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A Kronecker Limit Formula for Real Quadratic Fields*

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Contents

1. Introduction and Historical Survey

Let K be a number field, $\zeta_K(s)$ the Dedekind zeta-function of K. We can break up ζ_K into a finite sum

$$\zeta_K(s) = \sum_A \zeta(s, A),$$

where A runs over the ideal class group of K and

$$\zeta(s, A) = \sum_{a \in A} \frac{1}{N(a)^s} \quad (\mathrm{Re}(s) > 1).$$

Then $\zeta(s, A)$ is (after analytic continuation) a meromorphic function of s with a simple pole at $s = 1$ as its only singularity. Moreover, the residue of $\zeta(s, A)$ at $s = 1$ is independent of the ideal class A chosen; this fact, discovered by Dirichlet (for the case of quadratic fields) is at the basis of the analytic determination of the class number of K.

If we consider the Laurent expansion of $\zeta(s, A)$ at $s = 1$, however, say

$$\zeta(s, A) = \frac{\varkappa}{s-1} + \varrho(A) + \varrho_1(A)(s-1) + \cdots,$$

then it transpires that the constant term $\varrho(A)$ is no longer independent of the choice of A. In the easiest case (apart from $K = \mathbb{Q}$), namely when K is an imaginary quadratic field, the evaluation of $\varrho(A)$ was accomplished by Kronecker [4] (the so-called "first Kronecker limit formula"); we describe his result below in Section 2. The interest of determining $\varrho(A)$ is as follows. Let χ be any non-trivial character

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on the ideal class group; then for the corresponding L-series,

$$\begin{aligned} L(s,\chi) &= \sum_{a} \chi(a)\, N(a)^{-s} \\ &= \sum_{A} \chi(A)\, \zeta(s,A) \\ &= \sum_{A} \chi(A) \left(\frac{\varkappa}{s-1} + \varrho(A) + 0(s-1) \right) \\ &= \sum_{A} \chi(A)\, \varrho(A) + 0(s-1), \end{aligned}$$

so

$$L(1,\chi) = \sum_{A} \chi(A)\, \varrho(A) \quad (\chi \neq \chi_0).$$

Therefore knowing $\varrho(A)$ permits us to find $L(1,\chi)$ for all such characteres, and this in turn leads to an evaluation of the residue of the zeta-function (and hence of the class number) of unramified abelian extensions of K, since these zeta-functions are the product of $\zeta_K(s)$ and certain $L(s,\chi)$.

For number fields other than $\mathbb{Q}$ or imaginary quadratic fields, the analysis of $\zeta(s,A)$ is made very much harder by the presence of units of infinite order, and already the formula for the residue $\varkappa$ is much more complicated than in these two special cases, involving a determinant in the logarithms of certain units. As a result, an extension of Kronecker's theorem to other fields was not attempted for over half a century. Then, in 1917, Hecke [1] took up the problem again and showed how, by an ingenious trick, the series defining $\varrho(A)$ in the case of a real quadratic field K (a Dirichlet series involving an indefinite quadratic form) could be written as an integral of a series involving a *definite* form, whose Laurent expansion at $s=1$ could thus be evaluated by Kronecker's formula. This led to an expression for $\varrho(A)$ as an integral involving the transcendental function (namely $\log|\eta(z)|$) already occurring in Kronecker's formula for imaginary quadratic fields.[1] Hecke's argument is very beautiful, and will be reproduced in § 3. Nevertheless, the expression he obtains is far less satisfactory than Kronecker's for the imaginary case, since in place of a single, universal function of the element of a basis for some ideal, his evaluation of $\varrho(A)$ requires the integration of such a function and thus cannot be regarded as being in closed form (the same criticism was made by Schoeneberg, the editor of Hecke's collected works, whose footnote to Hecke's assertion „Es ist mir nun gelungen, das genannte Problem zu lösen..." is the laconic comment, „Diese Behauptung ist offenbar unzutreffend.").

The next progress was made by Meyer [6]. He used the same method as Hecke but applied to ideal classes in the narrow, rather than in the wide sense. Clearly, if B is such an ideal class, then we can define $\zeta(s,B)$ and $\varrho(B)$ by the same formulas as before; if K contains a unit of norm -1 this brings nothing new, but if all units have positive norm then each wide ideal class A is the union of two narrow ideal classes B and B^*. Clearly $\varrho(B)+\varrho(B^*)=\varrho(A)$; what Meyer did is to evaluate the difference $\varrho(B)-\varrho(B^*)$ in finite terms, this difference (apart

[1] In the same paper, Hecke considers the problem of determining $\varrho(A)$ for arbitrary number fields K and claims to have an analogous formula, based on Epstein's generalization of the Kronecker limit formula, in the general case; the details never appeared.

from a factor $\pi^2/\sqrt{D}$) turns out to be a rational number, given by the so-called "Dedekind sums" arising in the transformation theory of $\log\eta(z)$. Meyer's formula will be given in § 4.

It was recently observed by Hirzebruch and the author [3] that Meyer's result can be elegantly stated in terms of continued fractions. To each narrow ideal class B one can associate certain periodic continued fractions whose period (up to cyclic permutation) depends only on the class B. If we denote the length of this period by $l(B)$, then Meyer's formula can be stated as

$$\varrho(B)-\varrho(B^*)=-\frac{\pi^2}{6\sqrt{D}}\left(l(B)-l(B^*)\right).$$

This naturally suggested that the actual value of the Kronecker limit $\varrho(B)$ might be computable in terms of the continued fraction. Using Hecke's formula for $\varrho(A)=\varrho(B)+\varrho(B^*)$ and numerical integration on a computer, the values of $\varrho(B)$ were calculated for the ideal classes of a few real quadratic fields; the results showed a clear correlation between the values of $l(B)$ and $\frac{6\sqrt{D}}{\pi^2}\varrho(B^*)$, e.g.

$K=\mathbb{Q}(\sqrt{15})$:

Ideal class B	$\frac{6\sqrt{D}}{\pi^2}\varrho(B^*)$	$l(B)$
$B_0=(1,4+\sqrt{15})$	1.1692194	1
$B_0^*=(6,9+\sqrt{15})$	6.1692194	6
$B_1=(3,6+\sqrt{15})$	3.3986892	3
$B_1^*=(2,5+\sqrt{15})$	2.3986892	2

$K=\mathbb{Q}(\sqrt{105})$:

Ideal class B	$\frac{6\sqrt{D}}{\pi^2}\varrho(B^*)$	$l(B)$
$B_0=\left(1,\frac{11+\sqrt{105}}{2}\right)$	3.5268892	3
$B_0^*=\left(6,\frac{21+\sqrt{105}}{2}\right)$	11.5268892	11
$B_1=\left(2,\frac{13+\sqrt{105}}{2}\right)$	4.6101810	4
$B_1^*=\left(3,\frac{15+\sqrt{105}}{2}\right)$	8.6101810	8

With this encouragement, it was natural to look for an expression for $\zeta(s,B)$ involving the continued fractions associated to B; when this had been accomplished, it turned out that $\zeta(s,B)$ could be expressed as the sum of $l(B)$ simpler Dirichlet series in which the action of the fundamental unit of K no longer appeared and whose behaviour near $s=1$ could thus be studied by direct analytic means. The resulting formula is stated and proved in Sections 5 and 6. Like Kronecker's result, it involves a universal function of one variable, here (rather unoriginally) denoted $F(x)$; since this function does not seem to occur anywhere in the literature, we have also devoted a section (Section 7) to studying its properties (asymptotic expansion, special values, functional equations). These properties are used in Section 8 to deduce Meyer's theorem from our formula for $\varrho(B)$. In a final section, we discuss the identities arising by applying the formula $L(1,\chi)=\Sigma\chi(A)\,\varrho(A)$ to the special case of a "genus character", where the value of $L(1,\chi)$ is known by other considerations; the corresponding identities in the imaginary case are known under the name of "Kronecker's solution of the Pell equation".

2. Kronecker's Limit Formula

The basic idea for studying $\zeta(s,A)$ is always the same: if one picks a fixed ideal $\underline{b}$ belonging to the class A^{-1}, then the correspondence

$$a \mapsto a\underline{b} = \text{some principal ideal } (\lambda)$$

is a bijection between the set of ideals of A and the set of principal ideals (λ) divisible by $\underline{b}$, i.e. with $\lambda \in \underline{b}$. On the other hand, two numbers $\lambda_1, \lambda_2 \in \underline{b}$ define the same principal ideal iff $\lambda_1 = \varepsilon\lambda_2$ for some unit ε, i.e. iff they have the same image in $\underline{b}/U$ (U = group of units of K). Hence

$$\begin{aligned}\zeta(s,A) &= N(\underline{b})^s \sum_{a\in A} \frac{1}{N(a\underline{b})^s}\\ &= N(\underline{b})^s \sum_{\lambda\in\underline{b}/U}{}' \frac{1}{|N(\lambda)|^s},\end{aligned} \tag{2.1}$$

where (here and in the sequel) the prime on the summation sign indicates that the value 0 is to be omitted.

If now K is an imaginary quadratic field of discriminant $D<0$, then U is a finite group; its order is 2 for $D \neq -3, -4$. Hence

$$\zeta(s,A) = \frac{1}{|U|} N(\underline{b})^s \sum_{\lambda\in\underline{b}}{}' N(\lambda)^{-s}$$

(we can drop the abolute value sign since $N(\lambda)=\lambda\lambda'=\lambda\bar{\lambda}=|\lambda|^2>0$). This formula is unchanged if we replace $\underline{b}$ by $\alpha\underline{b}$ ($\alpha\in K-\{0\}$), so we can assume that $\underline{b}$ has a basis of the form $\{1,w\}$. We also suppose that this basis is *oriented*, i.e. $\mathrm{Im}(w)>0$ (here we have fixed an embedding of K in $\mathbb{C}$). Then

$$N(\underline{b}) = \frac{w-\bar{w}}{i\sqrt{|D|}} = \frac{2}{\sqrt{|D|}}\,\mathrm{Im}(w),$$

$$N(mw+n) = m^2|w|^2 + 2mn\,\mathrm{Re}(w) + n^2\,.$$

Therefore

$$\zeta(s,A) = \frac{|D|^{-s/2}}{|U|} \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{Q(m,n)^s} \tag{2.2}$$

(the $'$ means that m and n are not both 0), where

$$Q(m,n) = \frac{|mw+n|^2}{2\,\mathrm{Im}(w)} \tag{2.3}$$

is the binary form $N(m+nw)$, normalized to have determinant -1. Kronecker's theorem can now be stated.

Theorem. *Let*

$$Q(x,y)=ax^2+bxy+cy^2\,,\qquad a,c>0\,,\qquad b^2-4ac=-1\,,$$

be any positive definite quadratic form of discriminant -1, *and* w *the solution with positive imaginary part of the quadratic equation* $cw^2-bw+a=0$, *so that* Q *is given as in* (3). *Then the zeta-function of* Q, *defined by*

$$\zeta_Q(s)=\sum_{m,n}{}'\frac{1}{Q(m,n)^s}$$

if $\mathrm{Re}(s)>1$, *can be extended meromorphically to a neighbourhood of* $s=1$ *and has there a Laurent expansion*

$$\zeta_Q(s)=\frac{2\pi}{s-1}+C+0(s-1)$$

with residue independent of Q *and constant term given by*

$$C=4\pi(\gamma+\tfrac{1}{2}\log c-\log|\eta(w)|^2)\,. \tag{2.4}$$

Here γ *denotes Euler's constant and*

$$\eta(w)=e^{\frac{\pi i w}{12}}\prod_{n=1}^{\infty}(1-e^{2\pi i n w})\quad(\mathrm{Im}\,w>0)\,. \tag{2.5}$$

Dedekind's eta-function.

Proof. The terms with $m=0$ clearly give

$$\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1}{(cn^2)^s}=2c^{-s}\zeta(2s),$$

and the others are unchanged by $m\to -m$, so

$$\tfrac{1}{2}\zeta_Q(s)-c^{-s}\zeta(2s)=\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{Q(m,n)^s}\,. \tag{2.6}$$

Write

$$I(s)=\int_{-\infty}^{\infty}\frac{dt}{Q(1,t)^s}\,;$$

this is clearly a holomorphic function of s for $\mathrm{Re}(s)>\frac{1}{2}$. Then

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{Q(m,n)^s}&=\zeta(2s-1)\,I(s)\\&\quad+\sum_{m=1}^{\infty}\left(\sum_{n=-\infty}^{\infty}\frac{1}{Q(m,n)^s}-\frac{1}{m^{2s-1}}I(s)\right).\end{aligned} \tag{2.7}$$

The interior sum on the right equals

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\frac{1}{Q(m,n)^s}-\int_{-\infty}^{\infty}\frac{dy}{Q(m,y)^s}\quad(y=mt)\\ &=\sum_{n=-\infty}^{\infty}\left[\frac{1}{Q(m,n)^s}-\int_{n}^{n+1}\frac{dy}{Q(m,y)^s}\right]\\ &=\sum_{n=-\infty}^{\infty}\int_{n}^{n+1}\left(\frac{1}{Q(m,n)^s}-\frac{1}{Q(m,y)^s}\right)dy.\end{aligned}\tag{2.8}$$

By the mean value theorem, for $n\leqq y\leqq n+1$

$$\begin{aligned}\left|\frac{1}{Q(m,n)^s}-\frac{1}{Q(m,y)^s}\right|&\leqq\max_{n\leqq y\leqq n+1}\left|\frac{d}{dy}\frac{1}{Q(m,y)^s}\right|\\ &=0\left(\frac{\max(m,n)}{(m^2+n^2)^{s+1}}\right),\end{aligned}$$

where the constant implied in $0(\ )$ depends only on Q and on s, uniformly in the latter for $\frac{1}{2}<s<2$. Therefore expression (8) is $0\left(\frac{1}{m^{2s}}\right)$, from which it follows that the sum over m in (7) is absolutely convergent for $\mathrm{Re}(s)>\frac{1}{2}$ and that

$$\lim_{s\to1}\left(\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{Q(m,n)^s}-\zeta(2s-1)\,I(s)\right)$$

exists and equals

$$\sum_{m=1}^{\infty}\left(\sum_{n=-\infty}^{\infty}\frac{1}{Q(m,n)}-\frac{I(1)}{m}\right).$$

Since

$$\zeta(2s-1)=\frac{\frac{1}{2}}{s-1}+\gamma+0(s-1)\quad(s\to1),$$

$$\zeta(2s-1)\,I(s)=\frac{\frac{1}{2}I(1)}{s-1}+\left(\gamma I(1)+\tfrac{1}{2}I'(1)\right)+0(s-1),$$

so we obtain

$$\begin{aligned}\lim_{s\to1}\left(\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{Q(m,n)^s}-\frac{\frac{1}{2}I(1)}{s-1}\right)\\ =\gamma I(1)+\tfrac{1}{2}I'(1)+\sum_{m=1}^{\infty}\left(\sum_{n=-\infty}^{\infty}\frac{1}{Q(m,n)}-\frac{I(1)}{m}\right).\end{aligned}\tag{2.9}$$

It remains to evaluate $I(1)$, $I'(1)$ and the sum.

Now from (3) we get

$$\frac{1}{Q(m,n)}=\frac{1}{im}\left(\frac{1}{m\bar{w}+n}-\frac{1}{mw+n}\right).\tag{2.10}$$

Hence

$$I(1)=\int_{-\infty}^{\infty}\frac{dt}{Q(1,t)}=\frac{1}{i}\int_{-\infty}^{\infty}\left(\frac{1}{t+\bar{w}}-\frac{1}{t+w}\right)dt$$

$$=\frac{1}{i}\log\frac{t+\bar{w}}{t+w}\bigg|_{-\infty}^{\infty}=2\pi$$

and

$$\sum_{n=-\infty}^{\infty}\frac{1}{Q(m,n)}=\frac{1}{im}\sum_{n=-\infty}^{\infty}\left(\frac{1}{m\bar{w}+n}-\frac{1}{mw+n}\right)$$

$$=\frac{\pi}{i}\frac{1}{m}(\cot\pi m\bar{w}-\cot\pi mw), \tag{2.11}$$

where in the last equation we have used the standard expansion $\frac{\pi}{\tan\pi x}=\sum_n\frac{1}{n+x}$ (principal value). Therefore (9) becomes

$$\lim_{s\to 1}\left(\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{Q(m,n)^s}-\frac{\pi}{s-1}\right)$$

$$=2\pi\gamma+\tfrac{1}{2}I'(1)+\frac{\pi}{i}\sum_{m=1}^{\infty}\frac{1}{m}(\cot\pi m\bar{w}-i) \tag{2.12}$$

$$-\frac{\pi}{i}\sum_{m=1}^{\infty}\frac{1}{m}(\cot\pi mw+i);$$

the series converge because, for $\operatorname{Im}(w)>0$, $\cot\pi mw\to -i$ and $\cot\pi m\bar{w}\to +i$ with exponential rapidity as $m\to\infty$.

Now

$$\frac{1}{i}\cot\pi m\bar{w}=\frac{e^{\pi im\bar{w}}+e^{-\pi im\bar{w}}}{e^{\pi im\bar{w}}-e^{-\pi im\bar{w}}}$$

$$=1+2e^{-2\pi im\bar{w}}+2e^{-4\pi im\bar{w}}+\cdots$$

and so

$$\frac{1}{i}\sum_{m=1}^{\infty}\frac{1}{m}(\cot\pi m\bar{w}-i)$$

$$=2\sum_{m=1}^{\infty}\frac{1}{m}\sum_{n=1}^{\infty}e^{-2\pi imn\bar{w}}$$

$$=2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m}e^{-2\pi imn\bar{w}}$$

$$=-2\sum_{n=1}^{\infty}\log[1-e^{-2\pi in\bar{w}}]$$

$$=-\frac{\pi i\bar{w}}{6}-2\log\eta(-\bar{w}),$$

and similarly the second sum in (12) equals $-i\left(\frac{\pi i w}{6}-2\log\eta(w)\right)$. Substituting this into (12) and then into (6), we find $\left(\text{using } \frac{1}{c}=\frac{w-\bar{w}}{i}\right)$

$$\begin{aligned}\lim_{s\to 1}\left(\zeta_Q(s)-\frac{2\pi}{s-1}\right)&=\frac{\pi^2}{3c}+4\pi\gamma+I'(1)-\frac{\pi^2 i\bar{w}}{3}+\frac{\pi^2 i w}{3}\\&\quad-4\pi\log\eta(-\bar{w})-4\pi\log\eta(w)\\&=4\pi\gamma+I'(1)-4\pi\log|\eta(w)|^2\,.\end{aligned}$$

This proves the theorem, apart from verifying that

$$I'(1)=-\int_{-\infty}^{\infty}\frac{\log Q(1,t)}{Q(1,t)}\,dt$$

equals $2\pi\log c$; this is easily checked by substituting $x=2c(t+\mathrm{Re}(w))$, which gives

$$\begin{aligned}I(s)&=2^{2s-1}c^{s-1}\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)^s}\\&=2\pi\,\frac{\Gamma(2s-1)}{\Gamma(s)^2}\,c^{s-1}\,.\end{aligned}$$

For a more complete exposition of the Kronecker limit formula and its applications, the reader is referred to Siegel [8].

3. Hecke's Theorem

Now assume that K is a real quadratic field, $D>0$ its discriminant. Let ε be a fundamental unit (the smallest unit >1, where we have fixed once and for all an embedding $K\subset\mathbb{R}$); then $U=\{\pm\varepsilon^n\,|\,n\in\mathbb{Z}\}$ and hence (2.1) gives

$$\zeta(s,A)=\tfrac{1}{2}N(\underline{b})^s\sum_{\lambda\in\underline{b}/\varepsilon}{}'\frac{1}{|N(\lambda)|^s}\,.$$

Hence

$$2D^{s/2}\zeta(s,A)=\sum_{\lambda\in\underline{b}/\varepsilon}{}'\frac{\Delta^s}{|\lambda\lambda'|^s}\,,\tag{3.1}$$

where $\Delta=N(\underline{b})\,D^{\frac{1}{2}}$ is the discriminant of $\underline{b}$.

Hecke's trick is as follows. Consider the integral

$$c(s)=\int_{-\infty}^{\infty}\frac{dv}{(e^v+e^{-v})^s}\,.$$

Then clearly, for a, b real and $\neq 0$,

$$\int_{-\infty}^{\infty}\frac{dv}{(a^2e^v+b^2e^{-v})^s}=\frac{c(s)}{|ab|^s}\,;$$

indeed, the left-hand side depends only on the absolute values of a and b, is homogeneous of degree $-2s$, and only depends on the product ab ($a \to \lambda a, b \to b/\lambda$ corresponds to $v \to v - 2\log\lambda$). Therefore

$$2^{s+1} c(s) D^{s/2} \zeta(s, A) = \sum_{\lambda \in \underline{b}/\varepsilon}' \int_{-\infty}^{\infty} \frac{(2\Delta)^s \, dv}{(\lambda^2 e^v + \lambda'^2 e^{-v})^s}. \tag{3.2}$$

But replacing λ by $\varepsilon^n \lambda$ replaces λ^2 by $\varepsilon^{2n}\lambda^2$, λ'^2 by $\varepsilon^{-2n}\lambda'^2$, and this corresponds to $v \to v + 2n \log\varepsilon$; that is, the action of ε by multiplication on λ corresponds to an action on v by translation through $2\log\varepsilon$. Therefore the right-hand side of (2) equals

$$\sum_{\lambda \in \underline{b}}' \int_{-\log\varepsilon}^{\log\varepsilon} \frac{(2\Delta)^s \, dv}{(\lambda^2 e^v + \lambda'^2 e^{-v})^s}.$$

But now the summation and integration can be interchanged and then the sum

$$\sum_{\lambda \in \underline{b}}' \frac{(2\Delta)^s}{(\lambda^2 e^v + \lambda'^2 e^{-v})^s}$$

is of the type considered in § 2, namely the sum over the whole $\mathbb{Z}$-module $\underline{b}$ of a Dirichlet series with a *definite* quadratic form.

If we again assume that $\underline{b}$ has a basis of the form $\{1, w\}$, which as before we assume to be oriented (this now means $w > w'$), then writing $\lambda = mw + n$ we find

$$\Delta = w - w',$$
$$\lambda^2 e^v + \lambda'^2 e^{-v} = m^2[w^2 e^v + w'^2 e^{-v}] + 2mn[we^v + w' e^{-v}] + n^2[e^v + e^{-v}],$$

so the positive definite form

$$Q_v(m, n) = \frac{(mw+n)^2 e^v + (mw'+n)^2 e^{-v}}{2\Delta}$$

has determinant -1, and we have

$$2^{s+1} c(s) D^{s/2} \zeta(s, A) = \int_{-\log\varepsilon}^{\log\varepsilon} \zeta_{Q_v}(s) \, dv. \tag{3.3}$$

By the Kronecker limit formula,

$$\zeta_{Q_v}(s) = \frac{2\pi}{s-1} + 4\pi\left(\gamma + \tfrac{1}{2}\log\left(\frac{e^v + e^{-v}}{2\Delta}\right) - \log\left|\eta\left(\frac{w + iw'e^{-v}}{1 + i\varepsilon^{-v}}\right)\right|^2\right) + 0(s-1).$$

Substituting this into (3) and using the easily calculated values

$$c(1) = \frac{\pi}{2}, \quad c'(1) = -\pi \log 2,$$

we find

$$\operatorname{res}_{s=1} \zeta(s, A) = \frac{2\log\varepsilon}{\sqrt{D}}$$

and

$$\varrho(A)=\lim_{s\to 1}\left(\zeta(s,A)-\frac{2D^{-\frac{1}{2}}\log\varepsilon}{s-1}\right)$$

$$=\frac{2\log\varepsilon}{\sqrt{D}}\left(-\tfrac{1}{2}\log D+2\gamma\right)$$

$$+\frac{1}{\sqrt{D}}\int_{-\log\varepsilon}^{\log\varepsilon}\left(\log\left(\frac{e^{v}+e^{-v}}{w-w'}\right)-\log\left|\eta\left(\frac{w+iw'e^{-v}}{1+ie^{-v}}\right)\right|^{4}\right)dv\,.$$

4. Meyer's Theorem

Let K be a real quadratic field, $D>0$ its discriminant, and assume that K contains no unit of negative norm (this is the case, for instance, if some prime $p\equiv 3 \pmod 4$ divides D). Then each ordinary ideal class A is the disjoint union of two narrow ideal classes B and B^* (recall that two ideals $\underline{a}, \underline{b}$ are said to belong to the same narrow ideal class if $\underline{a}=(\alpha)\underline{b}$ for some principal ideal (α) with $N(\alpha)>0$); clearly $B^*=\Theta B$, where Θ is the narrow ideal class of principal ideals (α) with $N(\alpha)<0$.

We pick as before an ideal $\underline{b}\in B^{-1}$ having an oriented basis $\{1,w\}$ (i.e. $w>w'$). Then the action of the fundamental unit ε of K (for which $\varepsilon>1$, $\varepsilon\varepsilon'=1$) with respect to this basis is given by some matrix $M=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL_2\mathbb{Z}$:

$$\varepsilon w = aw+b\,,$$

$$\varepsilon = cw+d\,.$$

Then Meyer's theorem can be stated as:

$$\varrho(B)-\varrho(B^*)=\frac{\pi^2}{\sqrt{D}}\left(\frac{a+d-2(d,c)}{6c}-\frac{1}{2}\right), \tag{4.1}$$

where (d,c) denotes the "Dedekind sum"

$$(d,c)=6c\sum_{k=1}^{c-1}\left(\frac{k}{c}-\frac{1}{2}\right)\left(\frac{kd}{c}-\left[\frac{kd}{c}\right]-\frac{1}{2}\right)$$

(this is an integer; note that $c>0$ and d is prime to c).

The proof is given in Meyer's book [6] and a rather shorter exposition in a joint paper of Hirzebruch and the author [3]; here we only say that the same artifice used in Section 3 can be used to express $\zeta(s,B)-\zeta(s,B^*)$ as the integral from $-\log\varepsilon$ to $+\log\varepsilon$ of a complicated function of s and the variable of integration v; the value at $s=1$ again involves $\log\left|\eta\left(\frac{w+iw'e^{-v}}{1+ie^{-v}}\right)\right|$, but now in such a way that the integral can be explicitly evaluated by using the transformation law of the Dedekind η-function.

Now the number $w \in K$ can (since we have fixed an embedding of K in $\mathbb{R}$) be expanded in a unique way as a continued fraction

$$w = a_0 - \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \ddots}}}$$

with $a_i \in \mathbb{Z}$ (all i), $a_i \geqq 2$ ($i = 1, 2, 3, \ldots$). By the standard theory of continued fractions, the fact that w satisfies a quadratic equation over $\mathbb{Z}$ implies that the sequence $\{a_0, a_1, a_2, \ldots\}$ eventually becomes periodic, i.e. $a_{i+r} = a_i$ for all $i \geqq i_0$; the smallest such r is called the period of w, and the corresponding r-tuple $((a_{i_0+1}, \ldots, a_{i_0+r}))$ the cycle associated to w. If we choose a different number w with $w > w'$ and $\underline{b} = \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot w$, then the period is unchanged; moreover, this is also true if we replace $\underline{b}$ by another ideal in the same narrow ideal class B. Thus to each such class B we have associated an integer $r > 0$ and a cycle $((b_1, \ldots, b_r))$ of r integers $\geqq 2$, where the double parentheses indicate that the order of the b_i is only defined up to cyclic permutation. We denote the length r of the cycle by $l(B)$ and call it the *length* of the ideal class B.

The length of B^* is the period of the continued fraction of $-\dfrac{1}{w}$.

It is then an elementary exercise to show that (1) is equivalent to the formula

$$\varrho(B) - \varrho(B^*) = \frac{\zeta(2)}{\sqrt{D}} \left(l(B^*) - l(B)\right)$$

given in the introduction (cf. the paper [3] referred to above).

5. Statement of the Main Theorem

If $\{1, w\}$ and $\{1, w_1\}$ are oriented bases of fractional ideals in the same ideal class of an imaginary quadratic field K, then w and w_1 are points of the upper half-plane related by a Möbius transformation $w_1 = \dfrac{aw+b}{cw+d}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{Z}$. We therefore can, if we wish, assume that w lies in the standard fundamental region for the action of $SL_2\mathbb{Z}$ on the upper half-plane, i.e. that w satisfies the inequalities

$$-\tfrac{1}{2} < w + w' \leqq \tfrac{1}{2}, \qquad ww' \geqq 1 \quad (\text{equality only if } w + w' \geqq 0).$$

We call such an element $w \in K$ *reduced.* Then each ideal class of K contains a fractional ideal having a basis of the form $\{1, w\}$ with w reduced, and both the ideal and w are unique. (In other words, there is a $1:1$ correspondence between all ideal classes in all imaginary quadratic fields and all reduced imaginary quadratic irrationalities.)

For the case of a real quadratic field, we call a number $w \in K$ reduced if it satisfies the inequalities

$$w > 1, \qquad 0 < w' < 1 \tag{5.1}$$

(always with respect to a fixed embedding $K \subset \mathbb{R}$, e.g. $\sqrt{D} > 0$). This is equivalent to the condition that the continued fraction expansion of w [as in Eq. (4.2)] is *pure periodic*, i.e. satisfies $a_{i+r} = a_i$ for all $i \geqq 0$. It follows that, if B is a narrow ideal class of K with length $r = l(B)$ and cycle $((b_1, \ldots, b_r))$ $(b_i \in \mathbb{Z},\ b_i \geqq 2)$, then there are exactly r numbers $w \in K$ which are reduced and for which $\{1, w\}$ is a basis for some ideal in B, namely the numbers

$$w_k = b_k - \cfrac{1}{b_{k+1} - \cfrac{}{\ddots - \cfrac{1}{b_r - \cfrac{1}{b_1 - \ddots}}}} \qquad (k = 1, 2, \ldots, r). \tag{5.2}$$

We can now state the main theorem of this paper.

Theorem. *Let B be a narrow ideal class in a real quadratic field of discriminant D, and $\varepsilon > 1$ the smallest unit of K of norm 1. Then*

$$\lim_{s \to 1} \left(D^{s/2} \zeta(s, B) - \frac{\log \varepsilon}{s-1} \right) = \sum_{k=1}^{l(B)} P(w_k, w_k') \tag{5.3}$$

where the summation is over all $w \in K$ satisfying (1) *for which $\{1, w\}$ is a basis of some fractional ideal of B and $P(x, y)$ is a universal function of two variables, namely*

$$\begin{aligned} P(x, y) = {} & F(x) - F(y) + Li_2\left(\frac{y}{x}\right) - \frac{\pi^2}{6} \\ & + \log\frac{x}{y}\left(\gamma - \tfrac{1}{2}\log(x-y) + \tfrac{1}{4}\log\frac{x}{y}\right) \\ & (x > y > 0)\,. \end{aligned} \tag{5.4}$$

Here γ is Euler's constant, $\mathrm{Li}_2(t)$ *is the dilogarithm function* $\sum_{n=1}^{\infty} \frac{t^n}{n^2}$ $(0 < t < 1)$, *and*

$$\begin{aligned} F(x) &= \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\Gamma'(nx)}{\Gamma(nx)} - \log(nx) \right) \\ &= \int_0^{\infty} \left(\frac{1}{1-e^{-t}} - \frac{1}{t} \right) \log(1 - e^{-xt})\, dt \end{aligned} \tag{5.5}$$

is a function whose properties are given below (Section 7).

In fact, we will prove more, namely that $D^{s/2} \zeta(s, B)$ can be written as a sum of $l(B)$ functions each involving only s, w_j and w_j', so that in the Laurent expansion

$$D^{s/2} \zeta(s, B) = \sum_{n=-1}^{\infty} c_n (s-1)^n$$

we have

$$c_n = \sum_{k=1}^{l(B)} P_n(w_k, w_k')$$

with P_{-1}, $P_0 (= P)$, $P_1, \ldots$ certain universal functions of two variables.

6. Proof of the Main Theorem

We define w_k by (5.2) for $k=1,\dots,r=l(B)$ and extend the definition to all $k\in\mathbb{Z}$ by requiring w_k to depend only on $k(\operatorname{mod} r)$. We also fix the ideal $\underline{b}=\mathbb{Z}\cdot 1+\mathbb{Z}\cdot w_0\in B$.

Now define a sequence of numbers

$$0<\cdots<A_2<A_1<A_0=1<A_{-1}<A_{-2}<\cdots$$

by

$$A_k=\frac{1}{w_1\dots w_k}\qquad (k\geqq 1)$$

$$A_0=1 \tag{6.1}$$

$$A_{-k}=w_0w_{-1}\dots w_{-k+1}\quad (k\geqq 1),$$

so that $A_{k+1}=A_k/w_{k+1}$. From the continued fraction expansion we have

$$w_k=b_k-\frac{1}{w_k+1}, \tag{6.2}$$

from which it follows that

$$A_{k+1}=b_kA_k-A_{k-1}$$

and this implies that $\mathbb{Z}A_{k+1}+\mathbb{Z}A_k=\mathbb{Z}A_k+\mathbb{Z}A_{k-1}$, i.e. $\{A_{k-1}, A_k\}$ from a basis of $\underline{b}$ for each integer k. Moreover, the periodicity of the w_k implies

$$A_rA_k=A_{k+r}\quad (\forall k\in\mathbb{Z}), \tag{6.3}$$

and this means that multiplication by A_r is an automorphism of $\underline{b}$. It follows that A_r is a unit; also $A_r>0$ and $A_r'=\dfrac{1}{w_1'\dots w_r'}>0$, so A_r is a (negative) power of the unit ε (which was defined as the smallest unit >1 of norm $+1$). In fact, it can be shown that

$$A_r=\varepsilon^{-1}. \tag{6.4}$$

Now any number $\lambda\in\underline{b}$ can be written (for each $k\in\mathbb{Z}$) in the form

$$\lambda=pA_{k-1}+qA_k \tag{6.5}$$

with $p,q\in\mathbb{Z}$, and it is clear that, if $p,q\geqq 0$ (and not both are 0), then λ is totally positive (i.e. $\lambda>0$, $\lambda'>0$; we write $\lambda\gg 0$). Conversely; one can show that, if $\lambda\in\underline{b}$ is totally positive, then λ can always be written as $pA_{k-1}+qA_k$ with $p,q\geqq 0$ for some k. Moreover, this representation is unique unless $\lambda=nA_l$ $(n\in\mathbb{N})$ in which case we can take $k=l$, $p=0$, $q=n$ or $k=l+1$, $p=n$, $q=0$. Thus, if we make the restriction $p\geqq 1$, then to each λ is associated a triple $(k;p,q)$ of integers with $p\geqq 1$, $q\geqq 0$ and λ as in (5). This sets up a $1:1$ correspondence between $\{\lambda\in\underline{b}\,|\,\lambda\gg 0\}$ and $\{(k;p,q)\,|\,k,p,q\in\mathbb{Z},\ p\geqq 1,\ q\geqq 0\}$. Moreover, it is clear from (3) and (4) that if λ corresponds to the triple $(k;p,q)$, then the triple corresponding to $\lambda\varepsilon^n$ is $(k-nr;p,q)$; hence there is a $1:1$ correspondence between principal ideals (λ) with $\lambda\gg 0$, $\lambda\in\underline{b}$ and triples $(k(\operatorname{mod} r);p,q)$. (A reference for all of this is [2], § 2.3.)

Furthermore, $\lambda = A_k(q + pw_k)$, so

$$N(\lambda) = N(A_k)(q + pw_k)(q + pw_k').$$

Also, from $w_k = b_k - 1/w_{k+1}$ we obtain

$$w_k - w_k' = -\frac{1}{w_{k+1}} + \frac{1}{w_{k+1}'} = \frac{w_{k+1} - w_{k+1}'}{w_{k+1}w_{k+1}'}$$

or

$$(w_k - w_k')A_k A_k' = (w_{k+1} - w_{k+1}')A_{k+1}A_{k+1}';$$

this common value is found, by taking $k=0$, to be $w_0 - w_0'$. Therefore, for λ as in (5),

$$N(\lambda) = (w_0 - w_0')Q_k(p, q) \tag{6.6}$$

with

$$Q_k(x, y) = \frac{1}{w_k - w_k'}(y + xw_k)(y + xw_k'). \tag{6.7}$$

Hence

$$\begin{aligned} D^{s/2}\zeta(s, B^{-1}) &= D^{s/2} \sum_{a \in B^{-1}} \frac{1}{N(a)^s} \\ &= D^{s/2} N(\underline{b})^s \sum_{\substack{\lambda \in \underline{b}/U_+ \\ \lambda \gg 0}} \frac{1}{N(\lambda)^s} \end{aligned}$$

(where we have set $a\underline{b} = (\lambda)$ with $\lambda \gg 0$)

$$\begin{aligned} &= (w_0 - w_0')^s \sum_{k=1}^{r} \sum_{p=1}^{\infty} \sum_{q=0}^{\infty} \frac{1}{N(pA_{k-1} + qA_k)^s} \\ &= \sum_{k=1}^{r} \sum_{p=1}^{\infty} \sum_{q=0}^{\infty} \frac{1}{Q_k(p, q)^s}. \end{aligned}$$

We notice that $Q_k(x, y)$ is an indefinite binary quadratic form with positive coefficients, normalized to have determinant $+1$. Thus what we have proved so far can be summarized as:

Theorem. *For*

$$Q(x, y) = ax^2 + bxy + cy^2, \quad a, b, c > 0, \quad b^2 - 4ac = 1$$

an indefinite binary quadratic form with positive real coefficients and discriminant 1, *we define*

$$Z_Q(s) = \sum_{p=1}^{\infty} \sum_{q=0}^{\infty} \frac{1}{Q(p, q)^s};$$

then for the zeta-function of a narrow ideal class B of a real quadratic field of discriminant D, we have the decomposition

$$D^{s/2}\zeta(s, B^{-1}) = \sum_{k=1}^{r} Z_{Q_k}(s),$$

where $r = l(B)$ is the length of B and the quadratic forms Q_k are as in (7), *w_k being the elements of K whose continued fractions correspond to the various cyclic permutations of the cycle $((b_1, \ldots, b_r))$ associated to B.*

The main theorem follows immediately from this result and the following.

Theorem. *Let $Q(p, q)$ be as in the last theorem, and w, w' the roots of the quadratic equation $cw^2 - bw + a = 0$, labelled so that $w > w' > 0$. Then the function $Z_Q(s)$ has an analytic continuation to the half-plane* $\mathrm{Re}(s) > \frac{1}{2}$, *with a single pole at $s = 1$, and its Laurent expansion there is*

$$Z_Q(s) = \frac{\frac{1}{2}\log w/w'}{s-1} + P(w, w') + 0(s-1)$$

with P as in Eq. (5.4).

This theorem is the exact analogue of Kronecker's theorem as stated in Section 2 (with the added complication that the sum is over p, q positive rather than over all p, q), and the proof will follow the same lines. Before giving it, we make two remarks about the deduction of the main theorem. First, the fact that $\zeta(B, s)$ appears there in place of $\zeta(B^{-1}, s)$ does not matter since these functions are identically equal (since $B^{-1} = B' =$ class of conjugates of ideals of B, and an ideal and its conjugate have the same norms). Secondly, the stated formula for the residue is deduced by noting that, because of (4),

$$\sum_{k=1}^{r} \log \frac{w_k}{w'_k} = \log \frac{w_1 \dots w_r}{w'_1 \dots w'_r} = \log \frac{A'_r}{A_r} = \log \varepsilon^2 .$$

To prove the theorem, we define, in a manner completely analogous to the procedure of Section 2, a holomorphic function

$$I(s) = \int_0^\infty \frac{dt}{Q(1, t)^s} \quad (\mathrm{Re}\, s > \tfrac{1}{2}),$$

write

$$Z_Q(s) - \zeta(2s-1)\, I(s) = \sum_{p=1}^{\infty} \left[\sum_{q=0}^{\infty} \frac{1}{Q(p, q)^s} - \frac{1}{p^{2s-1}} I(s) \right],$$

and observe that the series is now absolutely convergent for $\mathrm{Re}(s) > \frac{1}{2}$, so that we have proved the analytic continuability of $Z_Q(s)$ and the equation

$$\begin{aligned} &\lim_{s \to 1} \left(Z_Q(s) - \frac{\frac{1}{2} I(1)}{s-1} \right) \\ &= \gamma I(1) + \tfrac{1}{2} I'(1) + \sum_{p=1}^{\infty} \left(\sum_{q=0}^{\infty} \frac{1}{Q(p, q)} - \frac{I(1)}{p} \right). \end{aligned} \tag{6.8}$$

Again, to evaluate $I(1)$ and $\sum_{q=0}^{\infty} \frac{1}{Q(p, q)}$, we write

$$\frac{1}{Q(x, y)} = \frac{1}{c(y + xw)(y + xw')} = \frac{1}{x} \left(\frac{1}{y + xw'} - \frac{1}{y + xw} \right) \tag{6.9}$$

$\left(\text{the normalization of } Q \text{ to have determinant 1 implies } c = \frac{1}{w-w'}\right)$; hence

$$\begin{aligned} I(1) &= \int_0^\infty \frac{dt}{Q(1,t)} \\ &= \int_0^\infty \left(\frac{1}{t+w'} - \frac{1}{t+w}\right) dt \\ &= \log \frac{t+w'}{t+w}\bigg|_0^\infty \\ &= \log \frac{w}{w'} \end{aligned} \tag{6.10}$$

and

$$\begin{aligned} \sum_{q=0}^\infty \frac{1}{Q(p,q)} &= \frac{1}{p} \sum_{q=0}^\infty \left(\frac{1}{q+pw'} - \frac{1}{q+pw}\right) \\ &= \frac{1}{p}(\psi(pw) - \psi(pw')) . \end{aligned} \tag{6.11}$$

Here we have used the (standard) notation

$$\begin{aligned} \psi(x) &= \lim_{N\to\infty} \left(\log N - \sum_{q=0}^{N} \frac{1}{q+x}\right) \\ &= \frac{\Gamma'(x)}{\Gamma(x)} \end{aligned} \tag{6.12}$$

for the logarithmic derivative of the gamma-function. Therefore

$$\begin{aligned} \sum_{q=0}^\infty \frac{1}{Q(p,q)} - \frac{I(1)}{p} &= \frac{1}{p}\left(\psi(pw) - \psi(pw') - \log\frac{w}{w'}\right) \\ &= \frac{1}{p}(\psi(pw) - \log(pw)) - \frac{1}{p}(\psi(pw') - \log(pw')) , \end{aligned}$$

and both terms are $0\left(\frac{1}{p^2}\right)$ for $p \to \infty$ $\left(\text{because } \psi(x) - \log x = 0\left(\frac{1}{x}\right)\right)$ and so can be summed over p. Putting this into (8), we obtain

$$\begin{aligned} &\lim_{s\to 1}\left(Z_Q(s) - \frac{\tfrac{1}{2}\log\frac{w}{w'}}{s-1}\right) \\ &\qquad = \gamma \log\frac{w}{w'} + \tfrac{1}{2} I'(1) + F(w) - F(w') , \end{aligned}$$

where $F(x)$ is the function appearing in Eq. (5.5). If we compare this with Eq. (5.4) for $P(x, y)$, we see that it remains to prove that

$$I'(1) = 2\,\mathrm{Li}_2\left(\frac{w'}{w}\right) - \frac{\pi^2}{3} + \log\frac{w}{w'}\left(\tfrac{1}{2}\log\frac{w}{w'} - \log(w-w')\right) . \tag{6.13}$$

Now

$$I'(1) = -\int_0^\infty \frac{\log Q(1,t)}{Q(1,t)}\,dt$$

$$= -I(1)\log\frac{w^2}{w-w'} - \int_0^\infty \frac{\log\left[Q(1,t)\dfrac{w-w'}{w^2}\right]}{Q(1,t)}\,dt\,,$$

and if we substitute $x = \frac{1}{w}t$ and note

$$Q(1,t)\cdot\frac{w-w'}{w^2} = \frac{1}{w^2}(t+w)(t+w') = (x+1)\left(x+\frac{w'}{w}\right),$$

$$\frac{dt}{Q(1,t)} = \left(\frac{1}{t+w'} - \frac{1}{t+w}\right)dt = \left(\frac{1}{x+w'/w} - \frac{1}{x+1}\right)dx\,,$$

this yields

$$I'(1) = -\left(\log\frac{w}{w'}\right)\left(\log\frac{w^2}{w-w'}\right) - g\left(\frac{w'}{w}\right)$$

with g defined by

$$g(\alpha) = \int_0^\infty \left(\frac{1}{x+\alpha} - \frac{1}{x+1}\right)\log[(x+1)(x+\alpha)]\,dx\,. \tag{6.14}$$

Comparing this with (13), we see that it remains only to show

$$g(\alpha) = -2\,\mathrm{Li}_2(\alpha) + \frac{\pi^2}{3} - 2\log\alpha\log(1-\alpha) - \tfrac{1}{2}(\log\alpha)^2 \qquad (0<\alpha\leqq 1)\,. \tag{6.15}$$

By means of the functional equation

$$\mathrm{Li}_2(\alpha) + \mathrm{Li}_2(1-\alpha) = \frac{\pi^2}{6} - \log\alpha\log(1-\alpha) \quad (0\leqq\alpha\leqq 1)$$

(cf. [5], formula (1.11)), this can be written in the form

$$g(\alpha) = 2\,\mathrm{Li}_2(1-\alpha) - \tfrac{1}{2}(\log\alpha)^2 \quad (0<\alpha\leqq 1)\,. \tag{6.16}$$

Now it is clear that the two sides of (16) agree for $\alpha = 1$ (both plainly vanish), so it suffices to prove (16) after differentiation, i.e. to show

$$g'(\alpha) = \frac{2\log\alpha}{1-\alpha} - \frac{\log\alpha}{\alpha} \quad (0<\alpha<1) \tag{6.17}$$

$\left(\text{we have used } \frac{d}{dt}\mathrm{Li}_2(t) = \sum_{n=1}^\infty \frac{t^{n-1}}{n} = -\frac{\log(1-t)}{t}\right)$. But

$$g'(\alpha) = -\int_0^\infty \frac{\log[(x+1)(x+\alpha)]}{(x+\alpha)^2}\,dx + \int_0^\infty \left(\frac{1}{x+\alpha} - \frac{1}{x+1}\right)\frac{dx}{x+\alpha}\,,$$

and integration by parts gives

$$-\int_0^\infty \frac{\log[(x+1)(x+\alpha)]}{(x+\alpha)^2}\,dx = \int_0^\infty \log[(x+1)(x+\alpha)]\,d\left(\frac{1}{x+\alpha}\right)$$

$$= -\frac{\log\alpha}{\alpha} - \int_0^\infty \frac{1}{x+\alpha}\left(\frac{1}{x+1}+\frac{1}{x+\alpha}\right)dx\,,$$

so

$$g'(\alpha) = -\frac{\log\alpha}{\alpha} - 2\int_0^\infty \frac{dx}{(x+1)(x+\alpha)}$$

$$= -\frac{\log\alpha}{\alpha} - \frac{2}{1-\alpha}\log\frac{x+\alpha}{x+1}\bigg|_0^\infty$$

$$= -\frac{\log\alpha}{\alpha} + \frac{2\log\alpha}{1-\alpha}\,.$$

This proves (17) and completes the proof of the main theorem.

7. Properties of the Function $F(x)$

Of the terms entering the formula for the Kronecker limit $P(w, w')$, all are familiar functions except Li_2 and F. The dilogarithm function Li_2 has been studied extensively, and there is a whole book ([5]) devoted to its properties; moreover, for $0 \leqq t \leqq 1$ $\mathrm{Li}_2(t)$ increases monotonously from 0 to $\frac{\pi^2}{6}$ and thus affects the behaviour of $P(w, w')$ very little. The function $F(x)$ appears to be new, however, and gives also by far the largest contribution to $P(w, w')$ when w is large or w' small; we therefore devote a section to describing its main properties.

First of all, for $\psi(x) - \log x$ we have the integral representation

$$\psi(x) - \log x = -\int_0^\infty \left(\frac{1}{1-e^{-t}} - \frac{1}{t}\right)e^{-xt}\,dt \tag{7.1}$$

(for this and all other facts used about the gamma-function and its derivatives, we refer the reader to [7] or other standard textbooks). If we replace x by px, divide by p, and sum, we immediately get

$$F(x) = \sum_{p=1}^\infty \frac{\psi(px) - \log(px)}{p} = \int_0^\infty \left(\frac{1}{1-e^{-t}} - \frac{1}{t}\right)\log(1-e^{-xt})\,dt \tag{7.2}$$

(a formula already quoted in § 5); an alternate form for the integral is

$$F(x) = \int_0^1 \left(\frac{1}{1-y} + \frac{1}{\log y}\right)\log(1-y^x)\frac{dy}{y}\,.$$

Secondly, for x large we have an asymptotic expansion

$$\psi(x)-\log x\sim-\frac{1}{2x}-\frac{B_2}{2x^2}-\frac{B_4}{4x^4}-\cdots$$

($B_2=\frac{1}{6}$, $B_4=-\frac{1}{30}$, $\cdots$ the Bernoulli numbers), and this immediately implies for $F(x)$ the asymptotic expansion

$$F(x)\sim-\frac{\pi^2}{12x}-\frac{B_2}{2}\frac{\zeta(3)}{x^2}-\frac{B_4}{4}\frac{\zeta(5)}{x^6}-\cdots \tag{7.3}$$

as $x\to\infty$.

To obtain the asymptotic behaviour of $F(x)$ near $x=0$, we use the functional equation

$$F(x)+F\left(\frac{1}{x}\right)=-\frac{\pi^2}{6}x-\frac{\pi^2}{6x}+\tfrac{1}{2}(\log x)^2+C_1 \tag{7.4}$$

($C_1=1.45738783\ldots$). We prove this as follows. Define

$$A(x,s)=\int_0^\infty\frac{t^s\,dt}{(e^t-1)(e^{xt}-1)}. \tag{7.5}$$

The integral converges for $\mathrm{Re}(s)>1$. Then

$$A(x,s)-\frac{\Gamma(s)\zeta(s)}{x^s}=\int_0^\infty\left(\frac{t}{e^t-1}-1\right)\frac{t^{s-1}\,dt}{e^{xt}-1}$$

with the integral on the right convergent for $\mathrm{Re}(s)>0$; hence $A(x,s)$ is meromorphic around $s=1$ and has there a Laurent expansion

$$A(x,s)=\frac{x^{-1}}{s-1}+a(x)+0(s-1), \tag{7.6}$$

with

$$a(x)=-\log x+\int_0^\infty\left(\frac{t}{e^t-1}-1\right)\frac{dt}{e^{xt}-1}$$

$$=-\log x-\frac{\pi^2}{6x^2}+\int_0^\infty\left(\frac{t}{1-e^{-t}}-1\right)\frac{dt}{e^{xt}-1}$$

$\left(\text{using}\int_0^\infty\frac{t\,dt}{e^{xt}-1}=\frac{\zeta(2)}{x^2}\right)$

$$=-\log x-\frac{\pi^2}{6x^2}+F'(x). \tag{7.7}$$

Now the substitution $t\to t/x$ in (5) immediately yields

$$A\left(\frac{1}{x},s\right)=x^{s+1}A(x,s),$$

and inserting the Laurent expansion from (6) and (7) we immediately get

$$F'(x)-\frac{1}{x^2}F'\left(\frac{1}{x}\right)=\frac{\pi^2}{6x^2}-\frac{\pi^2}{6}+\frac{\log x}{x},$$

which on integration yields (4).

Another functional equation for $F(x)$ can be deduced from (5)–(7) by writing

$$\begin{aligned}A(x,s)&=\int_0^\infty\left[\frac{1}{(e^{(x-1)t}-1)(e^t-1)}-\frac{1}{(e^{(x-1)t}-1)(e^{xt}-1)}-\frac{1}{e^{xt}-1}\right]t^s\,dt\\&=A(x-1,s)-x^{-s-1}A\left(\frac{x-1}{x},s\right)-\frac{\Gamma(s+1)\zeta(s+1)}{x^{s+1}},\end{aligned}$$

from which we get

$$a(x)=a(x-1)-\frac{\log x}{x(x-1)}-\frac{1}{x^2}a\left(\frac{x-1}{x}\right)-\frac{\pi^2}{6x^2}$$

or, on substituting in (7) and integrating,

$$F(x)-F(x-1)+F\left(\frac{x-1}{x}\right)=-\mathrm{Li}_2\left(\frac{1}{x}\right)+C_2 \tag{7.8}$$

($C_2=-0.91624015\ldots$). This equation will be used in Section 8 to derive Meyer's theorem on $\varrho(B)-\varrho(B^*)$ from our formula for $\varrho(B)$.

Yet a third application of (6) is to observe that (by integration by parts)

$$\begin{aligned}A(1,s)&=\int_0^\infty\frac{t^s\,dt}{(e^t-1)^2}=-\int_0^\infty t^s e^{-t}\,d\left(\frac{1}{e^t-1}\right)\\&=\int_0^\infty\frac{1}{e^t-1}(st^{s-1}e^{-t}-t^se^{-t})\,dt\\&=\Gamma(s+1)\,(\zeta(s+1)-\zeta(s));\end{aligned}$$

this yields

$$a(1)=1-\frac{\pi^2}{6}$$

or

$$F'(1)=1\,. \tag{7.9}$$

To evaluate the constants in the functional Eq. (4) and (8), we first set $x=1$ in (4), obtaining

$$C_1=\frac{\pi^2}{3}+2F(1) \tag{7.10}$$

and then set $x=2$ in (8) and use (4) to get

$$\begin{aligned}C_2&=F(2)-F(1)+F(\tfrac{1}{2})+\mathrm{Li}_2(\tfrac{1}{2})\\&=-F(1)+\left(\tfrac{1}{2}\log^2 2+C_1-\frac{\pi^2}{6}(2\tfrac{1}{2})\right)+\left(\frac{\pi^2}{12}-\tfrac{1}{2}\log^2 2\right)\\&=C_1-\frac{\pi^2}{3}-F(1)\\&=F(1)\,.\end{aligned} \tag{7.11}$$

In view of this, it seems interesting to calculate the value of $F(1)$. By definition, $2F(1) = \lim_{N\to\infty} S_N$, where

$$S_N = 2\sum_{k=1}^{N} \frac{\psi(k) - \log k}{k}$$

$$= \sum_{k=1}^{N} \left(\frac{1 + \frac{1}{2} + \cdots + \frac{1}{k-1} - \gamma}{k} \right)$$

$$+ \sum_{k=1}^{N} \left(\frac{\psi(N+1) - \frac{1}{k} - \frac{1}{k+1} - \cdots - \frac{1}{N}}{k} \right) - 2\sum_{k=1}^{N} \frac{\log k}{k}$$

$$= \sum_{1 \leqq n < k \leqq N} \frac{1}{kn} - \gamma\left(1 + \tfrac{1}{2} + \cdots + \frac{1}{N}\right)$$

$$+ \psi(N+1)\left(1 + \tfrac{1}{2} + \cdots + \frac{1}{N}\right) - \left(1 + \tfrac{1}{4} + \cdots + \frac{1}{N^2}\right)$$

$$- \sum_{1 \leqq k < n \leqq N} \frac{1}{kn} - 2\sum_{k=1}^{N} \frac{\log k}{k}.$$

The first two sums cancel, and we get

$$S_N = -\gamma\left(\log N + \gamma + 0\left(\frac{1}{N}\right)\right) + \left(\log N + 0\left(\frac{1}{N}\right)\right)\left(\log N + \gamma + 0\left(\frac{1}{N}\right)\right)$$

$$- \left(\frac{\pi^2}{6} + 0\left(\frac{1}{N}\right)\right) - 2\sum_{k=1}^{N} \frac{\log k}{k}$$

$$= -\gamma^2 - \frac{\pi^2}{6} - 2\left(\sum_{k=1}^{N} \frac{\log k}{k} - \tfrac{1}{2}(\log N)^2\right).$$

Hence

$$F(1) = -\tfrac{1}{2}\gamma^2 - \frac{\pi^2}{12} - \gamma_1 \tag{7.12}$$

where

$$\gamma_1 = \lim_{N\to\infty}\left(\sum_{k=1}^{N} \frac{\log k}{k} - \tfrac{1}{2}(\log N)^2\right) \tag{7.13}$$

$$= -0.07281588\ldots$$

is the higher analogue of Euler's constant. One can also prove (12) starting from the integral representation, in which case one obtains

$$F(1) = \lim_{s\to 1} \frac{d}{ds}\left(\Gamma(s)\,\zeta(s) - \frac{1}{s-1}\right)$$

and must use the Laurent expansion

$$\zeta(s) = \frac{1}{s-1} + \gamma - \gamma_1(s-1) + \cdots$$

of $\zeta(s)$ at $s = 1$ to two terms.

A table and a graph of the values of $F(x)$ are included.

Table of the function $F(x)$

x	$F(x)$	x	$F(x)$
0.02	−73.15376278	1.02	−0.896660251
0.04	−34.51812148	1.04	−0.877886440
0.06	−22.04952753	1.06	−0.859870568
0.08	−15.97979203	1.08	−0.842568209
0.10	−12.42224963	1.10	−0.825938309
0.12	−10.09989268	1.12	−0.809942873
0.14	− 8.472526648	1.14	−0.794546691
0.16	− 7.273308608	1.16	−0.779717085
0.18	− 6.355675387	1.18	−0.765423692
0.20	− 5.632637433	1.20	−0.751638258
0.22	− 5.049408063	1.22	−0.738334465
0.24	− 4.569823726	1.24	−0.725487765
0.26	− 4.169086118	1.26	−0.713075237
0.28	− 3.829644539	1.28	−0.701075455
0.30	− 3.538742462	1.30	−0.689468367
0.32	− 3.286894484	1.32	−0.678235191
0.34	− 3.066907290	1.34	−0.667358312
0.36	− 2.873231140	1.36	−0.656821199
0.38	− 2.701518919	1.38	−0.646608316
0.40	− 2.548319358	1.40	−0.636705058
0.42	− 2.410859210	1.42	−0.627097672
0.44	− 2.286885766	1.44	−0.617773207
0.46	− 2.174551113	1.46	−0.608719449
0.48	− 2.072325819	1.48	−0.599924872
0.50	− 1.978933690	1.50	−0.591378592
0.52	− 1.893301864	1.52	−0.583070320
0.54	− 1.814522189	1.54	−0.574990326
0.56	− 1.741821043	1.56	−0.567129399
0.58	− 1.674535519	1.58	−0.559478813
0.60	− 1.612094467	1.60	−0.552030297
0.62	− 1.554003284	1.62	−0.544776005
0.64	− 1.499831613	1.64	−0.537708492
0.66	− 1.449203328	1.66	−0.530820682
0.68	− 1.401788327	1.68	−0.524105854
0.70	− 1.357295756	1.70	−0.517557613
0.72	− 1.315468402	1.72	−0.511169876
0.74	− 1.276078000	1.74	−0.504936849
0.76	− 1.238921309	1.76	−0.498853013
0.78	− 1.203816805	1.78	−0.492913107
0.80	− 1.170601879	1.80	−0.487112113
0.82	− 1.139130455	1.82	−0.481445242
0.84	− 1.109270962	1.84	−0.475907923
0.86	− 1.080904589	1.86	−0.470495787
0.88	− 1.053923788	1.88	−0.465204660
0.90	− 1.028230984	1.90	−0.460030550
0.92	− 1.003737450	1.92	−0.454969638
0.94	− 0.980362340	1.94	−0.450018268
0.96	− 0.958031834	1.96	−0.445172940
0.98	− 0.936678402	1.98	−0.440430301
1.00	− 0.916240150	2.00	−0.435787136

Note: For $x > 2$ the values of $F(x)$ can be deduced using the functional equation (7.4).

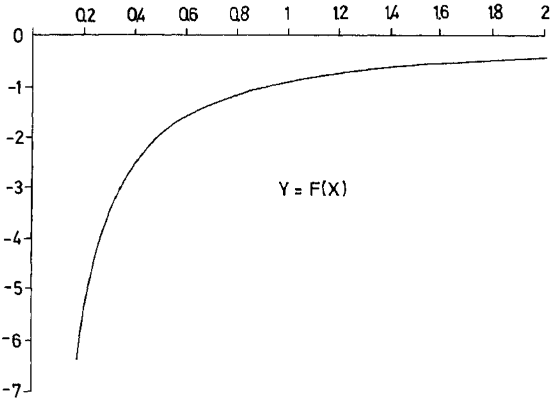

Fig. 1. Graph of the function $F(x)$

8. Deduction of Meyer's Theorem

We begin with a rather complicated identity involving the functions $F(x)$ and $\mathrm{Li}_2(x)$ appearing in the Kronecker limit formula.

Lemma. *For $x, y > 0$, define*

$$R(x,y) = F(x) - F(y) + \mathrm{Li}_2\left(\frac{-x}{y}\right) - \tfrac{1}{4}\log^2\left(\frac{x}{y}\right) + \tfrac{1}{2}\log\left(\frac{y}{x}\right)\log(x+y) + \frac{\pi^2}{12}\left(x + \frac{1}{x} - y - \frac{1}{y} - 1\right). \tag{8.1}$$

This function has the symmetry properties

$$R(x,y) = -R\left(\frac{1}{x}, \frac{1}{y}\right), \tag{8.2}$$

$$R(y,x) = -R(x,y). \tag{8.3}$$

Furthermore, for $w > 1 > w' > 0$, we have

$$\begin{aligned} R(w-1, 1-w') - R\left(1 - \frac{1}{w}, \frac{1}{w'} - 1\right) &= F(w) - F(w') + \mathrm{Li}_2\left(\frac{w'}{w}\right) + \tfrac{1}{4}\log^2\left(\frac{w'}{w}\right) + \tfrac{1}{2}\log\left(\frac{w'}{w}\right)\log(w - w') \\ &\quad + \frac{\pi^2}{12}\left(w + \frac{1}{w} - w' - \frac{1}{w'}\right). \end{aligned} \tag{8.4}$$

Proof. Equations (2) and (3) follow easily from the identities

$$F(x) + F\left(\frac{1}{x}\right) = -\frac{\pi^2}{6}\left(x + \frac{1}{x} - 2\right) + \tfrac{1}{2}\log^2 x + 2F(1) \tag{8.5}$$

[proved in the last section, Eq. (7.4)] and

$$\mathrm{Li}_2(-x)+\mathrm{Li}_2\left(-\frac{1}{x}\right)=-\tfrac{1}{2}\log^2 x-\frac{\pi^2}{6}\quad(x>0)\tag{8.6}$$

([5], Eq. (1.7); we recall that $\mathrm{Li}_2(x)$ is defined by the integral $-\int_x^1\log(1-t)\,dt/t$ for all $x\in(-\infty,1)$). To prove the crucial identity (4), one must apply the identity

$$F(x)-F(x-1)+F\left(\frac{x-1}{x}\right)=-\mathrm{Li}_2\left(\frac{1}{x}\right)+F(1)\quad(x>1)$$

of § 7 [Eq. (7.8)] with $x=w$ and with $x=\dfrac{1}{w'}$ and the identity

$$\mathrm{Li}_2(\theta\varphi)=\mathrm{Li}_2(\theta)+\mathrm{Li}_2(\varphi)+\mathrm{Li}_2\left(\frac{\theta\varphi-\theta}{1-\theta}\right)+\mathrm{Li}_2\left(\frac{\theta\varphi-\varphi}{1-\varphi}\right)+\tfrac{1}{2}\log^2\left(\frac{1-\theta}{1-\varphi}\right)$$
$$(0<\theta,\varphi<1)$$

([5], Eq. (1.24)) with $\theta=\dfrac{1}{w}$, $\varphi=w'$, as well as making repeated use of (5) and (6). The details are tedious but straightforward.

Therefore the function $P(w,w')$ appearing in the Kronecker limit formula and defined by Eq. (5.4) can be written in the form

$$\begin{aligned}P(w,w')=R(w-1,1-w')-R\left(1-\frac{1}{w},\frac{1}{w'}-1\right)\\ +\gamma\log\frac{w}{w'}-\frac{\pi^2}{12}\left(w+\frac{1}{w}\right)+\frac{\pi^2}{12}\left(w'+\frac{1}{w'}\right)-\frac{\pi^2}{6}.\end{aligned}$$

Therefore for the Kronecker limit $\varrho(B)$ of a narrow ideal class B in a real quadratic field, we have

$$\begin{aligned}\varrho(B)=\varrho_1(B)+\gamma\sum_{j=1}^{r}\log\frac{w_j}{w_j'}-\frac{\pi^2}{12}\sum_{j=1}^{r}\left(w_j+\frac{1}{w_j}\right)\\ +\frac{\pi^2}{12}\sum_{j=1}^{r}\left(w_j'+\frac{1}{w_j'}\right)-\frac{\pi^2}{6}\sum_{j=1}^{r}1,\end{aligned}\tag{8.7}$$

with

$$\varrho_1(B)=\sum_{j=1}^{r}\left[R(w_j-1,1-w_j')-R\left(1-\frac{1}{w_j},\frac{1}{w_j'}-1\right)\right].\tag{8.8}$$

We consider each term in (7) separately.

The term $\Sigma\log\dfrac{w_j}{w_j'}$ was already considered in § 6, where we showed that

$$\sum_{j=1}^{r}\log\frac{w_j}{w_j'}=2\log\varepsilon,$$

ε being the smallest totally positive unit >1.

As to $\Sigma\left(w_j+\frac{1}{w_j}\right)$, if we recall the definition of w_j as a continued fraction (5.2), then we see that $w_j=b_j-1/w_{j+1}$, and hence (using the periodicity property $w_{r+1}=w_1$)

$$\sum_{j=1}^{r}\left(w_j+\frac{1}{w_j}\right)=\sum_{j=1}^{r}\left(w_j+\frac{1}{w_{j+1}}\right)=\sum_{j=1}^{r}b_j.$$

By taking the conjugate of this equation, we find that

$$\sum_{j=1}^{r}\left(w'_j+\frac{1}{w'_j}\right)=\sum_{j=1}^{r}b_j$$

also. Finally, of course,

$$\sum_{j=1}^{r}1=l(B),$$

since $l(B)$ is defined as the length of the cycle $((b_1,\dots,b_r))$. Substituting the last four equations into (7), we find

$$\varrho(B)=\varrho_1(B)+2\gamma\log\varepsilon-\frac{\pi^2}{6}l(B). \tag{8.9}$$

It remains consider $\varrho_1(B)$. We first discuss the relationship between wide ideal classes and continued fractions. For narrow ideal classes B, we know that there is always a (fractional) ideal in B with a basis $\{1,w\}$ such that $w>1>w'>0$, and that the number w then has a continued fraction expansion $w=b_1-1/(b_2-1/\cdots)$ which is purely periodic. Similarly, in any *wide* ideal class A there is an ideal having a basis of the form $\{1,x\}$ with

$$x>1,\quad 0>x'>-1, \tag{8.10}$$

and such x are characterised as being those quadratic irrationalities whose ordinary continued fraction expansion

$$x=a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\ddots}}\qquad(a_i\geqq 1) \tag{8.11}$$

(with plus signs) is pure periodic. A different choice of ideal leads to an x with the same period up to cyclic permutation. Thus to A there corresponds a cycle of integers $[[a_1,\dots,a_m]]$ with $a_i\geqq 1$ (the cycle is defined only up to cyclical permutation) in just the same way as a cycle $((b_1,\dots,b_r))$ with $b_i\geqq 2$ was assigned to a narrow ideal class.

Now, clearly, if x satisfies (10), then $w=1+x$ is reduced in the old sense $(w>1>w'>0)$ and $w>2$, so w has a purely periodic "minus" continued fraction $b_1-\cfrac{1}{b_2-\cdots}$ with $b_1\geqq 3$. The corresponding narrow ideal class B then lies in the wide ideal class A corresponding to x (since both contain the ideal $\mathbb{Z}\cdot 1+\mathbb{Z}\cdot x=\mathbb{Z}\cdot 1+\mathbb{Z}\cdot w$). One can easily check that the expansion (11) of x is related to the expansion

$$w=b_1-\cfrac{1}{b_2-\cfrac{1}{b_3-\ddots}} \tag{8.12}$$

of w by

$$b_1 = a_1 + 2\,, \qquad b_2 = b_3 = \cdots = b_{a_2} = 2\,, \qquad b_{a_2+1} = a_3 + 2, \ldots .$$

That is,

$$((b_1, b_2, b_3, \ldots)) = ((a_1 + 2, \underbrace{2, \ldots, 2}_{a_2 - 1}, a_3 + 2, \underbrace{2, \ldots, 2}_{a_4 - 1}, \ldots))\,. \tag{8.13}$$

If instead of x, we use the cyclic permutation $[[a_2, a_3, \ldots]]$ $\left(\text{corresponding to } \dfrac{1}{a_1 - x}\right)$, we get instead $((a_2 + 2, \underbrace{2, \ldots, 2}_{a_3 - 1}, a_4 + 2, \ldots))$. This cycle corresponds to the opposite narrow ideal class ΘB. If we start with $[[a_3, a_4, \ldots]]$, we get a cycle which is the same as (13) (up to cyclic permutation).

If the primitive period $[[a_1, \ldots, a_m]]$ corresponding to A has *odd* length m, then we deduce that B and ΘB are the same narrow ideal class. This is the case when the quadratic field K under study has a unit of negative norm. In this case $A = B$ and $\varepsilon = \varepsilon_0^2$ (ε as above, ε_0 the fundamental unit). If, on the other hand, m is *even*, then the above procedure generates two cycles $((b_1, \ldots, b_r))$ starting from the given cycle $[[a_1, \ldots, a_m]]$, A is the disjoint union of two narrow ideal classes B and ΘB, and $\varepsilon = \varepsilon_0$. For convenience, in the first case we write the period twice, so that we always study the minimal even period $[[a_1, \ldots, a_{2k}]]$. Then we have $2k$ numbers $x_i \in K$ ($i = 1, \ldots, 2k$) satisfying (10) namely

$$x_i = a_i + \cfrac{1}{a_{i+1} + \cdots + \cfrac{1}{a_{2k} + \cfrac{1}{a_1 + \cdots}}}\,. \tag{8.14}$$

For B we have $r = a_2 + a_4 + \cdots + a_{2k}$ numbers w_i [r is the length of the cycle (13)], namely

$$\begin{aligned}
w_1 &= 1 + x_1 = a_1 + 2 - \frac{1}{w_2}\,,\\
w_2 &= 2 - \frac{1}{w_3}\,,\\
w_3 &= 2 - \frac{1}{w_4}\,,\\
&\;\vdots\\
w_{a_2} &= 2 - \frac{1}{w_{a_2+1}}\,,\\
w_{a_2+1} &= 1 + x_3 = a_3 + 2 - \frac{1}{w_{a_2+2}}\,,
\end{aligned} \tag{8.15}$$

etc. Now observe that the expression summed in (8) is contrived in such a way that the argument $1 - \dfrac{1}{w_j}$ of the second $R(\,,)$ is obtained from the argument $w - 1$ of the first by substituting $2 - \dfrac{1}{w}$ for w. Therefore, in computing the sum (8) for a string of w_j's as in (15) each related to the previous one by $w \to 2 - \frac{1}{w}$, we find that all intermediate terms cancel and we are left with only two terms. Explicitly, this means

$$\begin{aligned}\sum_{j=2}^{a_2+1}&\left[R(w_j-1,1-w_j')-R\left(1-\frac{1}{w_j},\frac{1}{w_j'}-1\right)\right]\\&=R(w_{a_2+1}-1,1-w'_{a_2+1})-R\left(1-\frac{1}{w_2},\frac{1}{w_2'}-1\right)\\&=R(x_3,-x_3')-R(x_1-a_1,a_1-x_1')\\&=R(x_3,-x_3')+R(x_2,-x_2'),\end{aligned} \tag{8.16}$$

where in the last line we have used $x_1 = a_1 + \frac{1}{x_2}$ [cf. (14)] and Eq. (2) of the Lemma. Summing this over all $a_2 + \cdots + a_{2k}$ values of w_j (rather than just $1 < j \leqq a_2 + 1$), we obtain $\varrho_1(B) = \Sigma_{i=1}^{2k} R(x_i, -x_i')$. We have proved:

Theorem. *Let B be a narrow ideal class in a real quadratic field $K = \mathbb{Q}(\sqrt{D})$ and A the wide ideal class containing it. Set $\tau = 1$ or 2 according as the fundamental unit $\varepsilon_0 > 1$ of K has norm $+1$ or -1 (so that the unit ε defined above is ε_0^τ), and let $[[a_1, \ldots, a_{2k}]]$ be τ times the smallest period of the (positive) continued fraction associated to A. Define $x_1, \ldots, x_{2k}$ by (14). Then the Kronecker limit $\varrho(B)$ is given by*

$$\varrho(B) = \sum_{i=1}^{2k} R(x_i, -x_i') + 2\gamma \log \varepsilon - \frac{\pi^2}{6} l(B), \tag{8.17}$$

where $R(x, y)$ is the function defined by (1).

Corollary 1 (Meyer's theorem). *Let Θ be the narrow ideal class of the ideal $(\sqrt{D})$. Then*

$$\varrho(B) - \varrho(\Theta B) = -\frac{\pi^2}{6}(l(B) - l(\Theta B)).$$

Proof. Immediate from (17), since the first two terms on the right only depend on the wide ideal class A.

Corollary 2 (Kronecker limit formula for wide ideal classes). *Let A be a wide ideal class in a real quadratic field K, $\varepsilon_0 > 1$ the fundamental unit of K, $x_1, \ldots, x_m$ the elements of K satisfying* (10) *and such that $\{1, x_i\}$ is a basis for some ideal in A* (thus $m = 2k/\tau$ in the notation of the theorem), *and $((a_1, \ldots, a_m))$ the corresponding cycle of integers. Then*

$$\begin{aligned}\lim_{s\to1}\left[\zeta(s,A)-\frac{2\log\varepsilon_0}{s-1}\right]&=2\sum_{i=1}^{m}R(x_i,-x_i')\\&\quad+4\gamma\log\varepsilon_0-\frac{\pi^2}{6}\sum_{i=1}^{m}a_i.\end{aligned} \tag{8.18}$$

Proof. If $\tau = 1$, then $A = B \cup \Theta B$ and $\zeta(s, A) = \zeta(s, B) + \zeta(s, \Theta B)$, so the residue is $2 \log \varepsilon = 2 \log \varepsilon_0$. If $\tau = 2$, then $A = B$, $\zeta(s, A) = \zeta(s, B)$, so the residue of

$\zeta(s, A)$ is $\log\varepsilon = 2\log\varepsilon_0$. For the constant term, we have, using (17),

$$\begin{aligned}\tau\cdot\lim_{s\to1}\left(\zeta(s,A)-\frac{2\log\varepsilon_0}{s-1}\right)&=\varrho(B)+\varrho(\Theta B)\\&=2\sum_{i=1}^{2k}R(x_i,-x_i')+4\gamma\log\varepsilon-\frac{\pi^2}{6}[l(B)+l(\Theta B)]\,,\end{aligned}\tag{8.19}$$

and, since $l(B)=a_2+a_4+\cdots+a_{2k}$ and $l(\Theta B)=a_1+a_3+\cdots+a_{2k-1}$, the term $[l(B)+l(\Theta B)]$ is just $\Sigma_{i=1}^{2k}a_i$. Now $m=2k$ if $\tau=1$, but if $\tau=2$ then $m=k$ and $a_{m+i}=a_i$, $x_{m+i}=x_i$ $(i=1,\ldots,m)$. Therefore the right-hand side of (19) is

$$2\tau\sum_{i=1}^{m}R(x_i,-x_i')+4\gamma\tau\log\varepsilon_0-\frac{\pi^2}{6}\tau\sum_{i=1}^{m}a_i\,,$$

and on dividing by τ we obtain the corollary.

Notice, that by arguments like those already used,

$$\sum_{i=1}^{m}a_i=\sum_{i=1}^{m}\left(x_i-\frac{1}{x_i}\right)$$

and

$$\begin{aligned}\log\varepsilon_0=\frac{1}{\tau}\log\varepsilon&=\frac{1}{\tau}\sum_{j=1}^{r}\log w_j\\&=\frac{1}{\tau}\sum_{j=1}^{r}\left[\log(w_j-1)-\log\left(1-\frac{1}{w_j}\right)\right]\\&=\frac{1}{\tau}\sum_{i=1}^{2k}\log x_i\\&=\sum_{i=1}^{m}\log x_i\,,\end{aligned}$$

and so the right-hand side of (18) can be replaced by $\Sigma_{i=1}^{m}Q(x_i,-x_i')$ with

$$Q(x,y)=2R(x,y)+4\gamma\log x-\frac{\pi^2}{6}\left(x-\frac{1}{x}\right).$$

In this form, Corollary 2 is the exact analogue of the main theorem of § 5 for narrow ideal classes.

9. On "Kronecker's Solution of Pell's Equation"

As was explained in the introduction, a knowledge of the Kronecker limits $\varrho(B)$ leads to the evaluation of $L(1,\chi)$ in the form

$$L(1,\chi)=\sum_{B}\chi(B)\varrho(B)\tag{9.1}$$

(sum over narrow ideal classes) for any character χ ($\neq \chi_0$) on the narrow ideal class group. For one type of character, the so-called genus characters, however, the left-hand side is explicitly known, and this leads to interesting identities.

A genus character is simply a real character on the narrow ideal class group, i.e. a character which assumes only the values ± 1 and is hence trivial on the squares of ideals. If D is the determinant of K, then there are 2^{t-1} genus characters, where t is the number of distinct primes dividing D. They are in 1 : 1 correspondence with all possible splittings $D = D_1 \cdot D_2$ with D_1, D_2 both discriminants of quadratic fields (and where one does not distinguish between the decompositions $D = D_1 \cdot D_2$ and $D = D_2 \cdot D_1$). If χ is the genus character corresponding to $D = D_1 \cdot D_2$, then

$$L(s,\chi) = L_{D_1}(s) L_{D_2}(s) \tag{9.2}$$

where

$$L_D(s) = \sum_{n=1}^{\infty} \frac{\left(\frac{D}{n}\right)}{n^s}.$$

It is a classical result that

$$L_D(1) = \begin{cases} \dfrac{2h\log\varepsilon}{\sqrt{D}} & \text{if} \quad D>0, \\[2ex] \dfrac{2\pi h}{w\sqrt{|D|}} & \text{if} \quad D<0, \end{cases} \tag{9.3}$$

where h, ε, w denote the class number, fundamental unit, and order of the group of units of $\mathbb{Q}(\sqrt{D})$, respectively.

If $K = \mathbb{Q}(\sqrt{D})$ is pure imaginary and $D = D_1 D_2$, then one of D_1, D_2, say D_1, is positive and the other negative. Denoting by a subscript i invariants of the field $\mathbb{Q}(\sqrt{D_i})$, we have

$$\begin{aligned} L(1,\chi) &= L_{D_1}(1) L_{D_2}(1) \\ &= \frac{4\pi h_1 h_2 \log\varepsilon_1}{w_2\sqrt{|D|}}. \end{aligned}$$

Comparing this with (1) and the Kronecker limit formula for $\varrho(B)$ (Section 2), we find

$$\frac{2h_1 h_2}{w_2}\log\varepsilon_1 = -\sum_B \chi(B)\Phi(B)$$

with $\Phi(B)$ defined by

$$\Phi(B) = \frac{w-\bar{w}}{2i}|\eta(w)|^2$$

(where, as usual, w such that $\{1, w\}$ is an oriented basis for some ideal in B^{-1}). Thus

$$(\varepsilon_1^{h_1})^{2h_2/w_2} = \prod_B \Phi(B)^{-\chi(B)}. \tag{9.4}$$

For instance, with $D_1 = 5, D_2 = -4\left(\text{so } \varepsilon_1 = \frac{1+\sqrt{5}}{2}, h_1 = h_2 = 1, w_2 = 2\right)$, one finds

$$
\begin{aligned}
1+\sqrt{5} &= \frac{\left|\eta\left(\frac{1+i\sqrt{5}}{2}\right)\right|^4}{|\eta(i\sqrt{5})|^4} \\
&= q^{-1/6}(1+q)^4\,(1+q^3)^4\,(1+q^5)^4 \ldots
\end{aligned}
\tag{9.5}
$$

$(q = e^{-\pi\sqrt{5}} = 0.00087\ldots)$[2].

Clearly it is something of a cheat to call (4) a solution of Pell's equation, since to really solve Pell's equation would mean to find the *least* solution of $x^2 - D_1 y^2 = +4$, i.e. the fundamental unit $\varepsilon_1 = \frac{x+y\sqrt{D_1}}{2}$ itself, rather than the power $\varepsilon_1^{h_1}$, which in any case is given by the analogous, but much simpler formula

$$
\varepsilon_1^{h_1} = \prod_{k=1}^{D_1-1}\left(\sin\frac{\pi k}{D_1}\right)^{-\left(\frac{D_1}{k}\right)}.
$$

In any case, it seems more sensible to regard an equation like (5) as a surprising identity for $\eta(z)$ rather than as a "formula" for $1+\sqrt{5}$.

If K is totally real, so that our formula rather than Kronecker's is needed in (1), then there are two very different cases, according as D_1 and D_2 are both negative or both positive.

If $D_1, D_2 < 0$, then the corresponding genus character χ satisfies $\chi(\Theta) = -1$ (where Θ, as in Section 4, denotes the narrow ideal class containing the principal ideals (α) with $\alpha\alpha' < 0$). Then on the one hand

$$
\begin{aligned}
L(1,\chi) &= \sum_B \chi(B)\varrho(B) \\
&= \tfrac{1}{2}\sum_B [\chi(B)\varrho(B) + \chi(B^*)\varrho(B^*)] \\
&= \tfrac{1}{2}\sum_B \chi(B)\,(\varrho(B) - \varrho(B^*)) \\
&= \frac{\pi^2}{12\sqrt{D}}\sum_B \chi(B)\,(l(B^*) - l(B)) \\
&= \frac{-\pi^2}{6\sqrt{D}}\sum_B \chi(B)\,l(B)
\end{aligned}
$$

by Meyer's theorem in the form given in Section 4; on the other hand, from (2) and (3) we get

$$
L(1,\chi) = \frac{4\pi^2}{\sqrt{D}}\,\frac{h_1 h_2}{w_1 w_2}.
$$

Hence

$$
\frac{h(D_1)}{w(D_1)} \cdot \frac{h(D_2)}{w(D_2)} = -\frac{1}{24}\sum_B \chi(B)\,l(B).
\tag{9.6}
$$

[2] The discussion and example are taken from Siegel [8], pp. 93—96.

For instance, if $p > 3$ is a prime congruent to 3 (mod 4), then (taking $D_1 = -p$, $D_2 = -4$) we find

$$h(-p) = -\tfrac{1}{3} \sum_B \chi(B) l(B).$$

If we assume for simplicity that $h(p) = 1$, so that the only terms of the sum are the principal ideal class and the class Θ, we find:

Theorem. *Let $p > 3$ be a prime $\equiv 3 \pmod 4$ such that $h(p) = 1$. Let l_+ and l_- be the lengths of the periods of the continued fractions of $\sqrt{p}$ and of $-\sqrt{p}$. Then*

$$l_- - l_+ = 3h(-p). \tag{9.7}$$

For example, with $p = 7$ we find

$$\sqrt{7} = 3 - \cfrac{1}{3 - \cfrac{1}{6 - \cfrac{1}{3 - \cfrac{1}{6 - \ddots}}}}$$

period

$$-\sqrt{7} = -2 - \cfrac{1}{2 - \cfrac{1}{3 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{3 - \cfrac{1}{3 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{3 - \ddots}}}}}}}}}}}$$

period

so $l_- - l_+ = 5 - 2 = 3$ With $p = 163$ one has $l_- = 21$, $l_+ = 18$, $h(-p) = 1$. It is not *a priori* clear either that $l_- > l_+$ or that $l_+ \equiv l_- \pmod 3$. For a further discussion of (6) and related formulas we refer the reader to [2] or [3].

Finally, consider the case when $D_1, D_2 > 0$. In this case $\chi(\Theta) = +1$, so we cannot use Meyer's theorem to simplify (1). Combining equations (1), (2) and (3), we find

$$\sqrt{D} \sum_B \chi(B) \varrho(B) = 4h_1 h_2 \log\varepsilon_1 \log\varepsilon_2, \tag{9.8}$$

and this in conjunction with the main theorem of this paper should give identities for $F(x)$ similar to the identities for $\eta(x)$ discussed above, but now involving a product of two logarithms of algebraic numbers rather than only one such number.

For example, with $D = 40$, $D_1 = 5$, $D_2 = 8$, the right-hand side is

$$4 \log(1 + \sqrt{2}) \log\left(\frac{1 + \sqrt{5}}{2}\right)$$

and the left-hand side is

$$F(4+\sqrt{10},4-\sqrt{10})+F\left(\frac{4+\sqrt{10}}{6},\frac{4-\sqrt{10}}{6}\right)+F\left(\frac{8+\sqrt{10}}{9},\frac{8-\sqrt{10}}{9}\right)$$
$$=F\left(\frac{10+\sqrt{10}}{10},\frac{10-\sqrt{10}}{10}\right)+F\left(\frac{10+\sqrt{10}}{9},\frac{10-\sqrt{10}}{9}\right)$$
$$+F\left(\frac{8+\sqrt{10}}{6},\frac{8-\sqrt{10}}{6}\right)-F\left(\frac{4+\sqrt{10}}{2},\frac{4-\sqrt{10}}{2}\right)$$
$$-F\left(\frac{4+\sqrt{10}}{3},\frac{4-\sqrt{10}}{3}\right)-F\left(\frac{5+\sqrt{10}}{5},\frac{5-\sqrt{10}}{5}\right)$$
$$-F\left(\frac{5+\sqrt{10}}{3},\frac{5-\sqrt{10}}{3}\right)$$

(corresponding to the two ideal classes of $\mathbb{Q}(\sqrt{40})$, with cycles $((8,2,2,2,2,2))$ and $((4,2,3,2))$, respectively), and the equality between these two expressions represents a certain–admittedly very complicated–identity for the function $F(x)$.

Remark. Since writing this paper, the author has discovered that the same problem has been studied and a similar result obtained by G. Herglotz, „Über die Kroneckersche Grenzformel für reelle, quadratische Körper. I, II" (Berichte über die Verhandl. d. Sächsischen Akad. der Wiss. zu Leipzig, **75** (1923) 3—14, 31—37).

The formula Herglotz obtains is related to Dedekind sums rather than continued fractions and also involves the function $F(x)$.

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