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Periods of modular forms and Jacobi theta functions

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1. Introduction and statement of theorem

The period polynomial of a cusp form $f(\tau)=\sum_{l=1}^{\infty} a_f(l)q^l$ ($\tau\in\mathfrak{H}$ = upper half-plane, $q=e^{2\pi i\tau}$) of weight k on $\Gamma=PSL_2(\mathbb{Z})$ is the polynomial of degree $k-2$ defined by

$$r_f(X)=\int_0^{i\infty} f(\tau)(\tau-X)^{k-2}\,d\tau \tag{1}$$

or equivalently by

$$r_f(X)=-\sum_{n=0}^{k-2}\frac{(k-2)!}{(k-2-n)!}\frac{L(f,n+1)}{(2\pi i)^{n+1}}X^{k-2-n}, \tag{2}$$

where $L(f,s)$ denotes the L-series of f (= analytic continuation of $\sum_{l=1}^{\infty} a_f(l)l^{-s}$). The Eichler-Shimura-Manin theory tells us that the map $f\mapsto r_f$ is an injection from the space S_k of cusp forms of weight k on Γ to the space of polynomials of degree $\leqq k-2$ and that the product of the nth and mth coefficients of r_f is an algebraic multiple of the Petersson scalar product (f,f) if f is a Hecke eigenform and n and m have opposite parity. More precisely, for each integer $l\geqq 1$ the polynomial in two variables

$$\sum_{\substack{f\in S_k\\ \text{eigenform}}}\frac{(r_f(X)r_f(Y))^-}{(2i)^{k-3}(f,f)}a_f(l), \tag{3}$$

has rational coefficients; here $(r_f(X)r_f(Y))^-=\frac{1}{2}(r_f(X)r_f(Y)-r_f(-X)r_f(-Y))$ is the odd part of $r_f(X)r_f(Y)$ and the sum is taken over a basis of Hecke eigenforms of S_k. A rather complicated expression for the coefficients of these polynomials was found in [3].

In this paper we will give a much more attractive formula for the expressions (3) by means of a generating function. First we multiply each expression (3) by q^l and sum over l, i.e., we replace $a_f(l)$ in (3) by the cusp form $f(\tau)$ itself. Secondly, we extend the definition of r_f (and of (f,f)) to non-cusp forms, the function $r_f(X)$ now being $1/X$ times a polynomial of degree k in X, and include the Eisenstein series in the sum (3). Then we define

$$c_k(X, Y; \tau) = \sum_{\substack{f \in M_k \\ \text{eigenform}}} \frac{(r_f(X) r_f(Y))^-}{(2i)^{k-3}(f,f)} f(\tau), \quad (k \geqq 2 \text{ even}, l \geqq 0) \tag{4}$$

where the sum is now over all Hecke eigenforms in the space M_k of modular forms of weight k on Γ. The function $c_k(X, Y; \tau)$ is identically 0 for $k=2$ or k odd and in general belongs to $M_k^{\mathbb{Q}} \otimes X^{-1} Y^{-1} \mathbb{Q}[X, Y]$, where $M_k^{\mathbb{Q}} = M_k \cap \mathbb{Q}[[q]]$; for instance,

$$c_4(X, Y; \tau) = -\tfrac{1}{3}((X^2-1)(Y^3+5Y+Y^{-1}) + (X^3+5X+X^{-1})(Y^2-1))G_4(\tau)$$

where $G_k(\tau) = -\dfrac{B_k}{2k} + \sum_{l=1}^\infty (\sum_{d|l} d^{k-1}) q^l$ (k even) denotes the normalized Eisenstein series of weight k on Γ. We combine all these functions into a single generating function

$$C(X, Y; \tau; T) = \frac{(XY-1)(X+Y)}{X^2 Y^2} T^{-2} + \sum_{k=2}^\infty c_k(X, Y; \tau) \frac{T^{k-2}}{(k-2)!}.$$

Then the result we will prove is

Main Theorem. *The function $C(X, Y; \tau; T) \in (XYT)^{-2} \mathbb{Q}[X, Y][[q, T]]$ is given by*

$$C(X, Y; \tau; T) = \theta'(0)^2 \frac{\theta((XY-1)T)\theta((X+Y)T)}{\theta(XYT)\theta(XT)\theta(YT)\theta(T)}, \tag{5}$$

where $\theta(u)$ denotes the classical Jacobi theta function

$$\theta(u) = \theta_\tau(u) = \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}(n+\frac{1}{2})^2} e^{(n+\frac{1}{2})u}. \tag{6}$$

From the Jacobi triple product formula

$$\theta(u) = q^{\frac{1}{8}}(e^{\frac{u}{2}} - e^{-\frac{u}{2}}) \prod_{n=1}^\infty (1-q^n)(1-q^n e^u)(1-q^n e^{-u})$$

one easily finds

$$\frac{\theta(u)}{u\theta'(0)} = \exp\left(-2 \sum_{k \geqq 2} G_k(\tau) \frac{u^k}{k!}\right), \tag{7}$$

so (5) can be rewritten in the form

Main Theorem (*Second version*).

$$C(X, Y; \tau; T) = \left(1 - \frac{1}{XY}\right)\left(\frac{1}{X} + \frac{1}{Y}\right) \frac{1}{T^2} \times$$

$$\times \exp\left(2 \sum_{k=2}^\infty [(X^k+1)(Y^k+1) - (XY-1)^k - (X+Y)^k] G_k(\tau) \frac{T^k}{k!}\right). \tag{8}$$

Notice that the coefficient of G_2, which is not a modular form, vanishes in (8). In fact, the right hand side of (5) is the simplest combination of theta-series in which

G_2 drops out and thus whose Taylor coefficients in each degree are modular forms in τ.

Formula (8) is surprisingly simple: the coefficient of T^k in the exponent on the right, which a priori could be an arbitrary polynomial of degree k in X and Y with coefficients in $M_k^{\mathbb{Q}}$, is just the product of the Eisenstein series $G_k(\tau)$ with a very simple polynomial. Yet either (8) or the equivalent formula (5) contains complete information about all modular forms on Γ and their periods: expanding the right-hand side of either formula as a power series in T, by hand or using a symbolic algebra package, we automatically obtain the Hecke eigenforms and their period polynomials in any desired weight.

The contents of the paper are as follows. In §2 we define the period functions r_f for $f \notin S_k$ and prove the basic properties of the extended period mapping. Section 3, which does not use the theory of periods and may be of independent interest, contains the construction of a certain simple function of three variables $\tau \in \mathfrak{H}$, $u, v \in \mathbb{C}$ which has nice transformation properties (modular in τ, elliptic in u and v) and nice expansions with respect to the variables q, u and v. This function is used in §4 to prove the main theorem, while §5 contains some consequences and numerical examples.

2. Periods of cusp forms and non-cusp forms

We begin by reviewing the classical theory of periods for cusp forms on $\Gamma = PSL_2(\mathbb{Z})$ (for more details, see [4], Chapter 5). Let k denote a positive even integer, S_k and M_k the spaces of cusp forms and modular forms of weight k on Γ, and V_k the space of polynomials of degree $\leq k-2$. The *periods* of $f \in S_k$ are the $k-1$ numbers

$$r_n(f) = \int_0^{i\infty} f(\tau)\tau^n \, d\tau \quad (0 \leq n \leq k-2)$$

and equal $i^{n+1} L^*(f, n+1)$, where

$$L^*(f, s) = \int_0^{\infty} f(iy) y^{s-1} \, dy = (-1)^{k/2} L^*(f, k-s)$$

is the L-series of f multiplied by its gamma-factor $(2\pi)^{-s}\Gamma(s)$. They can be assembled into the polynomial $r_f(X) = \sum_{n=0}^{k-2} (-1)^n \binom{k-2}{n} r_n(f) X^{k-2-n} \in V_k$ as in (1). The group Γ acts on the space V_k by

$$(\phi\,|\,\gamma)(X) = (\phi|_{2-k}\gamma)(X) = (cX+d)^{k-2}\phi\left(\frac{aX+b}{cX+d}\right) \quad \left(\phi \in V_k, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\right).$$

One checks easily that $r_f|\gamma$ is given by the same integral as in (1) but taken from $\gamma^{-1}(0)$ to $\gamma^{-1}(\infty)$. In particular,

$$r_f + r_f|S = \int_0^{i\infty} + \int_{i\infty}^{0} = 0, \quad r_f + r_f|U + r_f|U^2 = \int_0^{i\infty} + \int_{i\infty}^{1} + \int_1^{0} = 0\,,$$

where $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $U = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ are the standard generators of Γ of order 2 and 3, respectively. Therefore r_f belongs to the space

$$W_k = \{\phi \in V_k : \phi|(1+S) = \phi|(1+U+U^2) = 0\}\,,$$

where we have extended the action of the group Γ to one of the group ring $\mathbb{Z}[\Gamma]$ in the obvious way. If V_k^+ (resp. V_k^-) denotes the space of even (resp. odd) polynomials in V_k, then r_f can be written as $r_f^+ + r_f^-$ with $r_f^\pm \in W_k^\pm = W_k \cap V_k^\pm$. The map $r^-: f \mapsto r_f^-$ is an isomorphism from S_k to W_k^-, while r^+ is an isomorphism from S_k to a codimension 1 subspace of W_k^+ which was determined in [3], 4.2. Finally, if f is a normalized Hecke eigenform, then there are non-zero numbers $\omega_f^+ \in i\mathbb{R}$, $\omega_f^- \in \mathbb{R}$ such that the coefficients of $r_f^\pm(X)/\omega_f^\pm$ and the number $\omega_f^+\omega_f^-/i(f,f)$ belong to the number field $\mathbb{Q}_f$ generated by the Fourier coefficients of f and transform by σ if f is replaced by $f^\sigma = \sum a_f(l)^\sigma q^l$, $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. For instance, for $k=12$, $f = \Delta = q - 24q^2 + 252q^3 - \ldots$ we have

$$r_\Delta^+(X) = \left(\frac{36}{691}X^{10} - X^8 + 3X^6 - 3X^4 + X^2 - \frac{36}{691}\right)\omega_\Delta^+,$$

$$r_\Delta^-(X) = (4X^9 - 25X^7 + 42X^5 - 25X^3 + 4X)\omega_\Delta^-, \text{ where}$$

$$\omega_\Delta^+ = 0.114379\ldots i,\ \omega_\Delta^- = 0.00926927\ldots,\ \frac{\omega_\Delta^+\omega_\Delta^-}{i(\Delta,\Delta)} = 2^{10} \in \mathbb{Q} = \mathbb{Q}_\Delta.$$

Now suppose that f is a modular form of weight k but not a cusp form, say $f = \sum_{l=0}^\infty a_f(l)q^l$ with $a_f(0) \neq 0$. The function $L^*(f,s)$ is now defined for $Re(s) \gg 0$ by

$$L^*(f,s) = \int_0^\infty (f(iy) - a_f(0))y^{s-1}\,dy = (2\pi)^{-s}\Gamma(s)L(f,s),\ L(f,s) = \sum_{l=1}^\infty a_f(l)l^{-s};$$

it still has a meromorphic continuation to all s and satisfies the functional equation $L^*(f,s) = (-1)^{k/2}L^*(f,k-s)$, but now has (as its only singularities) simple poles of residue $-a_f(0)$ and $(-1)^{k/2}a_f(0)$ at $s=0$ and $s=k$, respectively. On the other hand, the binomial coefficient $\binom{k-2}{n}$, interpreted as $\frac{\Gamma(k-1)}{\Gamma(n+1)\Gamma(k-1-n)}$, has a simple zero at all $n \in \mathbb{Z}$, $n \notin \{0, 1, \ldots, k-2\}$, the values of its derivatives at $n=-1$ and $n=k-1$ being $1/(k-1)$ and $-1/(k-1)$, respectively. Hence the natural way to interpret the formula $r_f(X) = \sum_{n\in\mathbb{Z}} i^{1-n}\binom{k-2}{n}L^*(f,n+1)X^{k-2-n}$ (valid for cusp forms) is to define r_f by

$$r_f(X) = \frac{a_f(0)}{k-1}(X^{k-1} + X^{-1}) + \sum_{n=0}^{k-2} i^{1-n}\binom{k-2}{n}L^*(f,n+1)X^{k-2-n}. \quad (9)$$

This is no longer in V_k but instead in the bigger space

$$\hat{V}_k = \bigoplus_{-1\le n\le k-1} \mathbb{C}X^n = X^{-1}\cdot\{\text{polynomials of degree } \le k \text{ in } X\}.$$

Using the standard formula

$$L^*(f,s) = \int_{t_0}^\infty \Big(f(it) - a_f(0)\Big)t^{s-1}\,dt + \int_0^{t_0}\left(f(it) - \frac{a_f(0)}{(it)^k}\right)t^{s-1}\,dt$$
$$-a_f(0)\left[\frac{t_0^s}{s} + \frac{(-1)^{k/2}t_0^{k-s}}{k-s}\right] \qquad (t_0 > 0 \text{ arbitrary}),$$

we can give an alternative formulation of the definition as

$$r_f(X) = \int_{\tau_0}^{i\infty} \Big(f(\tau) - a_f(0)\Big)(\tau - X)^{k-2}\,d\tau + \int_0^{\tau_0} \left(f(\tau) - \frac{a_f(0)}{\tau^k}\right)(\tau - X)^{k-2}\,dy$$

$$+ \frac{a_f(0)}{k-1}\left[(X-\tau_0)^{k-1} + \frac{1}{X}\left(1 - \frac{X}{\tau_0}\right)^{k-1}\right] \quad (\tau_0 \in \mathfrak{H} \text{ arbitrary}) \qquad (10)$$

(that the right-hand side does not depend on τ_0 can be checked easily by differentiation). Note that wo do not have to write $\hat{r}_f$ for our new element of $\hat{V}_k$, since when f is a cusp form the new definition agrees with the old one. As before, we denote by $\hat{V}_k^+$ and $\hat{V}_k^-$ the even and odd parts of $\hat{V}_k$ and by $r_f^\pm$ the component of r_f in $\hat{V}_k^\pm$.

Theorem. *The function $r_f(X)$ belongs to the subspace*

$$\hat{W}_k = \{\phi \in \hat{V}_k : \phi|_{2-k}(1+S) = \phi|_{2-k}(1+U+U^2) = 0\}$$

of $\hat{V}_k$. This space is the direct sum of the two subspaces $\hat{W}_k^\pm = \hat{W}_k \cap \hat{V}_k^\pm$; $\hat{W}_k^+$ equals W_k^+, while $\hat{W}_k^-$ contains W_k^- with codimension 1 unless $k=2$, when $\hat{W}_k^\pm = W_k^\pm = \{0\}$. The maps $r^\pm : M_k \to \hat{W}_k^\pm$ are both isomorphisms.

Remarks. Note that the result here is *simpler* than the corresponding result for cusp forms, where only one of the two maps $r^\pm : S_k \to W_k^\pm$ was an isomorphism and the determination of the image of the other was a difficult problem. This simplification on passing from S_k to M_k is a main theme of this paper. We should also remark that $\hat{V}_k$ is not a Γ- or $\mathbb{Z}[\Gamma]$-module, since $\phi|_{2-k}\gamma$ for $\phi \in \hat{V}_k$ and $\gamma \in \Gamma$ is not in general in $\hat{V}_k$; nevertheless, $\phi|\gamma$ is a well-defined rational function and the definition of $\hat{W}_k$ makes sense.

To prove the relations $r_f|(1+S) = r_f|(1+U+U^2) = 0$ for $f \in M_k$ we could proceed as before, writing $r_f|\gamma$ as an integral from $\gamma^{-1}(0)$ to $\gamma^{-1}(\infty)$ via $\gamma^{-1}(\tau_0)$ and worrying about the contribution from $a_f(0)$. However, since $M_k = S_k \oplus \langle G_k \rangle$ and we will need the period polynomials of the Eisenstein series anyway, it is more convenient to simply check the assertions of the theorem directly for G_k. Thus we will deduce the theorem from

Proposition. (i) *For $k > 2$ the functions*

$$p_k^+(X) = X^{k-2} - 1, \quad p_k^-(X) = \sum_{\substack{-1 \le n \le k-1 \\ n \text{ odd}}} \frac{B_{n+1}}{(n+1)!}\frac{B_{k-n-1}}{(k-n-1)!} X^n \qquad (11)$$

belong to $\hat{W}_k^+$ and $\hat{W}_k^-$, respectively.

(ii) *The period polynomial of the Eisenstein series G_k is given by*

$$r_{G_k}(X) = \omega_{G_k}^- p_k^- + \omega_{G_k}^+ p_k^+, \quad \textit{where} \quad \omega_{G_k}^- = -\frac{(k-2)!}{2}, \quad \omega_{G_k}^+ = \frac{\zeta(k-1)}{(2\pi i)^{k-1}}\omega_{G_k}^-.$$

Proof. For (i) we must check that $p_k^\pm \in \hat{W}_k$, since $p_k^\pm \in \hat{V}_k^\pm$ is obvious. The condition $p_k^\pm|(1+S) = 0$ just says that the coefficients of X^n and X^{k-2-n} in $p_k^\pm$ differ by a factor $(-1)^{n+1}$, which is clear. Hence we need only check $p_k^\pm|(1+U+U^2) = 0$.

For p_k^+ this is immediate, since $p_k^+ | U = (X-1)^{k-2} - X^{k-2}$, $p_k^+ | U^2 = 1-(X-1)^{k-2}$. For p_k^- it is convenient to introduce the generating function

$$P(X,T) = \frac{1}{XT^2} + \sum_{\substack{k=2 \\ k \text{ even}}}^{\infty} p_k^-(X)T^{k-2} \tag{12}$$

$$= \left(\sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \frac{B_n}{n!}(XT)^{n-1} \right) \left(\sum_{\substack{m=0 \\ m \text{ even}}}^{\infty} \frac{B_m}{m!} T^{m-1} \right) = \frac{1}{4}\coth\frac{XT}{2}\coth\frac{T}{2}.$$

The addition law for the hyperbolic cotangent function, which can be written in the form

$$\alpha+\beta+\gamma = 0 \Rightarrow \coth\alpha\coth\beta + \coth\beta\coth\gamma + \coth\gamma\coth\alpha = -1\,,$$

now tells us that

$$P(X,T) + P\left(1-\frac{1}{X}, XT\right) + P\left(\frac{-1}{X-1}, (X-1)T\right) = \frac{1}{4},$$

and comparing the coefficients of T^{k-2} ($k \neq 2$) on both sides gives the desired conclusion. Note that for $p_2^-(X) = \frac{1}{12}(X+X^{-1})$ we have $p_2^- |_0 (1+U+U^2) = \frac{1}{4}$. Thus $p_2^+ \equiv 0$ and $p_2^- \notin \hat{W}_2^-$; for $k=0$, on the other hand, both functions $p_0^+(X) = X^{-2}-1$ and $p_0^-(X) = X^{-1}$ do satisfy the period relations.

For (ii) we use the definition (9) of r_f, observing that $a_{G_k}(0) = -B_k/2k$ and $L(G_k, s) = \zeta(s)\zeta(s-k+1)$. The assertions follow after a short calculation using the values $\zeta(1-n) = -B_n/n$, $\zeta(n) = -(2\pi i)^n B_n/2n!$ ($n>0$ even), $\zeta(1-n) = 0$ ($n>1$ odd).

The proof of the theorem is now immediate: $r_f \in \hat{W}_k$ for any $f \in M_k$ because M_k is the direct sum of S_k and $\langle G_k \rangle$, $\hat{W}_k^+ = W_k^+$ because $\hat{V}_k^+ = V_k^+$, W_k^- has codimension 1 in $\hat{W}_k^-$ for $k>2$ because $p_k^- \notin V_k$ and because the codimension of W_k in $\hat{W}_k$ is ≤ 1 (since $\phi|(1+S) = 0$ implies that the coefficients of X^{-1} and X^{k-1} in any $\phi \in \hat{W}_k$ are equal), and $r^\pm : M_k \to \hat{W}_k^\pm$ is an isomorphism because $r^+ : S_k \to W_k^+/\langle p_k^+ \rangle$ and $r^- : S_k \to W_k^-$ are isomorphisms.

We have now extended the definition of $r_f^\pm$ to all $f \in M_k$. In [7], §5, we defined the Petersson scalar product of arbitrary modular forms in M_k by Rankin's method, i.e.,

$$(f,g) = \frac{\pi}{3}\frac{(k-1)!}{(4\pi)^k}\operatorname{Res}_{s=k}\left[\sum_{l=1}^{\infty}\frac{a_f(l)\overline{a_g(l)}}{l^s}\right] \quad (f,g \in M_k)\,.$$

For the Petersson norm of the Eisenstein series this gives

$$\begin{aligned}(G_k, G_k) &= \frac{\pi}{3}\frac{(k-1)!}{(4\pi)^k}\operatorname{Res}_{s=k}\left[\frac{\zeta(s)\zeta(s-k+1)^2\zeta(s-2k+2)}{\zeta(2s-2k+2)}\right] \\ &= \frac{(k-1)!}{2^{2k-1}\pi^{k+1}}\zeta(k)\zeta'(2-k) \\ &= \frac{(k-1)!\,(k-2)!}{2^{3k-2}\pi^{2k-1}i^{k-2}}\zeta(k)\zeta(k-1) \\ &= \frac{(k-2)!}{(4\pi)^{k-1}}\frac{B_k}{2k}\zeta(k-1)\,. \end{aligned} \tag{13}$$

(The formula given on p. 435 of [7] contains a misprint: 2^{3k-3} should be 2^{3k-2}.) Comparing this with part (ii) of the proposition, we see that we have the same assertion $\omega_f^+\omega_f^-/i(f,f)\in\mathbb{Q}_f$ for G_k as for Hecke eigenforms $f\in S_k$.

It follows from (13) and part (ii) of the proposition that, if we decompose the expression (4) into a cuspidal part $c_k^0(X,Y;\tau)$ and an Eisenstein part $c_k^E(X,Y;\tau)$, then the latter is given by

$$c_k^E(X,Y;\tau)=\frac{2k(k-2)!}{B_k}\,(p_k^+(X)p_k^-(Y)+p_k^-(X)p_k^+(Y))\,G_k(\tau)\,.$$

Splitting up the generating function $C(X,Y;\tau;T)$ as a sum C^0+C^E in the corresponding way, we find for the value at the cusp $\tau=i\infty$, $q=0$, the value

$$\begin{aligned}C(X,Y;i\infty;T)&=C^E(X,Y;i\infty;T)\\&=\frac{(XY-1)(X+Y)}{X^2Y^2T^2}-\sum_{k=2}^{\infty}[(X^{k-2}-1)p_k^-(Y)\\&\quad+(Y^{k-2}-1)p_k^-(X)]\,T^{k-2}\\&=P(X,T)+P(Y,T)-P(Y,XT)-P(X,YT)\end{aligned}$$

(with $P(X,T)$ defined as in (12))

$$\begin{aligned}&=\frac{1}{4}\left(\coth\frac{XT}{2}+\coth\frac{YT}{2}\right)\left(\coth\frac{T}{2}-\coth\frac{XYT}{2}\right)\\&=\frac{\sinh\dfrac{(X+Y)T}{2}\sinh\dfrac{(XY-1)T}{2}}{4\sinh\dfrac{T}{2}\sinh\dfrac{XT}{2}\sinh\dfrac{YT}{2}\sinh\dfrac{XYT}{2}}\,.\end{aligned}$$

This proves (5) in the limit as $\tau\to i\infty$, since $\left.\dfrac{\theta(u)}{\theta'(0)}\right|_{q=0}=2\sinh\dfrac{u}{2}$.

3. A meromorphic Jacobi form

In this section we study the function of three variables $\tau\in\mathfrak{H}$, $u,v\in\mathbb{C}$ defined (for $\Re(u)<2\pi\Im(\tau)$ and $-\Re(v)<2\pi\Im(\tau)$) by

$$F_\tau(u,v)=\sum_{n=0}^{\infty}\frac{\eta^{-n}}{q^{-n}\xi-1}-\sum_{m=0}^{\infty}\frac{\xi^m\eta}{q^{-m}-\eta}\quad(q=e^{2\pi i\tau},\ \xi=e^u,\ \eta=e^v)\,.$$

We write $G_k(\tau)$ for the Eisenstein series defined in §1 if $k>0$ is even and set $G_k\equiv 0$ for k odd.

Theorem. *The function $F_\tau(u,v)$ has the following properties:*

(*i*) *(Symmetry) $F_\tau(u,v)=F_\tau(v,u)=-F_\tau(-u,-v)$.*

(*ii*) *(Analytic continuation) $F_\tau(u,v)$ extends meromorphically to all values of u,v. It has a simple pole in u of residue η^{-n} at $2\pi i(n\tau+s)$ $(n,s\in\mathbb{Z})$ and a simple pole in v of residue ξ^{-m} at $2\pi i(m\tau+r)$ $(m,r\in\mathbb{Z})$, and is holomorphic for $u,v\notin\Lambda=2\pi i(\mathbb{Z}\tau+\mathbb{Z})$.*

(*iii*) (*Fourier expansion*) *The coefficients of* F_τ *as a power series in* q *are elementary hyperbolic functions of* u *and* v:

$$F_\tau(u,v) = \frac{1}{2}\left(\coth\frac{u}{2} + \coth\frac{v}{2}\right) - 2\sum_{n=1}^{\infty}\left(\sum_{d|n}\sinh\left(du + \frac{n}{d}v\right)\right)q^n. \quad (14)$$

(*iv*) (*Laurent expansion*) *The Taylor coefficients of* $F_\tau(u,v) - \frac{1}{u} - \frac{1}{v}$ *are derivatives of Eisenstein series*:

$$F_\tau(u,v) = \frac{1}{u} + \frac{1}{v} - 2\sum_{r,s=0}^{\infty}\left(\frac{1}{2\pi i}\frac{d}{d\tau}\right)^{\min\{r,s\}} G_{|r-s|+1}(\tau)\frac{u^r v^s}{r!\,s!}. \quad (15)$$

(*v*) (*Elliptic property*) $F_\tau(u + 2\pi i(n\tau + s), v + 2\pi i(m\tau + r)) = q^{-mn}\xi^{-m}\eta^{-n}F_\tau(u,v)$ *for* $m, n, r, s \in \mathbb{Z}$.

(*vi*) (*Modular property*) $F_{\frac{a\tau+b}{c\tau+d}}\left(\frac{u}{c\tau+d}, \frac{v}{c\tau+d}\right) = (c\tau+d)e^{\frac{cuv/2\pi i}{c\tau+d}}F_\tau(u,v)$ *for* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

(*vii*) (*Relation to theta functions*) *Let* $\theta(u) = \theta_\tau(u)$ *be as in* (6). *Then*

$$F_\tau(u,v) = \frac{\theta'(0)\theta(u+v)}{\theta(u)\theta(v)}.$$

(*viii*) (*Logarithm*) $F_\tau(u,v) = \frac{u+v}{uv}\exp\left(\sum_{k>0}\frac{2}{k!}[u^k + v^k - (u+v)^k]\,G_k(\tau)\right).$

Proof. By expanding the fractions in the terms $n \neq 0$, $m \neq 0$ in the definition of F as geometric series, we can express F as a double series

$$F_\tau(u,v) = \frac{\xi\eta - 1}{(\xi-1)(\eta-1)} - \sum_{m,n=1}^{\infty}(\xi^m\eta^n - \xi^{-m}\eta^{-n})q^{mn};$$

this makes the symmetry properties (i) obvious and also gives the Fourier expansion (iii). The double series converges if $|\Re(u)|$ and $|\Re(v)|$ are less than $2\pi|\Im(\tau)|$. To get the analytic continuation in u and v, we choose a positive integer N, break up the double series into the terms with $n < N$ and those with $n \geqq N$, and sum over m in the former and n in the latter terms. This gives

$$F_\tau(u,v) = \frac{1}{\xi - 1} - \sum_{n=1}^{N-1}\left(\frac{\eta^n\xi}{1-q^n\xi} - \frac{\eta^{-n}\xi^{-1}}{1-q^n\xi^{-1}}\right)q^n$$
$$+ \frac{\eta}{\eta-1} - \sum_{m=1}^{\infty}\left(\frac{\xi^m\eta^N}{1-q^m\eta} - \frac{\xi^{-m}\eta^{-N}}{1-q^m\eta^{-1}}\right)q^{Nm}.$$

The infinite sum converges for $|\Re(u)| < 2\pi N|\Im(\tau)|$, since then ξq^N and $\xi^{-1}q^N$ are less than 1 in absolute value. Taking N large enough thus gives the meromorphic continuation to all values of u and v, the positions and residues of the poles being as stated in (ii) of the proposition. The elliptic property (v) is also easily deduced: taking $N = 1$ and replacing η by $q\eta$, we find

$$\begin{aligned}\xi F_\tau(u, v + 2\pi i\tau) &= \frac{\xi}{\xi - 1} + \frac{q\xi\eta}{q\eta - 1} - \sum_{m=1}^{\infty} \frac{\xi^{m+1} q^{m+1}\eta}{1 - q^{m+1}\eta} + \sum_{m=1}^{\infty} \frac{\xi^{-m+1} q^{m-1}\eta^{-1}}{1 - q^{m-1}\eta^{-1}} \\ &= \frac{\xi}{\xi - 1} + \frac{1}{\eta - 1} - \sum_{m=1}^{\infty} \frac{\xi^m q^m \eta}{1 - q^m\eta} + \sum_{m=1}^{\infty} \frac{\xi^{-m} q^m \eta^{-1}}{1 - q^m\eta^{-1}} = F_\tau(u, v),\end{aligned}$$

which proves (v) for $m = 1, n = 0$; the general case follows by interchanging u and v and by induction on m and n.

Inserting the Taylor expansions

$$\frac{1}{2}\coth\frac{u}{2} = \sum_{n=0}^{\infty} \frac{B_n}{n!} u^{n-1}, \quad \sinh(u + v) = \sum_{\substack{t=1\\ t \text{ odd}}}^{\infty} \frac{(u+v)^t}{t!} = \sum_{\substack{r,s \geqq 0\\ r+s \text{ odd}}} \frac{u^r}{r!}\frac{v^s}{s!},$$

into (14), we find

$$F_\tau(u, v) = \frac{1}{u} + \frac{1}{v} - 2 \sum_{\substack{r,s \geqq 0\\ r+s \text{ odd}}} \left(-\frac{B_{r+1}}{2r+2}\delta_{s,0} - \frac{B_{s+1}}{2s+2}\delta_{r,0} + \sum_{m,n \geqq 1} m^r n^s q^{mn} \right) \frac{u^r}{r!}\frac{v^s}{s!} \tag{16}$$

and the expression in brackets is clearly $(2\pi i)^{-\nu} G_k^{(\nu)}(\tau)$ with $\nu = \min\{r, s\}$ and $k = |r - s| + 1$. This proves formula (15). We can rewrite (15) as

$$F_\tau(u, v) = -2 \sum_{k \geqq 2} \tilde{G}_k\left(\tau, \frac{uv}{2\pi i}\right)(u^{k-1} + v^{k-1})$$

with

$$\tilde{G}_k(\tau, \lambda) = \sum_{\nu=0}^{\infty} \frac{\lambda^\nu}{\nu!(\nu + k - 1)!} G_k^{(\nu)}(\tau) - \frac{\delta_{k,2}}{2\lambda}.$$

A result of H. Cohen and N. Kuznetsov (cf. [2], p. 35) implies the transformation law $\tilde{G}_k\left(\frac{a\tau + b}{c\tau + d}, \frac{\lambda}{(c\tau + d)^2}\right) = (c\tau + d)^k e^{c\lambda/(c\tau+d)} \tilde{G}_k(\tau, \lambda)$ for $\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \Gamma$. The modular transformation property (vi) follows.

The closed formula (vii) is an easy consequence of the elliptic transformation properties of F_τ. Indeed, it is well-known (and elementary) that $\theta(u)$ has simple zeros at all points of the lattice Λ and no other poles. Since $F_\tau(u, v)$ has simple poles for u or v in Λ, is otherwise holomorphic, and vanishes for $u + v \in \Lambda$ (because of the antisymmetry property $F_\tau(u, -v) = -F_\tau(v, -u)$), the quotient $\theta(u)\theta(v)F_\tau(u, v)/\theta(u + v)$ is holomorphic in u and v. Now using (v) and the transformation properties $\theta(u + 2\pi i) = \theta(u)$, $\theta(u + 2\pi i\tau) = -e^{-\pi i\tau - u}\theta(u)$, both of which are obvious from the definition of θ as either a sum or a product, one finds that the quotient in question is invariant under $u \mapsto u + \omega$ or $v \mapsto v + \omega$ for all $\omega \in \Lambda$. It must therefore be a constant (for τ fixed); taking the limit as $u \to 0$, we find that this constant equals $\theta'(0)$. This proves (vii) and also—in view of the known modularity properties of $\theta(u)$—leads to another proof of (vi). Finally, the identity given in (viii) follows from the formula (7). This identity again makes the modular transformation properties of F_τ clear, since $G_k(\gamma\tau)$ is equal to $(c\tau + d)^k G_k(\tau)$ for $k > 2$ but to $(c\tau + d)^2 G_2(\tau) + ic(c\tau + d)/4\pi$ for $k = 2$.

Remark. Parts (v) and (vi) of the proposition say that the function $F_\tau(2\pi i z_1, 2\pi i z_2)$ ($\tau \in \mathfrak{H}$, $z_1, z_2 \in \mathbb{C}$) is a two-variable meromorphic Jacobi form of weight 1 and index $\mathbf{m} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$ (for the theory of Jacobi forms, see [2], where, however, only Jacobi forms of one variable were considered). Equation (14) says that this Jacobi form is singular in the sense that for each term $q^n \xi^{r_1} \eta^{r_2}$ occurring in its Fourier development the matrix $\begin{pmatrix} 2n & r \\ r^t & 2\mathbf{m} \end{pmatrix}$ ($r = (r_1 r_2)$) has determinant zero.

4. Proof of the main identity

In view of part (vii) of the proposition of the last section, the theorem stated in §1 is equivalent to the identity

$$C(X, Y; \tau; T) = F_\tau(T, -XYT) F_\tau(XT, YT). \tag{17}$$

Denote the coefficient of $\dfrac{T^{k-2}}{(k-2)!}$ in the expression on the right-hand side of (17) by $b_k(X, Y; \tau)$. We must show that $b_k = c_k$ for every $k \geqq 2$, the equality of the leading terms in (17) being obvious.

Because the term $k = 2$ dropped out in (8), and the functions G_k for $k > 2$ are modular forms, the right-hand side of (5) (or of (17)) is invariant under $\tau \mapsto \dfrac{a\tau + b}{c\tau + d}$, $T \mapsto (c\tau + d) T$ for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. This is equivalent to the assertion that $b_k(X, Y; \tau)$ is a modular form of weight k (with coefficients in $\mathbb{C}[X, X^{-1}, Y, Y^{-1}]$) for every $k \geqq 0$. But we already checked the correctness of (17) in the limit $\tau \to i\infty$ at the end of §2, so the Eisenstein parts of the modular forms $b_k(X, Y; \tau)$ and $c_k(X, Y; \tau)$ agree. We therefore need only check the cuspidal parts, i.e., the assertion that b_k and c_k have the same Petersson scalar product with each cusp form $f \in S_k$. In view of the definition of c_k, this is equivalent to proving that

$$(b_k(X, Y; \cdot), f(\cdot)) = \frac{1}{(2i)^{k-3}} (r_f(X) r_f(Y))^- \tag{18}$$

for each normalized Hecke eigenform $f \in S_k$.

For brevity of notation, write the Taylor expansion (15) as

$$F_\tau(u, v) = \sum_{h,l} g_{h,l}(\tau)(u^l v^{l+h-1} + u^{l+h-1} v^l)$$

with

$$g_{h,l} = \begin{cases} 1 & \text{if } h = 2,\ l = -1, \\ \dfrac{-2(2\pi i)^{-l}}{l!(l+h-1)!} G_h^{(l)}(\tau) & \text{if } h \geqq 2,\ l \geqq 0, \\ 0 & \text{otherwise}. \end{cases}$$

Then

$$\frac{1}{(k-2)!} b_k(X, Y; \tau) = \sum_{\substack{l,h,l',h' \\ h+h'+2(l+l')=k}} g_{h,l}(\tau) g_{h',l'}(\tau)$$
$$\times [(-XY)^{l+h-1} + (-XY)^l][X^{l'} Y^{l'+h'-1} + X^{l'+h'-1} Y^{l'}]. \tag{19}$$

The coefficients of $X^m Y^n$ with m or n equal to -1 or to $k-1$ involve only the Eisenstein series G_k and have already been taken care of. Also, it is clear that the coefficient of $X^m Y^n$ on the right of (19) is invariant under $m \leftrightarrow n$ and $(-1)^{m+1}$-invariant under $m \leftrightarrow k-2-m$, so we may assume $0 \leqq m < n \leqq \frac{1}{2}(k-2)$. (The middle relation is $<$ rather than $\leqq$ because m and n always have opposite parity.) For such m, n, the coefficient of $X^m Y^n$ on the right hand side of (19) equals

$$\sum_{\substack{l,l' \geqq -1 \\ l+l'=m}} (-1)^l g_{k-n-m-1,l}(\tau)\, g_{n-m+1,l'}(\tau) = \frac{4}{m!n!(k-n-2)!} F_m(G_{n-m+1}, G_{k-n-m-1}),$$

where δ_{ij} denotes the Kronecker delta and $F_m(G_h, G_{h'})$ for $h, h' \geqq 2$ is defined by

$$F_m(G_h, G_{h'}) = \frac{1}{(2\pi i)^m} \sum_{l=0}^{m} (-1)^{m-l} \binom{m}{l} \frac{(m+h-1)!(m+h'-1)!}{(l+h-1)!(m-l+h'-1)!} G_h^{(l)}(\tau) G_{h'}^{(m-l)}(\tau)$$
$$+ \frac{m!}{2(2\pi i)^{m+1}} \left(\frac{\delta_{h',2}}{h+m} G_h^{(m+1)}(\tau) + \frac{(-1)^m \delta_{h,2}}{h'+m} G_{h'}^{(m+1)}(\tau) \right).$$

If $m = 0$ and h and h' are both greater than or equal to 4, then $F_m(G_h, G_{h'})$ is simply the product of the Eisenstein series G_h and $G_{h'}$, and it was shown by Rankin ([9], Theorem 4) that (at least for $h \neq h'$) this product satisfies

$$(G_h G_{h'}, f) = \frac{1}{(2i)^{k-1}} r_{k-2}(f) r_{h'-1}(f)$$

for all normalized Hecke eigenforms in S_k, $k = h + h'$. If $m > 0$ and h and h' are both $\geqq 4$, then $F_m(G_h, G_{h'})$ is the result of applying to the Eisenstein series G_h and $G_{h'}$ the operator introduced by H. Cohen in [1] and is a cusp form of weight $k = h + h' + 2m$; here it was shown in [6] (Proposition 6, Corollary) that the scalar product of $F_m(G_h, G_{h'})$ with a normalized Hecke eigenform $f \in S_k$ is given by

$$(F_m(G_h, G_{h'}), f) = \frac{(k-2)!}{(2i)^{k-1}(k-2-m)!} r_{k-2-m}(f) r_{h'+m-1}(f).$$

The case when h or h' equals 2 is not mentioned explicitly in [1] or [6], but the above assertions remain true then also, as one proves by the same method as in the general case but using "Hecke's trick" to define G_2 as $\lim_{s \to 0} \sum (\cdots)^{-2} |\cdots|^{-s}$. Putting all this together gives the desired result (18).

Remarks. 1. The calculation we have given and the result we have proved are essentially restatements of Theorem 3 of [3] and its proof. The difference is that there we insisted on obtaining cusp forms and therefore had to modify $F_m(G_h, G_{h'})$ by subtracting a multiple of G_k when $m = 0$, with the consequence that the final formulas obtained were much more complicated and could not be combined conveniently into a generating function.

2. By combining (16) and (17), we get a simple closed formula for the Taylor coefficients of $C(X, Y; \tau; T)$ with respect to both q and T (i.e., for the expression (3) with "S_k" replaced by "M_k"). We state this as

Main Theorem (*Third version*). *For $k \geqq 2$ and $l \geqq 0$ let $c_{kl}(X, Y) \in \bigoplus_{\substack{-1 \leqq m,n \leqq k-1 \\ m \not\equiv n \pmod 2}} \mathbb{Q} X^m Y^n$ denote the coefficient of q^l in $c_k(X, Y; \tau)$. Then*

$$c_{k0}(X, Y) = -(k-2)!\,[p_k^+(X)p_k^-(Y) + p_k^-(X)p_k^+(Y)]$$

with $p_k^\pm$ as in the proposition of §2, while for $l > 0$

$$c_{kl}(X, Y) = \Bigg(\sum_{\substack{ad+bc=l \\ a,b,c,d>0}} (-cXY + dY + aX + b)^{k-2} - \frac{2}{k-1} \sum_{\substack{ad=l \\ a,d>0}} B_{k-1}^0(aX + dY) \Bigg) \Bigg| (1 - S_X)(1 - S_Y),$$

where $B_{k-1}^0(X) = \sum_{\substack{l=0 \\ l \text{ even}}}^{k-2} \binom{k-1}{l} B_l X^{k-1-l}$ is the $(k-1)$st Bernoulli polynomial with the term $(k-1)B_1X^{k-2}$ removed and $|S_X$, $|S_Y$ are the operators $\phi(X, Y) \mapsto X^{k-2}\phi(-X^{-1}, Y)$, $\phi(X, Y) \mapsto Y^{k-2}\phi(X, -Y^{-1})$.

The two formulas of this theorem give a closed formula for the polynomial (3), which is just the cuspidal part $c_{kl}^0 = c_{kl} + \frac{2k}{B_k}\sigma_{k-1}(l)c_{k0}$ of $c_{kl}(X, Y)$.

5. Properties of $C(X, Y; \tau; T)$ and examples

In this section we take the main theorem in the form (5) or (8) and discuss what consequences can be drawn from it.

In the first place, since $\theta_\tau(u)$ and $G_k(\tau)$ have rational coefficients as power series in u and q, it follows immediately from either version of the identity that all of the coefficients of $X^m Y^n$ in (3) are rational for all m and n lying between 0 and $k-2$ and hence that $r_m(f)r_n(f)/i(f,f) \in \mathbb{Q}_f$ for all Hecke eigenforms f and for all m and n of opposite parity. We also get integrality statements, e.g., that the coefficients of (4) with respect to X, Y and q are p-integral for all primes $p \geqq k$. (Numerical examples will be given at the end of this section.) Moreover, as already mentioned in the introduction, either (5) or (8) gives a completely algorithmic way of obtaining a basis of Hecke eigenforms for S_k and their period polynomials for any k. Specifically, either of these formulas gives a formula for

$$c_k^0(m, n, l) = \sum_{\substack{f \in S_k \\ \text{eigenform}}} \frac{1}{(2i)^{k-3}(k-2)!(f,f)} r_m(f)r_n(f)a_f(l) \tag{20}$$

for any four integers k, m, n and l satisfying $0 \leqq m, n \leqq k-2$, $m \not\equiv n \pmod 2$, $l \geqq 1$. The fact that knowing these numbers permits one to find the Fourier coefficients

$a_f(l)$ and periods $r_m(f)$ of the *individual* Hecke eigenforms $f(\tau)$, which at first sight may appear surprising, is a consequence of the following lemma, applied to $V_1 = S_k$, $V_2 = r^+(S_k) \subset W_k^+$, and $V_3 = W_k^-$:

Lemma. *Let $V_i (1 \leqq i \leqq 3)$ be three complex vector spaces of the same dimension d and suppose given an element ξ of $V_1 \otimes V_2 \otimes V_3$. If ξ has the form $\sum_{\nu=1}^{d} a_\nu^1 \otimes a_\nu^2 \otimes a_\nu^3$ with respect to some bases $\{a_\nu^i\}$ of V_i $(i = 1, 2, 3)$, then these bases are uniquely determined up to simultaneous permutation and scalar multiplication $a_\nu^i \mapsto \lambda_\nu^i a_\nu^i$, $\prod_{i=1}^{3} \lambda_\nu^i = 1$.*

The proof is easy, either by using the matrix representation of a transition between two bases with the given property or—more invariantly—by observing that the elements of the dual basis of $\{a_\nu^3\}$ are (up to scalars) the unique elements $\phi \in V_3^* = \mathrm{Hom}(V_3, \mathbb{C})$ for which $(1 \otimes 1 \otimes \phi)(\xi) \in V_1 \otimes V_2 \otimes \mathbb{C} \cong \mathrm{Hom}(V_1^*, V_2)$ corresponds to a homomorphism of rank 1. The process of finding the bases $\{a_\nu^i\}$ from ξ is algorithmic and will be illustrated numerically in our context at the end of the section.

Secondly, as already mentioned in §4, the fact that the non-modular form G_2 cancels out in substituting equation (7) into (5) implies that the right-hand side of (5) is multiplied by $(c\tau + d)^2$ under $\tau \mapsto \dfrac{a\tau + b}{c\tau + d}$, $T \mapsto (c\tau + d)^{-1} T$ for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and hence that the coefficient of T^{k-2} is a modular form of weight k in τ for every k. (One must also check that this coefficient contains no negative powers of q, but this is clear from (8).)

Thirdly, one sees directly from (8) that $C(X, Y; \tau; T)$ is the product of $c_0(X, Y)$ and a power series in T, XT, and YT which is invariant under $(X, Y) \mapsto (-X, -Y)$ or $(X, Y) \mapsto (Y, X)$ and is congruent to 1 modulo X or Y. This shows that each coefficient $c_k(X, Y; \tau)$ $(k > 0)$ is symmetric in X and Y and contains only monomials $X^m Y^n$ with $m \not\equiv n \pmod 2$ and $-1 \leqq m \leqq k - 1$, $-1 \leqq n \leqq k - 1$. Moreover, by looking at the extreme coefficients in the exponent in (8), we easily find the coefficients of all monomials with m or n equal to -1 or $k - 1$; these coefficients are multiples of G_k as calculated in §2. For instance, expanding (8) to the first two terms in Y gives

$$C(X, Y; \tau; T) =$$
$$-\frac{(1 - XY)(1 + X^{-1}Y)}{XY^2T^2}\left[1 - \sum_{k=2}^{\infty} \frac{2G_k(\tau)}{(k-1)!} X(X^{k-2} - 1) Y T^k + O(Y^2)\right],$$

so the coefficient of Y^{-1} in $c_k(X, Y; \tau)$ is $\dfrac{2G_k(\tau)}{(k-1)!}(X^{k-2} - 1)$ if $k \geqq 2$.

Fourthly, one can "see" the period relations $r_f|(1 + S) = r_f|(1 + U + U^2) = 0$ $(f \in M_k)$ directly from equation (5), although not without some work. These two relations are equivalent to the two identities

$$C(X, Y; \tau; T) + C\left(\frac{-1}{X}, Y; \tau; TX\right) = 0,$$
$$C(X, Y; \tau; T) + C\left(1 - \frac{1}{X}, Y; \tau; TX\right) + C\left(\frac{1}{1 - X}, Y; \tau; T(1 - X)\right) = 0.$$

The first of these is immediately obvious from (5) or (8) (using $\theta(-u) = -\theta(u)$ in the former case). The second, after multiplying through by the common denominator (a product of six theta functions), is the special case

$$\begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 \\ \beta_0 & \beta_1 & \beta_2 \end{pmatrix} = T\cdot\begin{pmatrix} 1 & X-1 & -X \\ 1 & Y-1 & -Y \end{pmatrix}$$

of the following theta series identity:

Proposition. *Let α_i, β_i $(i \in \mathbb{Z}/3\mathbb{Z})$ be six numbers satisfying $\sum_i \alpha_i = \sum_i \beta_i = 0$. Then*

$$\sum_i \theta(\alpha_i)\theta(\beta_i)\theta(\alpha_{i-1} + \beta_{i+1})\theta(\alpha_{i+1} - \beta_{i-1}) = 0\,.$$

Proof. One of Riemann's theta formulae (cf. [5], formula (R_5), p. 18) says

$$2\theta_{11}(x_1)\theta_{11}(y_1)\theta_{11}(u_1)\theta_{11}(v_1) = \sum_{i,j=0}^{1} (-1)^{i+j}\theta_{ij}(x)\theta_{ij}(y)\theta_{ij}(u)\theta_{ij}(v), \quad (21)$$

where x, y, u and v are arbitrary and

$$x_1 = \tfrac{1}{2}(x+y+u+v), \quad y_1 = \tfrac{1}{2}(x+y-u-v),$$

$$u_1 = \tfrac{1}{2}(x-y+u-v), \quad v_1 = \tfrac{1}{2}(x-y-u+v)\,.$$

The θ_{ij} are Jacobi theta functions whose definition is irrelevant here except that $\theta_{11}(u) = \theta(u)$ and that the other three $\theta_{ij}(u)$ are even functions of u. Therefore replacing v by $-v$ in (21), subtracting, and dividing by 2, we get

$$\theta(x_1)\theta(y_1)\theta(u_1)\theta(v_1) - \theta(x_2)\theta(y_2)\theta(u_2)\theta(v_2) = \theta(x)\theta(y)\theta(u)\theta(v),$$

where $x_2, \ldots, v_2$ are defined like $x_1, \ldots, v_1$ but with v replaced by $-v$. Up to renaming the variables, this identity is the same as the one in the proposition.

Finally, in support of the claim made in the introduction that the identity (5) contains all information about Hecke theory for $PSL_2(\mathbb{Z})$, we mention that it is possible to derive the Eichler-Selberg trace formula for the traces of Hecke operators on S_k from (5). The formula that comes out is rather different from the standard one and in some ways more elementary (for instance, no class numbers appear explicitly), and the calculation that relates it to the classical formula is rather amusing. Since the derivation is somewhat intricate, we have given it in a separate paper [8]. As a further application of (5), we also give in that paper an explicit formula for the action of Hecke operators on period polynomials of modular forms, generalizing a result of Manin on $r_0(f \mid T_l)$.

We end with a few numerical examples in weights $k \leqq 18$. Since dim $M_k \leqq 2$ in this range, we need only expand (5) up to terms in q^1, and the calculation of

$c_k(X, Y; \tau)$ up to this order is obtained immediately from the expansions given in §4. Subtracting the Eisenstein part c_k^E as given in §2, we find the values

k	$r_f^+(X)r_f^-(Y)/(2i)^{k-3}(f,f)$
12	$-2\left[\frac{36}{691}p_{12}^+(X) - q^+(X)\right]\left[q^-(Y)\right]$
16	$-\frac{13}{15}\left[\frac{360}{3617}p_{16}^+(X) - (2X^4 - X^2 + 2)q^+(X)\right]\left[(9Y^4 - 5Y^2 + 9)q^-(Y)\right]$
18	$\frac{4}{3}\left[\frac{18000}{43867}p_{18}^+(X) - (8X^4 - 9X^2 + 8)q_0^+(X)\right]\left[(6Y^4 - 7Y^2 + 6)q_0^-(Y)\right]$

for the unique normalized cusp form f of weights 12, 16 and 18 (we have given only $r_f^+(X)r_f^-(Y)$, since $(r_f(X)r_f(Y))^-$ is the sum of this polynomial and the one obtained by permuting X and Y). Here $p_k^+(X)$ denotes the polynomial $X^{k-2} - 1$ as in §2 and the polynomials $q^{\pm}$, $q_0^{\pm}$ are defined by

$$q^+(X) = X^2(X^2 - 1)^3, \quad q^-(X) = X(X^2 - 1)^2(X^2 - 4)(4X^2 - 1)$$

and $q_0^{\pm}(X) = (X^2 + 1)q^{\pm}(X)$. The fact that $r_f^+(X)$ modulo $p_k^+(X)$ and $r_f^-(X)$ are divisible by $q^+(X)$ and by $q^-(X)$, respectively, and also by $X^2 + 1$ if $k/2$ is odd, is an exercise in the use of the period relations $r_f|(1 + S) = r_f|(1 + U + U^2) = 0$ and is left to the reader. These properties can be translated using (5) into identities for theta series which can of course also be proved directly; for instance, the fact that $r_f(X)$ is divisible by $X - 1$ for all cusp forms f says that the function

$$C(1, Y; \tau; T) = \frac{\theta(XT - T)\theta(XT + T)\theta'(0)^2}{\theta(T)^2\theta(XT)^2}$$

has no cuspidal part, which is true because the elliptic function $\dfrac{\theta(u - v)\theta(u + v)}{\theta(u)^2\theta(v)^2}$ equals $\theta'(0)^{-2}(\wp(v) - \wp(u))$ (compare poles and zeros!) and because the Weierstrass $\wp$-function has a Laurent expansion involving only Eisenstein series. Notice also that the only large denominators occurring in the table are the numerators 691, 3617 and 43867 of B_k which occur as denominators in the coefficient of $p_k^+(X)$ in $r_f^+(X)$. These are cancelled by the Eisenstein part $\dfrac{2k}{B_k}\left(p_k^+(X)p_k^-(Y) + p_k^-(X)p_k^+(Y)\right)$, in accordance with the integrality properties mentioned at the beginning of the section.

We also give one example for $k = 24$, the first case with $\dim S_k > 1$, to illustrate the lemma at the beginning of the section. We will not give all of the periods of the two Hecke eigenforms, since there are many of them and they involve rather large numbers, but just enough data to show how the calculation works. If $\lambda = a_{f_1}(l)$ is the eigenvalue of one of the normalized Hecke eigenforms $f_1 \in S_{24}$ under some

Hecke operator T_l, then for any even integers m, m' and odd integers n, n' between 0 and 22, the matrix

$$\begin{pmatrix} c_{24}^0(m,n,l) & c_{24}^0(m,n',l) \\ c_{24}^0(m',n,l) & c_{24}^0(m',n',l) \end{pmatrix} - \lambda \begin{pmatrix} c_{24}^0(m,n,1) & c_{24}^0(m,n',1) \\ c_{24}^0(m',n,1) & c_{24}^0(m',n',1) \end{pmatrix} \tag{22}$$

(with $c_k^0(m,n,l)$ as in (20)) is proportional to the rank 1 matrix

$$\begin{pmatrix} r_{f_2}(m)r_{f_2}(n) & r_{f_2}(m)r_{f_2}(n') \\ r_{f_2}(m')r_{f_2}(n) & r_{f_2}(m')r_{f_2}(n') \end{pmatrix},$$

where f_2 is the other eigenform, and hence has determinant 0. Taking $m=0$, $m'=2$, $n=1$, $n'=3$, and $l=2$, we find from (5) and a computer algebra package that the matrix (22) equals

$$\begin{pmatrix} \dfrac{10165460749\lambda - 1426218953304}{922477119112821838325760000} & \dfrac{-2230037\lambda + 345194520}{307513728157618225152000} \\ \dfrac{-152819\lambda + 22554024}{7025004548610048000000} & \dfrac{12907\lambda - 2092008}{9016048618536960000} \end{pmatrix}.$$

The determinant of this is $\dfrac{569(\lambda^2 - 1080\lambda - 20468736)}{703674815509353343908721225236480000000}$, giving the two eigenvalues $\lambda = 540 \pm 12\sqrt{144169}$ of T_2 on S_{24}. Once one has these, the rest of the computation of the numbers $\dfrac{r_{f_v}(m)r_{f_v}(n)}{22!(2i)^{21}(f_v,f_v)} \in \mathbb{Q}(\sqrt{144169})$ $(v=1,2)$ is straightforward, although the numbers are too messy to give here.

Finally, we remark that a more natural way of writing period polynomials is as homogeneous polynomials in two variables. We can identify V_k with the space of homogeneous polynomials $\Phi(X_1,X_2)$ of degree $k-2$ (and similarly $\hat{V}_k$ with the space of functions $X_1^{-1}X_2^{-1}\Psi(X_1,X_2)$, Ψ a homogeneous polynomial of degree k) by

$$\phi(X) \mapsto \Phi(X_1,X_2) = X_2^{k-2}\phi(X_1/X_2), \quad \Phi(X_1,X_2) \mapsto \phi(X) = \Phi(X,1).$$

Under this identification the action $\phi \mapsto \phi|_{2-k}\gamma$ becomes simply $\Phi \mapsto \Phi \circ \gamma$, where γ acts as a matrix on the column vector $\binom{X_1}{X_2}$, and the formulas for $r_f^{\pm}$ and $p_k^{\pm}$, as well as the main theorem (in any of the three versions), become more symmetrical and more natural. We have preferred to use the inhomogeneous language because it is notationally somewhat simpler.

References

1. Cohen, H.: Sums involving the values at negative integers of L-functions of quadratic characters. Math. Ann. **217**, 271–285 (1975)
2. Eichler, M., Zagier, D.: The Theory of Jacobi Forms. Progr. Math. Vol. 55, Boston Basel Stuttgart: Birkhäuser 1985
3. Kohnen, W., Zagier, D.: Modular forms with rational periods. In: Rankin, R.A. (ed.) Modular Forms, pp. 197–249. Chichester: Ellis Horwood 1984
4. Lang, S.: Introduction to Modular Forms. Berlin Heidelberg New York: Springer 1976

5. Mumford, D.: Tata Lectures on Theta I. Prog. Math. Vol. 28, Boston Basel Stuttgart: Birkhäuser 1983
6. Zagier, D.: Modular forms whose Fourier coefficients involve zeta-functions of quadratic fields. In (eds) Serre, J-P., Zagier, D. Modular Forms of One Variable VI. (Lecture Notes Math. Vol. 627, pp. 105–169) Berlin Heidelberg New York: Springer 1977
7. Zagier, D.: The Rankin-Selberg method for automorphic functions which are not of rapid decay. J. Fac. Sci. Tokyo **28**, 415–438 (1982)
8. Zagier, D.: Hecke operators and periods of modular forms. Israel Mathematical Conference Proceedings **3**, 321–336 (1990)
9. Rankin, R.: The scalar product of modular forms. Proc. London Math. Soc. **2**, 198–217 (1982)