Lie groups

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Originally, the study of the Lie groups comes from a generalization of Galois theory: In the latter, one studies "symmetries" of polynomial equations to discuss their solutions, in the former, the "symmetries" of differential equations. Nowadays, Lie groups can be seen as symmetry groups for different topological objects. There are many aspects of Lie theory, and we will be able to learn only about a tiny piece of it. Our main goal is to discuss some strong classification results for Lie groups.

A first highlight in this direction is a quite complete description of abelian Lie groups. After this, our next goal is to show that well-known examples of Lie groups, namely matrix groups, are not as special as one might think: Indeed, a consequence of Peter-Weyl-Theorem is that any compact Lie group is isomorphic to a matrix group. On the way, we will have a first glance at representation theory of Lie groups, in other words, of Lie groups as symmetries of finite-dimensional vector spaces. Then we will show that we have a nice description for maximal connected abelian subgroups of compact connected Lie groups, and the consequences of this description include a step towards a classification theorem for compact connected Lie groups. Next, we will show that all the information about a simply connected Lie group is contained in its tangent space at the unit element (together with its Lie bracket), which contains also several of Lie's original insights. To do so, we will need to learn a bit of structure theory for Lie algebras. Last, we will have a short overview over the restrictions on cohomology of compact Lie groups.

Dates

The seminar will take place Tuesdays, 10-12 in 0.006 and Thursdays, 14-16 in Zeichensaal.

About this seminar

The requirements of this seminar are a regular participation in the seminar, a talk and a (possibly short) preliminary discussion of the talk with me about ten days before the actual talk. This preliminary discussion gives the opportunity to ask mathematical questions and also to fix the precise content of the talk.

Please feel free to contact me if you have any questions, both on formalities or on mathematical contents of your talk. Please plan your talk for 75-80 minutes. Take particular care of your target audience, consisting of your fellow students. For example, if you do not know a notion or do not understand a proof for a while, do not assume your audience to understand the same proof immediately or to know all the new notions. Personally, I would recommend to make a test run of your talk, either in front of a fellow student or even for yourself. The important point is to simulate your part as precisely as possible.

If you have any questions about the seminar, feel free to contact me in my office 4.014 or via email at

ozornova@math.uni-bonn.de.

I also recommend to read the following hints by Prof. M. Lehn on how to give a seminar talk (in German):

http://www.mathematik.uni-mainz.de/Members/lehn/le/seminarvortrag

Even if some of the expressed advice might be exaggerated, this tutorial is in any case helpful.

Prerequisites

General topological background (e.g. from Einführung in Geometrie und Topologie) will be assumed. In particular, you will need basic knowledge about smooth (or differntiable) manifolds, later on also basic covering theory and fundamental groups. Some basic algebra (e.g. from Einführung in Algebra) will also be needed. Background such as (co)homology groups, homotopy groups, fiber bundles, solvable groups, differential forms is helpful but not strictly necessary.

Talk 1: Basic Properties and First Examples- Apr 18

After a (very brief) reminder on the definition and examples of well-known matrix groups, discuss also the symplectic group and the Lorentz group. Discuss moreover Exercises 8 and 12 on p. 12 of [BtD95] concerning the relation of (special) orthogonal, special and general linear groups, cf. also [Tit83], II.4.6). Recall the different description of tangent spaces and the tangent bundle (and more generally vector bundles) of a manifold. Discuss the trivialization of the tangent bundle of a Lie group. Discuss the descriptions for a Lie algebra of a Lie group and give the first description of a Lie bracket. Follow [BtD95], Sections I.1-I.2 (up to and including (2.9)).

Talk 2: Exponential Map and More Examples- Apr 25

Give an alternative description of the Lie bracket on the Lie algebra of a Lie group in terms of the adjoint representation. Discuss that this bracket indeed satisfies the properties of a Lie bracket in the abstract sense. Identify the Lie algebras of matrix groups with some care. Define also the exponential map and discuss it in some detail in examples, also the non-surjectivity example of Exercise 1 on p. 30 of [BtD95]. Highlight that every connected topological group is generated by any neighborhood of the identity. Follow [BtD95], Sections I.2-I.3 (up to and including (3.4)). If you have time, discuss also the example of $O(n,1)$ (or the special case $n=1$ or $n=3$) following [Bak02], Chapter 6 – do not spend too much time on explicit matrix calculation though.

Talk 3: Abelian Lie Groups and Subgroups of Lie Groups - May 2

Your first goal is to show that an abelian Lie group which is either connected or compact is always of an easy form, namely product of a torus, $\mathbb{R}^n$ and a finite group. Next, discuss sub-Lie-groups and prove the criterion for checking an abstract subgroup to be a submanifold. Follow for this part [BtD95], Section I.3. Provide also an example of a Lie group which is not a subgroup of any (complex) general linear group. Follow [Bak02], Section 7.7.

Talk 4: Homogeneous Spaces and Examples - May 9

Your main goal is to discuss (basic properties of) quotients of Lie groups by closed subgroups. In particular, show that under appropriate conditions, the quotient is a manifold again, and also the quotient map "locally looks like a projection on the first factor in a product". To make this precise, recall/introduce the notion of a fiber bundle. Give also an example illustrating that the assumptions of the theorem are indeed necessary. Discuss in some detail examples relating to known spaces, such as e.g. Grassmannians. Prove also Kronecker's Theorem. Follow [BtD95], Section I.4.

Talk 5: Representations of Lie Groups I: Characters and Orthogonality Relations - May 16

Representations of Lie groups is a huge area of mathematics, so we can only touch the surface of these. Give basic definitions. Prove the existence of G-invariant inner product for representations of a compact Lie group G. (You will need to use Theorem I.5.13 of [BtD95] as a blackbox.) Prove Schur's Lemma and discuss the canonical decomposition. Discuss characters and orthogonality relations. Also mention some examples. Follow [BtD95], II.1, II.4 and II. 5. Especially concerning examples, please make sure to get in touch with the speaker of the next talk in advance.

Talk 6: Representations of Lie Groups II: Representations of Abelian Groups - May 23

The first objective of your talk is to give a complete description for representations of compact abelian groups. Moreover, discuss the relation of representations of Lie groups and Lie algebras, and illustrate the relation by the example of $\mathrm{sl}(2, \mathbb{C})$. Follow [BtD95], Sections II.8-II.10.

Talk 7: Theorem of Peter and Weyl - May 30

The aim of your talk is to study continuous complex-valued functions on compact Lie groups, and one of the main consequence of this studies is to show that any compact Lie group is isomorphic to a matrix group. You will need to recall some basic calculus/functional analysis. Discuss then representative functions and prove the Peter-Weyl-Theorem, claiming that the algebra of such is dense in continuous $\mathbb{C}$-valued functions on G. Then prove the mentioned (and possibly further) applications. Follow [BtD95], Sections III.1-III.4.

Talk 8: Maximal Tori - June 13

Your first goal is to show existence and uniqueness up to conjugacy of maximal tori in compact connected Lie groups. You will need to recall some of the theory of differential forms on Lie groups/smooth manifolds, e.g. as in [BtD95], I.5. Then give the (maybe sketch) proof of the Conjugacy theorem. Also define the Weyl group of a compact connected Lie group, and illustrate the Weyl group and the maximal tori in examples. Follow [BtD95], Section IV.1 and IV.3.

Talk 9: Weyl Groups and Root Systems - June 20

First, derive some consequences of the Conjugacy Theorem, including surjectivity of exponential map, identification of class functions and of representation ring in terms of the maximal torus. Follow for this part Section IV.2 of [BtD95]. Introduce as much of root systems and their Weyl groups as needed to formulate the relation to Weyl groups of Lie groups (Theorem V.3.12 of [BtD95]), and give examples. Sketch the way to classification of compact connected Lie groups using the remark at the beginning of V.5, Theorem V.8.1 and Exercise 6 on p. 238 of [BtD95] (without proofs; we will prove some bits of it later on).

Talk 10: First Lie Theorem - June 27

Your main aim is to show for simply connected analytic Lie groups that a map of their Lie algebras already induces a map of the groups themselves. For producing a local map, you will need to employ some analysis. Follow for this part [Tit83], Section III.4, mainly III.4.2 and III.4.3, or/and also have a look at [DK00], Section I.1.6. If you have time, discuss also the Campbell-Hausdorff(-Baker-Dynkin) formula, otherwise use it as a blackbox. Then discuss how the local map induces a global one in the simply-connected case, following [Tit83], Section II.4.3 (and parts of II.4.2). If you have time, discuss also the uniqueness and existence of an analytic structure, e.g. as in [DK00], Section I.1.6. If you have time, discuss the classification of connected 2-dimensional Lie groups ([Tit83], Section III.4.4, see also [Sti08], Section 9.6).

Talk 11: Realizability for Solvable Lie Algebras - July 4

Your main objective is to show that any finite-dimensional solvable Lie algebra is a Lie algebra of a Lie group ([Tit83], Section IV.1.7). For this, you need to discuss the notion of solvability. Show in particular that a Lie group is solvable if and only if it is solvable as an abstract group. Moreover, discuss the notion of solvability for Lie algebras, and the relation to such of Lie groups. Then prove the existence of the invariant flag (Lie's Theorem) for solvable groups acting on complex vector spaces. To conclude, you will also need semidirect products for Lie groups and Lie algebras. Follow [Tit83], Section IV.1, and also III.5.5-5.6 for semidirect products, and/or [Kna96], Sections I.4, I.5 and I.12.

Talk 12: Semisimple Lie Groups and Lie Algebras- July 11

The main goal of your talk is to study semisimple Lie groups and algebras. Recall the notion of semisimplicity for Lie groups and algebras and discuss the radical ideal more generally. Prove (or at least sketch a proof of) Cartan's semisimplicity and solvability criteria. Discuss some examples and also the relation to simple Lie algebras/groups. Show in particular that a Lie algebra is semisimple if and only if it is a product of simple Lie algebras. Follow [Kna96], I.7-I.8 and [Tit83], IV.3.2-IV.3.5.

Talk 13: Cartan-Lie-Theorem- July 18

Your main objective is to show that indeed any (and not only a solvable) Lie algebra comes from a Lie group. Prove Levi's theorem first and then conclude the existence of a corresponding Lie group. Show that any semisimple simply-connected Lie group is a product of Lie groups with simple Lie algebras. Give an example of a non-simple Lie group with a simple Lie algebra. Follow [Tit83], IV.3.6-IV.3.7 and/or Appendix B of [Kna96]. If you have time, discuss also which Lie algebras appear as Lie algebras of compact Lie groups, [DK00], Section 3.6.

Talk 14: Differential Forms on Lie Groups- July 25

Your main objective is to show that the cohomology of a (compact, connected) Lie group (as a topological space!) can be computed purely in terms of its Lie algebra, and to draw several more explicit consequences for the cohomology of Lie groups. Recall the de Rham cohomology and de Rham's theorem. Show that the inclusion of G-invariant differential forms into all is a quasi-isomorphism, and then translate to Lie algebra terms. Follow [Bre97], Section V.12. If you have time, give a survey on Hopf's theorem on cohomology of Lie groups, e.g. as in [Whi78], III.8.

References

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