Twisted Whitney towers and higher-order Arf invariants

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Recall that the following are equivalent:

- $L = \cup_{i=1}^{m} L_i \subset S^3$ is link-homotopically trivial.
- Non-repeating Milnor invariants $\mu_k(L)$ vanish for $k \leq m - 2$.
- L bounds an order $m - 1$ non-repeating Whitney tower $\mathcal{W} \subset B^4$.
- Intersection invariants $\lambda_k(\mathcal{W}) = 0 \in \Lambda_k$ for $k \leq m - 2$.
- L lifts to the mth level of the Goodwillie–Weiss link map tower.

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– Will generalize the above to give a classification of links bounding **order *n* twisted Whitney towers** in terms of **Milnor invariants** with repeated indices allowed (still torsion-free) and **higher-order Arf invariants** (2-torsion related to Whitney disk twistings).

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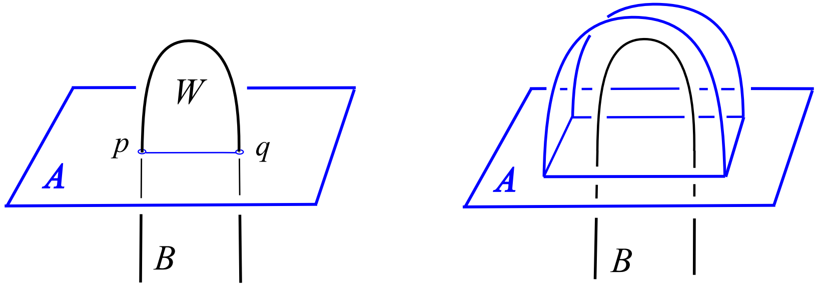
– We hope to find a corresponding relationship with the Goodwillie–Weiss concordance tower.

Outline (see 'Intro to Whitney towers' notes for ref's and details)

- Twisted Whitney towers and their trees
- Intersection invariants for order n twisted Whitney towers
- Classification of order n twisted Whitney towers in B^4
- The Higher-order Arf invariant Conjecture

Successful Whitney move: W is 'clean' and 'framed'

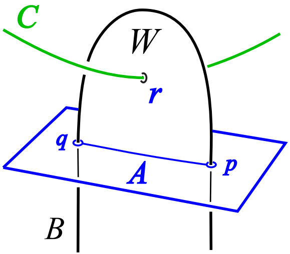
Eliminates $p, q \in A \pitchfork B$ without creating new intersections in A or B:

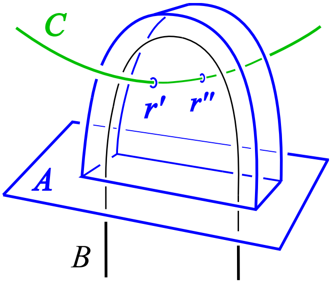


W is *clean* = embedded & interior disjoint from all surfaces.
W is *framed* = W has appropriate parallels.

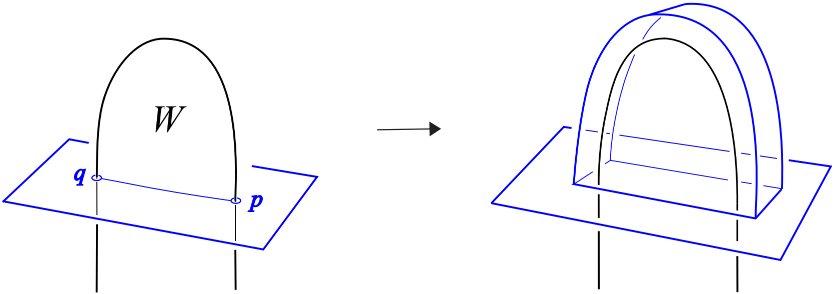
W not clean $\rightsquigarrow$ Whitney move creates new intersections:

$r \in W \pitchfork C \quad \rightsquigarrow \quad r', r'' \in A \pitchfork C$ after W-move on A:



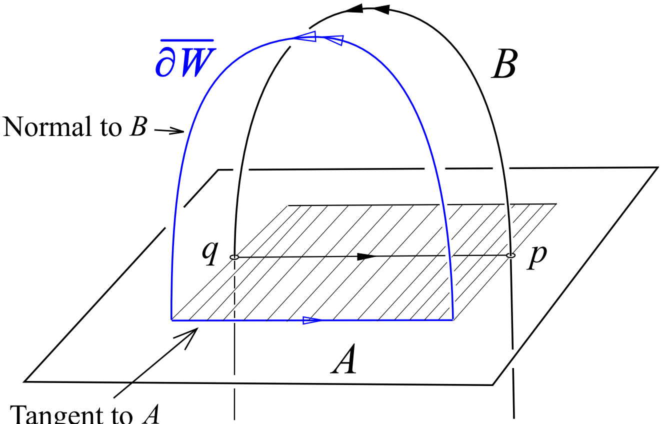


Whitney move uses two parallel copies of W:



Framed Whitney disks and twisted Whitney disks

The *twisting* $\omega(W) \in \mathbb{Z}$ of W is the relative Euler number of a normal section $\overline{\partial W}$ over ∂W determined by the sheets:



If $\omega(W) = 0$, then W is *framed*.
If $\omega(W) \neq 0$, then W is *twisted*.

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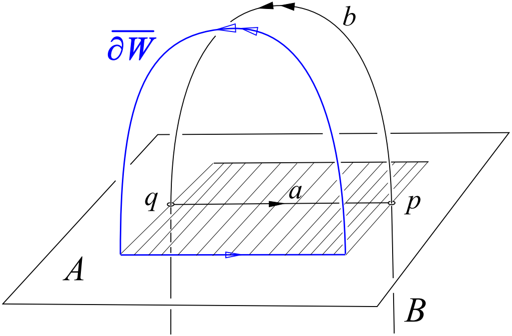

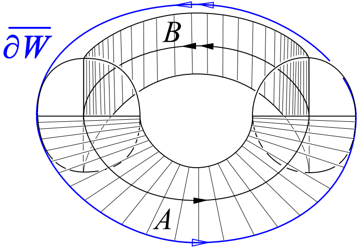

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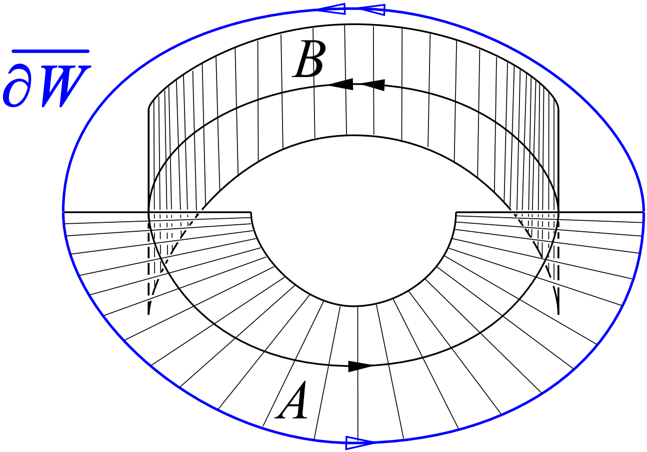
Close up of normal section $\overline{\partial W}$ in $\partial W \times D^2$:



Definition:

A *Whitney tower* on $A^2 \looparrowright X^4$ is defined by:

1. A itself is a Whitney tower.
2. If $\mathcal{W}$ is a Whitney tower and W is a Whitney disk pairing intersections in $\mathcal{W}$, then the union $\mathcal{W} \cup W$ is a Whitney tower.

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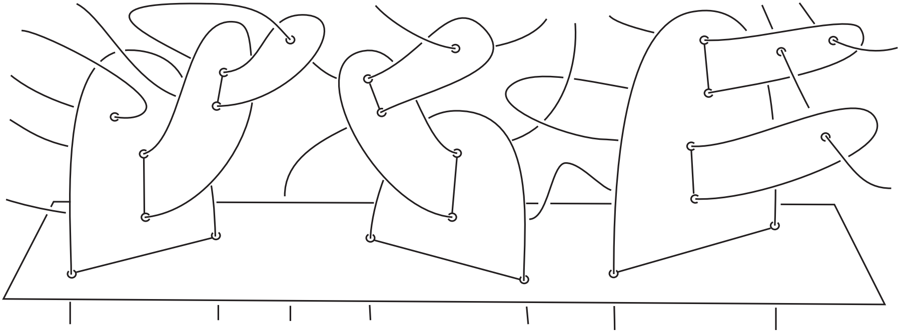
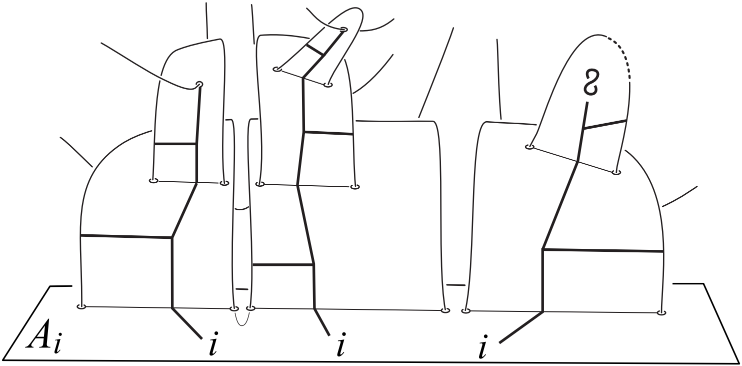


Part of a Whitney tower

The *intersection forest* multiset $t(\mathcal{W})$ of a Whitney tower $\mathcal{W}$

$$\mathcal{W} \mapsto t(\mathcal{W}) = \sum \epsilon_p \cdot t_p + \sum \omega(W_J) \cdot J^\infty$$



'framed tree' $t_p \leftarrow\!\!\dashv p$ unpaired intersection with sign $\epsilon_p = \pm 1$,
'twisted tree' $J^\infty := J \text{ --- } \infty \leftarrow\!\!\dashv W_J$ with twisting $\omega(W_J) \neq 0 \in \mathbb{Z}$.

Paired intersections $\longrightarrow$ rooted trees

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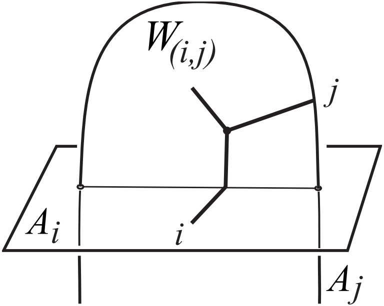
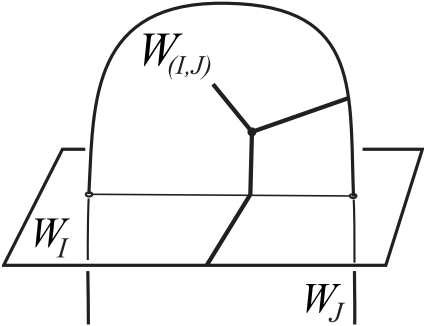
$W_{(i,j)}$ pairing $A_i \pitchfork A_j \quad \longmapsto \quad$ rooted tree $-\!\!<^{j}_{i} = (i,j)$



Paired intersections → rooted trees

Recursively: $W_{(I,J)}$ pairing $W_I \pitchfork W_J \quad \longmapsto \quad -\!\!<^{J}_{I} = (I, J)$



Rooted trees I, J = non-associative bracketings from $\{1, 2, 3, \ldots, m\}$
Notation convention: Singleton subscript W_i denotes component A_i.

Un-paired intersections → un-rooted trees

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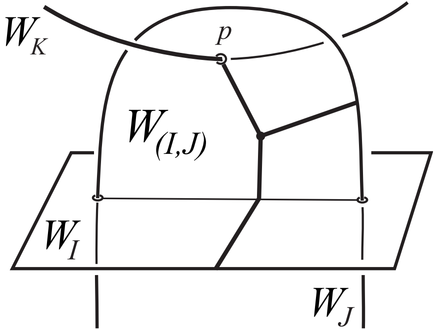
Inner product 'fuses' rooted edges into single edge:

$$p \in W_{(I,J)} \pitchfork W_k \quad \longmapsto \quad t_p = \langle (I,J), K \rangle = {}^{I}_{J} \!\!> \!\!- K$$



∞-trees ('twisted' trees) for twisted Whitney disks

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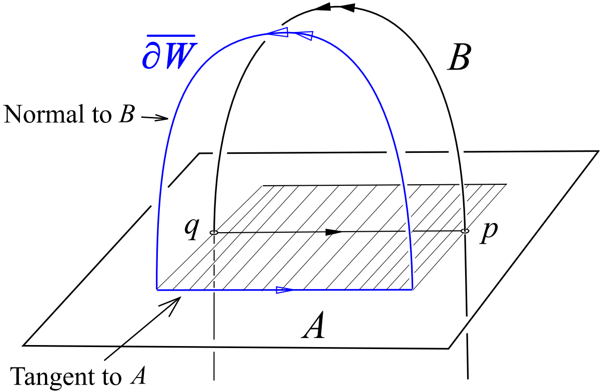
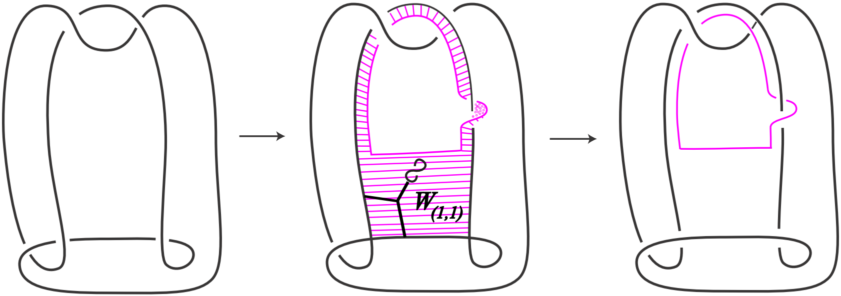
$$W_J \quad \mapsto \quad J^{\infty} := J \text{—}\infty \quad \text{if } \omega(W_J) \neq 0.$$

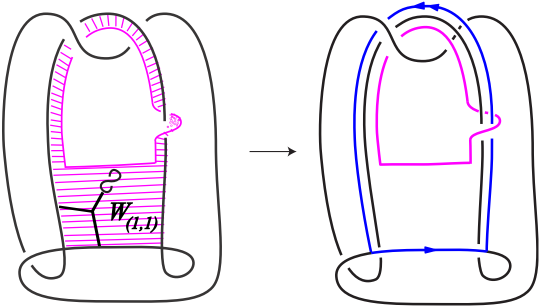


Example: Figure-8 knot bounds $\mathcal{W}$ with $t(\mathcal{W}) = (1,1)^{\infty} = \frac{1}{1} >\!\!\!- \infty$



The Whitney disk $W_{(1,1)}$ is <u>clean</u> (since right picture is an unlink).

Example: Figure-8 knot bounds $\mathcal{W}$ with $t(\mathcal{W}) = (1,1)^{\infty} = \frac{1}{1} >\!\!-\!\!- \infty$



The Whitney disk $W_{(1,1)}$ is twisted (since blue and purple link once).

Obstruction theory for links bounding twisted Whitney towers

- $\mathcal{W}$ is an *order n twisted Whitney tower* if $t(\mathcal{W})$ contains only framed trees of order $\geq n$ and twisted trees of order $\geq n/2$, where *order* := number of trivalent vertices.

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- Will define abelian groups $\mathcal{T}_n^\infty$ and intersection invariants $\tau_n^\infty(\mathcal{W}) := [t(\mathcal{W})] \in \mathcal{T}_n^\infty$ such that:

  L bounds an order n twisted $\mathcal{W}$ with $\tau_n^\infty(\mathcal{W}) = 0$ if and only if L bounds an order $n+1$ twisted Whitney tower.

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- $\tau_n^{\infty}(L) := \tau_n^{\infty}(\mathcal{W}) \leftrightarrow$ Milnor and higher-order Arf invariants

Towards intersection invariants $\tau_n^\infty(\mathcal{W}) = [t(\mathcal{W})] \in \mathcal{T}_n^\infty$ for order n twisted Whitney towers $\mathcal{W} \subset B^4$ bounded by $L \subset S^3$

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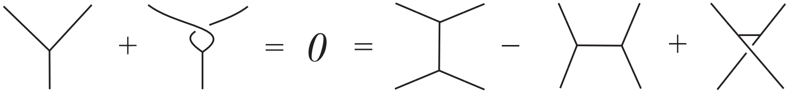
$\mathcal{T}_n :=$ free abelian group on order n framed trees modulo local *antisymmetry* (AS) and *Jacobi* (IHX) relations:



AS relations $\Rightarrow$ signs of the framed trees in $t(\mathcal{W})$ only depend on the orientation of $L = \cup_i \partial D^2 \subset \cup_i D^2 \overset{A_i}{\looparrowright} B^4$ after mapping to $\mathcal{T}_n$.

IHX trees can be created locally by controlled manipulations of Whitney disks.

The odd order target groups $\mathcal{T}^{\infty}_{2j-1}$

Obstructions to raising twisted order from $2j-1$ to $2j$:

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Obstructions to raising twisted order from $2j-1$ to $2j$:

Definition:
$\mathcal{T}^{\infty}_{2j-1}$ is the quotient of $\mathcal{T}_{2j-1}$ by *boundary-twist relations:*

$$i \,-\!\!<^{J}_{J} \, = \, 0$$

where J ranges over all order $j-1$ subtrees.

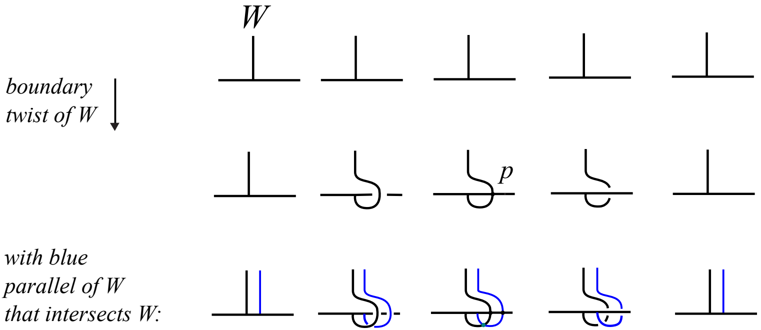
Since via <u>boundary-twisting</u> (see next frame):

$$i \,-\!\!<^{J}_{J} \quad \mapsto \quad i \,-\!\!<^{J}_{\infty} \quad + \text{ trees of order } \geq 2j$$

and the trees on the right are allowed in order $2j$ twisted $\mathcal{W}$.

Boundary twist on W changes $\omega(W)$ by ± 1, creates intersection p between W and a sheet paired by W

'Side view' near a point in ∂W:



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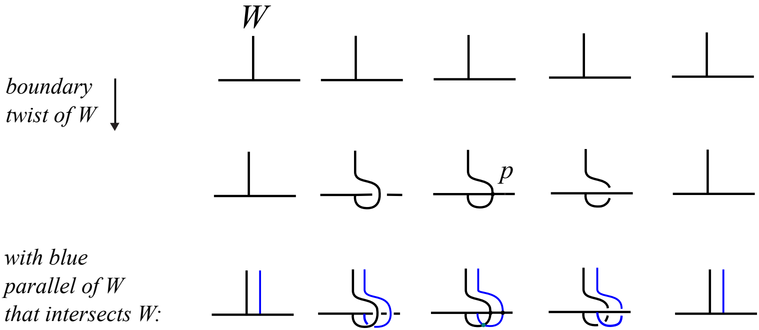

Can create any clean $W_{(I,J)}$ by finger moves,
then boundary twist into J-sheet changes $t(\mathcal{W})$ by:

$$I \prec^{J}_{J} \quad \pm \quad I \prec^{J}_{\infty}$$

The even order target groups $\mathcal{T}_{2j}^{\infty}$

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Obstructions to raising twisted order from $2j$ to $2j+1$:

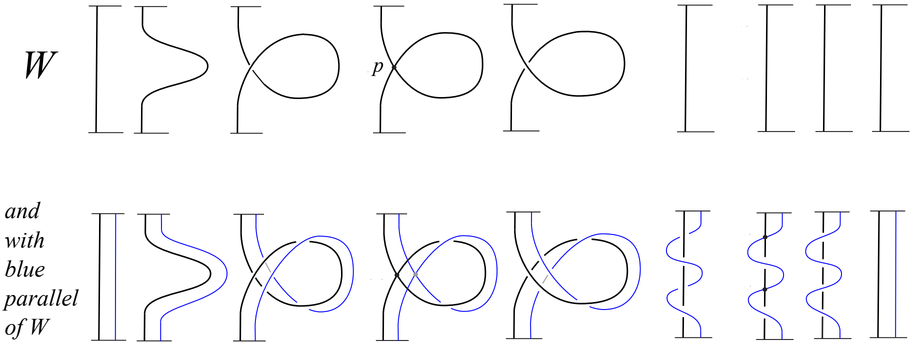
Definition:
$\mathcal{T}^{\infty}_{2j}$ is the quotient of the free abelian group on
framed trees of order $2j$ and ∞-trees of order j
by the following relations:

1. AS and IHX relations on order $2j$ framed trees
2. *symmetry* relations: $(-J)^{\infty} = J^{\infty}$
3. *twisted IHX* relations: $I^{\infty} = H^{\infty} + X^{\infty} - \langle H, X \rangle$
4. *interior-twist* relations: $2 \cdot J^{\infty} = \langle J, J \rangle$

Next frame shows how to realize interior-twist relation.
(See notes for realization of twisted IHX relation.)

$\pm$-*interior twist* on W changes $\omega(W)$ by ∓ 2 and creates $p \in W \pitchfork W$

After the interior twist,
near an arc in W that runs between the two sheets:



Can create any clean W_J by finger moves,
then $\pm$-interior twist changes $t(\mathcal{W})$ by:

$$\pm\langle J, J\rangle \quad \mp \quad 2 \cdot J^{\infty}$$

Definition:

For an order n twisted Whitney tower $\mathcal{W}$ define

$$\tau_n^{\infty}(\mathcal{W}) := [t(\mathcal{W})] \in \mathcal{T}_n^{\infty}$$

Intersection/obstruction theory for order n twisted Whitney towers

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Theorem:
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Idea of proof: Realize relations by geometric constructions to turn 'algebraic cancellation' in $\mathcal{T}_n^{\infty}$ into 'geometric cancellation' by new layer of Whitney disks.

Quick review of Milnor invariants

For $L = L_1 \cup L_2 \cup \cdots \cup L_m \subset S^3$ and $G = \pi_1(S^3 \setminus L)$:

$[L_i] \in G_{n+1}$ *(n+1)th lower central subroup* $\Longrightarrow \frac{G_{n+1}}{G_{n+2}} \cong \mathcal{L}_{n+1}$

$\mathcal{L} = \oplus_n \mathcal{L}_n$ the free $\mathbb{Z}$-Lie algebra on $\{X_1, X_2, \ldots, X_m\}$.

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Define the *order n Milnor invariant* $\mu_n(L)$:

$$\mu_n(L) := \sum_{i=1}^{m} X_i \otimes \ell_i \in \mathcal{L}_1 \otimes \mathcal{L}_{n+1}$$

where ℓ_i is the image in $\mathcal{L}_{n+1}$ of the i-th longitude $[L_i] \in \frac{G_{n+1}}{G_{n+2}}$.

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Turns out: $\mu_n(L) \in \mathcal{D}_n := \ker\{\mathcal{L}_1 \otimes \mathcal{L}_{n+1} \xrightarrow{\text{bracket}} \mathcal{L}_{n+2}\}$.

Summation maps η_n 'connect' $\tau_n^\infty(\mathcal{W})$ and $\mu_n(L)$

Definition:

The map $\eta_n : \mathcal{T}_n^\infty \to \mathcal{L}_1 \otimes \mathcal{L}_{n+1}$ is defined on generators by

$$\eta_n(t) := \sum_{v \in t} X_{\mathrm{label}(v)} \otimes \mathrm{Bracket}_v(t) \qquad \eta_n(J^\infty) := \frac{1}{2}\,\eta_n(\langle J, J\rangle)$$

Here J is a rooted tree of order j for $n = 2j$.

Examples of η_n for $n = 1, 2$

$$\begin{aligned}
\eta_1\left({}_1 \!\!-\!\!\!< {}^{3}_{2}\right) &= X_1 \otimes -\!\!\!< {}^{3}_{2} \;+\; X_2 \otimes {}_1\!\!-\!\!\!<{}^{3} \;+\; X_3 \otimes {}_1\!\!-\!\!\!<{}_{2} \\
&= X_1 \otimes [X_2, X_3] + X_2 \otimes [X_3, X_1] + X_3 \otimes [X_1, X_2].
\end{aligned}$$

$$\begin{aligned}
\eta_2\left({}_\infty \!\!-\!\!\!< {}^{2}_{1}\right) &= \tfrac{1}{2}\,\eta_2\left({}^{1}_{2} >\!\!-\!\!\!< {}^{2}_{1}\right) \\
&= X_1 \otimes {}_2\!>\!\!-\!\!\!< {}^{2}_{1} + X_2 \otimes {}^1\!>\!\!-\!\!\!< {}^{2}_{1} \\
&= X_1 \otimes [X_2, [X_1, X_2]] + X_2 \otimes [[X_1, X_2], X_1].
\end{aligned}$$

The summation maps η_n 'connect' $\tau_n^\infty(\mathcal{W})$ and $\mu_n(L)$

The image of η_n is equal to the bracket kernel $\mathcal{D}_n < \mathcal{L}_1 \otimes \mathcal{L}_{n+1}$.

Theorem:
If L bounds a twisted Whitney tower $\mathcal{W}$ of order n, then the order q Milnor invariants $\mu_q(L)$ vanish for $q < n$, and

$$\mu_n(L) = \eta_n \circ \tau_n^\infty(\mathcal{W}) \in \mathcal{D}_n$$

Proof idea: *Gropes* in $B^4 \setminus \mathcal{W}$ display longitudes of L as iterated commutators exactly according to $\eta_n \circ \tau_n^\infty(\mathcal{W})$...

The order n twisted Whitney tower filtration on links

$$\mathsf{W}_n^\infty := \frac{\{\text{links in } S^3 \text{ bounding order } n \text{ twisted Whitney towers in } B^4\}}{\text{order } n{+}1 \text{ twisted Whitney tower concordance}}$$

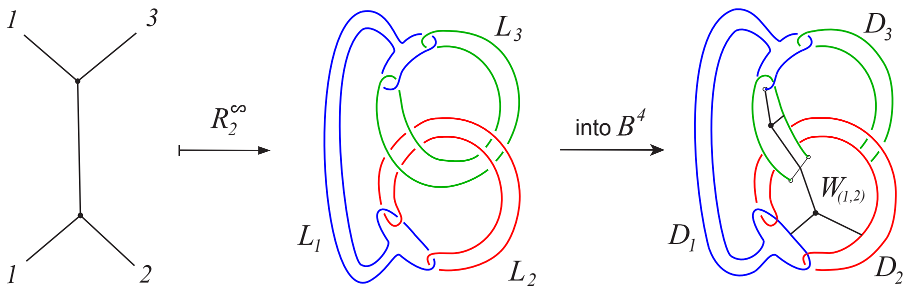
Obstruction theory $\Longrightarrow$ W_n^∞ is a finitely generated abelian group

Via Cochran's Bing-doubling techniques get epimorphisms

$$R_n^\infty : \mathcal{T}_n^\infty \twoheadrightarrow \mathsf{W}_n^\infty$$

which send $g \in \mathcal{T}_n^\infty$ to the equivalence class of links bounding an order n twisted Whitney tower $\mathcal{W}$ with $\tau_n^\infty(\mathcal{W}) = g$.

Example of $R_n^\infty : \mathcal{T}_n^\infty \twoheadrightarrow \mathsf{W}_n^\infty$ for $n = 2$



L bounds $\mathcal{W}$ with $\tau_2^\infty(\mathcal{W}) = \frac{1}{2} \succ\!\!-\!\!\prec \frac{1}{3}$

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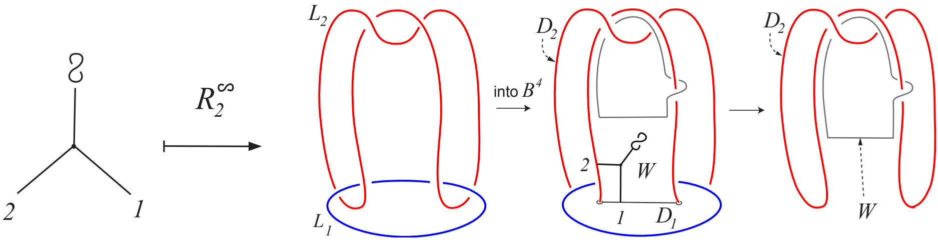


L bounds $\mathcal{W}$ with $\tau_2^\infty(\mathcal{W}) = {}^{2}_{1}\!\!>\!\!\!-\!\!-\, \infty$

Computation of W_n^∞ for $n \equiv 0, 1, 3 \mod 4$

Have commutative triangle diagram of epimorphisms:

$$\begin{array}{ccc} \mathcal{T}_n^\infty & \xrightarrow{R_n^\infty} & \mathrm{W}_n^\infty \\ & {\scriptstyle \eta_n} \searrow & \downarrow {\scriptstyle \mu_n} \\ & & \mathcal{D}_n \end{array}$$

Theorem:

The maps $\eta_n : \mathcal{T}_n^\infty \to \mathcal{D}_n$ are isomorphisms for $n \equiv 0, 1, 3 \mod 4$.

Corollary:

For $n \equiv 0, 1, 3 \mod 4$:

- $\mu_n : \mathrm{W}_n^\infty \to \mathcal{D}_n$ *and* $R_n^\infty : \mathcal{T}_n^\infty \to \mathrm{W}_n^\infty$ *are isomorphisms.*
- $\tau_n^\infty(\mathcal{W}) \in \mathcal{T}_n^\infty$ *only depends on* $L = \partial\mathcal{W}$.

Towards computation of W_n^∞ for remaining cases $n \equiv 2 \mod 4$

$\mathcal{D}_n$ is a free abelian group of known rank for all n, so have a complete computation of $\mathsf{W}_n^\infty \cong \mathcal{D}_n \cong \mathcal{T}_n^\infty$ in three quarters of the cases.

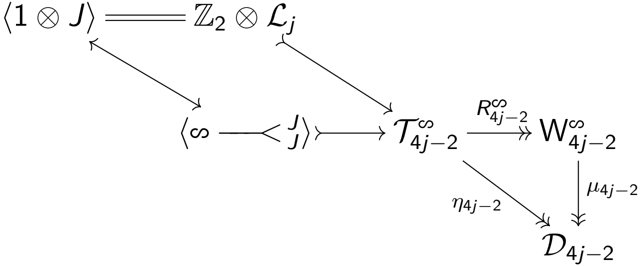
Towards understanding the remaining cases $n \equiv 2 \mod 4$:

Proposition:

The map $1 \otimes J \mapsto \infty \longrightarrow\!\!\!< {}^{J}_{J} \in \mathcal{T}^{\infty}_{4j-2}$ *induces an isomorphism:*

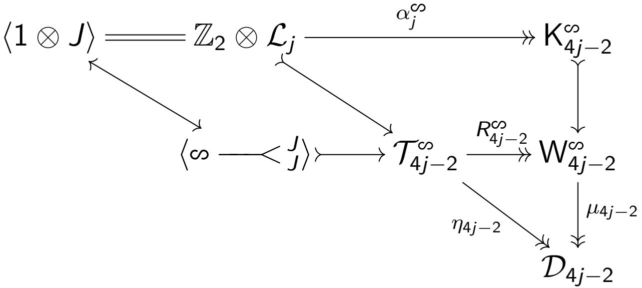
$$\mathbb{Z}_2 \otimes \mathcal{L}_j \cong \mathrm{Ker}(\eta_{4j-2} : \mathcal{T}^{\infty}_{4j-2} \to \mathcal{D}_{4j-2})$$

Extending the algebraic side of the triangle:



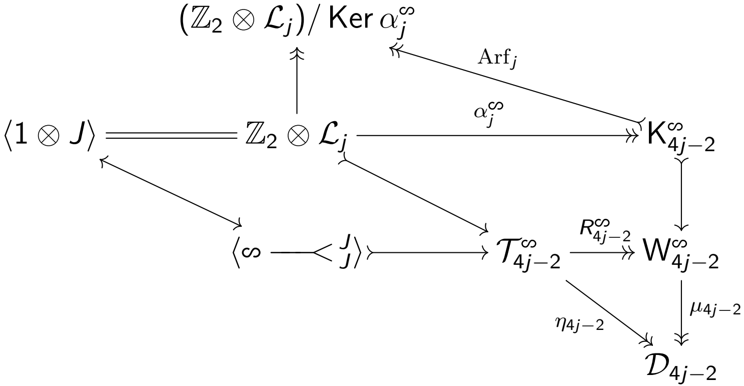
Towards defining higher-order Arf invariants

R^{∞}_{4j-2} induces $\alpha^{\infty}_{j} : \mathbb{Z}_2 \otimes \mathcal{L}_j \twoheadrightarrow \mathsf{K}^{\infty}_{4j-2} := \ker\{\mu_{4j-2} : \mathsf{W}^{\infty}_{4j-2} \twoheadrightarrow \mathcal{D}_{4j-2}\}$



Higher-order Arf invariant diagram

Also extending the topological side of the triangle:



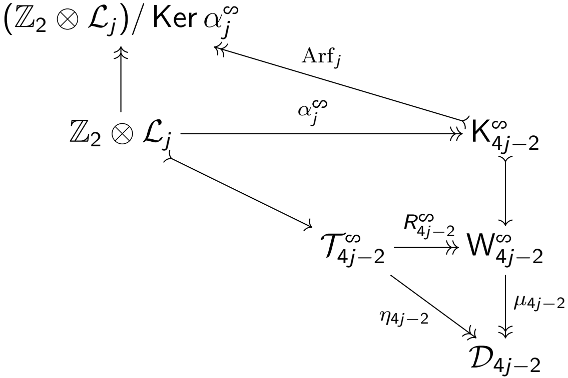
$$\mathrm{Arf}_j := \mathsf{K}_{4j-2}^\infty \to (\mathbb{Z}_2 \otimes \mathsf{L}_j)/\operatorname{Ker} \alpha_j^\infty$$

Corollary:

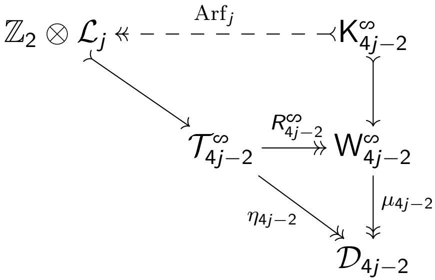
The groups W_n^∞ are classified by Milnor invariants μ_n and, in addition, higher-order Arf invariants Arf_j for $n = 4j - 2$.

In particular, a link bounds an order $n+1$ twisted $\mathcal{W}$ if and only if its Milnor invariants and higher-order Arf invariants vanish up to order n.

Higher-order Arf invariant diagram



Conjectured higher-order Arf invariant diagram



Conjecture: (Higher-order Arf invariant conjecture)

$\mathrm{Arf}_j : \mathsf{K}^{\infty}_{4j-2} \to \mathbb{Z}_2 \otimes \mathsf{L}_j$ *are isomorphisms for all* j.

This conjecture would imply $\mathsf{W}^{\infty}_n \xrightarrow{\tau^{\infty}_n} \mathcal{T}^{\infty}_n$ is an isomorphism for all n.

Determining the image of $2 \leq \mathrm{Arf}_j \leq \mathbb{Z}_2 \otimes \mathcal{L}_j$?

- Arf_1 corresponds to classical Arf invariants of the link components. Are the Arf_j for $j > 1$ also determined by finite type isotopy invariants?

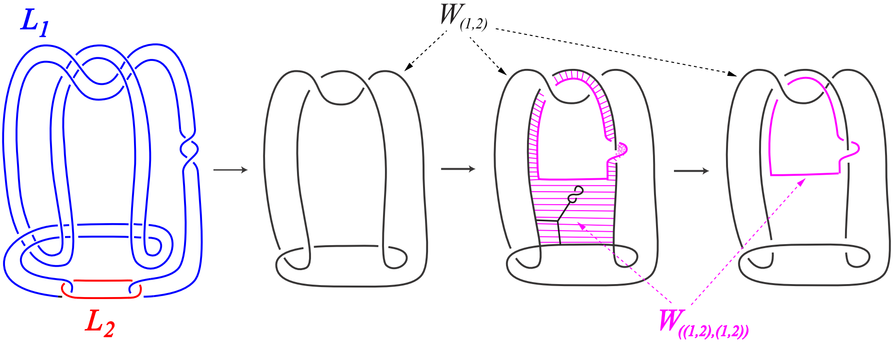
- The links $R^{\infty}_{4j-2}(\infty -\!\!\!< {}^{J}_{J}\,)$ realizing the image of Arf_j are known not to be *slice* by work of J.C. Cha.

- Fundamental first open test case: Does the Bing double of the Figure-8 knot $R^{\infty}_{6}(\infty -\!\!\!< {}^{(1,2)}_{(1,2)}\,) \in \mathsf{W}^{\infty}_{6}$ bound an order 7 twisted Whitney tower?

- *If* the Bing double of the Figure-8 knot *does* bound an order 7 twisted Whitney tower, then Arf_j are trivial for all $j \geq 2$.

Bing(Fig8) bounds $\mathcal{W}$ with $t(\mathcal{W}) = ((1,2),(1,2))^\infty$

$$\mathcal{W} = D_1 \cup D_2 \cup W_{(1,2)} \cup W_{(1,2),(1,2))}$$



Re-formulations of the higher-order Arf invariant Conjecture

- There does not exist $A : S^2 \cup S^2 \looparrowright B^4$ supporting $\mathcal{W}$ with

$$t(\mathcal{W}) = \infty \!\!-\!\!\!< \begin{smallmatrix}(1,2)\\(1,2)\end{smallmatrix}$$

(possibly + higher-order trees).

- The Bing double of any knot with non-trivial classical Arf invariant does not bound an order 6 *framed* Whitney tower.

- There does not exist $A : S^2 \cup S^2 \looparrowright B^4$ supporting $\mathcal{W}$ with

$$t(\mathcal{W}) = \langle((((((1,2),1),2),1),2),1)\rangle + \langle((((((1,2),2),1),2),1),2)\rangle$$

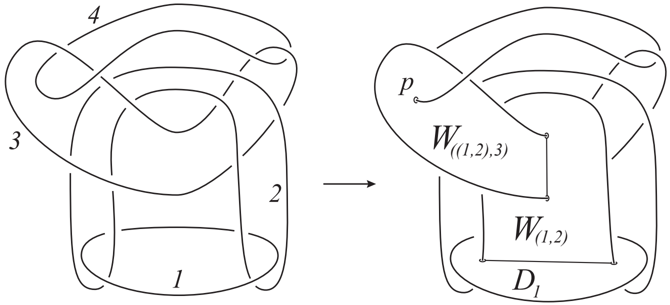
(possibly + higher-order trees).

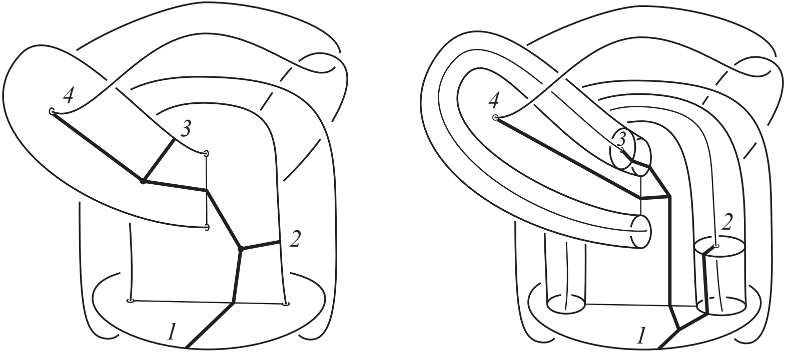
More questions/problems

- Equivariant Milnor and Arf invariant correspondence with π_1-decorated tree-valued intersection invariants for order n Whitney towers bounded by links in non-simply-connected 3-manifolds?

- Use $t(\mathcal{W})$ to efficiently formulate indeterminacies in Milnor invariants?

- Higher-order Arf invariants for 2-spheres supporting Whitney towers in 4-manifolds?


4
3
2
1
p
W((1,2),3)
W(1,2)
D1


4
3
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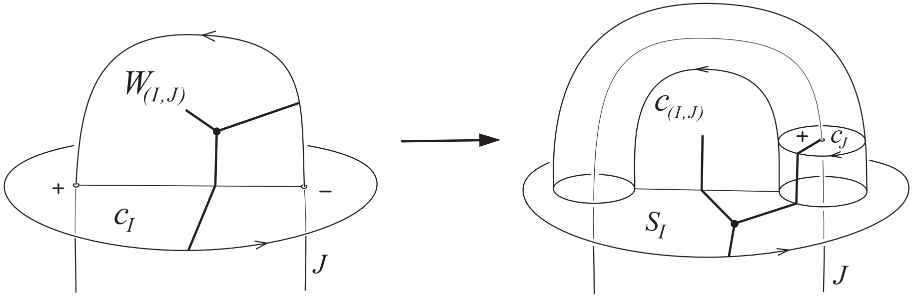

$W_{(I,J)}$
+
−
c_I
J
$c_{(I,J)}$
+
c_J
S_I
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