k}{f}\right)\qquad(\chi\text{ odd}).$$

Since K is abelian, it is either totally real ($r=[K:\mathbb{Q}]$, $s=0$) or totally imaginary ($r=0$, $s=\frac{1}{2}[K:\mathbb{Q}]$). In the first case all of the χ are even, so

$$\zeta_K(2m)=\frac{\pi^{2m[K:\mathbb{Q}]}}{\prod_\chi G_\chi}\prod_\chi\left(\sum_{k=1}^{f_\chi}\chi(k)\,b_{m,k,f_\chi}\right),$$

and this has the form $\dfrac{\pi^{2mr}}{\sqrt{|D|}}\times$(rational number) because $\prod_\chi G_\chi=\sqrt{D}$, $D>0$, and the set of χ is closed under the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. (We could also have appealed to the Klingen-Siegel theorem.) In the second case half of the χ are even and half are odd, so

$$\zeta_K(2m)=\frac{\pi^{2ms}i^s}{\prod_\chi G_\chi}\prod_{\chi\text{ even}}\left(\sum_{k=1}^{f_\chi}\chi(k)\,b_{m,k,f_\chi}\right)\prod_{\chi\text{ odd}}\left(\sum_{k=1}^{f_\chi-1}\chi(k)\,A_m\left(\cot\frac{\pi k}{f_\chi}\right)\right).$$

The factor in front equals $\pi^{2ms}/\sqrt{|D|}$ because $\prod G_\chi=\sqrt{D}$ and $(-1)^sD>0$; the second factor is rational for the same reason as before, and for the same reason the third factor is a rational (in fact, integral) linear combination of products of s values of $A_m(x)$ at arguments $x=\cot\dfrac{\pi n}{N}$. This completes the proof.

§ 5. Partial zeta-functions and decomposition of the volume

The zeta-function $\zeta_K(s)$ splits up naturally into h summands $\zeta_K(\mathscr{A},s)$, where h is the class number of K and for each ideal class $\mathscr{A}$ the *partial zeta-function* $\zeta_K(\mathscr{A},s)$ is defined as $\sum_{\mathfrak{a}\in\mathscr{A}}N(\mathfrak{a})^{-s}$. From a number-theoretical point of view, these partial zeta-functions are just as good as Dedekind zeta-functions, so it is natural to make

Conjecture 2. *Conjecture* 1 *remains true with* $\zeta_K(2m)$ *replaced by* $\zeta_K(\mathscr{A},2m)$ *for any ideal class* $\mathscr{A}$ *of* K.

This conjecture can be verified in some cases. For instance, if $K=\mathbb{Q}(\sqrt{D})$ is an imaginary quadratic field with class number 2, then the theory of genera gives

$$\zeta_K(\mathscr{A}_0,s)+\zeta_K(\mathscr{A}_1,s)=\zeta_K(s)=\zeta(s)\,L(s,\chi_D),$$
$$\zeta_K(\mathscr{A}_0,s)-\zeta_K(\mathscr{A}_1,s)=L(s,\chi_{D_1})\,L(s,\chi_{D_2}),$$

where $\mathscr{A}_0$ and $\mathscr{A}_1$ denote the trivial and non-trivial ideal classes and $D_1>0>D_2$ are fundamental discriminants with $D_1 \cdot D_2 = D$; the proof of Theorem 2 shows that for $s=2m$ the right-hand side of both expressions is $\pi^{2m}|D|^{-\frac{1}{2}}$ times a rational linear combination of numbers $A_m\left(\cot\frac{\pi n}{|D|}\right)$. Similar formulas hold for any imaginary quadratic field with one class per genus. A less trivial example is provided by the field $\mathbb{Q}(\sqrt{-55})$, whose class group is cyclic of order 4; here we can verify Conjecture 2 for $m=1$ using Eq. (13).

In the proof of Theorem 1, we obtained $\zeta_K(2)$ as (essentially) the volume of $\mathfrak{H}_3^s/\Gamma$, where Γ is a torsion-free group without parabolic elements contained in a totally definite quaternion algebra over K. However, the proof works even in the presence of elliptic or parabolic elements (provided we include cusps and elliptic fixed points as vertices of tetrahedra) and for quaternion algebras not ramified at the real places of K, except that then we have to take quotients of $\mathfrak{H}_2^t \times \mathfrak{H}_3^s$ $(0 \leqq t \leqq r)$ and these may be non-compact. In particular, we can take $\Gamma = SL_2(\mathcal{O}_K)$ acting on $\mathfrak{H}_2^r \times \mathfrak{H}_3^s$ (Hilbert modular group), in which case the quotient X has h cusps, but still has finite volume given as a simple multiple of $\zeta_K(2)$ (cf. [1], 7.4(1)). The fact that X has exactly the same number of cusps as the number of summands $\zeta_K(\mathscr{A}, 2)$ into which $\zeta_K(2)$ naturally decomposes suggests a possible geometric interpretation of Conjecture 2 for $m=1$: it may be possible to break up X into h pieces, each containing one cusp, in such a way that the volumes of the individual pieces are proportional to the $\zeta_K(\mathscr{A}, 2)$; then if the pieces can be triangulated by simplices with algebraic coordinates, Conjecture 2 follows. There are in fact various natural decompositions of X into h neighborhoods of cusps, but I have not been able to find any which gives the right volumes.

§ 6. Sharpening of Theorem 1 for imaginary quadratic fields

In Theorem 1, the arguments of $A(x)$ could in general be chosen to be of the form $\sqrt{-1}$ times a number quadratic over K_1, where K_1 was an arbitrarily chosen totally imaginary quadratic extension of K. Looking at the proof, we see that K_1 had to be introduced only in order to split the quaternion algebra used to define the group Γ. Hence if K is imaginary quadratic and we take Γ to be $SL_2(\mathcal{O}_K)$ or a torsion-free subgroup of finite index as in § 5, we do not need K_1, so the same proof yields a formula of the form

$$\zeta_K(2) = \frac{\pi^2}{\sqrt{|D|}} \sum_\nu c_\nu A(x_\nu)$$

with $c_\nu \in \mathbb{Q}$ and $x_\nu \sqrt{-1}$ of degree at most 2 over K. But in fact we can do better, namely we can find a representation of $\zeta_K(2)$ of this type with $x_\nu \sqrt{-1} \in K$ and hence (since x_ν is real) $x_\nu \in \mathbb{Q} \cdot \sqrt{|D|}$. More precisely, we have:

Theorem 3. *Let $K=\mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field of discriminant $-d$. Then*

$$\zeta_K(2)=\frac{\pi^2}{6d\sqrt{d}}\sum_\nu n_\nu A\left(\frac{\lambda_\nu}{\sqrt{d}}\right) \tag{14}$$

for some finite collection of numbers $n_\nu\in\mathbb{Z}$, $\lambda_\nu\in\mathbb{Q}$ with $\prod_\nu(\lambda_\nu+\sqrt{-d})^{n_\nu}\in\mathbb{Q}$.

Proof. We first prove (14) with "6" possibly replaced by a larger denominator and under the assumption that $M=\mathfrak{H}_3/\Gamma$ admits an ideal triangulation for some torsion-free subgroup Γ of $PSL_2(\mathcal{O}_K)$ of finite index. Here "ideal triangulation" means a triangulation of $M\cup\{\text{cusps}\}$ into geodesic tetrahedra all of whose vertices lie at the cusps, i.e., there is a fundamental domain for the action of Γ on $\mathfrak{H}_3$ which is a union of hyperbolic tetrahedra, disjoint except along their boundaries, with all vertices in $K\cup\{\infty\}\subset\mathbb{C}\cup\{\infty\}=\partial\mathfrak{H}_3$. An ideal tetrahedron is described by three angles α, β, γ with sum π which are the angles of the cross-section of the tetrahedron seen from any vertex (if one vertex is chosen at infinity, then the tetrahedron seen from above will look like a Euclidean triangle with angles α, β, γ). The volume of such a tetrahedron is $Л(\alpha)+Л(\beta)+Л(\gamma)=\frac{1}{2}[A(\cot\alpha)+A(\cot\beta)+A(\cot\gamma)]$ ([6], Theorem 7.2.1). Since the angles of a triangle with vertices in $K\cup\{\infty\}$ have cotangents in $\mathbb{Q}\cdot\sqrt{d}$ (by translating and rotating the triangle we can put its vertices at 0, 1 and $x+y\sqrt{-d}$ with $x,y\in\mathbb{Q}$, and then the cotangent of the angle at 0 is $x/y\sqrt{d}$), we can write this volume as $\frac{1}{2}\left[A\left(\frac{\lambda}{\sqrt{d}}\right)+A\left(\frac{\mu}{\sqrt{d}}\right)+A\left(\frac{\nu}{\sqrt{d}}\right)\right]$ with $\lambda,\mu,\nu\in\mathbb{Q}$, $(\lambda+\sqrt{-d})(\mu+\sqrt{-d})(\nu+\sqrt{-d})\in\mathbb{Q}$. Since the volume of $\mathfrak{H}_3/PSL_2(\mathcal{O}_K)$ equals $\frac{d^{3/2}}{4\pi^2}\zeta_K(2)$ [6, Theorem 7.4.1], this proves the theorem under the assumption stated and with 6 replaced by $\frac{1}{2}[PSL_2(\mathcal{O}_K):\Gamma]$.

Now in fact any hyperbolic 3-manifold with at least one cusp admits an ideal triangulation. This has been stated by Thurston and others and can be proved using the dual complex of a Ford domain subdivision, but we could not find a suitable reference. However, in [7] Thurston shows that any such manifold M admits a generalized ideal triangulation which is allowed to "fold back" on itself, so that some of the ideal tetrahedra are counted with multiplicity -1 instead of $+1$ (more precisely, given any geodesic triangulation of $\mathfrak{H}_3/\Gamma$ we can find a Γ-equivariant map from $\mathfrak{H}_3\cup\{\text{cusps}\}$ to itself, equivariantly homotopic to the identity, such that the images of the original simplices are ideal tetrahedra, possibly degenerate). This is sufficient for our purposes since it implies that $\mathrm{Vol}(M)$ is an integral linear combination of volumes of ideal tetrahedra with vertices at the cusps, and hence in our situation of expressions $\frac{1}{2}\left[A\left(\frac{\lambda}{\sqrt{d}}\right)+A\left(\frac{\mu}{\sqrt{d}}\right)+A\left(\frac{\nu}{\sqrt{d}}\right)\right]$ with λ, μ, ν and $(\lambda+\sqrt{-d})(\mu+\sqrt{-d})(\nu+\sqrt{-d})$ rational. The fact that some of the tetrahedra are counted with multiplicity -1 means that some of the coefficients n_ν in (14) may be negative, but we do not care about this and anyway could achieve $n_\nu>0$ (by changing the sign of λ_ν) or even $n_\nu=1$ (by repeating some λ_ν). Moreover, Thurston's construction has the property that it can be carried out equivariantly with respect to the (finite) group of isometries of M, at least if we allow ideal polyhedra rather than just ideal tetrahedra in the decomposition. Apply this to $M=\mathfrak{H}_3/\Gamma$ where Γ is a torsion-free normal subgroup of $PSL_2(\mathcal{O}_K)$ of finite index (e.g. the full congruence subgroup $\Gamma(n)$ for some $n>3$). Then the polyhedra of the subdivision of M are permuted by the finite group $G=PSL_2(\mathcal{O}_K)/\Gamma$ and we have

$$\mathrm{Vol}(\mathfrak{H}_3/PSL_2(\mathcal{O}_K))=\frac{1}{|G|}\sum_{\sigma}\pm\mathrm{Vol}(\sigma)=\sum_{[\sigma]}\pm\mathrm{Vol}(\sigma)/N_\sigma,$$

where σ ranges over all polyhedra of the subdivision in the first sum and over all G-orbits of such in the second, and where N_σ denotes the order of the stabilizer of σ in G or equivalently (since an element stabilizing a polyhedron has finite order and Γ is torsion-free) of the stabilizer in $PSL_2(\mathcal{O}_K)$ of a representative of σ in $\mathfrak{H}_3$. But the order of a torsion element of $PSL_2(\mathcal{O}_K)$ is at most 3 (this follows easily by looking at its trace), so this stabilizer is a finite group of $PSL_2(\mathbb{C})$ whose elements all have order 1, 2 or 3 and hence (by the classification of finite subgroups of $PSL_2(\mathbb{C})$) is isomorphic to a subgroup of the alternating group A_4 of order 12. It follows that $12\cdot\mathrm{Vol}(\mathfrak{H}_3/PSL_2(\mathcal{O}_K))$ is an integral linear combination of volumes of ideal polyhedra and hence (triangulating these polyhedra arbitrarily) of ideal tetrahedra. This proves Theorem 3 with the denominator stated.

As an example of the theorem, we can subdivide the polyhedra in Fig. 6 into three tetrahedra, and working out their dihedral angles we obtain the formula

$$\zeta_{\mathbb{Q}(\sqrt{-7})}(2)=\frac{2\pi^2}{21\sqrt{7}}\left(3A\left(\frac{1}{\sqrt{7}}\right)+3A\left(\frac{3}{\sqrt{7}}\right)+A\left(\frac{5}{\sqrt{7}}\right)\right)$$

with $\prod_\nu(\lambda_\nu+\sqrt{-d})^{n_\nu}=(1+\sqrt{-7})^{12}(3+\sqrt{-7})^{12}(5+\sqrt{-7})^4=2^{52}$. In this example the factor 1/6 in (14) can be replaced by 2/3, and this may well be true in general. Other questions one can ask about Theorem 3 are whether it extends to partial zeta-functions as discussed in § 5, whether there is a formula of the same type for $\zeta_K(4)$, $\zeta_K(6)$, etc. (with A replaced by A_m as in Conjecture 1), and whether Theorem 1 can be sharpened in general to have arguments $x_{\nu,j}\in\mathbb{R}\cap\sqrt{-1}\cdot\sigma_j(K)$ for arbitrary number fields K. Probably the answers to all of these questions will have to wait until there is a proof of Theorems 1 and 3 by the methods of analytic number theory, since it does not seem possible to push the geometric approach much further.

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