ALL 4-MANIFOLDS HAVE SPIN^c STRUCTURES

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1. Introduction

The recent developments in the theory of smooth 4-manifolds come from the so-called *monopole-equations* found by Seiberg and Witten [4]. They are the abelian version of Donaldson's instanton equations which had led to Donaldson's polynomial invariants in [1]. These invariants it possible to find exotic structures on many 4-manifold, may be most prominantly on Euclidean 4-space. The corresponding Seiberg-Witten invariants seem to contain the same information but are easier to compute due to the fact that the Gauge group is abelian.

In order to write down the monopole-equations on a smooth 4-dimensional manifold M one has to choose a Riemannian metric and a spin^c-structure on M. It turns out that the Seiberg-Witten invariants do not depend on the metric if $b_2^+(M) \geq 2$. But they depend crucially on the spin^c-structure, see for example [5].

In this note we prove that every orientable 4-manifold allows spin^c-structures. This was shown in the closed case by Hirzebruch and Hopf in [3]. They use Poincaré duality and a dimension counting argument which a priori does not apply in the non-compact setting.

We remark that the analogues result in the non-orientable case fails: $\mathbb{RP}^2 \times \mathbb{RP}^2$ does not have a pin^c-structure. (Such a structure is not sufficient in order to get the monopole equations since one needs the notions of positive spinors and positive 2-froms.)

The question whether or not non-compact 4-manifolds allow spin^c-structures arose in the Deninger-Schneider workshop on Seiberg-Witten invariants in Oberwolfach in October 1995.

2. Spin^c-structures

Recall that the group $Spin^c(n)$ is equal to $Spin(n) \times U(1)/\langle(-1,-1)\rangle$. Therefore, it fits into a central extension

$$1 \longrightarrow U(1) \longrightarrow Spin^c(n) \longrightarrow SO(n) \longrightarrow 1.$$

Given an $SO(n)$-pricipal bundle P over a space X one can thus ask for the existence of a reduction of the structure group to $Spin^c(n)$. Such a reduction exists for P if and only if the second Stiefel-Whitney class $w_2(P) \in H^2(X;\mathbb{Z}/2)$ is the mod 2 reduction of an integral cohomology class, see []. If X happens to be an oriented Riemannian manifold of dimension n then the bundle of oriented orthonormal frames is an $SO(n)$-principal bundle P_X. Note that $w_2(P_X)$ is independent of the orientation and Riemannian metric because it equals $w_2(TX)$, TX the tangent bundle of X. The result anounced in the introduction thus follows from the following

Proposition . *Let X be an orientable 4-manifold. Then $w_2(TX)$ is the reduction of a class in $H^2(X;\mathbb{Z})$.*

Proof. Consider the following commutative diagram of universal coefficient theorems induced by the projection $p : \mathbb{Z} \to \mathbb{Z}/2$:

$$\begin{array}{ccccc}
\mathrm{Ext}(H_1(X;\mathbb{Z}),\mathbb{Z}) & \longrightarrow & H^2(X;\mathbb{Z}) & \longrightarrow & \mathrm{Hom}(H_2(X;\mathbb{Z}),\mathbb{Z}) \\
\big\downarrow \mathrm{Ext}(p) & & \big\downarrow p & & \big\downarrow \mathrm{Hom}(p) \\
\mathrm{Ext}(H_1(X;\mathbb{Z}),\mathbb{Z}/2) & \longrightarrow & H^2(X;\mathbb{Z}/2) & \longrightarrow & \mathrm{Hom}(H_2(X;\mathbb{Z}),\mathbb{Z}/2)
\end{array}$$

Note that the induced map $\mathrm{Ext}(p)$ is an epimorphism since $\mathrm{Ext}^2_{\mathbb{Z}}(.,.) = 0$. Let $w \in \mathrm{Hom}(H_2(X;\mathbb{Z}),\mathbb{Z}/2)$ be defined by the Kronecker pairing

$$w(x) := \langle w_2(TX), x\rangle \in \mathbb{Z}/2.$$

It suffices to show that w is in the image of $\mathrm{Hom}(p)$. To this end we prove the following Lemma. In the closed case it follows from the Wu-formula which relates the Steenrod squares of the Wu-classes to the Stiefel-Whitney classes. But we will give a more elementary argument which also holds for non-compact manifolds.

Lemma . *In the above setting we have $w(x) \equiv x \cdot x \mod 2$ for all $x \in H_2(X;\mathbb{Z})$.*

Here $\cdot$ denotes the intersection pairing on the 4-manifold X which can be defined as follows: Represent $x_1, x_2 \in H_2(X;\mathbb{Z})$ by embeddings $x_i : F_i \hookrightarrow X$ in general position. Here F_i are closed oriented surfaces. The number $x_1 \cdot x_2 \in \mathbb{Z}$ is then the signed number of intersections of the images of x_i in X. Note that we have to choose an orientation on X to make this number an integer, otherwise we only get a number mod 2. This will be crucial in the next step of our proof.

Using the above Lemma we can finish the proof of our Proposition. Define T to be the kernel of the homomorphism

$$H_2(X;\mathbb{Z}) \longrightarrow \prod_y \mathbb{Z}$$

which sends $x \in H_2(X;\mathbb{Z})$ to the vector with components $x \cdot y$ for all $y \in H_2(X;\mathbb{Z})$. (In the closed case Poincaré duality implies that T is the torsion subgroup of $H_2(X;\mathbb{Z})$.) It is clear that our homomorphism w factors through the projection map $q : H_2(X;\mathbb{Z}) \to H_2(X;\mathbb{Z})/T$, i.e. $w = w' \circ q$. From [2] it follows that $H_2(X;\mathbb{Z})/T$ is a free group since it is a countable subgroup of the group $\prod_y \mathbb{Z}$. Therefore, the map w' may be lifted to a map $H_2(X;\mathbb{Z})/T \to \mathbb{Z}$ which proves that w lies in the image of $\mathrm{Hom}(p)$. ■

Proof of the Lemma. Start with an embedding $x : F \hookrightarrow X$ representing the class $x \in H_2(X;\mathbb{Z})$. Then

$$w(x) = \langle w_2(TX), x_*[F]\rangle = \langle w_2(TF \oplus NF), [F]\rangle = \langle w_2(NF), [F]\rangle.$$

Here NF is the normal bundle of the embedding $x : F \hookrightarrow X$, a 2-dimensional vector-bundle over F. We have used that F is orientable which implies $w_1(TF) = 0$ and also $w_2(TF) = w_1(TF)^2 = 0$. Note that X and thus NF are orientable and therefore $w_2(NF)$ is the mod 2 reduction of the Euler class $e(NF)$. The number $\langle e(NF), [F]\rangle$ is well known to be computed by picking any section s of NF, in general postion to the zero-section, and then counting the zeroes of s. But this is the same as counting the number of intersections of the zero-section with the image of s and thus we get by definition $w(x) \equiv \langle e(NF), [F]\rangle \equiv x \cdot x \mod 2$. ■

References

[1] Donaldson
[2] Fuchs
[3] Hirzebruch-Hopf
[4] Seiberg-Witten
[5] Taubes

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