ALL 4-MANIFOLDS HAVE SPIN$^c$ STRUCTURES

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1. Introduction

The recent developments in the theory of smooth 4-manifolds come from the so-called monopole-equations found by Seiberg and Witten [4]. They are the abelian version of Donaldson’s instanton equations which had led to Donaldson’s polynomial invariants in [1]. These invariants it possible to find exotic structures on many 4-manifold, may be most prominently on Euclidean 4-space. The corresponding Seiberg-Witten invariants seem to contain the same information but are easier to compute due to the fact that the Gauge group is abelian.

In order to write down the monopole-equations on a smooth 4-dimensional manifold $M$ one has to choose a Riemannian metric and a spin$^c$-structure on $M$. It turns out that the Seiberg-Witten invariants do not depend on the metric if $b_2^+(M) \geq 2$. But they depend crucially on the spin$^c$-structure, see for example [5].

In this note we prove that every orientable 4-manifold allows spin$^c$-structures. This was shown in the closed case by Hirzebruch and Hopf in [3]. They use Poincaré duality and a dimension counting argument which a priori does not apply in the non-compact setting.

We remark that the analogues result in the non-orientable case fails: $\mathbb{R}P^2 \times \mathbb{R}P^2$ does not have a pin$^c$-structure. (Such a structure is not sufficient in order to get the monopole equations since one needs the notions of positive spinors and positive 2-forms.)

The question whether or not non-compact 4-manifolds allow spin$^c$-structures arose in the Deninger-Schneider workshop on Seiberg-Witten invariants in Oberwolfach in October 1995.

2. Spin$^c$-structures

Recall that the group $Spin^c(n)$ is equal to $Spin(n) \times U(1)/\langle (-1, -1) \rangle$. Therefore, it fits into a central extension

$$1 \rightarrow U(1) \rightarrow Spin^c(n) \rightarrow SO(n) \rightarrow 1.$$ 

Given an $SO(n)$-pricipal bundle $P$ over a space $X$ one can thus ask for the existence of a reduction of the structure group to $Spin^c(n)$. Such a reduction exists for $P$ if and only if the second Stiefel-Whitney class $w_2(P) \in H^2(X; \mathbb{Z}/2)$ is the mod 2 reduction of an integral cohomology class, see []. If $X$ happens to be an oriented Riemannian manifold of dimension $n$ then
the bundle of oriented orthonormal frames is an $SO(n)$-principal bundle $P_X$. Note that $w_2(P_X)$ is independent of the orientation and Riemannian metric because it equals $w_2(TX)$, $TX$ the tangent bundle of $X$. The result announced in the introduction thus follows from the following

**Proposition.** Let $X$ be an orientable 4-manifold. Then $w_2(TX)$ is the reduction of a class in $H^2(X; \mathbb{Z})$.

**Proof.** Consider the following commutative diagram of universal coefficient theorems induced by the projection $p : \mathbb{Z} \to \mathbb{Z}/2$:

$$
\begin{array}{ccc}
\Ext(H_1(X; \mathbb{Z}), \mathbb{Z}) & \longrightarrow & H^2(X; \mathbb{Z}) \\
\downarrow \text{Ext}(p) & & \downarrow p \\
\Ext(H_1(X; \mathbb{Z}), \mathbb{Z}/2) & \longrightarrow & H^2(X; \mathbb{Z}/2) \\
& & \downarrow \text{Hom}(p)
\end{array}
$$

Note that the induced map $\text{Ext}(p)$ is an epimorphism since $\Ext^2_{\mathbb{Z}}(\ldots) = 0$. Let $w \in \text{Hom}(H_2(X; \mathbb{Z}), \mathbb{Z}/2)$ be defined by the Kronecker pairing

$$w(x) := \langle w_2(TX), x \rangle \in \mathbb{Z}/2.$$ 

It suffices to show that $w$ is in the image of $\text{Hom}(p)$. To this end we prove the following Lemma. In the closed case it follows from the Wu-formula which relates the Steenrod squares of the Wu-classes to the Stiefel-Whitney classes. But we will give a more elementary argument which also holds for non-compact manifolds.

**Lemma.** In the above setting we have $w(x) \equiv x \cdot x \mod 2$ for all $x \in H_2(X; \mathbb{Z})$.

Here $\cdot$ denotes the intersection pairing on the 4-manifold $X$ which can be defined as follows: Represent $x_1, x_2 \in H_2(X; \mathbb{Z})$ by embeddings $x_i : F_i \hookrightarrow X$ in general position. Here $F_i$ are closed oriented surfaces. The number $x_1 \cdot x_2 \in \mathbb{Z}$ is then the signed number of intersections of the images of $x_i$ in $X$. Note that we have to choose an orientation on $X$ to make this number an integer, otherwise we only get a number mod 2. This will be crucial in the next step of our proof.

Using the above Lemma we can finish the proof of our Proposition. Define $T$ to be the kernel of the homomorphism

$$H_2(X; \mathbb{Z}) \longrightarrow \prod_{y} \mathbb{Z}$$

which sends $x \in H_2(X; \mathbb{Z})$ to the vector with components $x \cdot y$ for all $y \in H_2(X; \mathbb{Z})$. (In the closed case Poincaré duality implies that $T$ is the torsion subgroup of $H_2(X; \mathbb{Z})$.) It is clear that our homomorphism $w$ factors through the projection map $q : H_2(X; \mathbb{Z}) \to H_2(X; \mathbb{Z})/T$, i.e. $w = w' \circ q$. From [2] it follows that $H_2(X; \mathbb{Z})/T$ is a free group since it is a countable subgroup of the group $\prod_{y} \mathbb{Z}$. Therefore, the map $w'$ may be lifted to a map $H_2(X; \mathbb{Z})/T \to \mathbb{Z}$ which proves that $w$ lies in the image of $\text{Hom}(p)$.
Proof of the Lemma. Start with an embedding $x : F \hookrightarrow X$ representing the class $x \in H_2(X; \mathbb{Z})$. Then

$$w(x) = \langle w_2(TX), x_*[F] \rangle = \langle w_2(TF \oplus NF), [F] \rangle = \langle w_2(NF), [F] \rangle.$$ 

Here $NF$ is the normal bundle of the embedding $x : F \hookrightarrow X$, a 2-dimensional vector-bundle over $F$. We have used that $F$ is orientable which implies $w_1(TF) = 0$ and also $w_2(TF) = w_1(TF)^2 = 0$. Note that $X$ and thus $NF$ are orientable and therefore $w_2(NF)$ is the mod 2 reduction of the Euler class $e(NF)$. The number $\langle e(NF), [F] \rangle$ is well known to be computed by picking any section $s$ of $NF$, in general position to the zero-section, and then counting the zeroes of $s$. But this is the same as counting the number of intersections of the zero-section with the image of $s$ and thus we get by definition $w(x) \equiv \langle e(NF), [F] \rangle \equiv x \cdot x \mod 2$. ■

References

[1] Donaldson
[2] Fuchs
[4] Seiberg-Witten
[5] Taubes

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