The Atiyah – Singer Theorem and Elementary Number Theory

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Foreword

Although you might never think so from the title, this book is about the relationship of certain corollaries of the Atiyah-Singer index theorem with some rather classical objects from the theory of numbers. That there is a connection was noticed more or less simultaneously by a good many people, including (independently) both of the authors. And since neither we nor any one else know why there should be one, it seemed an ideal subject for a book to present the enigma to the members of the mathematical community, for their puzzlement or entertainment as the case may be.

The book is largely based on a course given in 1970-71 by the senior author, but the writing-up was done by the junior author, and it is he who must take full responsibility for the mistakes, poor style, facetiousness and bad jokes, and for the regrettable fact that the final product is so much less coherent than the original lectures were.*

We would like to express our most hearty thanks to Mrs. Spanier for typing these notes and for showing so much patience with the bad manuscript and frequent alterations.

The first chapter is a discussion of a group of theorems in topology--the signature theorem, the holomorphic Lefschetz formula, and the G-signature theorem. These theorems, which we refer to collectively as "signature theorems," are all

* This is naturally not true. (F.H).

consequences of the Atiyah-Singer index theorem (though some of them preceded it historically), but the Atiyah-Singer theorem itself will not be formulated, nor are any of the signature theorems proved here. The whole first chapter contains no new results, and is essentially background for the remaining two chapters.

In Chapter II, we define and study certain elementary number-theoretical quantities, e.g.

$$s(q,p) = \frac{1}{4p} \sum_{k=1}^{p-1} \cot \frac{\pi k}{p} \cot \frac{\pi k q}{p} \tag{1}$$

$$t(a_1,\dots,a_n) = \sum_{k\geq 0} (-1)^k \sum_{\substack{0<x_1<a_1 \\ \vdots \\ 0<x_n<a_n \\ k<\frac{x_1}{a_1}+\dots+\frac{x_n}{a_n}<k+1}} 1 \tag{2}$$

$$(a_1,\dots,a_n \in \mathbb{Z},\ a_i>0).$$

These, and similar but more complicated expressions, occur in a variety of contexts in number theory and topology: $s(q,p)$, for instance, was studied by Dedekind in connection with modular forms, and $t(a_1,\dots,a_n)$ is the signature of a certain $(n-1)$-dimensional algebraic variety studied by Brieskorn. Moreover, despite the very different appearance of (1) and (2), they are in fact related: $s(q,p)$ can be rewritten in a rational form, and $t(a_1,\dots,a_n)$ can be expressed as a sum involving cotangents.

In Section 5, we describe some of the methods which can be used to study sums like (1) or (2). The remainder of Chapter II, which from the point of view of this book is really something of a digression, treats the relationship of these sums with other topics in number theory--the law of quadratic reciprocity, the study of modular forms, and the properties of Markoff triples (which arise in the theory of binary quadratic forms).

The rest of the book is then concerned with various topological situations where number-theoretical expressions of this sort arise. As we have already stated, sums like (1) or (2) can always be written in two ways: by an expression which is a rational number or an integer, and by a sum involving products of cotangents. This is reminiscent of the signature theorems of Chapter I, for in each of these theorems a topological invariant such as the signature is evaluated by means of characteristic classes whose definitions involve trigonometric functions. It is thus not particularly surprising that, by specialising the various signature theorems to specific manifolds and manifolds-with-group-action (e.g. complex projective space, lens spaces, and the above-mentioned Brieskorn varieties), one can obtain some of the number-theoretic results of Section 5. This is just what we do in Chapter III.

So the whole book consists of using the Atiyah-Singer theorem, one of the deepest and hardest results in mathematics, to prove a series of perfectly elementary identities which can be proved much more easily by direct means. This may seem to be a rather pointless course requiring justification. Of course, one can defend it simply by saying that both the number-theoretical and the topological ideas involved are important and far-reaching--Dedekind sums turn up in connection with topics ranging from class numbers of quadratic fields to logic and from the theory of modular forms to the problem of generating random numbers on computers, while the Atiyah-Singer index theorem probably has wider ramifications in topology and analysis than any other single result--and therefore any relationship between them, however nebulous, cannot fail to be of interest. Nevertheless, it would be nice, and would possibly have important consequences, if one could understand the real reasons for the relationship. Although no one yet has managed to explain, for instance, why the theory of the modular form $\Delta(z)$ and the index theorem for 4-manifolds should have anything to do with one another, at least some vague explanations of the connection are beginning to emerge. We will try to give some idea of this in the remainder of the introduction.

First of all, even on the formal level the link between the elementary number-theoretic identities and the Atiyah-Singer theorem has deeper roots than appears at first sight. All of the identities in question are based essentially on the Cauchy residue theorem (applied to rational functions on the Riemann sphere). The residue theorem is, so to speak, just a miniature version of the Atiyah-Singer theorem (as well as being a special case of it), for despite the great difference of depth the two theorems make essentially similar assertions: both express a global and topological invariant in terms of local differential data.

This "explanation" is, of course, rather metaphysical and unsatisfactory. But recently a series of quite concrete situations arose in which ideas related both to Dedekind sums and to signature theorems occurred. This culminated in the discovery that a certain invariant which had been defined using the signature theorem for 4-manifolds by Werner Meyer in Bonn is identical to another invariant which had been defined twenty years earlier by Curt Meyer (no relation) in Cologne using Dedekind sums. These two invariants, which assign rational numbers to elements of $SL(2,\mathbb{Z})$, had been invented for very different purposes--by W. Meyer to study certain torus bundles over the circle, and by C. Meyer to evaluate L(1) for certain L-functions associated to ideal classes of real quadratic fields (explicit Kronecker limit formula). These two seemingly unrelated aspects of the Meyer invariant turned out to tie up very closely with one another, in the following way:

To a real quadratic field is associated a certain discrete group (the Hilbert modular group) of automorphisms of $H \times H$ (where H is the upper half-plane). The quotient space of this action is a four-dimensional manifold which can be compactified by adding a certain number of cusps. The number of cusps needed is equal to the class number of the field, and each cusp is associated to an ideal class of the field and has an L-function associated to it. On the other hand, each cusp is a singular point of the compactified manifold and has a neighbourhood homeomorphic to the cone on a certain smooth 3-manifold which turns out to be a torus bundle over a circle. Then we can associate to the cusp an invariant by using either the topological or the number-theoretical Meyer definition, and the two agree. Similarly, the Hilbert modular group of a totally real algebraic number field K of degree $n > 2$ leads to an n-dimensional complex algebraic variety with $h(k)$ cusps. To each cusp corresponds a T^n-bundle over T^{n-1} (T^k = k-dimensional torus). To such a bundle is again associated a topological invariant (cf. Kreck [55]). But here it is not known how this invariant relates to the value at $s = 1$ of the corresponding L-series. (For a further discussion see [42], §3, and [46]).

Of course, even this only provides an example where the number theory and topology come together in a nice way: it does not provide any theoretical explanation why they should tie up. But now some very recent theorems of Atiyah, Singer and Patodi make it seem that such an explanation may have been found. We will try now to give some description of their ideas.

On the one hand, it has become increasing clear--both from the work of C. Meyer and from subsequent work on the Hilbert modular group--that the correct theoretical framework for the study of Dedekind sums is to consider them as giving the values of certain L-functions at $s = 1$. (It is also this point of view which explains the importance and ubiquitousness of Dedekind sums, for L-functions are among the basic objects of algebraic number theory.) On the other hand, it is possible to associate L-series to <u>elliptic operators:</u> instead of considering sums $\sum (N\mathfrak{a})^{-s}$ with $\mathfrak{a}$ ranging over some of the ideals of an algebraic number field, we consider sums $\sum \lambda^{-s}$, with λ ranging over some of the eigenvalues of an elliptic operator. Such Dirichlet series converge for s with $\mathrm{Re}(s)$ sufficiently large, and can be defined by analytic continuation for all s. Then the results of Atiyah, Singer and Patodi relate the values of these elliptic operator L-functions at $s = 0$

on the one hand to the values of certain of the number-theoretical L-functions at $s = 0$ (or $s = 1$, using the functional equations) and, on the other, to the index of the elliptic operator involved. To make this more precise, we state completely two of the Atiyah-Singer-Patodi results, though without explaining the meaning of the terms involved:

__THEOREM 1__ (Atiyah, Singer and Patodi [4]):
Let M be a $(4k-1)$-dimensional manifold and ρ a Riemannian metric on M. Then

$$\alpha(M,\rho) = \beta(m,\rho), \tag{3}$$

where the invariants α and β are defined as follows.

Choose $n \geq 1$ such that $n.M = \partial Y$ for some $4k$-manifold Y (the cobordism group Ω^{4k-1} is finite!) Extend ρ to a metric $\tilde{\rho}$ on Y which is a product metric in a tubular neighbourhood of ∂Y. Let $p_i(\tilde{\rho})$ $(i=1,\dots,k)$ be the Pontrjagin differential forms associated to this metric, and

$$L = L_k(p_1(\tilde{\rho}),\dots,p_k(\tilde{\rho})) \tag{4}$$

the corresponding L-polynomial (a differential form of degree $4k$ on Y). Then set

$$\alpha(M,\rho) = \frac{1}{n}\left[\int_Y L - \text{Sign } Y\right]. \tag{5}$$

That this is well defined, i.e. independent of the choice of n and Y, follows from the signature theorem and the additivity of the signature.

Now define a first-order self-adjoint operator $D: \Omega^{ev} \to \Omega^{ev}$ on M by $D = *d - d*$, so that $D^2 = \Delta$ (the Laplacian on M), and form the series

$$L_\rho(s) = \sum_{\lambda>0} 1/\lambda^s - \sum_{\lambda<0} 1/(-\lambda)^s ; \qquad (6)$$

here λ runs over the eigenvalues of D (which are real) with appropriate multiplicities. The series converges for $\mathrm{Re}(s) > \frac{1}{2} \dim M$ and defines a function which can be extended meromorphically to the entire complex plane and which is regular at $s = 0$. The invariant β is then defined by

$$\beta(M,\rho) = L_\rho(0). \qquad (7)$$

<u>THEOREM 2</u>: Let M be the T^2-bundle over S^1 defined by the automorphism of $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ induced by a matrix $A \in SL(2,\mathbb{Z})$ with $|\mathrm{Tr}\, A| > 2$. There is a natural metric ρ on M, and with respect to this metric

$$\beta(M,\rho) = 2\, L(0). \qquad (8)$$

Here $L(s)$ is the function defined by analytic continuation of the L-series

$$L(s) = \sum_{\mu \in \mathbb{Z}^2-\{0\}/A} \frac{\mathrm{sign}\, N(\mu)}{|N(\mu)|^s} \qquad (\mathrm{Re}\, s > 0), \quad (9)$$

where $N(\mu)$ is defined for μ represented by $(m,n) \in \mathbb{Z}^2-\{0\}$ as $\frac{1}{c}\{cm^2 + (d-a)mn - bn^2\}$.

Combining Theorems 1 and 2 with the functional equation for $L(s)$ and a formula for $L(1)$ due to C. Meyer, one obtains for the signature-related invariant $\alpha(M,\rho)$ an expression involving the Dedekind sum $s(d,c)$ (see [46]).

CHAPTER I: TOPOLOGICAL PRELIMINARIES

In this chapter we give a review of the topological material which will be needed in the following two chapters, namely those aspects of topology related to the signature theorem and its generalizations.

Subjects which are completely standard in topology, e.g. the theory of vector bundles and characteristic classes, will not be reviewed. We do, however, give the definitions and facts concerning sheaves over complex manifolds which are necessary to the statement of the Riemann-Roch theorem. We also describe, at the end of Section 1, the modifications required for the more general situation where a group acts on the manifolds and sheaves involved.

The signature theorems themselves are stated in the second section. We will only be interested in formulas expressing invariants associated to the cohomology ring of a manifold in terms of characteristic classes; thus there will be no mention of elliptic operators or of the general Atiyah-Singer index theorem.

We then discuss two definitions which are based on the various signature theorems: in Section 3 we give Thom's definition of an L-class for a space with the cohomological properties of a manifold, and show how to calculate this L-class for the quotient of a manifold by a finite group action, and in Section 4 we describe the properties of a number called the "α-invariant" which is associated to a free group action on an odd-dimensional manifold and whose existence is a consequence of the Atiyah-Singer G-signature theorem.

The contents of Section 1 and 2 are well known and can be found expounded at greater length in [36] and [3]. The contents of Sections 3 and 4 are also known, but are dispersed over a number of papers.

§1. Background on complex manifolds

This section contains fairly standard material on complex manifolds and their cohomology with coefficients in a sheaf or a holomorphic vector bundle. It is essentially to be regarded as a compendium of facts assembled for reference purposes; a complete exposition of the subjects covered (with the exception of the discussion in 1.4 of the equivariant situation) can be found in [36].

After a brief review of sheaves and of cohomology with coefficients in a sheaf (1.1), we give in 1.2 the main theorems on cohomology with coefficients in a bundle (Serre duality, etc.), using as a tool the fine resolution of the sheaf of germs of cross-sections of a bundle by means of sheaves of local differential forms. In 1.3 we discuss special properties of line bundles, such as the four-term formula and the Kodaira vanishing theorem, and define Hodge and Kähler manifolds. In 1.4, we restate some of the definitions and results for manifolds and bundles on which a group acts.

1.1. If $\mathfrak{S}$ is a sheaf over a topological space X (always assumed paracompact), we denote by $H^i(X,\mathfrak{S})$ the i^{th} cohomology group of X with coefficients in $\mathfrak{S}$, and by

$\Gamma(X,\mathfrak{S})$ the group of sections of $\mathfrak{S}$; then $\Gamma(X,\mathfrak{S})$ and $H^0(X,\mathfrak{S})$ are naturally isomorphic. If G is a group with the discrete topology we denote by $\underline{G}$ the trivial sheaf $X \times G$, and note that $H^i(X,\underline{G})$ is isomorphic to the usual cohomology group $H^i(X;G)$. If $\mathfrak{S}$ is a sheaf of $\mathbb{C}$-modules, then each $H^i(X,\mathfrak{S})$ is a complex vector space, and we say that $\mathfrak{S}$ is <u>of type (F)</u> if all of these vector spaces are finite-dimensional (over $\mathbb{C}$) and all but finitely many of them are zero. In that case we can define the <u>Euler characteristic</u> of $\mathfrak{S}$ by

$$\chi(X,\mathfrak{S}) = \sum_{i=0}^{\infty} (-1)^i \dim_{\mathbb{C}} H^i(X,\mathfrak{S}). \tag{1}$$

<u>THEOREM 1</u>: If

$$0 \to \mathfrak{S}' \xrightarrow{h'} \mathfrak{S} \xrightarrow{h} \mathfrak{S}'' \to 0 \tag{2}$$

is an exact sequence of sheaves over X, then there is an associated long exact sequence in cohomology:

$$0 \to H^0(X,\mathfrak{S}') \xrightarrow{h'_*} H^0(X,\mathfrak{S}) \xrightarrow{h_*} H^0(X,\mathfrak{S}'') \xrightarrow{\delta_*} H^1(X,\mathfrak{S}') \xrightarrow{h'_*} \cdots \tag{3}$$

<u>Corollary</u>: If two of the sheaves in (2) are of type (F), then so is the third and

$$\chi(X,\mathfrak{S}) = \chi(X,\mathfrak{S}') + \chi(X,\mathfrak{S}''). \tag{4}$$

<u>Definition</u> : The sheaf $\mathfrak{S}$ is <u>fine</u> if for any locally-finite covering $\{U_i\}_{i\in I}$ of X there are sheaf homomorphism $h_i:\mathfrak{S}\to\mathfrak{S}$ $(i\in I)$ with $\sum_{i\in I} h_i = \mathrm{id}$ such that $h_i | X-U_i$ is 0 (i.e. maps each stalk S_x to the 0 element of that stalk).

<u>THEOREM 2:</u> If $\mathfrak{S}$ is a fine sheaf, then

$$H^0(X,\mathfrak{S}) = \Gamma(X,\mathfrak{S}), \tag{5}$$

$$H^i(X,\mathfrak{S}) = 0 \qquad (i>0).$$

<u>Definition</u> : An exact sequence of sheaves

$$0 \longrightarrow \mathfrak{S} \xrightarrow{h} \mathfrak{S}_0 \xrightarrow{h^0} \mathfrak{S}_1 \xrightarrow{h^1} \mathfrak{S}_2 \longrightarrow \cdots \qquad (6)$$

is called a fine resolution of $\mathfrak{S}$ if each $\mathfrak{S}_p$ $(p \geq 0)$ is a fine sheaf. Then Theorem 2 and a diagram chase in the diagram of cohomology sequences induced by (6) yields immediately the following theorem, which is the main tool for computing the cohomology groups of a sheaf:

<u>THEOREM 3</u>: Let (6) be a fine resolution of the sheaf $\mathfrak{S}$ and

$$0 \longrightarrow \Gamma(X,\mathfrak{S}) \xrightarrow{h_*} \Gamma(X,\mathfrak{S}_0) \xrightarrow{h^0_*} \Gamma(X,\mathfrak{S}_1) \xrightarrow{h^1_*} \Gamma(X,\mathfrak{S}_2) \longrightarrow \cdots \qquad (7)$$

the induced sequence of groups of sections. Then

$$H^0(X,\mathfrak{S}) = \ker h^0_* , \qquad (8)$$

$$H^i(X,\mathfrak{S}) = (\ker h^i_*)/(\operatorname{im} h^{i-1}_*) \quad (i>0). \qquad (9)$$

We will be especially interested in the sheaf $\Omega(W)$ of germs of holomorphic sections of a holomorphic vector bundle W over a complex manifold X. In applying the notations introduced above to this sheaf, we omit the $\Omega(\ )$; thus we write

$$H^i(X,W) = H^i(X,\Omega(W)), \qquad (10)$$

and speak of "cohomology with coefficients in the bundle W," and similarly we write

$$\Gamma(X,W) = \Gamma(X,\Omega(W)) \qquad (11)$$

and (if $\Omega(W)$ is of type (F))

$$\chi(X,W) = \chi(X,\Omega(W)). \qquad (12)$$

$\Gamma(X,W)$ is the vector space of global holomorphic sections of the bundle W.

1.2 If X^m is an m-dimensional manifold, we define the tangent bundle ${}_{\mathbb{R}}\theta$ as the principal $GL(m,\mathbb{R})$-bundle given by coordinate transformations $\left(\partial x_r^{(i)} / \partial x_s^{(j)}\right)_{rs} \in GL(m,\mathbb{R})$ from the j^{th} to the i^{th} coordinate patch in X. The associated fibre bundles with fibre $\mathbb{R}^m$ and $\mathbb{C}^m$ (with the obvious action of $GL(m,\mathbb{R})$ on $\mathbb{C}^m$) are denoted ${}_{\mathbb{R}}T^*$ and ${}_{\mathbb{R}}T_{\mathbb{C}}^*$. Similarly, if X^n is a complex-analytic manifold of complex dimension n, we define a holomorphic tangent bundle θ by the coordinate transformations $f_{ij} = \left(\partial z_r^{(i)} / \partial z_s^{(j)}\right)_{rs} \in GL(n,\mathbb{C})$ and denote the associated C^n-bundle by T^* and the conjugate by $\overline{T}^*$ (coordinate functions $\overline{f}_{ij}$); the duals of these bundles coordinate transformations f_{ji}^{-1} and $\overline{f}_{ji}^{-1}$) are denoted T and $\overline{T}$ respectively. If we think of X as a real manifold (with dimension $m = 2n$) then ${}_{\mathbb{R}}T_{\mathbb{C}}^* = T^* \oplus \overline{T}^*$.

It follows from this description of the change of coordinate functions on a complex n-manifold X that a section ω of the p^{th} exterior power $\Lambda^p T$ is described in local coordinates by

$$\omega = \sum_{1 \leq i_1 < \dots < i_p \leq n} a_{i_1 \dots i_p}(z)\, dz_{i_1} \wedge \dots \wedge dz_{i_p} \tag{1}$$

so that $\Omega(\Lambda^p T)$ is the sheaf of germs of holomorphic p-forms on X and $\Gamma(X,\Lambda^p T)$ is the vector space of global holomorphic p-forms on X. We define

$$g_p = \dim_{\mathbb{C}} \Gamma(X,\Lambda^p T), \tag{2}$$

the number of linearly independent "forms of the first kind of degree p" on X. If X is a Riemann surface, g_1 is the genus.

The holomorphic bundle $\Lambda^p T$ just described will be especially important in the sequel. We apply all the previously introduced notations for a holomorphic bundle W to the tensor product $W \otimes \Lambda^p T$, simply adding a superscript p. Thus we write

$$\Omega^p = \Omega(\Lambda^p T), \quad (3)$$

$$\Omega^p(W) = \Omega(W \otimes \Lambda^p T) \approx \Omega(W) \otimes \Omega^p, \quad (4)$$

where $\mathcal{O} = \Omega(X \times \mathbf{C})$ is the structure sheaf of X (sheaf of germs of holomorphic complex-valued functions),

$$H^{p,q}(X,W) = H^q(X, W \otimes \Lambda^p T) = H^q(X, \Omega^p(W)), \quad (5)$$

$$h^{p,q}(X,W) = \dim H^{p,q}(X,W), \quad (6)$$

and, if $\Omega^p(W)$ is of type (F),

$$\chi^p(X,W) = \chi(X, \Omega^p(W)) = \sum_{q=0}^{\infty} (-1)^q h^{p,q}(X,W). \quad (7)$$

When we want to treat all of the numbers $\chi^p(W)$ simultaneously we introduce a dummy variable y and write (if all the $\Omega^p(W)$ are of type (F))

$$\chi_y(X,W) = \sum_{p \geq 0} y^p \chi^p(X,W) = \sum_{p,q} (-1)^q y^p h^{p,q}(X,W). \quad (8)$$

This is called the χ_y-characteristic of W. For $y = 0$ we have

$$\chi_0(X,W) = \chi^0(X,W) = \chi(X,W) = \sum_q (-1)^q h^{0,q}(X,W). \quad (9)$$

If $W = 1$ (trivial line bundle) we omit it from all notations; thus

$$\chi_0(X) = \chi^0(X) = \chi(X) = \sum_{q=0}^{\infty} (-1)^q h^{0,q}(X). \quad (10)$$

The integer (10) is the <u>arithmetic genus</u> of X.

Now let X be a real manifold, and $\mathfrak{A}^p$ the sheaf of germs of local differentiable sections of $\Lambda^p({}_{\mathbf{R}}T)$. There is then an exterior derivative

$$d: \mathfrak{A}^p \longrightarrow \mathfrak{A}^{p+1} \quad (11)$$

defined by

$$d\Big(\sum_{i_1<\dots<i_p} f_{i_1\dots i_p}(x)dx_{i_1}\wedge\dots\wedge dx_{i_p}\Big) = \sum_{i_1<\dots<i_p} df_{i_1\dots i_p}(x)\wedge dx_{i_1}\wedge\dots\wedge dx_{i_p}. \tag{12}$$

Now it is clear that $d^2 = 0$, and that the kernel of $d:\mathfrak{A}^0\to\mathfrak{A}^1$ is exactly the constant functions in $\mathfrak{A}^0$; this proves part of

THEOREM 1 (Poincaré lemma): The following sequence is exact:

$$0\to\mathbb{R}\xrightarrow{h}\mathfrak{A}^0\xrightarrow{h^0}\mathfrak{A}^1\xrightarrow{h^1}\mathfrak{A}^2\longrightarrow\cdots \tag{13}$$

Corollary (de Rham): Let $A^p = \Gamma(X,\mathfrak{A}^p)$ be the space of differentiable p-forms on X and $d:A^p\longrightarrow A^{p+1}$ the exterior derivative. Then

$$\begin{aligned} H^0(X,\mathbb{R}) &= \ker\,(d:A^0\to A^1),\\ H^p(X,\mathbb{R}) &= \ker\,(d:A^p\to A^{p+1})/\mathrm{im}(d:A^{p-1}\to A^p)\quad (p>0). \end{aligned} \tag{14}$$

The corollary is immediate since the sheaves $\mathfrak{A}^p$ are all fine. A similar result holds with $\mathbb{R}$ replaced by $\mathbb{C}$ if we consider forms with differentiable complex coefficients.

Now for X a complex manifold we have forms of type (p,q) which, expressed in local coordinates $z_1,\dots,z_n$, have p factors dz_i and q factors $d\bar{z}_j$. Let $\mathfrak{A}^{p,q}$ be the corresponding sheaf of germs of local differential forms of type (p,q), i.e. of local differentiable sections of the bundle $\Lambda^pT\otimes\Lambda^q\bar{T}$ (which is only a differentiable vector bundle, since $\bar{T}$ is not analytic). We define operators ∂ and $\bar{\partial}$ by formulas analogous to (12) but acting only on the z- or $\bar{z}$-variables, respectively; thus

$$\partial : \mathfrak{A}^{p,q} \to \mathfrak{A}^{p+1,q}, \tag{15}$$

$$\bar\partial : \mathfrak{A}^{p,q} \to \mathfrak{A}^{p,q+1}. \tag{16}$$

Then $\partial^2 = \bar\partial^2 = 0$ and $\partial\bar\partial = -\bar\partial\partial$, and so

$$d = \partial + \bar\partial \tag{17}$$

also has square zero. Since the $\bar\partial$-operator on a 0-form is just $\frac{\partial}{\partial \bar z}$, we find that $\ker(\bar\partial\, \mathfrak{A}^{p,0} \to \mathfrak{A}^{p,1})$ is given by germs $f(z)dz_{i_1} \ldots dz_{i_p}$ with $f(z)$ holomorphic; i.e. the sequence

$$0 \to \Omega(\wedge^p T) \to \mathfrak{A}^{p,0} \xrightarrow{\bar\partial} \mathfrak{A}^{p,1} \xrightarrow{\bar\partial} \mathfrak{A}^{p,2} \to \ldots \tag{18}$$

is exact at $\mathfrak{A}^{p,0}$. It can be shown that the whole sequence is in fact exact. Moreover, the $\mathfrak{A}^{p,q}$ are fine, so (18) is a fine resolution of Ω^p. Moreover, since $\bar\partial$ vanishes on holomorphic functions and since the transition functions of a complex analytic vector bundle W are holomorphic functions, we can tensor with $\Omega(W)$ without disturbing exactness. Therefore

<u>THEOREM</u> <u>2</u>: Let W be a holomorphic vector bundle over X and $\mathfrak{A}^{p,q}(W) = \mathfrak{A}^{p,q} \otimes_{\mathcal{O}} \Omega(W)$ be the sheaf of germs of local differential sections of the differentiable vector bundle $W \otimes \wedge^p T \otimes \wedge^q \bar T$. Then $\mathfrak{A}^{p,q}(W)$ is fine and

$$0 \to \Omega^p(W) \to \mathfrak{A}^{p,0}(W) \xrightarrow{\bar\partial} \mathfrak{A}^{p,1}(W) \xrightarrow{\bar\partial} \ldots \tag{19}$$

is a fine resolution of the sheaf $\Omega^p(W)$.

<u>Corollary:</u> If $p>n$ or $q>n$ ($n = \dim_{\mathbb{C}} X$), then $h^{p,q}(X,W) = 0$

The corollary follows immediately from Theorem 3 of 1.1 since $\mathfrak{A}^{p,q}(W) = \{0\}$ unless $p,q \leq n$.

We would now like to indicate what the Serre duality theorem says and how it is obtained. To do this we need another fine resolution of $\Omega^p(W)$, this time using the sheaf

$\mathfrak{K}^{p,q}$ of germs of differential forms of type (p,q) with distributional (rather than smooth) coefficients. The precise definition is as follows: if V is any complex differentiable vector bundle over X, we define $\mathcal{D}'(V|U)$ as the space of distributions on $V|U$, i.e. of linear functionals on the test space ${}_{0}\Gamma(X,V|U)$ of $\mathbb{C}^{\infty}$-sections of V with compact support contained in U. If $U' \subset U$, we let $i: {}_{0}\Gamma(X,V|U') \longrightarrow {}_{0}\Gamma(X,V|U)$ be the map induced by the inclusion, and $\Phi^{U}_{U'} : \mathcal{D}'(V|U) \longrightarrow \mathcal{D}'(V|U')$ be the adjoint of i. The spaces $\mathcal{D}'(V|U)$ and maps $\Phi^{U}_{U'}$, define a presheaf, and hence also a sheaf. If $V = \Lambda^{n-p}T \otimes \Lambda^{n-q}\bar{T}$, we denote this sheaf by $\mathfrak{K}^{p,q}$ (the sheaf of germs of local differential forms of type (p,q) with distributional coefficients); if $V = \Lambda^{n-p}T \otimes \Lambda^{n-q}\bar{T} \otimes W$ we denote it by $\mathfrak{K}^{p,q}(W)$. There is an isomorphism

$$\mathfrak{K}^{p,q}(W) \approx \Omega(W) \otimes_{\mathcal{O}} \mathfrak{K}^{p,q}. \tag{20}$$

Notice that there is a natural inclusion of $\mathfrak{U}^{p,q}$ in $\mathfrak{K}^{p,q}$, for on the presheaf level we can associate to a section ω of $\Lambda^{p}T \otimes \Lambda^{q}\bar{T}|U$ the distribution

$$\eta \longmapsto \int_X \omega \wedge \eta \quad (\eta \in {}_{0}\Gamma(X,\Lambda^{n-p}T \otimes \Lambda^{n-q}\bar{T}|U)).$$

(The integral is finite since η has compact support). The operator (16) extends to a map

$$\bar{\partial} : \mathfrak{K}^{p,q} \longrightarrow \mathfrak{K}^{p,q+1}, \tag{21}$$

and corresponding to Theorem 2 we have

<u>THEOREM</u> <u>3</u> : The sequence

$$0 \to \Omega^p(W) \to \mathfrak{K}^{p,0}(W) \xrightarrow{\bar\partial} \mathfrak{K}^{p,1}(W) \xrightarrow{\bar\partial} \mathfrak{K}^{p,2}(W) \to \dots \qquad (22)$$

is a fine resolution of $\Omega^p(W)$.

We now define a map

$$\wedge : \mathfrak{U}^{p,q} \otimes_{\mathcal{O}} \mathfrak{K}^{r,s} \longrightarrow \mathfrak{K}^{p+r,q+s} \qquad (23)$$

which extends the usual exterior product of differential forms with smooth coefficients. Over an open set U in X we define the exterior product as follows:

if $t \in \mathcal{D}'(\wedge^{n-r}T \otimes \wedge^{n-s}\bar{T}|U)$ and $a \in \Gamma(U, \wedge^p T \otimes \wedge^q \bar{T})$ then $a \wedge t \in \mathcal{D}'(\wedge^{n-r-p}T \otimes \wedge^{n-s-q}\bar{T}|U)$ is defined by

$$(a \wedge t)(\varphi) = t(a \wedge \varphi) \quad \text{for } \varphi \in {}_0\Gamma(X, \wedge^{n-r-p}T \otimes \wedge^{n-s-q}\bar{T}|U). \qquad (24)$$

This pairing on the presheaf level induces the map (23) of sheaves. We can also insert a holomorphic W as usual to obtain

<u>THEOREM 4</u> : Let W be a holomorphic vector bundle over X and W^* its dual. Then the exterior product $\wedge$ defines a pairing

$$\wedge : \mathfrak{U}^{p,q}(W) \otimes_{\mathcal{O}} \mathfrak{K}^{r,s}(W^*) \to \mathfrak{K}^{p+r,q+s} \qquad (25)$$

The definition of the exterior product has to be extended to distributional forms rather than just smooth ones in order that we obtain a duality map. Namely, integration over X

yields a map

$$K_c^{n,n} \longrightarrow \mathbf{C}, \tag{26}$$

where $K_c^{p,q}$ is the space of differential forms of type (p,q) with distributional coefficients and compact support. Combining this with (25) produces a map

$$\varepsilon: \quad A^{p,q}(W) \otimes K_c^{n-p,n-q}(W^*) \longrightarrow \mathbf{C}. \tag{27}$$

For $T \in K_c^{n-p,n-q}(W^*)$ we get from ε a linear mapping L_T from $\Gamma(X,\mathfrak{A}^{p,q}(W))$ to $\mathbf{C}$. Thus ε induces a map

$$L: K_c^{n-p,n-q}(W^*) \longrightarrow (A^{p,q}(W))'. \tag{28}$$

<u>THEOREM</u> <u>5</u> : The map L induced by the exterior product is an isomorphism. Furthermore, the diagram

$$\begin{array}{ccc} K_c^{n-p,n-q}(W^*) & \xrightarrow[\cong]{L} & (A^{p,q}(W))' \\ \bar{\partial} \downarrow & & \downarrow (\bar{\partial})' \\ K_c^{n-p,n-q+1}(W^*) & \xrightarrow[\cong]{L} & (A^{p,q-1}(W))' \end{array} \tag{29}$$

is commutative up to sign.

We are now ready to derive the Serre duality theorem. If we apply Theorem 3 of 1.1 to the fine resolutions (19) and (22 we obtain

$$H^{p,q}(X,W) \cong (\ker \bar{\partial}_q)/(\text{im } \bar{\partial}_{q-1}) \quad (\bar{\partial}_q = \bar{\partial}: A^{p,q}(W) \to A^{p,q+1}(W)) \tag{30}$$

and (using the isomorphism L and the commutativity of (29) to replace K_c^{**} by A^{**})

$$H^{n-p,n-q}(X,W^*) \cong (\ker \bar{\partial}'_{q-1})/(\text{im } \bar{\partial}'_q) . \tag{31}$$

But all of our spaces of cross-sections are Fréchet spaces, and it is known that if $L \xrightarrow{\mu} M \xrightarrow{\nu} N$ is a sequence of Fréchet spaces with $\nu\mu = 0$, and $N' \xrightarrow{\nu} M' \xrightarrow{\mu} L'$ the adjoint sequence, then $(\ker \nu)/(\mathrm{im}\,\mu) \cong (\ker \mu')/(\mathrm{im}\,\mu')$. Therefore

<u>THEOREM</u> <u>6</u>: (Serre duality): Let X be a connected paracompact complex manifold of (complex) dimension n, W a holomorphic vector bundle over X, and W^* the dual bundle. Then

$$H^{p,q}(X,W)' \cong H_c^{n-p,n-q}(X,W^*) \ . \tag{32}$$

If X is compact, we simply get a dual pairing between the spaces $H^{p,q}(X,W)$ and $H^{n-p,n-q}(X,W^*)$, the usual form of the Serre duality theorem. (In the compact case we do not need to ever introduce distributions). One can define an inner product on $A^{p,q}(W)$, and a map

$$\vartheta \ : \ A^{p,q}(W) \to A^{p,q-1}(W) \tag{33}$$

which is adjoint to $\bar\partial$ under this inner product (i.e. $(\alpha, \vartheta\beta) = (\bar\partial\alpha, \beta)$ for $\alpha, \beta \in A^{p,q}$). Then $\Box = \vartheta\bar\partial + \bar\partial\vartheta$ is an elliptic operator on $A^{p,q}$ (the complex Laplacian), the space $A^{p,q}$ decomposes as the orthogonal sum of the image of ϑ, the image of ∂, and the kernel of $\Box$, and the resolution (19) leads to an isomorphism $H^{p,q}(X,W) \cong \ker(\Box : A^{p,q}(W) \to A^{p,q}($
Since the kernel of an elliptic operator is finite-dimensional, we get (combining with corollary to Theorem 2):

<u>THEOREM</u> <u>7</u> (Kodaira): If X is compact, then $H^{p,q}(X,W)$ is isomorphic to the vector space of "complex harmonic forms" (elements in the kernel of $\Box$) of type (p,q) with coefficients in W. In particular it is finite-dimensional. Since $H^{p,q}(X,W) = \{0\}$ for $\max(p,q)>n$, the sheaves $\Omega^p(W)$ are always of type (F), so the numbers $\chi^p(X,W)$ are well-defined,

and $\chi_y(X,W)$ is a polynomial of degree at most n.
For X compact, the isomorphism (32) of finite-dimensional vector spaces gives also the equality of the dimensions:

$$h^{p,q}(X,W) = h^{n-p,n-q}(X,W^*). \tag{34}$$

Multiplying this by $(-1)^q$ and summing over q, we obtain

$$\chi^p(X,W) = (-1)^n \chi^{n-p}(X,W^*). \tag{35}$$

If we use the χ_y notation introduced in (8), we can express all of the equations (35) simultaneously as

$$\chi_y(X,W) = (-y)^n \chi_{1/y}(X,W^*), \tag{36}$$

an equality of polynomials of degree n.

1.3 In the framework of the ideas we have been discussing, and in their various applications, holomorphic line bundles play an especially important role. On the one hand, various special properties (the 1:1 correspondence of line bundles over X with elements of $H^2(X;\mathbb{Z})$, the property given in Theorem 1 below and the "four-term formula") make them especially easy objects to work with; on the other, they arise naturally in a variety of situations (Hopf bundle over $P_n(\mathbb{C})$; the line bundle of a divisor and its connection with the classical Riemann-Roch problem; the positive element in the second cohomology group of a Hodge manifold).

A line bundle is of course a principal $\mathbb{C}^*$-bundle, and is therefore (up to equivalence) classified by an element of $H^1(X,\mathbb{C}_c^*)$, where $\mathbb{C}_c^*$ is the sheaf of germs of local complex-valued continuous non-vanishing functions on X. Similarly, let $\mathbb{C}_c$ be the sheaf of germs of local continuous complex-valued functions, and $\underline{\mathbb{Z}}$ the constant sheaf (see 1.1); then clearly $\underline{\mathbb{Z}}$ is a subsheaf of $\mathbb{C}_c$, there is a map $\mathbb{C}_c \to \mathbb{C}_c^*$ locally given by $f \to e^{2\pi i f}$), and the sequence

$$0 \to \underline{\mathbb{Z}} \to \mathbb{C}_c \to \mathbb{C}_c^* \to 0 \qquad (1)$$

is exact. Moreover, $\mathbb{C}_c$ is a fine sheaf (define $h_i : \mathbb{C}_c \to \mathbb{C}_c$ by $h_i(f) = \alpha_i f$, where $\{\alpha_i\}_{i \in I}$ is a partition of unity subordinate to the covering $\{U_i\}_{i \in I}$), so that from Theorems 1 and 2 of 1.1 we get an isomorphism

$$\delta_* \; : \; H^1(X;\mathbb{C}_c^*) \xrightarrow{\cong} H^2(X;\mathbb{Z}). \qquad (2)$$

It can be checked (see [36], 4.3.1) that this is just the first Chern class

$$c_1 : \; H_1(X,\mathbb{C}_c^*) \longrightarrow H^2(X;\mathbb{Z})$$

when evaluated on the Hopf bundle over $X = P_n(\mathbb{C})$. Therefore (by naturality) the two maps agree for any X, proving the assertion that c_1 provides a 1:1 correspondence between line bundles and elements of $H^2(X;\mathbb{Z})$.

It is not generally true for holomorphic bundles that $\Lambda^p(W' \oplus W'') = \sum_{r+s=p} \Lambda^r(W') \otimes \Lambda^s(W'')$, but this does hold if one of the bundles is a line bundle. We can even say something about the exterior powers of an <u>extension</u> of a bundle by a line bundle:

<u>**THEOREM**</u> <u>1</u> : If

$$0 \longrightarrow F \longrightarrow W \longrightarrow W'' \longrightarrow 0 \qquad (4)$$

is an exact sequence of vector bundles (continuous, differentiable, or holomorphic) with F a line bundle, then there is a canonical exact sequence

$$0 \longrightarrow \Lambda^{p-1}W'' \otimes F \longrightarrow \Lambda^p W \longrightarrow \Lambda^p W'' \longrightarrow 0. \tag{5}$$

Similarly an exact sequence

$$0 \longrightarrow W' \longrightarrow W \longrightarrow F \longrightarrow 0 \tag{6}$$

with F a line bundle yields an exact sequence

$$0 \longrightarrow \Lambda^p W' \longrightarrow \Lambda^p W \longrightarrow \Lambda^{p-1}W' \otimes F \longrightarrow 0. \tag{7}$$

<u>Proof:</u> Given (4), there are natural homomorphisms

$$\alpha_1 : \Lambda^{p-1}W \otimes F \to \Lambda^p W, \tag{8}$$

$$\alpha_2 : \Lambda^{p-1}W \otimes F \to \Lambda^{p-1}W'' \otimes F, \tag{9}$$

and clearly α_2 is surjective and α_1 vanishes on ker α_2. This induces a map

$$\Lambda^{p-1}W'' \otimes F \longrightarrow \Lambda^p W, \tag{10}$$

and one easily checks that this map gives an exact sequence (5). The second assertion of the theorem follows from the first if we dualize (4) and (5) and write W', W, F for W''^*, W^*, F^*.

If X is a compact complex manifold, a <u>divisor</u> in X is a submanifold $S \subset X$ of (complex) codimension 1. The <u>associated line bundle</u> $\{S\}$ is given by transition functions $f_i/f_j : U_i \cap U_j \to \mathbb{C}$, where $f_i : U_i \to \mathbb{C}$ are holomorphic functions such that U_i has a system of local coordinates with f_i as one of the coordinates and such that $S \cap U_i = \{x \in U_i \mid f_i(x) = 0\}$. The bundle $\{S\}$ over X has a global section s given by

$$s(u) = (u, f_i(u)) \qquad (u \in U_i), \tag{11}$$

(this clearly transforms in the right way to give a section).

If W is any holomorphic vector bundle over X, we denote W_S the restriction of W to S. If $\mathfrak{S}$ is any sheaf over S, then $\hat{\mathfrak{S}}$ denotes the trivial extension of $\mathfrak{S}$ (= the unique sheaf with $\hat{\mathfrak{S}}|S = \mathfrak{S}$ and $\hat{\mathfrak{S}}|(X-S) = 0$).

<u>THEOREM 2</u>: The sequence

$$0 \longrightarrow \Omega(W) \xrightarrow{h'} \Omega(W \otimes \{S\}) \xrightarrow{h} \widehat{\Omega(W_S \otimes \{S\}_S)} \longrightarrow 0 \qquad (12)$$

is exact, where the map h' sends a section of W to its tensor product with s.

<u>Proof</u>: Clearly h' is a monomorphism since s does not vanish identically in any open set of X. Since s is non-zero on $X-S$, h' is an isomorphism there while the third sheaf is 0; therefore we only have to prove exactness on S. At a point $x \in S$, choose a neighbourhood U of x contained in a coordinate neighbourhood U_i and such that $W|U$ and $\{S\}|U$ are trivial. Identify them with $U \times \mathbf{C}^q$ and $U \times \mathbf{C}$, respectively, and identify $\mathbf{C}^q \otimes \mathbf{C}$ with $\mathbf{C}^q$ in the obvious way. Then an element of $\Omega(W)|U$ is a q-tuple of functions $(g_1, \ldots, g_q)$, and h' sends this to $(s_i g_1, \ldots, s_i g_q)$. The map h sends a q-tuple $(f_1, \ldots f_q)$ to its restriction to S. Therefore h is onto (the germ of a holomorphic function on S is the restriction of the germ of a holomorphic function on X, since in local coordinates S is given by the vanishing of one coordinate) and $\ker h = \{(f_1, \ldots f_q) | f = \ldots = f_q = 0 \text{ whenever } s_i = 0\}$

$$= \{(f_1, \ldots f_q) | \text{ each } f_r \text{ is divisible by } s_i\} = \operatorname{im} h'.$$

<u>Corollary:</u> $$\chi(S, W_S) = \chi(X, W) - \chi(X, W \otimes \{S\}^{-1}). \qquad (13)$$

<u>Proof</u> : Replace W by $W\otimes\{S\}^{-1}$ in (12) and apply 1.1(4).

Now the restriction $\{S\}_S$ is just the normal bundle of S in X, so there is an exact sequence of bundles

$$0 \longrightarrow T(S) \longrightarrow T(X)_S \longrightarrow \{S\}_S \longrightarrow 0. \qquad (14)$$

The dual sequence is

$$0 \longrightarrow \{S\}_S^{-1} \longrightarrow T^*(X)_S \longrightarrow T^*(S) \longrightarrow 0. \qquad (15)$$

Applying Theorem 1 gives an exact sequence

$$0 \longrightarrow \Lambda^{p-1}(T^*(S))\otimes\{S\}_S^{-1} \longrightarrow \Lambda^p(T^*(X))_S \longrightarrow \Lambda^p(T^*(S)) \longrightarrow 0. \quad (16)$$

Tensoring this with W_S and applying 1.1(4) gives

<u>THEOREM 3</u> : $\chi(S, \Lambda^p(T^*(X))_S\otimes W_S) = \chi^{p-1}(S, (W\otimes\{S\}^{-1})_S) + \chi^p(S, W_S)$.

(17)

Combining this with (13) applied to $\Lambda^p(T^*)\otimes W$ gives the <u>four-term formula:</u>

$$\chi^p(X,W) - \chi^p(X, W\otimes\{S\}^{-1}) = \chi^{p-1}(S, W_S\otimes\{S\}_S^{-1}) + \chi^p(S, W_S). \quad (18)$$

<u>Corollary:</u>

$$\chi^p(S, W_S) = \sum_{0\leq i\leq p} (-1)^i \left\{\chi^{p-i}(X, W\otimes\{S\}^{-i}) - \chi^{p-i}(X, W\otimes\{S\}^{-i-1})\right\}. \quad (19)$$

<u>Proof</u> : The corollary follows by iterating (18). Note that the case p = 0 is just the corollary to Theorem 2; thus (13) expresses the Euler characteristic of W_S in terms of information about X and W, and (19) does the same for the χ^p-characteristic. We can combine the equations (19) for all p by using the χ_y-characteristic, obtaining

$$\chi_y(S, W_S) = \chi_y(X,W) - (1+y) \sum_{i=1}^{\infty} (-y)^{i-1} \chi_y(X, W\otimes\{S\}^{-i}). \quad (20)$$

The last thing which we wish to discuss in this section is the Kodaira vanishing theorem. We first have to define a Kähler metric on a compact complex manifold X. A <u>Hermitian metric</u> on X is given in local coordinates by

$$ds^2 = 2\sum g_{\alpha\beta}(z,\bar z)dz_\alpha d\bar z_\beta \,. \tag{21}$$

(where $g_{\alpha\beta} = \bar g_{\beta\alpha}$). To it is associated an exterior differential form

$$\omega = i\sum g_{\alpha\beta}(z,\bar z)dz_\alpha \wedge d\bar z_\beta \,. \tag{22}$$

We write this as a real form (using $i\, dz\wedge d\bar z = dx\wedge dy$); then the condition that the metric is a <u>Kähler metric</u> is that $d\omega = 0$. In this case the class $\omega \in H^2(X;\mathbb{R})$ is called the fundamental class of the metric, and X (always assumed to be compact) is called a <u>Kähler manifold</u>. In this case the complex Laplace operator $\Box$ (see the remarks preceding Theorem 7 of 1.2) equals $\frac{1}{2}\Delta$, where Δ is the real Laplace operator, so $\Box$ commutes with conjugation and there is an anti-automorphism from $\ker(\Box: A^{p,q} \to A^{p,q})$ to $\ker(\Box: A^{q,p} \to A^{q,p})$ and hence (by Theorem 7 of 1.2) from $H^{p,q}(X)$ to $H^{q,p}(X)$. Moreover, the theory of de Rham and Hodge gives an isomorphism from $H^r(X;\mathbb{C})$ to $\bigoplus_{p+q=r} \ker(\Box: A^{p,q} \to A^{p,q})$. Therefore

<u>THEOREM 4</u>: Let X be a Kähler manifold. Then

$$h^{p,q}(X) = h^{q,p}(X) \tag{23}$$

and the r^{th} Betti number is given by

$$b_r(X) = \dim_{\mathbb{C}} H^r(X;\mathbb{C}) = \sum_{p+q=r} h^{p,q}(X) \tag{24}$$

<u>Corollary 1</u>: Let X be a Kähler manifold. Then

$$\chi_{-1}(X) = e(X), \tag{25}$$

where $e(X)$ is the ordinary Euler-Poincaré characteristic of the space X and $\chi_{-1}(X)$ is the value at $y=-1$ of $\chi_y(X)$.

<u>Proof</u>: From the theorem,

$$e(X) = \sum_r (-1)^r b_r(X) = \sum_{p,q} (-1)^{p+q} h^{p,q}(X)$$

$$= \sum_p (-1)^p \chi(X, \Omega^p). \tag{26}$$

However, the corollary holds for an arbitary compact complex manifold, as we see by applying 1.1(4) repeatedly to the following exact sequence of sheaves of type (F):

$$0 \longrightarrow \underline{C} \longrightarrow \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \quad \cdots \longrightarrow \Omega^n \longrightarrow 0. \tag{27}$$

<u>Corollary 2</u>: The arithmetic genus of a Kähler manifold X (defined in 1.2(10)) is related to the numbers g_p (defined in 1.2(2)) by

$$\chi(X) = \sum_p (-1)^p g_p. \tag{28}$$

Now on a Kähler manifold X, consider the sequence

$$0 \longrightarrow \underline{\mathbf{Z}} \longrightarrow \mathbf{C}_\omega \longrightarrow \mathbf{C}^*_\omega \longrightarrow 0, \tag{29}$$

defined just as was the sequence (1) but with $\mathbf{C}^*_\omega$ and $\mathbf{C}_\omega$ being sheaves of holomorphic rather than continuous complex functions. Since $\mathbf{C}_\omega = \Omega = \Omega^0$ is not fine, we do not get an isomorphism corresponding to (2), but the long exact sequence induced by (29) still yields a map from line bundles to $H^2(X;\mathbf{Z})$ which can be shown to equal the Chern class:

$$H^1(X,\Omega) \longrightarrow H^1(X,\mathbf{C}^*_\omega) \xrightarrow{c_1} H^2(X;\mathbf{Z}) \longrightarrow H^2(X,\Omega). \tag{30}$$

Now $H^q(X,\Omega) = H^q(X,1) = H^{0,q}(X)$, and Kodaira and Spencer have proved that $a \in H^2(X;\mathbf{Z})$ is mapped to zero in $H^2(X,\Omega)$ if and only if a is of type $(1,1)$. Therefore

THEOREM 5: An element $a \in H^2(X;\mathbb{Z})$, X a Kähler manifold, is the first Chern class of a complex analytic line bundle if and only if it is of type $(1,1)$.

Definition : An element $x \in H^{1,1}(X;\mathbb{R})$ is positive if x can be chosen as the fundamental class of a Kähler metric on X. An element x in $H^{1,1}(X;\mathbb{Z})$ is positive if its image in $H^{1,1}(X;\mathbb{R})$ is. A holomorphic line bundle F is positive if $c_1(F)$, which by Theorem 5 lies in $H^{1,1}(X;\mathbb{Z})$, is positive. A Hodge manifold X is a Kähler manifold admitting a Kähler metric with fundamental class in $\mathrm{im}(H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{R}))$.

THEOREM 6: A compact complex manifold is a Hodge manifold if and only if it is an algebraic manifold.

This fundamental theorem is due to Kodaira. One direction is easy, since to find a Hodge manifold structure on any algebraic manifold it suffices to find one on complex projective space (because an algebraic manifold is a submanifold of some complex projective space, and we can restrict the class x).

THEOREM 7: (Kodaira vanishing theorem): If F is a positive holomorphic line bundle over a Kähler manifold X of dimension n, then $H^i(X,F^{-1})$ vanishes for $i \neq n$.

An important line bundle over X is the canonical line bundle $K = \Lambda^n T$. The case $p = 0$ of the Serre duality theorem (Theorem 6 of 1.2) yields a duality pairing between $H^q(X;W)$ and $H^{n-q}(X,K\otimes W^*)$. Therefore we can reformulate Theorem 7:

THEOREM 8 : If X is a Kähler manifold with canonical line bundle K and F is a line bundle over X with $F\otimes K^{-1}$ positive, then $H^i(X,F) = 0$ $(i>0)$. In particular,

$$\dim \Gamma(X,F) = \chi(X,F). \tag{31}$$

1.4 As above, let W be a holomorphic vector bundle over a compact complex manifold X. A map g is said to be an <u>automorphism of (X,W)</u> if $g:X \to X$ is a biholomorphic map of manifolds and $g':W \to W$ a biholomorphic map of bundles (thus $g'|W_x$ is a vector space isomorphism $W_x \xrightarrow{\cong} W_{gx}$). Then g induces a map from $\Gamma(X,W)$ to itself by taking a section f of W to the section g^*f defined by

$$g^*f(x) = g'^{-1}f(gx) \tag{1}$$

Clearly if g_1 and g_2 are both isomorphisms then $(g_1g_2)^*$ equals $g_2^*g_1^*$, from which it follows that, since g is invertible, g^* also is. Similarly g induces automorphisms of the higher cohomology groups $H^q(X,W)$. We define

$$\chi(X,W;g) = \sum_q (-1)^q \operatorname{Tr}(g^* | H^q(X,W)). \tag{2}$$

Since a map g from X to itself induces a map dg on T, we see that an automorphism of (X,W) induces an automorphism of $(X, \Lambda^p T \otimes W)$. We define

$$\chi^p(X,W;g) = \chi(X, \Lambda^p T \otimes W; g), \tag{3}$$

$$\chi_y(X,W;g) = \sum_p y^p \chi^p(X,W;g). \tag{4}$$

If V is a real or complex vector space on which a group G acts, we denote by V^G the subspace of vectors invariant under the action of all the elements of G. We recall two simple facts:

<u>THEOREM</u>: If G is a finite group acting linearly on a finite-dimensional real or complex vector space V, then

$$\dim V^G = \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}(g|V). \tag{5}$$

__Proof:__ Let i be the inclusion of V^G in V, and $f: V \to V$ be the map

$$f(v) = \frac{1}{|G|} \sum_{g \in G} gv \tag{6}$$

Clearly $f(v)$ always lies in V^G, i.e. the map f factors as $i\pi$, where $\pi: V \to V^G$. But if $v \in V^G$, then $gv = v$ for all $g \in G$, so $f(v) = v$. Therefore $\pi i = \mathrm{id}_{V^G}$, so

$$\dim V^G = \mathrm{Tr}(\mathrm{id}_{V^G}) = \mathrm{Tr}(\pi i) = \mathrm{Tr}(i\pi) = \mathrm{Tr}\, f. \tag{7}$$

(Note: the theorem and proof are also valid for compact groups, if we replace the average over G by an integral with respect to the normalized Haar measure on G.)

__Proposition:__ If

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{8}$$

is an exact sequence of G-vector spaces and G-equivariant maps, then there is an exact sequence

$$A^G \xrightarrow{f|A^G} B^G \xrightarrow{g|B^G} C^G. \tag{9}$$

__Proof:__ Clearly $f(A^G) \subset B^G$, $g(B^G) \subset C^G$, and the composite map in (9) is zero. If $b \in B^G$ maps to zero, then $b = f(a)$ for some $a \in A$ (because (8) is exact), and then $a' = \pi(a)$ (in the notation of the proof just given) lies in A^G and satisfies $f(a') = b$.

We now return to our vector bundle W over X and suppose that G is a group of automorphisms of (X,W). We associate to this situation a number

$$\chi(X,W)^G = \sum_q (-1)^q \dim H^q(X,W)^G, \qquad (10)$$

and make similar definitions for $\chi^p(X,W)^G$ and $\chi_y(X,W)^G$. According to the theorem just proved, these numbers are not essentially new, but can be obtained from the numbers (2)-(4) by averaging over G.

All of the material covered in 1.1-1.3 now carries over to the equivariant case, because the required fine resolutions can be made equivariant. For example, we find (using the proposition above) that, if $0 \to W' \to W \to W'' \to 0$ is an exact sequence of G-vector bundles over X,

$$\chi(X,W)^G = \chi(X,W')^G + \chi(X,W'')^G. \qquad (11)$$

When quoting the equivariant version of any of the formulas of this section, we will add a superscript "G" to the number; thus $1.2(36)^G$ refers to the equivariant Serre duality theorem

$$\chi_y(X,W)^G = (-y)^n \chi_{1/y}(X,W^*)^G \quad (X \text{ compact}) \qquad (12)$$

and $1.3(18)^G$ refers to the equivariant four-term formula

$$\chi^p(X,W)^G - \chi^p(X,W \otimes \{S\}^{-1})^G = \chi^{p-1}(S,W_S \otimes \{S\}_S^{-1})^G + \chi^p(S,W_S)^G. \qquad (13)$$

§ 2. Signature theorems

We will discuss in this section various theorems expressing invariants of the cohomology of a manifold in terms of characteristic classes. The first theorem of this type, discovered in 1953, was the signature theorem, which states that the signature of a manifold (an integer defined by the intersection pairing in the middle dimension of the homology of the manifold) is given by a certain polynomial in the Pontrjagin classes of the tangent bundle of the manifold. This was followed by the Riemann-Roch theorem (also in 1953) expressing $\chi(X,W)$ and $\chi_y(X,W)$ (W a holomorphic vector bundle over a Hodge manifold X) in terms of the Chern classes of W and of the tangent bundle of X. Later the Atiyah-Singer index theorem made it possible to prove the Riemann-Roch theorem and the equality Sign $(X) = \chi_1(X)$ for any complex manifold, and also gave the G-signature theorem, which for g in a compact group G acting on X gives Sign(g,X) (a number defined by the action of g on the middle homology group of X) in terms of the Chern classes of X and of the normal bundle in X of the fixed-point set of g (and of the action of g on this normal bundle).

We define Sign(X) and Sign(g,X) in 2.1 and give their main properties (cobordism invariance, multiplicativity, Novikov additivity). In 2.2 we discuss multiplicative sequences, the device which produces from the Chern or Pontrjagin class the more complicated characteristic classes appearing in the statement of the various signature theorems. The signature themselves are stated in section 2.3. At the end of 2.3 the most general formula of this type (giving a formula for $\chi_y(X,W;g)$) is formulated; this is given for completeness' sake and because it is not written out explicitly in [3].

2.1 Let X^{2n} be a closed (=compact and without boundary) oriented smooth manifold. The bilinear form

$$B(x,y) = \langle (x \cup y), [X] \rangle \qquad (x,y \in V = H^n(X;\mathbb{R})) \qquad (1)$$

is, by Poincaré duality, a non-degenerate form on the real vector space V and is symmetric or skew-symmetric according as n is even or odd. In the former case, let p_+ and p_- be the number of positive and negative eigenvalues of B and define

$$\mathrm{Sign}\,(X) = p_+ - p_-.$$

If n is odd or is X is odd-dimensional, define $\mathrm{Sign}(X)$ to be 0.

If X has a boundary, we take $V = H^n(X,\partial X;\mathbb{R})$. Poincaré duality and the exact sequence of the pair $(X,\partial X)$ give

$$V = H^n(X,\partial X;\mathbb{R}) \xrightarrow{i^*} H^n(X;\mathbb{R}) = \mathrm{Hom}(H_n(X),\mathbb{R})$$

$$= \mathrm{Hom}(H^n(X,\partial X;\mathbb{R}),\mathbb{R}), \qquad (3)$$

i.e. a bilinear form B on V (B is just the intersection number if we use Poincaré duality to identify V with $H_n(X)$). This form can be degenerate, but it induces a non-degenerate form B' on $V' = \mathrm{im}\, i^* \cong V/\ker i^*$, where

$$\ker i^* = \{x \in V : B(x,y) = 0 \text{ for all } y \in V\}.$$

Clearly if n is even then B and B' have the same signature, and we take this as $\mathrm{Sign}(X)$; if $\dim X$ is not a multiple of 4 we define $\mathrm{Sign}(X) = 0$ again.

We summarize the main properties of $\mathrm{Sign}(X)$ in the next three theorems. The first is that Sign(X) depends only on the cobordism class of X:

<u>THEOREM 1</u>: If $X = \partial Y$ is the boundary of a differentiable manifold, then $\mathrm{Sign}(X) = 0$.

<u>Proof</u> : Let $j: X \subset Y$ be the inclusion map, and consider the diagram

$$\begin{array}{ccccc} H^{2k}(Y) & \xrightarrow{\;j^*\;} & H^{2k}(X) & \xrightarrow{\;\delta\;} & H^{2k+1}(Y,X) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ H_{2k+1}(Y,X) & \xrightarrow{\;\partial\;} & H_{2k}(X) & \xrightarrow{\;j_*\;} & H_{2k}(Y) \end{array}$$

(all cohomology with real coefficients), where the vertical maps are Poincaré duality maps and $\dim X = 4k$. Then $A = \mathrm{im}\, j^*$ is mapped isomorphically to $K = \ker j_*$, while under the duality between $H^{2k}(X)$ and $H_{2k}(X)$, A corresponds to $H_{2k}(X)/K$. Therefore

$$\dim A = \dim K = \dim H_{2k}(X) - \dim K$$

$$= \frac{1}{2} \dim H_{2k}(X). \qquad (5)$$

But for $a \in A$, $\langle a^2, [X]\rangle = \langle (j^* b)^2, [X]\rangle = \langle b^2, (j_* [X])\rangle = 0$, since $j_*[X]$ is 0 in $H_{4k}(Y)$. Therefore A cannot intersect the p_+-dimensional subspace (sum of the eigenspaces of all positive eigenvalues) on which B is positive definite, so we must have $(\dim A) + p_+ \leq \dim H_{2k}(X)$. The corresponding inequality for p_-, the fact $p_+ + p_- = \dim H_{2k}(X)$, and eq. (5) then give $p_+ = p_-$.

<u>THEOREM 2</u> : If X^n and Y^m are closed oriented manifolds,

$$\mathrm{Sign}(X \times Y) = \mathrm{Sign}(X)\, \mathrm{Sign}(Y) \qquad (6)$$

In particular $X \times Y$ has 0 signature if m and n are not both divisible by 4.

The proof of Theorem 2 is given in [36], and is quite simple. From the last two theorems we find that Sign induces a ring homomorphism from the cobordism ring Ω^* (cobordism classes of closed orientable manifolds; the addition, negative, and multiplication are given by disjoint union, reversed orientation, and Cartesian product, respectively) to the integers. By results of Thom, Ω^* is given up to torsion by the Pontrjagin numbers (i.e. $\Omega^* \otimes Q = \tilde{\Omega}^* \otimes Q$, where $\tilde{\Omega}^*$ is the ring of equivalence classes of manifolds, two manifolds being equivalent if they are of the same dimension and all their Pontrjagin numbers agree), so that the signature of a manifold must be a rational linear combination of its Pontrjagin numbers. This is the content of the signature theorem, but the explicit formula must wait till 2.3 since we need the notation of multiplicative sequences to state it.

Another property of Sign which we will need is the Novikov additivity law. The proof is again elementary and will be omitted; it is given in [3].

<u>THEOREM 3</u> ; (Novikov): Let X, X' be orientable manifolds with boundary, and $h: \partial X \to \partial X'$ an orientation-reversing diffeomorphism. Let $Z = X \cup_h X'$ be the manifold obtained by gluing X and X' together along the boundary. Then

$$\mathrm{Sign}(Z) = \mathrm{Sign}(X) + \mathrm{Sign}(X'). \tag{7}$$

To define the equivariant signature Sign(g,X), we first consider the situation of a group G acting on (V,B), where V is a finite-dimensional real vector space and B is a non-degenerate symmetric or skew-symmetric bilinear form on V. Non-degenerate means that the map from V to $V^* = \mathrm{Hom}(V,\mathbb{R})$ sending x to $B(x,-)$ is bijective. To say that G acts on (V,B) means that $B(gx,gy) = B(x,y)$ for all $g \in G$. We choose an equivariant scalar product $\langle\, ,\rangle$ on V (this is always possible if G is a compact Lie group, by integrating over G the translates of any scalar product), and define a map A from V to itself by

$$\langle Ax,y\rangle = B(x,y). \tag{8}$$

Then

$$\langle A(gx),gy\rangle = B(gx,gy) = B(x,y) = \langle Ax,y\rangle = \langle gAx,gy\rangle, \tag{9}$$

so A is equivariant, and

$$\langle A^*x,y\rangle = \langle x,Ay\rangle = \langle Ay,x\rangle = B(y,x) = \pm B(x,y)$$

$$= \pm\langle Ax,y\rangle, \tag{10}$$

so that A is self-adjoint or $A = -A^*$ according as B is symmetric or skew-symmetric. Since A is G-equivariant, G operates on each eigenspace of A ($Ax = \lambda x$ implies $A(gx)=g(Ax)= \lambda(gx)$). If B is symmetric we define V^+ as the direct sum of the eigenspaces of positive eigenvalues and V^- the same for negative eigenvalues and define

$$\mathrm{Sign}(g,V) = \mathrm{tr}(g|V^+) - \mathrm{tr}(g|V^-). \tag{11}$$

Thus in this case

$$\mathrm{Sign}(1,V) = \mathrm{Sign}\ B. \tag{12}$$

If however, B is skew-symmetric, then we must proceed differently as we can no longer diagonalize A. The operator AA^* is (strictly) positive-definite since $\langle AA^*x,x\rangle = \langle A^*x,A^*x\rangle > 0$ for $x \neq 0$, so we can form its positive-definite square root and define

$$J = \frac{A}{\sqrt{AA^*}}. \tag{13}$$

Then $A^* = -A$ implies $J^2 = -1$, so that we obtain on V the structure of a <u>complex</u> vector space V_J, and since the operations of G clearly commute with J we find that the action of G on V_J is complex linear. Thus we have a trace $\mathrm{tr}(g|V_J) \in \mathbb{C}$, and can define

$$\mathrm{Sign}(g,V) = \mathrm{tr}(g|V_J) - \overline{\mathrm{tr}(G|V_J)}$$

$$= 2i\ \mathrm{Im}(\mathrm{tr}(g|V_J)). \tag{14}$$

In this case the signature vanishes for $g = \mathrm{id}$. One can check that the two definitions (12) and (14) depend only on g, V, and B and not on the scalar product chosen. One can also easily check that, if (V_1,B_1) and (V_2,B_2) are as described above, then the signature of g on $V = V_1 \otimes V_2$, $B = B_1 \otimes B_2$ is given by

$$\mathrm{Sign}(g,V_1 \otimes V_2) = \mathrm{Sign}(g,V_1)\mathrm{Sign}(g,V_2), \tag{15}$$

regardless of the symmetry properties of B_1, B_2, and B.

Now if G acts on a compact manifold X^{2n} (smoothly and preserving the orientation), then it acts on the real vector space $V' = \mathrm{im}(i^*: H^n(X, \partial X) \longrightarrow H^n(X))$ described at the beginning of the section, and leaves invariant the symmetric or skew-symmetric non-degenerate intersection form B', so we can define

$$\mathrm{Sign}(g,X) = \mathrm{Sign}(g,V'). \tag{16}$$

If X is odd-dimensional we define $\mathrm{Sign}(g,X)$ as zero. As a minor point, we notice that G acts on X to the left and therefore on V' to the right (i.e. $(g_1 g_2)(x) = g_2(g_1(x))$) since cohomology is contravariant; to remedy this we replace the action by $x \to g^{-1}x$ which is a left action (this corresponds to identifying the cohomology group with a homology group by Poincaré duality and considering g_* on homology). This changes nothing if $\dim X \equiv 0 \pmod 4$, since $\mathrm{tr}(g^{-1}) = \mathrm{tr}(g)$ for a finite-dimensional <u>real</u> representation of a finite or compact group. But for a complex representation we have $\mathrm{tr}(g^{-1}) = \overline{\mathrm{tr}(g)}$, so for $\dim X \equiv 2 \pmod 4$ the sign of (16) changes under $g \to g^{-1}$.

All of the theorems on $\mathrm{Sign}(X)$ go through in the equivariant case; thus $\mathrm{Sign}(g,X)$ is 0 if X is an equivariant boundary,

$$\mathrm{Sign}(g,X_1 \times X_2) = \mathrm{Sign}(g,X_1)\,\mathrm{Sign}(g,X_2) \tag{17}$$

(now we have more possible combinations of dimensions than for $g = 1$, since X_i can have any even dimension), and

$$\mathrm{Sign}(g,X_1 \cup_h X_2) = \mathrm{Sign}(g,X_1) + \mathrm{Sign}(g,X_2) \tag{18}$$

for the union of two G-manifolds glued along a common boundary by an equivariant orientation-reversing diffeomorphism h.

If g is an involution we can simplify the definition of Sign(g,V), and do not need to introduce an invariant scalar product at all. Indeed, if B is skew-symmetric then

$$\mathrm{Sign}(g,V) = \mathrm{tr}(g|V_J) - \mathrm{tr}(g^{-1}|V_J) = 0 \qquad (19)$$

since $g^{-1} = g$. If B is symmetric, then so is

$$\widetilde{B}(x,y) = B(gx,y) \qquad (20)$$

(because $g^2 = \mathrm{id}$), and (by a trivial calculation, since $g|V$ can be diagonalized),

$$\mathrm{Sign}(g,V) = \mathrm{Sign}\ \widetilde{B}. \qquad (21)$$

This defines Sign(g,V) even when B is degenerate, and shows that it is an integer.

Now assume that G is finite and acts freely on X (i.e. $gx = x$ for some $x \in X$ implies $g = 1$); then X/G is an oriented manifold (with orientation class $\frac{1}{|G|}\ \pi_*[X]$, where $\pi : X \to X/G$ is the projection), and we have

<u>THEOREM</u> <u>4</u> : Let G be a finite group acting freely on a closed manifold X. Then

$$\mathrm{Sign}(X/G) = \frac{1}{|G|} \sum_{g \in G} \mathrm{Sign}(g,X). \qquad (22)$$

<u>Proof</u>: If $\dim X$ is odd, both sides are 0 by definition If $\dim X \equiv 2 \pmod 4$ then the left-hand side is 0 by definition, while

$$\mathrm{Sign}(g,X) = -\mathrm{Sign}(g^{-1},X) \qquad (23)$$

by the remark made above. If $\dim X$ is divisible by 4, then the right-hand side is

$$\frac{1}{|G|} \sum_{g \in G} \left[tr(g|V^+) - tr(g|V^-)\right] = \dim(V^+)^G - \dim(V^-)^G \qquad (24)$$

by the theorem of 1.4 (eq. 1.4(5)). Theorem 4 then follows from

THEOREM 5: Let G be a finite group acting on a closed manifold X, not necessarily freely, and let $\pi : X \to X/G$ be the projection. Then the induced map

$$\pi^* : \quad H^*(X/G) \longrightarrow H^*(X) \qquad (25)$$

is an isomorphism from $H^*(X/G)$ onto the invariant subspace $H^*(X)^G$, if the cohomology is taken with coefficients in $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$ (or any field of characteristic 0 on which G acts trivially).

This was proved by Grothendieck [31] and is now quite standard (it is quoted in [37] and proved in [5]).

If G does not act freely, then X/G is not a manifold but (essentially by Theorem 5) has the local real or rational cohomology of a manifold. Such objects will be discussed in §4, but it is already clear that they have a cup product and fundamental class, and therefore that $\mathrm{Sign}(X/G)$ can be defined. Then the identical proof shows that Theorem 4 still is valid with the assumption of freeness dropped.

We give two more properties of the signature. One is a fairly special fact which we give here only because we will need it for an application in Chapter 3. The other is a basic result, namely:

THEOREM 6: Let X be a compact complex manifold and g a biholomorphic automorphism of X. Then

$$\mathrm{Sign}(g,X) = \chi_1(X;g) \qquad (26)$$

The case $g = \mathrm{id}$, X a Kähler manifold, was proved by Hodge[47]; the proof is given as Theorem 15.8.2 of [36]. The extensions to arbitrary complex manifolds and to the equivariant case follow from the work of Atiyah and Singer, and will be discussed in 2.3.

We now state and prove the proposition mentioned above:

THEOREM 7 : Let X be a closed oriented n-manifold and W an oriented real m-dimensional vector bundle over X, with m and n even and $m+n$ divisible by four. Let B_W be the disc bundle associated to W (an $(m+n)$-dimensional manifold with boundary), $e \in H^m(X;\mathbb{Z})$ the Euler class of W, and $\mathrm{Sign}(X,e)$ the signature of the quadratic form

$$x,y \longrightarrow \langle x \cup y \cup e\,,[X]\rangle \quad (x,y \in H^{(n-m)/2}(X;\mathbb{R})). \qquad (27)$$

Then

$$\mathrm{Sign}(B_W) = \mathrm{Sign}(X,e) \qquad (28)$$

Proof: Over $\mathbb{Z}$, we have the Thom isomorphism

$$\Phi : H^*(X) \xrightarrow{\cong} H^*(B_W, \partial B_W). \qquad (29)$$

The space on the right is a module over $H^*(X) = H^*(B_W)$, free of rank 1 with generator $U \in H^m(B_W, \partial B_W)$. The map (29) is given by cup product with U. Then the definition of the Euler class $e(W)$ is

$$e = \Phi^{-1}(U \cup U), \qquad (30)$$

or $e \cup U = U \cup U$. Under the Thom isomorphism the quadratic form defining $\mathrm{Sign}(X,e)$ is carried into that defining $\mathrm{Sign}(B_W)$, because

$$\langle (x \cup y \cup e), [X] \rangle = \langle (x \cup y \cup e \cup U), [B_W, \partial B_W] \rangle$$
$$= \langle (x \cup y \cup U \cup U), [B_W, \partial B_W] \rangle$$
$$= \langle (-1)^m (\Phi(x) \cup \Phi(y)), [B_W, \partial B_W] \rangle. \qquad (31)$$

Therefore the signatures are the same.

2.2 Let $\underline{S}_n$ be the set of finite unordered n-tuples of non-zero complex numbers. We define a semiring structure on $\underline{S} = \bigcup \underline{S}_n$ by defining

$$\{\alpha_1, \ldots, \alpha_r\} + \{\beta_1, \ldots, \beta_s\} = \{\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s\} \qquad (1)$$

$$\{\alpha_1, \ldots, \alpha_r\} \times \{\beta_1, \ldots, \beta_s\} = \{\alpha_i \beta_j : 1 \leq i \leq r, 1 \leq j \leq s\}. \qquad (2)$$

This clearly makes $\underline{S}$ a commutative semiring and the map $\underline{S} \longrightarrow Z^+$ which maps $\underline{S}_n$ to n is a homomorphism of semirings. Let $\underline{S}'$ be the set of polynomials with complex coefficients with leading coefficient 1. The $\underline{S}'$ and $\underline{S}$ can be put in 1:1 correspondence by

$$f(x) = \prod_{i=1}^{n} (1 + \alpha_i x) \longleftrightarrow \{\alpha_1, \ldots, \alpha_n\} \in \underline{S}_n \qquad (3)$$

(we had to exclude $\alpha_i = 0$ in defining $\underline{S}$ to make this correspondence unique), so we obtain a semiring structure on $\underline{S}'$. Denote its operations by $\boxplus$ and $\boxtimes$; then if

$$f(x) = 1 + \sum_{i=1}^{n} a_i x^i = \prod_{i=1}^{n} (1 + \alpha_i x), \qquad (4)$$

$$g(x) = 1+\sum_{j=1}^{m} b_j x^j = \prod_{j=1}^{m} (1 + \beta_j x), \tag{5}$$

we have

$$f \boxplus g(x) = \prod_i (1+ \alpha_i x) \prod_j (1+ \beta_j x) = f(x)g(x), \tag{6}$$

$$f \boxtimes g(x) = \prod_{i,j} (1+ \alpha_i \beta_j x)$$

$$= 1 + (a_1 b_1)x + (a_1^2 b_2 + a_2 b_1^2 - 2a_2 b_2)x^2 +$$

$$(a_1^3 b_3 + a_1 a_2 b_1 b_2 + a_3 b_1^3 - 3a_1 a_2 b_3 - 3a_3 b_1 b_2 + 3a_3 b_3)\, x^3 + \dots \tag{7}$$

The coefficient of x^r in $f \boxplus g$ or $f \boxtimes g$ is thus given by a certain universal polynomial $k_r(a_1,\dots,a_r;b_1,\dots,b_r)$ in the first r coefficients of f and g. This polynomial is clearly independent of the degrees n and m, and has rational (indeed, integral) coefficients.

Since (7) can also be written

$$f \boxtimes g(x) = \prod_{j=1}^{m} f(\beta_j x), \tag{8}$$

we also have a definition of $f \boxtimes g(x)$ as a power series in x when f is a power series and g is a polynomial. But using the polynomials $k_r(\underline{a};\underline{b})$ above, it is clearly possible to define $f \boxtimes g$ even when f and g are both infinite series. A more direct way to do this is to take logarithms and write

$$\log f(x) = \sum_{i=1}^{n} \log(1+ \alpha_i x) = \sum_{i=1}^{n} \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r} \alpha_i^r x^r$$

$$= \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r} s_r(f)x^r, \tag{9}$$

where $s_r(f) = \sum \alpha_i^r$ can be written in a well-known way as a polynomial in $a_1,\ldots,a_r$. Then clearly

$$s_r(f \boxplus g) = s_r(f) + s_r(g), \tag{10}$$

$$s_r(f \boxtimes g) = s_r(f)s_r(g), \tag{11}$$

Thus our semiring structure on $1 + x\mathbb{C}[[x]]$ = {formal power series in x with leading coefficient 1) is given by using

$$f(x) = \exp\left[\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r} s_r x^r\right] \longmapsto \{s_1,s_2,\ldots\} \tag{12}$$

to identify $1 + x\mathbb{C}[[x]]$ with the set of sequences of complex numbers, and giving the latter set the ring structure of the direct product of countably many copies of $\mathbb{C}$ (i.e. coordinatewise addition and multiplication).

For fixed f, the map

$$g \longrightarrow \hat{g} = f \boxtimes g \tag{13}$$

defines a map from $1 + x\mathbb{C}[[x]]$ to itself such that the coefficient of x^r in $\hat{g}$ is given as a polynomial in the coefficients of $x,x^2,\ldots,x^r$ in g and such that

$$\widehat{g_1 g_2} = (g_1 \boxplus g_2) \boxtimes f = (g_1 \boxtimes f) + (g_2 \boxtimes f) = \hat{g}_1\hat{g}_2. \tag{14}$$

(using the distributive law of $\boxtimes$ over $\boxplus$ and (6)). That is, (13) defines a "multiplicative sequence with characteristic power series f" in the sense of [36]. However, the old notation obscures the symmetry between the function f defining the multiplicative sequence and the function g to which it is applied.

(Digression: Observe that (9) can be written in the form

$$\sum_{r=1} s_r(f)\, x^r = -x \frac{d}{dx} \log f(-x) \qquad (15)$$

a formula that will be familiar to topologists as identical to that defining the Adams operations ψ^k from the λ^k-operations. Then the usual equations

$$\psi^k(\xi \oplus \eta)) = \psi^k(\xi) + \psi^k(\eta), \qquad (16)$$

$$\psi^k(\xi \otimes \eta)) = \psi^k(\xi) \times \psi^k(\eta), \qquad (17)$$

stating that ψ^k is a ring homomorphism can be translated, using the formalism given above, to the statement that the power series

$$\lambda_y(\xi) = \sum_i \lambda^i(\xi)\, y^i \qquad (18)$$

satisfy

$$\lambda_y(\xi \oplus \eta) = \lambda_y(\xi) \boxplus \lambda_y(\eta) = \lambda_y(\xi)\ \lambda_y(\eta) \qquad (19)$$

$$\lambda_y(\xi \otimes \eta) = \lambda_y(\xi) \boxtimes \lambda_y(\eta) \qquad (20)$$

Thus our multiplication $\boxtimes$ has appeared before in a topological setting.)

Note that if f only involves powers of x^r, then so does $f \boxtimes g$; this can be seen from (8) or directly from (7) since a polynomial in x^r is exactly one whose set of roots is invariant under multiplication by $e^{2\pi i/r}$.

Finally, note that the definition (8) makes sense (i.e. gives a result only depending on f and g) even if $f(x)$ does not have leading coefficient 1. However, in the previous definition we could introduce spurious factors 1 by writing $f(x) = \prod(1 + \alpha_i x)$ with some of the α_i equal to 0 (i.e. with the number of indices i greater than the degree of f) without changing $f \boxtimes g$, while in (8) a factor $(1 + 0x)$ multiplies $f \boxtimes g(x)$ by $f(0)$. Also it is clearly not possible to define $f \boxtimes g$ if $f(0) \neq 1$ and g is an infinite power series. Thus we can allow a multiplicative sequence $g \to f \boxtimes g$ defined by a power series f with $f(0) \neq 1$ so long as it is only applied to g of finite and specified "degree" (not necessarily the largest power of x appearing in g with a non-0 coefficient). Such a multiplicative sequence will be called <u>non-stable</u>.

We now define characteristic classes. Recall that a $U(q)$- bundle ξ has <u>Chern classes</u>

$$c_i(\xi) \in H^{2i}(X;\mathbb{Z}) \tag{21}$$

which are natural (i.e. $c_i(g^*\xi) = g^*c_i(\xi)$ for any map $g:Y \longrightarrow X$) and satisfy $c_0(\xi) = 1$, $c_i(\xi) = 0$ for $i>q$.

Now let K be one of the fields $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$, and $f(x)$ a power series with coefficients in K and leading coefficient 1. Define

$$\Phi_i(\xi) \in H^{2i}(X;K) \tag{22}$$

by

$$\sum_{i=0} \Phi_i(\xi)\, x^i = f(x)\otimes \left\{\sum_i c_i(\xi)\, x^i\right\} \qquad (23)$$

The map Φ_i from {complex bundles over X} to $H^{2i}(X)$ is called a <u>characteristic class</u>; we denote by $\Phi(\xi)$ the total characteristic class $\sum \Phi_i(\xi) \in H^{ev}(X)$. From the multiplicativity and naturality of the Chern classes, we deduce (using (14)) the corresponding properties for Φ, namely

$$\Phi(\xi \oplus \eta) = \Phi(\xi)\Phi(\eta), \qquad (24)$$

$$\Phi(g^*\xi) = g^*\Phi(\xi). \quad (g:Y \longrightarrow X). \qquad (25)$$

If η is a line bundle, then it follows from (23) that

$$\Phi(\eta) = f(c_1(\eta)). \qquad (26)$$

Properties (24)-(26) constitute the usual axiomatic definition of a characteristic class Φ. They define Φ uniquely because of the splitting principle, which asserts that, for any bundle ξ over X, there is a continuous map $g:Y \to X$ such that $g^*\xi$ splits into a sum of line bundles and such that $g^*: H^*(X) \to H^*(Y)$ is a monomorphism.

Notice that the polynomial $\sum_i c_i(\xi)\, x^i$ has a well-defined degree $q = \dim \xi$, even though the "top" coefficient $c_q(\xi)$ may vanish. Therefore we can also allow power series $f(x)$ with $f(0) \neq 1$ in (23), obtaining non-stable characteristic classes.

To take two trivial examples, the characteristic class c given by the power series $1+x$ is simply the Chern class, and the characteristic class c^* given by $1-x$ is simply the Chern class of the dual, since $c_i(\xi^*) = (-1)^i c_i(\xi) = c_i^*($

The characteristic classes we will need are:

i) The <u>Pontrjagin class</u> $\tilde{p}(\xi)$, given by $f(x) = 1 + x^2$. Since $f(x)$ is a polynomial in x^2, we deduce from the remark following (20) that $\tilde{p}_i(\xi) = 0$ for i odd. We write $p_i(\xi) = \tilde{p}_{2i}(\xi) \in H^{4i}(X)$, in agreement with the usual notation.

ii) The <u>Euler class</u> $e(\xi)$, given by $f(x) = x$. If ξ is a $U(q)$-bundle, then

$$e(\xi) = c_q(\xi) \quad \varepsilon \quad H^{2q}(X). \tag{27}$$

iii) The <u>L-class</u> $\mathcal{L}(\xi)$, given by $f(x) = \dfrac{x}{\tanh x}$.

iv) The <u>Todd class</u> $td(\xi)$, given by

$$f(x) = \frac{x}{1 - e^{-x}} = e^{x/2}\left[\frac{x/2}{\sinh x/2}\right]. \tag{28}$$

v) The class $T_y(\xi)$, given by

$$f_y(x) = \frac{x(1+y)}{1 - e^{-x-xy}} - xy = \frac{x(1+y)}{e^{x+xy} - 1} + x$$

$$= x\,\frac{1 + ye^{-x-xy}}{1 - e^{-x-xy}}. \tag{29}$$

Here y is any real number. Since $f_y(x)$ is a power series in x and xy, the coefficient of x^n in $f_y(x)$ is a polynomial in y of degree $\leqslant n$, so the component in $H^{2i}(X)$ of $T_y(\xi)$ is a polynomial in y of degree at most i. By setting $y = 0, 1, -1$ in (29), we find

$$T_0(\xi) = td(\xi), \tag{30}$$

$$T_1(\xi) = \mathcal{L}(\xi), \tag{31}$$

$$T_{-1}(\xi) = c(\xi), \tag{32}$$

vi) The class $\widetilde{T}_y(\xi)$, given by

$$\widetilde{f}_y(x) = (1+y)\, f_y\left(\frac{x}{1+y}\right) = x\, \frac{1+ye^{-x}}{1-e^{-x}} . \tag{33}$$

The components of degree $2r$ of $T_y(\xi)$ and $\widetilde{T}_y(\xi)$ differ only by a factor $(1+y)^{r-q}$. Thus for $y \neq -1$ the two characteristic classes are essentially the same (up to a scalar factor in each dimension), while for $y = -1$ they are different. We have

$$\widetilde{T}_0(\xi) = td(\xi) \tag{34}$$

$$\widetilde{T}_{-1}(\xi) = e(\xi) \tag{35}$$

vii) The class $\mathcal{L}_\theta(\xi)$, given by $f(x) = \coth(x + i\theta/2) = (e^{i\theta}e^{2x} + 1)/(e^{i\theta}e^{2x} - 1)$, where θ is a real number not divisible by 2π. In particular $\mathcal{L}_\pi(\xi)$ is given by the power series

$$(e^{2x}-1)/(e^{2x}+1) = x\,(x/\tanh x)^{-1}, \text{ so}$$

$$\mathcal{L}_\pi(\xi) = e(\xi)\,\mathcal{L}(\xi)^{-1} \tag{36}$$

(this makes sense since $\mathcal{L}(\xi)$ has leading coefficient 1 and is therefore invertible).

viii) The class $ch\,\lambda_y(\xi)$, given by $f(x) = 1+ye^x$. (The notation refers to the fact that this is the Chern character

applied to the multiplicative sequence (18) in K-theory.)
From the last equality in (33) we obtain

$$\widetilde{T}_y(\xi) = td(\xi)\cdot ch\ \lambda_y(\xi^*). \tag{37}$$

ix) The class $\mathcal{U}_\theta(\xi)$, given by $f(x) = (1-e^{-x-i\theta})^{-1}$, where θ is a real number not divisible by 2π.

x) The class $T_y^\theta(\xi)$, given by

$$f(x) = \frac{1 + ye^{-i\theta-x(1+y)}}{1 - e^{-i\theta-x(1+y)}}, \tag{38}$$

with y and θ as before. Thus

$$T_1^\theta(\xi) = \mathcal{L}_\theta(\xi) \tag{39}$$

$$T_{-1}^\theta(\xi) = 1 \tag{40}$$

$$T_0^\theta(\xi) = \mathcal{U}_\theta(\xi) \tag{41}$$

xi) The class $\widetilde{T}_Y^\theta(\xi)$, given by

$$f(x) = \frac{1 + ye^{-i\theta-x}}{1 - e^{-i\theta-x}}. \tag{42}$$

Of all the characteristic classes we have defined, only the Pontrjagin class, Todd class, and T_y-class are stable (i.e. defined by power series with leading coefficient one),

Observe that any characteristic class given by a power series in x^2 can be obtained from the Pontrjagin class rather than the Chern class; for instance, the L-class can be obtained from the Pontrjagin class by applying to it the multiplicative sequence with characteristic power series

$\sqrt{x}/\tanh \sqrt{x}$. This is important because the Pontrjagin class can be defined for real bundles: if ξ is an $O(q)$-bundle, then $\xi \otimes \mathbb{C}$ is a $U(q)$-bundle and we set $p_i(\xi) = (-1)^{i/2} \times c_i(\xi \otimes \mathbb{C})$. This is consistent with the previous definition, i.e. a complex q-dimensional bundle can be thought of as a real 2q-dimensional bundle, and then the two definitions of the Pontrjagin class agree. It follows that the L-class is also defined for real bundles.

The Euler class is also defined for oriented real bundles (i.e. $SO(q)$-bundles), by a definition too familiar to be repeated here. If ξ is non-oriented, the Euler class is also defined but is an element of a cohomology group with "twisted coefficients". It then follows from (36) that $\mathcal{L}_\pi(\xi)$ can be defined for $O(q)$-bundles (lying in $H^*(X;\mathbb{Q})$ if ξ is an $SO(q)$-bundle, in a group with twisted coefficients otherwise). However, $\mathcal{L}_\theta(\xi)$ for θ not an odd multiple of π is defined only if ξ is a complex bundle.

2.3 In this section we shall state the signature, Riemann-Roch, and G-signature theorems, which give formulas for Sign(X), $\chi_y(X,W)$, and Sign(g,X) respectively. Since Sign(X) = Sign(1,X) and (for X a complex manifold) Sign(X) = $\chi_1(X)$, the signature theorem is a special case of both the R.R. and the G-signature theorems. However, we shall still state it separately, since we wish to discuss it in more detail than the generalizations which follow it:

THEOREM 1: For a closed oriented manifold X with tangent bundle ${}_{\mathbb{R}}\theta$ (cf. 1.2),

$$\mathrm{Sign}(X) = \langle \mathcal{L}(X), [X] \rangle \qquad (1)$$

Here $\mathcal{L}$ is the characteristic class defined in 2.2, where it was pointed out that it is defined for real bundles, and $\mathcal{L}(X)$ denotes $\mathcal{L}({}_{\mathbb{R}}\theta(X)) \in H^*(X;\mathbb{Q})$.

Notice that the fact that Sign(X) is a cobordism invariant (Theorem 1 of 2.1) follows from (1). Indeed, if $X = \partial Y$ then the double Z of Y (= $Y \cup -Y$ glued along X) is a smooth manifold in which X has trivial normal bundle, so

$$\mathrm{Sign}(X) = \mathcal{L}({}_{\mathbb{R}}\theta(X))[X] = \mathcal{L}(i^*{}_{\mathbb{R}}\theta(Z))[X]$$

$$= \mathcal{L}({}_{\mathbb{R}}\theta(Z))i_*([X]) = 0, \qquad (2)$$

since the cycle represented by X is homologous to zero in Z. In fact this calculation is used in the reverse direction one uses the fact that Sign(X) is a cobordism invariant and defines a ring homomorphism from the cobordism ring to the integers (cf. 2.1) to prove that it can be written as $\Phi({}_{\mathbb{R}}\theta(X))[X]$ for some characteristic class Φ.

Since ${}_{\mathbb{R}}\theta$ is a real bundle, Φ can only depend on the Pontrjagin class i.e. Φ is given by an even power series $f(x)$. To find $f(x)$, one uses the special case $X = P_n(\mathbb{C})$. Here there is a class $x \in H^2(X)$ with $x^n[X] = 1$ and with $c(X) = (1+x)^{n+1}$. Therefore $\mathcal{L}({}_{\mathbb{R}}\theta(X)) = f(x)^{n+1}$ and we require coefficient of x^n in $f(x)^{n+1} = \mathrm{Sign}(P_n(\mathbb{C}))$.

But x generates the cohomology of X, so the signature of X is 1 if n is even and 0 if n is odd. Since $f(x)$ must begin with 1, we can define a power series g by

$$x = g(y), \qquad y = x/f(x). \tag{4}$$

Then (3) can be written

$$\operatorname{res}_{x=0}\left(\frac{f(x)^{n+1}}{x^{n+1}}\,dx\right) = \operatorname{res}_{y=0}\left(\frac{g'(y)}{y^{n+1}}\,dy\right)$$
$$= \begin{cases} 1 & (n \text{ even}) \\ 0 & (n \text{ odd}). \end{cases} \tag{5}$$

(Here $\operatorname{res}_{x=0} w$ denotes the residue at $x=0$ of a differential w). Therefore

$$g'(y) = 1 + y^2 + y^4 + \dots,$$

$$x = g(y) = y + y^3/3 + y^5/5 + \dots = \tanh^{-1} y, \tag{6}$$

$$f(x) = x/y = x/\tanh x.$$

We next state the Riemann-Roch theorem:

<u>THEOREM 2</u>: Let X be a Hodge manifold with complex tangent bundle θ, and W a holomorphic vector bundle over X. Then

$$\chi(X,W) = (\operatorname{ch} W \cdot \operatorname{td}(\theta))\,[X]. \tag{7}$$

We recall that the Chern character of a $U(q)$-bundle ξ over X is defined by

$$\operatorname{ch} \xi = e^{\alpha_1} + \dots + e^{\alpha_q} \in H^{ev}(X;\mathbb{Q}), \tag{8}$$

where

$$\sum_{i=0}^{q} c_i(\xi)x^i = \prod_{i=1}^{q} (1 + \alpha_i x). \tag{9}$$

If ξ' is a second bundle with

$$\sum_{i=0}^{q'} c_i(\xi')x^i = \prod_{j=1}^{q'} (1 + \beta_j x), \tag{10}$$

then

$$\sum_{i=0}^{q+q'} c_i(\xi \oplus \xi')x^i = \prod_{i=1}^{q} (1+\alpha_i x) \prod_{j=1}^{q'} (1 + \beta_j x), \tag{11}$$

$$\sum_{i=0}^{qq'} c_i(\xi \otimes \xi')x^i = \prod_{i=1}^{q} \prod_{j=1}^{q'} (1 + (\alpha_i + \beta_j)x), \tag{12}$$

from which we easily deduce that ch is a ring homomorphism from $K(X)$ to $H^{ev}(X)$. (In fact, when tensored with $\mathbb{Q}$ it becomes an isomorphism of rings). The Chern classes of the p^{th} exterior power of ξ are given by

$$\sum_{i} c_i(\Lambda^p \xi)x^i = \prod_{1 \leq i_1 < \dots < i_p \leq q} (1 + (\alpha_{i_1} + \dots + \alpha_{i_p}) \tag{13}$$

from which it follows that

$$\sum_{p=0}^{q} y^p \operatorname{ch}(\Lambda^p \xi) = \sum_{p} \sum_{i_1, \dots, i_p} y^p e^{(\alpha_{i_1} + \dots + \alpha_{i_p})x}$$

$$= \prod_{i=1}^{q} (1 + ye^{\alpha_i}) \tag{14}$$

$$= (\operatorname{ch} \lambda_y)(\xi), \tag{15}$$

where in the last line we use the notation given in the last section (cf. 2.2(viii)) and which is now justified. Notice that (12) and (14) give

$$\begin{aligned} ch\, \lambda_y(\xi \otimes \xi') &= \prod_{i=1}^{q} \prod_{j=1}^{q'} (1 + y e^{\alpha_i} e^{\beta_j}) \\ &= (ch\, \lambda_y(\xi)) \circledast (ch\, \lambda_y(\xi')), \qquad (16) \end{aligned}$$

with the $\circledast$ defined by $ch\, \lambda_y(\xi)$ as a polynomial in y; since ch is a ring isomorphism we recover eq. 2.2(20).

We can now use this to extend the Riemann-Roch theorem, obtaining a formula for $\chi_y(X,W)$, the χ_y-characteristic of W (cf. 1.2 (8)). Indeed, using (15), 2.2(37), and the fact that $T^* = \Theta$, we obtain from (7) the equalities

$$\begin{aligned} \chi_y(X,W) &= \sum_p y^p \ \chi(X, W \otimes \Lambda^p T) \\ &= \sum y^p \ (ch\, W \cdot ch(\Lambda^p T) \cdot td\, \Theta)\, [X] \\ &= (ch\, W \cdot ch\, \lambda_y \Theta^* \cdot td\, \Theta)\ [X] \\ &= (ch\, W \cdot \tilde{T}_y(\Theta))\ [X]. \qquad (17) \end{aligned}$$

With $W = 1$ we get (setting $y = 0$ and using 2.2(34)) the formula

$$\chi(X) = (td(\Theta))\, [X] \qquad (18)$$

for the arithmetic genus of X, and (setting $y = 1$ and using 2.2(35))

$$\chi_{-1}(X) = e(\Theta)\, [X] = e(X) \qquad (19)$$

for the Euler characteristic, the latter in agreement with 1.3(25). Finally, if we replace $\text{ch } W = e^{\delta_1} + \ldots + e^{\delta_q}$ by $\sum_{i=1}^{q} e^{(1+y)\delta_i}$, and $\widetilde{T}_y(\theta)$ by $T_y(\theta)$, then we do not change (17) since the 2q-dimensional component of the cohomology class is unchanged. We can thus state the generalised Riemann-Roch theorem:

<u>THEOREM 3</u> [36]: Let X,W be as in Theorem 2, and

$$\sum_{i=0}^{q} c_i(W)x^i = \prod_{i=1}^{q} (1 + \delta_i x)$$

the formal splitting of the Chern classes of W. Then

$$\chi_y(X,W) = \left(\sum_{i=1}^{q} e^{(1+y)\delta_i} \cdot T_y(\theta)\right) [X] \quad (20)$$

Setting $y = 1$ and W = trivial line bundle, we find from 2.2(31) and Theorem 1 that the equality $\text{Sign}(X) = \chi_1(X)$ (Theorem 6 of 2.1) is a consequence of Theorem 3.

It is known that the Riemann-Roch theorem is true for arbitrary compact complex manifolds rather than just for Hodge manifolds, but this requires the Atiyah-Singer theorem The Riemann-Roch theorem for Hodge manifolds is more elementary and ultimately depends on the special case given by the signature theorem.

We now state the G-signature theorem for Sign(g,X), which for the sort of applications we will be interested in is the most useful of the theorems of this section. Let G be a finite or a compact Lie group acting on a closed oriented manifold X (the action being smooth and orientation-preserving). Then $X^g = \{x: gx = x\}$ is a submanifold of X, and g acts on the normal bundle N^g of X^g in X. Moreover, this action is effective (leaves no non-zero vector fixed), since N^g can be equivariantly identified with a tubular neighbourhood of X^g in X and g acts freely on $X - X^g$. Therefore standard representation theory for compact groups tells us that on a fibre N^g_x at a point $x \in X^g$, the action of g can be represented as a direct sum of 1×1 matrices (-1) and 2×2 matrices

$$A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \qquad (21)$$

Since A_θ and $A_{-\theta}$ are equivalent, we can assume $0<\theta<\pi$ and split N^g_x up into a sum of vector spaces $N^g_{\pi,x}$ on which g acts as multiplication by -1 and spaces $N^g_{\theta,x}$ on which g acts by the representation (21). For $\theta \neq 0,\pi$, A_θ defines an orientation and even a complex structure on $N^g_{\theta,x}$, with respect to which it is multiplication by $e^{i\theta}$. Moreover, the splitting of N^g_x extends to the whole bundle N^g, so we have:

<u>THEOREM 4</u>: Let G be a compact Lie group acting (differentiably and preserving the orientation) on an oriented closed manifold X. Then X^g is a submanifold of X, and its normal bundle N^g splits as a direct sum

$$N^g = N^g_\pi \oplus \sum_{0<\theta<\pi} N^g_\theta, \qquad (22)$$

where g acts on N^g_π by multiplication with -1 and where N^g_θ has a complex structure with respect to which g acts as multiplication by $e^{i\theta}$.

Since X and the bundles N^g_θ are oriented, so is the total space of N^g_π, i.e. N^g_π and $T(X)^g$ are "twisted" in the same way. Therefore N^g_π is an $SO(q)$-bundle if and only if X^g is orientable and if not, the Euler classes of N^g_π and of X^g lie in the cohomology of X with the same twisted coefficients $\tilde{\mathbb{Q}}$. We can now state the G-signature theorem:

<u>THEOREM</u> 5 (Atiyah-Singer [3])†: Let X, G be as in Theorem 4. If X^g is connected and orientable, then

$$\operatorname{Sign}(g,X) = \left(\mathcal{L}(X^g)\ \mathcal{L}_\pi(N^g_\pi) \prod_{0<\theta<\pi} \mathcal{L}_\theta(N^g_\theta) \right) [X^g]. \qquad (23)$$

Here $\mathcal{L}_\theta$ is the non-stable characteristic class for complex bundles introduced in 2.2 and $\mathcal{L}_\pi$ is the characteristic class for orientable bundles given by 2.2(36). If X^g is not orientable, then the $e(N^g)$ in 2.2(36) and evaluation on $[X^g]$ in (23) have to be taken in the sense of twisted coefficients (i.e. the system of coefficients locally isomorphic to $\mathbb{Q}$ defined by the local orientations of X^g). If X^g is not connected, we replace (23) by a sum of the corresponding expression evaluated for the various components.

Although we will not need it in the sequel, we state for completeness the result obtained by Atiyah and Singer for $\chi(X,W;g)$, the equivariant Euler characteristic of a holomorphic bundle W over a compact complex manifold X with respect to an automorphism g of (X,W) (see 1.4). Since X is complex, we have

$$N^g = \sum_{0<\theta<2\pi} N^g_\theta, \qquad (24)$$

† An elementary proof of Theorem 5 for G finite was given by Ossa [78]. A generalization to certain topological manifolds and group actions is given in Chapter 14B of Wall [96].

where each subbundle N^g_θ inherits a complex structure from X, and g acts as $e^{i\theta}$ on N^g_θ. Of course X^g also has a complex structure and is in particular oriented.

If ξ is a complex bundle on which G acts, then under the identification

$$K_G(X) = K(X) \otimes R(G) \tag{25}$$

(where $R(G)$ is the complex representation ring of G) we can write

$$\xi = \sum_i a_i \otimes \chi_i \qquad (a_i \in K(X)). \tag{26}$$

where χ_i is the character of the i^{th} irreducible representation in $R(G)$, and define

$$\mathrm{ch}\, \xi\,(g) = \sum_i \chi_i(g) \cdot \mathrm{ch}\, a_i \in H^*(X;\mathbf{C}). \tag{27}$$

Thus, for example, since g acts on N^g_θ by $e^{i\theta}$ we have

$$\mathrm{ch}\, N^g_\theta\,(g) = e^{i\theta} \cdot \mathrm{ch}\, N^g_\theta. \tag{28}$$

Then the formula which Atiyah and Singer give for $\chi(X,W;g)$, called the "holomorphic Lefschetz theorem" in [3], is

<u>THEOREM</u> <u>6</u> (Atiyah-Singer [3]): Let W be a holomorphic bundle over a compact complex manifold X and g an automorphism of (X,W). Then

$$\chi(X,W;g) = \Big\{ \mathrm{ch}(W|X^g)(g) \cdot \prod_{0<\theta<2\pi} \mathcal{U}_\theta(N^g_\theta) \cdot \mathrm{td}(\theta(X^g)) \Big\} [X^g]. \tag{29}$$

Finally, if we want to write down the most general possible formula of this type, we note that $T(X)$ is the dual of the tangent bundle, so

$$T(X) \mid X^g = T(X^g) \oplus \sum_{0<\theta<2\pi} N_\theta^{g*}. \tag{30}$$

Therefore, using 2.2(19) and the multiplicativity of the Chern character,

$$\sum_p y^p \operatorname{ch}(W \otimes \Lambda^p T(X) \mid X^g)(g) = \operatorname{ch}(W|X^g)(g) \cdot \operatorname{ch} \lambda_y T(X^g) \cdot \prod_{\theta \neq 0} \operatorname{ch} \lambda_y N_\theta^{g*}. \tag{31}$$

From (28) we deduce

$$\operatorname{ch} \Lambda^p N_\theta^{g*}(g) = e^{-ip\theta} \operatorname{ch} \Lambda^p N_\theta^{g*},$$

so

$$\operatorname{ch} \Lambda_y N_\theta^{g*}(g) = \prod_{i=1}^{q} (1 + ye^{-i\theta-\alpha_j}), \tag{32}$$

where

$$\sum_i c_i(N_\theta^g)\, x^i = \prod_{i=1}^{q} (1 + \alpha_i x). \tag{33}$$

Putting all this into (29) gives finally

<u>THEOREM 7</u> : Let X, W, g be as in Theorem 6. Then

$$\chi_y(X, W; g) = \Big\{\operatorname{ch}(W|X^g)(g) \cdot \widetilde{T}_y(\Theta(X^g)) \cdot \prod_{0<\theta<2\pi} \widetilde{T}_y^\theta(N_\theta^g)\Big\} [X^g] \tag{34}$$

As before, if W is the trivial line bundle or if we modify $\operatorname{ch} W$ by multiplying its $2i$-dimensional component by $(1+y)^i$

then we can rewrite (34) using T_y and T_y^θ rather than $\widetilde{T}_y$ and $\widetilde{T}_y^\theta$.

If we take W = trivial line bundle and $y = 1$, then (using 2.2(31) and 2.2(39)) the right-hand side of (34) is just the expression which appears on the right side of the G-signature theorem, but the two theorems were proved independently by Atiyah and Singer (applying their index theorem to different elliptic operators), so that we deduce the equality between $\chi_1(X;g)$ and Sign(g,X) that was asserted at the end of 2.1.

If on the other hand we take W = trivial line bundle and $y = -1$, then by 2.2(35) and 2.2(40) the right-hand side of (34) is just

$$e(\theta(X^g))[X^g] = e(X^g), \tag{35}$$

where $e(X^g)$ is the ordinary Euler-Poincaré characteristic of the manifold X^g. Therefore

<u>THEOREM</u> <u>8</u> : If g is an automorphism of a complex manifold X, then

$$\sum_i (-1)^i \operatorname{tr}(g^*|H^i(X;\mathbb{C})) = e(X^g). \tag{36}$$

<u>Corollary</u>: If G is a finite group of automorphisms, then the Euler characteristic of X/G is

$$e(X/G) = \frac{1}{|G|} \cdot \sum_{g \in G} e(X^g), \tag{37}$$

(This theorem holds for all differential manifolds X .)

<u>Proof of the corollary</u>: We use Theorem 5 of 2.1 and the

theorem of 1.4 to get

$$e(X/G) = \sum_i (-1)^i \dim H^i(X/G;\mathbb{C})$$

$$= \sum_i (-1)^i \dim H^i(X;\mathbb{C})^G$$

$$= \sum_i (-1)^i \left(\frac{1}{|G|} \sum_{g \in G} \operatorname{tr}(g^* | H^i(X;\mathbb{C})) \right) \tag{38}$$

and the corollary then follows immediately from the theorem.

§ 3. The L-class of a rational homology manifold

In 1958, Thom [95] showed that the L-class of a differentiable manifold is a combinatorial rather than just a differentiable invariant. To do this, he gave a definition of the L-class of a rational homology manifold (a topological space having locally the rational cohomology of Euclidean space) generalizing the usual definition for a differentiable manifold. This definition of Thom, and a simpler formulation given later by Milnor, will be discussed in 3.1 In section 3.2 we consider a quotient space X/G, where G is a finite group acting on a manifold X (that such a space is a rational homology manifold was mentioned in 2.1), and compute its L-class in terms of the data involved in the G-signature theorem. We also generalize the G-signature theorem to such quotient spaces. A reference for the contents of § 3 is [99].

3.1 A rational homology manifold is a finite simplicial complex X such that for each point $x \in X$,

$$H^i(X, X-\{x\}; \mathbb{Q}) = \begin{cases} 0, & \text{if } i \neq n, \\ \mathbb{Q}, & \text{if } i = n. \end{cases} \qquad (1)$$

Such "manifolds" were first studied by Thom [95], who showed that one could define for them the concepts of orientability, submanifolds (still in the sense of rational homology), and transverse regularity. If X is oriented, then (by definition) there is a fundamental class

$$[X] \in H_n(X; \mathbb{Q}), \qquad (2)$$

and cap products with X lead to the usual Poincaré isomorphisms.

Now let K be a manifold and

$$f: X \longrightarrow K \qquad (3)$$

be a simplicial map transverse regular (in Thom's sense) along a submanifold N of K. Then $f^{-1}(N) = Y$ is a rational homology submanifold of X, and we define

$$\nu = f^*(\text{normal bundle of } N \text{ in } K). \qquad (4)$$

This is an orthogonal bundle over Y which, if X and Y were differentiable manifolds, would be simply the normal bundle of Y in X. Moreover, in that case we would have

$$\mathrm{Sign}(Y) = \langle \mathcal{L}(Y), [Y] \rangle = \langle i^* \mathcal{L}(X) \cdot \mathcal{L}(\nu)^{-1}, [Y] \rangle, \qquad (5)$$

where (as in 2.3) we write $\mathcal{L}(X)$ for the L-class of the tangent bundle of X. What Thom did was to show that even in the non-differentiable case, these submanifolds have a "good" real normal bundle (i.e. the real bundle ν over Y depends only on the rational homology manifolds X and Y, and not on the map f), and that there are so many of these "good" submanifolds-with-normal-bundle Y that the homology classes they represent generate $H_*(X;\mathbb{Q})$. Then for each such Y, we can define $\mathrm{Sign}(Y)$ (since Y is an oriented rational homology manifold and therefore has a fundamental class and an intersection form in the middle dimension) and $\mathcal{L}(\nu)$ (since ν is a well-defined $SO(q)$-bundle over Y), and we try to find a cohomology class

$$\mathcal{L}(X) \in H^{4*}(X;\mathbb{Q}) \tag{6}$$

such that (5) is satisfied for all good submanifolds Y. The equations (5) certainly determine $\mathcal{L}(X)$ completely (since one knows that the homology classes of the good Y's generate $H_*(X;\mathbb{Q})$), and it can also be proved that they do not conflict with one another, so that one gets a consistent definition for the L-class of X. If Y is a good submanifold of X with normal bundle ν, then

$$i^*\mathcal{L}(X) = \mathcal{L}(Y)\mathcal{L}(\nu) \tag{7}$$

just as in the differentiable case. The other important property of the L-class, namely the signature theorem

$$\mathrm{Sign}\, X = \langle \mathcal{L}(X), [X] \rangle, \tag{8}$$

is of course an immediate consequence of the definition. It is not possible to define in a natural way a tangent bundle ${}_{\mathbb{R}}\theta(X)$ in such a way that the L-class of the bundle ${}_{\mathbb{R}}\theta(X)$ agrees with the Thom L-class of X; nevertheless, Thom's definition gives a well-defined class (6) agreeing with $\mathcal{L}({}_{\mathbb{R}}\theta(X))$ if X is smooth.

One can now also define Pontrjagin classes for X. Indeed, it is clear from 2.2(11) that a multiplicative sequence $g \mapsto f \boxtimes g$ is invertible (i.e. there is a power series h with $f \boxtimes h(x) = 1+x$) if and only if no $s_r(f)$ is 0. Since the power series

$$f(x) = \frac{\sqrt{x}}{\tanh \sqrt{x}} \tag{9}$$

defining the L-class from the Pontrjagin class has this property (the $s_r(f)$ are certain expressions involving Bernoulli numbers, given in [36], and are all non-zero), it follows that the Pontrjagin class of a bundle can be deduced from its L-class. It seems, however, that for rational homology manifolds that do not carry any differentiable structure, the L-class defined above is a very natural class to consider, whereas the resulting definition of the Pontrjagin class is artificial. In other words, the L-class is natural in situations where there is a signature, the Pontrjagin class only when there is a tangent bundle. The observation that the L-class determines the Pontrjagin class does have an important consequence, though, namely that Thom's result implies the combinatorial invariance of the Pontrjagin class (i.e. two piecewise-linear-homeomorphic differentiable manifolds have the same Pontrjagin class).

A slightly simpler formulation of the definition of $\mathcal{L}(X)$ was given by Milnor [71], who pointed out that to generate the rational homology of X, at least in low dimensions, it is sufficient to take good submanifolds Y with trivial normal bundle, for which (5) takes on the simpler form

$$\text{Sign } (Y) = \langle i^* \mathcal{L}(X), [Y] \rangle . \tag{10}$$

Moreover, the concept of a good submanifold with trivial normal bundle can be defined without introducing the concept of transverse regularity. If X were smooth, and $f: X \to S^r$ a smooth map, then for almost every point $p \in S^r$, $f^{-1}(p)$ would be a submanifold of X with trivial normal bundle. Thus, if X is a rational homology manifold and $f: X \longrightarrow \Sigma^r$ a simplicial map (where Σ^r is the boundary of an $(r+1)$-simplex), then $Y_p = f^{-1}(p) \subset X$ for a point $p \in \Sigma^r$ in general position should play the role of submanifolds-with-trivial-normal-bundle in the differentiable case.
Milnor showed that, for almost all p in Σ^r, Y_p is a rational homology manifold. Since Y_p acquires an orientation from those on X and Σ^r, the number Sign (Y_p) is then defined, and it turns out to be independent of p, for almost all $p \in \Sigma^r$. Thus it is possible to associate an integer $S(f)$ to the map f. Replacing f by a homotopic map does not change $S(f)$, because Y_p is replaced by a cobordant manifold, and cobordant manifolds have the same signature. Therefore S defines a map

$$S : \pi^r(X) \to \mathbb{Z}, \tag{11}$$

where $\pi^r(X)$ denotes the set of homotopy classes of maps from X to S^r. Let $n = \dim X$. It is known from the work of Serre [89] that $\pi^r(X)$ has an abelian group structure if $n \leq 2r-2$, and that the natural map

$$\phi^* : \quad \pi^r(X) \longrightarrow H^r(X;\mathbb{Z}) \tag{12}$$

sending the homotopy class of a map

$$f : \quad X \to \Sigma^r \tag{13}$$

to $f^*(\sigma)$ (where σ is the generator of $H^r(\Sigma^r;\mathbb{Z})$) is an isomorphism in these dimensions after tensoring with $\mathbb{Q}$. It is easily checked that (11) is a homomorphism. Therefore, if we tensor everything with $\mathbb{Q}$ and use (12) to identify $\pi^r(X)\otimes\mathbb{Q}$ with $H^r(X;\mathbb{Q})$, we obtain a homomorphism $H^r(X;\mathbb{Q}) \to \mathbb{Q}$. By Poincaré duality, there is a unique class $\ell_r \in H^{n-r}(X;\mathbb{Q})$ such that this map consists of cupping with ℓ_r and evaluating on the fundamental class of X. We define

$$\mathcal{L}_j(X) \in H^{4j}(X;\mathbb{Q}) \tag{14}$$

as ℓ_{n-4j} if $n \leq 2(n-4j)-2$; thus $\mathcal{L}_j(X)$ is characterized by the equality

$$\mathrm{Sign}(f^{-1}(p)) = \left\{ \mathcal{L}_j(X)\cdot f^*(\sigma) \right\} [X] \tag{15}$$

for all maps (13) and almost all $p \in \Sigma^r$ $(r = n-4j)$. If the condition $n \geq 8j+2$ is not satisfied, then we take the Cartesian product of X and Σ^r with a sphere of large dimension, obtaining a definition of $\mathcal{L}_j(X\times S^N)$ for N large, and then get $\mathcal{L}_j(X)$ by identifying $H^{4j}(X\times S^N)$ with $H^{4j}(X)$. Thus $\mathcal{L}_j(X)$ can be defined for all j, and we set $\mathcal{L}(X) = \sum_j \mathcal{L}_j(X)$.

For concrete problems, Milnor's definition is easier to work with than Thom's, since it is usually not easy to tell whether a given subcomplex of a rational homology manifold is a submanifold with good normal bundle. We will use Milnor's definition in 3.2 to find the L-class of a quotient space X/G, where G is a finite group acting cn a smooth manifold X.

3.2 If a finite group G acts orientably on an oriented closed manifold X, then by Theorem 5 of 2.1, the projection π from X to X/G induces an isomorphism from $H^*(X/G)$ to $H^*(X)^G$, where all homology and cohomology is to be taken with rational coefficients. The quotient or orbit space X/G is a rational homology manifold, and is oriented: the fundamental class is given by

$$[X/G] = \frac{1}{\deg \pi} \pi_* [X]. \tag{1}$$

Therefore X/G has an L-class in the sense of Thom; we wish to compute it.

Let X^g be the fixed-point set of an element $g \in G$, and

$$\mathcal{L}'(g,X) = \mathcal{L}(X^g) \cdot \mathcal{L}_\pi(N_\pi^g) \cdot \prod_{0<\theta<\pi} \mathcal{L}_\theta(N_\theta^g)$$

$$\in H^*(X^g;\widetilde{\mathbb{C}}) \tag{2}$$

be the class defined in the statement of the G-signature theorem ($\tilde{\mathbb{C}}$ denotes the twisted coefficients, locally isomorphic to $\mathbb{C}$ and twisted by the local orientations of X^g, that were mentioned in 2.3). Now if we have an embedding i of one oriented manifold A in another B, there is a <u>covariant</u> map $i_!$ (the Gysin homomorphism) from the cohomology of A to that of B, obtained by using duality to identify the cohomology of a manifold with its homology and then applying i_* in homology, i.e.

$$i_! = D_B^{-1} i_* D_A : H^*(A) \to H^*(B) \qquad (3)$$

(here i_* is the induced map in homology and D_A is the Poincaré duality isomorphism from $H^*(A)$ to $H_*(A)$ defined by cap products with the fundamental class $[A]$). If the submanifold A is not oriented, then (3) still defines a map $i_!$ to the cohomology of B, but the group $H^*(A)$ in (3) must be replaced by the group with coefficients twisted by the local orientations of A. Applying this to the class $\mathcal{L}'(g,X)$, we can define

$$\mathcal{L}(g,X) = i_!^g \, \mathcal{L}'(g,X) \in H^*(X;\mathbb{C}), \qquad (4)$$

where i^g is the inclusion of X^g in X and $i_!^g$ the homomorphism just defined. When X^g has several components, the classes (2) are defined for each, as are the maps $i_!^g$, and the class $\mathcal{L}(g,X)$ is the sum of the expressions (4). We can now state:

<u>THEOREM 1</u> : Let G be a finite group acting orientably on an oriented closed manifold X, and let π be the projection of X onto X/G. Then

$$\left(\frac{1}{\deg \pi}\right) \pi^* \mathcal{L}(X/G) = \frac{1}{|G|} \sum_{g \in G} \mathcal{L}(g,X), \qquad (5)$$

where $\mathcal{L}(g,X)$ is the class defined by (2)-(4).

Corollary: The sum on the right of (5) is a rational rather than just a complex cohomology class (this generalizes the consequence of Theorem 4 of 2.1 that the expression on the right of 2.1(22) is an integer).

Note that if G acts effectively (i.e. no element of G except the identity acts trivially on X), we have $\deg \pi = |G|$, so that the numerical factors in (5) can be dropped. Since any action can be replaced by an effective one (by factoring it through G/H, where $H \subset G$ is the normal subgroup consisting of elements acting trivially), we will assume in future that the action of G is effective

Before discussing the proof of Theorem 1, we should point out why the right-hand side is a reasonable expression for $\pi^*\mathcal{L}(X/G)$. First of all, if $h \in G$, then one easily checks that

$$h^* \mathcal{L}(g,X) = \mathcal{L}(h^{-1}gh,X). \qquad (6)$$

Therefore the right-hand side of (5) is invariant under the action of G and therefore (by Theorem 5 of 2.1) is equal to $\pi^* L$ for a unique class $L \in H^*(X/G)$. To be a candidate for $\mathcal{L}(X/G)$, the class L should be 1 in dimension 0, Sign(X/G) in the top dimension, and zero in dimensions not divisible by 4. We can easily check these properties. If $i: A \to B$ is an embedding of manifolds as described above, then the map $i_!$ of (3) raises dimensions by b-a (b = dim B, a = dim A), so $i_! x$ can only have a 0-dimensional component if a = b. Since (for an effective action) $g \neq id \Longrightarrow \dim X^g < \dim X$, we deduce that $\mathcal{L}(g,X)$ has no component in $H^0(X)$ for $g \neq 1$. On the other hand,

$$\mathcal{L}(1,X) = \mathcal{L}(X) \tag{7}$$

has leading coefficient 1, and we find that the 0-dimensional component of the sum in (5) is indeed 1. That the top dimensional component is Sign(X/G) follows from Theorem 4 of 2.1 combined with the G-signature theorem and the fact that, for $i : A \to B$ a map of manifolds,

$$(i_! x)[B] = (x)[A]. \quad (x \in H^*(A)). \tag{8}$$

Finally, it is not hard to see that L is zero except in dimensions divisible by four. It is certainly even-dimensional because the expression (2) is in $H^{ev}(X)$ and the codimension of X^g in X is even (the bundles N^g_θ in Theorem 4 of 2.3 are complex bundles and hence even-dimensional, and the bundle N^g_π on which g acts as -1 must be even dimensional since otherwise the action of g would be orientation-reversing). Using $X^g = X^{g^{-1}}$, and $N^g = N^{g^{-1}}$ (as bundles, with g and g^{-1} acting in opposite ways), one can easily check that each sum

$$\mathcal{L}(g,X) + \mathcal{L}(g^{-1},X) \tag{9}$$

has non-zero components only in dimensions which are multiples of four.

To prove Theorem 1, we must show that $L = \pi^{*-1}(\sum_{g \in G} \mathcal{L}(g,X))$ satisfies formula 3.1(15). Now, for almost all points $p \in \Sigma^r$, the set $A = f^{-1}(p)$ is a submanifold of X with trivial normal bundle, and, for the same reasons, the set $A^g = A \cap X^g = (f|X^g)^{-1}(p)$ has trivial normal bundle in X^g. It follows that the normal bundle of A^g in A is the restriction $j_g^* N^g$ of the normal bundle of X^g in X, where j_g is the inclusion map $A^g \subset X^g$. Since f and all inclusion maps are G-equivariant, this even holds for the normal bundle considered as a G-bundle, i.e. the action of g on the normal bundle of A^g in A is the restriction of the action of g on N^g. It then follows immediately from formula (2) that $\mathcal{L}'(g,A)$ is the restriction of $\mathcal{L}'(g,X)$, and therefore also that $\mathcal{L}(g,A) = j^*\mathcal{L}(g,X)$ (where j is the inclusion of A in X). Moreover, the homology class in X represented by the submanifold A is the Poincaré dual of $f^*(\sigma)$ (where σ is as usual the generator of $H^r(\Sigma^r)$), so we have

$$(j^*x)[A] = (x)(j_*[A]) = (x \cup f^*(\sigma))[X] \qquad (10)$$

for all $x \in H^*(X)$. Using all of this, we can now verify that L satisfies 3.1(15). Namely, let $f': X/G \to \Sigma^r$ be an arbitrary simplicial map, so that $f = f' \circ \pi$ is an equivariant map from X to Σ^r. Then (using in succession (1), the definition of L, (10), the equation $j^*\mathcal{L}(g,X) = \mathcal{L}(g,A)$, (4), (8), 2.3(23), and 2.1(22)) we find

$$\begin{aligned}(L \cup f'^*(\sigma))[X/G] &= \left(\frac{1}{\deg \pi}\right)(L \cup f'^*(\sigma))\pi_*[X] \\ &= \left(\frac{1}{\deg \pi}\right)(\pi^*L \cup \pi^* f'^*(\sigma))[X] \\ &= \left(\frac{1}{|G|} \sum_{g \in G} \mathcal{L}(g,X) \cup f^*(\sigma)\right)[X]\end{aligned}$$

$$= \frac{1}{|G|} \sum_{g \in G} j^* \mathcal{L}(g,X) \; [A]$$

$$= \frac{1}{|G|} \sum_{g \in G} \mathcal{L}(g,A) \; [A]$$

$$= \frac{1}{|G|} \sum_{g \in G} (i^g_! \mathcal{L}'(g,A)) \; [A]$$

$$= \frac{1}{|G|} \sum_{g \in G} \mathcal{L}'(g,A)[A^g]$$

$$= \frac{1}{|G|} \sum_{g \in G} \text{Sign}(g,A)$$

$$= \text{Sign}(A/G)$$

$$= \text{Sign}(f'^{-1}(p)). \qquad (11)$$

Therefore L fulfills Milnor's definition for $L(X/G)$ and Theorem 1 is proven.

We stated earlier that the Pontrjagin class of a rational homology manifold cannot be naturally defined, since a Pontrjagin class essentially requires a bundle for its definition, while an L-class is defined in terms of signatures. Similarly, the Atiyah-Singer class $\mathcal{L}'(g,X) \in H^*(X^g)$ does not naturally extend to rational homology manifolds because it depends on the normal bundle of $X^g \subset X$, but the lifted class $\mathcal{L}(g,X) \in H^*(X)$ <u>can</u> be generalized because it has a definition in terms of signatures. Indeed, from the last calculation we get

$$< \mathcal{L}(g,X) \cup f^*(\sigma) \, , \; [X] > = \text{Sign}(g, f^{-1}(p)) \qquad (12)$$

for a G-invariant map $f: X \to S^r$ (X a smooth manifold).

This exactly corresponds to (15) of 3.1 for differentiable manifolds without G-actions, and we can try to imitate Milnor's proof to define a class

$$\mathcal{L}(g,X) \in H^*(X;\mathbb{C}) \tag{13}$$

for rational homology G-manifolds in such a way that (12) holds. The G-invariant-homotopy classes of G-invariant maps f from X to S^r are in 1:1 correspondence with homotopy classes of maps f' from X/G to S^r. Just as in Milnor's definition, the map $f' \mapsto \mathrm{Sign}(g, \pi^{-1}(f'^{-1}(p)))$, where $p \in S^r$ is a point in general position, gives a well-defined homomorphism from $\pi^r(X/G)$ to $\mathbb{C}$, even if X is only a rational homology manifold. For $\dim X \leq 2r-2$, we have a commutative diagram

$$\begin{array}{ccc} H^r(X/G;\mathbb{C}) & \xrightarrow{\ \pi^*\ } & H^r(X;\mathbb{C})^G \\ \uparrow \approx & & \uparrow \approx \\ \pi^r(X/G) \otimes \mathbb{C} & \xrightarrow{\ \pi^*\ } & \pi^r(X)^G \otimes \mathbb{C} \end{array} \tag{14}$$

and therefore obtain a well-defined homomorphism $H^r(X/G;\mathbb{C}) \to \mathbb{C}$. Since X/G is an oriented rational homology manifold if X is, we can use Poincaré duality to obtain from this homomorphism an element $\ell \in H^*(X/G;\mathbb{C})$ which satisfies

$$\mathrm{Sign}(g, f^{-1}(p)) = \langle (\ell \cup f'^*(\sigma)), [X/G] \rangle$$

$$(f = f' \circ \pi) \tag{15}$$

for all $f' : X/G \longrightarrow \Sigma^r$. Then if we define $\mathcal{L}(g,X)$ as $(\deg \pi)^{-1} \pi^* \ell$, the equation (12) is satisfied, as one sees by a calculation similar to (11). This definition automatically gives a G-invariant class $\mathcal{L}(g,X)$, since $\mathcal{L}(g,X)$ lies in the image of π^*. But from (6) we see that the class $\mathcal{L}(g,X)$, in the differentiable case, is not in general G-invariant, unless G is abelian. Therefore the definition of $\mathcal{L}(g,X)$ just proposed is only reasonable (i.e. consistent with (4) if X is differentiable) if the group G is abelian. This is no restriction, though, because the class $\mathcal{L}(g,X)$ should depend only on g and not on the group G, so we can always replace G by the abelian group generated by g and then use the above definition. Equation (15) defines ℓ uniquely and π^* is an isomorphism, so this definition uniquely defines a class $\mathcal{L}(g,X)$. This class must agree with the old one if X is differentiable, since for G abelian the class $\mathcal{L}(g,X)$ defined by (4) is G-invariant and therefore in the image of π^*. We have thus proved:

<u>THEOREM 2</u>: Let X be an oriented rational homology manifold on which a finite abelian group G acts orientably. Then there is a unique class $\ell \in H^*(X/G;\mathbb{C})$ such that (15) holds for any map f' from X/G to a sphere. The class

$$\mathcal{L}(g,X) = \left(\frac{1}{\deg \pi}\right) \pi^* \ell \in H^*(X;\mathbb{C}) \qquad (16)$$

satisfies (12) and agrees with the class defined by (4) if X is a differentiable manifold. In particular, from (12) follows the formula

$$\mathrm{Sign}(g,X) = \mathcal{L}(g,X)[X] \qquad (17)$$

which, for X a differentiable manifold, is just the G-signature theorem.

One can check without difficulty that, with this definition, Theorem 1 and also formula (6) for the operation of G on $\mathcal{L}(g,X)$ remain valid when X is an arbitrary rational homology manifold. In fact, one proves essentially as before that the following generalization of Theorem 1 holds:

<u>THEOREM 3</u> : Let G be a group acting on an oriented rational homology manifold X and preserving the orientation. Let H be a finite normal subgroup of G, and π the projection from X to X/H. Then G/H acts naturally on X/H, and if ξ is an element of G/H (i.e. a coset of H in G) we have

$$\left(\frac{1}{\deg \pi}\right) \pi^* \mathcal{L}(\xi, X/H) = \frac{1}{|H|} \sum_{g\in\xi} \mathcal{L}(g,X) \qquad (18)$$

Again the numerical factors can be omitted if H acts effectively on X, for then $\deg \pi = |H|$.

§4. <u>The α-invariant of Atiyah and Singer</u>

In this section we will consider some of the consequences of the G-signature theorem stated in section 2. In particular, we use this theorem to define an interesting invariant of a free group action on an odd-dimensional manifold, and give methods for evaluating this invariant when the group acting is $\mathbb{Z}_2$ or S^1.

4.1 The assertion of the G-signature theorem is that Sign(g,X) is equal to a certain number L(g,X) defined in terms of the fixed point set of the action of g on a closed manifold X. This equality is false if X is a manifold with boundary, and the deviation turns out to be an invariant of the action of g on the boundary of X (an invariant first defined in general by Atiyah and Singer, and now known as the α-invariant). Because Sign(g,X) and L(g,X) will no longer be equal, we will want to deduce several properties of these numbers (e.g. under what circumstances they are real, rational, or integral) directly from the two definitions, rather than proving the property for one of the two numbers and appealing to the G-signature theorem, as we could in the closed case.

We begin by recalling the enunciation of the G-signature theorem:

$$\mathrm{Sign}(g,X) = L(g,X), \tag{1}$$

where

$$L(g,X) = \left\{\mathcal{L}(X^g)e(N^g_\pi)\,\mathcal{L}(N^g_\pi)^{-1}\prod_{0<\theta<\pi}\mathcal{L}_\theta(N^g_\theta)\right\}[X^g]. \tag{2}$$

Here the bundles N^g_θ are those of Theorem 4 of 2.3, the various characteristic classes appearing are defined by the multiplicative sequences given in 2.2, and (as usual) $\mathcal{L}(X)$ denotes the L-class of the (real) tangent bundle of a differentiable manifold X. We assume X is even-dimensional (Sign(g,X) is zero for odd-dimensional manifolds) and write $2k = \dim X$,

$2r = \dim X^g$, $2m(\theta) = \dim N_\theta^g$ (we use real dimensions even for the complex bundles N_θ^g). Then

$$k = r + m(\pi) + \sum_{0<\theta<\pi} m(\theta) \qquad (3)$$

The class $\mathcal{L}_{-\theta}$ is given by the power series $\coth(x + \frac{i\theta}{2}) = i\cot(\frac{\theta}{2} - ix)$, so the cohomology class $i^{-m(\theta)}\mathcal{L}_\theta(N_\theta^g)$ is zero in odd dimensions, real in dimensions divisible by 4, and pure imaginary in dimensions $4j + 2$. Since the L-class is a real cohomology class and is zero in dimensions not divisible by four, and since $e(N_\pi^g) \in H^{2m(\pi)}(X^g)$ we deduce using (3): $\mathcal{L}(g,X)$ is i^k times a real number, i.e. real if k is even and pure imaginary if k is odd. Moreover, we have proved this for the expression in (2) on each component, rather than just for the number $L(g,X)$ in (1) with its implied summation over components. From the definitions in 2.1, Sign(g,X) is also real if k is even and pure imaginary if k is odd.

In general we can say little about the values of Sign(g,X) and L(g,X) besides this. However, if g is of finite order n, then it follows from the definition of Sign(g,X) as a difference of traces of g and of L(g,X) in terms of the action of g on the fibres of the normal bundle of X^g that both of these numbers lie in the field $\mathbb{Q}(e^{2\pi i/n})$.

An even more trivial property is that, if X_1 and X_2 are G-manifolds, then

$$(X_1 \times X_2)^g = X_1^g \times X_2^g \qquad (4)$$

and the normal bundle is also a product, so

$$L(g,X_1\times X_2) = L(g,X_1)L(g,X_2), \tag{5}$$

Equation (17) of 2.1 states the same equality for Sign(g,X

We now turn to the definition of the "α-invariant." We begin by stating the following basic theorem, which is a trivial consequence of the G-signature theorem but surprisingly has not been given a proof independent of the theory of elliptic operators:

<u>THEOREM 1</u> : Let G be a compact Lie group acting orientably, smoothly, and freely on a closed oriented manifold X. Then for $g\in G-\{1\}$

$$\mathrm{Sign}(g,X) = 0 \tag{6}$$

More generally, this holds for any action of G if the element g acts on X without leaving any point fixed.

Now assume that X_1 and X_2 are oriented manifolds with boundary on which G acts freely, and that

$$h:\quad \partial X_1 \xrightarrow{\cong} \partial X_2 \tag{7}$$

an equivariant and orientation-preserving diffeomorphism of their boundaries. Then G acts freely on

$$Z = X_1 \underset{h}{\cup} (-X_2), \tag{8}$$

and combining the (equivariant) Novikov additivity theorem with Theorem 1 gives

$$\mathrm{Sign}(g,X_1) - \mathrm{Sign}(g,X_2) = \mathrm{Sign}(g,Z) = 0 \quad (g\neq 1), \tag{9}$$

so that the number $\mathrm{Sign}(g,X_i)$ is independent of the free

G-manifold X_i with a given free G-manifold as boundary. More generally, suppose that Y is a free closed G-manifold and that X_1, X_2 are free G-manifolds with

$$N_1 \cdot Y = \partial X_1 \tag{10}$$

$$N_2 \cdot Y = \partial X_2, \tag{11}$$

where $N \cdot Y$ means N disjoint copies of the manifold Y (N a positive integer). Then $N_2 \cdot X_1$ and $N_1 \cdot X_2$ (with the obvious action of G) are free G-manifolds with the same boundary $N_1 N_2 \cdot Y$, so by (9)

$$\mathrm{Sign}(g, N_2 \cdot X_1) = \mathrm{Sign}(g, N_1 \cdot X_2) \quad (g \neq 1) \tag{12}$$

Since obviously $\mathrm{Sign}(g, N \cdot X)$ is N times $\mathrm{Sign}(g, X)$, we get

$$\frac{1}{N_1} \mathrm{Sign}(g, X_1) = \frac{1}{N_2} \mathrm{Sign}(g, X_2). \tag{13}$$

The cobordism theory of Conner and Floyd [20] shows that for an odd-dimensional manifold Y with a free action of a <u>finite</u> group G, some multiple of Y can always be expressed as the equivariant boundary of a free manifold X. (The case of even-dimensional Y is of course uninteresting, since $\mathrm{Sign}(g, X)$ is 0 for odd-dimensional X.) Therefore

<u>THEOREM 2</u>: Let Y be a closed odd-dimensional oriented manifold on which a finite group G acts freely. Then there exists a positive integer N such that $N \cdot Y$ is the equivariant boundary of a free G-manifold X, and the number

$$\alpha(g, Y) = \frac{1}{N} \mathrm{Sign}(g, X) \quad (g \in G - \{1\}) \tag{14}$$

is independent of N and X, and depends only on Y and the action of g on Y. The assertion that (14) is independent of N and X also holds if G is any compact Lie group, but then it may not be possible to find a free G-manifold X with boundary a multiple of Y. Equation (14) also remains valid even if G does not act freely, if the element g has no fixed points on Y or on X.

Note that the definition of the invariant α given here differs in sign from that given for the invariant σ in [3].

If G is not a finite group, we cannot in general find a free manifold X with the required boundary, and even in the finite group case we may often have X represented as the boundary of a G-manifold with fixed points but not know how to find the free manifold X of whose existence we are assured by Theorem 2. However, if some multiple N of the free G-manifold Y bounds a G-manifold X, even with fixed points, we can form the expression

$$\frac{1}{N}\left(\mathrm{Sign}(g,X) - L(g,X)\right) \qquad (g \neq 1). \tag{15}$$

Indeed, L(g,X) is still defined for a manifold with boundary, as long as g has no fixed points on the boundary (because in that case the fixed-point set X^g is disjoint from ∂X, and since (2) only depends on X^g and its normal bundle, it is not affected by the boundary). Moreover, it is trivial that

$$L(g,Z) = L(g,X_1) - L(g,X_2) \tag{16}$$

for Z as in (8), since the fixed point set Z^g avoids the submanifold along which X_1 and X_2 are glued. Then an argument exactly like that above shows that (15) is independent of N and X. In particular, if X is a free manifold then (15) reduces to (14), so that the new definition coincides with the old one if such a free bounding manifold exists. Therefore $\alpha(g,Y)$ can be defined whenever g acts on Y without fixed points and some multiple of Y bounds a manifold X to which the action of g extends, possibly with fixed points. Moreover, the various properties which were proved independently for $L(g,X)$ and $\mathrm{Sign}(g,X)$ (i.e. without using (1)) carry over automatically to $\alpha(g,Y)$, so we have:

<u>THEOREM 3</u>: Let Y be a closed oriented manifold of dimension $2k-1$ on which a compact Lie group G acts orientably. If $g \in G$ acts on Y without fixed points, and if some multiple $N>0$ of Y bounds a G-manifold X, then

$$\alpha(g,Y) = \frac{1}{N}\Big(\mathrm{Sign}(g,X) - L(g,X)\Big) \tag{17}$$

is an invariant of the action of g on Y and is independent of N and X. The number $\alpha(g,Y)$ is real if k is even and pure imaginary if k is odd; it lies in $\mathbb{Q}(e^{2\pi i/n})$ if g^n acts as the identity on Y; and it satisfies

$$\alpha(g,Y\times Z) = \alpha(g,Y)\cdot \mathrm{Sign}(g,Z) \tag{18}$$

for any closed even-dimensional G-manifold Z.

We will need one other property of the α-invariant in the case of free finite group actions (i.e. the special case of Theorem 2), namely:

<u>THEOREM 4</u> : Let Y and G be as in Theorem 2, and H a normal subgroup of G. Then G/H acts naturally on Y/H and for a coset $\xi \in G/H$ we have

$$\alpha(\xi, Y/H) = \frac{1}{|H|} \sum_{g \in \xi} \alpha(g, Y). \tag{19}$$

<u>Proof</u>: By Theorem 2, we can assume (taking if necessary a multiple N of Y) that Y bounds a free G-manifold X. Since X is then also a free H-manifold, we can consider the pairs (X,Y) and $(X/H, Y/H)$, each consisting of a manifold and its boundary. Using the usual identification (Theorem 5 of 2.1) of the cohomology of the quotient with the invariant part of the cohomology of the original space, we find that the bilinear forms B, B' on $H^*(X,Y)$ and on $H^*(X/H, Y/H)$ which are used to define the numbers $\operatorname{Sign}(g,X)$ and $\operatorname{Sign}(\xi, X/H)$ (see 2.1) correspond, and the theorem is reduced to the following algebraic theorem:

<u>THEOREM 5</u>: Let V be a real vector space and B a symmetric or skew-symmetric non-degenerate bilinear form on V. Let G be a finite group, H a normal subgroup, and assume that G acts (linearly) on V preserving B. Let V^H denote the subspace of V left fixed by H, and B' the restriction of B to V^H. Then G/H acts on V^H preserving the form B', and for $\xi \in G/H$,

$$\operatorname{Sign}(\xi, V^H) = \frac{1}{|H|} \sum_{g \in \xi} \operatorname{Sign}(g, V). \tag{20}$$

Proof : This generalizes Theorem 4 of 2.1, and the proof is similar. That G/H acts on V^H is clear, since for $v \in V^H$ we have $g_1 v = g_2 v$ if g_1 and g_2 are in the same coset of H. B' is invariant with respect to this action, for

$$B'(\xi x, \xi y) = B(gx, gy) = B(x,y) = B'(x,y)$$

$$(\xi \in G/H,\ x,y \in V^H) \qquad (21)$$

since B is G-invariant (here g is any element in ξ). To prove (20), we need the following generalization of the theorem of 1.4:

THEOREM 6 : Let G,H be as above, and assume that G acts on a real or complex vector space V. Then for $\xi \in G/H$,

$$tr(\xi|V^H) = \frac{1}{|H|}\sum tr(g|V), \qquad (22)$$

where the summation is to be taken over all $g \in \xi$.

Proof : We have already observed that G/H acts on V^H (this is clearly valid also in the complex case). Define a map h from V to itself by

$$h(v) = \frac{1}{|H|}\sum_{g \in \xi} g \cdot v . \qquad (23)$$

Clearly $h(v) \in V^H$, and therefore $tr(h|V) = tr(h|V^H)$. But $h|V^H$ is just the action of ξ, and $tr(h|V)$ is the desired average in (22). This proves Theorem 6.

Returning to the proof of Theorem 5, we have to distinguish between two cases, according as B is symmetric or skew-symmetric. In the first case, we introduced (in 2.

a G-invariant scalar product $\langle\ ,\ \rangle$ on V, a map $A: V \to V$ defined by $\langle Ax, y\rangle = B(x,y)$, and the decomposition $V = V^+ \oplus V^-$ of V into the $+1$ and -1 eigenspaces of A, and defined $\mathrm{Sign}(g,V)$ as $\mathrm{tr}(g|V^+) - \mathrm{tr}(g|V^-)$ Since A commutes with the action of G, the restriction A' of A to V^H takes V^H into itself. The $(+1)$- eigenspaces of A' is $\{x \mid hx = x,\ \text{all}\ h \in H,\ Ax = x\} = (V^+)^H$, and similarly the (-1)-eigenspace is $(V^-)^H$. Then Theorem 6 applied to the real vector spaces V^+ and V^- gives

$$
\begin{aligned}
\mathrm{Sign}(\xi, V^H) &= \mathrm{tr}(\xi \mid (V^H)^+) - \mathrm{tr}(\xi \mid (V^H)^-) \\
&= \mathrm{tr}(\xi \mid (V^+)^H) - \mathrm{tr}(\xi \mid (V^-)^H) \\
&= \frac{1}{|H|} \sum_{g \in \xi} \mathrm{tr}(g|V^+) - \frac{1}{|H|} \sum_{g \in \xi} \mathrm{tr}(g|V^-) \\
&= \frac{1}{|H|} \sum_{g \in \xi} \mathrm{Sign}(g,V). \qquad (24)
\end{aligned}
$$

Similarly, if B is skew-symmetric one shows that the G-invariant complex structure J on V induces a (G/H)-invariant complex structure J' on V^H. Then (20) is obtained by applying the complex version of Theorem 6 to the complex G-space V_J and to its invariant part $(V_J)^H = (V^H)_{J'}$.

4.2 We want to study the α-invariant of the non-trivial element of a free $(\mathbb{Z}/2\mathbb{Z})$-action, i.e. of an involution with no fixed points. We know that the α-invariant of an element of order n lies in $\mathbb{Q}(e^{2\pi i/n})$ (Theorem 3 of 4.1), so the

α-invariant for an involution is certainly rational. We want to show that it is an integer. As in 4.1, we first must show that Sign(g,X) and L(g,X) are both integers (even if X has a non-empty boundary and g acts freely).

In fact, however, we know from 2.1(19)-(21) that, if g is an involution, then Sign(g,X) is 0 if k is odd and equals the Sign(B_g) if k is even, where

$$B_g(x,y) = (x \cup gy)[X] \quad (x,y \in H^k(X)). \qquad (1)$$

Therefore Sign(g,X) is certainly an integer. Another proof of this is to let G be the group consisting of 1 and g; then X/G is a rational homology manifold and it follows from Theorem 4 of 2.1 (which as we pointed out, also holds for non-free actions) that

$$\mathrm{Sign}(g,X) = 2\,\mathrm{Sign}(X/G) - \mathrm{Sign}(X) \qquad (2)$$

and is therefore an integer.

We want to show that the contribution to L(g,X) from each component of X^g for an involution g is also an integer having a natural interpretation as a signature, and that it is 0 if k is odd. An involution can only have eigenvalues ± 1, so $N^g = N^g_\pi$ and 4.1(2) becomes

$$L(g,X) = \{\mathcal{L}(X^g)e(N^g)\mathcal{L}(N^g)^{-1}\}[X^g]. \qquad (3)$$

We claim that there is a submanifold A of X^g whose normal bundle in X^g is i^*N^g (where $i\colon A \subset X^g$, and N^g is the normal bundle of X^g in X) and such that the homology cycle $i_*[A]$ represented by A is the Poincaré dual of $e(N^g)$. (This is at least consistent with the following fact (cf. [36], 4, 11.3) : if $i : A \subset B$ is the inclusion of one oriented manifold in another, and $h \in H^*(B)$ is the Poincaré dual of $i_* [A] \in H_*(B)$, then the Euler class of the normal bundle of A in B is precisely $i^* h$.) Then

$$(y \cup e(N^g))[X^g] = (y)(i_* [A]) = (i^* y)[A] \qquad (4)$$

for any $y \in H^*(X^g)$, so from (3) we obtain

$$\begin{aligned} L(g,X) &= (i^* \{ \mathcal{L}(X^g) \mathcal{L}(N^g)^{-1} \}) [A] \\ &= (i^* \mathcal{L}(X^g) \cdot \mathcal{L}(i^* N^g)^{-1}) [A] \\ &= \mathcal{L}(A) [A] \\ &= \text{Sign } A . \qquad (5) \end{aligned}$$

To find A, we use Thom's transversality theory. Thus we replace the inclusion map of $X^g \longrightarrow X$ by a homotopic smooth map f which is transverse regular on X^g, and let $A = f^{-1}(X^g)$. Then A is oriented, because the normal bundle of A in X^g is $i^* N^g$ and its normal bundle in X is therefore $i^* N^g \oplus i^* N^g$, which is orientable, while the manifold X is orientable by assumption. Moreover, by the usual Thom theory, the oriented cobordism class of A is independent of the choice of f. The manifold A (determined up to cobordism) is called the <u>self-intersection manifold</u> of X^g in X and is denoted $X^g \circ X^g$. We have thus far proved

<u>THEOREM 1</u>: Let g be an orientation-preserving involution of an oriented closed manifold X^{2k}. Let B_g denote the

quadratic form (1), and $X^g \circ X^g$ the oriented cobordism class of the self-intersection manifold of X^g in X. Then we have the following three formulas for Sign(g,X):

$$\mathrm{Sign}(g,X) = \begin{cases} 0, & \text{if } k \text{ is odd.} \\ \mathrm{Sign}\, B_g, & \text{if } k \text{ is even.} \end{cases} \qquad (6)$$

$$\mathrm{Sign}(g,X) = 2\,\mathrm{Sign}(X/g) - \mathrm{Sign}(X). \qquad (7)$$

$$\mathrm{Sign}(g,X) = \mathrm{Sign}(X^g \circ X^g). \qquad (8)$$

From any of these it follows that Sign(g,X) is an integer and vanishes for k odd.

We now wish to show that $\alpha(g,Y)$ is always an integer; to see this, we will give several alternate definitions for $\alpha(g,Y)$, just as we did for Sign(g,X) in Theorem 1. Choose N so that $NY = \partial X$, as in 4.1 (17). We know that Sign(g,X) and L(g,X) are integers, since they are equal to Sign B_g and to $\mathrm{Sign}(X^g \circ X^g)$, respectively. Burdick [10] has shown that N can be taken to be 2, so in any case $2\alpha(g,Y)$ is an integer (we are always assuming that k is even; otherwise α is 0).

We now proceed to describe, without proof, two alternate descriptions of $\alpha(g,Y)$, g an involution, which show that it is an integer. Both can be found in [43], though the first definition had already been given by Browder and Livesay [9] for Y a homotopy sphere.

The Browder-Livesay definition is as follows. A <u>characteristic</u> <u>submanifold</u> for (g, Y^{2k-1}) is a submanifold $W \subset Y$ of codimension one, invariant under g, and such that Y-W has two components which are interchanged by

g; thus $W = \partial A$ for a compact submanifold-with-boundary $A \subset X$ such that $A \cap gA = W$ and $A \cup gA = Y$. Since $gW = g(\partial A) = \partial(gA) = -\partial A = -W$, g is orientation-reversing on W. Such a characteristic submanifold always exists. Let $i: W \to A$ be the inclusion map and

$$K = \ker\,(i_*: H_{k-1}(W;\mathbb{Q}) \to H_{k-1}(A;\mathbb{Q})\,) \tag{9}$$

We define a quadratic form on K (analogous to the form B_g of Theorem 1) by

$$f(x,y) = x \cdot g_* y \tag{10}$$

The f is symmetric: (the dot indicates the intersection form)

$$\begin{aligned} f(y,x) &= y \cdot g_* x = -g_*(y \cdot g_* x) = -g_* y \cdot x \\ &= -(-1)^{k-1}\, x \cdot g_* y = f(x,y). \end{aligned} \tag{11}$$

(because k is even and g reverses the orientation of W)

<u>THEOREM 2</u> : (Hirzebruch und Jänich [43]) : Let g, Y, W, A, K and f be as above. Then

$$\alpha(g,Y) = \operatorname{Sign} f. \tag{12}$$

so in particular the signature of the bilinear form f is independent of W.

The other definition for $\alpha(g,Y)$ given by Hirzebruch and Jänich depends on a construction of Dold [25] (not given here) producing a certain smooth 2k-manifold $\mathfrak{D}$. The boundary of $\mathfrak{D}$ is $Y - 2(Y/g)$. There is an involution T of $\mathfrak{D}$ such that $T|\partial\mathfrak{D}$ interchanges the two components Y/g and equals the involution g on Y.

The quotient $\mathfrak{Y}/T \approx I \times Y/g$, so $\mathfrak{Y}$ is a double branched covering of $(Y/g)\times I$, with branching locus $\mathfrak{D}^g$ and covering transformation T. Moreover, $\mathfrak{D}^g$ has trivial normal bundle in $\mathfrak{Y}$, so $L(g,\mathfrak{Y})$ is 0. Therefore $\mathrm{Sign}(T,\mathfrak{Y})$ $= \alpha(T,\partial\mathfrak{Y}) = \alpha(g,Y) - \alpha(t,2(Y/g))$, where t is the interchange of the two components. But

<u>THEOREM 6</u> : Let M be any oriented manifold and t the interchange on $2\cdot M$. Then

$$\alpha(t,2\cdot M) = 0 \qquad (13)$$

<u>Proof</u> : It is well known that the elements of the oriented cobordism ring Ω_* in dimensions not divisible by 4 are all of finite order. Let X be a manifold with boundary $N\cdot M$ $(N > 0)$. Let t' be the trivial involution (interchange) on $2\cdot X$. Then t', restricted to $\partial(2\cdot X) = N\cdot M$, is just N copies of t, so by definition $\alpha(t,N\cdot M) = \frac{1}{N}\mathrm{Sign}(t',2\cdot X)$ (t' is free, so $L(t', 2X) = 0$). But the intersection matrix giving $\mathrm{Sign}(t',2\cdot X)$ is of the form $\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$ with E a symmetric matrix, and such a form has signature 0. Q.E.D

It follows that

$$\alpha(g,Y) = \mathrm{Sign}(T,\mathfrak{Y}), \qquad (14)$$

giving another proof that α is an integer. Moreover, from (2) and the fact $\mathrm{Sign}(\mathfrak{Y}/T) = \mathrm{Sign}(I\times Y/g) = 0$, we deduce

$$\mathrm{Sign}(T,\mathfrak{Y}) = -\mathrm{Sign}(\mathfrak{Y}), \qquad (15)$$

This gives yet another formula for α which is obviously integral.

4.3 We now consider the actions of the circle group S^1 on a compact manifold X. Since every closed proper subgroup of S^1 is a finite cyclic group, every point in X is either a fixed point of the whole group or is left fixed by only finitely many $t \in S^1$. Let Z be the fixed point set X^{S^1} and assume $Z \cap \partial X = \phi$. Then for all but finitely many $t \in S^1$, $X^t = Z$. The G-signature theorem yields for such t

$$L(t,X) = \left\{ \prod_k \left[\prod_j \frac{t^k e^{2x_j} + 1}{t^k e^{2x_j} - 1} (N_k) \right] \mathcal{L}(Z) \right\} [Z] , \quad (1)$$

where t is thought of as a complex number of norm one, k ranges over all integers, N_k is the complex subbundle of the normal bundle of Z in X on which $e^{i\theta} \in S^1$ acts as multiplication by $e^{ik\theta}$, and the x_j's are as usual the "roots" of the Chern class of N_k. This expression is clearly a meromorphic function of t (defined for all $t \in \mathbb{C}$), holomorphic in $\mathbb{C}$ except at the finitely many values of $t \in S^1$ which have fixed points in $X - Z$.

Expression (1) only makes sense if Z avoids $Y = \partial X$, i.e. if the action of S^1 on Y is <u>fixed-point</u> free (no point left fixed by the whole group). Conversely, if Y is an odd-dimensional closed manifold with a fixed-point free circle action, then it was shown by Ossa [79] that some multiple of Y (which can always be taken as a power of two) bounds an S^1-manifold X, not necessarily fixed-point free. Now $\mathrm{Sign}(t,X) = \mathrm{Sign}(1,X) = \mathrm{Sign}\, X$ for all $t \in S^1$, because S^1 is connected, so the action of t on X is homotopic to the identity and $t^*: H^*(X) \to H^*(X)$ is the identity. Therefore the α-invariant of the action on Y is essentially the function (1), and in particular is a meromorphic function of t with no poles except at values of $t \in S^1$ which have a larger fixed point set than Z, In fact, a more general argument of Atiyah and Singer [3] shows that the only poles of $\alpha(t,Y)$ can be at the values of t which have fixed-points on Y itself.

Two easy consequences of formula (1) are

$$L(t^{-1},x) = (-1)^k L(t,X) \qquad (2)$$

where $2k = \dim X$ and t is not a pole of (1)), and

$$L(\infty,X) = \mathrm{Sign}(Z). \qquad (3)$$

If the action on Y is free rather than just semi-free, we can calculate $L(t,Y)$. In this case, Y is the total space of an S^1-bundle ξ whose base space Y/S^1 is also a manifold. Let X be the associated D^2-bundle; then X is an S^1-manifold with boundary Y and fixed-point set $Z = X^{S^1}$ = zero-section of ξ. We identify the zero-section of the bundle with the base space, and write $x \in H^2(Z;\mathbb{Z})$ for the first Chern class of ξ. The normal bundle of Z in X is, of course, simply ξ with S^1 acting by the first power. Therefore, only $k = 1$ occurs in (1), and $N_1 \cong \xi$, so

$$L(t,X) = \left(\frac{te^{2x} + 1}{te^{2x} - 1} \cdot \mathcal{L}(Z)\right) [Z]. \qquad (4)$$

Moreover, we can apply Theorem 7 of 2.1 to the calculation of $\mathrm{Sign}(X)$, and we get:

<u>THEOREM 1</u> (Atiyah and Singer [3]): Let Y^{2k-1} be an odd-dimensional manifold on which S^1 acts <u>freely,</u> and let $Z = Y/S^1$ be the base space of the corresponding

complex line bundle ξ. Let $x = c_1(\xi)$. Then

$$\alpha(t,Y) = -\left(\frac{te^{2x}+1}{te^{2x}-1}\mathcal{L}(Z)\right)[Z] + \mathrm{Sign}(Z,x), \qquad (5)$$

where $\mathrm{Sign}(Z,x)$ is the signature of the quadratic form

$$a,b \longmapsto (abx)[Z] \qquad (a,b \in H^{k-2}(Z)). \qquad (6)$$

<u>Corollary:</u> Let Z be a manifold of even dimension, and $x_1,\ldots,x_r \in H^2(Z)$ be two-dimensional integral cohomology classes. Then the number

$$\tau(x_1,\ldots,x_r) = \langle \tanh x_1 \ldots \tanh x_r\, \mathcal{L}(Z), [Z] \rangle \qquad (7)$$

is an integer.

<u>Proof of corollary</u>: First suppose that $r = 1$. Let ξ be the complex line bundle with $c_1(\xi) = x_1$ and Y the associated principal S^1-bundle. Then S^1 acts freely on Y with quotient space Z, and from (5) we deduce

$$\alpha(-1,Y) = -\tau(x_1) + \mathrm{Sign}(Z,x_1). \qquad (8)$$

Since the action of $-1 \in S^1$ is an involution, it follows from 4.2 that $\alpha(-1,Y) \in \mathbb{Z}$. But $\mathrm{Sign}(Z,x)$ is an integer by definition; hence $\tau(x_1) \in \mathbb{Z}$.

Now let r be arbitrary, ξ_i the line bundle classified by x_i $(1 \leq i \leq r)$, X the disc bundle associated to the vector bundle $\xi_1 \oplus \ldots \oplus \xi_r$, and Y the associated sphere bundle. The torus group T^r acts freely on Y by a smooth action extending to X (namely $t = (t_1,\ldots,t_r) \in T$ acts by multiplication with t_i on ξ_i). It is clear that for $t = (t_1,\ldots,t_r)$ with all $t_i \neq 1$,

$$L(t,X) = \left\langle \prod_{i=1}^{r} \frac{t_i e^{2x_i}+1}{t_i e^{2x_i}-1}\, \mathcal{L}(Z),[Z] \right\rangle, \tag{9}$$

since the fixed-point set X^t is precisely the zero-section Z of the bundle X. On the other hand,

$$\mathrm{Sign}(t,X) = \mathrm{Sign}(X) \in \mathbb{Z} \tag{10}$$

(because the group T^r is connected, so $t \in T^r$ acts as the identity on $H^*(X)$). We now set $t = (-1,\ldots,-1) \in T^r$; this is the involution of Y defined as the antipodal map on each fibre S^{2r-1}. Again by 4.2, $\alpha(t,Y) \in \mathbb{Z}$, while from (9) and (10) we obtain

$$\alpha(t,Y) = \mathrm{Sign}\, X + \tau(x_1,\ldots,x_n). \tag{11}$$

The desired conclusion follows.

The number $\tau(x_1,\ldots,x_r)$ defined in (7) is called the <u>virtual index</u> (cf. [36]). That it is always an integer was known before the G-signature theorem had been proved and the α-invariant defined. One can, namely, interpret $\tau(x_1,\ldots,x_r)$ in the following way: According to Thom [94], the homology class dual to $x_1 \in H^2(Z;\mathbb{Z})$ can be represented by a submanifold V_1 of Z (of codimension 2). Applying this theorem repeatedly, we obtain $Z \supset V_1 \supset \ldots \supset V_r$ with V_i a submanifold of codimension 2 in V_{i-1} representing in $H_*(V_{i-1})$ the class dual to the restriction of x_i to $H^*(V_{i-1})$. Then

$$\tau(x_1,\ldots,x_r) = \mathrm{Sign}\, V_r. \tag{12}$$

To see this, say, for $r = 1$, we calculate

$$\text{Sign } V_1 = \langle \mathcal{L}(V_1), [V_1] \rangle$$

$$= \langle j^*\mathcal{L}(Z)\cdot\mathcal{L}(\nu)^{-1}, [V_1] \rangle$$

(where $j: V_1 \subset Z$ is the inclusion and ν the normal bundle of V_1 in Z)

$$= \langle j^*\mathcal{L}(Z)\cdot \frac{\tanh j^*x_1}{j^*x_1}, [V_1] \rangle$$

(since ν is a line bundle with first Chem class j^*x_1)

$$= \langle \mathcal{L}(Z)\cdot \frac{\tanh x_1}{x_1}, \; j_* [V_1] \rangle$$

$$= \langle \mathcal{L}(Z) \tanh x_1, [Z] \rangle$$

(since $j_*[V_1] = x_1 \cap [Z]$ by assumption). Equation (12) for $r > 1$ is proved by induction. It is clear from (12) that $\tau(x_1,\ldots,x_r)$ is an integer.

We now give a further formula relating the virtual index and the α-invariant of an S^1-action.

<u>THEOREM 2</u>: Let Y, Z, x be as in Theorem 1. Then for $r \in \mathbb{Z}$, the virtual index of rx is given by

$$\tau(rx) = \text{res}_{t=1}\left[\frac{t^r-1}{t^r+1}\,\alpha(t,Y)\,\frac{dt}{2t}\right]$$

$$= \text{res}_{x=0}\left[\tanh rx\cdot\alpha(e^{2x},Y)\cdot dx\right] \tag{13}$$

<u>Proof:</u> The function $\alpha(t) = \alpha(t,Y)$ has a pole at $t = 1$ and is regular elsewhere. Therefore, applying the residue theorem to the rational function $\frac{t^r-1}{t^r+1}\;\frac{\alpha(t)}{2t}$, we obtain

$$\text{res}_{t=1}\left[\frac{t^r-1}{t^r+1}\,\alpha(t)\frac{dt}{2t}\right] = -(\text{res}_0+\text{res}_\infty+\sum_{t^r=-1}\text{res}_t)\left[\frac{t^r-1}{t^r+1}\alpha(t)\frac{dt}{2t}\right]$$

$$= \tfrac{1}{2}\alpha(0) + \tfrac{1}{2}\alpha(\infty) - \frac{1}{r}\sum_{t^r=-1}\alpha(t). \qquad (14)$$

For convenience, we define

$$\bar{\alpha}(t) = \alpha(t) - \tfrac{1}{2}(\alpha(0) + \alpha(\infty)). \qquad (15)$$

Thus $\bar{\alpha}(t)$ differs from $\alpha(t)$ only by a constant. The right-hand side of (14) is now

$$-\frac{1}{r}\sum_{t^r=-1}\bar{\alpha}(t). \qquad (16)$$

But, by Theorem 1,

$$\bar{\alpha}(t) = -\left\langle \frac{te^{2x}+1}{te^{2x}-1}\,\mathcal{L}(Z),[Z]\right\rangle. \qquad (17)$$

Therefore we obtain

$$\text{res}_{t=1}\left[\frac{t^r-1}{t^r+1}\,\alpha(t)\frac{dt}{2t}\right] = \frac{1}{r}\sum_{t^r=-1}\left\langle \frac{te^{2x}+1}{te^{2x}-1}\,\mathcal{L}(Z),[Z]\right\rangle$$

$$= \left\langle \frac{e^{2rx}-1}{e^{2rx}+1}\,\mathcal{L}(Z),\ Z\right\rangle$$

$$= \tau(rx), \qquad (18)$$

as was to be proved.

Theorem 2 is only interesting if the dimension of Y is congruent to 3 (modulo 4), since otherwise both sides vanish. If this is the case, then it gives considerable information about the L-class of Y/S^1 in terms of the α-invariant of the S^1-action on Y: namely, since the coefficient of x^m in tanh x is non-zero for all odd m, we obtain from (13) the value of $\langle x^m \mathcal{L}(Z),[Z]\rangle$ for all m in terms of $\alpha(t,Y)$. We deduce, in fact, that

$$\langle f(x)\mathcal{L}(Z),[Z]\rangle = \operatorname{res}_{x=0}\left[f(x)\alpha(e^{2x},Y)dx\right] \tag{19}$$

for any $f \in \mathbb{C}[[x]]$.

The question now arises whether there is any analogous relation between $\alpha(t,Y)$ and $L(Y/S^1)$ also for S^1-actions which are not free. Of course, in general this question does not make sense, since Y/S^1 will not in general be a manifold. However, if we make the additional assumption that S^1 acts fixed-point freely on Y, so that $\alpha(t,Y)$ is defined, then the quotient space Y/S^1 will be a rational homology manifold and hence have an L-class (cf. §3.1). We wish to state an analogue of Theorem 2. First we must define a class $x \in H^2(Z)$ which replaces the first Chern class of the S^1-bundle $Y \longrightarrow Y/S^1 = Z$ in the free case. The points $t \in S^1$ which have a non-empty fixed-point set lie in some cyclic subgroup $G_N \subset S^1$. We use the exact sequence

$$0 \longrightarrow G_N \longrightarrow S^1 \underset{t \longmapsto t^N}{\longrightarrow} S^1 \longrightarrow 0 \tag{20}$$

to define

$$Y \longrightarrow \hat{Y} = Y/G_N \longrightarrow Z = Y/S^1 ; \tag{21}$$

then $Y \longrightarrow \hat{Y}$ is the projection of a manifold onto its quotient by a finite group action, and $\hat{Y} \longrightarrow Z$ is the projection onto the quotient space of a free S^1-action. Because the group G_N is finite and acts smoothly and orientably on Y, the space $\hat{Y}$ is a rational homology manifold (cf. 3.2). Then S^1 acts freely and $\hat{Y} \longrightarrow Z$ is a principal S^1-bundle over Z. Let ξ be the corresponding complex line bundle and set

$$x = \frac{1}{N} c_1(\xi) \in H^2(Z;\mathbb{Q}). \tag{22}$$

This class will serve as "first Chern class of $Y \longrightarrow Z$;" that the map $Y \longrightarrow Z$ is not actually a bundle is reflected in the fact that x is not an integral cohomology class. It is easy to see that (22) is a sensible definition: if we had chosen a different multiple N' of the orders of the (finite) isotropy groups of the points of Y, then $\frac{1}{N'} c_1(\xi')$ would be the same as x. We can now formulate:

<u>THEOREM 3</u>: Let Y be a smooth, oriented manifold whose dimension is congruent to 3 (modulo 4), and suppose that S^1 acts on Y with $Y^{S^1} = \emptyset$. Let $Z = Y/S^1$ and $x \in H^2(Z)$ be defined as in (22). Then, for any $r \in \mathbb{Z}$,

$$\langle \tanh Nrx \cdot \mathcal{L}(Z), [Z] \rangle = \sum_{\alpha(\zeta,Y)=\infty} \operatorname{res}_\zeta \left[\frac{t^{Nr}-1}{t^{Nr}+1} \alpha(t,Y) \frac{dt}{2t} \right] \tag{23}$$

(sum over all poles of the rational function $\alpha(t,Y)$).

This theorem has been proved by Atiyah, using the methods of [2]. Again we can deduce that

$$\langle f(x)\mathcal{L}(Z), [Z] \rangle = \sum_{\alpha(e^{2x},Y)=\infty} \operatorname{res}_x \left[f(x)\alpha(e^{2x},Y)dx \right] \tag{24}$$

for any power series f(x), and in particular for f(x) = tanh kx, even if $N \nmid k$. But the statement given is more interesting because, since Nx is an integral cohomology class, and since Thom's transversality theory holds also for rational homology manifolds, the left-hand side equals Sign V ($V \subset Z$ a rational homology submanifold representing the homology class dual to Nrx) and is therefore an integer.

Observe that, just as in the free case, we can rewrite the right-hand side of (23) as a linear combination of values of $\alpha(t,Y)$, namely as

$$\frac{1}{2}\,\alpha(0,Y) + \frac{1}{2}\alpha(\infty,Y) - \frac{1}{Nr} \sum_{t^{Nr}=-1} \alpha(t,Y), \tag{25}$$

because the poles of $\alpha(t,Y)$ all occur at $t^N = 1$ and are thus disjoint from the poles of $\frac{1}{2t}\,\frac{t^{Nr}-1}{t^{Nr}+1}$.

We can also rewrite the right-hand side of (23), by using the fact that all poles of $\alpha(t,Y)$ are N^{th} roots of unity and replacing t by ζt:

$$\begin{aligned}\tau(rNx) &= \sum_{\zeta^N=1} \operatorname{res}_{t=\zeta}\left[\frac{t^{Nr}-1}{t^{Nr}+1}\,\alpha(t,Y)\,\frac{dt}{2t}\right] \\ &= \sum_{\zeta^N=1} \operatorname{res}_{t=1}\left[\frac{t^{Nr}-1}{t^{Nr}+1}\,\alpha(\zeta t,Y)\,\frac{dt}{2t}\right] \\ &= \operatorname{res}_{y=0}\left\{\tanh ry \left[\frac{1}{N}\sum_{\zeta^N=1} \alpha(\zeta e^{2y/N},Y)\right]dy\right\}\end{aligned} \tag{26}$$

This becomes nearly identical to equation (13) if we define the α-invariant of the free S^1-action on $\hat{Y}$ by

$$\alpha(t,\hat{Y}) =_{DEF} \frac{1}{N} \sum_{s^N=t} \alpha(s,Y). \tag{27}$$

This formal definition is consistent with Theorem 4 of 4.1, and can be justified in the framework of the theory described in [2].

CHAPTER II: COTANGENT SUMS AND RELATED NUMBER THEORY

The form of the G-signature formula of Atiyah and Singer makes it natural to study sums of terms which are products of cotangents (or hyperbolic cotangents). Another motivation for the study of such sums was the observation that they can be used to rewrite results obtained by Brieskorn [7] for the signatures of certain algebraic varieties.

These trigonometric sums, whose appearance in topology is still rather mysterious, are not new. Although not particularly well known, even among number theorists, they have turned up before in contexts so varied as to make their appearance in topology less startling: in connection with the law of quadratic reciprocity, the theory of modular forms, class invariants of quadratic number fields and, more recently, the problem of generating random numbers. In this chapter we will study the elementary properties of these sums and discuss some of their relations to other topics in number theory, postponing to the following chapter the discussion of their applications in topological problems.

§ 5. Elementary properties of cotangent sums

5.1 Let p,q be coprime integers, p>0. We define the Dedekind sum s(q,p) by

$$s(q,p) = \sum_{k=1}^{p} \left(\left(\frac{k}{p}\right)\right) \left(\left(\frac{kq}{p}\right)\right), \qquad (1)$$

where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \in \mathbb{R},\ x \in \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}, \end{cases} \tag{2}$$

(here $[x]$ is the greatest integer $\leq x$).

<u>THEOREM 1</u>: (Dedekind reciprocity law) : Let p,q be coprime positive integers. Then

$$s(q,p) + s(p,q) = \frac{p^2 + q^2 + 1 - 3pq}{12pq}. \tag{3}$$

<u>Proof</u> : Since $\{kq \mid 0 < k < p\}$ runs over a system of non-zero residues modulo p, and since $((x))$ is a periodic function with period 1, we have

$$\sum_{k=1}^{p-1} \left(\left(\frac{kq}{p}\right)\right)^r = \sum_{k=1}^{p-1} \left(\left(\frac{k}{p}\right)\right)^r \tag{4}$$

for any integer r. But $((\frac{k}{p})) = \frac{k}{p} - \frac{1}{2}$ for $0 < k < p$, so the right-hand side of (4) can be evaluated explicitly. The cases $r = 1,2$ give

$$\sum_{k=1}^{p-1} \left(\left(\frac{kq}{p}\right)\right) = 0, \tag{5}$$

$$\sum_{k=1}^{p-1} \left(\left(\frac{kq}{p}\right)\right)^2 = \frac{(p-1)(q-1)}{12p}. \tag{6}$$

Expanding the left-hand side of (5) gives

$$\sum_{k=1}^{p-1} \left[\frac{kq}{p}\right] = \frac{(p-1)(q-1)}{2}, \tag{7}$$

a statement which we will use often. Expanding the left-hand side of (6), and using (7), we obtain

$$\sum_{k=1}^{p-1}\left[\frac{kq}{p}\right]^2 - 2\,\frac{q}{p}\sum_{k=1}^{p-1} k\left[\frac{kq}{p}\right] = (1-q^2)\frac{(p-1)(2p-1)}{6p}. \tag{8}$$

We introduce a new symbol, which is equivalent to the Dedekind sum but which is easier to work with and has the advantage of being an integer:

$$f_p(q) = \sum_{k=1}^{p-1} k\left[\frac{kq}{p}\right], \qquad (p,q) = 1. \tag{9}$$

Then (using (5))

$$6p\ s(q,p) = 6p\sum_{k=1}^{p-1}\left\{\left(\left(\frac{k}{p}\right)\right) + \frac{1}{2}\right\}\cdot\left(\left(\frac{kq}{p}\right)\right)$$

$$= 6\sum_{k=1}^{p-1} k\left\{\frac{kq}{p} - \left[\frac{kq}{p}\right] - \frac{1}{2}\right\}$$

$$= (p-1)(2pq-q-\frac{3p}{2}) - 6f_p(q). \qquad (10$$

Equation (8) relates $f_p(q)$ to $\Sigma\left[\frac{kq}{p}\right]^2$. We can relate $f_q(p)$ to this same sum by the simple trick of expanding $[x]$ as $\sum_{0<k<x} 1$:

$$f_q(p) = \sum_{\ell=1}^{q-1} \ell\left[\frac{\ell p}{q}\right] = \sum_{\ell=1}^{q-1} \ell \sum_{0<k<\ell p/q} 1$$

$$= \sum_{0<\frac{k}{p}<\frac{\ell}{q}<1} \ell = \sum_{k=1}^{p-1}\ \sum_{\ell=[kq/p]+1}^{q-1} \ell$$

$$= \sum_{k=1}^{p-1} \left\{ \frac{q^2-q}{2} - \frac{[kq/p]^2 + [kq/p]}{2} \right\}$$

$$= \frac{1}{4}(p-1)(q-1)(2q-1) - \frac{1}{2} \sum_{k=1}^{p-1} \left[\frac{kq}{p}\right]^2 . \qquad (11$$

Combining this with (8) gives the reciprocity law

$$p\ f_q(p) + q\ f_p(q) = \frac{1}{12}\ (p-1)(q-1)(8pq-p-q-1), \qquad (12$$

and we can then use (10) to translate this into the desired reciprocity law for $s(q,p)$.

We can now deduce other properties of Dedekind sums. It follows immediately from (10) that

$$6p\ s(q,p) \in \mathbb{Z}; \qquad (13$$

indeed, we can even deduce from (10) that

$$6p\ s(q,p) \equiv 6p\ s(1,p) = \frac{(p-1)(p-2)}{2} \pmod 2 \qquad (14$$

(the second equality is equation (6)), a congruence which will receive an interpretation in Section 6.2.

Next, we can deduce from equations (12) and (3) that

$$12q\ f_p(q) \equiv q^2 - 1 \pmod p \qquad (15$$

$$12qp\ s(q,p) \equiv q^2 + 1 \pmod p. \qquad (16$$

If we define the "socius" q' (mod p) by

$$qq' \equiv 1 \pmod p \qquad (17$$

then we can rewrite (15) and (16) as

$$q' \equiv q - 12f_p(q) \pmod p, \tag{18}$$

$$q' \equiv 12p\, s(q,p) - q \pmod p. \tag{19}$$

(The right-hand side of (19) is an integer, by (13).) This gives elegant formulas for the socius in terms of the sum (9) or the Dedekind sum (cf. Meyer [61], Lerch [58], Rademacher [81]).

We now define a generalized Dedekind sum

$$s(q,r;p) = \sum_{k=1}^{p} \left(\left(\frac{kq}{p}\right)\right)\left(\left(\frac{kr}{p}\right)\right), \tag{20}$$

where $p > 0$ and q,r are integers prime to p. By the same sort of argument as that preceding (4), we have

$$s(q,r;p) = s(qx,rx;p) \tag{21}$$

for any x prime to p. In particular,

$$s(q,r;p) = s(qq',rq';p) = s(1,rq';p) = s(rq',p), \tag{22}$$

so the sum (20) can be reduced to the ordinary Dedekind sum. Nevertheless, the expression (20) is of interest because it satisfies a reciprocity law generalizing Thèorem 1:

<u>THEOREM 2</u> (Rademacher reciprocity law): Let p,q,r be pairwise coprime positive integers. Then

$$s(q,r;p) + s(p,r;q) + s(p,q;r) = \frac{p^2 + q^2 + r^2 - 3pqr}{12pqr}. \tag{23}$$

(The special case $r = 1$ is just Dedekind reciprocity).

<u>Lemma:</u> Let p,q,r be as in the theorem. Then

$$\sum_{x=1}^{p-1} \left[\frac{xq}{p}\right]\left[\frac{xr}{p}\right] + \sum_{y=1}^{q-1} \left[\frac{yp}{q}\right]\left[\frac{yr}{q}\right] + \sum_{z=1}^{r-1} \left[\frac{zp}{r}\right]\left[\frac{zq}{r}\right]$$

$$= (p-1)(q-1)(r-1). \qquad (24)$$

Proof: This generalises equation (7). To prove it, consider

$$\sum_{\substack{0<x<p\\0<y<q\\0<z<r}} 1 = (p-1)(q-1)(r-1) \qquad (25)$$

We break this up into six sums, according to the relative sizes of the three numbers $\frac{x}{p}, \frac{y}{q}, \frac{z}{r} \in (0,1)$. (These numbers are certainly unequal, because p,q,r are mutually prime to one another). One of these sums is

$$\sum_{0<\frac{x}{p}<\frac{y}{q}<\frac{z}{r}<1} 1 = \sum_{y=1}^{q-1} \left(\sum_{0<x<yp/q} 1\right)\left(\sum_{yr/q<z<r} 1\right)$$

$$= \sum_{y=1}^{q-1} \left[\frac{yp}{q}\right]\left(r-1-\left[\frac{yr}{q}\right]\right)$$

$$= \frac{1}{2}(p-1)(q-1)(r-1) - \sum_{y=1}^{q-1} \left[\frac{yp}{q}\right]\left[\frac{yr}{q}\right]. \qquad (26)$$

If we plug (26) and its five permutations into (25), we get (24

Proof of Theorem 2 : Expanding (20) and using (7) and (9), we find

$$s(q,r;p) = \sum_{k=1}^{p-1} \left\{\frac{kq}{p} - \left[\frac{kq}{p}\right] - \frac{1}{2}\right\}\left\{\frac{kr}{p} - \left[\frac{kr}{p}\right] - \frac{1}{2}\right\}$$

$$= qr\frac{(p-1)(2p-1)}{6p} - \frac{p-1}{4} - \frac{q}{r} f_p(r) - \frac{r}{p} f_p(q)$$

$$+ \sum_{k=1}^{p-1} \left[\frac{kq}{p}\right]\left[\frac{kr}{p}\right].$$

Therefore, by virtue of (12) and the lemma, the expression $s(q,r;p) + s(p,r;q) + s(p,q;r)$ can be evaluated completely as a rational function of p, q and r. The result of the computation is (23).

5.2 As explained in the introduction to the book, the main interest to us of sums like the Dedekind sum is that they can also be given by trigonometric expressions. We now give one way of doing this.

The function $((x))$ is odd and periodic, therefore has a Fourier series of the form

$$((x)) = \sum_{n=1}^{\infty} a_n \sin 2\pi nx,$$

$$a_n = 2 \int_0^1 ((x)) \sin 2\pi nx \, dx$$

$$= \int_0^1 (2x-1) \sin 2\pi nx \, dx$$

$$= -\frac{1}{\pi n}.$$

Hence for integers a and p

$$((\tfrac{a}{p})) = -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{2\pi na}{p}$$

$$= -\frac{1}{\pi} \lim_{N\to\infty} \sum_{k=1}^{p-1} \sin \frac{2\pi ka}{p} \sum_{\substack{0<n<N \\ n\equiv k \bmod p}} \frac{1}{n}$$

Since $\sin (2\pi ka/p)$ is odd, this is unchanged if we replace the last sum by

$$\frac{1}{2} \lim_{N\to\infty} \left[\sum_{\substack{0<n<N \\ n\equiv k \bmod p}} \frac{1}{n} - \sum_{\substack{0<n<N \\ n\equiv -k \bmod p}} \frac{1}{n} \right] = \frac{1}{2} \lim_{S\to\infty} \left[\sum_{-S<s<S} \frac{1}{k+ps} \right]$$

$$= \frac{\pi}{2p} \cot \frac{\pi k}{p}$$

because

$$\pi \cot \pi x = \lim_{S \to \infty} \left[\sum_{s=-S}^{S} \frac{1}{x+s} \right].$$

Hence

$$((\frac{a}{p})) = -\frac{1}{2p} \sum_{k=1}^{p-1} \sin \frac{2\pi k a}{p} \cot \frac{\pi k}{p}.$$

In other notation,

$$((\frac{a}{p})) = -\frac{1}{2p} \sum_{\substack{\lambda^p=1 \\ \lambda \neq 1}} \left(\frac{\lambda^a - \lambda^{-a}}{2i} \right) \frac{\lambda+1}{\lambda-1}.$$

Since $\frac{\lambda+1}{\lambda-1}$ is replaced by its negative under $\lambda \to \lambda^{-1}$, this gives

$$((\frac{a}{p})) = -\frac{1}{2pi} \sum_{\substack{\lambda^p=1 \\ \lambda \neq 1}} \lambda^a . \frac{\lambda+1}{\lambda-1} .$$

Formulas (1) and (2) are due to Eisenstein [27].

Now we can easily evaluate

$$s(q,p) = \sum_{k=1}^{p} ((\frac{k}{p})) ((\frac{kq}{p}))$$

$$= \frac{-1}{4p^2} \sum_{k=1}^{p} \sum_{\substack{\lambda^p = \zeta^p = 1 \\ \lambda, \zeta \neq 1}} \lambda^k . \zeta^{qk} . \frac{\lambda+1}{\lambda-1} \frac{\zeta+1}{\zeta-1}$$

Since

$$\sum_{k=1}^{p} \lambda^k . \zeta^{qk} = \begin{cases} 0 & \lambda\zeta^q \neq 1 , \\ p & \lambda\zeta^q = 1 , \end{cases}$$

this equals

$$-\frac{1}{4p}\sum_{\substack{\zeta^p=1\\ \zeta\neq 1}} \frac{\zeta^{-q}+1}{\zeta^{-q}-1}\cdot\frac{\zeta+1}{\zeta-1},$$

so

$$s(q,p) = \frac{1}{4p}\sum_{k=1}^{p-1} \cot\frac{\pi k}{p}\cot\frac{\pi kq}{p} \qquad (3)$$

We can use 5.1(22) to write this in a symmetric form:

THEOREM 1: Let p be a positive integer, and q,r prime to p. Then

$$s(q,r;p) = \frac{1}{4p}\sum_{k=1}^{p-1} \cot\frac{\pi kq}{p}\cot\frac{\pi kr}{p}. \qquad (4)$$

This is a special case of the following theorem which we do not prove.

THEOREM 2 [100]: Let a be a positive integer, and $a_1,\ldots,a_n$ prime to a (n even). Define

$$\delta(a;a_1,\ldots,a_n) = \sum_{\substack{0<b_1,\ldots b_n<a\\ a\mid(a_1b_1+\ldots+a_nb_n)}} \left(\frac{2b_1}{a}-1\right)\ldots\left(\frac{2b_n}{a}-1\right). \qquad (5)$$

Then

$$\delta(a;a_1,\ldots,a_n) = \frac{(-1)^{n/2}}{a}\sum_{j=1}^{a-1} \cot\frac{\pi j a_1}{a}\ldots\cot\frac{\pi j a_n}{a}. \qquad (6)$$

The numbers $\delta(a;a_1,\ldots,a_n)$, which directly generalize the Dedekind sums ($\delta(a;b,c) = -4\, s(b,c;a)$), satisfy a similar reciprocity, generalizing Theorem 2 of 5.1:

THEOREM 3 [100]: Let $a_0,\ldots,a_n$ (n even) be mutually coprime positive integers. Then

$$\sum_{k=0}^{n} \delta(a_k; a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n) = 1 - \frac{\ell_n(a_0, \ldots, a_n)}{a_0 \ldots a_n} \quad (7)$$

where ℓ_n is a certain polynomial in $n+1$ variables which is symmetric in its variables, even in each variable, and homogeneous of degree n. For instance

$$\ell_2(a,b,c) = \tfrac{1}{3}(a^2 + b^2 + c^2), \quad (8)$$

$$\ell_4(a,b,c,d,e) = \tfrac{1}{18}(a^2 + b^2 + c^2 + d^2 + e^2)^2$$

$$- \tfrac{7}{90}(a^4 + b^4 + c^4 + d^4 + e^4). \quad (9)$$

About the proofs of these two formulas we only remark that (6) is proved just like the special case (3), by the use of the Eisenstein formula for $((\frac{a}{p}))$, and that the reciprocity law (7) is proved using equation (6) and the calculus of residues.

The interest of Theorem 3 is that the polynomials ℓ_n which occur are essentially the same as the polynomials giving the L-class (cf. 2.3) in terms of the Pontrjagin classes. To be more precise, let p_i $(i = 1,2,\ldots,n/2)$ be the i^{th} elementary symmetric function in $a_0^2, \ldots, a_n^2$; then $\ell_n(a_0, \ldots, a_n$ equals a certain polynomial $L_k(p_1, \ldots, p_k)$ in the p_i's $(k=n/2)$ This polynomial L_k is exactly the polynomial expressing the component of $\mathcal{L}(\xi)$ in $H^{4k}(X)$ in terms of $p_1(\xi), \ldots, p_k(\xi)$, where ξ is a bundle over a space X, $\mathcal{L}(\xi)$ the L-class of ξ, and $p_i(\xi) \in H^{4i}(X)$ the i^{th} Pontrjagin class.

5.3 In 5.2 we indicated a method of using Fourier analysis to obtain identities between rational and cotangent expressions. Here we give a slightly different method, based more directly on the calculus of residues, which is useful for expressions like the Brieskorn sums defined in the introduction to the book

Let $a_1,\dots a_n$ and p be positive integers, and set

$$c_i = \#\{k_1,\dots,k_n \mid 0< k_1,\dots,k_n<p,\ a_1k_1+\dots+a_nk_n=i\}, \tag{1}$$

$$\begin{aligned} t_p^{\text{even}}(a_1,\dots,a_n) &= c_0 + c_{2p} + c_{4p} + \dots \\ &= \#\{k_1,\dots,k_n \mid 0<k_1,\dots,k_n<p,\ \frac{a_1k_1+\dots+a_nk_n}{p} \\ &\qquad \text{is an even integer}\}, \end{aligned} \tag{2}$$

$$\begin{aligned} t_p^{\text{odd}}(a_1,\dots,a_n) &= c_p + c_{3p} + c_{5p} + \dots \\ &= \#\{k_1,\dots,k_n \mid 0<k_1,\dots,k_n<p,\ \frac{a_1k_1+\dots+a_nk_n}{p} \\ &\qquad \text{is an odd integer}\}, \end{aligned} \tag{3}$$

$$\begin{aligned} t_p(a_1,\dots,a_n) &= \sum_{r\geq 0} (-1)^r c_{rp} \\ &= t_p^{\text{even}}(a_1,\dots,a_n) - t_p^{\text{odd}}(a_1,\dots,a_n). \end{aligned} \tag{4}$$

<u>THEOREM 1</u>: Let p, $a_1,\dots,a_n$ (n even) be positive integers with each a_i an odd integer not divisible by p. Then

$$t_p(a_1,\dots,a_n) = \frac{(-1)^{n/2}}{p} \sum_{\substack{j=1 \\ j \text{ odd}}}^{2p-1} \cot\frac{\pi j a_1}{2p}\cdots\cot\frac{\pi j a_n}{2p}. \tag{5}$$

By virtue of 5.2(6), this can also be written

$$t_p(a_1,\ldots,a_n) = 2\,\delta(2p;a_1,\ldots,a_n) - \delta(p;a_1,\ldots,a_n). \qquad (6)$$

Proof: Write

$$f(t) = \sum_{i\geq 0} c_i\, t^i\ ; \qquad (7)$$

this is a polynomial because $c_i = 0$ for $i>(p-1)(a_1+\ldots+a_n)$.
Then

$$t_p(a_1,\ldots,a_n) = \sum_{r>0} (-1)^r \operatorname{res}_{t=0} \left\{ f(\tfrac{1}{t})\, t^r\, \frac{dt}{t} \right\}$$

$$= \operatorname{res}_{t=0} \left\{ \frac{f(\frac{1}{t})}{1+t^p}\, \frac{dt}{t} \right\}. \qquad (8)$$

This is a legitimate expression because, since f is a polynomial, the function $f(1/t)$ is meromorphic. For the same reason, $f(1/t)$ has no poles for $t\neq 0$ and is holomorphic at infinity, so the only poles of the expression in (8) are $t=0$, $t^p = -1$. The residue theorem therefore gives

$$t_p(a_1,\ldots,a_n) = - \sum_{\zeta^p=-1} \operatorname{res}_{t=\zeta} \left[\frac{f(1/t)\ dt}{t(1+t^p)} \right]$$

$$= \frac{1}{p} \sum_{\zeta^p=-1} f(\zeta^{-1}). \qquad (9)$$

But it is clear that

$$f(t) = \prod_{k=1}^{n} (t^{a_k} + t^{2a_k} + \ldots t^{(p-1)a_k})$$

$$= \prod_{k=1}^{n} \frac{t^{pa_k} - t^{a_k}}{t^{a_k} - 1}. \qquad (10$$

If we set $t = \zeta^{-1} = e^{-\pi ij/p}$ $(0<j<2p,\ j \text{ odd})$ in (10), then the k^{th} factor is clearly p if $2p|a_k$, -1 if $2p\nmid a_k$ and a_k is even, and $i \cot \frac{\pi j a_k}{2p}$ if $p\nmid a_k$ and a_k is odd. We are assuming that the last possibility is always the case; then (9) reduces to (5).

Let $a_1,\ldots,a_n$, $b_1,\ldots,b_n$ be integers $(a_j>0)$ and N a common multiple of the a_i's. In the above proof, replace $f(t)$ by

$$\prod_{k=1}^{n} (t^{Nb_k/a_k} + t^{2Nb_k/a_k} + \ldots + t^{(a_k-1)Nb_k/a_k}) \tag{11}$$

and $1/(1+t^p)$ by

$$g(t) = \frac{t - t^N}{1-t} \cdot \frac{1}{1+t^N} \tag{12}$$

in (8). Then exactly the same proof as before gives:

THEOREM 2: *Let n be odd, $a_1,\ldots,a_n > 0$ and $b_1,\ldots,b_n$ be integers with b_j odd. Let N be any common multiple of the a_i's. Then*

$$t(a_1,\ldots,a_n;\ b_1,\ldots,b_n) = \frac{(-1)^{\frac{n-1}{2}}}{N} \sum_{\substack{k=1 \\ k \text{ odd}}}^{2N-1} \cot\frac{\pi k}{2N} \times \cot\frac{\pi k b_1}{2a_1} \ldots \cot\frac{\pi k b_n}{2a_n}, \tag{13}$$

where by definition

$$t(a_1,\ldots,a_n;\ b_1,\ldots,b_n) =$$

$$= \#\{0<x_1<a_1,\dots,0<x_n<a_n \mid 0<\frac{b_1x_1}{a_1}+\dots+\frac{b_nx_n}{a_n}<1 \mod 2\}$$

$$- \#\{0<x_1<a_1,\dots,0<x_n<a_n \mid <\frac{b_1x_1}{a_1}+\dots+\frac{b_nx_n}{a_n}<2 \mod 2\} \tag{14}$$

Notice that if we define

$$h(x) = \begin{cases} (-1)^x & (x \in \mathbb{Z}) \\ 0 & (x \in \mathbb{Z}), \end{cases} \tag{15}$$

then equation (14) states that

$$t(a_1,\dots,a_n;b_1,\dots,b_n) = \sum_{x_1=1}^{a_1-1}\dots\sum_{x_n=1}^{a_n-1} h\left(\frac{b_1x_1}{a_1}+\dots+\frac{b_nx_n}{a_n}\right). \tag{16}$$

If we represent $h(x)$ by its Fourier series

$$h(x) = \frac{2}{i\pi}\sum_{\substack{k=-\infty \\ k \text{ odd}}}^{\infty} \frac{e^{i\pi kx}}{k},$$

we obtain for (14) the formula

$$\frac{2}{i\pi}\sum_{\substack{k=-\infty \\ k \text{ odd}}}^{\infty} \frac{1}{k}\left(\sum_{x_1=1}^{a_1-1}\dots\sum_{x_n=1}^{a_n-1} e^{\pi ik\left(\frac{b_1x_1}{a_1}+\dots+\frac{b_nx_n}{a_n}\right)}\right)$$

$$= \frac{2}{i\pi}\sum_{\substack{k=-\infty \\ k \text{ odd}}}^{\infty} \frac{1}{k}\prod_{j=1}^{n}\left(\frac{e^{\pi ikb_j/a_j} - e^{\pi ikb_j}}{1-e^{\pi ikb_j/a_j}}\right),$$

from which we can prove our theorem just as 5.2(1) was proved, by using the same expansion of $\pi\cot\pi x$.

From (13) we see that $t(a_1,\dots,a_n;b_1,\dots,b_n)$ only depends on N and the numbers $b_i(N/a_i)$:

$$t(a_1,\dots,a_n;b_1,\dots,b_n) = t(N,\dots,N;\ b_1 \frac{N}{a_1},\dots,b_n \frac{N}{a_n}). \qquad (17)$$

$$= t_N(1,b_1 \frac{N}{a_1},\dots,b_n \frac{N}{a_n}). \qquad (18)$$

In one special case $t_N(d_0,\dots,d_n)$ can be evaluated explicitly:

THEOREM 3: If $d_0,\dots,d_n$ are relatively prime integers (n odd) and $N > 0$ is a multiple of $d_0,\dots,d_n$, then $t_N(d_0,\dots,d_n)$ is given by a rational expression in N and the d_i's, namely:

$$t_N(d_0,\dots,d_n) = 1 - \frac{1}{d_0\cdots d_n} \sum_{k\geq 1} \frac{2^{2k}(2^{2k}-1)}{(2k)!} \times$$

$$\times B_{2k} N^{2k-1} \ell_{n-2k+1}(d_0,\dots,d_n), \qquad (19)$$

where B_{2k} is the $(2k)^{th}$ Bernoulli number and ℓ_{n-2k+1} the polynomial discussed in 5.2.

Proof: By (7), the left-hand side equals

$$\frac{1}{N} \sum_{\zeta^N=-1} \left(\frac{\zeta^{d_0}+1}{\zeta^{d_0}-1} \cdots \frac{\zeta^{d_n}+1}{\zeta^{d_n}-1}\right)$$

$$=\frac{1}{2} \sum_{\zeta^N=-1} \operatorname{res}_{z=\zeta} \left\{\frac{z^{d_0}+1}{z^{d_0}-1} \cdots \frac{z^{d_n}+1}{z^{d_n}-1} \cdot \frac{z^N-1}{z^N+1} \quad \frac{dz}{z}\right\}.$$

Since the d_i's are relatively prime to one another, if $z \neq 1$ then at most one of the factors $z^{d_i} - 1$ can vanish, and therefore(because of the factor $z^N - 1$ in the numerator) the function in brackets is regular at $z^{d_i} = 1$, $z \neq 1$. Therefore (setting $z = e^{2x}$),

$$t_N^{even}(d_0,\dots,d_n) - t_N^{odd}(d_0,\dots,d_n) =$$

$$= -\frac{1}{2}\left[\mathrm{res}_{z=0} + \mathrm{res}_{z=\infty} + \mathrm{res}_{z=1}\right]\left\{\frac{z^{do}+1}{z^{do}-1}\cdots\frac{dz}{z}\right\}$$

$$= -\frac{1}{2}\left[-1-1+2\,\mathrm{res}_{x=0}\left\{\prod_{j=0}^{n}(\coth d_j x)\cdot\tanh Nx\cdot dx\right\}\right]$$

$$= 1 - \frac{1}{d_0\cdots d_n}\left[\text{coeff. of } x^n \text{ in } \tanh Nx\cdot\prod_{j=0}^{n}\frac{d_j x}{\tanh d_j x}\right]$$

The theorem then follows from the expansions

$$\tanh Nx = \sum_{k\geq 1}\frac{2^{2k}(2^{2k}-1)}{(2k)!}\; B_{2k}\cdot(Nx)^{2k-1}, \qquad (20$$

$$\prod_{j=0}^{n}\frac{d_j x}{\tanh d_j x} = \sum_{i\geq 0}\ell_i(d_0,\dots,d_n)\,x^i. \qquad (21$$

In particular, if one defines

$$t(a_1,\dots,a_n) = t(a_1,\dots,a_n;\,1,\dots,1), \qquad (22$$

in agreement with equation (2) of the introduction of the book then the theorem gives the value of $t(a_1,\dots,a_n)$ for $a_1,\dots,a_n$ integers such that $N/a_1,\dots,N/a_n$ are relatively prime integers. For example, if $n=3$ the formula obtained is

$$t(pq,pr,qr) = \frac{-2+p^2+q^2+r^2-p^2q^2r^2}{3}. \qquad (23$$

(p,q,r pairwise coprime)

Finally, to take the mysterious cotangents off their high pedestal, we will give expressions for some of the previous quantities involving tangents. Write (5) as

$$t_p(a_1,\ldots,a_n) = \frac{1}{p} \sum_{\zeta^p=-1} \prod_{j=1}^{n} \left(\frac{\zeta^{a_j}+1}{\zeta^{a_j}-1} \right). \tag{24}$$

Since $\zeta^p = -1$ implies $(\zeta^{-1})^p = -1$, we can replace ζ by ζ^{-1} in the sum, and we find that the right-hand side equals $(-1)^n$ times itself, and hence is zero if n is odd (which we already knew). Similarly we can replace ζ by $-\zeta$, which is a p^{th} root of -1 if p is even and of 1 if p is odd Since the a_i's are odd, $\frac{(-\zeta)^{a_j}+1}{(-\zeta)^{a_j}-1} = \frac{\zeta^{a_j}-1}{\zeta^{a_j}+1}$. Therefore the right-hand side of (1) also equals

$$\frac{1}{p} \sum_{\zeta} \prod_{j=1}^{n} \left(\frac{\zeta^{a_j}-1}{\zeta^{a_j}+1} \right),$$

where the sum is over the p^{th} roots of 1 or −1 according as p is odd or even. Returning to the notation of trigonometric functions rather than complex roots of unity, we have the following statement equivalent to Theorem 1:

<u>THEOREM 4</u>: If $a_1,\ldots,a_n$ <u>are odd integers, then for</u> n <u>even</u>,

$$t_p(a_1,\ldots,a_n) = \begin{cases} \dfrac{(-1)^{\frac{n}{2}}}{p} \displaystyle\sum_{k=1}^{p} \tan\frac{\pi k a_1}{p} \cdots \tan\frac{\pi k a_n}{p} & (p \text{ odd}) \\ \dfrac{(-1)^{\frac{n}{2}}}{p} \displaystyle\sum_{\substack{k=1 \\ k \text{ odd}}}^{2p-1} \tan\frac{\pi k a_1}{2p} \cdots \tan\frac{\pi k a_n}{2p} & (p \text{ even}) \end{cases} \tag{25}$$

If n is odd, both sides of eq. (2) are zero.

5.4 The next property of the cotangent sums which we will consider is their behaviour when the integers $a_1,\dots,a_n$ are large. If we set $b_1=\dots=b_n=1$ in 5.3(14), we find that

$$t(a_1,\dots,a_n) = \sum_{k=0}^{n-1} (-1)^k N_k, \tag{1}$$

with

$$N_k = \#\left\{0<x_1<a_1,\dots,0<x_n<a_n \,\middle|\, k<\frac{x_1}{a_1}+\dots+\frac{x_n}{a_n}<k+1\right\} \tag{2}$$

It is then clear that

$$\lim_{a_1,\dots,a_n\to\infty}\left[\frac{t(a_1,\dots,a_n)}{a_1\cdots a_n}\right] = \sum_{k=0}^{n-1} (-1)^k V_k^{(n)}, \tag{3}$$

where

$$V_k^{(n)} = \underset{k<t_1+\dots+t_n<k+1}{\int_0^1 \cdots \int_0^1} dt_1\cdots dt_n. \tag{4}$$

Thus $V_k^{(n)}$ is the volume of a diagonal slice of the n-dimensional unit cube. From 5.3(13), we have (using symmetry

$$t(a_1,\dots,a_n) = \frac{(-1)^{\frac{n-1}{2}}}{N} \sum_{\substack{k=1\\ k \text{ odd}}}^{2N-1} \cot\frac{\pi k}{2N}\cot\frac{\pi k}{2a_1}\cdots\cot\frac{\pi k}{2a_n}$$

$$= \frac{2\,(-1)^{\frac{n-1}{2}}}{N} \sum_{\substack{k=1\\ k \text{ odd}}}^{N-1} \cot\frac{\pi k}{2N}\cot\frac{\pi k}{2a_1}\cdots\cot\frac{\pi k}{2a_n} \tag{5}$$

Each cotangent is large when its argument is near a multiple of π ($\frac{\pi k}{2a}$ cannot actually be a multiple of π for k odd), so the very large terms of the formula are those with k small, when each of the cotangents has an argument near zero. This is especially true for $a_1,\ldots,a_n$ large, but even for $n = 3$ and $(a_1,a_2,a_3) = (3,4,5)$, with $N = 60$, the **term** $k=1$ of (5) is -16.382 and no other term is bigger than 0.6; the value of $t(3,4,5,)$ is -16 (in the notation of equation (4), $N_0=N_2=2$, $N_1 = -20$). Thus with large a_i's the value of $t(a_1,\ldots,a_n)$ is approximately given by the sum of the first few terms. Since $\cot x$ is approximately $\frac{1}{x}$ for x small, we can approximate (5) by

$$\frac{i(-1)^{\frac{n-1}{2}}}{N} \sum_{\substack{k\geq 1\\ k \text{ odd,small}}} \left(\frac{2N}{\pi k}\right)\left(\frac{2a_1}{\pi k}\right)\cdots\left(\frac{2a_n}{\pi k}\right)$$

$$= \frac{2^{n+2}\cdot(-1)^{\frac{n-1}{2}}\cdot a_1\cdots a_n}{\pi^{n+1}} \sum_{\substack{k=1\\ k \text{ odd}}}^{\infty} \frac{1}{k^{n+1}}.$$

Hence

$$\lim_{a_1,\ldots,a_n\to\infty} \left[\frac{t(a_1,\ldots,a_n)}{a_1\cdots a_n}\right] =$$

$$= (-1)^{\frac{n-1}{2}} \frac{2^{n+2}}{\pi^{n+1}} a_1\cdots a_n \left(1-\frac{1}{2^{n+1}}\right)\zeta(n+1). \qquad (6)$$

Since n is odd, we have

$$\zeta(n+1) = (-1)^{\frac{n-1}{2}}\cdot 2^n\pi^{n+1}\frac{B_{n+1}}{(\ldots)}, \qquad (7)$$

so finally

$$\lim_{a_1,\ldots,a_n\to\infty}\left[\frac{t(a_1,\ldots,a_n)}{a_1\ldots a_n}\right] = \frac{2^{n+1}(2^{n+1}-1)}{(n+1)!}\quad B_{n+1} \tag{8}$$

(here B_n is the n^{th} Bernoulli number: $B_0 = 1$, $B_1 = -\frac{1}{2}$ $B_2 = 1/6$, $B_3 = 0$, etc.) For instance, if $n = 1$ then $t(a) = a-1$ and the right-hand side is $\frac{4\times 3}{2!}\times\frac{1}{6} = 1$.

Others of the results obtained previously can similarly be made to yield results about volumes. In particular, we can get an expression for $V_k^{(n)}$. If we replace the function of 5.3(12) by $\frac{t-t^N}{1-t}\cdot\frac{1}{1-xt^N}$ and set $b_j = 1$ in 5.3(11), then the usual study of $\mathrm{res}_{t=0}(f(t^{-1})g(t)\,dt/t)$ gives, just as with $x = -1$,

$$f_n(x) \underset{\mathrm{DEF}}{=} \sum_{k=0}^{n-1} x^k\cdot N_k = \frac{1}{N}\sum_{t^N=x}\frac{t^{-1}-x^{-1}}{1-t^{-1}}\ f(t)$$

$$= \frac{1}{N}\sum_{t^N=x}\frac{t^{-1}-x^{-1}}{1-t^{-1}}\ \prod_{j=1}^{n}\left(\frac{t^{c_j}-x}{1-t^{c_j}}\right)$$

(N as in 5.3(11), $c_j = N/a_j$). The large terms come from t near 1, or $\log t$ small. Since $\log t = (\log x + 2\pi ir)/N$, we get

$$\lim_{a_1,\ldots,a_n\to\infty}\left[\frac{f_n(x)}{a_1\ldots a_n}\right]$$

$$= \lim\left[\frac{1}{Na_1\ldots a_n}\sum_{r=-\infty}^{\infty}\frac{1-x^{-1}}{\frac{1}{N}(\log x + 2\pi ir)}\ \prod_{j=1}^{n}\left(\frac{-1+x}{\frac{c_j}{N}(\log x + 2\pi ir)}\right.\right.$$

$$= -(x^{-1}-1) \sum_{r=-\infty}^{\infty} \frac{(x-1)^n}{(\log x + 2\pi ir)^{n+1}}$$

$$= -(x^{-1}-1)\cdot(x-1)^n \cdot \frac{(-1)^n}{n!} \frac{d^n}{dy^n} \left[\sum_{r=-\infty}^{\infty} \frac{1}{y+2\pi ir} \right]_{y=\log x}$$

$$= -(x^{-1}-1)\frac{(1-x)^n}{n!} \frac{d^n}{dy^n} \left[\frac{1}{2} \cdot \frac{e^y+1}{e^y-1} \right]_{y=\log x} . \qquad (9)$$

That is,

$$\sum_{k=0}^{n-1} x^k V_k(n) = -\frac{(1-x)^{n+1}}{2n!x} \left(x\frac{d}{dx}\right)^n \left(\frac{x+1}{x-1}\right) . \qquad (10)$$

Applying Taylor's theorem to (9), we get finally

$$\sum_{n=1}^{\infty} \sum_{k=0}^{n-1} x^k z^n V_k(n) = \frac{e^{(1-x)z}-1}{1-xe^{(1-x)z}} . \qquad (11)$$

With $x = -1$ this says

$$\sum_{n=1} \left[\lim_{a_i \to \infty} \frac{t(a_1,\ldots,a_n)}{a_1 \ldots a_n} \right] z^n = \tanh z, \qquad (12)$$

which agrees with eq. (8).

Equation (11), giving the generating function for the volumes $V_k^{(n)}$, was proved by Meyer and by van Randow [70]. Because this formula is quite an interesting one, we give another proof (based on the idea of van Randow's, but much simpler). Let

$$\Omega_k = \left\{(t_1,\ldots,t_n) \in R_+^n \;\middle|\; k \le t_1+\ldots+t_n < k+1\right\}. \qquad (13)$$

Clearly the n-dimensional volume of Ω_k is $((k+1)^n-k^n)/n!$. Therefore

$$\sum_{k=0}^{\infty} \frac{(k+1)^n-k^n}{n!} x^k = \sum_{k=0}^{\infty} x^k \int_{\Omega_k}\!\!\cdots\!\int d\theta_1 \ldots d\theta_n$$

$$= \int_0^{\infty}\!\!\cdots\!\int_0^{\infty} x^{[\theta_1+\ldots+\theta_n]} d\theta_1 \ldots d\theta_n, \qquad (14$$

where the square brackets denote the greatest-integer function We now write $\theta_i = r_i + t_i$ $(i = 1,\ldots,n)$, where r_i is a non-negative integer and $0 \leq t_i < 1$. Then (14) becomes

$$\sum_{r_1,\ldots,r_n \geq 0} \int_0^1\!\!\cdots\!\int_0^1 x^{[r_1+\ldots+r_n + t_1 + \ldots + t_n]} dt_1 \ldots dt_n$$

$$= \left[\sum_{r_1>0}\cdots\sum_{r_n>0} x^{r_1+\ldots+r_n}\right]\left[\int_0^1\!\!\cdots\!\int_0^1 x^{[t_1+\ldots+t_n]} dt_1 \ldots dt\right.$$

$$= (1-x)^{-n} \sum_{k=0}^{n-1} V_k^{(n)} x^k. \qquad (15$$

If we multiply this by $z^n(1-x)^n$ and sum over n, we obtain

$$\sum_{n=1}^{\infty} \sum_{k=0}^{n-1} V_k^{(n)} x^k z^n = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} x^k \frac{z^n(1-x)^n}{n!} \{(k+1)^n-k^n\}$$

$$= \sum_{k=0}^{\infty} x^k \{e^{(k+1)z(1-x)} - e^{kz(1-x)}\}$$

$$= (e^{z(1-x)} - 1)/(1-xe^{z(1-x)}), \qquad (16$$

in agreement with (11).

The same argument which we used to get from (5) to (6) works for other sums studied in this section. Thus, from the definition of $\delta(a;a_1,\ldots,a_n)$, equation 5.2(5), we have

$$\lim\,(a^{-n+1}\,\delta(a;a_1,\ldots,a_n)) = 2^n\int_0^1\cdots\int_0^1 \delta_{\mathbb{Z}}(a_1x_1+\ldots+a_nx_n)$$

$$\cdot((x_1))\ldots((x_n))\,dx_1\ldots dx_n, \qquad (17)$$

for any $a_1,\ldots,a_n \in \mathbb{Z}$, where $\delta_{\mathbb{Z}}$ is the distribution

$$\delta_{\mathbb{Z}}(y) = \sum_{n\in\mathbb{Z}} \delta(y-n). \qquad (18)$$

But from 5.2(6) we obtain, for a large,

$$\delta(a;a_1,\ldots,a_n) \sim 2\,\frac{(-1)^{n/2}}{a}\sum_{j\geq 1}\left(\frac{a}{\pi j a_1}\right)\cdots\left(\frac{a}{\pi j a_n}\right)$$

$$= \frac{2(-1)^{n/2}a^{n-1}\zeta(n)}{\pi^n\,a_1\ldots a_n} = \frac{-2^n}{a_1\ldots a_n}\,\frac{B_n}{n!}\,a^{n-1}, \qquad (19)$$

and a comparison of (18) and (19) leads to an evaluation of the integral (which, because of the presence of the δ-function, is essentially an $(n-1)$-dimensional volume integral). Of course, the integral can also be evaluated directly: one uses the Fourier expansions $\delta_{\mathbb{Z}}(y) = \sum_r e^{2\pi iry}$, $((x)) = \sum_{r\neq 0}\frac{e^{2\pi irx}}{2\pi ir}$ to obtain the Fourier expansion of the integrand in (18) (which is a function of n variables, periodic in each one), and then just picks out the constant term by hand.

5.5 We will now discuss the number-theoretical properties of the numbers $t(a_1,\dots,a_n)$. We recall that these are zero for even n, and can trivially be evaluated for $n = 1$. We will prove that $t(a_1,\dots a_n)$, as a function of (say) a_1, is the sum of a linear function and a periodic function, the period of the latter being the least common multiple of $a_2,\dots,a_n$. We then use this to prove that $t(p,q,2) \equiv 0 \pmod 8$, for p and q odd and mutually prime. (In fact, $t(a_1,\dots,a_n) \equiv 0 \pmod 8$ whenever the a_i's are all prime to one another (and even under somewhat less restrictive conditions), but the proof uses topology and will not be given here.)

To study the dependence of $t(a_1,\dots,a_n)$ on a_1, we use eq. 5.3(16). For x real and $a>0$ an integer, we define $F_a(x) = \sum_{i=1}^{a-1} h(x+\frac{i}{a})$, with h the function introduced in 5.3(15). Then

$$F_a(x) = (-1)^{[x]} \sum_{i=1}^{a-1} h(x - [x] + \tfrac{i}{a})$$

$$= (-1)^{[x]} \left\{ \sum_{\frac{i}{a} + x - [x] < 1} 1 - \sum_{\frac{i}{a} + x - [x] > 1} 1 \right\}$$

$$= \begin{cases} (-1)^{[x]} (a+2a[x] - 2[ax]) & ax \in \mathbb{Z},\ x \notin \mathbb{Z} \\ (-1)^{[x]} (a-1 + 2a[x] - 2[ax]) & ax \notin \mathbb{Z} \text{ or } x \in \mathbb{Z} \end{cases} \tag{1}$$

It follows that if N is an integer such that $Nx \in \mathbb{Z}$, then

$$F_{a+N}(x) - F_a(x) = N\cdot(-1)^{[x]} (1+2[x] - 2x). \tag{2}$$

Putting this in eq. 5.3(16), which can be stated as

$$t(a_1,\dots a_n) = \sum_{x_2=1}^{a_2-1} \cdots \sum_{x_n=1}^{a_n-1} F_{a_1}\left(\frac{x_2}{a_2}+\dots+\frac{x_n}{a_n}\right), \qquad (3)$$

we find that, if N is a multiple of $a_2,\dots a_n$, then

$$\frac{1}{N}\left[t(a_1+N, a_2,\dots,a_n) - t(a_1,a_2,\dots a_n)\right] = d(a_2,\dots,a_n), \qquad (4)$$

where

$$d(a_2,\dots,a_n) = \sum_{x_2=1}^{a_2-1} \cdots \sum_{x_n=1}^{a_n-1} (-1)^{\left[\frac{x_2}{a_2}+\dots+\frac{x_n}{a_n}\right]} \times$$

$$\times\left(1+2\left[\frac{x_2}{a_2}+\dots+\frac{x_n}{a_n}\right] -2\left(\frac{x_2}{a_2}+\dots+\frac{x_n}{a_n}\right)\right). \qquad (5)$$

For example, if r is prime to 12, we have the following values for $t(3,4,r)$:

$$t(3,4,12k+1) = -40k$$
$$t(3,4,12k+5) = -40k - 16$$
$$t(3,4,12k+7) = -40k - 24$$
$$t(3,4,12k+11) = -40k - 40.$$

Even this table can be shortened. In 5.3(16) we can allow the a_i's to be negative by introducing the usual convention for interpreting sums $\sum_{k=p}^{q}$ with $p>q$. We easily find that

$$t(a_1,\dots,-a_i,\dots,a_n) = -\,t(a_1,\dots,a_n). \qquad (6)$$

(This can also be seen from eq. 5.3(13)). Hence the table above can be shortened to

$$t(3,4,12k\pm1) = -40k$$

$$t(3,4,12k\pm5) = -40k \mp 16.$$

On the following page we give tables of $t(p,q,r)$ for p,q,r relatively prime numbers with p and q small, following the same scheme. Of the table we have just given for $(p,q) = (3,4)$, the fact $t(3,4,1) = 0$ is a special case of

$$t(1,a_2,\ldots,a_n) = 0 \tag{7}$$

which is a trivial consequence of the definition. That $t(3,4,5) = -16$ was mentioned in 5.4. Finally, the fact that $d(3,4) = -40/12$ is a case of the following [97]

THEOREM 1: *Let p and q be relatively prime integers. Then*

$$d(p,q) = -\frac{(p^2-1)(q^2-1)}{3pq}; \tag{8}$$

that is

$$t(p,q,r+npq) = t(p,q,r) - \frac{n}{3}(p^2-1)(q^2-1) \tag{9}$$

Note that $(p,q) = 1$ gives in particular that either p or q is not a multiple of 3 and hence that p^2-1 or q^2-1 is 0 (mod 3), so the right-hand side of (9) is an integer. Moreover either p or q is odd so that $(p^2-1)(q^2-1)/3$ is not only an integer but a multiple of 8.

Proof: By definition

$$d(p,q) = \sum_{i=1}^{p-1}\sum_{j=1}^{q-1}(-1)^{\left[\frac{i}{p}+\frac{j}{q}\right]}\left(1+2\left[\frac{i}{p}+\frac{j}{q}\right]-\frac{2i}{p}-\frac{2j}{q}\right)$$

$$= \sum_{\substack{0<i<p\\0<j<q\\\frac{i}{p}+\frac{j}{q}<1}}\left(1-\frac{2i}{p}-\frac{2j}{q}\right) + \sum_{\substack{0<i<p\\0<j<q\\\frac{i}{p}+\frac{j}{q}>1}}\left(\frac{2i}{p}+\frac{2j}{q}-3\right)$$

TABLE: The function t(p,q,r)

f = - t(p,q,r)/8

p	q	r	f
2	3	6k±1	k
2	5	10k±1	3k
		10k±3	3k±1
2	7	14k±1	6k
		14k±3	6k±1
		14k±5	6k±2
2	9	18k±1	10k
		18k±5	10k±3
		18k±7	10k±4
2	11	22k±1	15k
		22k±3	15k±2
		22k±5	15k±3
		22k±7	15k±5
		22k±9	15k±6
2	13	26k±1	21k
		26k±3	21k±2
		26k±5	21k±4
		26k±7	21k±6
		26k±9	21k±7
		26k±11	21k±9
2	15	30k±1	28k
		30k±7	28k±6
		30k±11	28k±10
		30k±13	28k±12
2	17	34k±1	36k
		34k±3	36k±3
		34k±5	36k±5
		34k±7	36k±7
		34k±9	36k±10
		34k±11	36k±12
		34k±13	36k±14
		34k±15	36k±16
2	19	38k±1	45k
		38k±3	45k±3
		38k±5	45k±6
		38k±7	45k±8
		38k±9	45k±10
		38k±11	45k±13
		38k±13	45k±15
		38k±15	45k±18
		38k±17	45k±20
2	21	42k±1	55k
		42k±5	55k±6
		42k±11	55k±15
		42k±13	55k±17
		42k±17	55k±22
		42k±19	55k±25
2	23	46k±1	66k
		46k±3	66k±4
2	23	46k±5	66k±7
		46k±7	66k±10
		46k±9	66k±13
		46k±11	66k±15
		46k±13	66k±19
		46k±15	66k±22
		46k±17	66k±24
		46k±19	66k±27
		46k±21	66k±30
3	4	12k±1	5k
		12k±5	5k±2
3	5	15k±1	8k
		15k±2	8k±1
		15k±4	8k±2
		15k±7	8k±4
3	7	21k±1	16k
		21k±2	16k±1
		21k±4	16k±3
		21k±5	16k±4
		21k±8	16k±6
		21k±10	16k±8
3	8	24k±1	21k
		24k±5	21k±4
		24k±7	21k±6
		24k±11	21k±10
3	10	30k±1	33k
		30k±7	33k±8
		30k±11	33k±12
		30k±13	33k±14
4	5	20k±1	15k
		20k±3	15k±2
		20k±7	15k±5
		20k±9	15k±7
4	7	28k±1	30k
		28k±3	30k±3
		28k±5	30k±5
		28k±9	30k±10
		28k±11	30k±12
		28k±13	30k±14
4	9	36k±1	50k
		36k±5	50k±7
		36k±7	50k±10
		36k±11	50k±15
		36k±13	50k±18
		36k±17	50k±24
5	6	30k±1	35k
		30k±7	35k±8
		30k±11	35k±13
		30k±13	35k±15

$$= \sum_{i=1}^{p-1} \left\{ \sum_{j=1}^{\left[q-\frac{qi}{p}\right]} \left(1-\frac{2i}{p}-\frac{2j}{q}\right) + \sum_{j=\left[q-\frac{qi}{p}\right]+1}^{q-1} \left(\frac{2i}{p}+\frac{2j}{q}-3\right) \right\}$$

Replacing the first sum by $\sum_{j=1}^{q-1} - \sum_{j=\left[q-\frac{qi}{p}\right]+1}^{q-1}$ and replacing j by $q-j$ in both sums, we get

$$d(p,q) = \sum_{i=1}^{p-1} \left\{ -\frac{2i}{p}(q-1) + \sum_{j=1}^{\left[\frac{qi}{p}\right]} \left(\frac{4i}{p} - \frac{4j}{q}\right) \right\}$$

$$= \sum_{i=1}^{p-1} \left\{ \frac{2i}{p}\left(2\left[\frac{qi}{p}\right]-q+1\right) - \frac{2\left[\frac{qi}{p}\right]\left(\left[\frac{qi}{p}\right]+1\right)}{q} \right\}$$

$$= \frac{4}{p} f_p(q) - (p-1)(q-1) - \frac{2}{q} \sum_{i=1}^{p-1} \left[\frac{qi}{p}\right]^2 - \frac{(q-1)(p-1)}{q},$$

using eqs. 5.1(7), 5.1(9). Using 5.1(8), this is equal to

$$\frac{(q^2-1)(p-1)(2p-1)}{3pq} - (p-1)(q-1) - \frac{(q-1)(p-1)}{q} = -\frac{(q^2-1)(p^2-1)}{3pq}$$

QED.

Another proof, more in the spirit of 5.3, is the following Just as we derived 5.4(5), one can show that

$$d(a_2,\dots,a_n) = \frac{-(-1)^{\frac{n-1}{2}}}{N} \sum_{\substack{k=1 \\ k \text{ odd}}}^{2N-1} \csc^2 \frac{\pi k}{2N} \cot \frac{\pi k}{2a_2} \cdots \cot \frac{\pi k}{2a_n} \qquad (10)$$

[This formula is rather amusing. Equation (4) says that $d(a_2,\dots,a_n)$ is some sort of a derivative of $t(a_1,\dots,a_n)$. But the last equation is also a sort of derivative of 5.4(5), since $\frac{d}{dx}\left(\frac{1}{N}\cot\frac{x}{N}\right) = -\frac{1}{N^2}\csc^2\frac{x}{N}$.] From (10), we get

$$d(p,q) = \frac{1}{pq}\sum_{\substack{k=1\\ k\text{ odd}}}^{2pq-1} \csc^2\frac{\pi k}{2pq}\cot\frac{\pi k}{2p}\cot\frac{\pi k}{2q}$$

$$= \frac{1}{pq}\sum_{t^{pq}+1=0}\left(\frac{4t}{(t-1)^2}\cdot\frac{t^q+1}{t^q-1}\cdot\frac{t^p+1}{t^p-1}\right)$$

$$= \sum_{t^{pq}+1=0} \text{res}_{z=t}\left[\frac{2}{(z-1)^2}\cdot\frac{z^q+1}{z^q-1}\cdot\frac{z^p+1}{z^p-1}\cdot\frac{z^{pq}-1}{z^{pq}+1}\cdot dz\right]$$

Since the differential form in brackets has no poles except $z = 1$ and $z^{pq} = -1$, (the factor z^p-1 creates no poles since if $z^p = 1$ then $z^{pq} = 1$), this gives

$$d(p,q) = -\text{res}_{z=1}\left[\frac{2z}{(z-1)^2}\cdot\frac{z^q+1}{z^q-1}\cdot\frac{z^p+1}{z^p-1}\cdot\frac{z^{pq}-1}{z^{pq}+1}\cdot\frac{dz}{z}\right]$$

or, setting $z = e^{2x}$,

$$= -\text{res}_{x=0}\left[\text{csch}^2 x\cdot\coth qx\cdot\coth px\cdot\tanh pqx\,dx\right]$$

$$= -\text{res}_{x=0}\Big[\left(\frac{1}{x}-\frac{x}{6}+\dots\right)^2\left(\frac{1}{qx}+\frac{qx}{3}+\dots\right)\left(\frac{1}{px}+\frac{px}{3}+\dots\right)$$

$$\times\quad\left(pqx-\frac{p^2q^2}{3}x^3+\dots\right)dx\Big]$$

$$= -\frac{(p^2-1)(q^2-1)}{3pq}$$

We will now find an explicit formula for $t(p,q,2)$, where p and q are odd and relatively prime. As in the evaluation of $d(p,q)$, one can proceed either directly or by the use of cotangent sums. Yet another approach (which we describe here because a similar technique will be necessary in §7 to prove the relation between Dedekind sums and the symbol $s(p,q)$) is to use the following theorem:

THEOREM 2: For p,q *odd coprime integers,*

$$\begin{aligned} &\text{i)} & t(p,q,2) &= t(q,p,2) \\ &\text{ii)} & t(p,-q,2) &= -t(p,q,2) \\ &\text{iii)} & t(1,p,2) &= 0 \\ &\text{iv)} & t(p,q+2p,2) &= t(p,q,2) - (p^2-1) \end{aligned}$$

These four properties characterize the function $t(p,q,2)$.

Proof: Statement i) is trivial, ii) and iii) are special cases of eqs. (6) and (7), and iv) is a special case of Theorem 1. Conversely, given any function with properties i)-iv), we can compute its value on a pair of coprime odd integers p,q, as follows: use ii) and i) if necessary to obtain $q>p>0$. Then apply iv) repeatedly until q is reduced to a number in $(-p,p)$. If necessary use ii) to make p and q positive and repeat the whole process. Eventually one of the two numbers becomes one (because p,q are odd and coprime; this is a modified Euclidean algorithm) and one can apply iii).

To show how this works out in practise, we compute an example (writing $b(p,q)$ for $t(p,q,2)$):

$$\begin{aligned} b(413,79) &= b(79,413-6\times79) - 3(79^2-1) \\ &= b(79,-61) - 18720 \\ &= -b(61,79-2\times61) + (61^2-1) - 18720 \\ &= b(43,61) - 15000 \end{aligned}$$

$$
\begin{aligned}
&= \quad b(43,61-2\times 43) - (43^2-1)-15000 \\
&= -b(25,43) - 16848 \\
&= -b(25,43-2\times 25) + (25^2-1) - 16848 \\
&= \quad b(7,25) - 16224 \\
&= \quad b(7,25-4\times 7) - 2(7^2-1) -16224 \\
&= -b(3,7) - 16320 \\
&= -b(3,7-2\times 3) + (3^2-1) -16320 \\
&= -b(1,3) -16312 \\
&= -16312.
\end{aligned}
$$

This may seem laborious, but is certainly easier than working out the $412\times 78 = 32136$ terms of the sum defining $t(413,79,2)$.

Since p^2-1 is a multiple of 8 for odd p, it is clear that the repeated use of i)-iv) can only yield multiples of 8, so we obtain

COROLLARY: *For* p,q *as in the theorem,* $t(p,q,2) \equiv 0 \pmod 8$.

Now we apply the theorem to prove the formula for $t(p,q,2$

THEOREM 3: For p,q odd integers prime to one another, we have

$$t(p,q,2) = -\frac{pq}{2} + \frac{2}{3}\left(\frac{p}{q} + \frac{q}{p}\right) - 1 + \frac{1}{6pq} - 4s(2q,p) - 4\, s(2p,q). \tag{11}$$

which can be rewritten, using the reciprocity 5.1(3), as

$$t(p,q,2) = -q\cdot \frac{p^2-1}{2p} - 4\, s(2q,p) + 4\, s(q,2p) \tag{12}$$

Proof: The expression (11) is clearly symmetric in q and p, while properties ii)-iv) can be seen from formula (12)--the right-hand side is clearly an odd function of q and vanishes if $p = 1$, while increasing q by $2p$ adds $-(p^2-1)$ to the first term and leaves the other two terms unchanged.

This is a rather indirect proof of eqs. (11) and (12). A more direct proof proceeds starting with 5.3(16):

$$t(p,q,2) = \sum_{i=1}^{p-1} \sum_{j=1}^{q-1} h\left(\frac{1}{2} + \frac{i}{p} + \frac{j}{q} \right)$$

$$= \sum_{0<\frac{i}{p}+\frac{j}{q}<\frac{1}{2}} 1 - \sum_{\frac{1}{2}<\frac{i}{p}+\frac{j}{q}<\frac{3}{2}} 1 + \sum_{\frac{3}{2}<\frac{i}{p}+\frac{j}{q}<2} 1$$

$$= 4 \sum_{0<\frac{i}{p}+\frac{j}{q}<\frac{1}{2}} 1 - (p-1)(q-1)$$

$$= 4 \sum_{i=1}^{\frac{p-1}{2}} \left[\frac{q}{2} - \frac{qi}{p}\right] - (p-1)(q-1)$$

$$= 4 \sum_{i=1}^{(p-1)/2} \left\{\frac{q-1}{2} - \left[\frac{qi}{p} + \frac{1}{2}\right]\right\} - (p-1)(q-1)$$

$$= -4 \sum_{i=1}^{\frac{p-1}{2}} \left[\frac{qi}{p} + \frac{1}{2}\right]. \qquad (13)$$

from which it is clear at least that 4 divides $t(p,q,2)$.

Using 5.1(2), we can rewrite this as

$$t(p,q,2) = -4 \sum_{i=1}^{\frac{p-1}{2}} \left\{\frac{qi}{p} - \left(\left(\frac{qi}{p} + \frac{1}{2}\right)\right)\right\}$$

$$= -\frac{4q}{p} \cdot \frac{\left(\frac{p-1}{2}\right)\left(\frac{p+1}{2}\right)}{2} + 4 \sum^{\frac{p-1}{2}} \left(\left(\frac{qi}{p} + \frac{1}{2}\right)\right).$$

Comparing this with (12), we obtain as the formula to be proved

$$\sum_{i=1}^{\frac{p-1}{2}} \left(\left(\frac{qi}{p} + \frac{1}{2}\right)\right) = s(q,2p) - s(2q,p)$$

$$= \sum_{x=1}^{2p-1} \left(\left(\frac{x}{2p}\right)\right)\left(\left(\frac{qx}{2p}\right)\right) - \sum_{x=1}^{p-1} \left(\left(\frac{x}{p}\right)\right)\left(\left(\frac{2xq}{p}\right)\right) \quad (14)$$

In the first sum the term $x = p$ is zero. Split the first sum into sums over even x $(x = 2y,\ 1 \leq y \leq p-1)$ and odd x $(x = 2y+p,\ 1 \leq y \leq p-1)$. Then the right hand side of (14) equals

$$\sum_{y=1}^{p-1} \left(\left(\frac{y}{p}\right)\right)\left(\left(\frac{qy}{p}\right)\right) + \sum_{y=1}^{p-1} \left(\left(\frac{y}{p}+\frac{1}{2}\right)\right)\left(\left(\frac{qy}{p}+\frac{1}{2}\right)\right) - \sum_{x=1}^{p-1} \left(\left(\frac{x}{p}\right)\right)\left(\left(\frac{2xq}{p}\right)\right)$$

$$= \sum_{y=1}^{p-1} \left\{\left(\left(\frac{y}{p}+\frac{1}{2}\right)\right) - \left(\left(\frac{y}{p}\right)\right)\right\}\left(\left(\frac{qy}{p}+\frac{1}{2}\right)\right)$$

$$+ \sum_{y=1}^{p-1} \left(\left(\frac{y}{p}\right)\right)\left\{\left(\left(\frac{qy}{p}\right)\right) - \left(\left(\frac{2qy}{p}\right)\right) + \left(\left(\frac{qy}{p} + \frac{1}{2}\right)\right)\right\}$$

But $((x + \frac{1}{2})) - ((x)) = ((2x)) - 2((x)) = \frac{1}{2}(-1)^{[2x]}$ (x not an integer) so the second sum vanishes and the first equals

$$\sum_{y=1}^{p-1} \left\{\frac{1}{2}\cdot(-1)^{\left[\frac{2y}{p}\right]}\right\}\left(\left(\frac{qy}{p}+\frac{1}{2}\right)\right) = \frac{1}{2}\sum_{y=1}^{\frac{p-1}{2}}\left(\left(\frac{qy}{p}+\frac{1}{2}\right)\right) - \frac{1}{2}\sum_{y=\frac{p+1}{2}}^{p-1}\left(\left(\frac{qy}{p}+\frac{1}{2}\right)\right)$$

$$= \sum_{y=1}^{\frac{p-1}{2}} \left(\left(\frac{qy}{p}+\frac{1}{2}\right)\right).$$

Finally, one can give a third proof of (12) using the cotangent formulas for $s(q,p)$ and $t(p,q,2)$. One simply applies the residue theorem to $\frac{y+1}{y-1}\frac{y^{2p}+1}{y^{2p}-1}\frac{y^{2q}+1}{y^{2q}-1}\frac{dy}{y^{pq}+1}$,

which has simple poles at $y^{2p}=1$ and $y^{2q}=1$ ($y \neq \pm 1$; if $y^{2p}=y^{2q}=1$ then $y^2=1$ since p and q are coprime) and at $y^{pq}=-1$, a pole of second order at $y=-1$ and one of third order at $y=1$.

This proof has the advantage of generalizing to a proof of the following formula for $t(p,q,r)$, which is essentially due to Mordell [73]:

$$t(p,q,r) = 1 - \frac{1+p^2r^2+p^2q^2+q^2r^2-p^2q^2r^2}{3pqr} + 4\,s(pr,q) + 4s(pq,r) + 4\,s(qr,p), \tag{15}$$

where p,q,r are relatively coprime integers. Indeed, by 5.3(13)

$$t(p,q,r) = \frac{-1}{pqr}\sum_{t^{pqr}=-1}\frac{t+1}{t-1}\cdot\frac{t^{qr}+1}{t^{qr}-1}\cdot\frac{t^{pr}+1}{t^{pr}-1}\cdot\frac{t^{pq}+1}{t^{pq}-1}$$

$$= -\frac{1}{2}\sum_{t^{pqr}=-1}\operatorname{res}_{z=t}\left[\frac{z+1}{z-1}\,\frac{z^{qr}+1}{z^{qr}-1}\,\frac{z^{pr}+1}{z^{pr}-1}\,\frac{z^{pq}+1}{z^{pq}-1}\,\frac{z^{pqr}-1}{z^{pqr}+1}\,\frac{dz}{z}\right]$$

Since p,q,r are pairwise relatively prime, the only poles of the function in brackets are at $z=0$, $z=\infty$, $z=1$, $z^{pqr}=-1$, $z^p=1$, $z^q=1$, $z^r=1$. Applying the residue theorem, one easily obtains (15). **Then (15) and Dedekind reciprocity imply** $t(p,q,r) = -\frac{(p^2-1)(q^2-1)}{3pq}r + 4\{s(rp,q)+s(rq,p)-s(r,pq)\}$, **providing another proof of Theorem 1.**

A proof of equation (15) using the index theorem can be found in [40].

Literature for §5: There is a very large literature on Dedekind sums; for which we refer the reader to the papers of Carlitz, Dieter, Lang, C. Meyer and Rademacher listed in the bibliography as well as to the recent book [85] of Rademacher and Grosswald.

§6. Quadratic reciprocity and cotangent sums

In 6.1 we define and study the Legendre-Jacobi symbol $(\frac{a}{n})$, following the approach of Frobenius, and give a direct proof of the law of quadratic reciprocity. In 6.2 we relate the Legendre-Jacobi symbol to the symbol $s(a,n)$ studied in 5.1 and deduce the quadratic reciprocity law from the Dedekind reciprocity law 5.1(3).

6.1 The usual definition of the Legendre-Jacobi symbol $(\frac{q}{p})$, with p prime and q not divisible by p, is

$$(\frac{q}{p}) = \begin{cases} +1 & \text{if } x^2 \equiv q \pmod p \text{ is solvable,} \\ -1 & \text{if } x^2 \equiv q \pmod p \text{ is insoluble.} \end{cases} \quad (1)$$

This definition has several disadvantages. It is somewhat unnatural (why should one use ± 1 to distinguish residues and non-residues?), is only valid for prime p, and is not very suitable for proving the main properties of $(\frac{q}{p})$ (Gauss' lemma, reciprocity law).

We will therefore use a different definition of $(\frac{q}{p})$ which has none of these defects. It seems to be due to Zolatareff [101] and Frobenius [29].

Let $n \geq 1$ be an integer, and a an integer prime to n. Let π_a be the map from $(\mathbb{Z}/n\mathbb{Z})$ to itself defined by multiplication of residue classes by a. Since $(a,n) = 1$, this is a permutation of the n elements of $(\mathbb{Z}/n\mathbb{Z})$. We define

$$\left(\frac{a}{n}\right) = \text{sign } \pi_a \in \{+1,-1\}. \tag{2}$$

Clearly $\left(\frac{a}{n}\right)$ only depends on a (mod n), so that $\left(\frac{-}{n}\right)$ defines a function from the group $(\mathbb{Z}/n\mathbb{Z})^*$ to the group $\mathbb{Z}/2\mathbb{Z}$. This function is clearly a homomorphism, that is:

$$\left(\frac{a}{n}\right)\left(\frac{b}{n}\right) = \left(\frac{ab}{n}\right) \tag{3}$$

for a and b prime to n. Moreover,

THEOREM: *If m and n are odd and mutually prime, and a is prime to both m and n, then*

$$\left(\frac{a}{mn}\right) = \left(\frac{a}{m}\right)\left(\frac{a}{n}\right) \tag{4}$$

Proof: It is obvious that, if $\alpha : A \to A$ and $\beta : B \to B$ are two permutations, then $\alpha\times\beta$ is a permutation of $A\times B$ with sign

$$\text{sign}(\alpha\times\beta) = (\text{sign}\,\alpha)^{|B|}\cdot(\text{sign }\beta)^{|A|} \tag{5}$$

Since $(m,n) = 1$, any number can be written as $xn + ym$ where x (mod m) and y (mod n) are uniquely determined by the residue class of the original number (mod mn) and uniquely determine it. This identifies $(\mathbb{Z}/mn\mathbb{Z})$ with $(\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})$. Since $a(xn+ym) = (ax)n + (ay)m$, the permutation π_a on $(\mathbb{Z}/mn\mathbb{Z})$ is the product of π_a on $\mathbb{Z}/m\mathbb{Z}$ and π_a on $\mathbb{Z}/n\mathbb{Z}$ under this identification. Therefore

$$\left(\frac{a}{mn}\right) = \left(\frac{a}{m}\right)^n \left(\frac{a}{n}\right)^m,$$

and for m and n odd this gives eq. (4),

From the same formula, we find that $\left(\frac{a}{2^k m}\right) = \left(\frac{a}{2^k}\right)$ for $k>0$ and m odd. It is easy to show that $\left(\frac{a}{2^k}\right)$ is always $+1$ if $k\geq 3$. For $k = 2$ we have $\left(\frac{3}{4}\right) = -1$. Therefore

$$n \text{ even} \Longrightarrow \left(\frac{a}{n}\right) = \begin{cases} -1, \text{if } n \equiv 4 \pmod 8 \\ \qquad a \equiv 3 \pmod 4 \\ +1, \quad \text{otherwise} \end{cases} \tag{6}$$

This is uninteresting, and in future we will only consider $\left(\frac{a}{n}\right)$ with n odd.

By the theorem, it is (in principle) only necessary to evaluate $\left(\frac{a}{n}\right)$ for $n = p$ or $n = p^k$ (p prime). For $n = p$, we have such a formula if we show that definitions (1) and (2) agree. To see this, recall that $(\mathbb{Z}/p\mathbb{Z})^*$ is a cyclic group (of order $p-1$), and let g be a generator. Then

$$\left(\frac{g}{p}\right) = \operatorname{sign}\begin{pmatrix} g & g^2 & \cdots & g^{p-2} & g^{p-1} \\ g^2 & g^3 & \cdots & g^{p-1} & g \end{pmatrix} = (-1)^{p-2} = -1 \text{ if } p \text{ is an}$$

odd prime, so, by eq. (3), $\left(\frac{g^k}{p}\right)$ equals $(-1)^k$. On the other hand, it is clear that g^k is a quadratic residue (mod iff k is even. This proves the equivalence of (1) and (2 for prime moduli. To complete the determination of the symbol $(\frac{a}{n})$, we have:

<u>THEOREM 1</u>: <u>If</u> p <u>is an odd prime, then for</u> $k \geq 1$,

$$\left(\frac{a}{p^k}\right) = \left(\frac{a}{p}\right)^k . \tag{7}$$

<u>Proof</u>: We can write $(\mathbb{Z}/p^k\mathbb{Z})$ as the disjoint union of A and B, where $A = (\mathbb{Z}/p^k\mathbb{Z})^*$ and B is the set of multiples of p. Then π_a takes both A and B into themselves, and sign $(\pi_a|B) = \left(\frac{a}{p^{k-1}}\right)$. Therefore $\left(\frac{a}{p^k}\right) = \left(\frac{a}{p^{k-1}}\right) \cdot$ sign $(\pi_a|A)$, so the theorem is proved by induction if we can show that sign $(\pi_a|A) = (\frac{a}{p})$.

To do this, let f and g be the functions from A to itself defined by $f(z) = z^{p^{k-1}}$ and $g(z) = z^{1-p^{k-1}}$. Let $C = \operatorname{im} f$, $D = \operatorname{im} g$. Now Fermat's theorem states that $z^p \equiv z$ (mod p), and iterating this one obtains $z^{p^{k-1}} \equiv z$ (mod p), or $f(z) \equiv z$ (mod p), $g(z) \equiv 1$ (mod p). In particular, if $f(z) = f(z')$ then $z \equiv z'$ (mod p). The converse of this is also true, for if we write $z = rz'$ (mod p^k), possible since z' is invertible mod p^k, then $z \equiv z'$ implies $r \equiv 1$ (mod p) and therefore that $r^{p^{k-1}} \equiv 1$ (mod p^k), the last being a consequence of the binomial theorem applied to $r = 1 + cp$.

We have therefore proved that C has exactly p-1 elements and D at most p^{k-1} (since every element of D equals 1 mod p). But $f \times g$: $A \to C \times D$ must be one-one since $z = f(z)g(z)$. Therefore D must have exactly p^{k-1} elements and $f \times g$ is a set isomorphism, since $|A| = p^{k-1}(p-1)$.

Identifying A and $C\times D$ by the isomorphism $f\times g$, we see that the permutation $\pi_a|A$ is identified with a permutation of $C\times D$ which is a product of a permutation σ of A (namely $z^{p^{k-1}} \to a^{p^{k-1}} \cdot z^{p^{k-1}}$) and a permutation τ of B (namely $y \to a^{1-p^{k-1}} \cdot y$). Applying (5) and noting that $|C| = p-1$ is even and $|D| = p^{k-1}$ odd, we get that sign $(\pi_a|A)$ = sign σ. But we showed above that the elements of C are in one-one correspondence with elements of $(\mathbb{Z}/p\mathbb{Z})^*$, the correspondence being effected by reduction mod p. Since $a^{p^{k-1}} \equiv a \pmod p$, the permutation σ corresponds under this identification to the permutation π_a of $(\mathbb{Z}/p\mathbb{Z})^*$, and so sign $\sigma = (\frac{a}{p})$.

Q.E.D.

<u>THEOREM 2</u>: Let $n>1$ be odd. Then

$$(\frac{-1}{n}) = (-1)^{\frac{n-1}{2}} \tag{8}$$

$$(\frac{2}{n}) = (-1)^{\frac{n^2-1}{8}} \tag{9}$$

<u>Proof</u> : The permutation π_{-1} is $\begin{pmatrix} 1 & 2 \ldots & n-1 \\ n-1 & \ldots 2 & 1 \end{pmatrix}$, with $\frac{n-1}{2}$ transpositions, and π_{-2} is $\begin{pmatrix} 1 & 2 & \ldots & \frac{n-1}{2} & \frac{n+1}{2} & \ldots n-1 \\ 2 & 4 & \ldots n-1 & 1 & \ldots n-2 \end{pmatrix}$, with $1+2+\ldots+(n-1)/2 = \frac{n^2-1}{8}$ transpositions.

<u>THEOREM 3</u>: (Generalized Gauss' lemma):

<u>Let</u> $n>1$ <u>be odd</u>, $(a,n) = 1$. <u>Then</u>

$$(\frac{a}{n}) = (-1)^{N_{a,n}}, \tag{10}$$

<u>where</u>

$$N_{a,n} = \#\{1 \leq x \leq \tfrac{n-1}{2} : ((\tfrac{ax}{n})) > 0\}. \qquad (11)$$

<u>Proof</u> : By definition $(\frac{a}{n}) = (-1)^k$, where k is the number of inversions in the permutation $(1,2,\ldots,n) \to (a,2a,\ldots,na \bmod n)$. That is, $k = |S|$ where $S = \{0<x<y<n : ax > ay \pmod n\}$. Here $ax > ay \pmod n$ means that the inequality holds for the "canonical" representatives (those between 0 and n) of ax and ay. Clearly if $ax > ay \pmod n$ then $a(n-y) > a(n-x) \pmod n$, and therefore if (x,y) is in S, so is $(n-y,n-x)$. If $x+y \neq n$, this pairs (x,y) to another element. Therefore the number of elements of S, reduced mod 2, is equal to the number of x such that $(x,n-x)$ is in S. These x are easily seen to be those with $0<x<\frac{n}{2}$ and $((ax)) > 0$.

<u>THEOREM 4</u>: (Quadratic Reciprocity Law): <u>If</u> p <u>and</u> q <u>are odd and mutually prime</u>, <u>then</u>

$$(\tfrac{p}{q})(\tfrac{q}{p}) = (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}} \qquad (12)$$

<u>Proof</u>: Define four sets:

$$S_1 = \{1 \leq x \leq \tfrac{p-1}{2},\ 1 \leq y \leq \tfrac{q-1}{2} : py - qx > \tfrac{p}{2}\},$$

$$S_2 = \{1 \leq x \leq \tfrac{p-1}{2},\ 1 \leq y \leq \tfrac{q-1}{2} : \tfrac{p}{2} > py - qx > 0\}.$$

$$S_3 = \{1 \leq x \leq \tfrac{p-1}{2},\ 1 \leq y \leq \tfrac{q-1}{2} : 0 > py - qx > -\tfrac{q}{2}\},$$

$$S_4 = \{1 \leq x \leq \tfrac{p-1}{2},\ 1 \leq y \leq \tfrac{q-1}{2} : -\tfrac{q}{2} > py - qx\}.$$

It is clear that $|S_1| + |S_2| + |S_3| + |S_4| = \frac{p-1}{2}\,\frac{q-1}{2}$.

Moreover, $|S_1| = |S_4|$, as one sees by replacing the element (x,y) of S_1 by $(\frac{p+1}{2} - x, \frac{q+1}{2} - y)$. Finally, $|S_3| = |\{1 \leqslant y \leqslant \frac{q-1}{2} : \exists\, x \text{ with } \frac{py}{q} + \frac{1}{2} > x > \frac{py}{q}\}|$ $= |\{1 \leqslant y \leqslant \frac{q-1}{2} : ((\frac{py}{q})) > 0\}| = N_{p,q}$, and similarly $|S_2|$ $= N_{q,p}$. Therefore $N_{p,q} + N_{q,p} \equiv (\frac{p-1}{2})(\frac{q-1}{2}) \bmod 2$, which gives the theorem on applying Gauss' lemma.

6.2 We will use Gauss' lemma to relate $(\frac{a}{n})$ to the various sums treated in §5.

The third line of 5.5(13) states

$$t(p,q,2) = -(p-1)(q-1) + 4 \sum_{\substack{0<i<p,\,0<j<q \\ \frac{i}{p} + \frac{j}{q} < \frac{1}{2}}} 1 .$$

Clearly in this sum $1 \leqslant i \leqslant \frac{p-1}{2}$ and $1 \leqslant j \leqslant \frac{q-1}{2}$. Writing $x = \frac{p+1}{2} - i$, $y = j$,

$$t(p,q,2) = -(p-1)(q-1) + 4 \sum_{\substack{1 \leqslant x \leqslant \frac{p-1}{2} \\ 1 \leqslant y \leqslant \frac{q-1}{2} \\ -\frac{x}{p} + \frac{y}{q} < \frac{-1}{2p}}} 1$$

$$= -(p-1)(q-1) + 4\,|S_4|,$$

in the notation of the proof of the quadratic reciprocity law

In that proof we showed that $|S_4| = \frac{1}{2}(|S_1| + |S_4|)$ $= \frac{1}{2}(\frac{p-1}{2}\frac{q-1}{2} - N_{q,p} - N_{p,q})$. Therefore

<u>THEOREM 1</u> : For p,q odd and coprime,

$$t(p,q,2) = -2\left[\frac{p-1}{2}\cdot\frac{q-1}{2} + N_{q,p} + N_{p,q}\right]. \tag{1}$$

<u>COROLLARY</u> 1:

$$t(p,q,2) < 0 \tag{2}$$

<u>COROLLARY 2</u>:

$$N_{q,p} + N_{p,q} \equiv -\frac{p-1}{2}\frac{q-1}{2} \pmod 4. \tag{3}$$

The second corollary follows from the fact $t(p,q,2) \equiv 0 \pmod 8$, which was proved in 5.5. Corollary 2 is a sharpening of the quadratic reciprocity law (which only asserts the same equality mod 2); it was first discovered in 1914 by Frobenius [30].

To get a formula for $N_{p,q}$ itself, we use the following relations, already used in 5.5:

$$((x + \tfrac{1}{2})) - ((x)) = ((2x)) - 2((x))$$

$$= \begin{cases} \frac{1}{2} & \text{if } ((x)) < 0 \\ -\frac{1}{2} & \text{if } ((x)) > 0 \end{cases}$$

$$= \frac{1}{2}\cdot(-1)^{[2x]} \tag{4}$$

if $2x \notin \mathbb{Z}$. Putting (4) into the definition of $N_{p,q}$ (p,q odd, coprime) gives

$$N_{q,p} = \{1 \leq x \leq p-1 : ((\tfrac{xq}{p})) > 0,\ ((\tfrac{x}{p})) < 0\}$$

$$= \sum_{x=1}^{p-1} \left\{\tfrac{1}{2} + ((\tfrac{x}{p} + \tfrac{1}{2})) - ((\tfrac{x}{p}))\right\}\left\{\tfrac{1}{2} + ((\tfrac{xq}{p})) - ((\tfrac{xq}{p} + \tfrac{1}{2}))\right\}$$

$$= \sum_{x=1}^{p-1} \{\tfrac{1}{4} - ((\tfrac{x}{p}))((\tfrac{xq}{p})) - ((\tfrac{x}{p} + \tfrac{1}{2}))((\tfrac{xq}{p} + \tfrac{1}{2}))$$

$$+((\tfrac{x}{p} + \tfrac{1}{2}))((\tfrac{xq}{p})) + ((\tfrac{x}{p}))((\tfrac{xq}{p} + \tfrac{1}{2}))\}, \qquad (5)$$

where in the second line we have used

$$\sum_{x=1}^{p-1} ((\tfrac{qx}{p})) = 0.$$

Setting $y = 2x$, we get

$$N_{q,p} = \sum_{\substack{y=1 \\ y \text{ even}}}^{2p-1} \left\{\tfrac{1}{4} - ((\tfrac{y}{2p}))((\tfrac{yq}{2p})) - ((\tfrac{y+p}{2p}))((q\cdot\tfrac{y+p}{2p}))\right.$$

$$\left. + ((\tfrac{y+p}{2p}))((\tfrac{yq}{2p})) + ((\tfrac{y}{2p}))((q\cdot\tfrac{y+p}{2p}))\right\}$$

since q is odd. Since $y + p$ runs over all odd residue clas classes (mod $2p$) as y runs over all even ones, this is

$$N_{q,p} = \frac{p-1}{4} - \sum_{y=1}^{2p-1} ((\tfrac{y}{2p}))((\tfrac{yq}{2p})) + \sum_{y=1}^{2p-1} ((\tfrac{y}{2p} + \tfrac{1}{2}))((\tfrac{qy}{2p})),$$

or, on applying eq. (4) to $((\frac{y}{2p} + \frac{1}{2})) - ((\frac{y}{2p}))$,

$$N_{q,p} = \frac{p-1}{4} + \frac{1}{2}\sum_{y=1}^{p-1}((\frac{qy}{2p})) - \frac{1}{2}\sum_{y=p+1}^{2p-1}((\frac{qy}{2p}))$$

$$= \frac{p-1}{4} + \sum_{y=1}^{p-1}((\frac{qy}{2p})) \tag{6}$$

The sum in this formula was studied by Dedekind. It is sometimes denoted by

$$S(\frac{q}{2p}) = \sum_{x=1}^{p-1}((\frac{qx}{2p})) \qquad (q,2p) = 1 \tag{7}$$

Therefore we have proved

$$N_{q,p} = \frac{p-1}{4} + S(\frac{q}{2p}). \qquad (q,2p) = 1. \tag{8}$$

To express this result in terms of the quantities studied in §5, we will prove that

$$t(p;q) = -4S(\frac{q}{2p}), \qquad (q,2p) = 1 \tag{9}$$

where $t(p;q)$ is defined by 5.3(14). Indeed, by definition

$$t(p;q) = \{0<x<p : 0 < \frac{xq}{p} < 1 \mod 2\}$$

$$- \{0<x<p : 1 < \frac{xq}{p} < 2 \mod 2\}$$

$$= \sum_{\substack{x=1 \\ ((\frac{xq}{2p}))<0}}^{p-1} 1 - \sum_{\substack{x=1 \\ ((\frac{xq}{2p}))>0}}^{p-1} 1$$

$$= 2\sum_{x=1}^{p-1} \left\{ ((\tfrac{xq}{2p} + \tfrac{1}{2})) - ((\tfrac{xq}{2p})) \right\}$$

$$= 2\sum_{x=1}^{p-1} ((q\cdot \tfrac{x+p}{2p})) - 2\sum_{x=1}^{p-1} ((\tfrac{xq}{2p}))$$

$$= -4\sum_{x=1}^{p-1} ((\tfrac{xq}{2p})).$$

We have now evaluated $N_{q,p}$, namely:

$$N_{q,p} = \frac{p-1}{4} - \frac{1}{4}\, t(p;q). \qquad (10$$

However, for the study of the Legendre-Jacobi symbol we need much less, i.e. only the value of $N_{q,p}$ (mod 2). Using the notations of 5.1, we have for q and p prime to one another, p odd,

$$f_p(q) = \sum_{k=1}^{p-1} k \cdot \left[\frac{kq}{p}\right]$$

$$\equiv \sum_{\substack{k=1 \\ k \text{ odd}}}^{p-1} \left[\frac{kq}{p}\right] \qquad (\text{mod } 2)$$

$$= \frac{(p-1)(q-1)}{2} - \sum_{\substack{k=1 \\ k \text{ even}}}^{p-1} \left[\frac{kq}{p}\right]$$

by 5.1 (7). That is,

$$f_p(q) \equiv (q-1)\cdot\frac{p-1}{2} - \sum_{x=1}^{\frac{p-1}{2}} \left[\frac{2xq}{p}\right] \qquad (\text{mod } 2)$$

$$\equiv \frac{(q-1)(p-1)}{2} - \sum_{\substack{0<x<p/2 \\ \left[\frac{2xq}{p}\right] \text{ odd}}} 1$$

$$\equiv \frac{(q-1)(p-1)}{2} - \sum_{\substack{0<x<p/2 \\ \left(\left(\frac{qx}{p}\right)\right) > 0}} 1$$

$$= \frac{(p-1)(q-1)}{2} - N_{q,p} .$$

Therefore we have proved

$$f_p(q) \equiv \frac{(p-1)(q-1)}{2} - N_{q,p} \quad (\text{mod } 2) \tag{11}$$

and, since $(\frac{q}{p}) = (-1)^{N_{q,p}} \equiv 2N_{q,p} + 1 \pmod 4$,

$$(\tfrac{q}{p}) \equiv 1 + (p-1)(q-1) - 2f_p(q) \quad (\text{mod } 4). \tag{12}$$

Using 5.1(10) to translate to $s(q,p)$, this becomes

$$(\tfrac{q}{p}) + 6ps(q,p) \equiv \frac{p+1}{2} \quad (\text{mod } 4). \tag{13}$$

Then the Dedekind reciprocity law 5.1(3) becomes, for p and q odd,

$$2\left(q(\tfrac{q}{p}) + p(\tfrac{p}{q})\right) \equiv q(p+1) + p(q+1) - 12pq\left\{s(q,p) + s(p,q)\right\} \quad (\text{mod } 8)$$

$$= 2pq + p + q + 3pq - p^2 - q^2 - 1$$

$$\equiv 5\,pq + p + q - 3, \qquad (\text{mod } 8)$$

since $p^2 \equiv q^2 \equiv 1 \pmod 8$. Since q and $(\frac{q}{p})$ are odd, $q(\frac{q}{p}) \equiv (\frac{q}{p}) + q-1 \pmod 4$, and similarly with p and q interchanged, so we find

$$2(\tfrac{q}{p}) + 2(\tfrac{p}{q}) \equiv 5pq - p - q + 1 \qquad (\text{mod } 8)$$

$$\equiv (p-1)(q-1) + 4 \qquad (\text{mod } 8),$$

or

$$\frac{1}{2}\left[(\tfrac{q}{p}) - 1\right] + \frac{1}{2}\left[(\tfrac{p}{q}) - 1\right] \equiv \frac{p-1}{2}\cdot\frac{q-1}{2} \qquad (\text{mod } 2),$$

which is a restatement of the law of quadratic reciprocity.

An even more direct connection between $s(q,p)$ and $(\frac{q}{p})$ is the fact that the number of inversions of the permutation π_q of $\mathbb{Z}/p\mathbb{Z}$ (i.e. the number $k = |S|$ of our proof of Gauss' lemma in 6.1) equals $3p\left[s(1,p) - s(q,p)\right] = \frac{(p-1)(p-2)}{4} - 3p\,s(q,p)$. This is proved in C. Meyer [61].

Indeed, the number k is given by

$$k = \#\{\, 0<x<y<p:\ qx > qy \mod p\}$$

$$= \sum_{x=1}^{p-1} \sum_{\substack{y=1\\ y\neq x}}^{p-1} \left\{((\tfrac{x-y}{p})) - ((\tfrac{x}{p})) + ((\tfrac{y}{p})) + \tfrac{1}{2}\right\} \times$$

$$\times \left\{((\tfrac{qy-qx}{p})) - ((\tfrac{qy}{p})) + ((\tfrac{qx}{p})) + \tfrac{1}{2}\right\},$$

where we have used

$$((\alpha-\beta))-((\alpha))+((\beta))+\frac{1}{2}=\begin{cases}1 & 1>\beta>\alpha>0 \pmod 1\\ 0 & 1>\alpha>\beta>0 \pmod 1\\ \frac{1}{2} & \alpha\equiv 0 \text{ or } \beta\equiv 0 \text{ or } \alpha\equiv\beta \pmod 1\end{cases} \tag{14}$$

If we expand this expression for k, we get 16 terms. Using the oddness of the function $((x))$ and the fact that q is prime to p, we get

$$k=\sum_{\substack{0<x,y<p\\ x\neq y}}\Big\{-((\tfrac{x-y}{p}))((\tfrac{qx-qy}{p}))-((\tfrac{x}{p}))((\tfrac{qx}{p}))-((\tfrac{y}{p}))((\tfrac{qy}{p}))$$

$$+((\tfrac{x-y}{p}))((\tfrac{qx}{p}))-((\tfrac{x-y}{p}))((\tfrac{qy}{p}))-((\tfrac{x}{p}))((\tfrac{qy-qx}{p}))$$

$$+((\tfrac{x}{p}))((\tfrac{qx}{p}))+((\tfrac{y}{p}))((\tfrac{qx}{p}))+((\tfrac{y}{p}))((\tfrac{qy-qx}{p}))+\tfrac{1}{4}\Big\}.$$

Each of the first three terms in brackets clearly contributes $-(p-2)s(q,p)$, while each of the following six terms contribute $-s(q,p)$ (for example, summing $((\frac{x-y}{p}))((\frac{qx}{p}))$ over all x and y would give zero, but here the omitted terms $y\equiv 0$ mod p give $-s(q,p)$), and the last term gives $\frac{(p-1)^2-(p-1)}{4}$.

This proves

THEOREM 2: Let p,q be coprime integers, $p>0$. Then

$$\frac{(p-1)(p-2)}{2}-6ps(q,p)$$

$$=2\cdot\#\{x,y \pmod p \mid x<y,\ qx>qy \pmod p\}.$$

§7. Cotangent sums and modular forms

In the introduction to this chapter, we mentioned the relationship between the cotangent sums we have been studying and the theory of modular forms. In fact, it was in this connection that the cotangent sums were first discovered and studied by Dedekind in a famous commentary [21] to incomplete notes of Riemann about the boundary behaviour of modular functions.

Dedekind studied the function (now named after him)

$$\eta(z) = e^{\pi i z/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}) \quad (\mathrm{Im}\ z > 0) \tag{1}$$

$$= q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \tag{2}$$

where

$$q = e^{2\pi i z} \qquad (|q| < 1). \tag{3}$$

This function is related to other standard functions of elliptic function theory as follows:

$$\Delta(z) = (2\pi)^{12} \ \eta(z)^{24}, \tag{4}$$

where $\Delta(z)$ is defined by

$$\omega^{-12}\Delta(\tfrac{\omega'}{\omega}) = g_2(\omega,\omega')^3 - 27\ g_3(\omega,\omega')^2, \quad (\mathrm{Im}\ (\tfrac{\omega'}{\omega}) > 0) \tag{5}$$

$$g_2(\omega,\omega') = 60 \sum_{(m,m') \neq (0,0)} (m\omega + m'\omega')^{-4}, \tag{6}$$

$$g_3(\omega,\omega') = 140 \sum_{(m,m')\neq(0,0)} (m\omega + m'\omega')^{-6}. \qquad (7)$$

To show that (4) and (5) really do define the same function it suffices to show that (4) defines a modular form Δ of weight 12 (precise definitions will be given in 7.1). For then (4) and (5) both define cusp forms of weight 12 and with the same leading terms, and therefore agree, since the space of cusp forms of weight 12 is one-dimensional (a reference for all this is Gunning [32] or Serre [92]).

The statement that $\Delta(z)$ is a modular form means that $\Delta(\frac{az+b}{cz+d})$ is related to $\Delta(z)$ by a certain transformation formula given below. Thus from (4) we deduce that the extent to which $\eta(z)$ fails to obey such a formula is given by a 24th root of unity. That is, to every transformation $z \to \frac{az+b}{cz+d}$ there is associated a certain 24th root of unity. If we study just the power $\Delta(z) = \eta(z)^{24}$, we lose the information given by this root; this is the reason that Dedekind studied η. Hermite [34] had already studied the cube of η (which he thought of in terms of theta-functions, namely $\eta(\tau)^3 = \frac{1}{2\pi}\vartheta_1'(0)$, where τ is the ratio of the periods of the theta-function ϑ_1). He evaluated the eighth root of unity which arises in terms of the Legendre-Jacobi symbol. Dedekind went beyond this by considering the function $\log \eta(z)$ (with an appropriate choice of branch for the logarithm), which contains much more information; now to each $\binom{a\ b}{c\ d}$ is associated an integral multiple of $2\pi i$, and Hermite's eighth root is given by reducing this integer modulo 8. The value of this integer turns out to be given by the Dedekind sum studied in §5. In particular the relation just mentioned, combined with Hermite's evaluation of the eighth root of 1, gives the connections between Dedekind sums and the quadratic reciprocity symbol which was considered in §6.

The proof that $\Delta(z)$ is a modular form of weight 12 (following Siegel) occupies 7.1; in 7.2 we show how cotangent sums arise in the study of $\log \eta(z)$.

7.1 Given a non-singular matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with complex coefficients, the map

$$z \to \frac{az + b}{cz + d} \tag{1}$$

is a holomorphic automorphism of $\bar{\mathbb{C}} = \mathbb{C} \cup \infty$ $(= P_1(\mathbb{C}))$. Indeed, it is well-known that these "linear fractional transformations" (or "Möbius transformations") are the only biholomorphic automorphisms of $\bar{\mathbb{C}}$. We will be interested only in those preserving the upper half-plane

$$H = \{z: \ \mathrm{Im}\ z > 0\}. \tag{2}$$

One can easily show that (1) takes $\bar{\mathbb{R}}$ onto itself only if a,b,c,d are all real (after multiplication, if necessary, of all four by a common factor), and when this is the case it obviously must take H onto itself or onto the lower half-plane. Since (for a,b,c,d real)

$$\mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \mathrm{Im}\left(\frac{(az+b)(c\bar{z}+d)}{|cz+d|^2}\right) = \frac{ad-bc}{|cz+d|^2} \times \mathrm{Im}\,(z),$$

H is mapped onto itself only if $ad - bc$ is positive. We are now only interested in matrices with integer coefficients whose inverses are also integral; then $\det A = ad-bc = +1$, i.e. $A \in SL(2,\mathbb{Z})$. Finally, since we can multiply a,b,c,d in (1) by a common factor without changing the map, we do not distinguish between A and -A. Therefore we define the <u>modular group</u> Γ by

$$\Gamma = SL(2,\mathbb{Z}) \,/\, \{\pm 1\}. \tag{3}$$

Associating to any matrix A the map (1) defines an action of Γ on H. We define a <u>modular form of weight 2r</u> as a meromorphic function g on H such that

$$g\left(\frac{az+b}{cz+d}\right) = g(z)\cdot(cz+d)^{2r} \quad \left(z \in H, \pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\right). \tag{4}$$

(There is some variation of notation here; for instance, Gunning in [32] calls such a function an "unrestricted modular form" and defines its weight as r rather than 2r).

Since (using $ad-bc = 1$) we have

$$\frac{d}{dz}\left(\frac{az+b}{cz+d}\right) = \frac{1}{(cz+d)^2},$$

eq. (4) can be rewritten as

$$g(Az)\cdot\left[d(Az)\right]^r = g(z)\cdot\left[dz\right]^r, \tag{5}$$

so g corresponds to a Γ-invariant differential form of degree r (this is the reason that we refer to the function g as a modular form; the term modular function is used if $r = 0$). Since H is contractible, the cotangent bundle T^*H and its rth tensor product $(T^*H)^r$ are trivial bundles, so that we can identify forms with functions by $g(z) \to g(z)\,(dz)^r$; however, (5) is a better way to think of the definition of modular form. In particular, it is clear from (5) that, if equation (4) holds for matrices A and B, it also holds for AB (and A^{-1}). Therefore, in checking whether a given form is a modular form, it suffices to check (4) for the generators of Γ.

The main result of this section is

THEOREM: *The function $\Delta(z)$ defined by 7(1),(3) is a modular form of weight 12.*

Proof: This is a classical result, but the proof we shall give is fairly new (due to Siegel [93]). An even more elementary proof, based on the partial fractions identity

$$\frac{1}{e^{2\pi iz}-1} = -\frac{1}{2} - \frac{1}{2i}\cot \pi z = -\frac{1}{2} - \frac{1}{2\pi iz} + \frac{z}{\pi i}\sum_{n=1}^{\infty}\frac{1}{n^2-z^2} \qquad (6)$$

can be found in Chandrasekharan [17].

It is well known that $SL(2,\mathbb{Z})$ is generated by the transformations

$$S = \begin{pmatrix}1&1\\0&1\end{pmatrix}, \qquad S(z) = z+1, \qquad (7)$$

$$T = \begin{pmatrix}0&-1\\1&0\end{pmatrix}, \qquad T(z) = -1/z. \qquad (8)$$

Therefore by the remarks above, it suffices to prove that $\Delta(z)$

satisfies (4) for the two matrices S and T. The first is trivial, since $\Delta(z)$ only depends on $e^{2\pi iz}$ and therefore satisfies

$$\Delta(z+1) = \Delta(z). \tag{9}$$

It therefore only remains to prove that

$$\Delta(-1/z) = z^{12}\,\Delta(z). \tag{10}$$

To rewrite this in terms of η, we must take a twenty-fourth root, and this requires a convention about many valued functions We cut the complex plane along the negative real axis, and define log on $\mathbb{C} - \mathbb{R}_-$ by requiring it to be real on $\mathbb{R}_+$. For $z \in \mathbb{C}-\mathbb{R}_-$, z^a will denote $e^{a \log z}$. Then eq. 7(3) becomes

$$\eta(z) = \frac{1}{\sqrt{2\pi}}\,\Delta(z)^{1/24}, \tag{11}$$

and (10) certainly will follow from

$$\eta(-\tfrac{1}{z}) = \eta(z)\cdot(z/i)^{1/2} \quad (z \in H). \tag{12}$$

We prove rather more by looking at the logarithm of η rather than η itself. Thus we define

$$f(z) = 24 \log \eta(z) = \log\left((2\pi)^{-12}\Delta(z) \right), \tag{13}$$

and write

$$2\pi iz - f(z) = -24 \sum_{n=1}^{\infty} \log\,(1-e^{2\pi inz})$$

$$= - 24\sum_n \log(1-q^n) \quad (q = e^{2\pi i z}, \quad |q| < 1)$$

$$= \quad 24\sum_n\sum_k \frac{q^{nk}}{k} \;=\; 24\sum_k\sum_n \frac{q^{nk}}{k}$$

$$= \quad 24\sum_k \frac{1}{k}\;\frac{q^k}{1-q^k} \tag{14}$$

since the absolute convergence of the double series for $|q| < 1$ permits the interchange of summations. Using

$$\frac{q^k}{1-q^k} = \frac{e^{2\pi ikz}}{1-e^{2\pi ikz}} = -\frac{1}{2} - \frac{1}{2i}\cot \pi kz, \tag{15}$$

we obtain from (14)

$$2\pi i\left(z + \frac{1}{z}\right) - f(z) + f\left(-\frac{1}{z}\right)$$

$$= 12i \sum_{k=1}^{\infty} \frac{1}{k}\left(\cot\frac{\pi k}{z} + \cot \pi kz\right), \tag{16}$$

and the series is convergent because (14) is.

To evaluate the sum in (16), we will consider the residues of the function

$$g(z) = \frac{1}{z}\cot(az)\cot(bz). \qquad (a,b \in \mathbb{C} - \{0\}). \tag{17}$$

Clearly, $g(z)$ has a triple pole at $z = 0$ and poles whenever az or bz is a multiple of π. The latter poles are simple if there are no z for which az/π and bz/π are simultaneously integers, and thus if $a/b \notin \mathbb{Q}$. We assume $a/b \notin \mathbb{R}$. To find the residue of g at $z = 0$ we substitute the expansion

$$\cot x = \frac{1}{x} - \frac{x}{3} + O(x^3) \qquad (x \to 0)$$

into (17); we then get

$$\text{res}_{z=0} (g(z)\, dz) = -\frac{a^2 + b^2}{3ab} .$$

The residues at the simple poles are, of course, even simpler to evaluate, and are given by

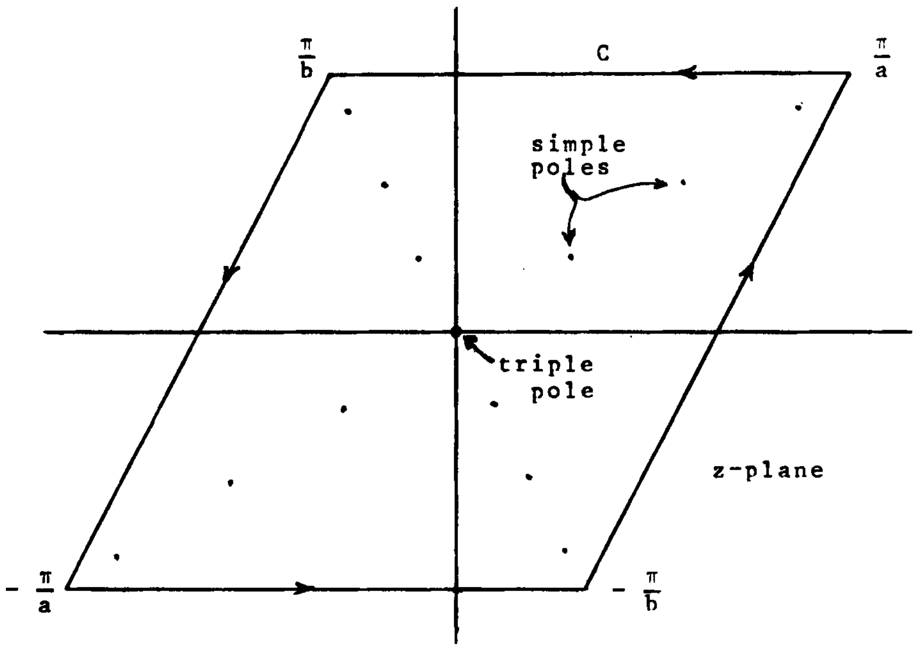
$$\text{res}_{z=k\pi/a}(g(z)\, dz) = \frac{1}{k\pi} \cot \frac{\pi k b}{a} \quad (k \in \mathbb{Z},\ k \neq 0),$$

$$\text{res}_{z=k\pi/b}(g(z)\, dz) = \frac{1}{k\pi} \cot \frac{\pi k a}{b} \quad (k \in \mathbb{Z},\ k \neq 0).$$

From these formulas, it is clear that the residues only depend on a/b, and so are unchanged if g is replaced by

$$g_\nu(z) = \nu g(\nu z) = \frac{1}{z} \cot (a\nu z) \cot (b\nu z). \quad (\nu \in \mathbb{R},\ \nu > 0). \tag{18}$$

The poles of g_ν are of course now located at $z = k\pi/a\nu$, $z = k\pi/b\nu$. In the parallelogram C with vertices $\pm \pi/a$, $\pm \pi/b$, therefore, the poles of g_ν are 0, $\pm k\pi/a$, $\pm k\pi/b$ $(k = 1,2,\ldots, \nu)$. We take $\nu = n + \frac{1}{2}$, for n a positive integer (the picture shows $n = 3$). Then applying the residue theorem to C (which has a non-empty interior



since a/b is not real) gives

$$-\frac{a^2+b^2}{3ab}+\frac{2}{\pi}\sum_{k=1}^{n}\frac{1}{k}\left(\cot\frac{\pi kb}{a}+\cot\frac{\pi ka}{b}\right)$$

$$=\frac{1}{2\pi i}\oint_C g_\nu(z)\,dz. \qquad (19)$$

Now for $x \in \mathbb{C} - \mathbb{R}$,

$$\lim_{k\to\infty}(\cot\ kx)=\lim_{k\to\infty}\left(i\,\frac{e^{2\pi ikx}+1}{e^{2\pi ikx}-1}\right)=\begin{cases} i & \text{if Im } x < 0\\ -i & \text{if Im } x > 0.\end{cases} \qquad (20)$$

Therefore the function

$$g_\infty(z) = \frac{-1}{z}\text{ sign (Im } az\text{) sign (Im } bz\text{)}, \qquad (21)$$

which equals $+\frac{1}{z}$ on two sides of C and $-\frac{1}{z}$ on the other two, and is undefined on the corners) is the limit $\lim_{n\to\infty} g_\nu(z)$, uniformly on C except near the corners. Also $g_\nu(z)$ is uniformly bounded on C (i.e. $|\nu(z)| < M$ for all $z \in C$ and all ν), because we chose $\nu = n + \frac{1}{2}$. It follows that

$$\lim_{n\to\infty} \left[\oint_C g_\nu(z)\, dz \right] = \oint_C g_\infty(z)\, dz.$$

But

$$\oint_C g_\infty(z)\, dz = \left[-\int_{\pi/a}^{\pi/b} + \int_{\pi/b}^{-\pi/a} - \int_{-\pi/a}^{-\pi/b} + \int_{-\pi/b}^{\pi/a} \right] \frac{dz}{z}$$

$$= -\log\frac{a}{b} + \log\frac{-b}{a} - \log\frac{a}{b} + \log\frac{-b}{a}$$

$$= 4\log\frac{a}{bi}, \quad \text{if} \quad \operatorname{Im}\left(\frac{a}{b}\right) > 0 \tag{21}$$

(we have assumed $a/b \notin \mathbb{R}$, and the sign in (21) can be checked by taking $b = 1$, $a = i$). Therefore (19) gives

$$-\frac{a^2+b^2}{3ab} + \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{1}{k}\left(\cot\frac{\pi k a}{b} + \cot\frac{\pi k b}{a}\right)$$

$$= \frac{2}{\pi i} \log\frac{a}{bi} \quad \left(\operatorname{Im}\frac{a}{b} > 0\right). \qquad (22$$

Writing z for a/b gives the equality

$$12i \sum_{k=1}^{\infty} \frac{1}{k}\left(\cot\frac{\pi k}{z} + \cot \pi k z\right) = 2\pi i\left(z + \frac{1}{z}\right) + 12\log\frac{z}{i}$$

$$(\operatorname{Im} z > 0), \qquad (23$$

and substituting this into (16), we obtain

$$f(-\tfrac{1}{z}) \quad = \quad f(z) \quad + \quad 12 \log \tfrac{z}{i}. \qquad (24)$$

and the theorem is proved.

7.2 We now want to discuss the behaviour of $f(z)$ under general modular transformations and show how this relates to the cotangent sums of §5. By the theorem of 7.1,

$$\Delta(\tfrac{az+b}{cz+d}) \quad = \quad \Delta(z)\cdot(cz+d)^{12}, \qquad (1)$$

and taking logarithms (according to the convention for defining log in $\mathbb{C} - \mathbb{R}_-$ given in 7.1) we find

$$f(\tfrac{az+b}{cz+d}) \quad = \quad f(z) + 6 \log\left[-(cz+d)^2\right] + 2\pi i(a,b,c,d) \qquad (2)$$

for some integer* (a,b,c,d) depending on the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We use $\log\left[-(cz+d)^2\right]$ because, if $c \neq 0$, then $cz+d \notin \mathbb{R}$ and therefore $\left[-(cz+d)^2\right] \notin \mathbb{R}_-$, so that $\log\left[-(cz+d)^2\right]$ is defined by our convention. In using (2) we will always assume that $c \neq 0$. (If $c = 0$, then $A = \pm\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, and this is a trivial case since $f(z+b) = f(z) + 2\pi ib$ for $b \in \mathbb{Z}$).

We have therefore associated an integer (a,b,c,d) to

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*The notation (a,b,c,d) and the notation (d,c) used later are those of Dedekind [21]; we have avoided using the latter in §§5-6 to avoid confusion with the greatest-common-divisor function. However, in all of 7.2, (d,c) will denote the Dedekind symbol.

every $A \in SL(2, \mathbb{Z})$ with $c \neq 0$. In studying its properties, we shall follow the original paper of Dedekind [21]. First note that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} a' & b' \\ c & d \end{pmatrix}$ are both in $SL(2,\mathbb{Z})$, then $ad-bc = a'd - b'c = 1$ and therefore $c(a-a') = d(b-b')$. Since c and d are mutually prime, we deduce $a \equiv a' \pmod{c}$ and therefore

$$a' = a + nc, \qquad b' = b + nc$$

$$\frac{a'z + b'}{cz + d} = \frac{az + b}{cz + d} + n.$$

Hence

$$f\left(\frac{a'z + b'}{cz + d}\right) = f\left(\frac{az + b}{cz + d}\right) + 2\pi i n,$$

from which we deduce

$$(a',b',c,d) = (a,b,c,d) + n = (a,b,c,d) + \frac{a' - a}{c}.$$

Therefore $c\cdot(a,b,c,d) - a$ is independent of a and b for fixed c and d (subject to $ad - bc = 1$). For symmetry, we subtract the trace $a+d$ instead of just a. It will follow from the discussion below that $c\cdot(a,b,c,d) - (a+d)$ is in fact always an even integer. We therefore define

$$(d,c) = \frac{1}{2}\left[a + d - c\cdot(a,b,c,d)\right], \tag{3}$$

and summarize what we have already proved as

THEOREM 1 (Dedekind): For $z \in H$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$, $c \neq 0$,

$$f\left(\frac{az+b}{cz+d}\right) = f(z) + 6\log\left[-(cz+d)^2\right] + 2\pi i\,\frac{a+d-2(d,c)}{c}, \tag{4}$$

<u>where</u> $2(d,c)$ <u>is an integer depending only on</u> c <u>and</u> d <u>and defined for all relatively prime</u> c <u>and</u> d <u>with</u> $c \neq 0$.

We now study the properties of the Dedekind symbol (d,c). Since $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as an element of Γ is only determined up to a factor ± 1, we have $(-a,-b,-c,-d) = (a,b,c,d)$, from which it follows that

$$(-d,-c) = -(d,c). \tag{5}$$

Therefore (since $c \neq 0$) we can restrict ourselves to the case $c > 0$. Next, since $(a,b,c,d) \in \mathbb{Z}$) and $ad \equiv 1 \pmod c$ we have

$$2(d,c) \equiv d + d^{-1} \pmod c, \tag{6}$$

where d^{-1} denotes the residue class $x \pmod c$ with $xd \equiv 1$

Our main goal is to prove

<u>THEOREM 2</u>: <u>Let</u> c,d <u>be relatively prime integers</u>, $c>0$. <u>Then</u>

$$(d,c) = \frac{3}{2}\sum_{k=1}^{c-1} \cot\frac{\pi k}{c}\,\cot\frac{\pi k d}{c}. \tag{7}$$

<u>COROLLARY:</u> <u>The Dedekind symbol</u> (d,c) <u>is an integer</u>.

We can avoid the restriction $c>0$ by writing (7) as

$$(d,c) = \frac{3}{2}\sum_{\substack{k \pmod c \\ k \not\equiv 0}} \cot\frac{\pi k}{c}\,\cot\frac{\pi k d}{c}, \tag{8}$$

This makes sense (because cot is a periodic function), and satisfies equation (5). Using the results of §5, we can rewrite (7) as

$$(d,c) = 6c\, s(d,c) \tag{9}$$

$$= 6c \sum_{k=1}^{c-1} ((\tfrac{k}{c}))\,((\tfrac{kd}{c})) \tag{10}$$

for $c>0$ (this is eq. 5.2(3)). The corollary then follows from 5.1(13). Formula (10) was known to Dedekind.

To prove Theorem 2, we proceed as in the proof of 5.5(11), namely by characterizing (d,c) by its properties under simple changes of argument. Specifically, we prove

<u>THEOREM 3</u>: <u>The Dedekind symbol</u> (d,c) <u>has the properties</u>

$$(-d,-c) = -(d,c) \tag{11}$$

$$(d,-c) = (d,c) \tag{12}$$

$$(-d,c) = -(d,c) \tag{13}$$

$$(d+c,c) = (d,c) \tag{14}$$

$$(0,\pm 1) = 0 \tag{15}$$

$$c(c,d) + d(d,c) = \tfrac{1}{2}\left[1 + c^2 + d^2 - 3|cd|\right]. \tag{16}$$

<u>These properties uniquely define</u> (d,c) <u>for all relatively prime</u> c,d ($c \neq 0$).

To prove the last statement of this theorem, we notice that (11)-(16) suffice to define (d,c) for relatively prime d and c by a Euclidean algorithm. We can use (11)-(13) to make c and d positive. Then (14) says that (d,c) only depends on the residue class of d(mod c), so we can ensure

$0 \le d < c$. Then applying (16) we can find (d,c) if we know (c,d) and d is smaller than c. Continuing in this way, we must get down to $(0,1)$, which is evaluated in (15). This also shows that (d,c) is an integer, since the right-hand side of (16) is obviously integral for relatively prime c and d. Define a symbol $(d,c)'$ as $6|c|\ s(d,|c|)$. Because of the uniqueness clause of the theorem, we can deduce Theorem 2 as soon as we check that $(d,c)'$ satisfies properties (11-(16). Properties (11)-(14) are self-evident from the definition of $s(d,c)$, while (15) is also trivial since the sum in (10) is empty for $c = 1$. Finally, the reciprocity law (16) for $(d,c)'$ is just the Dedekind reciprocity law 5.1(3), of which several proofs were given in §5.

It remains to check (11)-(16) for the Dedekind symbol itself. Consider the map $z \to -\bar{z}$, which takes the upper half plane H onto itself. On the one hand, substituting $-\bar{z}$ for z in (4) gives

$$f\left(\frac{-a\bar{z} + b}{-c\bar{z} + d}\right) = f(-\bar{z}) + 6 \log\left[-(-c\bar{z}+d)^2\right] + 2\pi i\frac{a+d-2(d,c)}{c} . \tag{17}$$

On the other hand, it is clear from the definition that $\eta(-\bar{z}) = \overline{\eta(z)}$, and therefore that $f(-\bar{z}) = \overline{f(z)}$. Therefore applying (4) to the matrix $\begin{pmatrix} +a & -b \\ -c & d \end{pmatrix}$ and conjugating,

$$f\left(\frac{-a\bar{z} + b}{-c\bar{z} + d}\right) = \overline{f\left(\frac{+az - b}{-cz + d}\right)}$$

$$= \overline{f(z)} + \overline{6 \log -[(-cz+d)^2]} + \overline{2\pi i\ \frac{a+d-2(d,-c)}{-c}}$$

$$= f(-\bar{z}) + 6 \log\left[-(-c\bar{z}+d)^2\right] - 2\pi i \frac{a+d-2(d,-c)}{-c} . \qquad (18)$$

Comparing these two formulas gives (12). Equation (11) is the sameaas (5), and (13) follows from (11) and (12).

We can use the transformation $z \to z+1$ (which also takes H onto itself) in a similar way: on the one hand, replacing z by $z+1$ in (4) gives

$$f\left(\frac{az+a+b}{cz+c+d}\right) = f(z+1) + 6 \log\left[-(cz+c+d)^2\right] + 2\pi i \frac{a+d-2(d,c)}{c}; \qquad (19)$$

on the other, replacing $\binom{a\ b}{c\ d}$ by $\binom{a\ \ a+b}{c\ \ c+d}$ in (4) yields

$$f\left(\frac{az+a+b}{cz+c+d}\right) = f(z) + 6 \log\left[-(cz+c+d)^2\right] + 2\pi i \frac{a+d+c-2(d+c,c)}{c}. \qquad (20)$$

Since $f(z+1) = f(z) + 2\pi i$, a comparison of these two equations yields (14).

Finally, we can use the transformation $z \to -\frac{1}{z}$ from H onto itself to deduce (16), just as $z \to -\bar{z}$ and $z \to z+1$ were used to deduce (12) and (14). Thus replacing z by $-\frac{1}{z}$ in (4) gives

$$f\left(\frac{a(-1/z)+b}{c(-1/z)+d}\right) = f\left(\frac{-1}{z}\right) + 6 \log\left[-\left(\frac{-c}{z} + d\right)^2\right] + 2\pi i \frac{a+d-2(d,c)}{c}, \qquad (21)$$

while replacing $\binom{a\ b}{c\ d}$ by $\binom{b\ \ -a}{d\ \ -c}$ in (4) gives

$$f\left(\frac{bz-a}{dz-c}\right) = f(z) + 6 \log\left[-(dz-c)^2\right] + 2\pi i \frac{b-c-2(-c,d)}{d} \qquad (22)$$

Subtracting equation (22) from (21) and using 7.1(24) gives

$$6 \log\left[-(-\tfrac{c}{z} + d)^2\right] + 12 \log \tfrac{z}{i} - 6 \log\left[-(dz - c)^2\right]$$

$$= 2\pi i \left(\frac{b - c - 2(-c,d)}{d} - \frac{a + d - 2(d,c)}{c} \right). \qquad (23)$$

With our conventions for evaluating the logarithms, the value of the left-hand side of (23) is $-6\pi i$ if cd is positive and $+6\pi i$ if cd is negative, (this is most easily checked by setting $z = i$). Therefore, multiplying the whole equation by $\frac{cd}{2\pi i}$, we get

$$3|cd| = c(b-c-2(-c,d)) - d(a+d-2(d,c))$$

$$= -1 - c^2 - d^2 + 2c(c,d) + 2d(d,c). \qquad (24)$$

This gives (16). Setting $d = 0$, $c = 1$ in (16) gives (15) (which is just a restatement of 7.1(24)). Theorem 3 is thus completely proved.

It is also possible to prove Theorem 2 directly in a way exactly analogous to our previous proof of the special case 7.1(24). The function $g(z)$ defined by 7.1(17) must be replaced by the more complicated function

$$h(z) = g(z) + \frac{4}{z} \sum_{k=1}^{c-1} \frac{e^{2ikbz/c}}{1 - e^{2ibz}} \frac{e^{-2ik^*az/c}}{1 - e^{-2iaz}}, \qquad (25)$$

where, as before, a and b are complex numbers whose ratio is not real, and where k^* is defined by

$$0 < k^* < c, \quad k^* \equiv -dk \pmod{c}. \qquad (26)$$

Again we define $h_\nu(z)$ by $h_\nu(z) = \nu h(\nu z)$, where $\nu = n + \frac{1}{2}$, and integrate $h_\nu(z)$ around the parallelogram C. The function h_ν has the same poles with the same orders as does g_ν, but the residues are more complicated; for example, the triple pole at $z = 0$ has residue $-(a^2+b^2)/3abc + 4s(-d,c)$. As before, h_ν tends as $\nu \to \infty$ to an elementary function h_∞, the convergence being uniform away from the corners of C and uniformly bounded, and on evaluating $\int_C h_\infty(z)dz$ we obtain (4). The details can be found in Lehmer [57], but since they are a little tedious we have preferred an indirect proof of (7) using the transformation properties of f(z).

The transformation law (4) for $\log \Delta(z)$ is very important, and has been given many proofs. Dedekind's proof [21] is essentially the one we have given. A second proof was given by Rademacher [81] in 1932 using Mellin transforms. A third proof, also given by Rademacher [84], is the proof we just mentioned, using Siegel's contour-integration trick. A fourth proof, due to Iseki [49], uses incomplete theta functions; a fifth, also due to Iseki [50] uses the partial-fraction expansion of $\frac{e^{az}}{1-e^z}$ for $0<a<1$ (cf. (6)). A sixth proof, due to Meyer [66], is similar to Iseki's first proof but uses a certain functional equation for the incomplete theta function, involving a confluent hypergeometric function. A very recent proof of Lewittes [59] proceeds by using analytic continuations to define the Eisenstein series $G_k(\tau)$ (classically defined if $k = 4,6,\ldots$) for $k = 0$.

§8. Cotangent sums and Markoff triples

In Section 5 we studied

$$s(q,r;p) = \frac{1}{4p} \sum_{k=1}^{p-1} \cot \frac{\pi k q}{p} \cot \frac{\pi k r}{p} \tag{1}$$

$$= \sum_{k=1}^{p-1} \left(\left(\frac{kq}{p}\right)\right) \left(\left(\frac{kr}{p}\right)\right) \tag{2}$$

for q and r prime to p (eqs. 5.1(20), 5.2(4)), and proved the reciprocity law

$$s(q,r;p) + s(p,r;q) + s(p,q;r) = \frac{p^2 + q^2 + r^2 - 3pqr}{12pqr} \tag{3}$$

for mutually prime positive integers p,q,r. When we study the topological interpretations of cotangent sums (in Chapter III), $s(q,r;p)$ will measure the contribution from an isolated fixed point of a $\mathbb{Z}_p$-action on a four-dimensional manifold to a certain "defect" which in some sense measures the deviation of the action from a free action. Then the left-hand side of (3) will be the total defect (summed over all isolated fixed points), and it will therefore be of interest to know if and when it can vanish. This gives some motivation for looking for solutions (in positive and mutually prime integers) of the Diophantine equation

$$p^2 + q^2 + r^2 = 3pqr. \tag{4}$$

From this topological point of view, (4) is the first of a series of Diophantine equations in $2n + 1$ variables corresponding to the total defect of a finite group action with isolated fixed points on a 4n-dimensional manifold; these equations are obtained from the reciprocity law given in 5.2 (Theorem 3) for the expressions

$$\delta(p;a_1,\ldots,a_{2n}) = \frac{(-1)^{n/2}}{p}\sum_{k=1}^{p-1} \cot\frac{\pi k a_1}{p}\cdots\cot\frac{\pi k a_{2n}}{p}. \qquad (5)$$

For example, with $n = 2$ the reciprocity law states (by 5.2(7), (10))

$$\sum_{k=0}^{4} \delta(a_k;a_0,\ldots,\hat{a}_k,\ldots,a_4) = 1 - \frac{5(a_0^2+\ldots+a_4^2)^2-7(a_0^4+\ldots+a_4^4)}{90\, a_0\cdots a_4} \qquad (6)$$

and we want relatively prime solutions of

$$5(p^2+q^2+r^2+s^2+t^2)^2 - 7(p^4+q^4+r^4+s^4+t^4) = 90pqrts. \qquad (7)$$

Of course, (1,...,1) and (2,1,...,1) are solutions of (7), (and indeed of all the corresponding Diophantine equations in more variables since (5) vanishes for $p = 1$ or $p = 2$). The only non-trivial solution of (7) in integers ≤ 100 (and the only one we know at all) is

$$(p,q,r,s,t) = (2,7,19,47,59), \qquad (8)$$

found by computer.* There seems to be no general theory of our Diophantine equation in $2n + 1$ variables for $n \geq 2$, despite its natural occurence in a topological problem.

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*The IBM 7090 at Bonn

But the situation is quite different for the equation (4). Here there is not only a complete and satisfactory theory giving the solutions, but the equation is a classical and famous equation, first studied by Markoff in a context which we will discuss briefly below. First we make an elementary deduction from eq. (3).

LEMMA: *If*

$$q^2 + r^2 \equiv 0 \pmod{p}, \tag{9}$$

then

$$s(q,r;p) = 0. \tag{10}$$

Proof: Let $rx \equiv q \pmod{p}$; this is soluble since r is prime to p. Then by 5.1(22)

$$s(q,r;p) = s(qr^{-1},1;p) = 4p\ s(x,p).$$

But (9) says that $x^2 \equiv -1 \pmod{p}$, or $x^{-1} \equiv x$, and so

$$\begin{aligned} s(q,r;p) &= s(r,q;p) \\ &= s(rq^{-1},1;p) \\ &= 4p\ s(x^{-1},p) \\ &= 4p\ s(-x,p) \\ &= -4p\ s(x,p). \end{aligned}$$

Equation (10) follows.

THEOREM: *For* p,q,r *mutually prime positive integers, the following five statements are all equivalent:*

i) $p^2 + q^2 + r^2 = npqr$ (for some $n \in \mathbb{Z}$)

ii) $p^2 + q^2 \equiv 0 \pmod{r}$, $p^2 + r^2 \equiv 0 \pmod{q}$, $q^2 + r^2 \equiv 0 \pmod{p}$

iii) $s(q,r;p) = s(p,r;q) = s(p,q;r) = 0$

iv) $s(q,r;p) + s(p,r;q) + s(p,q;r) = 0$

v) $p^2 + q^2 + r^2 = 3pqr.$

Proof: The lemma gives ii) $\Rightarrow$ iii), and iii) $\Rightarrow$ iv) $\Rightarrow$ v) $\Rightarrow$ i) $\Rightarrow$ ii) is obvious using eq. (3).

One might conjecture that a corresponding result holds for the Diophantine equation in $2n + 1$ variables discussed above, i.e. that if

$$\sum_{i=0}^{n} \frac{1}{a_i} \delta(a_i; a_0, \ldots, \hat{a}_i, \ldots, a_{2n}) \tag{11}$$

vanishes, then each of the summands must be zero. However, this is not true for the example (8), where we have (after laborious calculations)

$$\frac{1}{2}\delta(2;\ 7,19,47,59) = 0$$

$$\frac{1}{7}\delta(7;\ 2,19,47,59) = -2$$

$$\frac{1}{19}\delta(19;\ 2.7,47,59) = +6$$

$$\frac{1}{47}\delta(47;\ 2,7,19,59) = +14$$

$$\frac{1}{59}\delta(59;\ 2,7,19,47) = -18$$

The sum of these numbers is zero but the individual terms are not.

If we have a solution of

$$p^2 + q^2 + r^2 = npqr \tag{12}$$

with p,q,r not all zero, then clearly n,p,q,r are non-zero and can be taken to be positive. If d is the greatest common divisor of p and q, then d^2 divides $npqr - p^2 - q^2 = r^2$ so $d|r$ also. Then

$$\left(\frac{p}{d}\right)^2 + \left(\frac{q}{d}\right)^2 + \left(\frac{r}{d}\right)^2 = nd\left(\frac{p}{d}\right)\left(\frac{q}{d}\right)\left(\frac{r}{d}\right)$$

with $\frac{p}{d}, \frac{q}{d}, \frac{r}{d}$ relatively prime, so by the last theorem $nd = 3$. Thus

THEOREM: _If_ n,p,q,r _are non-zero integers satisfying_ (12), _then either_ $n = 3$ _and_ p,q,r _are pairwise prime_, _or_ $n = 1$ _and any two of_ p,q,r _have greatest common divisor exactly_ 3.

This theorem can also be proved quite easily by the method of descent, without using cotangent sums. The number 3 seems to play a very special role, but in fact this is not the case: for any $n>2$ it is not too hard to show that the equation

$$x_1^2 + \ldots + x_n^2 = kx_1\ldots x_n \quad (x_1,\ldots,x_n \in \mathbb{Z}) \tag{13}$$

is only soluble for finitely many values of k, the largest of which is n (thus for $n = 4$, k can only be 1 or 4; for $n = 7$, k can be 1,2,3,5 or 7); cf. Hurwitz [48].

The special interest in the Markoff equation (4) is therefore not that it is essentially the only soluble equation of the form (12), but rather its appearance in a completely different context (discovered by Markoff in [60]), namely the theory of the minimum value attained by an indefinite quadratic form (a problem important in the geometry of numbers; for instance, Markoff's equation is treated in Cassels [15], [16]).

What Markoff proved is

<u>THEOREM</u> (Markoff): <u>Let</u>

$$f = f(x,y) = ax^2 + bxy + cy^2 \quad (a,b,c \text{ real}) \tag{14}$$

<u>be an indefinite quadratic form</u>:

$$D = b^2 - 4ac > 0. \tag{15}$$

<u>Let</u>

$$M = \inf \{ |f(x,y)| : x,y \in \mathbb{Z},\ (x,y) \neq (0,0) \}. \tag{16}$$

<u>Then</u>

$$\frac{M}{\sqrt{D}} \leq \frac{1}{3} \tag{17}$$

<u>unless</u> f <u>(multiplied by a suitable factor</u> k) <u>is equivalent to a form</u>

$$\phi = \phi(x,y) = px^2 + (3p-2a)xy + (b-3a)y^2, \tag{18}$$

<u>where</u> a <u>and</u> b <u>are determined by</u>

$$0 < a < \frac{p}{2}, \qquad \pm aq \equiv r \pmod p, \tag{19}$$

$$bp - a^2 = 1$$

<u>for some solution</u> (p,q,r) <u>of</u> (4). <u>For the form</u> ϕ,

$$D = 9p^2 - 4, \quad M = p, \quad \frac{M}{\sqrt{D}} = \frac{1}{3\sqrt{(1-4/9p^2)}} > \frac{1}{3}. \tag{21}$$

In particular, if the ratios $a:b:c$ *are not all rational, then* (17) *holds*.

This remarkable theorem has been the starting point for a large amount of research on the Markoff triples (p,q,r) satisfying (4) and the Markoff numbers p,q,r themselves (the latter are interesting since by (21) they determine the possible values of $MD^{-1/2}$ above 1/3). There are only 13 Markoff numbers smaller than 1000, namely

1,2,5,13,29,34,89,169,194,233,433,610,985.

That eight of these are also Fibonacci numbers is no coincidence, since it is easy to show that the only solutions of (4) with $p = 1$ are $(1,F_{2k-1},F_{2k+1})$ with F_n the n^{th} Fibonacci number ($F_0=0$, $F_1=1$, $F_{n+1} = F_n + F_{n-1}$).

A discussion of many elementary properties of the Markoff numbers can be found in Frobenius [28]. For example, it is shown there that the roots of the quadratic equation

$$ax^2 + bx + c = 0$$

for a form (14) equivalent to a Markoff form have continued fraction expansions containing only ones and twos in their periods (although not all continued fractions whose periods contain only ones and twos actually occur). This means that, of all irrational numbers, these Markoff quadratic irrationalities are the worst approximable by rationals (see, for instance the remarks at the beginning of 11.10 of Hardy and Wright [33]; the numbers $\sqrt{5}$ and $2\sqrt{2}$ occurring there are equivalent to the first two Markoff irrationalities), and partly explains why the Markoff forms do not represent small numbers. These continued fraction expansions, in a modified form, were used in a recent paper (Cohn [19]) relating Markoff triples to the problem of identifying primitive words in the free group on two generators.

CHAPTER III: APPLICATIONS

In this chapter we will study problems involving both the general topological theorems of Chapter I and the specific number-theoretical formulas of Chapter II. The signature theorems (Riemann-Roch, G-signature, etc.) discussed in § 2 of Chapter I yield formulas in terms of elementary trigonometric functions of characteristic classes, for certain invariants of manifolds. Our aim now is to construct manifolds for which these invariants can be calculated directly as well as by means of the corresponding signature theorem. The resulting equality is then an identity of the type investigated in Chapter II.

In § 10 we study the G-manifold $P_n(\mathbb{C})$ (complex projective space), where G is the product of $n+1$ finite cyclic groups acting in the natural way. We first compute directly the equivariant χ_y-characteristic of $P_n(\mathbb{C})$ with coefficients in any power of the Hopf bundle. This is possible because of our very detailed knowledge of the topology of the space and the action of the group. This is then used to compute the L-class of the rational homology manifold $P_n(\mathbb{C})/G$, a result due to Bott. We also derive the signature of the Brieskorn variety V_a. These results are then all reproved using the signature theorems of Chapter I; for example, the calculation of $\mathcal{L}(P_n(\mathbb{C})/G)$ is based on the theorem of 3.2.

In § 11 we study the Brieskorn variety V_a and related manifolds (namely the S^1-manifold Σ_a and quotient space $Z_a = \Sigma_a/S^1$) and calculate some of the invariants arising (L-class of Z_a, α-invariant of the S^1-action on Σ_a). We also try to understand the topological significance of a fact proved in § 5 as a purely number-theoretical statement, namely a certain periodicity phenomenon exhibited by the numbers $\mathrm{Sign}(V_a)$.

In § 12 we define a natural involution on lens spaces of any dimension with odd rotation numbers, and calculate their Browder-Livesay invariant using the G-signature theorem. It again turns out to be one of the arithmetical functions studied in § 5. In this case we did not succeed in doing the calculation "by hand" as well, although we do describe how one might begin a direct calculation.

The first section of the chapter, Section 9, is somewhat out of line with the other sections in that we do not look at specific manifolds, but rather consider the forms which the signature theorems assume for manifolds of low dimension. Thus we write out very explicitly the assertion of the equivariant Riemann-Roch theorem for surfaces and of the G-signature theorem for 4-manifolds. The latter is of especial interest since here the number theory arising is classical: the G-signature theorem on 4-manifolds leads to the classical Dedekind sums, and the involution on 3-dimensional lens spaces to the Legendre-Jacobi symbol.

Other places where the interplay between signature theorems and number theory in the spirit of this chapter arises are: Zagier [98], Chapter III; Hirzebruch [40], [38], [39]; Neumann [74]; Meyer [68]; Kreck [55]; Hirzebruch-Neumann-Koh [45].

§ 9. The signature theorems on low-dimensional manifolds

In applying the equivariant signature theorem of 2.3, considerable simplification results if the manifolds involved have small dimension. Recall that for closed, orientable and connected manifolds and orientation-preserving group actions, any component of a fixed-point set has even codimension. In applying the signature theorems, the case of codimension zero is trivial since there is no normal bundle, and the case of codimension two also is easy to handle since the action on the normal bundle is given by a single eigenvalue $e^{i\theta}$. The case of a zero-dimensional fixed-point set is also easy to deal with, since now the process of evaluating on the fundamental class of the fixed-point set is trivial.

It follows from these considerations that essentially any signature problem for group actions on a two-dimensional manifold (Riemann surface) can be solved completely by the application of the G-signature theorem or holomorphic Lefschetz theorem, and that the same holds for group actions on four-dimensional manifolds if we make certain assumptions about the action of the group (namely that all isotropy groups are abelian). We describe the holomorphic Lefschetz theorem in 9.1, discuss the G-signature theorem for 4-manifolds in 9.2, and give some examples of the latter in 9.3.

9.1 Let X be a compact connected Riemann surface and G a finite group acting effectively (and holomorphically) on X. The canonical line bundle K (see 1.3) is just the dual T of the tangent bundle. For $a \in \mathbb{Z}$, we can form the a^{th} power K^a, and $g \in G$ is an automorphism of X, K^a (cf. 1.4). The statement of the holomorphic Lefschetz theorem in this case (Theorem 6 of 2.3) is

$$\operatorname{tr}(g|H^0(X,K^a)) - \operatorname{tr}(g|H^1(X,K^a)) = \chi(X,K^a;g) \qquad (1)$$

$$= \sum_{x\in X^g} \mu_{g,x}^a \cdot \frac{1}{1-\mu_{g,x}} \quad (g \neq 1). \qquad (2)$$

Here X^g is a finite set of points since $g \neq 1$ and G acts effectively, and $\mu_{g,x}$ $(x \in X^g)$ is the complex number giving the action of g on T_x (so $\mu_{g,x} = e^{-i\theta}$ in the notation of 2.3(29), since T is the dual of the tangent bundle $\theta(X)$). Note that $\mu_{g,x} \neq 1$ since x is an isolated fixed point, so that equation (2) makes sense.

At a point x of X, the isotropy group G_x of elements of G leaving x fixed is necessarily cyclic, since it is mapped injectively into $U(1) = S^1$ by its representation on $T_x \cong \mathbb{C}$. Let its order be $b(x)$ and the order of G/G_x be $d(x)$; thus $d(x) = |G|/b(x)$ and there are $d(x)$ points of X in the same orbit of G as x, at each of which the isotropy group is a conjugate in G of G_x and therefore is also a cyclic group of order $b(x)$. Thus at each point y of the quotient $Y = X/G$ there is a well-defined order b_y at y defined by

$$b_y = b(x) \quad (x \in \pi^{-1}(y)), \qquad (3)$$

where

$$\pi : X \to X/G = Y \qquad (4)$$

is the projection. We call b_y the <u>branching number</u> at y;

then $d_y = |G|/b_y$ points of X lie over Y and at each of them the map π looks like the Riemann surface of z^{b_y}. Moreover, the quotient Y is also a Riemann surface, because the quotient of $\mathbb{C}$ by a finite cyclic group G_b is naturally isomorphic to $\mathbb{C}$ by $z \to z^b$.

We now calculate the invariant Euler characteristic of K^a (cf. 1.4):

$$\begin{aligned}
\chi(X,K^a)^G &= \frac{1}{|G|} \sum_{g\in G} \chi(X,K^a;g) \\
&= \frac{1}{|G|}\left[\chi(X,K^a) + \sum_{\substack{g\in G\\ g\neq 1}} \sum_{x\in X^g} \frac{\mu^a_{g,x}}{1-\mu_{g,x}}\right] \\
&= \frac{1}{|G|}\chi(X,K^a) + \frac{1}{|G|}\sum_{y\in Y}\left[\sum_{\pi(x)=y} \sum_{g\in G_x-\{1\}} \frac{\mu^a_{g,x}}{1-\mu_{g,x}}\right] \\
&= \frac{\chi(X,K^a)}{|G|} + \frac{1}{|G|}\sum_{y\in Y} d_y \sum_{\substack{\mu^{b_y}=1\\ \mu\neq 1}} \frac{\mu^a}{1-\mu}\,. \qquad (5)
\end{aligned}$$

<u>Lemma</u>: Let b 0 be an integer. Then, for any $a \in \mathbb{Z}$,

$$\frac{1}{b}\sum_{\substack{\mu^b=1\\ \mu\neq 1}} \frac{\mu^a}{1-\mu} = \left(\left(\frac{2a-1}{2b}\right)\right). \qquad (6)$$

<u>Proof:</u> Recall that $((x))$ for $x \in \mathbb{R} - \mathbb{Z}$ is defined as $x - \frac{1}{2}$ if $0 < x < 1$ and by the requirement that it be a periodic function of period one otherwise. Thus both sides of (6) only dspend on $a \pmod b$, and we can assume $0 < a < b$. Let the left-hand side of (6) be denoted λ_a; then the rational

function

$$f(t) = \frac{t^{a-1}}{(1-t)(t^b - 1)} \tag{7}$$

has no residue at $t = \infty$ (since $a < b$) or at $t = 0$ (since $a \geqslant 1$), while the sum of its residues at $t^b = 1$, $t \neq 1$ is clearly λ_a. The only other pole is $t = 1$, so

$$\lambda_a = -\mathrm{res}_{t=1}(f(t)\,dt) = +\mathrm{res}_{x=0}\left[\frac{(1+x)^{a-1}}{(1+x)^b - 1}\;\frac{dx}{x}\right]$$

$$= \mathrm{res}_{x=0}\left[\frac{1 + (a-1)x + \dots}{bx + \frac{1}{2}b(b-1)x^2 + \dots}\;\frac{dx}{x}\right] = \frac{1}{b}(a-1 - \frac{b-1}{2})$$

$$= ((\frac{2a-1}{2b})). \tag{8}$$

A shorter, but less direct, proof of (6) is to notice that

$$\lambda_{a+1} - \lambda_a = -\frac{1}{b}\sum_{\substack{\mu^b=1\\ \mu\neq 1}} \mu^a = \begin{cases} 1/b & \text{if } a \not\equiv 0 \pmod b, \\ -(b-1)/b & \text{if } a \equiv 0 \pmod b, \end{cases} \tag{9}$$

from which (6) follows immediately by summation.

Substituting $d_y = |G|/b_y$ and equation (6) into (5) gives

$$\chi(X,K^a)^G = \frac{1}{|G|}\chi(X,K^a) + \sum_{y\in Y}((\frac{2a-1}{2b_y})). \tag{10}$$

This also gives the χ_y-characteristic, since

$$\chi_y(X,K^a)^G = \sum_{p=0} y^p\,\chi(X,K^a \times \Lambda^p T)^G = \chi(X,K^a)^G + y\,\chi(X,K^{a+1})^G \tag{11}$$

In particular,

$$\chi_1(X,K^a)^G = \frac{1}{|G|}\chi_1(X,K^a) + \sum_{y\in Y}\left[\left(\left(\frac{2a-1}{2b_y}\right)\right) + \left(\left(\frac{2a+1}{2b_y}\right)\right)\right]$$

$$= \frac{(X,K^a)}{|G|} + 2\sum_{y\in Y}\left(\left(\frac{a}{b_y}\right)\right) \tag{12}$$

[This corresponds to using

$$\frac{1}{2b}\sum_{\substack{\mu^b=1\\ \mu\neq 1}} \mu^a.\frac{1+\mu}{1-\mu} = \left(\left(\frac{a}{b}\right)\right) \tag{13}$$

instead of (6), since the left-hand side of (13) is just $\frac{1}{2}(\lambda_a + \lambda_{a+1})$. Formula (13) in trigonometric notation becomes

$$\left(\left(\frac{a}{b}\right)\right) = \frac{-1}{2b}\sum_{k=1}^{b-1} \sin\frac{2\pi ka}{b}\cot\frac{\pi k}{b}, \tag{14}$$

a formula of Eisenstein given in §5 (equations 5.2(1), (2))].

Since $((\frac{a}{b}))$ is an odd function of a, equation (12) implies

$$\chi_1(X,K^a)^G = -\chi_1(X,K^{-a})^G, \tag{15}$$

which is a special case of the equivariant Serre duality formula 1.2(36)^G. For $a = 0$ this gives

$$\chi_1(X)^G = \chi(X)^G + \chi(X,K)^G = 0, \tag{16}$$

which is also clear since, by Theorems 6 and 4 of 2.1, we know that

$$\chi_1(X)^G = \mathrm{Sign}(X/G), \tag{17}$$

and $Y = X/G$ is a two dimensional manifold and therefore has signature zero.

Substituting $y = -1$ in (11) and using equations (16) and (10), we find

$$\begin{aligned}\chi_{-1}(X)^G &= \chi(X)^G - \chi(X,K)^G \\ &= 2\,\chi(X)^G \\ &= \frac{2\,\chi(X)}{|G|} + 2\sum_{y\in Y}\left(\left(\frac{-1}{2b_y}\right)\right) \\ &= \frac{2\,\chi(X)}{|G|} + \sum_{y\in Y}\left(1 - \frac{1}{b_y}\right), \qquad (18)\end{aligned}$$

which corresponds to the case $a = 0$ of (9). Another proof of (18) is given by Theorem 8 of 2.3, from which

$$\begin{aligned}\chi_{-1}(X)^G = e(Y) &= \frac{1}{|G|}\sum_{g\in G} e(X^g) \\ &= \frac{1}{|G|}\left(e(X) + \sum_{\substack{g\in G\\ g\neq 1}}\ \sum_{gx=x} 1\right) \\ &= \frac{e(X)}{|G|} + \frac{1}{|G|}\sum_{x\in X}\ \sum_{\substack{g\in G_x\\ g\neq 1}} 1 \\ &= \frac{e(X)}{|G|} + \frac{1}{|G|}\sum_{x\in X}(b(x) - 1) \\ &= \frac{e(X)}{|G|} + \sum_{y\in Y}\frac{b_y - 1}{b_y} \qquad (19)\end{aligned}$$

since there are $|G|/b_y$ isolated fixed points over $y\in Y$, each having Euler characteristic one. In particular, since Y is a compact Riemann surface, the expression on the right of (19) must be an even integer ≤ 2, so that we get some information about the possible sizes b_y of the isotropy groups. Formula (19), of course, is just the classical Hurwitz formula.

If X has genus > 1 and $a > 1$, then by Theorem 8 of 1.3 the group $H^1(X,K^a)$ vanishes, so that

$$\chi(X,K^a)^G = \dim H^0(X,K^a)^G \tag{20}$$

is the number of G-invariant sections of K^a, i.e. is the (complex) dimension of the vector space of forms of weight a on Y. The evaluation of this number is the classical Riemann-Roch theorem.

9.2 The G-signature theorem has an especially simple form for an element $g \in G$ with isolated fixed points: if x is an isolated fixed point of g, then the contribution of x to $L(g,X)$ is (from 4.1(2) and the definition of $\mathcal{L}_\theta$)

$$L(g,X)_x = \prod_{0<\theta<\pi} \left(\coth \frac{i\theta}{2}\right)^{m(\theta)} = \prod_{\theta} \left(-i \cot \frac{\theta}{2}\right)^{m(\theta)}, \tag{1}$$

where $m(\theta)$ is the complex dimension of the eigenspace of $e^{-i\theta}$ of the action of g on the tangent bundle of X at x. If there are any eigenvalues equal to -1, then $L(g,X)_x$ is 0.

The next most difficult case of the G-signature theorem is that of a fixed-point set component Y of codimension 2 in X (of course for X^g of codimension 0 we simply have $L(g,X) = \mathrm{Sign}(X)$). In this case N^g is either a real bundle of dimension 2 on which g acts as -1 or a complex bundle of dimension 1 on which g acts as $e^{i\theta} \neq 1$; in the latter case Y is oriented. In the former case we have

$$L(g,X) = (\mathcal{L}(X^g)e(N^g))[X^g] = \text{Sign}(X^g \circ X^g) \qquad (2)$$

by the argument of 4.2, while in the latter case we obtain from 4.1(2) that

$$L(g,X) = (\mathcal{L}(X^g)\coth(x + \tfrac{1}{2}i\theta))[X^g], \qquad (3)$$

where x is the first Chern class of the complex line bundle N^g over X^g.

Now if X has real dimension 4, an oriented action can only have fixed-point sets of dimensions 0,2, or 4. If Y is a two-dimensional component of X^g and g acts by $e^{i\theta}$ on its normal bundle, then (3) gives for the contribution from Y to $L(g,X)$ the value

$$\begin{aligned} L(g,X)_Y &= (\mathcal{L}(Y)\coth(\tfrac{i\theta}{2} + x))[Y] \\ &= \coth(\tfrac{i\theta}{2} + x)[Y] \\ &= (\coth\tfrac{i\theta}{2} + \frac{x}{\sin^2(\theta/2)} + \ldots)[Y] \\ &= (\csc^2\tfrac{\theta}{2})\, x\, [Y] \\ &= (\csc^2\theta/2)\, Y \circ Y, \end{aligned} \qquad (4)$$

since Y is two-dimensional and therefore has a trivial L-class Here $Y \circ Y$ is the oriented cobordism class of the self-intersection manifold of Y (cf. 4.2), which can be thought of as an integer since the cobordism class of a 0-dimensional oriented manifold in a connected manifold is clearly determined by the (algebraic) number of points. Since the signature of a point is 1, we also have

$$\mathrm{Sign}(Y \circ Y) = Y \circ Y, \tag{5}$$

so that (2) is just the special case of (4) with eigenvalue $e^{i\theta} = e^{i\pi} = -1$.

If G acts effectively on X, then only $g = 1$ has a fixed-point set of dimension four, so we obtain from (1) and (4) the result:

<u>THEOREM 1:</u> Let G act orientably and effectively on a connected four-manifold X. Then

$$L(g,X) = \begin{cases} \mathrm{Sign}(X) & \text{if } g = 1, \\ -\sum_x \cot(\tfrac{1}{2}\alpha_{x,g})\cot(\tfrac{1}{2}\beta_{x,g}) + \sum_Y \csc^2(\tfrac{1}{2}\theta_{Y,g}) Y \circ Y & \text{if } g \neq 1. \end{cases} \tag{6}$$

Here x ranges over all isolated fixed points of g (with $e^{i\alpha_{x,g}}$ and $e^{i\beta_{x,g}}$ the eigenvalues of g on the tangent space of X at x) and Y ranges over all two-dimensional components of X^g (with $e^{i\theta_{Y,g}}$ the eigenvalue of g on the normal bundle of Y in X).

Notice that the formula (6) is well-defined; the angles $\theta_{Y,g}$ are determined, modulo 2π and up to sign, by the action of g, and $\csc^2(\theta/2)$ is an even function of θ with period 2π; while the angles $\alpha_{g,X}$ and $\beta_{g,X}$ are well-defined modulo 2π and up to $\alpha,\beta \to \beta,\alpha$ or $\alpha,\beta \to -\alpha,-\beta$ (we can distinguish between α,β and $\alpha,-\beta$ because X is oriented), and $(\cot \frac{\alpha}{2} \cot \frac{\beta}{2})$ is invariant under these changes. Of course the sums in (6) are finite since X is compact.

Let Y be any connected two-dimensional submanifold of

X and

$$G_Y = \{g\in G:\ g|_Y \text{ is the identity}\}. \tag{7}$$

Let

$$G(Y) = G_Y - \{1\}. \tag{8}$$

Clearly $G(Y)$ is empty for all but finitely many $Y\subset X$, and consists of all $g\in G$ for which Y is one of the two-dimensional components of X^g appearing in (6). Similarly, we define

$$G(x) = \{g\in G:\ x \text{ is an isolated fixed point of } g\} \tag{9}$$

and note that the isotropy group G_x is the disjoint union

$$G_x = \{1\} \cup G(x) \cup \sum_{Y\ni x} G(Y). \tag{10}$$

We now consider the formula 2.1(22):

$$\mathrm{Sign}(X/G) = \frac{1}{|G|} \sum_{g\in G} \mathrm{Sign}(g,X). \tag{11}$$

Because of Theorem 1, this can be rewritten

$$|G|\ \mathrm{Sign}(X/G) = \mathrm{Sign}(X) + \sum_x \mathrm{def}_x + \sum_Y \mathrm{def}_Y, \tag{12}$$

where x ranges over all points of X and

$$\mathrm{def}_x = - \sum_{g\in G(x)} \cot(\tfrac{1}{2}\alpha_{g,x})\cot(\tfrac{1}{2}\beta_{g,x}), \tag{13}$$

and where Y ranges over all two-dimensional submanifolds of X, connected but not necessarily orientable, with

$$\mathrm{def}_Y = (Y\circ Y) \sum_{g\in G(Y)} \csc^2(\tfrac{1}{2}\theta_{g,Y}). \tag{14}$$

The notation "def" is an abbreviation for "defect": the sums in (12) measure the defect from the formula Sign(X/G) $= \frac{1}{|G|}$ Sign X which would hold for a free finite group action or covering space (cf. [40]).

Consider first equation (14). The action of G_Y on the normal bundle of Y gives an effective one-dimensional complex representation of the group G_Y (unless $\theta_{g,Y} = \pi$, when it gives an effective two-dimensional real representation) so we get a monomorphism from G_Y to $U(1) = S^1$. Therefore if n is the order of G_Y, the map $g \to e^{i\theta_{g,Y}}$ gives an isomorphism from G_Y to the cyclic group G_n, and

$$\text{def}_Y = (Y \circ Y) \sum_{\substack{e^{in\theta} = 1 \\ e^{i\theta} \neq 1}} \csc^2 \frac{\theta}{2}$$

$$= \frac{n^2 - 1}{3} (Y \circ Y) \qquad (15)$$

by a simple trigonometric calculation (e.g. from 5.1(4) and 5.2(3) we deduce $\sum_\theta \cot^2(\frac{1}{2}\theta) = (n-1)(n-2)/3$, and since $\csc^2 x = \cot^2 x + 1$ we obtain (15)).

The situation (13) at an isolated fixed point is more complicated. We now have an effective representation of G_x on $\mathbb{C}^2$. We make the assumption that G (or at least each isotropy subgroup G_x of G) is abelian; then we can find a single splitting of T_x into a sum of two subspaces invariant under the actions of all $g \in G_x$, so that mapping g to its eigenvalues on these two subspaces defines a homomorphism from G_x to $S^1 \times S^1$. This map is injective since G acts effectively, so G_x is isomorphic to its image $H \subset S^1 \times S^1$.

Let n and m be the orders of the two finite cyclic groups $H \cap (S^1 \times 1)$ and $H \cap (1 \times S^1)$. Then $G_n \times G_m$ is a subgroup of H, and the map $S^1 \times S^1 \to S^1 \times S^1$ defined by $(x,y) \to (x^n, y^m)$ gives an isomorphism from $H/G_n \times G_m$ onto its image H' in $S^1 \times S^1$. The image H' is a finite subgroup of $S^1 \times S^1$ whose intersection with $S^1 \times 1$ or $1 \times S^1$ consists of just the identity, and such a subgroup must be of the form

$$\left\{(e^{2\pi ik/p},\ e^{2\pi ikq/p}),\quad k \bmod p\right\}, \tag{16}$$

where $p = |H'|$ and q is prime to p. Therefore

$$G_x \cong H = \left\{(e^{i\theta}, e^{i\phi}) \,\middle|\, e^{in\theta} = e^{2\pi ik/p},\ e^{im\phi} = e^{2\pi ikq/p}\right\} \tag{17}$$

where k runs over all residues (mod p). Of course

$$G(x) \cong \left\{(e^{i\theta},\ e^{i\phi}) \in H \text{ such that } e^{i\theta} \neq 1,\ e^{i\phi} \neq 1\right\}. \tag{18}$$

Therefore

$$\mathrm{def}_x = \sum_{g \in G(x)} \frac{e^{i\alpha_{g,x}} + 1}{e^{i\alpha_{g,x}} - 1} \cdot \frac{e^{i\beta_{g,x}} + 1}{e^{i\beta_{g,x}} - 1}$$

$$= \sum_{k=1}^{p} \left[\sum_{\substack{a^n = e^{2\pi ik/p} \\ a \neq 1}} \ \sum_{\substack{b^m = e^{2\pi ikq/p} \\ b \neq 1}} \frac{a+1}{a-1}\ \frac{b+1}{b-1} \right]. \tag{19}$$

We now use the identity

$$\frac{1}{n} \sum_{a^n = y} \frac{az + 1}{az - 1} = \frac{yz^n + 1}{yz^n - 1} \tag{20}$$

(proof: both sides are rational functions of z, holomorphic except for a simple pole of residue $\frac{2z}{n}$ at points z with $yz^n = 1$, and equal to -1 at $z = 0$) to find:

$$\sum_{\substack{a^n = y \\ a \neq 1}} \frac{a+1}{a-1} = n\,\frac{y+1}{y-1} \quad (y \neq 1), \tag{21}$$

$$\begin{aligned}
\sum_{\substack{a^n = 1 \\ a \neq 1}} \frac{a+1}{a-1} &= \lim_{z\to 1}\left(n\,\frac{z^n+1}{z^n-1} - \frac{z+1}{z-1}\right) \\
&= \lim_{x\to 0}\left(n\,\frac{(1+x)^n+1}{(1+x)^n-1} - \frac{2+x}{x}\right) \\
&= \lim_{x\to 0}\left(\frac{2+nx+\dots}{x+\frac{1}{2}(n-1)x+\dots} - \frac{2+x}{x}\right) \\
&= 0.
\end{aligned} \tag{22}$$

Therefore (19) becomes

$$\begin{aligned}
\mathrm{def}_x &= \sum_{k=1}^{p-1} nm\,\frac{e^{2\pi ik/p}+1}{e^{2\pi ik/p}-1}\cdot\frac{e^{2\pi ikq/p}+1}{e^{2\pi ikq/p}-1} \\
&= -nm\sum_{k=1}^{p-1}\cot\frac{\pi k}{p}\,\cot\frac{\pi kq}{p}.
\end{aligned} \tag{23}$$

By 5.2(3), this is equivalent to

$$\mathrm{def}_x = -4nmp\,s(q,p) = -4nmp\sum_{k=1}^{p-1}\left(\left(\frac{k}{p}\right)\right)\left(\left(\frac{kq}{p}\right)\right). \tag{24}$$

We know from 5.1(13) that $6ps(q,p)$ is an integer (indeed $2p\,s(q,p)$ is already an integer unless p is a multiple of 3

Therefore from (23) and (15) we find that the numbers def_x and def_Y in (12) are not only rational numbers, but in fact integral multiples of 1/3 (from the definition they are only <u>a priori</u> real numbers). From (12) we then find that the sum of Sign(X) and all the numbers def_x and def_Y is not only a multiple of 1/3 but an integer, and indeed divisible by the order of the group G. We summarize our results:

<u>THEOREM 2</u>: Let G,X be as in Theorem 1, G finite abelian. For each connected two-dimensional submanifold Y of X let n_Y be the order of the cyclic group G_Y. For each $x \in X$ let n_x, m_y, p_x, and $q_x > 0$ (q_x prime to p_x) be the numbers determining G_x as in (17); thus $n_x m_x p_x$ is the order of the isotropy group G_x. Then

$$|G| \ \text{Sign}(X/G) = \text{Sign}(X) + \sum_Y (n_Y^2 - 1)\frac{Y\circ Y}{3} - \sum_x 4 n_x m_x p_x \ s(q_x, p_x), \tag{25}$$

and all of the numbers appearing on the right are integral multiples of 1/3.

9.3 We give two examples illustrating Theorem 2 of 9.2.

For the first, we take $X = P_2(\mathbb{C})$ and

$$G = G_a \times G_b \times G_c = \{(\alpha,\beta,\gamma) \in \mathbb{C}^3 \mid \alpha^a = \beta^b = \gamma^c = 1\}, \quad \text{(a,b,c mutually coprime integers)} \tag{1}$$

(G_a = group of a^{th} roots of unity), acting on X by

$$(\alpha,\beta,\gamma)\circ(z_0 : z_1 : z_2) = (\alpha z_0 : \beta z_1 : \gamma z_2). \tag{2}$$

This is a special case of the situation which will be considered--in arbitrary dimensions and from a somewhat different point of view--in Section 10.

We dsnote by Y_i the surfacs given by $z_i = 0$ and by x_i the point with $z_j = \delta_{ij}$ $(i=0,1,2)$; it is clear that Y_0, Y_1, Y_2, x_0, x_1 and x_2 are the only components of fixed-point sets of elements in $G - \{1\}$. For $z \in Y_0$ and $g = (\alpha,\beta,\gamma) \in G$, the condition that z is a fixed-point of g is

$$(0:\beta z_1:\gamma z_2) = (0:z_1:z_2) \; ; \tag{3}$$

this can be satisfied for all $z \in Y_0$ only if $\beta = \gamma$, and then β and γ must both be 1 since $(b,c) = 1$. Thus the isotropy group G_{Y_0} is isomorphic to G_a. Since clearly $Y_0 \circ Y_0 = 1$ (Y_0 is $P_1(\mathbb{C})$ canonically embedded in $P_2(\mathbb{C})$), equation (15) of 9.2 gives

$$\operatorname{def}_{Y_0} = \frac{a^2 - 1}{3} . \tag{4}$$

Similarly Y_1 and Y_2 have defects $\frac{b^2-1}{3}$, $\frac{c^2-1}{3}$.

Now consider x_0. The point x_0 is left fixed by the whole group G, and since

$$(\alpha,\beta,\gamma)\circ(1:t_1:t_2) = (1:\alpha^{-1}\beta t_1:\alpha^{-1}\gamma t_2), \tag{5}$$

the action of $(\alpha,\beta,\gamma) \in G$ on $T_{x_0}X$ is given by eigenvalues $\alpha^{-1}\beta$, $\alpha^{-1}\gamma$. We can thus identify G_{x_0} with

$$H = \{(\alpha^{-1}\beta,\alpha^{-1}\gamma) \in S^1 \times S^1 \mid \alpha \in G_a,\ \beta \in G_b,\ \gamma \in G_c\}. \tag{6}$$

Then $H \cap (S^1 \times 1)$ is given by $\alpha^{-1}\gamma = 1$, from which we deduce $\alpha = \gamma = 1$, i.e. $H \cap (S^1 \times 1) = G_b \times 1$. Similarly $H \cap (1 \times S^1) = 1 \times G_c$. Therefore in the notations of 9.2 we have

$$n_{x_0} = b, \quad m_{x_0} = c. \tag{7}$$

Then $(e^{i\theta}, e^{i\phi}) = (\alpha^{-1}\beta, \alpha^{-1}\gamma) \in H$ implies

$$(e^{in_{x_0}\theta}, e^{im_{x_0}\phi}) = (\alpha^{-b}, \alpha^{-c}) ,$$

and therefore p_{x_O}, q_{x_O} are given by

$$p_{x_O} = a , \qquad q_{x_O} b \equiv c \pmod a. \tag{8}$$

Now (24) gives

$$\begin{aligned} \mathrm{def}_{x_O} &= -4\, n_{x_O} m_{x_O} p_{x_O}\, s(q_{x_O}, p_{x_O}) \\ &= -4\, bca\, s(b^{-1}c, a) \\ &= -4\, abc\, s(b,c;a) , \end{aligned} \tag{9}$$

where $s(b,c;a)$ is the symbol defined in 5.1(20). The defects of x_1, x_2 are given by cyclical permutation of a, b, c. Therefore by Theorem 2 of 9.2,

$$\begin{aligned} abc\ \mathrm{Sign}(X/G) &= \mathrm{Sign}(X) + \frac{a^2-1}{3} + \frac{b^2-1}{3} + \frac{c^2-1}{3} \\ &\quad - 4\, abc\, \{s(b,c;a) + s(c,a;b) + s(a,b;c)\}. \end{aligned} \tag{10}$$

Now $\mathrm{Sign}(X) = \mathrm{Sign}(P_2(\mathbb{C})) = 1$. We claim that $\mathrm{Sign}(X/G)$ is also equal to 1 . Indeed, since a,b,c are coprime,

$$G = G_a \times G_b \times G_c \cong \mathbb{Z}/a \times \mathbb{Z}/b \times \mathbb{Z}/c \cong \mathbb{Z}/abc \cong G_{abc} , \tag{11}$$

and under this identification G_{abc} acts on X by

$$\zeta \circ (z_0 : z_1 : z_2) = (\zeta^{bc} z_0 : \zeta^{ac} z_1 : \zeta^{ab} z_2) \qquad (\zeta^{abc} = 1). \tag{12}$$

But (12) makes sense for any $\zeta \in S^1$, i.e. the action of G embeds in an S^1-action. Since S^1 is connected and $\mathrm{Sign}(g,X)$ is defined by the action of g on $H^*(X)$ and is therefore invariant under homotopy, we deduce that $\mathrm{Sign}(g,X) = \mathrm{Sign}(1,X) = \mathrm{Sign}(X)$ for all g in G, so

$$\mathrm{Sign}(X/G) = \frac{1}{|G|}\sum_{g\in G} \mathrm{Sign}(g,X) = \mathrm{Sign}(X) = 1. \tag{13}$$

Substituting this into (10) gives the Rademacher reciprocity law 5.1(23).

For our second example, we consider the non-singular affine algebraic surface

$$W = \{(z_1,z_2,z_3)\in \mathbb{C}^3 \mid z_1^{bc} + z_2^{ac} + z_3^{ab} = 1\}, \tag{14}$$

with a,b,c as before, and let the group $G = G_{abc}$ act by

$$\zeta\circ(z_1,z_2,z_3) = (\zeta^a z_1,\zeta^b z_2,\zeta^c z_3) \qquad (\zeta^{abc} = 1). \tag{15}$$

It is fairly clear that this action has no fixed surfaces Y_i. It has bc points of the form

$$x_i = (z_1,0,0), \quad z_1^{bc} = 1 \tag{16}$$

with isotropy group G_a, an element $\zeta\in G_a$ acting on the tangent space of W at x_i with eigenvalues ζ^b, ζ^c. Thus each of the points (16) has defect $-4as(b,c;a)$. Similarly there are ac points of the form $(0,z_2,0)$, each with defect $-4bs(a,c;b)$, and ab points of the form $(0,0,z_3)$ with defect $-4cs(a,b;c)$. Therefore the sum of the defects over all fixed-point components of elements of $G - \{1\}$ is

$$\sum_x \mathrm{def}_x = -4abc\{s(b,c;a)+s(a,c;b)+s(a,b;c)\}.$$

In particular, the sum of the defects vanishes if (a,b,c) is a Markoff triple (cf. Theorem, §8).

However, even if this is the case, we cannot use Theorsm 2 of 9.2 to deduce that $\mathrm{Sign}(W) = |G|\,\mathrm{Sign}(W/G)$, because W is not compact. In fact, the signature of W/G is 0 (the middle rational cohomology group of W/G is in fact zero, since W/G can be identified with the orbit space of $\mathbb{C}^2$ under a certain finite group action), whereas the signature of W is given by the formula

$$\text{Sign } W = \frac{-2 + a^2 + b^2 + c^2 - a^2b^2c^2}{3} . \qquad (17)$$

Equation (17) can be proved in the spirit of this section as follows: We let V_{abc} be the standard hypersurface of degree abc in $P_3(\mathbb{C})$ (cf. Exercise 1 below) and let the group $G_a \times G_b \times G_c$ act on V_{abc} by

$$(\alpha,\beta,\gamma)\circ(z_0:z_1:z_2:z_3) = (z_0:\alpha z_1:\beta z_2:\gamma z_3) ; \qquad (18)$$

then the quotient $V_{a,b,c}/G_a\times G_b\times G_c$ is a smooth manifold in which the open subset given by $z_0 \neq 0$ can be identified with W. Now by applying Theorem 2, 9.2, to the action of $G_a \times G_b \times G_c$ and using the known formula for $\text{Sign}(V_{abc})$ (Exercise 1), we deduce (17). The reader can look up the details in [40] or work them out as an exercise. The amusing point about (17) is that W is a very special case of the Brieskorn variety

$$\{(z_1,\ldots,z_n) \in \mathbb{C}^n \mid z_1^{a_1} + \ldots + z_n^{a_n} = 1\}$$

whose signature is given by the function $t(a_1,\ldots,a_n)$ studied in §5 (this fact is due to Brieskorn [7]; we give a different proof in 10.2). Thus (17) provides a topological proof of the identity

$$t(bc,ac,ab) = \tfrac{1}{3}(-2+a^2+b^2+c^2-a^2b^2c^2) \quad (a,b,c \text{ coprime})$$

proved in 5.3 (eq. (23)).

<u>Exercises</u>: 1. Let n be a positive integer and

$$V_n = \{(z_0:z_1:z_2:z_3) \in P_3(\mathbb{C}) \mid z_0^n + z_1^n + z_2^n + z_3^n\}$$

the standard hypersurface of degree n in $P_3(\mathbb{C})$. Evaluate $\mathrm{Sign}(V_n)$ in two ways:

i) as $\tau(nx)$, where $x \in H^2(P_3(\mathbb{C});\mathbb{Z})$ is the standard generator and $\tau(nx)$ the "virtual index" discussed in 4.3 (eqs. 4.3(7),(12)) ;

ii) by applying Theorem 2 of 9.2 to the action of G_n on V_n given by

$$\zeta\circ(z_0:z_1:z_2:z_3) = (\zeta z_0:z_1:z_2:z_3) \qquad (\zeta^n = 1);$$

the quotient V_n/G_n can be identified with $P_2(\mathbb{C})$.

2. The analogue of Theorem 2, 9.2, for the Euler characteristic is

$$|G|\ e(X/G) = e(X) + \sum_Y (n_Y-1)\ e(Y) + \sum_x n_x m_x p_x$$

(in the notation of that theorem), because the equivariant Euler characteristic $e(g,X)$ corresponding to $\mathrm{Sign}(g,X)$ is simply the Euler number $e(X^g)$ of the fixed-point set. Evaluate both sides of this equation for all the manifolds and group actions considered in this section.

§ 10. The action of T^{n+1} on $P_n(\mathbb{C})$

The basic situation considered in this section is the action of the torus group $T^{n+1} = S^1\times\ldots\times S^1$ on $P_n(\mathbb{C})$ defined by

$$(\zeta_0,\ldots,\zeta_n)\bullet(z_0:\ldots:z_n) = (\zeta_0 z_0:\ldots:\zeta_n z_n) \qquad (1)$$

for $(\zeta_0,\ldots,\zeta_n)\in T^{n+1}$ and $(z_0:\ldots:z_n)\in(\mathbb{C}^{n+1}-\{0\})/\mathbb{C}$ $= P_n(\mathbb{C})$. We will look especially at the action of the finite subgroup

$$G = G_{b_0}\times\ldots\times G_{b_n} \subset S^1\ \ldots\ S^1 = T^{n+1}, \qquad (2)$$

where the b_k are positive integers (G_b = group of b^{th} roots of unity). This action is closely related to the topology of the Brieskorn manifold

$$V_a = \{(z_1,\ldots,z_n)\in\mathbb{C}^n \mid z_1^{a_1}+\ldots+z_n^{a_n} = 1\}, \qquad (3)$$

where $a_k = N/b_k$ (N being any common multiple of the b_i's).

In 10.1 - 10.3 we use the techniques of §1 only. We compute $\chi_y(P_n(\mathbb{C}),H^k)^G$ for all integers k (H=Hopf bundle over $P_n(\mathbb{C})$) in 10.1 and the function $\chi_y(S)^G$ in 10.2, where S is the divisor in $P_n(\mathbb{C})$ given by $\sum z_i^N = 0$, N as above We then use these results in 10.3 to calculate the L-class of the quotient $P_n(\mathbb{C})/G$, at least for n odd (the result was first obtained by Bott). In 10.2 we also use the value of $\chi_1(S)^G$ to calculate $\mathrm{Sign}(V_a)$. In 10.4 we use the signature theorems to redo some of these calculations. We first find the fixed-point sets and their normal bundles for the elements of G, and then rederive the results on $\mathcal{L}(P_n(\mathbb{C}))/G$ (this time for all n) and on $\chi_y(P_n(\mathbb{C}),H^k)^G$ by applying the theorem of 3.1 and the holomorphic Lefschetz theorem.

10.1 Let $P(n,k)$ be the complex subspace of $\mathbb{C}[z_0,\dots,z_n]$ consisting of homogeneous polynomials of degree k. We claim that

$$\dim P(n,k) = \binom{n+k}{k}, \tag{1}$$

Indeed, in the expansion,

$$\frac{1}{(1-tz_0)\dots(1-tz_n)} = (1+tz_0+t^2z_0^2+\dots)\dots(1+tz_n+tz_n^2+\dots) \tag{2}$$

the coefficient of t^k is the sum of all monomials $z_0^{k_0} \dots z_n^{k_n}$ with $k_0 +\dots+ k_n = k$ and therefore, if we set $z_0 =\dots=z_n=1$, gives the number of such monomials. Since these monomials are clearly a basis for the complex vector space $P(n,k)$ we find

$$\frac{1}{(1-t)^{n+1}} = \sum_{k=0}^{\infty} t^k \dim P(n,k), \tag{3}$$

from which (1) follows.

The group G of 10(1) clearly acts on $\mathbb{C}^{n+1}$ and therefore on $P(n,k)$, and the invariant subspace $P(n,k)^G$ obviously has as a basis the set of monomials $z_0^{k_0}\dots z_n^{k_n}$ with $k_0 +\dots+ k_n = k$ and k_i divisible by b_i, so the argument just used yields

$$\sum_{k=0}^{\infty} t^k \dim P(n,k)^G = \prod_{i=0}^{n} \frac{1}{1 - t^{b_i}} . \qquad (4)$$

We now consider

$$P^p_{n,k} = \left\{ \sum_{0 \leqslant i_1 < \ldots < i_p \leqslant n} a_{i_1 \ldots i_p} \, dz_{i_1} \wedge \ldots \wedge dz_{i_p} \;\middle|\; a_{i_1 \ldots i_p} \in P(n,k-p) \right\}. \qquad (5)$$

Since $i_1, \ldots, i_p$ can be chosen in $\binom{n+1}{p}$ ways, we deduce from (1) that

$$\dim P^p_{n,k} = \binom{n+1}{p}\binom{n+k-p}{k-p}, \qquad (6)$$

or

$$\sum_{k=0}^{\infty} t^k \dim P^p_{n,k} = \frac{\binom{n+1}{p} t^p}{(1-t)^{n+1}} . \qquad (7)$$

Following the usual formalism of χ, χ^p and χ_y we define

$$p_y(n,k) = \sum_{p=0}^{n+1} y^p \dim P^p_{n,k} ; \qquad (8)$$

then (7) can be rewritten

$$\sum_{k=0}^{\infty} t^k p_y(n,k) = \left(\frac{1 + yt}{1 - t}\right)^{n+1} . \qquad (9)$$

More generally,

<u>THEOREM 1</u>: Let

$$p_y^G(n,k) = \sum_{p=0}^{n+1} y^p \dim (P^p_{n,k})^G. \tag{10}$$

Then

$$\sum_{k=0}^{\infty} t^k\, p_y^G(n,k) = \prod_{i=0}^{n} \left(\frac{1+yt^{b_i}}{1-t^{b_i}}\right). \tag{11}$$

Proof : We assign weight r to a power z^r or to a differential $d(z^r)$. Thus the elements of $P^p_{n,k}$ are homogenous of weight k. Note that

$$b\, z^{mb-1}\, dz = z^{(m-1)b}\, d(z^b), \tag{12}$$

which is consistent with this weighting. A monomial in $P^p_{n,k}$ is G-invariant if and only if it can be written as

$$c_{i_1\ldots i_p}\, d(z^{b_{i_1}}) \ldots d(z^{b_{i_n}}) \tag{13}$$

with $c_{i_1 \ldots i_p}$ an invariant monomial in $z_0,\ldots,z_n$ of weight $k-b_{i_1}-\ldots-b_{i_p}$.

Therefore the coefficient of $y^p t^k$ in

$$\prod_{i=0}^{n} \frac{1 + yt^{b_i}\, d(z_i)^{b_i}}{1 - t^{b_i}\, z_i^{b_i}} \tag{14}$$

is exactly the sum of all invariant monomials (13), and setting z_i and $d(z_i^{b_i})$ equal to one gives the number of such monomials, i.e. $\dim(P^p_{n,k})^G$.

We now relate these calculations to the X_y-characteristic

$\chi_y(P_n(\mathbb{C}),H^k)^G$. The Hopf bundle over $P_n(\mathbb{C})$ is the principal $\mathbb{C}^*$-bundle

$$\psi : \mathbb{C}^{n+1} - \{0\} \longrightarrow P_n(\mathbb{C}), \tag{15}$$

and H is defined as the associated line bundle

$$\psi : (\mathbb{C}^{n+1} - \{0\}) \times_{\mathbb{C}^*} \mathbb{C} \longrightarrow P_n(\mathbb{C}), \tag{16}$$

which can be thought of geometrically as "blowing up" the origin of $\mathbb{C}^{n+1}$ into $P_n(\mathbb{C})$. The k^{th} power H^k is the bundle with total space $(\mathbb{C}^{n+1} - \{0\}) \times_{\mathbb{C}^*} \mathbb{C}$, where $\mathbb{C}^*$ now acts on $\mathbb{C}$ by $\lambda . z = \lambda^k z$ ($\lambda \in \mathbb{C}^*$, $z \in \mathbb{C}$). This is defined for negative k also (H is a line bundle and therefore invertible); we sometimes use $\tilde{H}$ to denote H^{-1}.

Let s be an element of $H^0(P_n(\mathbb{C}),H^k)$, i.e. a section of H^k. This lifts to a section s' in the diagram

$$\begin{array}{ccc} \mathbb{C}^{n+1} - \{0\} & \overset{s'}{\longleftrightarrow} & (\mathbb{C}^{n+1} - \{0\}) \times \mathbb{C} \\ \downarrow & & \downarrow \\ P_n(\mathbb{C}) & \underset{\psi}{\overset{s}{\longleftrightarrow}} & (\mathbb{C}^{n+1} - \{0\}) \times_{\mathbb{C}^*} \mathbb{C} = H^k, \end{array} \tag{17}$$

where the vertical maps consist of dividing out by the actions of $\mathbb{C}^*$. Since s' is a section, it is given by a map $f: \mathbb{C}^{n+1} - \{0\} \rightarrow \mathbb{C}$, and from the commutativity of (17) we deduce that, for $z \in \mathbb{C}^{n+1} - \{0\}$ and $\lambda \in \mathbb{C}^*$, $s'(z)$ and $s'(\lambda z)$ are in the same orbit of $\mathbb{C}^*$. But $s'(z) = (z,f(z))$ and $s'(\lambda z) = (\lambda z, f(\lambda z)) = \lambda o(z, \lambda^k f(\lambda z))$ by definition of the action of $\mathbb{C}^*$ on H^k, so

$$f(\lambda z) = \lambda^{-k} f(z) \qquad (z \in \mathbb{C}^{n+1} - \{0\}, \lambda \in \mathbb{C}^*). \tag{18}$$

But by a well-known theorem of Hartogs, a function of more than one variable which is holomorphic in a deleted neighbourhood of a point is holomorphic at that point also, so that the function f extends to $\tilde{f}: \mathbb{C}^{n+1} \to \mathbb{C}$ and

$$\tilde{f}(\lambda z) = \lambda^{-k} f(z) \qquad (z \in \mathbb{C}^{n+1}, \lambda \in \mathbb{C}^*) \tag{19}$$

if $n \geqslant 1$. This establishes a 1:1 correspondence between the elements of $H^0(P_n(\mathbb{C}), H^k)$ and functions $\tilde{f}$ of $n+1$ variables which are homogenous of degree $-k$. Since any homogenous function is a polynomial, we have proved:

<u>THEOREM 2:</u> Assume that $n \geqslant 1$, and write $\tilde{H}$ for H^{-1}. Then

$$H^0(P_n(\mathbb{C}), \tilde{H}^k) \cong \begin{cases} 0, & \text{if } k < 0. \\ P(n,k), & \text{if } k \geqslant 0. \end{cases} \tag{20}$$

By a similar proof (Serre [91]),

$$H^0(P_n(\mathbb{C}), \Lambda^p(n+1) \otimes \tilde{H}^{k-p}) \cong \begin{cases} 0, & \text{if } k < p \\ P^p_{n,k}, & \text{if } k \geqslant p. \end{cases} \tag{21}$$

Moreover, these isomorphisms are G-equivariant, and therefore induce isomorphisms of the G-invariant parts of the vector spaces. Here $(n+1)$ is the trivial bundle $P_n(\mathbb{C}) \times \mathbb{C}^{n+1}$. There is an obvious bundle map from H to $(n+1)$.

Let the quotient bundle be V, with fibre $V_x = \mathbb{C}^{n+1}/H_x$, so that there is an exact sequence

$$0 \longrightarrow H \longrightarrow (n+1) \longrightarrow V \longrightarrow 0. \tag{22}$$

<u>THEOREM 3</u>: If T^* is the tangent bundle of $P_n(\mathbb{C})$, then there is an isomorphism

$$T^* \cong V \otimes H^{-1}. \tag{23}$$

<u>Proof</u>: Let $\underline{G}$ denote the group $GL(n+1;\mathbb{C})$, which acts transitively on $P_n(\mathbb{C})$. Choose a basepoint $e = (1:0:\ldots:0)$ in $P_n(\mathbb{C})$, and let $\underline{H}$ be the isotropy group of e:

$$\underline{H} = \underline{G}_e = GL(1,n;\mathbb{C}) = \left\{ B = \begin{bmatrix} a_0 & a_1 \ldots a_n \\ 0 & \\ \vdots & A \\ 0 & \end{bmatrix}, \quad a_i \in \mathbb{C},\ a_0 \neq 0,\ A \in GL(n;\mathbb{C}) \right\}. \tag{24}$$

Clearly $P_n(\mathbb{C})$ is the homogenous space $\underline{G}/\underline{H}$. It follows that the bundle T^* is given in terms of its fibre T^*_e at e (on which $\underline{H}$ acts) as $\underline{G} \times_{\underline{H}} T^*_e$. By mapping an element $B \in \underline{H}$ as in (24) to a_0 or to A, we get homomorphisms from $\underline{H}$ to the groups $\mathbb{C}^*$ and $GL(n;\mathbb{C})$. The tangent space at e is identified with an infinitesimally small neighbour- of $e \in P_n(\mathbb{C})$ by sending $(z_1,\ldots,z_n)$ to the point $(1:\varepsilon z_1:\ldots:\varepsilon z_n) \in P_n(\mathbb{C})$; therefore the action of B on T^*_e is given by $(z_1,\ldots,z_n) \longrightarrow (w_1,\ldots,w_n)$, where

$$B(1:\varepsilon z_1:\ldots:\varepsilon z_n) = (1:\varepsilon w_1:\ldots:\varepsilon w_n), \tag{25}$$

$$w_i = \frac{a_{i1}z_1 + \ldots + a_{in}z_n}{a_0 + \varepsilon a_1 z_1 + \ldots + \varepsilon a_n z_n}. \tag{26}$$

Letting $\varepsilon \to 0$, we see that B acts on T^*_e as the matrix

A/a_0. Since $\mathbf{B}$ acts on V_e by A and on H by a_0, this implies the assertion (23).

Substituting (23) into (22) gives an exact sequence

$$0 \longrightarrow H \longrightarrow (n+1) \longrightarrow T^* \otimes H \longrightarrow 0, \tag{27}$$

or, tensoring with $\widetilde{H}$,

$$0 \longrightarrow 1 \longrightarrow (n+1) \otimes H^{-1} \longrightarrow T^* \longrightarrow 0, \tag{28}$$

or, dualizing,

$$0 \longrightarrow T \longrightarrow (n+1) \otimes H \longrightarrow 1 \longrightarrow 0, \tag{29}$$

where 1 is the trivial line bundle. We apply to this the second statement of Theorem 1 of 1.3 to obtain

$$0 \longrightarrow \Lambda^p T \longrightarrow \Lambda^p((n+1) \otimes H) \longrightarrow \Lambda^{p-1} T \longrightarrow 0, \tag{30}$$

or equivalently

$$0 \longrightarrow \Lambda^p T \longrightarrow \Lambda^p (n+1) \otimes H^p \longrightarrow \Lambda^{p-1} T \longrightarrow 0. \tag{31}$$

In particular, if $p = n+1$, the first term vanishes and the last is the canonical line bundle K of $P_n(\mathbb{C})$ (defined in 1.3), so we get an isomorphism

$$K \cong H^{n+1}. \tag{32}$$

If we tensor (31) with $\widetilde{H}^k = H^{-k}$ and apply 1.4(11), we obtain

$$\chi^p(P_n(\mathbb{C}), \widetilde{H}^k)^G + \chi^{p-1}(P_n(\mathbb{C}), \widetilde{H}^k)^G = \chi(P_n(\mathbb{C}), \Lambda^p(n+1) \otimes \widetilde{H}^{k-p})^G. \tag{33}$$

Now it is known that

$$H^*(P_n(\mathbb{C})) = \mathbb{Z}[x]/x^{n+1} \tag{34}$$

with

$$x = c_1(\widetilde{H}) = -c_1(H); \tag{35}$$

this is the usual convention for x as the Poincaré dual of $[P_{n-1}(\mathbb{C})] \in H_{2n-2}(P_n(\mathbb{C}))$. It is also known that an element of $H^2(P_n(\mathbb{C});\mathbb{Z})$ is positive (cf. 1.3) if and only if it is a positive integral multiple of x. Therefore $c_1(\widetilde{H}^k \times K^{-1})$ $= (k+n+1)x$ (by (32) and (35)) is positive if $k+n \geqslant 0$, so the Kodaira vanishing theorem (Theorem 8 of 1.3) tells us that the higher cohomology groups of $P_n(\mathbb{C})$ with coefficients in the sheaf of sections of $\widetilde{H}^k$ are 0 in this case. We deduce

$$\chi(P_n(\mathbb{C}),\ \Lambda^p(n+1)\times \widetilde{H}^{k-p})^G = \dim H^0(P_n(\mathbb{C}),\Lambda^p(n+1)\times \widetilde{H}^{k-p})^G \tag{36}$$

if $k-p+n \geqslant 0$. Since $n-p \geqslant -1$, it suffices to have $k > 0$ (if $p > n+1$ both sides of (36) are trivially zero). We substitute (21) into (36) and the result into (33), multiply by y^p, sum over p, and use Theorem 1, obtaining

<u>THEOREM 4</u>: For $k > 0$,

$$(1+y)\ \chi_y(P_n(\mathbb{C}),\widetilde{H}^k)^G = \text{coefficient of } t^k \text{ in } \prod_{i=0}^{n} \frac{1+yt^{b_i}}{1-t^{b_i}}. \tag{37}$$

To take care of $k < 0$, we apply $1.2(36)^G$ (equivariant

Serre duality) to get

$$\chi_y(P_n(\mathbb{C}),\tilde{H}^k)^G = (-y)^n \chi_{1/y}(P_n(\mathbb{C}),\tilde{H}^{-k})^G, \tag{38}$$

so that for $k<0$ we deduce from (37) the equation

$$\chi_y(P_n(\mathbb{C}),\tilde{H}^k)^G = \frac{(-y)^n}{1+y^{-1}} \operatorname{res}_{t=0}\left[t^{k-1}\,dt\cdot \prod_{i=0}^{n} \frac{1+y^{-1}t^{b_i}}{1-t^{b_i}}\right]$$

$$= \frac{1}{1+y} \operatorname{res}_{t=\infty}\left[\frac{dt}{t}\, t^{-k} \prod_{i=0}^{n} \frac{1+yt^{b_i}}{1-t^{b_i}}\right], \tag{39}$$

where we have replaced t by t^{-1}, using $\frac{d(t^{-1})}{t^{-1}} = -\frac{dt}{t}$.

<u>THEOREM 5</u>: For any integer k,

$$(1+y)\,\chi_y(P_n(\mathbb{C}),H^k)^G = (\operatorname{res}_{t=0} + \operatorname{res}_{t=\infty})\left[t^k \frac{dt}{t} \prod_{i=0}^{n} \frac{1+yt^{b_i}}{1-t^{b_i}}\right]$$

$$= -\sum_{|z|=1} \operatorname{res}_{t=z}\left[t^k \frac{dt}{t} \prod_{i=0}^{n} \frac{1+yt^{b_i}}{1-t^{b_i}}\right]. \tag{40}$$

<u>Proof</u>: The equality of the first and second lines follows from the residue theorem. For negative k the residue at infinity vanishes and we are left with equation (37), while similarly for positive k there is no residue at zero and the equation reduces to (39); notice that we have replaced k by $-k$ and $\tilde{H}$ by H in (40). It remains to prove the equation for $k = 0$, namely (evaluating the residues)

$$(1+y)\,\chi_y(P_n(\mathbb{C}))^G = 1-(-y)^{n+1}, \tag{41}$$

or

$$\chi_y(P_n(\mathbb{C}))^G = 1-y+\ldots+(-y)^n. \tag{42}$$

Since G acts trivially on $H^*(P_n(\mathbb{C}))$ (because the action of G embeds in that of the connected group T^{n+1}), it suffices to prove (42) without the G. Since $P_n(\mathbb{C})$ is certainly a Kähler manifold, we deduce from Theroem 4 of 1.3 that the numbers $h^{p,q}$ for $P_n(\mathbb{C})$ satisfy $h^{p,q} = h^{q,p}$ and $\sum_{p+q=r} h^{p,q} = b_r$. But from (34) we find that b_r is one for r even and 0 for r odd. We deduce that $h^{p,p} = 1$ for $0 \leqslant p \leqslant n$ and that $h^{p,q} = 0$ for $p \neq q$. Equation (42) then follows.

10.2 In this section we will apply the theorems of 1.3 to the non-singular divisor

$$S = \{(z_0:\ldots:z_n) \in P_n(\mathbb{C}) : z_0^N + \ldots + z_n^N = 0\}. \tag{1}$$

The line bundle associated to this divisor is $\{S\} = H^{-N}$. We assume that N is a multiple of all the b_i's; then the group G of 10(2) acts on S. From $1.3(20)^G$ (a consequence of the equivariant four-term formula) with $W = 1$, we obtain

$$\chi_y(S)^G = \chi_y(P_n(\mathbb{C}))^G - (1+y)\sum_{i=1}^{\infty}(-y)^{i-1}\,\chi_y(P_n(\mathbb{C}),H^{iN})^G. \tag{2}$$

Using 10.1(40), we can rewrite this as

$$\chi_y(S)^G = \frac{-1}{1+y}\sum_{|z|=1} \operatorname{res}_z\left[\frac{f(t)dt}{t}\right] + \sum_{i=1}^{\infty}(-y)^{i-1}\sum_{|z|=1}\operatorname{res}_z\left[\frac{t^{iN}f(t)dt}{t}\right] \tag{3}$$

where

$$f(t) = \prod_{i=0}^{n} \frac{1+yt^{b_i}}{1-t^{-b_i}} . \tag{4}$$

By the Weierstrass theorem on double series, we can interchange the order of summation in (3) if $|y|$ is sufficiently small, and then perform the inner sum over i (a geometric series in yt^N), obtaining

$$(1+y)\chi_y(S)^G = -\sum_{\substack{|z|=1 \\ z \text{ a pole of } f}} \operatorname{res}_{t=z}\left[\frac{1-t^N}{1+yt^N}\,\frac{f(t)}{t}\,dt\right] \tag{5}$$

This must be valid for all y, since a polynomial identity which we have proved to hold for sufficiently small y.

The poles of $f(t)$ are given by $t^{b_i} = 1$ (some i), so since N is a multiple of all the b_i's they all satisfy $t^N = 1$. In particular, if $y \neq -1$ the poles of f are disjoint from the zeroes of $1 + yt^N$, so applying the residue theorem to (5),

$$(1+y)\chi_y(S)^G = \left(\operatorname{res}_0 + \operatorname{res}_\infty + \sum_{1+yz^N=0}\operatorname{res}_z\right)\left[\frac{1-t^N}{1+yt^N}\,\frac{f(t)}{t}\,dt\right] \tag{6}$$

$$= 1 - (-y)^n - \frac{1+y^{-1}}{N}\sum_{1+yz^N=0} f(z). \tag{7}$$

We can rewrite this using the techniques of 5.3. Let

$$a_k = N/b_k; \tag{8}$$

then if $1+yz^N=0$,

$$\frac{1+yz^{b_i}}{1-z^{b_i}} = \frac{z^{-b_i}-z^{-a_ib_i}}{z^{-b_i}-1} = -(z^{-b_i}+z^{-2b_i}+\dots+z^{-(a_i-1)b_i}), \tag{9}$$

and substituting this into (7) gives

$$\chi_y(S)^G = \frac{1-(-y)^n}{1+y} + \frac{(-1)^n}{Ny}\sum_{t^N+y=0}\prod_{i=0}^{n}(t^{b_i}+\dots+t^{(a_i-1)b_i}), \tag{10}$$

where we have written t for z^{-1}. Since

$$\frac{1}{N}\sum_{t^N+y=0} t^r = \begin{cases} (-y)^k, & \text{if } r = kN, \\ 0, & \text{if } N\nmid r, \end{cases} \tag{11}$$

we can rewrite (10) as

$$\chi_y(S)^G = \sum_{k=1}^{n} (-y)^{k-1}(1 - (-1)^n \tilde{N}_k), \tag{12}$$

where

$$\tilde{N}_k = \#\{0<j_0<a_0,\dots,0<j_n<a_n:\ j_0b_0+\dots+j_nb_n = kN\}. \tag{13}$$

We now use a result of Lefschetz. Given an n-dimensional complex submanifold V of $P_k(\mathbb{C})$ one can find a hyperplane $E\subset P_k(\mathbb{C})$ such that $V\cap E$ is a submanifold of V of complex codimension one, and the assertion is that

$$H^i(V;\mathbb{Z}) \longrightarrow H^i(V\cap E;\mathbb{Z}) \tag{14}$$

is bijective for $i<n-1$ and injective for $i=n-1$. Indeed, $V-V\cap E$ has the homotopy type of a CW complex of dimension n (see [1] or [6]), so $H^i(V,V\cap E) = H_{2n-i}(V-V\cap E)$ vanishes for $i<n$, and the result of Lefschetz follows.

If we take $V = P_n(\mathbb{C})$ and $k = \binom{n+N}{n} - 1$, there is an embedding of V in $P_k(\mathbb{C})$ given by $w_i = z_0^{i_0}\dots z_n^{i_n}$ for $(z_0:\dots:z_n)\in V$, where $i=(i_0,\dots,i_n)$ ranges over the k multi-indices with $i_0,\dots,i_n \geqslant 0$ and $i_0+\dots+i_n = N$. Then the divisor S of (1) is precisely $V\cap E$, where E is the hyperplane

$$w_{(N,0,\dots,0)} + w_{(0,N,0,\dots,0)} + \dots + w_{(0,\dots,0,N)} = 0, \tag{15}$$

and so from the Lefschetz theorem just quoted,

$$H^i(P_n(\mathbb{C})) \longrightarrow H^i(S) \tag{16}$$

is an isomorphism for $i<n-1$. Since S is a closed manifold of real dimension $2n-2$, we can deduce all but the middle Betti number of S:

$$b_i(S) = \begin{cases} 0, & \text{if } i<0 \text{ or } i>2n-2,\\ 0, & \text{if } 0\leqslant i\leqslant 2n-2,\ i\neq n-1,\ i \text{ odd},\\ 1, & \text{if } 0\leqslant i\leqslant 2n-2,\ i\neq n-1,\ i \text{ even}.\end{cases} \tag{17}$$

Since S is a Kähler manifold (indeed algebraic), we deduce just as for $P_n(\mathbb{C})$ (see the proof of Theorem 5 of 10.1) that for $p+q \neq n-1$ the only non-zero $h^{p,q}$ of S are the numbers $h^{p,p} = 1$ $(0 \leq p \leq n-1,\ 2p \neq n-1)$. Therefore

$$\chi^p(S) = \sum_q (-1)^q h^{p,q}$$

$$= \begin{cases} (-1)^p + (-1)^{n-1-p} h^{p,n-1-p} & (p \neq \frac{n-1}{2}), \\ (-1)^p h^{p,p} & (p = \frac{n-1}{2}) \end{cases} \tag{18}$$

cf. Appendix of [36], theorem 22.1.2). Comparing this with (12) (in which we take $b_0=\ldots=b_n=1$ and omit the "G"), we obtain the only remaining values of $h^{p,q}$, namely

$$h^{p,n-1-p} = \begin{cases} 1 + \tilde{N}_{p+1}, & 0 \leq p \leq n-1,\ p \neq \frac{1}{2}(n-1), \\ \tilde{N}_{p+1} & p = \frac{1}{2}(n-1), \end{cases} \tag{19}$$

with $\tilde{N}_{p+1}$ given by (13) with $b_0=\ldots=b_n=1$, namely

$$\tilde{N}_{p+1} = \#\{0 < j_0,\ldots,j_n < N:\ j_0+\ldots+j_n = (p+1)N\}$$

$$= \#\{0 < j_1,\ldots,j_n < N:\ p < \frac{j_1+\ldots\ j_n}{N} < p+1\}. \tag{20}$$

All of this works in the equivariant case, for the map (16) is G-equivariant (it is induced by the inclusion) and therefore gives an isomorphism of the G-invariant parts of the cohomology groups for $i = n-1$, and G acts trivially on $H^*(P_n(\mathbb{C}))$, so that (18) and (19) still hold, with $\tilde{N}_k$ given by (13). If $b_0= 1$ we can do the same trick as that leading to (20):

$$\tilde{N}_{p+1} = \#\{0<j_0<N,\ 0<j_1<a_1,\ldots,0<j_n<a_n:\ j_0 + b_1j_1+\ldots+b_nj_n = (p+1)N\}$$

$$=\#\{0<j_1<a_1,\ldots,0<j_n<a_n:\ pN<b_1j_1+\ldots+b_nj_n<(p+1)N\} \qquad (21)$$

$$= N_p, \qquad (22)$$

with N_p defined by 5.4(2). We have proved

<u>THEOREM 1</u> : Let $S \subset P_n(\mathbb{C})$ be the divisor (1). Then

$$h^{p,q}(P_n(\mathbb{C}))^G = h^{p,q}(P_n(\mathbb{C})) = \begin{cases} 1, & \text{if } p=q=0,\ldots,n, \\ 0, & \text{otherwise,} \end{cases} \qquad (23)$$

and for $p+q \leq n-1$,

$$h^{p,q}(S)^G = h^{p,q}(P_n(\mathbb{C}))^G + \begin{cases} \tilde{N}_{p+1}, & \text{if } p+q = n-1, \\ 0, & \text{otherwise,} \end{cases} \qquad (24)$$

where $\tilde{N}_k$ is given by (13) and $\tilde{N}_{k+1} = N_k$ if $b_0 = 1$ (N_k as in 5.4(2)).

We now shall apply the results of this section to the Brieskorn variety 10(3). We assume n is an <u>odd</u> number $2k + 1$. Let E be the hypersurface $z_0 = 0$ in $P_n(\mathbb{C})$, so that G leaves E as well as S invariant, and let T be a closed equivariant tubular neighbourhood of $E \cap S$ in S. We claim

$$\mathrm{Sign}(g,T) = 1 \quad (g \in G). \qquad (25)$$

Indeed, by the Lefschetz theorem given above, $H^{n-3}(E\cap S) \cong H^{n-3}(S)$ (we take $V = S$ in (14)), and by (16) this is isomorphic to $H^{n-3}(P_n(\mathbb{C})) \cong \mathbb{Z}$. But T has $E \cap S$ as deformation retract, and T has real dimension $2n-2$, so the middle cohomology group $H^{n-1}(T,\partial T) \cong H_{n-1}(T) \cong H_{n-1}(E\cap S)$ $= H^{n-3}(E\cap S) = \mathbb{Z}$ since $E\cap S$ has dimension $2n-4$. Therefore Sign(g,T) must be ± 1 or 0. Since T is the total space of the normal bundle ν of $E \cap S$ in S, we can apply Theorem 7 of 2.1 to get

$$\mathrm{Sign}(T) = \mathrm{Sign}(E\cap S, e) = \mathrm{Sign}(f),$$

$$f(x,y) = (xye)[E \cap S] \quad (x,y \in H^{n-3}(E\cap S)), \tag{26}$$

where e is the Euler class of ν. We have shown that the group $H^{n-3}(E \cap S)$ is $\mathbb{Z}$; its generator is e^{k-1} and $(e^{k-1}\cdot e^{k-1}\cdot e)[E\cap S] = e^{n-2}[E\cap S] = N > 0$, so $\mathrm{Sign}(f) = +1$. A similar argument in the equivariant case proves (25).

Now we apply Novikov additivity (Theorem 3 of 2.1) to deduce

$$\mathrm{Sign}(g,S) = \mathrm{Sign}(g,\overline{S-T}) + \mathrm{Sign}(g,T) = \mathrm{Sign}(g,\overline{S-T}) + 1, \tag{27}$$

so from Theorems 4 and 6 of 2.1 we obtain

$$\begin{aligned}\mathrm{Sign}(\overline{S-T}/G) &= \frac{1}{|G|}\sum_{g\in G}(\mathrm{Sign}(g,S) - 1)\\ &= \chi_1(S)^G - 1.\end{aligned} \tag{28}$$

Recall that the signature of a non-compact manifold is defined by the intersection pairing in homology or the cup product followed by evaluation on the fundamental class in cohomology with compact supports, and that with this definition $\mathrm{Sign}(\overset{\circ}{X}) = \mathrm{Sign}(X)$ for a compact manifold with boundary X, where $\overset{\circ}{X}$ is the interior of X. It follows that $\mathrm{Sign}(\overline{S-T}/G) = \mathrm{Sign}((S-T)/G) = \mathrm{Sign}((S-E\cap S)/G)$. But

$$S-(E\cap S) = \{(z_0:\dots:z_n) \in P_n(\mathbb{C}) : z_0 \neq 0, z_0^N + z_1^N + \dots + z_n^N = 0\}$$

$$= \{(z_1,\dots,z_n) \in \mathbb{C}^n : 1 + z_1^N + \dots + z_n^N = 0\}, \qquad (29)$$

and under this identification G acts by

$$(\zeta_0,\dots,\zeta_n)\circ(z_1,\dots,z_n) = (\zeta_0^{-1}\zeta_1 z_1,\dots,\zeta_0^{-1}\zeta_n z_n)$$

$$(\zeta_i \in G_{b_i}). \qquad (30)$$

If $b_0 = 1$ we can omit ζ_0, and then the map $z_i \to z_i^{b_i}$ defines an isomorphism

$$(S - E\cap S)/G \cong \{(z_1,\dots,z_n) \in \mathbb{C}^n : 1 + z_1^{a_1} + \dots + z_n^{a_n} = 0\}$$

$$\cong V_a, \qquad (31)$$

where V_a is the Brieskorn manifold defined in 10(1). Combining this with (28) and (7) (with $y = 1$, $b_0 = 1$, and n odd), we deduce

$$\mathrm{Sign}(V_a) = -\frac{1}{N} \sum_{1+z^N=0} \frac{1+z}{1-z} \prod_{i=1}^{n} \frac{1+z^{b_i}}{1-z^{b_i}} \quad (n \text{ odd}). \qquad (32)$$

10.3 We recall formula 10.2(7) for the invariant χ_y-characteristic of the divisor S:

$$\chi_y(S)^G = \frac{-1}{1+y}\sum_z \operatorname{res}_{t=z}\left[\frac{1-t^N}{1+yt^N}\prod_{i=0}^{n}\frac{1+yt^{b_i}}{1-t^{b_i}}\frac{dt}{t}\right], \tag{1}$$

where the sum is over all z with $|z| = 1$ and $z^{b_i} = 1$ for some i. Write $t = e^{-2x}$ and $y = 1$; then (1) becomes

$$\chi_1(S)^G = +\sum_\xi \operatorname{res}_{x=\xi}\left[\frac{1-e^{-2Nx}}{1+e^{-2Nx}}\prod_{i=0}^{n}\frac{1+e^{-2b_ix}}{1-e^{-2b_ix}}\,dx\right], \tag{2}$$

with the sum now over all ξ with $0 \leq i\xi < \pi$ and $\xi = \pi ir/b_k$ for some k and some $r \in \mathbb{Z}$. Then making the substitution $x \to x + \xi$, and observing that $N\xi \equiv 0 \pmod{i\pi}$ since b_k divides N for all k, we find:

$$\begin{aligned}\chi_1(S)^G &= \sum_\xi \operatorname{res}_{x=\xi}\left[\tanh Nx \cdot \prod_{i=0}^{n}\coth b_ix \cdot dx\right]\\ &= \sum_\xi \operatorname{res}_{x=0}\left[\tanh N(x+\xi)\cdot\prod_{i=0}^{n}\coth b_i(x+\xi)\cdot dx\right]\\ &= \sum_\xi \operatorname{res}_{x=0}\left[\tanh Nx\cdot\prod_{i=0}^{n}\coth b_i(x+\xi)\cdot dx\right].\end{aligned} \tag{3}$$

If we replace ξ by $i\xi$, this becomes

$$\chi_1(S)^G = \operatorname{res}_{x=0}\left[\tanh Nx\, F(x)\frac{dx}{x^{n+1}}\right], \tag{4}$$

with

$$F(x) = \sum_{\substack{0 \leq \xi < \pi \\ b_k\xi = 0 \pmod{\pi} \\ \text{for some } k}} \prod_{k=0}^{n} \frac{x}{\tanh b_k(x+i\xi)} . \tag{5}$$

We have introduced the factor x^{n+1} in $F(x)$ to make it holomorphic at $x = 0$; thus $F(x)$ is a certain power series in x (in fact in x^2, since $F(-x) = F(x)$ as we see by replacing ξ by $\pi - \xi$). From Theorems 4 and 6 of 2.1 we find

$$\chi_1(S)^G = \text{Sign}(S/G), \tag{6}$$

and therefore (4) becomes

<u>THEOREM 1</u>: Let S be the divisor 10.2(1) and G the group 10(2). Then

$$\text{Sign}(S/G) = \text{coefficient of } x^n \text{ in } (\tanh Nx)\, F(x), \tag{7}$$

where $F(x)$ is the power series given by (5).

This is only of interest if n is odd, since for even n both sides are zero (to see this for the right-hand side make the substitutions $x \to -x$ and $\xi \to \pi-\xi$ in equations (4) and (5)).

If a finite group G acts on a manifold X, the quotient is a rational homology manifold (see 3.1). If $Y \subset X$ is a submanifold which is mapped into itself by G, and if Y is transverse to all the fixed-point sets X^g, we get a bundle ${}_G\nu$ over Y/G whose fibre over Gy is ν_y/G_y (for $y \in Y$, the isotropy group G_y acts on the fibre ν_y of the normal bundle ν of Y in X). If, as well as the transversality assumption, we assume that G acts trivially on ν (i.e.

that G_y acts trivially on ν_y for all $y \in Y$), then the bundle ${}_G\nu$ is a vector bundle and Y/G has a normal bundle in X/G in the sense of Thom (cf. 1.3). These assumptions are satisfied for $Y = S$, $X = P_n(\mathbf{C})$, so we get a bundle ${}_G\nu$ on S/G. Consider

$$\begin{array}{ccc} S & \xrightarrow[(\nu)]{i} & P_n(\mathbf{C}) \\ \bar{\pi}\downarrow & & \downarrow\pi \\ S/G & \xrightarrow[({}_G\nu)]{\bar{i}} & P_n(\mathbf{C})/G \end{array} \qquad (8)$$

Let $y \in H^2(S/G)$ be the first Chern class of the complex line bundle ${}_G\nu$. We will take cohomology with rational coefficients, so $\bar{\pi}^*: H^2(S/G) \cong H^2(S)$ by Theorem 5 of 2.1 (since G acts trivially on $H^2(S)$). The first Chern class of ν is then $\bar{\pi}^*(y)$. But $\{S\} = H^{-N}$ so $\nu = i^*(H^{-N})$ and therefore

$$\bar{\pi}^*(y) = N\, i^*x, \qquad (9)$$

where x is the standard generator of $P_n(C)$ given in 10.1(35). Let d denote the degree of π, with $[S/G] = \frac{1}{d}\bar{\pi}_*[S]$. Then from 3.1(5) we deduce:

$$\mathrm{Sign}(S/G) = (\bar{i}^*\mathcal{L}(P_n(\mathbf{C})/G) \cdot \mathcal{L}({}_G\nu)^{-1})[S/G]$$

$$= \frac{1}{d}(\bar{\pi}^*\, \bar{i}^*\mathcal{L}(P_n(\mathbf{C})/G) \cdot \frac{\tanh \bar{\pi}^*y}{\bar{\pi}^*y})\ [S]$$

$$= \frac{1}{d}(i^*\pi^*\mathcal{L}(P_n(\mathbf{C})/G) \cdot i^*(\frac{\tanh Nx}{Nx}))\,[S]$$

$$= \frac{1}{d}(\pi^*\mathcal{L}(P_n(\mathbb{C})/G) \cdot \frac{\tanh Nx}{Nx})(i_*[S])$$

$$= \frac{1}{d}(\pi^*\mathcal{L}(P_n(\mathbb{C})/G) \cdot \tanh Nx)[P_n(\mathbb{C})], \tag{10}$$

since $Nx \in H^*(P_n(\mathbb{C}))$ is the Poincaré dual of $i_*[S]$. Since $H^*(P_n(\mathbb{C}))$ is generated by x, we must have

$$\pi^*\mathcal{L}(P_n(\mathbb{C})/G) = T(x) \tag{11}$$

for some polynomial T (with rational coefficients, since Thom's definition gives an L-class in rational cohomology). Since evaluation on the fundamental class in $P_n(\mathbb{C})$ corresponds to picking out the coefficient of x^n, (10) becomes

$$\mathrm{Sign}(S/G) = \text{coefficient of } x^n \text{ in } \frac{1}{d}\,T(x)\tanh Nx. \tag{12}$$

Now $T(x)$ is an even polynomial since the L-class of a manifold only has non-zero components in dimensions divisible by four. The right-hand side of (12) is a polynomial in N, identically zero if n is even (since $\tanh t$ is an odd function). However, since $\tanh t$ has an expansion in which the coefficient of t^r for r odd is non-zero (it is essentially a Bernoulli number), the coefficients of the various powers of N in this polynomial completely determine the coefficients of the various even powers of x in $T(x)$. Therefore if $T(x)$ is an even polynomial in x of degree at most n and we know the value of the right-hand side of (12 for infinitely many N, the polynomial $T(x)$ is completely determined. This is our situation (the infinitely many values of N being all common multiples of $b_0,\ldots,b_n$), and therefore comparison of (12) and (7) tells us that $\frac{1}{d}\,T(x)$ and $F(x)$ have the same coefficients of $1, x^2, \ldots, x^{n-1}$ if n is odd. But $x^{n+1} = 0$ in $H^*(P_n(C))$ and $F(x)$ is even, so we have an equality $T(x) = d\,F(x)$.

<u>THEOREM 2</u> (Bott): Let G be the group defined in 10.(2). Assume n is <u>odd</u>. Let $x \in H^2(P_n(\mathbb{C}))$ be the standard generator and

$$\pi : P_n(\mathbb{C}) \to P_n(\mathbb{C})/G \tag{13}$$

the projection map. Let $(b_0,\ldots,b_n)$ be the greatest common divisor of the b's. Then

$$\pi^*\mathcal{L}(P_n(\mathbb{C})/G) = \frac{1}{(b_0,\ldots,b_n)} \sum_{0 \leq \xi < \pi} \prod_{k=0}^{n} \frac{b_k x}{\tanh b_k(x+i\xi)} . \tag{14}$$

<u>Proof:</u> This is simply the equation $T(x) = dF(x)$, except that we must prove that

$$d = b_0 \ldots b_n/(b_0,\ldots,b_n) \tag{15}$$

and that the condition "$b_k\xi \equiv 0 \pmod{\pi}$, for some k" can be omitted in (5). The latter is clear, since if ξ is such that no $b_k\xi$ is $0 \pmod{\pi}$, then each factor in the product in (14) is a power series beginning with a multiple of x, so the product is a power series whose first non-zero coefficient is that of x^{n+1}. Since $x^{n+1} = 0$ in $H^*(P_n(\mathbb{C}))$, such ξ do not contribute to the sum. In particular, only finitely many ξ in (14) give a non-zero contribution, so the infinite sum makes sense.

To prove (15), recall that the degree d of π is equal to $|G/H|$, where H is the subgroup of G acting trivially on $P_n(\mathbb{C})$. From the definition of the action and of $P_n(\mathbb{C})$, we see that H consists of the elements $(\zeta_0,\dots,\zeta_n)$ of G with $\zeta_0 = \dots = \zeta_n$. Therefore $H \cong \{\zeta | \zeta^{b_k} = 1$ for $k = 0,\dots,n\}$ is just a cyclic group of order $(b_0,\dots,b_n)$. Since $|G| = b_0 \dots b_n$, equation (15) follows. We could also have deduced formula (15) from the equality $T(x) = dF(x)$. For $T(x)$ has leading term 1 (since it is on L-class), while from (5) we see that $F(0)$ is $\frac{1}{b_0 \cdots b_n}$ $\#\{0<\xi<\pi \mid b_k\xi \equiv 0 \pmod{\pi}$ for all $k\}$.

Theorem 2 is especially interesting when the b_i's are mutually coprime. In that case, for $\xi \neq 0$ at most one of the $b_k\xi$ can be $0 \pmod{\pi}$, so the product in (14) begins with a multiple of x^n. Since x^{n+1} is 0 and the coefficient of x^n in $F(x)$ vanishes ($F(x)$ is an even power series, n is odd we can drop all the terms with $\xi \neq 0$. Therefore

$$\pi^*\mathcal{L}(P_b(\mathbb{C})/G) = \prod_{k=0}^{n} \frac{b_k x}{\tanh \ b_k x} \tag{16}$$

in this case, and we can (exceptionally) solve explicitly for the Pontrjagin class, obtaining

<u>Corollary:</u> Let G be as in the theorem, n <u>odd</u>, and assume that all of the b_i's are prime to one another. Then the rational Pontrjagin class in the sense of Thom of the quotient $P_n(\mathbb{C})/G$ is given by

$$\pi^* p(P_n(\mathbb{C})/G) = \prod_{k=0}^{n} (1 + b_k^2 x^2). \tag{17}$$

10.4. The methods of calculation used up to now in §10 have been very explicit, essentially going back to the definitions of the quantities involved. Thus to calculate $\chi_y(P_n(\mathbb{C}),H^k)^G$ in 10.1 we related it to a situation where Kodaira's vanishing theorem applied, so that it was only necessary to calculate the dimension of the 0^{th} dimensional cohomology group (or space of sections) as a certain vector space of homogeneous polynomials. However, there are relatively few situations which can be treated this concretely, and it is therefore worthwhile to show how the results of 10.1-10.3 can be derived by using the theorems of 2.3. This would be the only feasible approach in most problems, and even in the situation considered here gives stronger results than calculating "by hand": we can prove the Theorem 2 of 10.3 for $\mathcal{L}(P_n(\mathbb{C})/G)$ without the assumption that n is odd.

We will denote $P_n(\mathbb{C})$ by X and let

$$g = (\zeta_0,\ldots,\zeta_n) \tag{1}$$

be an element of T^{n+1}, acting on X as in 10(1). Then a point $z \in X$ is in X^g if

$$(\zeta_0 z_0:\ldots:\zeta_n z_n) = (z_0:\ldots:z_n), \tag{2}$$

i.e.

$$\zeta_k z_k = \zeta z_k \quad (k = 0,\ldots,n) \tag{3}$$

for some $\zeta \in \mathbb{C}^*$. Clearly (3) determines ζ uniquely (since at least one of the z_k's is non-zero), and indeed ζ must equal one of the ζ_k's. Therefore

$$X^g = \bigsqcup_{\zeta \in S^1} X(\zeta) \tag{4}$$

with

$$X(\zeta) = \{(z_0:\ldots:z_n) \in X: \zeta_k \neq \zeta \Rightarrow z_k = 0\}, \tag{5}$$

and the union (4) is a finite disjoint union. Since each $X(\zeta)$ is clearly a projective space and hence connected, (4) exactly gives the decomposition of X^g into connected components.

The normal bundle of $X(\zeta)$ in X will be denoted $N(\zeta)$. We want to study the eigenvalues of the action of g on $N(\zeta)$ and the Chern classes of the corresponding subbundles. Fix ζ, and renumber the z_i's so that the ζ_k's equal to ζ are $\zeta_0,\ldots,\zeta_s$. Then (5) becomes

$$X(\zeta) = \{(z_0:\ldots:z_s:0:\ldots:0) \in X\} \cong P_s(\mathbb{C}). \tag{6}$$

If we pick coordinates $z_0,\ldots,z_s$ to represent $z \in X(\zeta)$ (rather than just an equivalence class under the $\mathbb{C}^*$-action), the fibre $N(\zeta)_z$ can be identified with a neighbourhood of z by

$$\mathbb{C}^n \ni (y_1,\ldots,y_{n-s}) \leftrightarrow (z_0:\ldots:z_n:y_1:\ldots:y_{n-s}) \in X, \tag{7}$$

so the action of g on $N(\zeta)_z$ is

$$go(y_1,\dots,y_s) \leftrightarrow$$

$$g(z_0:\dots z_s:y_1:\dots:y_{n-s}) = (\zeta z_0:\dots:\zeta z_s:\zeta_{s+1}y_1:\dots:\zeta_n y_{n-s})$$

$$= (z_0:\dots z_s: \zeta^{-1}\zeta_{s+1}y_1:\dots: \zeta^{-1}\zeta_n y_{n-s})$$

$$\leftrightarrow (\zeta^{-1}\zeta_{s+1}\, y_1,\dots,\zeta^{-1}\zeta_n y_{n-s}).$$

That is, g acts on $N(\zeta)$ with eigenvalues $\zeta^{-1}\zeta_i$ $(i = s+1,\dots,n)$, and the corresponding eigenbundles are the line bundles given by the coordinate z_i or y_{i-s} $(s<i\leqslant n)$. Let

$$a \in H^2(X(\zeta)) \tag{9}$$

be the standard generator (defined as in 10.1(35) for $P_s(\mathbb{C})$ and transferred to $X(\zeta)$ by the isomorphism (6)). Then the Chern class of the line bundle on which g acts by $\zeta^{-1}\zeta_i$ $(i = s+1,\dots,n)$ is $1 + a$.

We apply this first to the calculation of the numbers $\chi_y(X,H^k;g)$. By the holomorphic Lefschetz theorem (theorem 7 of 2.3), the contribution of $X(\zeta)$ to $\chi_y(X,H^k;g)$ equals

$$\left\{ch(H^k|X(\zeta))\ (g)\cdot\left(\frac{a}{1+y}\cdot\frac{1+ye^{-a}}{1-e^{-a}}\right)^{s+1}\cdot(1+y)^s\cdot\right.$$

$$\left.\prod_{i=s+1}^{n}\left(\frac{1+y\zeta\zeta_i^{-1}e^{-a}}{1-\zeta\zeta_i^{-1}e^{-a}}\right)\right\}[X(\zeta)], \tag{10}$$

since the tangent bundle $\theta(X(\zeta))$ has Chern class $(1+a)^{s+1}$ and the complex manifold $X(\zeta)$ has dimension s.

But g acts on the bundle $H|X(\zeta)$ as multiplication by ζ (for H is the dual of the normal bundle of X in $P_{n+1}(\mathbb{C})$, and just as in (8) one finds that g acts on the normal bundle by ζ^{-1}), so it follows from definition 2.3 (28) that

$$\begin{aligned} ch(H^k|X(\zeta))\ (g) &= \zeta^k\ ch(H^k|X(\zeta)) \\ &= \zeta^k\ i^*\ ch(H^k) \\ &= \zeta^k\ i^*(e^{-kx}) \\ &= \zeta^k\ e^{-ka}, \end{aligned} \tag{11}$$

where i is the inclusion of $X(\zeta)$ in X. Here we have used 2.3(8), 10.1(35), and the obvious equality $i^* x = a$. Finally, we see from (6) that evaluation on the fundamental class of $X(\zeta)$ corresponds to finding the coefficient of a^s. Therefore (10) becomes

$$\frac{1}{1+y}\ \mathrm{res}_{a=0}\left[\frac{da}{a^{s+1}}\left\{\zeta^k e^{-ka}\ .a^{s+1}\prod_{i=0}^{n}\frac{1+y\zeta\zeta_i^{-1}\ e^{-a}}{1-\zeta\zeta_i^{-1}e^{-a}}\right\}\right]$$

$$= \frac{-1}{1+y}\ \mathrm{res}_{z=\zeta}\left[z^k\ \frac{dz}{z}\prod_{i=0}^{n}\frac{1+y\zeta_i^{-1}\ z}{1-\zeta_i^{-1}\ z}\right], \tag{12}$$

where we have written z for ζe^{-a}. Combining this with (4), we get

<u>THEOREM 1</u>: Let g be the element (1) and k any integer. Then

$$\chi_y(P_n(\mathbb{C}),H^k;g) = \frac{-1}{1+y} \sum_{|\zeta|=1} \operatorname{res}_{z=\zeta} \left[z^k \frac{dz}{z} \prod_{i=0}^{n} \frac{1+y\zeta_i^{-1} z}{1-\zeta_i^{-1} z} \right].$$

Notice that the sum is well-defined since the only ζ giving non-zero residues are those with $\zeta = \zeta_i$ for some i. Theorem 5 of 10.1 is an immediate corollary of (13) if we average over G, using the identity

$$\frac{1}{b} \sum_{\zeta^b=1} \frac{1+y\zeta^{-1}z}{1-\zeta^{-1}z} = \frac{1+yz^b}{1-z^b}. \tag{14}$$

In the same way, from eq. (2) of 3.2 we find that the contribution from the component $X(\zeta)$, of X^g to $\mathcal{L}'(g,X)$ is

$$\mathcal{L}(X(\zeta)) \prod_{\theta\neq 0} \mathcal{L}_\theta(N_\theta^g(\zeta))$$

$$= \left(\frac{a}{\tanh a}\right)^{s+1} \prod_{i=s+1}^{n} \frac{\zeta^{-1}\zeta_i e^{2a}+1}{\zeta^{-1}\zeta_i e^{2a}-1}. \tag{15}$$

Now since $a^r \in H^{2r}(P_s(\mathbb{C}))$ is dual to $[P_{s-r}(\mathbb{C})]$ in $P_s(\mathbb{C})$, and since $[P_{s-r}(\mathbb{C})]$ is dual in $P_n(\mathbb{C})$ to x^{n-s+r}, we deduce from the definition of the Umkehr map that

$$i_!(a^r) = x^{n-s+r}. \qquad \text{(all r)}. \tag{16}$$

Therefore $i_!$ applied to (15) gives

$$x^{n-s}\left(\frac{x}{\tanh x}\right)^{s+1} \prod_{i=s+1}^{n} \frac{\zeta^{-1}\zeta_i e^{2x}+1}{\zeta^{-1}\zeta_i e^{2x}-1}. \tag{17}$$

as the contribution of $X(\zeta)$ to $\mathcal{L}(g,X)$ and therefore

THEOREM 2: Let g be the element (1) and X the standard generator of $H^2(P_n\mathbb{C})$. Then

$$\mathcal{L}(g,P_n(\mathbb{C})) = \sum_{\zeta\in S^1} \prod_{i=0}^{n} \left(x\cdot\frac{\zeta^{-1}\zeta_i e^{2x}+1}{\zeta^{-1}\zeta_i e^{2x}-1}\right). \tag{18}$$

Only finitely many terms in the sum are non-zero, since $x^{n+1} = 0$.

Corollary 1: Theorem 2 of 10.3 is true for even n as well as for n odd.

Proof of corollary: By Theorem 1 of 3.2, the left-hand side of 10.3(14) is equal to the average over $g\in G$ of (18). This average can be performed easily by using identity (14) with $y=1$. Then the substitution $\zeta = e^{-2i\xi}$ in (18) gives the expression on the right-hand side of 10.3(14).

The Corollary to Theorem 2 of 10.3, however, namely the equation $\pi^*p(P_n(\mathbb{C})/G) = \Pi(1+b_k^2x^2)$ for the Pontrjagin class of $P_n(\mathbb{C})/G$ in case the b_k are pairwise coprime, is not in general true for n even. The argument used to prove that corollary again gives

$$\pi^*\mathcal{L}(P_n(\mathbb{C})/G) \equiv \prod_{k=0}^{n} \frac{b_k x}{\tanh b_k x} \quad (\mathrm{mod}\ x^n) , \tag{19}$$

but for n even we no longer know that the coefficient of x^n in $\pi^*\mathcal{L}(P_n(\mathbb{C})/G)$ vanishes and therefore cannot deduce that the two sides of (19) are equal. What we do know is that the signature of $P_n(\mathbb{C})/G$ equals 1, because the action of G embeds in an action of the connected group T^{n+1} and therefore $\mathrm{Sign}(g,P_n(\mathbb{C})) = \mathrm{Sign}(P_n(\mathbb{C})) = 1$

for all $g \in G$. Therefore the equation

$$\pi^* \mathcal{L}(P_n(\mathbb{C})/G) = \prod_{k=0}^{n} \frac{b_k x}{\tanh b_k x} \tag{20}$$

holds if and only if the coefficient of x^n in the product on the right equals $\prod_{k=0}^{n} b_k$. We thus have:

<u>Corollary 2</u>: Let n be even and $b_0,\dots,b_n$ mutually coprime positive integers. Then the formula

$$\pi^* p(P_n(\mathbb{C})/G) = \prod_{k=0}^{n} (1 + b_k^2 x^2) \tag{21}$$

for the Pontrjagin class of $P_n(\mathbb{C})/G$ holds if and only if the integers $b_0,\dots,b_n$ satisfy the Diophantine equation

$$b_0 \dots b_n = \ell_n(b_0,\dots,b_n) \tag{22}$$

considered in §8, where ℓ_n is the polynomial defined by eq. 5.3(21). In particular, for $n=2$ equation (21) holds if and only if (b_0,b_1,b_2) is a Markoff triple, and for $n=4$ we have the solution

$b_0 = 2$, $b_1 = 7$, $b_2 = 19$, $b_3 = 47$, $b_4 = 59$.

As a final remark, we observe that the equation

$$\begin{aligned} &\text{coefficient of } x^n \text{ in } \pi^* \mathcal{L}(P_n(\mathbb{C})/G) \\ &= (\deg \pi) \times \operatorname{Sign}(P_n(\mathbb{C})/G) \\ &= \frac{b_0 \dots b_n}{(b_0,\dots,b_n)} \times 1 \end{aligned} \tag{23}$$

for n even in combination with the formula 10.3(14) for $\pi^* \mathcal{L}(P_n(\mathbb{C})/G)$ provides a "topological proof" of the generalised Rademacher reciprocity which was given without proof as Theorem 3 of 5.2. (Cf. [39], [40].)

§ 11. Brieskorn manifolds

Let $a_o,\dots,a_n \geq 2$ be integers, and

$$X_a = \{z \in \mathbb{C}^{n+1} \mid z_o^{a_o} + \dots + z_n^{a_n} = 0\} \qquad (1)$$

the (singular) Brieskorn variety in $\mathbb{C}^{n+1}$. Write

$$\Sigma_a = X_a \cap S^{2n+1}. \qquad (2)$$

Then Σ_a is a smooth manifold of dimension $2n-1$, and X_a is homeomorphic to the cone on Σ_a. Let N be any common multiple of the numbers a_i, and

$$b_i = N/a_i \qquad (i = 0,\dots,n). \qquad (3)$$

We define an action of S^1 on Σ_a by

$$t\circ(z_o,\dots,z_n) = (t^{b_o}z_o,\dots,t^{b_n}z_n). \qquad (4)$$

We will study this action and the quotient space

$$Z_a = \Sigma_a/S^1$$

in 11.1; in particular, when the integers b_i are coprime, Z_a is a complex manifold and we compute the (integral) cohomology and the Chernclass of Z_a. In 11.2 we calculate the α-invariant $\alpha(t, \Sigma_a)$ of the S^1-action. Finally, in 11.3 we discuss periodicity phenomena in the topology of the Brieskorn varieties; this is related to the periodicity of the numbers $t(a_1,\dots,a_n)$ discussed in 5.5.

11.1 In this section we study the manifold Z_a defined above. The first observation is that

$$Z_a = S/G, \qquad (1)$$

where S and G are defined as in § 10 (eqs. 10.2(1) and 10(2)); namely, S is the hypersurface of degree N in $P_n(\mathbb{C})$ and G is a product of cyclic groups of order b_i, with the obvious action on S. To see this, we write S as

$$S = \{(w_o,\dots,w_n)\in \mathbb{C}^{n+1} - \{0\} \mid w_o^N + \dots + w_n^N = 0\}/\underset{1}{\sim} \tag{2}$$

where $\underset{1}{\sim}$ is the equivalence relation

$$(w_o,\dots,w_n) \underset{1}{\sim} (tw_o,\dots,tw_n) \quad (t\in \mathbb{C}^*). \tag{3}$$

The action of G on S consists of multiplication of w_i by some b_i'th root of unity; hence a point in S/G is described by coordinates $z_i = w_i^{b_i}$ and we have

$$S/G = \{(z_o,\dots,z_n)\in \mathbb{C}^{n+1} - \{0\} \mid z_o^{a_o} + \dots + z_n^{a_n} = 0\}/\underset{2}{\sim} \tag{4}$$

where $\underset{2}{\sim}$ is the equivalence relation

$$(z_o,\dots,z_n)\underset{2}{\sim}(t^{b_o}z_o,\dots,t^{b_n}z_n) \qquad (t\in\mathbb{C}^*). \tag{5}$$

This shows that $S/G = (X_a - \{0\})/\mathbb{C}^* = Z_a$.

We thus have two representations of Z_a: as the quotient of a smooth manifold by a fixed-point free S^1-action, and as the quotient of a smooth manifold by a finite group action. Both representations show that Z_a is a rational homology manifold; we can also use (1) to compute the rank of $H^r(Z_a)$ and the L-class of Z_a. Namely, by using theorem 1 of 10.2 (eq. 10.2 (24)), we find

$$\begin{aligned}\dim_{\mathbb{Q}} H^r(Z_a;\mathbb{Q}) &= \dim_{\mathbb{Q}} H^r(S/G;\mathbb{Q})\\ &= \dim_{\mathbb{Q}} H^r(S;\mathbb{Q})^G\\ &= \sum_{p+q=r} h^{p,q}(S)^G\\ &= \begin{cases}1 & r \text{ even}\\ 0 & r \text{ odd}\end{cases} + \begin{cases}R & r = n-1\\ 0 & r \neq n-1,\end{cases}\end{aligned} \tag{6}$$

where

$$R = \sum_{k=1}^{n} \tilde{N}_k \tag{7}$$

($\tilde{N}_k$ defined in 10.2 (13)). As to the L-class, if

$$p : S \longrightarrow S/G = Z_a \tag{8}$$

denotes the projection, then a calculation like that of 10.4 gives

$$p^*\mathcal{L}(Z_a) = \frac{\deg p}{|G|} \sum_{g\in G} L(g,S)$$

$$= \frac{1}{(b_0,\dots,b_n)} \sum_{\zeta_0^{b_0}=1} \cdots \sum_{\zeta_n^{b_n}=1} \sum_{\zeta\in S^1} \frac{\tanh Nx}{Nx} \prod_{i=0}^{n} \left(x \frac{\zeta^{-1}\zeta_i e^{2x}+1}{\zeta^{-1}\zeta_i e^{2x}-1} \right)$$

$$= \frac{1}{(b_0,\dots,b_n)} \frac{\tanh Nx}{Nx} \sum_{0\leq\xi<\pi} \prod_{i=0}^{n} \frac{b_i x}{\tanh\ b_i(x+i\xi)}, \qquad (9)$$

where $x \in H^2(S)$ is the restriction to S of the standard generator of $H^2(P_n(\mathbb{C}))$. This generalizes Bott's theorem (Theorem 2 of 10.3) and reproves our previous formula (10.3(12)) for the signature of Z_a.

Everything we have said up to now applies for arbitrary exponents a_i. From now on we make the additional assumption that <u>the integers</u> $b_i = N/a_i$ <u>are pairwise coprime</u>. Then the action 11(4) of S^1 on Σ_a is free, so Z_a has the structure of a smooth manifold. We can now give more precise information about the topology of Z_a than in the general case; this is contained in the following three theorems.

<u>Theorem 1</u>: Let $Z_a = \Sigma_a/S^1$ be defined as above, with the integers b_i mutually coprime. Then Z_a has a complex manifold structure such that the map (8) is a ramified covering.

<u>Theorem 2</u>: The cohomology of Z_a is given by

$$H^i(Z_a) = \begin{cases} \mathbb{Z} & i \text{ even} \\ 0 & i \text{ odd} \end{cases} \oplus \begin{cases} H^{n-1}(\Sigma_a) & i = n-1, \\ 0 & i \neq n-1, \end{cases} \qquad (10)$$

and $H^{n-1}(\Sigma_a)$ is free abelian of rank R (R as in (7)). Moreover, if y denotes the integral cohomology class

$$y = -c_1(W) \in H^2(Z_a) \tag{11}$$

with W the complex line bundle associated to the principal S^1-bundle $\Sigma_a \longrightarrow Z_a$, then y^k is a generator of the summand $\mathbb{Z}$ of $H^{2k}(Z_a)$ if $2k \leq n-1$ and is d times a generator of $H^{2k}(Z_a)$ if $2k \geq n$, where

$$d = N/\, b_o \cdots b_n \tag{12}$$

(d is an integer since the b_i are by assumption coprime).

Theorem 3: The Chernclass of Z_a is given by

$$c(Z_a) = (1+Ny)^{-1} \prod_{k=0}^{n} (1 + b_k y). \tag{13}$$

Remark: That Z_a has a natural complex manifold structure when the b_i are coprime was observed by Brieskorn and van de Ven [8]; in fact, they showed that Z_a bears a complex structure (as projective algebraic complex space) for arbitrary exponents a_i, and gave necessary and sufficient conditions on the a_i in order that this structure be non-singular. Later, Neumann [74] showed that these are the only exponents for which Z_a can be given the structure of a differentiable manifold. Although exponents a_k satisfying these conditions need not have coprime b_k's, the manifolds Z_a obtained are always isomorphic to other Z_a for which the b_k are coprime; thus there is no loss of generality in our assumption. As to Theorems 2 and 3, both results were communicated to us by Neumann. However, in our proofs of Theorems 1-3 we follow the exposition of Tetsuro Kawasaki, who was kind enough to place a preprint of his paper "Free S^1-actions on Brieskorn varieties" at our disposal.

Proof of Theorem 1: From (1) we see that Z_a can be considered as the hypersurface of degree N in the twisted projective space $P_n(\mathbb{C})/G$ (quotient of the Bott action); namely, we have a diagram

$$\begin{array}{ccc} S & \overset{i}{\subset} & P_n(\mathbb{C}) \\ \Big\downarrow p & & \Big\downarrow \bar{p} \\ Z_a = S/G & \overset{i}{\subset} & P_n(\mathbb{C})/G . \end{array} \tag{14}$$

We denote by $P_n(\mathbb{C})'$ the space

$$P_n(\mathbb{C})' = P_n(\mathbb{C}) - \{(1:0:\ldots:0), (0:1:\ldots:0),\ldots,(0:\ldots:0:1)\}. \tag{15}$$

We assert that, if the b_i's are coprime, the space $P_n(\mathbb{C})'/G$ is a complex manifold. To show this, one must show that, for any point $w \in P_n(\mathbb{C})'$, the quotient of a neighborhood of w by the isotropy group G_w is a disc. Write

$$w = (w_o:\ldots:w_s:0:\ldots:0) \tag{16}$$

with $w_o,\ldots,w_s \neq 0$, $s \geqslant 1$. Then $\zeta = (\zeta_o,\ldots,\zeta_n)$ is in G_w only if $\zeta_o = \ldots = \zeta_s$, (and since $s \geqslant 1$ and $b_o,\ldots,b_s$ are coprime) this common value must be 1, i.e.

$$G_w = 1\times \ldots \times 1 \times G_{b_{s+1}} \times\ldots\times G_{b_n}. \tag{17}$$

We define a coordinate chart $\mathbf{t}$ from a neighbourhood of w to $\mathbb{C}^n$ by $t_i = w_i/w_o$ $(i=1,\ldots,n)$; then G_w acts by

$$(1,\ldots,1,\zeta_{s+1},\ldots,\zeta_n)\cdot(t_1,\ldots,t_n)$$

$$= (t_1,\ldots,t_s,\zeta_{s+1}t_{s+1},\ldots,\zeta_n t_n), \tag{18}$$

so

$$(u_1,\ldots,u_n) = (t_1,\ldots,t_s, t_{s+1}^{b_{s+1}},\ldots,t_n^{b_n}) \tag{19}$$

is a coordinate chart for $P_n(\mathbb{C})/G$ at the point $\bar{p}(w)$.

Since $S \subset P_n(\mathbb{C})'$, the manifold Z_a is represented by (14) as a subset of the complex manifold $P_n(\mathbb{C})'/G$. At the point $\bar{p}(w) \in P_n(\mathbb{C})'/G$, the equation defining $Z_a = S/G$ is (in terms of the local coordinates (19))

$$1 + u_1^N +\ldots+ u_s^N + u_{s+1}^{a_{s+1}} +\ldots+ u_n^{a_n} = 0. \tag{20}$$

This defines a non-singular algebraic variety in $\mathbb{C}^n$, and it follows that Z_a is a complex submanifold of $P_n(\mathbb{C})'/G$.

Proof of Theorem 2: Let

$$V_a(u) = \{z \in \mathbb{C}^{n+1} \mid z_o^{a_o} + \ldots + z_n^{a_n} = u\} \tag{21}$$

for $u \neq 0$. For u, u' small enough, $V_a(u)$ and $V_a(u')$ are hiholomorphically equivalent complex manifolds; thus we write simply V_a for $V_a(u)$ with u small. We denote by $\overline{V}_a$ the intersection of V_a with the unit disc in $\mathbb{C}^{n+1}$; then $\partial\overline{V}_a$ is diffeomorphic to Σ_a. But it is known (cf. Milnor [ ]) that the 2n-manifold-with-boundary $\overline{V}_a$ is homotopy equivalent to a bouquet of n-spheres. Thus $H^i(\overline{V}_a) = 0$ if $i \neq 0, n$ and $H^{i+1}(\overline{V}_a, \partial\overline{V}_a) = H_{2n-i-1}(\overline{V}_a) = 0$ if $i \neq n-1, 2n-1$. It then follows from the exact sequence of the pair $(\overline{V}_a, \partial\overline{V}_a)$ that

$$H^i(\partial\overline{V}_a) = 0 \quad \text{if} \quad i \neq 0, n-1, n, 2n-1, \tag{22}$$

and that $H^{n-1}(\partial\overline{V}_a)$, $H^n(\partial\overline{V}_a)$ are given by

$$\begin{array}{ccccccccc} 0 \longrightarrow H^{n-1}(\partial\overline{V}_a) \longrightarrow & H^n(\overline{V}_a, \partial\overline{V}_a) & \longrightarrow & H^n(\overline{V}_a) & \longrightarrow H^n(\partial\overline{V}_a) \longrightarrow 0, \\ & \wr\!\wr\!\wr & \overset{\sigma}{\nearrow} & & \\ & H_n(\overline{V}_a) & & & \end{array} \tag{23}$$

i.e.

$$H^{n-1}(\partial\overline{V}_a) \cong \ker \sigma, \quad H^n(\partial\overline{V}_a) \cong \operatorname{coker} \sigma \tag{24}$$

Since $H_n(\overline{V}_a)$ is free abelian, the group $H^{n-1}(\partial\overline{V}_a)$ is also free abelian, say of rank r. Since the intersection matrix σ is known and has been diagonalized (cf. Pham [80], Brieskorn [7]), one can explicitly calculate r and show that

$$\begin{aligned} r &= \#\{j_o, \ldots, j_n \in \mathbb{Z} \mid 0 < j_k < a_k, \quad N \mid j_o b_o + \ldots + j_n b_n\} \\ &= \tilde{N}_1 + \ldots + \tilde{N}_n = R \end{aligned} \tag{25}$$

in the notation of (7) and 10.2 (13). However, we will be able to avoid appealing to the calculation of the matrix of σ by using equation (6).

To obtain the cohomology of Z_a, we use the Gysin sequence of the S^1-bundle $\Sigma_a \longrightarrow Z_a$:

$$\cdots \longrightarrow H^{i-2}(Z_a) \xrightarrow{\cdot y} H^i(Z_a) \longrightarrow H^i(\Sigma_a) \longrightarrow \cdots \quad . \tag{26}$$

It then follows from (22) that multiplication by y gives an isomorphism $H^{i-2}(Z_a) \cong H^i(Z_a)$ for $1 \leq i \leq n-2$ and $n+2 \leq i \leq 2n-2$. Since $H^0(Z_a) = \mathbb{Z}$, $H^{2n-2}(Z_a) = \mathbb{Z}$, we deduce that $H^i(Z_a) = 0$ for i odd, $i \neq n-1$, that $H^{2k}(Z_a) = \mathbb{Z}$ with generator y^k for $0 \leq k < \frac{n-1}{2}$, and that $H^{2k}(Z_a) = \mathbb{Z}$ with y^k equal to d' times a generator for $\frac{n-1}{2} < k \leq n-1$, where d' is independent of k. Taking $k = n-1$, we find

$$\begin{aligned} d' &= \langle y^{n-1}, [Z_a] \rangle \\ &= \langle y^{n-1}, [S/G] \rangle \\ &= \frac{1}{\deg p} \langle y^{n-1}, p_*[S] \rangle \\ &= \frac{1}{\deg p} \langle (p^*y)^{n-1}, [S] \rangle \quad . \end{aligned} \tag{27}$$

Here p is the map (8), and $p^*y = x \in H^2(S)$ (notation as above). Since the hypersurface S represents the homology class in $H_{2n-2}(P_n(\mathbb{C}))$ dual to N times the standard generator of $H^2(P_n(\mathbb{C}))$, we have

$$\langle x^{n-1}, [S] \rangle = N. \tag{28}$$

On the other hand, the group G acts effectively on S, so

$$\deg p = |G| = b_0 \cdots b_n. \tag{29}$$

Therefore,

$$d' = \frac{1}{b_0 \cdots b_n} N = d. \tag{30}$$

The only assertion of Theorem 2 still left to prove is the calculation of the middle cohomology group of Z_a. The middle part of the Gysin sequence is

$$
\begin{array}{ccccccccc}
0 \longrightarrow H^{n-1}(Z_a) \longrightarrow & H^{n-1}(\Sigma_a) & \longrightarrow & H^{n-2}(Z_a) & \xrightarrow{\cdot y} & H^n(Z_a) \\
 & \wr\| & & \wr\| & & \wr\| \\
 & \ker\sigma & & \mathbb{Z} & & \mathbb{Z}
\end{array}
$$

$$
\begin{array}{ccccc}
\longrightarrow & H^n(\Sigma_a) & \longrightarrow & H^{n-1}(Z_a) & \longrightarrow 0 \\
 & \wr\| & & & \\
 & \operatorname{coker}\sigma & & &
\end{array}
$$

if n is even; by what we have already proved, the map $\mathbb{Z} \longrightarrow \mathbb{Z}$ given by $H^{n-2} \xrightarrow{\cdot y} H^n$ sends a generator to d times a generator; therefore

$$H^{n-1}(Z_a) \cong H^{n-1}(\Sigma_a) \cong \mathbb{Z}^r, \tag{31}$$

$$H^n(\Sigma_a) \cong H^{n-1}(Z_a) \oplus \mathbb{Z}_d \cong \mathbb{Z}^r \oplus \mathbb{Z}_d \tag{32}$$

in this case. If n is odd, the relevant part of the Gysin sequence is

$$
\begin{array}{ccccc}
0 \longrightarrow & H^{n-3}(Z_a) & \xrightarrow{\cdot y} H^{n-1}(Z_a) \longrightarrow & H^{n-1}(\Sigma_a) & \longrightarrow 0, \\
 & \wr\| & & \wr\| & \\
 & \mathbb{Z} & & \ker\sigma &
\end{array}
$$

$$
\begin{array}{ccccc}
0 \longrightarrow & H^n(\Sigma_a) & \longrightarrow H^{n-1}(Z_a) \xrightarrow{\cdot y} & H^{n+1}(Z_a) & \longrightarrow 0, \\
 & \wr\| & & \wr\| & \\
 & \operatorname{coker}\sigma & & \mathbb{Z} &
\end{array}
$$

and we deduce that

$$H^{n-1}(Z_a) \cong \mathbb{Z} \oplus H^{n-1}(\Sigma_a) \cong \mathbb{Z} \oplus \mathbb{Z}^r \tag{33}$$

with the first factor generated by $y^{\frac{n-1}{2}}$, and that

$$H^n(\Sigma_a) = H^{n-1}(\Sigma_a). \tag{34}$$

Comparing equations (31) and (33) with our previous calculation of the rank of $H^{n-1}(Z_a)$ for arbitrary exponents a_i (equation (6)) we deduce that $r = R$. This completes the proof of Theorem 2, as well as providing a complete evaluation of the groups $H^i(\Sigma_a;\mathbb{Z})$.

Proof of Theorem 3: We will in fact show that the (stable) complex tangent bundle of Z_a is given by

$$\theta(Z_a) \oplus W^{-N} \oplus 1 = W^{-b_o} \oplus \dots \oplus W^{-b_n}, \tag{35}$$

where 1 denotes the trivial complex line bundle and W^k the k^{th} power of the line bundle W associated to $\Sigma_a \longrightarrow Z_a$. This clearly implies equation (13).

To prove (35) we use the embedding of Z_a in $P_n(\mathbb{C})'/G$ described in the proof of Theorem 1. We can describe the quotient space $P_n(\mathbb{C})/G$ alternatively as the quotient space

$$P_n(\mathbb{C})/G = S^{2n+1}/S^1 \tag{36}$$

where S^1 acts on $S^{2n+1} \subset \mathbb{C}^{n+1}$ by

$$t \circ (u_o,\dots,u_n) = (t^{b_o}u_o,\dots,t^{b_n}u_n). \tag{37}$$

The orbit through a point $u \in S^{2n+1}$ is thus $\{(e^{ib_o\theta}u_o,\dots,e^{ib_n\theta}u_n) | e^{i\theta} \in S^1\}$, and

$$\frac{d}{d\theta}(e^{ib_o\theta}u_o,\dots,e^{ib_n\theta}u_n)\Big|_{\theta=0} = (ib_ou_o,\dots,ib_nu_n). \tag{38}$$

Hence the tangent space to the orbit at u is spanned by the vector $B_u = (ib_ou_o,\dots,ib_nu_n)$, and the slice at u of the action is the orthogonal complement. If

$$u \in (S^{2n+1})' = \{(u_o,\dots,u_n) \in S^{2n+1} | \text{ at least two } u_j \neq 0\}. \tag{39}$$

then the action of S^1 at u is free and we obtain

$$\theta(P_n(\mathbb{C})'/G) = \{(u,v) \in (S^{2n+1})' \times \mathbb{C}^{n+1} \mid v \perp B_u\}/S^1, \tag{40}$$

where S^1 acts by

$$\lambda \circ (u,v) = (\lambda \circ u, \lambda \circ v) \qquad (\lambda \in S^1). \tag{41}$$

We define a line bundle η_k $(k \in \mathbb{Z})$ on $P_n(\mathbb{C})'/G$ by

$$\eta_k = ((S^{2n+1})' \times \mathbb{C})/S^1 \tag{42}$$

where S^1 acts by

$$\lambda \circ (u,t) = (\lambda \circ u, \lambda^k t) \qquad (\lambda \in S^1). \tag{43}$$

Then

$$\eta_{b_o} \oplus \dots \oplus \eta_{b_n} = ((S^{2n+1})' \times \mathbb{C}^{n+1})/S^1 \tag{44}$$

where S^1 acts by (41). Thus $\theta(P_n(\mathbb{C})'/G)$ is a subbundle of $\eta_{b_o} \oplus \dots \oplus \eta_{b_n}$, and its complement is the line bundle

$$\{(u, tB_u) \mid u \in (S^{2n+1})', \ t \in \mathbb{C}\}/S^1. \tag{45}$$

This bundle is clearly trivial since it has a non-degenerate cross-section $u \longmapsto (u, B_u)$ (this is well-defined because B and the action commute), so

$$\theta(P_n(\mathbb{C})'/G) \oplus 1 = \eta_{b_o} \oplus \dots \oplus \eta_{b_n}. \tag{46}$$

We now write i for the inclusion of Z_a in $P_n(\mathbb{C})'/G$ and ν for the normal bundle. We assert that

$$\nu \cong i^* \eta_N. \tag{47}$$

This will imply (35), for it is clear that

$$i^* \eta_k = W^{-k} \quad (k \in \mathbb{Z}). \tag{48}$$

To prove (47), we define an open covering $\bigcup_{i=0}^{n} U_i$ of $P_n(\mathbb{C})'/G$ by

$$U_i = \{(u_o : \dots : u_n) \in P_n(\mathbb{C})' \mid u_i \neq 0 \}/G. \tag{49}$$

On the intersection of U_i and a tubular neighbourhood of Z_a we have the trivialization of ν given by

$$f_i : \bar{p}(u_o : \dots : 1 : \dots : u_n) \longmapsto u_o^N + \dots + 1 + \dots + u_n^N \tag{50}$$

($\bar{p}$ denotes the projection from $P_n(\mathbb{C})$ to $P_n(\mathbb{C})/G$). Then on $U_o \cap U_1$,

$$\bar{p}(1 : u_1 : \dots : u_n) \xrightarrow{\quad f_o \quad} 1 + u_1^N + u_2^N + \dots + u_n^N$$

$$\|$$

$$\bar{p}(u_1^{-1} : 1 : u_1^{-1} u_2 : \dots u_1^{-1} u_n) \xrightarrow{\ f_1\ } u_1^{-N} + 1 + u_1^{-N} u_2^N + \dots + u_1^{-N} u_n^N.$$

That is, we have the coordinate transformation

$$f_1 = u_1^{-N} f_o \tag{51}$$

(note that this makes sense because the ambiguity of u_1 lies in the group of N^{th} roots of unity). On the other hand, for the bundle η_N we have the trivialization

$$g_i : \pi((z_o,\ldots,1,\ldots,z_n),t) \longmapsto t \tag{52}$$

on U_i (where $\pi : (S^{2n+1})' \times \mathbb{C} \longrightarrow \eta_N$ is the projection onto the quotient), and on $U_o \cap U_1$

$$\pi((1, u_1^{b_1},\ldots, u_n^{b_n}), t) \xrightarrow{\ g_o\ } t$$
$$\parallel$$
$$\pi((u_1^{-b_o},1, u_1^{-b_2} u_2^{b_2},\ldots,u_1^{-b_n} u_n^{b_n}), u_1^{-N} t) \xrightarrow{\ g_1\ } u_1^{-N} t,$$

i.e.

$$g_1 = u_1^{-N} g_o. \tag{53}$$

Comparing (51) and (53) shows that the line bundles ν and $i^*\eta_N$ are isomorphic. This completes the proof of Theorem 3.

An immediate corollary of Theorem 3 is that the L-class of Z_a (in the case of a free action of S^1 on Σ_a) is given by

$$\mathcal{L}(Z_a) = \frac{\tanh Ny}{Ny} \prod_{k=0}^{n} \frac{b_k y}{\tanh b_k y}. \tag{54}$$

This agrees with our previous calculation of $L(Z_a)$ for arbitrary a_k's (eq. (9)), since, if the b_i's are relatively prime, then $(b_o,\ldots,b_n) = 1$ and

$$\frac{\tanh Nx}{Nx} \prod_{k=0}^{n} \frac{b_k x}{\tanh b_k(x+i\xi)} = 0$$

for $\xi \neq 0$ since then at most one of the numbers $\tanh (ib_k\xi)$ can vanish, while $x^n = 0$ (compare the proof of the corollary to Theorem 2 of 10.3).

A further consequence of Theorem 3 is that we can calculate the χ_y-characteristic of Z_a with coefficients in W^k $(k \in \mathbb{Z})$. Indeed, by the Riemann-Roch Theorem (eq. 2.3 (17)),

$$\chi_\lambda(Z_a, W^k) = \{e^{-ky} \tilde{T}_\lambda(\theta(Z_a))\}[Z_a] \tag{55}$$

(we write χ_λ rather than χ_y to avoid confusion with $y \in H^2(Z_a)$)
But it follows from Theorem 3 that

$$\tilde{T}_\lambda(\theta(Z_a)) = \frac{1}{1+\lambda} \cdot \frac{1-e^{-Ny}}{Ny(1+\lambda e^{-Ny})} \prod_{i=0}^{n} \frac{b_k y(1+\lambda e^{-b_i y})}{1-e^{-b_i y}} . \tag{56}$$

Using

$$y^r[Z_a] = 0 \quad (r \neq n-1), \quad y^{n-1}[Z_a] = d = N/b_0 \ldots b_n \tag{57}$$

(cf. (30)), we find

$$\begin{aligned} \chi_\lambda(Z_a, W^k) &= d \times \text{coefficient of } y^{n-1} \text{ in } e^{-ky}\tilde{T}_y(\theta(Z_a)) \\ &= \frac{1}{1+\lambda} \operatorname{res}_{y=0}\left[e^{-ky} \frac{1-e^{-Ny}}{1+\lambda e^{-Ny}} \prod_{i=0}^{n} \frac{1+\lambda e^{-b_i y}}{1-e^{-b_i y}} \, dy\right] \\ &= -\frac{1}{1+\lambda} \operatorname{res}_{x=1}\left[\frac{1-x^N}{1+\lambda x^N} \prod_{i=0}^{n} \frac{1+\lambda x^{b_i}}{1-x^{b_i}} \, x^{k-1} dx\right] \end{aligned} \tag{58}$$

Since the b_k's are coprime, $x^{b_k} = x^{b_\ell} = 1$ implies $x = 1$; thus all poles of $\prod_i \frac{1+\lambda x^{b_i}}{1-x^{b_i}}$ other than $x = 1$ are simple poles occurring for x an N^{th} root of unity, and hence are cancelled by the factor $1-x^N$ in the numerator of (58). Thus the function in brackets in (58) has, besides the pole of order n at $x = 1$, poles only at $x = 0$, $x = \infty$ and the roots of $1 + \lambda x^N = 0$, and we obtain

$$\begin{aligned} \chi_\lambda(Z_a, W_k) &= \frac{1}{1+\lambda} \left(\operatorname{res}_{x=0} + \operatorname{res}_{x=1} + \sum_{1+\lambda x^N=0} \operatorname{res}_x\right)(\ldots) \\ &= \frac{1}{1+\lambda} \cdot \left(\text{coefficient of } x^{-k} \text{ in } \frac{1-x^N}{1+\lambda x^N} \prod_{i=0}^{n} \frac{1+\lambda x^{b_i}}{1-x^{b_i}}\right) \end{aligned}$$

$$+ \frac{1}{1+\lambda} \text{ (coefficient of } x^k \text{ in } \frac{1-x^N}{1+\lambda x^N} \prod_{i=0}^{n} \frac{1+\lambda x^{b_i}}{1-x^{b_i}})$$

$$- \frac{1}{N\lambda} \sum_{\rho^N=-\lambda^{-1}} \rho^k \prod_{i=0}^{n} \frac{1+\lambda\rho^{b_i}}{1-\rho^{b_i}} \quad . \tag{59}$$

The first term is 0 if $k > 0$, the second if $k < 0$. For $k=0$, we find

$$\chi_y(Z_a) = \frac{1-(-y)^n}{1+y} - \frac{y^{-1}}{N} \sum_{1+y\rho^N=0} \prod_{k=0}^{n} \frac{1+y\rho^{b_k}}{1-\rho^{b_k}} \quad . \tag{60}$$

This agrees with equation 10.2 (7), since

$$\chi_y(Z_a) = \chi_y(S/G) = \chi_y(S)^G. \tag{61}$$

11.2 In this section we discuss the α-invariant of the S^1-action on Σ_a. We first suppose, as in 11.1, that the integers b_i are mutually coprime, so that the action is free. Then, by Theorem 1 of 4.3,

$$\alpha(t, \Sigma_a) = \text{sign } f - \left\langle \frac{te^{2y}+1}{te^{2y}-1} \cdot \mathcal{L}(Z_a), [Z_a] \right\rangle, \tag{1}$$

where

$$f : H^{n-2}(Z_a) \times H^{n-2}(Z_a) \longrightarrow \mathbb{Z}$$

is the form

$$f(\alpha, \beta) = \langle \alpha\beta y, [Z_a] \rangle . \tag{2}$$

By Theorem 2 of 11.1, $H^{n-2}(Z_a)$ is $\mathbb{Z}$ if n is even and 0 if n is odd, and is generated by $y^{\frac{n-2}{2}}$ in the former case. Since then

$$f(y^{\frac{n-2}{2}}, y^{\frac{n-2}{2}}) = \langle y^{n-1}, [Z_a] \rangle = d > 0, \tag{3}$$

we have sign $f = 1$ for n even, i.e.

$$\text{sign } f = \frac{1+(-1)^n}{2} . \tag{4}$$

Now, by virtue of (54) and (57) of 11.1,

$$\frac{te^{2y}+1}{te^{2y}-1}\ \mathcal{L}(Z_a),[Z_a]\rangle = \operatorname{res}_{y=0}\Big(\frac{te^{2y}+1}{te^{2y}-1}\cdot\tanh Ny\prod_{k=0}^{n}\coth b_k y\cdot dy\Big)$$

$$= \operatorname{res}_{z=1}\Big(\frac{tz+1}{tz-1}\cdot\frac{z^N-1}{z^N+1}\prod_{k=0}^{n}\frac{z^{b_k}+1}{z^{b_k}-1}\ \frac{dz}{2z}\Big). \qquad (5)$$

Notice that the function in (5) has residue $-\frac{1}{2}$ at $z=\infty$ and $\frac{(-1)^{n+1}}{2}$ at $z=0$; thus from (1), (4), (5) we get

$$\alpha(t,\Sigma_a) = -(\operatorname{res}_{z=0}+\operatorname{res}_{z=\infty}+\operatorname{res}_{z=1})\ (\ldots.)$$

$$= (\operatorname{res}_{z=t^{-1}}+\sum_{\zeta^N=-1}\operatorname{res}_{z=\zeta})\ (\ldots.)$$

$$= \frac{1-t^N}{1+t^N}\prod_{k=0}^{n}\frac{1+t^{b_k}}{1-t^{b_k}}+\frac{1}{N}\sum_{\zeta^N=-1}\frac{t\zeta+1}{t\zeta-1}\cdot\prod_{k=0}^{n}\frac{\zeta^{b_k}+1}{\zeta^{b_k}-1}\ . \qquad (6)$$

It is clear from this expression that the only pole of $\alpha(t,\Sigma_a)$ is $t=1$, which is as it should be for a free S^1-action.

If we do not make the assumption that the b_i's are pairwise coprime, then the S^1-action on Σ_a is still fixed-point free, so the α-invariant is still well-defined. Formula (6) is presumably valid in this more general situation. In any case, the function on the right-hand side of (6) has poles precisely for those values of t which have a non-empty fixed-point set on Σ_a, namely all t with $t^{b_i}=t^{b_j}=1$ for some $i\neq j$. Furthermore, (6) is compatible with Theorem 3 of 4.3, as the reader can check using formula 11.1(9) for the L-class of Z_a. Finally, (6) is valid for $t^N=1$ (provided t acts freely on Σ_a; otherwise both sides have a pole at t). To see this, we observe that such t operate on the manifold $\bar V_a$ defined in the proof of Theorem 2 of 11.1, and that the diffeomorphism

$$\Sigma_a \cong \partial\bar V_a \qquad (7)$$

is equivariant with respect to this action. Thus for $t \in G_N \subset S^1$ we can use (7) rather than $\Sigma_a = \partial(\Sigma_a \times_{S^1} D^2)$ to compute the α-invariant. The calculation is given in § 18 of [98] and agrees with (6).

11.3 In section 10.2, we considered the Brieskorn variety

$$V_a = \{(z_o,\ldots,z_n) \in \mathbb{C}^{n+1} \mid z_o^{a_o} + \ldots + z_n^{a_n} = 1\} \tag{1}$$

and showed (eq. 10.2 (32)) that its signature is given by

$$\text{Sign } V_a = t(a). \tag{2}$$

Here $t(a) = t(a_o,\ldots,a_n)$ is the function studied in § 5, namely

$$t(a) = -\frac{1}{N} \sum_{z^N=-1} \frac{1+z}{1-z} \sum_{k=0}^{n} \frac{1+z^{N/a_k}}{1-z^{N/a_k}} \tag{3}$$

(where N is any common multiple of $a_o,\ldots,a_n$) or equivalently

$$t(a) = t^{odd} - t^{even}$$

$$t^{odd(even)} = \{(j_o,\ldots,j_n) \in \mathbb{Z}^{n+1} \mid 0 < j_k < a_k,$$
$$r < \frac{j_o}{a_o} + \ldots + \frac{j_n}{a_n} < r+1 \text{ for some odd(even) integer } r\}. \tag{4}$$

Our proof yielded the trigonometric formula (3) for Sign V_a, whereas Brieskorn's calculation [7] produces the expression (4); in fact, Brieskorn explicitly computes the intersection pairing on $H^n(V_a)$ and finds its number of positive (resp. negative) eigenvalues to be precisely t^{odd} (resp. t^{even}).

When we studied the number-theoretical properties of $t(a)$ in § 5, we proved the following two results:

$$t(a_o,a_1,\ldots,a_n, 2,2) = -t(a_o,\ldots,a_n), \tag{5}$$

$$t(a_o + M,a_1,\ldots,a_n) - t(a_o,\ldots,a_n) = Md(a_1,\ldots,a_n), \tag{6}$$

where, in formula (6), M denotes any common multiple of

$a_1,\dots,a_n$ and $d(a_1,\dots,a_n) \in \mathbb{Q}$ is independent of a_o and of M.

In this section, we will try to see whether these number-theoretical statements are merely accidental properties of the function $t(a)$ or whether they reflect properties of the topological situation. Thus we will be interested

1) in the relationship between the varieties

$$V_{a_o,\dots,a_n} \quad \text{and} \quad V_{a_o,\dots,a_n,2,2} \quad (\text{"suspension"});$$

and

2) in the relationship between $V_{a_o,a_1,\dots,a_n}$ and $V_{a'_o,a_1,\dots,a_n}$ where $a'_o \equiv a_o \pmod{a_i}$ for $1 \leq i \leq n$ ("periodicity").

To discuss these questions, we will put them into the framework of the theory of singularities on complex hypersurfaces, whose main results we now briefly review. For a complete exposition, Milnor's book [72] is extremely highly recommended.

Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ be a polynomial in n variables with $f(0) = 0$. We will always use the following notations:

$$D_\varepsilon = D_\varepsilon^{2n} = \{z \in \mathbb{C}^n : \|z\| < \varepsilon\}, \tag{7}$$

$$S_\varepsilon = S_\varepsilon^{2n-1} = \partial D_\varepsilon = \{z \in \mathbb{C}^n : \|z\| = \varepsilon\}, \tag{8}$$

$$K = f^{-1}(0) \cap S_\varepsilon, \tag{9}$$

$$V = f^{-1}(\delta) \cap \operatorname{int} D_\varepsilon, \tag{10}$$

$$\bar{V} = f^{-1}(\delta) \cap D_\varepsilon. \tag{11}$$

Here $\varepsilon > 0$ is a sufficiently small positive real number (the homotopy type of K is then independent of ε) and, once ε is chosen, $\delta \neq 0$ is a sufficiently small complex number (then V is a smooth open $(2n-2)$-manifold whose diffeomorphism type is independent of δ).

The map

$$\phi : S_\varepsilon - K \longrightarrow S^1 \tag{12}$$

defined by

$$\phi(z) = f(z)/|f(z)| \tag{13}$$

is the projection of a smooth fibre bundle called the <u>Milnor fibering</u>; each fibre $F_\theta = \phi^{-1}(e^{i\theta})$ is a parallelizable (2n-2)-manifold diffeomorphic to V. If we choose some continuous one-parameter family of maps $\alpha_\theta : F_o \longrightarrow F_\theta$ with α_o homotopic to the identity, then the map $\alpha = \alpha_{2\pi}: F_o \longrightarrow F_o$, called a <u>characteristic map</u> for f, has a well-defined homotopy type. Thus the induced map

$$\alpha_* : H_*(F_o; \mathbb{Z}) \longrightarrow H_*(F_o;\mathbb{Z}), \tag{14}$$

called the <u>monodromy map</u> of f, is well-defined. (We sometimes use the homotopy equivalences $F_o \sim V \sim \bar{V}$ to consider α_* as an automorphism of H_*V or $H_*\bar{V}$). The space K is (n-3)-connected.

If 0 is an isolated critical point of f, we can say much more. In this case K is a smooth (2n-3)-manifold and $\bar{V}$ is an (n-2)-connected (2n-2)-manifold with boundary diffeomorphic to K, and has the homotopy type of a bouquet of (n-1)-spheres. Then $H_*(V)$ is non-trivial only in dimension n-1, and we define

$$\Delta(t) = \det(I - t\alpha_* : \tilde{H}_{n-1}(V;\mathbb{Q}) \longrightarrow \tilde{H}_{n-1}(V;\mathbb{Q})). \tag{15}$$

The polynomial $\Delta(t) \in \mathbb{Q}[t]$, called the <u>Alexander polynomial</u> of f, is an important invariant. For instance,

$$\begin{aligned} \Delta(1) \neq 0 &\Longleftrightarrow K \text{ is a rational homology sphere} \\ &\Longleftrightarrow f^{-1}(0) \text{ is a rational homology manifold} \end{aligned} \tag{16}$$

and, if $n \neq 3$,

$$\Delta(1) = \pm 1 \iff K \text{ is a topological sphere} \iff f^{-1}(0) \text{ is a topological manifold.} \qquad (17)$$

In the latter case, the differentiable structure (exotic or standard) on K is determined by the Arf invariant or signature of $\bar{V}$, depending whether n is even or odd; if n is even, then K is the standard sphere (and hence $f^{-1}(0)$ can be given the structure of a smooth manifold) iff $\Delta(-1) \equiv \pm 1 \pmod 8$, while if n is odd, K is the standard sphere iff Sign V is divisible by a certain integer depending on n (namely 8 times the order of the finite cyclic group bP_{2n-2}).

Having given this background material, we can now discuss for general hypersurface singularities the questions posed at the beginning of this section for the Brieskorn polynomials.

Suspension: The question is what relationship exists between the varieties V and K for the polynomial $f(z_1,\dots,z_n)$ and the corresponding varieties defined by the polynomial $f(z_1,\dots,z_n) + z_{n+1}^2 + z_{n+2}^2$.

More generally, we consider two polynomials

$$f' : \mathbb{C}^n \longrightarrow \mathbb{C}, \qquad f'' : \mathbb{C}^m \longrightarrow \mathbb{C} \qquad (18)$$

and define

$$f : \mathbb{C}^{n+m} = \mathbb{C}^n \times \mathbb{C}^m \longrightarrow \mathbb{C} \qquad (19)$$

by

$$f(z,w) = f'(z) + f''(w) \qquad (z \in \mathbb{C}^n, \ w^{o} \in \mathbb{C}^m). \qquad (20)$$

Then we can ask for the relationship between the fibres of the Milnor fiberings of f', f" and f. The answer is provided by Join Theorem (Sebastiani-Thom [88] for isolated singularities; Oka [76] for weighted homogeneous polynomials; Sakamoto [87] in general case). Let $f(z,w) = f'(z) + f''(w)$ as above be a sum of polynomials in disjoint sets of variables. Let F', F", F be the(homotopy types of) the fibres of the Milnor fiberings for f', f'' and f, and $\alpha', \alpha'', \alpha$ the corresponding characteristic maps. Then F is homotopy-equivalent to the join of F' and F'' and the homotopy equivalence $r : F' * F'' \longrightarrow F$ can be chosen so as to make the diagram

$$\begin{array}{ccc} F' * F'' & \xrightarrow{\quad r \quad} & F \\ {\scriptstyle \alpha' * \alpha''} \downarrow & & \downarrow {\scriptstyle \alpha} \\ F' * F'' & \xrightarrow{\quad r \quad} & F \end{array}$$

commute.

<u>Corollary</u>: If f' and f'' both have isolated singularities at the origin (in which case f also does), then the Alexander polynomials Δ', Δ'' and Δ of f', f'', f are related by

$$\Delta(-t) = \Delta'(-t) \boxtimes \Delta''(-t), \tag{21}$$

where the operation $\boxtimes$ on polynomials is the one described in Seotion 2.2.

<u>Example 1:</u> The map

$$f : \mathbb{C} \longrightarrow \mathbb{C}, \qquad f(z) = z^a, \qquad (22$$

where $a \geqslant 2$ is an integer, clearly has a variety V consisting of a points, i.e. of the homotopy type of the group G_a of a^{th} roots of unity, and Alexander polynomial

$$\Delta(t) = \frac{1-t^a}{1-t} = \prod_{\substack{\omega^a=1 \\ \omega\neq 1}} (1 - \omega t). \tag{23}$$

Thus by iterating the theorem, we deduce that the variety $V = V_a$ of the Brieskorn polynomial

$$f(z_1,\ldots,z_n) = z_1^{a_1} + \ldots + z_n^{a_n} \quad (a_i \geqslant 2) \tag{24}$$

has the homotopy type of $G_{a_1} * \ldots * G_{a}$, and that the Alexander polynomial is given by

$$\Delta(t) = \prod_{\substack{\omega_1^{a_1}=1 \\ \omega_1 \neq 1}} \cdots \prod_{\substack{\omega_n^{a_n}=1 \\ \omega_n \neq 1}} (1 - \omega_1 \cdots \omega_n t) \tag{25}$$

Example 2: Taking $a = 2$ in (23), we see that the map $f''(w) = w^2$ has Alexander polynomial $\Delta''(t) = 1+t$. Here $V'' = f''^{-1}(\delta) \cap D_\varepsilon = \{w : w^2 = \delta\}$ consists of two points, so $F'' \sim V'' \sim S^0$ and $F \sim F' * F'' \sim S^0 * F' = \Sigma F'$. Thus the variety V for a polynomial of the form $f(z_1,\ldots,z_{n+1}) = f'(z_1,\ldots,z_n) + z_{n+1}^2$ is homotopy equivalent to the suspension of the corresponding variety for the polynomial f', and the Alexander polynomials of f and f' are related by $\Delta(t) = \Delta'(-t)$. If we iterate this, taking $f''(w_1,w_2) = w_1^2 + w_2^2$, we find: if

$$f(z_1,\ldots,z_n,z_{n+1}, z_{n+2}) = f'(z_1,\ldots,z_n) + z_{n+1}^2 + z_{n+2}^2, \tag{26}$$

then the varieties $\bar{V} = f^{-1}(\delta) \cap D_\varepsilon$ and $\bar{V}' = f'^{-1}(\delta) \cap D_\varepsilon$ are related by

$$\bar{V} \sim \Sigma^2 \bar{V}' \tag{27}$$

(double suspension), and the Alexander polynomials of f' and f are equal:

$$\Delta(t) = \Delta'(t). \tag{28}$$

Unfortunately, in studying the relationship between the manifolds with-boundary $\bar{V}'$ and $\bar{V}$ (of dimension $2n-2$ and $2n+2$, respectively), one has to use homotopy equivalences between (non-closed!) manifolds of different dimensions. Thus all information about intersection numbers is lost, and we cannot show that

$$\text{Sign } \bar{V} = -\text{Sign } \bar{V}', \tag{29}$$

although presumably, in view of the special case (5), this equation is true generally for isolated hypersurface singularities.

Periodicity: We now turn to periodicity phenomena. Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ be any polynomial with an isolated critical point at the origin. For g a positive integer, define $f_g : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ by

$$f_g(z_o, z_1, \ldots, z_n) = z_o^g + f(z_1, \ldots, z_n). \tag{30}$$

We distinguish the varieties and invariants by a subscript g; thus: V_g, K_g, Δ_g . We want to investigate any properties of the topology of V_g or K_g which show a periodicity with respect to g.

What is a natural period? In the case of the Brieskorn polynomial (24), the period is the least common multiple of the a_i, and this is precisely the order of the monodromy map α_* of the polynomial. In general, the monodromy map α_* for a polynomial with isolated singularity at 0 need not be periodic, but it will have the property that for some positive integer N, $\alpha_*^N - I$ is nilpotent. The smallest such N is called the monodromy period of f. An equivalent definition is that N is the least common multiple of the orders of the eigenvalues of α_* (=roots of $\Delta(t)$), which are all roots of unity.

(Note: In many cases, the monodromy map actually is periodic. Thus, if f is a weighted homogeneous polynomial, i.e. if there exist positive integers $q_1, \ldots, q_n$ and N such that $q_1 i_1 + \ldots + q_n i_n = N$ for every monomial $z_1^{i_1} \ldots z_n^{i_n}$ in f, then for $\alpha_\theta : F_o \to F_\theta$ we can take the map

$$(z_1, \ldots, z_n) \longmapsto (e^{iq_1\theta/N} z_1, \ldots, e^{iq_n\theta/N} z_n);$$

then $\alpha : F_o \longrightarrow F_o$ is $(z_1, \ldots, z_n) \longmapsto (\zeta^{q_1} z_1, \ldots, \zeta^{q_n} z_n)$ with

$\zeta = e^{2\pi i/N}$, and clearly $\alpha^N = 1$. More generally, if $f = f_o + f_1$ with f_o weighted homogeneous as above and also having an isolated singularity at the origin and f_1 of smaller order at the origin, i.e. f_1 a sum of monomials $c z_1^{i_1} \ldots z_n^{i_n}$ with $q_1 i_1 + \ldots + q_n i_n < N$, then Oka [77] has shown that f and f_o have homotopy equivalent fibres and the same monodromy maps, so that $\alpha_*^N = \mathrm{id}$ for f also.)

We can now state a theorem about singularities which generalizes the number-theoretical assertion (6).

<u>Theorem</u> (W. Neumann [75]). Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}$, $f(0) = 0$ be a polynomial with an isolated critical point at the origin and N its monodromy period. For any positive integer g, define f_g by (30) and let $\bar{V}_g = f_g^{-1}(\delta) \cap D_\varepsilon^{2n+2}$ be the corresponding $2n$-manifold with boundary. Then the function

$$\tau(g) = \operatorname{Sign} \bar{V}_g \tag{31}$$

is the sum of a linear and a periodic function, i.e.

$$\tau(g) = \frac{8\lambda}{N} g + \tau'(g) \tag{32}$$

with $\lambda \in \mathbb{Z}$ independent of g and $\tau'(g)$ a function of g (mod N).

This theorem follows from a general theorem (also due to Neumann) on the non-multiplicativity of the signature for finite coverings of manifolds with boundary.

Neumann's theorem tells us that the number $\operatorname{Sign}(\bar{V}_g)$ shows a periodicity phenomenon as a function of g. We can ask whether there is any periodicity in the homology, homotopy type, or diffeomorphism type of the "neighbourhood boundary"

$$K_g = f_g^{-1}(0) \cap S_\varepsilon^{2n+1} \tag{33}$$

of f_g. We first have

<u>Theorem</u>: Let $F = F_o$ be the fibre of the Milnor fibering of f, and K_g as in (33) the neighbourhood boundary for $f_g = f(z_1,\ldots,z_n) + z_o^g$. Then there is an exact sequence

$$H_{n-1}(F) \xrightarrow{1+\alpha_*+\ldots+\alpha_*^{g-1}} H_{n-1}(F) \xrightarrow{i_*} H_{n-1}(K_g) \longrightarrow 0 , \qquad (34)$$

where $\alpha_*: H_{n-1}(F) \longrightarrow H_{n-1}(F)$ is the monodromy map.

<u>Corollary</u> : If K is a homotopy sphere, then the homology group $H_{n-1}(K_g)$ only depends on $\pm g$ (mod N), where N is the monodromy period of f.

This theorem was stated by Durfee for f equal to the Brieskorn polynomial (24) ([26], Theorem 11.6). The proof uses the fact that $K_g^{2n-1} - K^{2n-3}$ is (in an obvious way) a g-fold covering of $S_\varepsilon^{2n-1} - K^{2n-3}$ (i.e. K_g is a g-fold branched covering of S_ε^{2n-1} with branching locus K). We have a map $\psi : K_g - K \longrightarrow S^1$ given by

$$\psi(z_o,z_1,\ldots,z_n) = z_o /|z_o| , \qquad (35)$$

and a commutative diagram

$$\begin{array}{ccc} K_g - K & \xrightarrow{\psi} & S^1 \\ {\scriptstyle (z_o,\ldots,z_n)\mapsto (z_1,\ldots,z_n)} \downarrow & & \downarrow {\scriptstyle z \mapsto z^g} \\ S_\varepsilon - K & \xrightarrow{\phi} & S^1 \end{array} \qquad (36)$$

whose vertical maps are g-fold coverings and whose horizontal maps are fibre bundles with fibre F. The theorem is now proved by the methods of branched cyclic covers.

To prove the corollary, we first write down the Wang sequence of ϕ,

$$0 \to H_n(S_\varepsilon - K) \to H_{n-1}F \xrightarrow{1-\alpha_*} H_{n-1}F \to H_{n-1}(S_\varepsilon - K) \to 0 \qquad (37)$$

$$H_n(S_\varepsilon - K) \cong H_n(K), \qquad H_{n-1}(S_\varepsilon - K) \cong H_{n-2}(K)$$

(we have used Alexander and Poincaré duality to get $H_i(S_\varepsilon - K) \cong H^{2n-2-i}(K) \cong H_{i-1}(K)$). Thus K is a homology sphere iff $1 - \alpha_*$ is invertible (i.e. iff 1 is not an eigenvalue of α_*, i.e. iff $\Delta(1) \neq 0$; cf. (16)). Then $\alpha_*^N = 1$ implies

$$1 + \alpha_* + \ldots + \alpha_*^{N-1} = 0, \qquad (38)$$

from which it follows that the map

$$\alpha(g) = 1 + \alpha_* + \ldots + \alpha_*^{g-1} \qquad (39)$$

only depends on $g \pmod N$ and that $\alpha(N-g) = -\alpha(g)$. Since by the theorem

$$H_{n-1}(K_g) \cong \text{Coker } \alpha(g), \qquad (40)$$

this implies the corollary.

Note that the theorem itself follows from the Wang sequence of ψ in case K is a homotopy sphere. For, since the characteristic map of the fibre bundle ψ is clearly α^g, the Wang sequence for ψ is

$$H_{n-1}(F) \xrightarrow{1-\alpha_*^g} H_{n-1}(F) \to H_{n-1}(K_g - K) \to 0, \qquad (41)$$

and if K is a homotopy sphere then $H_{n-1}(K_g - K) \cong H_{n-1}(K_g)$ and $1-\alpha_*$ is invertible, so

$$H_{n-1}(K_g) \cong \text{Coker } (1-\alpha_*^g) \cong \text{Coker } \alpha(g). \qquad (42)$$

It follows from another theorem of Durfee ([26], Theorem 8.1) that (for n even, $n \geqslant 4$) the diffeomorphism type of K_g only depends on $H_{n-1}(K_g)$, Sign $\overline{V}_g$, and the quadratic form on $H_{n-1}(K_g)$. If K is a homotopy sphere, then we have shown that the first two show a periodicity in g. If one could show that the quadratic form on $H_{n-1}(K_g)$ also depends only on $g \pmod N$, we would obtain from Durfee's theorem the result that

$$K_{g+N} \approx K_g \# \lambda \Sigma_8 \tag{43}$$

(K_{g+N} is diffeomorphic to the connected sum of K_g and λ copies of the Milnor generator Σ_8 of bP_{2n}), where $\lambda \in \mathbb{Z}$ is the number appearing in (32). In particular, the homeomorphism type of K_g would only depend on $g \pmod N$, strengthening the above corollary, and the diffeomorphism type of K_g would only depend on $g \pmod{NK_n}$, where K_n is the order of the finite cyclic Kervaire-Milnor group bP_{2n} ($K_4 = 28$, $K_6 = 992$, $K_8 = 8128, \ldots$; cf [54] or [44]). In the case of Brieskorn varieties one __can__ calculate the quadratic form and obtain

__Theorem__ (Durfee [26], 11.7). Let $n \geq 4$ be even and $a_1, \ldots, a_n \geq 2$ be such that the Brieskorn manifold

$$\Sigma_{a_1,\ldots,a_n} = \{ z \in S^{2n-1} : z_1^{a_1} + \ldots + z_n^{a_n} = 0 \} \tag{44}$$

is a homotopy sphere. Suppose that $g > 0$ and $H_{n-1}(\Sigma_{g,a_1,\ldots,a_n})$ has no summands of order 2 or 4. Let N be any common multiple of $a_1, \ldots, a_n$. Then

$$\Sigma_{g+N,a_1,\ldots,a_n} \approx \Sigma_{g,a_1,\ldots,a_n} \# c\, \Sigma_8, \tag{45}$$

where Σ_8 is the Milnor generator of bP_{2n} and

$$c = \frac{N}{8} d(a_1, \ldots, a_n)$$

(in the notation of eq. (6)) is an integer independent of g.

§ 12 The Browder-Livesay invariant of lens spaces

If we represent S^{2n-1} as the unit sphere in $\mathbb{C}^n$, we can define an action of G_p on it by

$$\zeta \circ (z_1,\ldots,z_n) = (\zeta^{q_1} z_1,\ldots,\zeta^{q_n} z_n) \quad (\zeta \in G_p), \tag{1}$$

where $q_1,\ldots,q_n$ are integers. If each of the q_i is prime to p, this action is free, so the quotient

$$\mathcal{L}(p;q_1,\ldots,q_n) = S^{2n-1}/G_p \quad , \tag{2}$$

is a manifold with fundamental group $\mathbb{Z}/p\mathbb{Z}$, called a $(2n-1)$ dimensional lens space.

If the integers $q_1,\ldots,q_n$ are odd as well as prime to p, we can define $\mathcal{L}(2p;q_1,\ldots,q_n)$ in the same way, and the inclusion of G_p in G_{2p} defines a map

$$\mathcal{L}(p;q_1,\ldots,q_n) \longrightarrow \mathcal{L}(2p;q_1,\ldots,q_n). \tag{3}$$

This is a double covering, so its covering translation

$$T : \mathcal{L}(p;q_1,\ldots,q_n) \longrightarrow \mathcal{L}(p;q_1,\ldots,q_n) \quad , \tag{4}$$

is a free involution on the lens space $\mathcal{L}(p;q_1,\ldots,q_n)$. We wish to compute its Browder-Livesay invariant (cf. 4.2). In 12.1 we compute this invariant using the G-signature theorem and discuss its relation to the quadratic reciprocity law in case $n = 2$, (following [38]).

In 12.2, we show how to obtain a characteristic submanifold for T and compute the dimension of its middle cohomology group; unfortunately we cannot evaluate the quadratic form of the Browder-Livesay definition (eq. 4.2 (10)), and so cannot complete this second calculation of $\alpha(T,\mathcal{L}(p;q_1,\ldots,q_n))$.

12.1 The lens space $\mathcal{L}(p;q_1,\ldots,q_n)$ will be denoted X and the projection from S^{2n-1} will be denoted π. The involution

T on X is induced by thè action of μ on S^{2n-1}, where μ is any element of $G_{2p} - G_p$, i.e. where

$$\mu^p = -1. \tag{1}$$

We thus get a commutative diagram

$$\begin{array}{ccc} S^{2n-1} & \xrightarrow{\; z \longmapsto \mu \circ z \;} & S^{2n-1} \\ \pi \downarrow & & \downarrow \pi \\ X & \xrightarrow{\quad T \quad} & X \end{array} \tag{2}$$

If p is odd, we can take $\mu = -1$, so that in this case (2) shows that T lifts to an involution (the antipodal map) of S^{2n-1}, and we get a diagram

$$\begin{array}{ccc} S^{2n-1} & \longrightarrow & P_{2n-1}(\mathbb{R}) \\ \pi \downarrow & & \downarrow \\ X & \longrightarrow & X/T = L(2p;q_1,\dots,q_n). \end{array} \tag{3}$$

We now calculate $\alpha(T,X)$ by relating it to the numbers $\alpha(\zeta,S^{2n-1})$ $(\zeta \in G_{2p})$ and calculating the latter directly from the Atiyah-Singer definition of the α-invariant.

The action 12(1) is defined on all of $\mathbb{C}^n$ and is free except for an isolated fixed-point at the origin, since we are assuming that all the q_i are prime to p. Since the complex eigenvalues on the tangent space at $z = 0$ (i.e. on the normal bundle of the fixed-point set) are $\zeta^{q_1},\dots,\zeta^{q_n}$, we find from 9.2(1) that

$$L(\zeta,\ D^{2n}) = \prod_{i=1}^{n} \frac{\zeta^{q_i} + 1}{\zeta^{q_i} - 1} \quad . \tag{4}$$

On the other hand, the middle cohomology group of D^{2n} vanishes, so

$$\mathrm{Sign}(\zeta,\ D^{2n}) = 0. \tag{5}$$

It follows from the definition of the α-invariant that

$$\alpha(\zeta,\ S^{2n-1}) = -\prod_{i=1}^{n} \frac{\zeta^{q_i}+1}{\zeta^{q_i}-1}\ . \tag{6}$$

We now apply Theorem 4 of 4.1, with $H = G_p$, $G = G_{2p}$, ξ the non-trivial coset in $G/H = \mathbb{Z}/2\mathbb{Z}$, and $Y = S^{2n-1}$, obtaining

$$\begin{aligned}\alpha(T,X) &= \alpha(\xi,\ Y/H)\\ &= \frac{1}{|H|}\sum_{g\in\xi}\alpha(g,Y)\\ &= \frac{1}{p}\sum_{\zeta^p=-1}\alpha(\zeta,\ S^{2n-1})\\ &= -\frac{1}{p}\sum_{\zeta^p=-1}\prod_{i=1}^{n}\frac{\zeta^{q_i}+1}{\zeta^{q_i}-1}\ .\end{aligned} \tag{7}$$

If n is odd then both sides are zero (for the left-hand side this holds because the α-invariant of an involution is always zero except in dimensions $4k-1$; for the right-hand side one makes the substitution $\zeta \longrightarrow \zeta^{-1}$). If n is even, we can write (7) in trigonometric notation as

$$\alpha(T,X) = \frac{(-1)^{\frac{n}{2}-1}}{p}\sum_{\substack{k=1\\ k\ \text{odd}}}^{2p-1} \cot\frac{\pi k q_1}{2p}\cdots\cot\frac{\pi k q_n}{2p}\ . \quad (n\ \text{even}) \tag{8}$$

But now we can use the formulas of §5 to rewrite this in several other forms. If we combine (8) with 5.3(5) we obtain immediately the following formula for $\alpha(T,X)$:

$$\alpha(T,X) = t_p(q_1,\ldots,q_n) = t_p^{\text{odd}}(q_1,\ldots,q_n) - t_p^{\text{even}}(q_1,\ldots,q_n), \tag{9}$$

where

$$t_p^{\substack{\text{odd}\\ \text{even}}}(q_1,\ldots,q_n) = \#\left\{0 < k_1,\ldots,k_n < p \;\middle|\; \frac{q_1k_1+\cdots+q_nk_n}{p}\ \text{is an}\ {\substack{\text{odd}\\ \text{even}}}\ \text{integer}\right\}. \tag{10}$$

If $q_1 = 1$ (which can always be arxanged by replacing each q_i by $q_i r$, where $q_1 r \equiv 1 \pmod p$) we can use 5.3(18) to write this as

$$\alpha(T;\mathcal{L}(p;1,q_2,\ldots,q_n)) = t(p,\ldots,p;q_2,\ldots,q_n) \tag{11}$$

$$= \sum_i (-1)^i \#\{0<k_2,\ldots,k_n<p : i < \frac{k_2q_2+\ldots+k_nq_n}{p} < i+1\}. \tag{12}$$

For n even, we can use the theorem of 5.4 to rewrite (8) with tangents:

$$\alpha(T,x) = \frac{(-1)^{\frac{1}{2}n-1}}{p} \sum_{\substack{k(\mathrm{mod}\ 2p)\\ k+p\ \mathrm{odd}}} \tan\frac{\pi k q_1}{2p} \cdots \tan\frac{\pi k q_n}{2p} \quad (n\ \text{even}). \tag{13}$$

For the classical (3-dimensional) lens space

$$L(p,q) = \mathcal{L}(p;1,q), \tag{14}$$

we have from (11) and 6.2(9) the formulas

$$\alpha(T,L(p,q)) = t(p;q)$$

$$= \sum_i (-1)^i \#\{0<k<p \mid i < \frac{kq}{p} < i+1\}$$

$$= -4\ S(\frac{q}{2p}), \tag{15}$$

where, for q prime to $2p$,

$$S(\frac{q}{2p}) = \sum_{x=1}^{p-1} ((\frac{qx}{2p})). \tag{16}$$

Finally, if p as well as q is odd, we get from 6.2(8) the relation

$$\alpha(T,L(p,q)) = p - 1 - 4N_{q,p} \quad (p,q\ \text{odd, coprime}) \tag{17}$$

relating the Browder-Livesay invariant of an involution on the classical lens space $L(p,q)$ to the numbers $N_{q,p}$ of Gauss' lemma.

The law of quadratic reciprocity in terms of these numbers is the statement

$$N_{q,p} + N_{p,q} \equiv -\frac{p-1}{2}\cdot\frac{q-1}{2} \pmod 2 \tag{18}$$

(in fact Corollary 2 to the theorem of 6.2 tells us that this congruence even holds modulo 4). It would be pleasing to prove (18) directly from (17), i.e. to prove that

$$p\alpha(T,L(q,p)) + q\alpha(T,L(p,q)) \equiv 1 - pq \pmod 8. \qquad (19)$$

We would thus like to find a four-manifold whose boundary consists of disjoint copies of $L(p,q)$ and $L(q,p)$, and having a free involution restricting to the involution T on the lens spaces. Such a manifold can be constructed as follows.

Let V_N be the hypersurface of degree N in $P_3(\mathbb{C})$:

$$V_N = \{(z_0:z_1:z_2:z_3) \in \mathbb{C}P^3 \mid z_0^N + z_1^N + z_2^N + z_3^N = 0\} \qquad (20)$$

Then V_N is a smooth 4-manifold whose signature equals $N(4-N^2)/3$ (Exercise 1, 9.3). We set $N = 2pq$ and let G_{pq} act on V_{2pq} by

$$\zeta \circ (z_0:z_1:z_2:z_3) = (z_0:\zeta^{px-p}z_1:\zeta z_2:\zeta^{px+qy^2}z_3), \qquad (21)$$

where x and y satisfy

$$xp + yq = 1. \qquad (22)$$

We further assume that the numbers p and q, as well as being odd and coprime, satisfy

$$(p-1,q) = 1, \quad (q-1,p) = 1 \qquad (23)$$

(this is the case, for instance, if $2 < p < q$ are primes, $q \not\equiv 1 \pmod p$). Then it is easy to check that the action (21) has $2pq$ points

$$(1:\lambda:0:0), \qquad \lambda^{2pq} = -1 \qquad (24)$$

with isotropy group G_p, and $2pq$ points

$$(0:0:1:\lambda), \qquad \lambda^{2pq} = -1 \qquad (25)$$

with isotropy group G_q, and is free otherwise. Let V' be the manifold with boundary obtained by deleting a small equivariant disc centred around each of the points (24),(25). Then V' has $4pq$ disjoint copies of S^3 as boundary, and G_{pq} acts freely on V'. The manifold

$$W' = V'/G_{pq} \qquad (26)$$

has boundary

$$\partial W' = 2p \cdot L(p,q) + 2q\, L(q,p). \qquad (27)$$

It is clear that the involution

$$(z_0:z_1:z_2:z_3) \longrightarrow (z_0:z_1: -z_2: -z_3) \qquad (28)$$

is compatible with the action (21), and so induces an involution T of W'. Since each of the points (24),(25) is fixed by (28), the boundary components of W' are mapped into themselves by T. Indeed it is clear that the only fixed points of (28) on V_{2pq} are the points (24), (25), so that T is a free involution on W', and one also sees easily that the restriction of T to the lens spaces bounding W' is precisely the involution studied in this section. Therefore

$$\begin{aligned} 2p\alpha(T,L(p,q)) &+ 2q\alpha(T,L(q,p)) \\ &= \text{Sign}\,(T,W') - L(T,W') \\ &= \text{Sign}\,(T,W') \qquad (29) \end{aligned}$$

the latter because T acts freely on W'. We can reduce the calculation of the equivariant signature Sign(T,W') to a calculation of ordinary signatures by using the relation

$$\text{Sign}\,(T,W') = 2\,\text{Sign}\,(W'/T) - \text{Sign}(W'). \qquad (30)$$

Moreover, if W denotes the variety V/G_{pq} then W is obtained from W' be pasting on the cones on the lens spaces making W'. These cones are contractible and so have zero signature, so by Novikov additivity we have Sign W = Sign W'. Therefore

$$2p\alpha(T,L(p,q)) + 2q\alpha(T,L(q,p)) = 2\,\text{Sign}\,W/T - \text{Sign}\,W. \qquad (31)$$

The problem of calculating the left hand side (mod 16) is thus reduced to the problem of computing the signatures of the closed rational homology manifolds W and W/T modulo sixteen and eight, respectively. The calculation of these signatures in $\mathbb{Z}$ probably requires the G-signature theorem. However, there exist theorems giving information on the signatures of 4-manifolds modulo 8 or 16 (theorems of Kervaire-Milnor and of Rochlin; see e.g. [45]) and by applying these theorems to W and W/T, it should be possible to deduce the quadratic reciprocity law from the formula for the α-invariant of T. We leave this as an open problem for the reader.

12.2 We now attempt to calculate $\alpha(T,X)$ ($X = \mathcal{L}(p;q_1,\dots,q_n)$ as before) directly from the Browder-Livesay definition. Recall that this requires the calculation of the signature of the form

$$Q(x,y) = x \cdot T_* y \qquad (x,y \in K) \tag{1}$$

with

$$K = \operatorname{Ker}(H_{n-1}W \longrightarrow H_{n-1}A), \tag{2}$$

W being a characteristic manifold for the involution T — i.e.

$$A^{2n-1} \subset X, \quad A \cup TA = X, \quad A \cap TA = \partial A = W^{2n-2} \tag{3}$$

In the case $n = 2$, $X = L(p,q)$, a characteristic manifold was found by Neumann [74], using the description of 3-dimensional lens spaces as the union of two solid tori $D^2 \times S^1$ glued together by a diffeomorphism of their boundaries. These 3-dimensional lens spaces fit into the general cadre of 3-manifolds admitting a fixed-point-free S^1-action; for the involution on such manifolds obtained from $-1 \in S^1$, Neumann computed the α-invariant in terms of the Seifert-Raymond classification of the manifold.

We will construct a characteristic manifold in a different way.

Define a function $f : \mathbb{C}^n \longrightarrow \mathbb{R}$ by

$$f(z) = \operatorname{Re}(z_1^p + \dots + z_n^p). \tag{4}$$

Let

$$A' = \{ z \in S^{2n-1} : f(z) \geq 0\}, \tag{5}$$

$$W' = \{ z \in S^{2n-1} : f(z) = 0\}. \tag{6}$$

Then A' is a manifold with boundary W' (one must check that W' is a submanifold of S^{2n-1}, i.e. that the gradients of f and of $\sum_{j=1}^{n} |z_j|^2$ are linearly independent on W'). Since the free action of G_p maps these manifold into themselves

$$A = A'/G_p \tag{7}$$

is a submanifold of X with boundary

$$W = W'/G_p. \tag{8}$$

Clearly the action of μ (12.1(2)) is a diffeomorphism from A' to $\overline{S^{2n-1} - A'}$, so the involution T takes A to $\overline{X-A}$, and therefore W is a characteristic submanifold for T. To apply the Browder-Livesay definition of $\alpha(T,X)$, we first need to know

$$K = \ker(H_{n-1}(W) \longrightarrow H_{n-1}(A)). \tag{9}$$

Here we can take homology with real coefficients (since we will want to compute the signature of a certain form on K), so we can apply Theorem 5 of 2.1 to identify K with the G_p-invariant part of

$$K' = \ker(H_{n-1}(W') \longrightarrow H_{n-1}(A')). \tag{10}$$

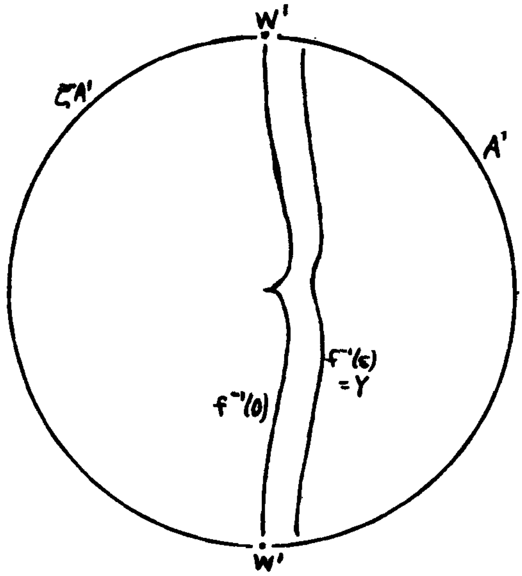
To compute K', we use a fibration theorem for real singularities of Milnor (Lemma 11.3 of [72]; in Milnor's notation we have $k=1$ and he assumes $k > 1$, but the proof is the same). Thus we choose $\varepsilon > 0$ and define

$$Y = f^{-1}(\varepsilon) \cap D^{2n}. \tag{11}$$

Let U be the region

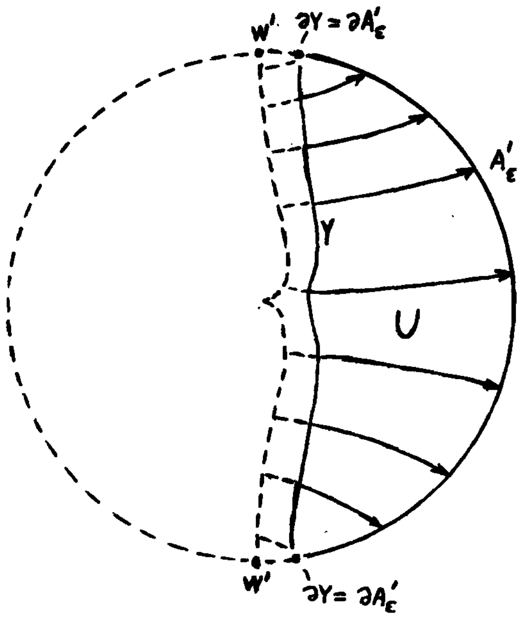
$$U = \{ z \in D^{2n} : f(z) > 0\}. \tag{12}$$

We define a vector field $v(z)$ on U with real coordinates



THE CHARACTERISTIC SUBMANIFOLD OF T IS THE QUOTIENT

$$W = W'/G_p$$



MOVING ALONG THE TRAJECTORIES OF THE VECTOR FIELD v

YIELDS A DIFFEOMORPHISM $Y \longrightarrow A'_\varepsilon$

$(u_1,v_1,\ldots,u_n,v_n)$, where

$$u_j(z) + iv_j(z) = \bar{z}_j^{p-1} \quad (z \in U). \tag{13}$$

Then, writing $z_j = x_j + iy_j$, we compute

$$\langle v(z), \text{grad} \|z\|^2 \rangle = 2\langle v(z), z\rangle = 2 \sum_{j=1}^{n} (u_j x_j + v_j y_j)$$

$$= 2 \sum_{j=1}^{n} \text{Re}((u_j - iv_j)(x_j + iy_j)) = 2 \sum_{j=1}^{n} \text{Re}(z_j^{p-1} z_j) = 2f(z) > 0, \tag{14}$$

and similarly,

$$\langle v(z), \text{grad } f(z)\rangle = \sum_{j=1}^{n} \left(u_j \frac{\partial f}{\partial x_j} + v_j \frac{\partial f}{\partial y_j}\right)$$

$$= \frac{1}{2} \sum_{j=1}^{n} \left(u_j \frac{\partial}{\partial x_j} + v_j \frac{\partial}{\partial y_j}\right)(z_j^p + \bar{z}_j^p)$$

$$= \frac{1}{2} \sum_{j=1}^{n} \left[u_j (p z_j^{p-1} + p\bar{z}_j^{p-1}) + v_j (ipz_j^{p-1} - ip\bar{z}_j^{p-1})\right]$$

$$= \sum_{j=1}^{n} p z_j^{p-1} \bar{z}_j^{p-1} > 0. \tag{15}$$

It follows that both $\|z\|$ and $f(z)$ are monotone increasing as we move outwards along any trajectory of v. Therefore there is a well-defined map from U to A' sending a point p to the intersection with S^{2n-1} of the trajectory of v passing through p, and this map takes Y diffeomorphically onto

$$A'_\varepsilon = \{z \in S^{2n-1} \mid f(z) \geqslant \varepsilon\}. \tag{16}$$

Clearly $\partial Y = \partial A'_\varepsilon \approx W'$ if ε is small enough, and clearly K' is isomorphic to the kernel of the homology map induced by $\partial A'_\varepsilon \subset A'_\varepsilon$, so with the diffeomorphism we get

$$K' \cong \ker(H_{n-1}(\partial Y) \longrightarrow H_{n-1}(Y)). \tag{17}$$

This isomorphism is G-equivariant since all of the diffeomorphisms were.

We now need to know the homology of Y. Essentially, Y is a Brieskorn variety V_a with $a = (p,\dots,p)$, the only difference being that we fix only the real part of $\Sigma z_j{}^p$. In fact V_a is a deformation retract of Y : the calculation of the homology of V_a (Pham [80]) proceeds by first showing that

$$U_a \quad = \quad \{z \in V_a \mid z_j^p \text{ is real and } \geq 0 \text{ for all } j\} \tag{18}$$

is a deformation retract of V_a, and the same proof in our case shows it to be a deformation retract of Y as well. But the homology of V_a is given by

$$\begin{aligned} H_i(V_a) &= 0 \qquad (i \neq n-1), \\ H_{n-1}(V_a) &\cong \mathbb{Z}[G]/\langle 1+w_j+\dots+w_j^{p-1} = 0,\ j=1,\dots,n\rangle \end{aligned} \tag{19}$$

(cf. [44], §12, for an exposition of the homology of Brieskorn varieties), where $G = G_p \times \dots \times G_p$ (n copies), $\mathbb{Z}[G]$ is the group ring, and w_j $(j=1,\dots,n)$ are the generators of the various copies of G_p. From now on we work with real homology, since this suffices for studying the signature of a quadratic form. Clearly $H_{n-1}(Y;\mathbb{R})$ is $\mathbb{R}[G]$ divided by the same relations as in (19).

Since Y is a $(2n-1)$-dimensional manifold with boundary, we now deduce that the only non-trivial part of the homology sequence of $(Y,\partial Y)$ is

$$0 \longrightarrow H_n(Y,\partial Y) \longrightarrow H_{n-1}(\partial Y) \longrightarrow H_{n-1}(Y) \longrightarrow 0, \qquad (20$$

and from this and (17) we get

$$K' \otimes \mathbb{R} \;\cong\; H_n(Y,\partial Y;\mathbb{R}) \;\cong\; H^{n-1}(Y;\mathbb{R}) \;\cong\; (H_{n-1}(Y;\mathbb{R}))^* . \qquad (21$$

We denote by I the ideal in $\mathbb{R}[G]$ generated by the elements $1 + w_j + w_j^2 + \ldots + w_j^{p-1}$ $(j = 1,\ldots,n)$; then, by (19) and (21),

$$K' \otimes \mathbb{R} \cong (\mathbb{R}[G]/I)^* \cong \operatorname{Ker}(\mathbb{R}[G]^* \longrightarrow I^*).$$

We let $y_i \in \mathbb{R}[G]$ $(i=1,\ldots,n)$ be the dual basis to w_i (i.e. $y_i(w_j) = \delta_{ij}$) ; then $\mathbb{R}[G]$ has a basis consisting of all monomials $y_1^{k_1}\ldots y_n^{k_n}$ with $0 \leq k_1,\ldots,k_n \leq p-1$, and clearly an element

$$y = \sum_{\substack{0\leq k_1<p \\ \vdots \\ 0\leq k_n<p}} a(k_1,\ldots,k_n)\, y_1^{k_1}\ldots y_n^{k_n}$$

is in $\operatorname{Ker}(\mathbb{R}[G]^* \to I^*)$ if and only if for each i

$$a(k_1,\ldots,\underset{i}{0},\ldots,k_n) = -\sum_{k=1}^{p-1} a(k_1,\ldots,\underset{i}{k},\ldots,k_n) .$$

Therefore $K' \otimes \mathbb{R}$ has a basis consisting of all monomials $y_1^{k_1}\ldots y_n^{k_n}$ with $1 \leq k_i \leq p-1$. The action of a generator of G_p on K' is given by

$$y_1^{k_1}\ldots y_n^{k_n} \mapsto y_1^{k_1+q_1} \ldots y_n^{k_n+q_n} ,$$

so $K \otimes \mathbb{R} = (K' \otimes \mathbb{R})^{G_p}$ is spanned by the vectors

$$\left(1 + y_1^{q_1}\ldots y_n^{q_n} + (y_1^{q_1}\ldots y_n^{q_n})^2 + \ldots + (y_1^{q_1}\ldots y_n^{q_n})^{p-1}\right) \times y_1^{k_1}\ldots y_n^{k_n} . \tag{22}$$

From this we deduce easily that

$$\dim_{\mathbb{R}}(K \otimes \mathbb{R}) = \#\{(k_1,\ldots,k_n) \mid 0<k_i<p,\ \sum_{i=1}^{n} q_i k_i \equiv 0 \pmod p\}, \tag{23}$$

i.e. the rank of K is $t_p^{odd}(q_1,\ldots,q_n) + t_p^{even}(q_1,\ldots,q_n)$, where t_p^{odd} and t_p^{even} have the same meaning as in 5.3, namely

$$t_p^{\substack{even\\odd}}(q_1,\ldots,q_n) = \#\{(k_1,\ldots,k_n) \mid 0 < k_i < p \text{ and } \frac{1}{p}(q_1k_1 + \ldots + q_nk_n) \text{ an } \substack{even\\odd} \text{ integer}\}. \tag{24}$$

The Browder-Livesay invariant is given by

$$\alpha(T,X) = \text{Sign } Q , \tag{25}$$

with Q the quadratic form (1) on K. From 12.1 we know that

$$\alpha(T,X) = t_p^{odd} - t_p^{even} . \tag{26}$$

The numbers t_p^{odd} and t_p^{even} must therefore give precisely the numbers of positive and negative eigenvalues, respectively, of the Browder-Livesay form for the characteristic manifold we have chosen. Unfortunately, the homotopy equivalences used to determine K do not give any information about intersection numbers in Y, so the problem of calculating geometrically the intersection form Q with respect to the given basis of K remains open.

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