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Universal quadratic forms and Whitney tower intersection invariants

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A general algebraic theory of quadratic forms is developed and then specialized from the non-commutative to the commutative to, finally, the symmetric settings. In each of these contexts we construct *universal* quadratic forms. We then show that the intersection invariant for twisted Whitney towers in the 4–ball is such a *universal* symmetric refinement of the framed intersection invariant. As a corollary, we obtain a short exact sequence, Theorem 11, that has been essential in a sequence of papers by the authors on the classification of Whitney towers in the 4–ball.

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Dedicated to Mike Freedman, on the occasion of his 60th birthday

1 Introduction

This paper is about an algebraic theory of quadratic forms. For example, we construct various *universal* quadratic refinements of a given hermitian form as in Theorem 21. This kind of algebra became necessary to formalize our intersection theory of Whitney towers in 4–manifolds, see our survey [4]. Another important new result here is Theorem 11 which has already been used in our main paper [5] on the subject.

We begin the paper by explaining the first appearance of these higher-order intersection invariants. This is not directly relevant for the rest of the paper but serves as a motivation and a homage to Mike Freedman's work.

Let M be a closed oriented simply connected 4–manifold, not necessarily smooth. The intersection form λ_M can be defined on $H^2(M)$ using cup-products or on $H_2(M)$ using geometric intersections: Any class in $H_2(M)$ can be represented by a (topologically generic) immersed sphere $S\colon S^2 \looparrowright M$. This means that S looks locally like $\mathbb{R}^2 \times 0 \subset \mathbb{R}^4$, except for finitely many double points around which S looks like $\mathbb{R}^2 \times 0 \cup 0 \times \mathbb{R}^2 \subset \mathbb{R}^4$. Similarly, any two classes in $H_2(M)$ can be represented by a

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pair $S, S'\colon S^2 \looparrowright M$ which intersect generically in the same sense and $\lambda_M(S, S') \in \mathbb{Z}$ just counts their (oriented) intersection points.

Given $S\colon S^2 \looparrowright M$, one can add local self-intersection points to S until their algebraic sum is zero. This operation is a sequence of cusp homotopies, not changing the homotopy (hence homology) class $[S] \in H_2(M)$ but changing the Euler number of the normal bundle of S to become equal to $\lambda_M(S, S)$. Pick a pairing of the $\{\pm 1\}$ self-intersection points of S and choose *Whitney disks* W_i as in Figure 1, one for each such pair of self-intersections. The Whitney disks are (topologically generic) immersed disks $W_i\colon D^2 \looparrowright M$ whose boundary consists of two arcs, each going between the two intersection points but on different sheets of S.

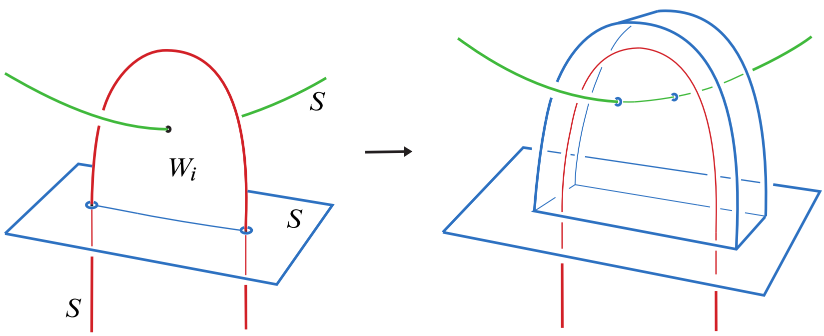

Figure 1: A (framed) Whitney disk and a Whitney move

We obtain an intersection invariant $\tau_1(S, W_i) \in \mathbb{Z}_2$, computed by summing the (topologically generic) intersections between S and (the interiors of) framed Whitney disks W_i:

$$\tau_1(S, W_i) := \sum_i \#\{S \pitchfork W_i\} \quad \text{mod } 2$$

Remark 1 Figure 1 shows a *framed* Whitney disk W_i in the sense that there are two *disjoint* parallel copies of W_i, as needed for the Whitney move on the right hand side. In general, a Whitney disk comes with a framing of its boundary and hence admits a well defined Euler number in $\mathbb{Z}$, its *twist*. The operation of *boundary twisting* (Freedman and Quinn [9]) allows us to assume that all Whitney disks are framed, ie have twist zero. Moreover, we can also assume that the W_i are (disjointly) embedded disks, by pushing all (self)-intersections off the boundary.

If $[S] \in H_2(M)$ is represented by an embedding then obviously $\tau_1(S, W_i) = 0$ for some choices of S, W_i. In fact, one can either say that no Whitney disks W_i are needed or that they are embedded with interiors disjoint from S and hence a sequence of Whitney moves leads to an embedding. As a consequence, the following result implies that τ_1 is an obstruction to representing characteristic elements $[S] \in H_2(M)$ by embeddings. We will explain in Lemma 6 how to relate this to the original approach in Freedman and Kirby [8].

Theorem 2 (Freedman–Kirby) *Let $c \in H_2(M)$ be* characteristic *in the sense that*

$$\lambda_M(c,x) \equiv \lambda_M(x,x) \mod 2 \quad \forall x \in H_2(M).$$

Then $\tau_M(c) := \tau_1(S, W_i) \in \mathbb{Z}_2$ does not depend on the choices S, W_i discussed above. Moreover, the following generalization of Rokhlin's theorem holds:

$$\mathrm{KS}(M) \equiv \tau_M(c) + \frac{\lambda_M(c,c) - \mathrm{signature}(\lambda_M)}{8} \mod 2$$

Here $\mathrm{KS}(M)$ *is the* Kirby–Siebenmann invariant *of the simply connected 4–manifold M.*

Rokhlin's original theorem is the case where M is smooth and $c = 0$ (implying that $\tau_M(c) = 0 = \mathrm{KS}(M)$ and hence that signature(λ_M) is divisible by 16). In [9], the invariant $\tau_M(c)$ was called the *Kervaire–Milnor invariant* because these authors [10] first generalized Rokhlin's formula to the case where M is smooth and c is represented by an embedded sphere (implying that $\tau_M(c) = 0 = \mathrm{KS}(M)$ but possibly with $\lambda_M(c,c) \neq 0$).

The set $C(\lambda_M)$ of characteristic elements is a $H_2(M)$-torsor via the action $(c,x) \mapsto c+2x$. Rokhlin's theorem above implies that $\tau_M\colon C(\lambda) \to \mathbb{Z}_2$ is a *quadratic refinement* of λ_M in the sense that:

$$\tau_M(c+2x) \equiv \tau_M(c) + \frac{\lambda_M(c,x) - \lambda_M(x,x)}{2} \mod 2$$

This formula implies that τ_M is completely determined by *one* of its values, knowing λ_M modulo 4. One should think of the pair (λ_M, τ_M) as the basic quadratic form of M which is a purely algebraic invariant characterized by the above condition.

A beautiful consequence of Mike Freedman's disk embedding theorem is the existence of non-smoothable 4–manifolds. In the simply connected setting, we can use the discussion above to formulate it as follows:

Theorem 3 (Freedman) *Any* odd *unimodular symmetric form $\lambda\colon \mathbb{Z}^m \otimes \mathbb{Z}^m \to \mathbb{Z}$ is realized as the intersection form of exactly* two *closed simply connected oriented 4–manifolds (up to homeomorphism). These 4–manifolds are homotopy equivalent and are distinguished by the following (equivalent) criteria: Exactly one of the two manifolds M with $\lambda_M \cong \lambda$ …*

(i) *…is smoothable after crossing with* $\mathbb{R}$.

(ii) *…is smoothable after connected sum with finitely many copies of* $S^2 \times S^2$.

(iii) *…has a linear reduction of its normal micro bundle.*

(iv) *…has vanishing Kirby–Siebenmann invariant* $\mathrm{KS}(M) \in \mathbb{Z}_2$.

(v) *…exhibits the following formula for its quadratic refinement* τ_M *of* λ_M:

$$\tau_M(c) \equiv \frac{\lambda_M(c,c) - \text{signature}(\lambda_M)}{8} \mod 2 \quad \forall \textit{ characteristic elements } c.$$

From our current point of view, the beauty of the invariant τ_M is that it has a simple geometric definition and at the same time carries deep information about (stable) smoothability of M and its normal micro bundle. It follows from the above theorem that τ_M is not invariant under homotopy equivalences (even though λ_M is). As a consequence, the quadratic refinement of λ *cannot* be defined for Poincaré complexes.

Remark 4 For every *even* unimodular symmetric form λ, Freedman showed that there is a *unique* closed simply connected topological 4–manifold realizing it. A particular case is the Poincaré conjecture.

Remark 5 By Donaldson's Theorem A [6], *exactly* the diagonalizable odd forms λ are realized by closed *smooth* 4–manifolds. Diagonal forms are realized by connected sums of complex projective planes (with varying orientations); in fact, most such forms are now known to admit infinitely many smooth representatives (all being homeomorphic by the above theorem), see eg Fintushel, Park and Stern [7].

To connect with our theory of Whitney towers in [4; 5], we recall that the 2–complex $\mathcal{W} := S \cup W_i$ in M is referred to as a *Whitney tower* of order 1 supported by S with order 1 Whitney disks W_i. The invariant $\tau_1(\mathcal{W}) = \tau_1(S, W_i)$ used above is the *order 1 intersection invariant* of such Whitney towers, the order zero intersection invariants being given by the intersection form λ_M. In a sequence of papers, we generalized this invariant to higher orders, see for example our survey [4].

The idea is that if $\tau_1(\mathcal{W})$ vanishes then all intersections between S and W_i can be paired by *order 2* Whitney disks $W_{i,j}$ and there should be a second order intersection invariant $\tau_2(\mathcal{W}, W_{i,j})$ measuring the obstruction for finding order 3 Whitney disks, and so on.

In [5] we worked out this higher-order intersection theory in detail for Whitney towers built on immersed disks in the 4–ball bounded by framed links in the 3–sphere. In this simply connected setting the invariant $\tau_n(\mathcal{W})$ of an order n (framed) Whitney tower

$\mathcal{W}$ takes values in an abelian group $\mathcal{T}_n(m)$ generated by trivalent trees (where m is number of link components), and the vanishing of $\tau_n(\mathcal{W})$ implies that the link bounds an order $n+1$ Whitney tower. For links bounding *twisted* Whitney towers there is an analogous obstruction theory and intersection invariant $\tau_n^\infty(\mathcal{W}) \in \mathcal{T}_n^\infty(m)$, and in the main Section 4 of this paper we develop an algebraic theory of quadratic forms, leading to a beautiful relation between these framed and twisted obstruction groups, spelled out in Theorem 11. This result is used in the computation of the Whitney tower filtration on classical links described in [5]. The groups $\mathcal{T}_n(m)$ and $\mathcal{T}_n^\infty(m)$ are recalled in Section 3, after the introductory exposition of the origins of the first order intersection theory is completed in Section 2.

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2 A combinatorial approach to the Kirby–Siebenmann invariant

Freedman and Kirby proved the generalized Rokhlin formula from Theorem 2 also in the non-simply connected setting, see Kirby [11, XI, Theorem 2]. They considered a *characteristic surface* in an oriented 4–manifold M, ie an embedded oriented surface $\Sigma \subset M$ together with a spin structure on $M \smallsetminus \Sigma$ that does not extend across Σ. Let $\pi\colon S\nu(\Sigma, M) \to \Sigma$ be the projection map of the boundary of a normal disk bundle for Σ. This 3–manifold inherits a spin structure from that of $M \smallsetminus \Sigma$ and so do any codimension one submanifolds of it. In particular, taking the inverse image torus $\pi^{-1}(a)$ for a circle a in Σ one sees that a comes equipped with a canonical spin structure (because the fiber circle of π has the non-bounding spin structure). Varying the circles a gives a canonical spin structure σ on Σ.

Freedman and Kirby define $\phi(M, \Sigma) \in \mathbb{Z}_2$ to be the spin bordism class of (Σ, σ) (which is detected by its Arf invariant). They prove the same Rokhlin formula that was stated in the simply connected setting in Theorem 2 (with $\tau_M(c)$ replaced by $\phi(M, \Sigma)$ as explained by Lemma 6). We note that Rokhlin's formula implies that $\phi(M, \Sigma)$ does not depend on the original spin structure on $M \smallsetminus \Sigma$.

Now assume in addition that $[\Sigma] \in H_2(M)$ is represented by $S\colon S^2 \looparrowright M$ and that the self-intersection points of S are paired by Whitney disks $W_1, \dots, W_g$. As explained in the introduction, this means that $[\Sigma]$ is represented by a Whitney tower of order 1.

It is not hard to see that this condition is equivalent to saying that $[\Sigma]$ is represented by a *capped surface*, see the proof below. Note that a surface $\Sigma \subset M$ admits caps if and only if the induced map $\pi_1(\Sigma) \to \pi_1(M)$ is trivial.

These equivalent conditions are always satisfied if M is simply connected as assumed in the introduction. Exactly as explained there, we can define $\tau_1(S, W_i) \in \mathbb{Z}_2$ to be the sum of all intersections between the immersed sphere S and the interiors of the Whitney disks W_i. We then get the same result as in the simply connected setting:

Lemma 6 *If Σ is a characteristic surface represented by a Whitney tower (S, W_i) of order 1 (or equivalently, by a capped surface) then $\tau_1(S, W_i) = \phi(M, \Sigma)$.*

Proof In [8] the following definition of $\phi(M, \Sigma)$ is used: Assume that the characteristic surface Σ comes equipped with (immersed, framed) *caps*. These are (immersed, framed) disks $A_1, \dots, A_g, B_1, \dots, B_g$ in M bounding a hyperbolic basis $a_1, \dots, a_g, b_1, \dots, b_g$ of embedded circles in Σ.

Freedman and Kirby show that the spin structure σ' on Σ is equivalent to the quadratic refinement $q\colon H_1(\Sigma) \to \mathbb{Z}_2$ (of the intersection form on $H_1(\Sigma)$) given by $q(a_i) =$ number of intersections between the interior of the cap A_i and Σ, and similarly for $q(b_i)$. By definition of the Arf invariant, one gets that

$$\phi(M, \Sigma) = \sum_{i=1}^{g} q(a_i) \cdot q(b_i)\,.$$

Assume now that $[\Sigma]$ is represented by an immersed sphere S whose self-intersection points are paired by (immersed, framed) Whitney disks $W_1, \dots, W_k$. We can get into the capped surface situation as follows: For each pair of self-intersection points of S, add a tube T_i on one sheet going from one self-intersection to the other. That turns S into an embedded surface Σ with half of the caps A_i given by small normal disks to Σ that bound the generating circles on T_i. Moreover, the Whitney disks W_i can serve as the dual caps B_i, preserving the framing, as illustrated in Figure 2.

By construction, $q(a_i) = 1$ since each normal disk A_i intersects Σ in a single point. Therefore, the required formula follows:

$$\phi(M, \Sigma) = \sum_{i=1}^{g} q(b_i) = \tau_1(S, W_i) \qquad \square$$

Remark 7 In the simply connected case, it is not hard to see that $\tau_1 \in \mathbb{Z}_2$ is well-defined exactly on characteristic elements. One thing to check is that it does not depend on the choices of the Whitney disks W_i. Once we fix the boundary, any two such

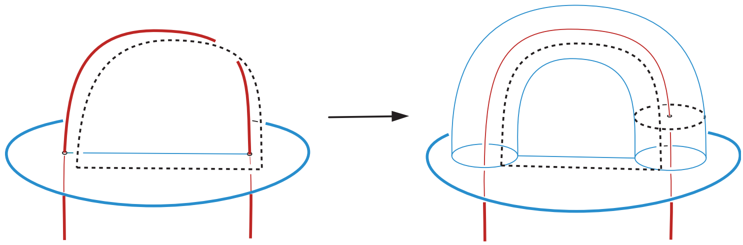

Figure 2: Turning an immersed sphere with Whitney disks into a capped surface

choices differ by a connected sum into a sphere S_i. If we require the Whitney disks to be (stably) framed then S_i needs to be (stably) framed and hence it intersects a characteristic sphere in an even number of points, leaving the count τ_1 unchanged modulo two.

Similar considerations can be found in Chapter 10 of the book [9] by Freedman and Quinn and we claim no originality. Unfortunately, the results in [9] don't hold as stated for 4–manifolds with fundamental groups that contain 2–torsion elements. The problem arises from different choices of pairings of intersection points, as pointed out by Richard Stong in [14]. Taking this into consideration, the last two authors gave a complete discussion of an enhancement of the invariant τ_1 which takes values in an infinitely generated group if $\pi_1 M$ is non-trivial [13].

3 Abelian groups generated by trees

This section recalls various algebraic aspects of our intersection theory of Whitney towers, without explaining the background. We refer the reader to our survey [4] and our paper [5] for more details.

All trees considered in this paper are *unitrivalent, oriented and labelled*. This means that they are equipped with *vertex orientations*, ie, cyclic orderings of the edges incident to each trivalent vertex, and the univalent vertices are labeled by elements of the index set $\{1, 2, \ldots, m\}$. (Indices may used more than once as labels on the same tree.) A *rooted* tree has a single designated univalent vertex called the *root* which is usually left unlabeled. All trees are considered up to label-preserving isomorphism.

The *order* of a tree is the number of trivalent vertices.

Given rooted trees I and J, the *rooted product* (I, J) is the rooted tree gotten by identifying the two roots to a vertex and adjoining a rooted edge to this new vertex, with the orientation of the new trivalent vertex given by the ordering of I and J in

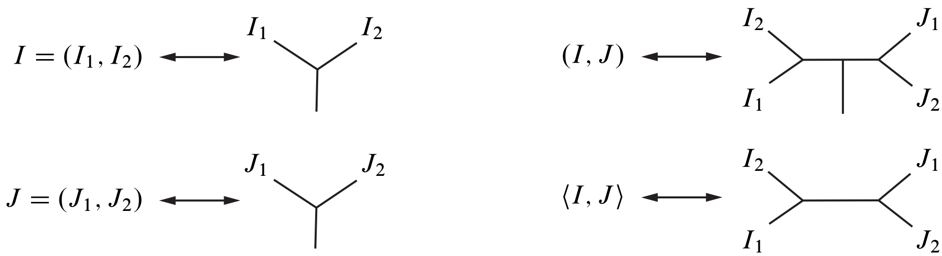

Figure 3: Rooted and inner products

(I, J). The *inner product* $\langle I, J\rangle$ of two rooted trees I and J is defined to be the unrooted tree gotten by identifying the two rooted edges to a single edge. We observe that the two products interact well in the sense of Figure 4.

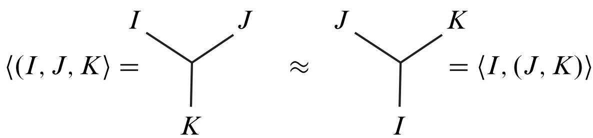

Figure 4: Invariance of the inner product

Let $\mathbb{L}(m) = \bigoplus_{n=0}^{\infty} \mathbb{L}_n(m)$ be the free abelian group generated by (isomorphism classes of) rooted trees as above. It is graded by order and the rooted product can be extended linearly to a pairing:

$$(\,\cdot\,,\,\cdot\,)\colon\ \mathbb{L}(m) \otimes \mathbb{L}(m) \longrightarrow \mathbb{L}(m)$$

This is grading preserving on $\mathbb{L}(m)[1]$, ie it preserves the grading when shifted up by one (so order is replaced by the number of univalent non-root vertices). On the other hand, the inner product

$$\langle\,\cdot\,,\,\cdot\,\rangle\colon\ \mathbb{L}(m) \otimes \mathbb{L}(m) \longrightarrow \mathbb{T}(m)$$

is grading preserving via order. Here $\mathbb{T}(m) = \bigoplus_{n=0}^{\infty} \mathbb{T}_n(m)$ is the free abelian group generated by unrooted trees as above.

Note that rotating the relevant planar trees by 180 respectively 120 degrees shows that the inner product is both *symmetric* and *invariant*: $\langle I, J\rangle = \langle J, I\rangle$ and $\langle (I, J), K\rangle = \langle I, (J, K)\rangle$, see Figure 4 for the proof of invariance.

Definition 8 The graded abelian groups $\mathcal{L}(m)$ respectively $\mathcal{T}(m)$ are defined as quotients of $\mathbb{L}(m)$ respectively $\mathbb{T}(m)$ by the AS and IHX relations as in Figure 5.

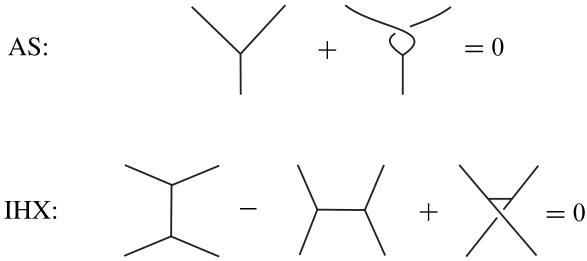

Figure 5: Local *antisymmetry* (AS) and *Jacobi* (IHX) relations in $\mathcal{L}(m)$ and $\mathcal{T}(m)$. All trivalent orientations come from an orientation of the plane, and univalent vertices extend to subtrees which are fixed in each equation.

It is well known that $\mathcal{L}(m)$ is the free (quasi) Lie algebra over $\mathbb{Z}$ on m generators with Lie bracket induced by the rooted product. Here the word *quasi* refers to the fact that we only require the antisymmetry relations $[Z,Y]=-[Y,Z]$. So $[Z,Z]$ is not necessarily zero in these Lie algebras. In our previous papers, we needed to consider both versions of Lie algebras and used the notation $\mathsf{L}'_{n+1}(m)$ for $\mathcal{L}_n(m)$ (recall that one gets a *graded* Lie algebra only when shifting the order by one). In this paper we will only study one type of Lie algebra and usually omit the adjective 'quasi'.

Remark 9 The inner product extends uniquely to a bilinear, symmetric, invariant pairing:

$$\langle\cdot,\cdot\rangle\colon\ \mathcal{L}(m)\times\mathcal{L}(m)\longrightarrow\mathcal{T}(m)$$

This follows simply from observing that the AS and IHX relations hold on both sides and are preserved by the inner product. We will show in Lemma 12 that this inner product is in fact *universal*.

Definition 10 The group $\mathcal{T}^{\infty}_{2n}(m)$ is gotten from $\mathcal{T}_{2n}(m)$ by including new order n ∞-trees as additional generators. These are rooted trees of order n as above, except that the root carries the label ∞. In addition to the IHX– and AS–relations on unrooted trees in $\mathcal{T}_{2n}(m)$, these ∞–trees are involved in the following *symmetry*, *interior twist* and *twisted IHX* relations. Here J is a rooted tree and the letters I,H,X stand for rooted trees differing locally as in Figure 5 above.

$$J^{\infty}=(-J)^{\infty}\qquad 2\cdot J^{\infty}=\langle J,J\rangle\qquad I^{\infty}=H^{\infty}+X^{\infty}-\langle H,X\rangle$$

As their names suggest, these new relations arose from geometric considerations for twisted Whitney towers in [5]. They will be explained algebraically in Section 4.8 via the theory of universal quadratic refinements.

Roughly speaking, the universal symmetric pairing $\langle\cdot,\cdot\rangle$ will be shown to admit a universal quadratic refinement $q\colon \mathcal{L}_n(m)\to\mathcal{T}^\infty_{2n}(m)$ defined by $q(J):=J^\infty$. In particular, with the right algebraic notion of 'quadratic refinement', the group $\mathcal{T}^\infty_{2n}(m)$ is completely determined by the pairing $\langle\cdot,\cdot\rangle$. The rest of this paper is devoted to finding this notion.

As a consequence, we will prove the following exact sequence at the very end of this paper. It was used substantially in [5] for the classification of Whitney towers in the 4–ball.

Theorem 11 *For all m,n, the maps $t\mapsto t$ respectively $J^\infty\mapsto 1\otimes J$ give an exact sequence:*

$$0\longrightarrow\mathcal{T}_{2n}(m)\longrightarrow\mathcal{T}^\infty_{2n}(m)\longrightarrow\mathbb{Z}_2\otimes\mathcal{L}_n(m)\longrightarrow 0$$

4 Invariant forms and quadratic refinements

In this section we explain an algebraic framework into which our groups $\mathcal{T}(m)$ and $\mathcal{T}^\infty(m)$ fit naturally. In Lemma 12 we show that the $\mathcal{T}(m)$–valued inner product $\langle\cdot,\cdot\rangle$ on the free Lie algebra is universal. Then a general theory of quadratic refinements is developed and specialized from the non-commutative to the commutative to, finally, the symmetric setting. In Corollary 35 we show that $\mathcal{T}^\infty_{2n}(m)$ is the home for the universal quadratic refinement of the $\mathcal{T}_{2n}(m)$–valued inner product $\langle\cdot,\cdot\rangle$.

We work over the ground ring of integers but all our arguments go through for any commutative ring. We also only discuss the case of finite generating sets $\{1,\dots,m\}$, even though everything holds in the infinite case.

4.1 A universal invariant form

The following lemma shows that the $\mathcal{T}(m)$–valued inner product $\langle\cdot,\cdot\rangle$ is *universal* for Lie algebras with m generators.

Lemma 12 *Let $\mathfrak{g}$ be a Lie algebra together with a bilinear, symmetric, invariant pairing $\lambda\colon\mathfrak{g}\times\mathfrak{g}\to M$ into some abelian group M. If $\alpha\colon\mathcal{L}(m)\to\mathfrak{g}$ is a Lie homomorphism (given by m arbitrary elements in $\mathfrak{g}$) there exists a* unique *linear map $\Psi\colon\mathcal{T}(m)\to M$ such that for all $X,Y\in\mathcal{L}(m)$*

$$\lambda(\alpha(X),\alpha(Y))=\Psi(\langle X,Y\rangle).$$

Proof The uniqueness of Ψ is clear since the inner product map is onto. For existence, we first construct a map $\psi\colon \mathbb{T}(m) \to M$ as follows. Given a tree $t \in \mathbb{T}(m)$ pick an edge in t to split $t = \langle X, Y\rangle$ for rooted trees $X, Y \in \mathbb{L}(m)$. Then set:

$$\psi(t) := \lambda(\alpha(X), \alpha(Y))$$

If we split t at an adjacent edge, this expression stays unchanged because of the symmetry and invariance of λ. However, one can go from any given edge to any other by a sequence of adjacent edges, showing that $\psi(t)$ does not depend on the choice of splitting.

It is clear that ψ can be extended linearly to the free abelian group on $\mathbb{T}(m)$ and since α preserves AS and IHX relations by assumption, this extension factors through a map Ψ as required. □

Remark 13 Recall that $\mathcal{L}(m)[1]$ is actually a graded Lie algebra, ie, the Lie bracket preserves the grading when shifted up by one (so order is replaced by the number of univalent non-root vertices). Let's assume in the above lemma that the groups $\mathfrak{g}$, M are $\mathbb{Z}$–graded, $\mathfrak{g}[1]$ is a graded Lie algebra and that λ, α preserve those gradings. Then the proof shows that the resulting linear map Ψ also preserves the grading.

4.2 Non-commutative quadratic groups

The rest of this section describes a general setting for relating our groups $\mathcal{T}^{\infty}_{2n}(m)$ that measure the intersection invariant of twisted Whitney towers to a universal (symmetric) quadratic refinement of the $\mathcal{T}_{2n}(m)$–valued inner product. We first give a couple of definitions that generalize those introduced by Hans Baues in [1] and [2, Sectin 8], and Andrew Ranicki in [12, page 246]. These will lead to the most general notion of quadratic refinements for which we construct a universal example. Later we shall specialize the definitions from *non-commutative* to *commutative* and finally, to *symmetric* quadratic forms and construct universal examples in all cases.

Definition 14 A *(non-commutative) quadratic group*

$$\mathfrak{M} = (M_e \xrightarrow{h} M_{ee} \xrightarrow{p} M_e)$$

consists of two groups M_e, M_{ee} and two homomorphisms h, p satisfying

(i) M_{ee} is abelian,

(ii) the image of p lies in the center of M_e,

(iii) $hph = 2h$.

$\mathfrak{M}$ will serve as the range of the (non-commutative) quadratic forms defined below. We will write both groups additively since in most examples M_e turns out to be commutative. A morphism $\beta\colon \mathfrak{M} \to \mathfrak{M}'$ between quadratic groups is a pair of homomorphisms

$$\beta_e\colon M_e \to M_e' \quad \text{and} \quad \beta_{ee}\colon M_{ee} \to M_{ee}'$$

such that both diagrams involving h, h', p, p' commute.

Examples 15 The example motivating the notation comes from homotopy theory, see eg [1]. For $m < 3n-2$, let $M_e = \pi_m(S^n)$, $M_{ee} = \pi_m(S^{2n-1})$, h be the Hopf invariant and p be given by post-composing with the Whitehead product $[\iota_n, \iota_n]\colon S^{2n-1} \to S^n$.

This quadratic group satisfies $php = 2p$ which is part of the definition used in [1], where M_e is also assumed to be commutative. As we shall see, these additional assumptions have the disadvantage that they are not satisfied for the universal Example 20.

Another important example comes from an abelian group with involution $(M, *)$. Then we let

$$(M) \qquad M_{ee} := M, \quad M_e := M/\langle x - x^* \rangle, \quad h([x]) := x + x^*$$

and p be the natural quotient map. For example, if M is a ring with involution $r \mapsto \overline{r}$, then we get two possible involutions on the abelian group M: $r^* = \pm\overline{r}$. The choice of sign determines whether we study symmetric respectively skew-symmetric pairings.

We note that in this example $hp - \mathrm{id} = *$ and in the homotopy theoretic example $hp - \mathrm{id} = (-1)^n$. In fact, we have the following lemma:

Lemma 16 *Given a quadratic group, the endomorphism $hp - \mathrm{id}$ gives an involution on M_{ee} (which we will denote by $*$). Moreover, the formula $\dagger(x) := ph(x) - x$ defines an anti-involution on M_e. These satisfy*

(i) $* \circ h = h$,

(ii) $php = p + p \circ *$,

(iii) $p \circ * = \dagger \circ p$.

The proof of this lemma is straightforward and will be left to the reader. To show that $\dagger$ is an anti-homomorphism one uses that $\mathrm{Im}(p)$ is central and that $x \mapsto -x$ is an anti-homomorphism.

Definition 17 A quadratic group $\mathfrak{M}$ is a *quadratic refinement* of an abelian group with involution $(M, *)$ if

$$M_{ee} = M \quad \text{and} \quad * = hp - \mathrm{id}.$$

It follows from (i) in Lemma 16 that in this case, the image of h lies in the fixed point set of the involution: $h\colon M_e \to M^{\mathbb{Z}_2} = H^0(\mathbb{Z}_2; M)$.

The example (M) above gives one natural choice of a quadratic refinement, however, there are other canonical (and non-commutative) ones as we shall see in Example 20.

It follows from (ii) in Lemma 16 that the additional condition $php = 2p$ used in [1] is satisfied if and only if $p = p \circ *$, or equivalently, if p factors through the cofixed point set of the involution:

$$p\colon M_{ee} \twoheadrightarrow (M_{ee})_{\mathbb{Z}_2} = H_0(\mathbb{Z}_2; M_{ee}) \to M_e$$

It follows that the notion in [12, page 246] is equivalent to that in [1], except that M_{ee} is assumed to be the ground ring R in the former. In that case, our involution is simply $r^* = \epsilon\overline{r}$, where $\epsilon = \pm 1$ and $r \mapsto \overline{r}$ is the given involution on the ring R.

In this setting, ϵ–symmetric forms in the sense of Ranicki become hermitian forms in the sense defined below. In particular, Ranicki's $(+1)$–symmetric forms are different from the notion of *symmetric* form in this paper: We reserve it for the easiest case where both involutions, $*$ and $\dagger$, are trivial.

4.3 Non-commutative quadratic forms

Definition 18 A *(non-commutative) quadratic form* on an abelian group A with values in a (non-commutative) quadratic group

$$\mathfrak{M} = \left(M_e \xrightarrow{h} M_{ee} \xrightarrow{p} M_e\right)$$

is given by a bilinear map $\lambda\colon A \times A \to M_{ee}$ and a map $\mu\colon A \to M_e$ satisfying

(i) $\mu(a + a') = \mu(a) + \mu(a') + p \circ \lambda(a, a')$ and

(ii) $h \circ \mu(a) = \lambda(a, a)$ $\forall a, a' \in A$.

We say that μ is a *quadratic refinement* of λ: Property (i) says that μ is quadratic and property (ii) means that it "refines" λ. The notation M_e and M_{ee} was designed (by Baues) to reflect the number of variables (**e**ntries) of the maps μ and λ respectively. He also writes $\lambda = \lambda_{ee}$ and $\mu = \lambda_e$, however, we decided not to follow that part of the notation.

We write $(\lambda,\mu)\colon A\to\mathfrak{M}$ for such quadratic forms and we always assume that the quadratic group $\mathfrak{M}$ is part of the data for (λ,μ). This means that the morphisms in the category of quadratic forms are pairs of morphisms

$$\alpha\colon A\to A' \quad \text{and} \quad \beta=(\beta_e,\beta_{ee})\colon \mathfrak{M}\to\mathfrak{M}'$$

such that both diagrams involving $\lambda,\lambda',\mu,\mu'$ commute.

Lemma 19 *Let $(\lambda,\mu)\colon A\to\mathfrak{M}$ be a quadratic form as above. Then λ is* hermitian *with respect to the involution $*=hp-\mathrm{id}$ on M_{ee}:*

$$\lambda(a',a)=\lambda(a,a')^*$$

and μ is hermitian *with respect to the anti-involution $\dagger=ph-\mathrm{id}$ on M_e:*

$$\mu(-a)=\mu(a)^\dagger$$

Proof As a consequence of conditions (i) and (ii) we get

$$\begin{aligned}\lambda(a,a)+\lambda(a',a')+\lambda(a',a)+\lambda(a,a')&=\lambda(a+a',a+a')\\&=h\circ\mu(a+a')=h(\mu(a)+\mu(a')+p\circ\lambda(a,a'))\\&=\lambda(a,a)+\lambda(a',a')+hp(\lambda(a,a'))\end{aligned}$$

or equivalently, $\lambda(a',a)=(hp-\mathrm{id})\lambda(a,a')=\lambda(a,a')^*$. Similarly,

$$\begin{aligned}0=\mu(0)=\mu(a-a)&=\mu(a)+\mu(-a)+p\circ\lambda(a,-a)\\&=\mu(a)+\mu(-a)-p\circ h\circ\mu(a)=\mu(-a)+(\mathrm{id}-ph)\mu(a)\end{aligned}$$

or equivalently, $\mu(-a)=\dagger\circ\mu(a)=:\mu(a)^\dagger$. □

Starting with a hermitian form λ with values in a group with involution $(M,*)$, the first step in finding a quadratic refinement for λ is to find a quadratic refinement $\mathfrak{M}$ of $(M,*)$ in the sense of Definition 17, motivating our terminology.

4.4 Universal quadratic refinements

Example 20 Given a hermitian form $\lambda\colon A\times A\to(M,*)$, one gets a quadratic refinement μ_λ of λ as follows. Set $M_{ee}:=M$ and define the universal target $M_e:=M_{ee}\times_\lambda A$ to be the group consisting of pairs (m,a) with $m\in M_{ee}$ and $a\in A$ and multiplication given by

$$(m,a)+(m',a'):=(m+m'-\lambda(a,a'),a+a')\,.$$

In other words, M_e is the central extension

$$1 \longrightarrow M_{ee} \longrightarrow M_{ee} \times_\lambda A \longrightarrow A \longrightarrow 1$$

determined by the cocycle λ, compare Section 4.7. It follows that M_e is commutative if and only if λ is *symmetric* in the naive sense that $\lambda(a',a) = \lambda(a,a')$. Set

$$p_\lambda(m) := (m,0), \quad h_\lambda(m,a) := m + m^* + \lambda(a,a)\,.$$

We claim that $\mathfrak{M}_\lambda := (M_{ee} \overset{p_\lambda}{\to} M_e \overset{h_\lambda}{\to} M_{ee})$ is a quadratic group as in Definition 14. It is clear that p_λ is a homomorphism with image in the center of M_e. The homomorphism property of h_λ follows from the fact that λ is bilinear and hermitian:

$$\begin{aligned}
h_\lambda((m,a)+(m',a')) &= h_\lambda(m+m'-\lambda(a,a'),a+a') \\
&= (m+m'-\lambda(a,a')) + (m+m'-\lambda(a,a'))^* + \lambda(a+a',a+a') \\
&= (m+m^*+\lambda(a,a)) + (m'+m'^*+\lambda(a',a')) = h_\lambda(m,a)+h_\lambda(m',a')
\end{aligned}$$

Condition (iii) of a quadratic group is also checked easily:

$$\begin{aligned}
h_\lambda p_\lambda h_\lambda(m,a) &= h_\lambda(m+m^*+\lambda(a,a),0) \\
&= (m+m^*+\lambda(a,a)) + (m+m^*+\lambda(a,a))^* \\
&= 2(m+m^*+\lambda(a,a)) = 2h_\lambda(m,a)
\end{aligned}$$

We also see that

$$(h_\lambda p_\lambda - \mathrm{id})(m) = h_\lambda(m,0) - m = (m+m^*) - m = m^*$$

which means that $\mathfrak{M}_\lambda$ "refines" (in the sense of Definition 17) the group with involution $(M,*)$. Finally, setting $\mu_\lambda(a) := (0,a)$, we claim that $(\lambda,\mu_\lambda)\colon A \to \mathfrak{M}_\lambda$ is a quadratic refinement of λ. We need to check properties (i) and (ii) of a quadratic form (Definition 18): (i) is simply $h_\lambda \circ \mu_\lambda(a) = h_\lambda(0,a) = \lambda(a,a)$, and (ii) explains why we used a sign in front of λ in our central extension:

$$\begin{aligned}
\mu_\lambda(a) + \mu_\lambda(a') + p_\lambda \circ \lambda(a,a') &= (0,a) + (0,a') + (\lambda(a,a'),0) \\
&= (-\lambda(a,a'),a+a') + (\lambda(a,a'),0) = (0,a+a') = \mu_\lambda(a+a')
\end{aligned}$$

The following result will show that μ_λ is indeed a *universal* quadratic refinement of λ. This is the content of the first statement in the theorem below. It follows from the second statement because for any quadratic refinement μ of λ it shows that forgetting the quadratic data gives canonical isomorphisms

$$\mathsf{QF}(L \circ R(\lambda,\mu),(\lambda,\mu)) \cong \mathsf{HF}(R(\lambda,\mu),R(\lambda,\mu)) = \mathsf{HF}(\lambda,\lambda)$$

where QF respectively HF are (the morphisms in) the categories of quadratic respectively hermitian forms. Since

$$L \circ R(\lambda, \mu) = L(\lambda) = (\lambda, \mu_\lambda)$$

and the morphisms in the category QR_λ of quadratic refinements of λ by definition all lie over the identity of λ, the set $\mathsf{QR}_\lambda(\mu_\lambda, \mu)$ contains a unique element, namely the required universal morphism $\mu_\lambda \to \mu$.

Theorem 21 *The quadratic form (λ, μ_λ) is initial in the category of quadratic refinements of λ. In fact, the forgetful functor $R(\lambda, \mu) = \lambda$ from the category of quadratic forms to the category of hermitian forms has a left adjoint $L\colon \mathsf{HF} \to \mathsf{QF}$ given by $L(\lambda) := (\lambda, \mu_\lambda)$.*

Proof We have to construct natural isomorphisms

$$\mathsf{QF}((\lambda, \mu_\lambda), (\lambda', \mu')) = \mathsf{QF}(L(\lambda), (\lambda', \mu')) \cong \mathsf{HF}(\lambda, R(\lambda', \mu')) = \mathsf{HF}(\lambda, \lambda')$$

for any quadratic form (λ', μ') and hermitian form λ. Recall that the morphisms in QF are pairs $\alpha\colon A \to A'$ and $\beta = (\beta_e, \beta_{ee})\colon \mathfrak{M} \to \mathfrak{M}'$ such that the relevant diagrams commute. This implies that forgetting about the quadratic datum β_e gives a natural map from the left to the right hand side above.

Given a morphism $(\alpha, \beta_{ee})\colon \lambda \to \lambda'$ consisting of homomorphisms $\alpha\colon A \to A'$ and $\beta_{ee}\colon (M_{ee}, *) \to (M'_{ee}, *')$ such that

$$\lambda'(\alpha(a_1), \alpha(a_2)) = \beta_{ee} \circ \lambda(a_1, a_2) \in M'_{ee} \quad \forall a_i \in A$$

we need to show that there is a *unique* homomorphism $\beta_e\colon M_e \to M'_e$ such that the following 3 diagrams commute:

$$\begin{tikzcd} M_e \arrow[r, "h"] \arrow[d, "\beta_e"'] \arrow[dr, phantom, "(1)"] & M_{ee} \arrow[d, "\beta_{ee}"] \\ M'_e \arrow[r, "h'"] & M'_{ee} \end{tikzcd} \qquad \begin{tikzcd} M_{ee} \arrow[r, "p"] \arrow[d, "\beta_{ee}"'] \arrow[dr, phantom, "(2)"] & M_e \arrow[d, "\beta_e"] \\ M'_{ee} \arrow[r, "p'"] & M'_e \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\mu_\lambda"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "(3)"] & M_e \arrow[d, "\beta_e"] \\ A' \arrow[r, "\mu'"] & M'_e \end{tikzcd}$$

We will now make use of the fact that $M_e = M_{ee} \times_\lambda A$ because μ_λ is given as in Example 20. In this case, diagrams (2) and (3) are equivalent to

$$\beta_e(m, 0) = p' \circ \beta_{ee}(m) \quad \text{and} \quad \beta_e(0, a) = \mu' \circ \alpha(a)$$

because $p(m) = (m, 0)$ and $\mu_\lambda(a) = (0, a)$. This implies directly the uniqueness of β_e. For existence, we only have to check that the formula

$$\beta_e(m, a) := p' \circ \beta_{ee}(m) + \mu' \circ \alpha(a)$$

gives indeed a group homomorphism $M_e \to M'_e$ that makes diagram (1) commute. Note that the image of p' is central in M'_e and hence the order of the summands does not matter. We have:

$$\begin{aligned}\beta_e((m,a)+(m',a'))\beta_e(m+m'-\lambda(a,a'),a+a')\\ &= p'\circ\beta_{ee}(m+m'-\lambda(a,a'))+\mu'\circ\alpha(a+a')\\ &= p'\circ\beta_{ee}(m)+p'\circ\beta_{ee}(m')-p'\circ\lambda'(\alpha(a),\alpha(a'))+\mu'\circ\alpha(a+a')\\ &= p'\circ\beta_{ee}(m)+p'\circ\beta_{ee}(m')+\mu'\circ\alpha(a)+\mu'\circ\alpha(a')\\ &= \beta_e(m,a)+\beta_e(m',a')\end{aligned}$$

To get to the fourth line, we used property (ii) of a quadratic form to cancel the term $p'\circ\lambda'(\alpha(a),\alpha(a'))$. For the commutativity of diagram (1) we use property (i) of a quadratic form, as well as the fact that β_{ee} preserves the involution $*$:

$$\begin{aligned}h'\circ\beta_e(m,a) &= h'(p'\circ\beta_{ee}(m)+\mu'\circ\alpha(a))\\ &= h'p'(\beta_{ee}(m))+\lambda'(\alpha(a),\alpha(a))\\ &= \beta_{ee}(m)^{*'}+\beta_{ee}(m)+\beta_{ee}\circ\lambda(a,a)\\ &= \beta_{ee}(m^*+m+\lambda(a,a))=\beta_{ee}\circ h(m,a)\end{aligned}$$

This finishes the proof of left adjointness of $L\colon \mathsf{HF}\to\mathsf{QF}$. □

If the bilinear form λ happens to be *symmetric*, or more precisely, if it takes values in a group M_{ee} with *trivial* involution $*$, then the above construction still gives a quadratic refinement μ_λ. Its target quadratic group $\mathfrak{M}_\lambda$ has the properties that M_e is abelian and $h_\lambda p_\lambda = 2\,\mathrm{id}$. It is not hard to see that our construction above leads to the following result.

Theorem 22 *For any symmetric form λ one can functorially construct a quadratic form (λ,μ_λ) that is initial in the category of quadratic refinements of λ with trivial involution $*$. In fact, the forgetful functor $R(\lambda,\mu)=\lambda$ from the category of quadratic forms with trivial involution $*$ to the category of symmetric forms has a left adjoint $L(\lambda)=(\lambda,\mu_\lambda)$.*

Remark 23 It follows from the above considerations that a quadratic form (λ,μ) is universal if and only if the homomorphism

$$M_{ee}\times_\lambda A\to M_e \quad \text{given by} \quad (m,a)\mapsto p(m)+\mu(a)$$

is an isomorphism. This in turn is equivalent to

(i) $p\colon M_{ee}\to M_e$ is injective and

(ii) $\mu\colon A\to M_e/\operatorname{Im}(p)$ is an isomorphism.

4.5 Commutative quadratic groups and forms

The case where $*$ is non-trivial but the anti-involution $\dagger$ on M_e is trivial is even more interesting. In this case, λ is still hermitian with respect to $*$ but one is only interested in quadratic refinements μ that are symmetric in the sense that $\mu(-a) = \mu(a)$. This case deserves its own definition:

Definition 24 A *commutative quadratic group*

$$\mathfrak{M} = (M_e \xrightarrow{h} M_{ee} \xrightarrow{p} M_e)$$

consists of two abelian groups M_e, M_{ee} and two homomorphism h, p satisfying $ph = 2\,\mathrm{id}$.

In fact, a commutative quadratic group is the same thing as a non-commutative quadratic group with trivial anti-involution $\dagger$. This comes from the fact that the squaring map $x \mapsto 2x$ is a homomorphism if and only if M_e is commutative. Our universal example $\mathfrak{M}_\lambda$ is in general *not* commutative because one gets in this case:

$$\begin{aligned}\dagger_\lambda(m,a) &= p_\lambda \circ h_\lambda(m,a) - (m,a) = p_\lambda(m + m^* + \lambda(a,a)) - (m,a)\\ &= (m + m^* + \lambda(a,a), 0) + (-m - \lambda(a,a), -a) = (m^*, -a)\end{aligned}$$

However, we shall see in Theorem 27 that we can just divide by these relations $(m,a) = (m^*, -a)$ to obtain another universal quadratic refinement of a given hermitian form λ but this time with values in a *commutative* quadratic group. Before we work this out, let us mention the essential example from topology.

Example 25 Consider a manifold X of dimension $2n$ and let $\mathfrak{M}$ be as in (M) from Examples 15 with $M = \mathbb{Z}[\pi_1 X]$. In particular, we have $ph - \mathrm{id} = \dagger = \mathrm{id}$ but in general the involution $*$ is non-trivial. On group elements, it is given by

$$g^* := (-1)^n w_1(g) g^{-1}$$

with w_1 (induced by) the first Stiefel–Whitney class of X. Then the equivariant intersection form $\lambda = \lambda_X$ on $\pi_n X$ is bilinear and hermitian as required. Moreover, the self-intersection invariant μ_X defined by Wall [15] gives a quadratic refinement of λ_X, at least on the subgroup A of elements represented by immersed n spheres with vanishing normal Euler number.

As discussed in the introduction, one can change an immersion by (non-regular) cusp homotopies. Each of these introduces one self-intersection point and changes the normal Euler number by ± 2. Wall's μ_X was originally defined only on *regular* homotopy classes of immersed n–spheres in X. By requiring the normal Euler number to be zero, one can also define it on the subgroup A of $\pi_n(X)$. Note that A is the kernel of the n^{th} Stiefel–Whitney class $w_n\colon \pi_n(X) \to \mathbb{Z}_2$.

In our main Theorem 27 below, we shall use the following lemma:

Lemma 26 *If $(\lambda, \mu)\colon A \to \mathfrak{M}$ is a commutative quadratic form, then $\mu(n \cdot a) = n^2 \cdot \mu(a)$ for all integers $n \in \mathbb{Z}$.*

Here we say that a quadratic form $(\lambda, \mu)\colon A \to \mathfrak{M}$ is *commutative* if the target quadratic group $\mathfrak{M}$ is commutative, ie if the anti-involution $\dagger$ is trivial (compare Definition 18).

Proof Since the involution $\dagger = ph - \mathrm{id}$ is trivial by assumption, we already know that $\mu(-a) = \mu(a)$ from Lemma 19. Thus it suffices to prove the claim for positive $n > 1$ by induction:

$$\begin{aligned}\mu((n+1)\cdot a) &= \mu(n \cdot a) + \mu(a) + p \circ \lambda(n \cdot a, a)\\ &= n^2 \cdot \mu(a) + \mu(a) + n \cdot p \circ h \circ \mu(a)\\ &= (n^2+1)\cdot \mu(a) + n \cdot 2 \cdot \mu(a) = (n+1)^2 \cdot \mu(a)\end{aligned}$$

Here we again used the fact that $p \circ h = 2\,\mathrm{id}$. □

Theorem 27 *Any hermitian bilinear form λ has a universal commutative quadratic refinement. In fact, the forgetful functor $R(\lambda, \mu) = \lambda$ from the category CQF of commutative quadratic forms to the category HF of hermitian forms has a left adjoint $L\colon \mathsf{HF} \to \mathsf{CQF}$, $L(\lambda) = (\lambda, \mu^c_\lambda)$.*

Proof As hinted to above, we will force the anti-involution $\dagger$ to be trivial in the universal construction of Theorem 21. This means that we should define the universal (commutative) group M^c_e as the quotient of our previously used group $M_{ee} \times_\lambda A$ by the relations:

$$\begin{aligned}0 &= (m^*, -a) - (m, a) = (m^*, -a) + (-m - \lambda(a,a)), -a)\\ &= (m^* - m - 2\lambda(a,a), -2a)\end{aligned}$$

By setting a respectively m to zero, these relations imply

$$(m^*, 0) = (m, 0) \quad \text{and} \quad (-2\lambda(a,a), -2a) = 0\,.$$

Vice versa, these two types of equations imply the general ones and hence we see that M^c_e is the quotient of the centrally extended group

$$1 \longrightarrow M_{ee}/(m^* = m) \longrightarrow M_{ee}/(m^* = m) \times_\lambda A \longrightarrow A \longrightarrow 1$$

by the relations $(-2\lambda(a,a),-2a)=0$. We write elements in M_e^c as $[m,a]$ with the above relations understood. It then follows that $p_\lambda^c(m):=[m,0]$ is a homomorphism $M_{ee}\to M_e^c$ (which is in general not any more injective). Moreover, our original formula leads to a homomorphism $h_\lambda^c\colon M_e^c\to M_{ee}$ given by:

$$h_\lambda^c[m,a]:=h_\lambda(m,a)=m+m^*+\lambda(a,a)$$

To see that this is well defined, observe $h_\lambda(m^*,0)=m+m^*=h_\lambda(m,0)$ and

$$h_\lambda(-2\lambda(a,a),-2a)=-4\lambda(a,a)+\lambda(-2a,-2a)=0\,.$$

Finally, we set $\mu_\lambda^c(a):=[0,a]$ to obtain a commutative quadratic refinement of λ which is proven exactly as in Theorem 21.

To show that μ_λ^c is universal, or more generally, that $L(\lambda):=(\mu_\lambda^c,\lambda)$ is a left adjoint of the forgetful functor R, we proceed as in the proof of Theorem 21: We are given a morphism $(\alpha,\beta_{ee})\colon\lambda\to\lambda'$ consisting of homomorphisms $\alpha\colon A\to A'$ and $\beta_{ee}\colon(M_{ee},*)\to(M'_{ee},*')$ such that

$$\lambda'(\alpha(a_1),\alpha(a_2))=\beta_{ee}\circ\lambda(a_1,a_2)\in M'_{ee}\quad\forall a_i\in A.$$

We need to show that there is a *unique* homomorphism $\beta_e\colon M_e^c\to M'_e$ such that the three diagrams from the proof of Theorem 21 commute. We can use the same formulas as before, if we check that they vanish on our new relations in M_e^c. For this we'll have to use that the given quadratic group $\mathfrak{M}'$ is *commutative*. Recall the formula:

$$\beta_e(m,a)=p'\circ\beta_{ee}(m)+\mu'\circ\alpha(a)$$

Splitting our relations into two parts as above, it suffices to show that

$$p'\circ\beta_{ee}(m^*)=p'\circ\beta_{ee}(m)\qquad\text{and}\qquad\beta_e(-2\lambda(a,a),-2a)=0\,.$$

The first equation follows from part (iii) of Lemma 16 and the fact that we are assuming that $\dagger'=\mathrm{id}$:

$$p'\circ\beta_{ee}(m^*)=(p'\circ *')(\beta_{ee}(m))=(\dagger'\circ p')(\beta_{ee}(m))=p'\circ\beta_{ee}(m)$$

For the second equation we compute:

$$\begin{aligned}\beta_e(-2\lambda(a,a),-2a)&=p'\circ\beta_{ee}(-2\lambda(a,a))+\mu'\circ\alpha(-2a)\\&=-2(p'\circ\lambda'(\alpha(a),\alpha(a)))+\mu'\circ\alpha(-2a)\\&=-2(\mu'(\alpha(a)+\alpha(a))-\mu'(\alpha(a))-\mu'(\alpha(a)))+\mu'(-2\alpha(a))\\&=-2(4\mu'(\alpha(a))-2\mu'(\alpha(a)))+4\mu'(\alpha(a))\\&=-4\mu'(\alpha(a))+4\mu'(\alpha(a))=0\end{aligned}$$

We used Lemma 26 for $n = \pm 2$ and hence the commutativity of $\mathfrak{M}$. $\square$

4.6 Symmetric quadratic groups and forms

The simplest case of a quadratic group is where both $*$ and $\dagger$ are trivial. Let's call such a quadratic group

$$\mathfrak{M} = \left(M_e \xrightarrow{h} M_{ee} \xrightarrow{p} M_e\right)$$

symmetric. Equivalently, this means that $hp = 2\,\mathrm{id} = ph$ (and hence M_e is commutative). Then a quadratic form $(\lambda, \mu)\colon A \to \mathfrak{M}$ will automatically be *symmetric* in the sense that

$$\lambda(a, a') = \lambda(a', a) \quad \text{and} \quad \mu(-a) = \mu(a) \quad \forall\, a \in A.$$

We call μ a *symmetric quadratic refinement* of λ and obtain a category of symmetric quadratic forms with a forgetful functor R to the category of symmetric forms. It is not hard to show that the construction in Theorem 27 gives a universal symmetric quadratic refinement μ^c_λ for any given symmetric bilinear form λ. More precisely, we have:

Theorem 28 *Any symmetric bilinear form λ has a universal symmetric quadratic refinement. In fact, the forgetful functor $R(\lambda, \mu) = \lambda$ from the category SQF of symmetric quadratic forms to the category SF of symmetric forms has a left adjoint $L\colon \mathsf{HF} \to \mathsf{CQF}$, $L(\lambda) = (\lambda, \mu^c_\lambda)$.*

Remark 29 We observe that the map $p^c_\lambda\colon M_{ee} \to M^c_e$ is a monomorphism in this easiest, symmetric, case, just like it was in the hardest, non-commutative, case (compare Remark 23). This can be seen by noting that the first set of relations $(m^*, 0) = (m, 0)$ is redundant if the involution $*$ is trivial. Therefore, if $0 = p^c_\lambda(m) = [m, 0]$ then $(m, 0)$ must lie in the span of the second set of relations, ie, since λ is symmetric it must be of the form

$$(m, 0) = (-2\lambda(a, a), -2a) \quad \text{for some } a \in A.$$

This implies that $2a = 0$ and hence $\lambda(2a, a) = 0$ which in turn means $m = 0$.

Corollary 30 *There is an exact sequence:*

$$1 \longrightarrow M_{ee} \xrightarrow{p} M^c_e \longrightarrow \mathbb{Z}_2 \otimes A \longrightarrow 1$$

Examples 31 If $M_{ee} = M_e$ then $h = \mathrm{id}$ and $p = 2\,\mathrm{id}$ is a canonical choice for which μ is determined by λ. Another canonical choice is $p = \mathrm{id}$ and $h = 2\,\mathrm{id}$. Then a quadratic refinement of (M_e, h, p) with this choice exists exactly for even forms, at least for free groups A. Moreover, if M_{ee} has no 2–torsion then a quadratic refinement is uniquely determined by the given even form.

At the other extreme, consider $M_{ee} = M_e = \mathbb{Z}_2$. If A is a finite dimensional $\mathbb{Z}_2$–vector space then non-singular symmetric bilinear forms λ are classified by their rank and their *parity*, ie whether they are even or odd, or equivalently, whether they admit a quadratic refinement or not. In the even case, quadratic forms (λ, μ) are classified by rank and *Arf invariant*. This additional invariant takes values in $\mathbb{Z}_2$ and vanishes if and only if μ takes more elements to zero than to one (thus the Arf invariant is sometimes referred to as the "democratic invariant").

If λ is odd then the following trick of Brown [3] allows one to still define Arf invariants and it motivates the introduction of M_e. Let again A be a finite dimensional $\mathbb{Z}_2$–vector space, $M_{ee} = \mathbb{Z}_2$ and $M_e = \mathbb{Z}_4$ with the unique nontrivial homomorphisms h, p. Then any non-singular symmetric bilinear form λ has a quadratic refinement μ and quadratic forms (λ, μ) are classified by rank and an Arf invariant with values in $\mathbb{Z}_8$. If λ is even, this agrees with the previous Arf invariant via the linear inclusion $\mathbb{Z}_2 \subset \mathbb{Z}_8$.

4.7 Presentations for universal quadratic groups

Consider a central group extension

$$1 \to M \to G \xrightarrow{\pi} A \to 1$$

and assume that M and A have presentations $\langle m_i \mid n_j \rangle$ respectively $\langle a_k \mid b_\ell \rangle$. To avoid confusion, we write groups multiplicatively for a while and switch back to additive notation when returning to hermitian forms.

It is well known how to get a presentation for G: Pick a section $s\colon A \to G$ with $s(1) = 1$ which is not necessarily multiplicative. Write a relation in A as $b_\ell = a'_1 \cdots a'_r$, where a'_i are generators of A or their inverses, then

$$1 = s(1) = s(b_\ell) = s(a'_1) \cdots s(a'_r)\, w_\ell$$

where $w_\ell = w_\ell(m_i)$ is a word in the generators of M. This equation follows from the fact that the projection π is a homomorphism and for simplicity we have identified M with its image in G. We obtain the presentation

$$G = \langle m_i, \alpha_k \mid n_j, [m_i, \alpha_k], \beta_\ell\, w_\ell \rangle$$

where $\alpha_k := s(a_k)$ and $\beta_\ell := s(a'_1) \cdots s(a'_r)$ is the same word in the α_k as b_ℓ is in the a_k. The commutators $[m_j, \alpha_k]$ arise because we are assuming that the extension is central, in a more general case one would write out the action of A on M.

It will be useful to rewrite this presentation as follows. Observe that the section s satisfies

$$s(a_1 a_2) = s(a_1) s(a_2) c(a_1, a_2)$$

for a uniquely determined *cocycle* $c\colon A \times A \to M$. By induction one shows that:

$$\begin{aligned} s(a_1 \cdots a_r) &= s(a_1 \cdots a_{r-1}) s(a_r) c(a_1 \cdots a_{r-1}, a_r) = \cdots \\ &= s(a_1) \cdots s(a_r) c(a_1, a_2) c(a_1 a_2, a_3) c(a_1 a_2 a_3, a_4) \cdots c(a_1 \cdots a_{r-1}, a_r) \end{aligned}$$

Comparing this expression with the definition of the word w_ℓ in the presentation of G, it follows that

$$w_\ell = c(a'_1, a'_2) c(a'_1 a'_2, a'_3) \cdots c(a'_1 \cdots a'_{r-1}, a'_r) \ \in M$$

so that the above presentation of G is entirely expressed in terms of the cocycle c (and does not depend on the section s any more).

Now assume that $\lambda\colon A \times A \to M$ is a hermitian form with respect to an involution $*$ on M. Then the universal (non-commutative) quadratic group M_e from Example 20 is a central extension as above with cocycle $c = \lambda$. Reverting to additive notation, we see that

$$\begin{aligned} w_\ell &= \lambda(a'_1, a'_2) + \lambda(a'_1 + a'_2, a'_3) + \cdots + \lambda(a'_1 + \cdots + a'_{r-1}, a'_r) \\ &= \sum_{1 \le i < j \le r} \lambda(a'_i, a'_j) \end{aligned}$$

where the ordering of the summands is irrelevant because M is central in M_e. Summarizing the above discussion, we get the following lemma:

Lemma 32 *The universal (non-commutative) quadratic group M_e corresponding to the hermitian form λ has a presentation*

$$M_e = \left\langle m_i, \alpha_k \;\middle|\; n_j, [m_i, \alpha_k], \beta_\ell + \sum_{1 \le i < j \le r} \lambda(a'_i, a'_j) \right\rangle$$

where the generators m_i, α_k and words n_j, β_ℓ are defined as above. Moreover, the universal quadratic refinement $\mu\colon A \to M_e$ is a (in general non-multiplicative) section of the central extension and hence $\alpha_k = \mu(a_k)$ for the generators a_k of A.

As discussed in Theorem 27, we get the universal *commutative* quadratic group M_e^c for λ by adding the relations $(m^*, 0) = (m, 0)$ and $(-2\lambda(a,a), -2a) = 0$. The latter can be rewritten in the form $2(0, a) = (\lambda(a,a), 0)$. In the current notation, where $(m, 0)$ is identified with $m \in M$, we obtain the relations

$$m^* = m \quad \text{and} \quad 2\mu(a) = \lambda(a,a) \ \in M_e^c \quad \forall m \in M, a \in A.$$

Recalling that A, M and M_e^c are *commutative* groups, we can write our presentation in that category to obtain the following:

Lemma 33 *The universal (commutative) quadratic group M_e^c corresponding to the hermitian form $\lambda\colon A\times A\to M$ has a presentation*

$$M_e^c=\left\langle m_i,\mu(a_k)\;\middle|\; n_j,\beta_\ell+\sum_{1\le i<j\le r}\lambda(a_i',a_j'),m^*=m,2\cdot\mu(a)=\lambda(a,a)\right\rangle$$

Here $\langle m_i\mid n_j\rangle$ is a presentation of M and a_k are generators of A. Moreover, for every relation $b_\ell=\sum_{i=1}^r a_i'$ in A, we use the word $\beta_\ell:=\sum_{i=1}^r\mu(a_i')$.

4.8 Twisted intersection invariants and a universal quadratic group

If we apply this construction to the universal inner product on order n rooted trees

$$\langle\cdot,\cdot\rangle\colon\ \mathcal{L}_n(m)\times\mathcal{L}_n(m)\longrightarrow\mathcal{T}_{2n}(m)=:\mathcal{T}_{2n}(m)_{ee}$$

we obtain a universal symmetric quadratic refinement:

$$q:=\mu^c_{\langle\cdot,\cdot\rangle}\colon\ \mathcal{L}_n(m)\to\mathcal{T}_{2n}(m)_e^c$$

Let us compute the presentation from Lemma 33 in this case. Recall that the generators of $\mathcal{L}_n(m)$ are rooted trees J of order n and the relations are the AS and IHX relations from Figure 5. Similarly, $\mathcal{T}_{2n}(m)$ is generated by unrooted trees t of order $2n$, modulo the same relations. Putting these together, we see that $\mathcal{T}_{2n}(m)_e^c$ is generated by unrooted trees t of order $2n$ and elements $q(J)$, one for each rooted tree J of order n. The three types of relations from Lemma 33 are:

n_j : Relations in $M=\mathcal{T}_{2n}(m)$ are ordinary AS and IHX relations for unrooted trees t,

β_ℓ : Every relation b_ℓ in $A=\mathcal{L}_n(m)$ is an AS–relation $J+\overline{J}=0$ or an IHX–relation $I-H+X=0$. We obtain the following *twisted* AS– respectively IHX–relations:

$$0=q(J)+q(\overline{J})+\langle J,\overline{J}\rangle$$
$$0=q(I)+q(H)+q(X)-\langle I,H\rangle+\langle I,X\rangle-\langle H,X\rangle$$

c : $2\cdot q(J)=\langle J,J\rangle$

The last relation c builds in the commutativity of the universal group as discussed above because we are in the easiest, symmetric, setting where the involution $*$ is trivial. Using relation c, the twisted AS relation simply becomes

$$q(\overline{J})=q(-J)=q(J)$$

which was expected since we are in the symmetric case. This relation means that the orientation of J is irrelevant when forming $q(J)$ and in fact, with some care one can see that the twisted IHX–relation makes sense for unoriented trees.

Lemma 34 *This is a presentation for the target group $\mathcal{T}^{\infty}_{2n}(m)$ of twisted Whitney towers from Definition 10.*

Proof The translation comes from setting $J^{\infty} = q(J)$ for rooted trees J (and keeping unrooted trees unchanged). We need to show that the twisted IHX–relations in the original definition of $\mathcal{T}^{\infty}_{2n}(m)$ are equivalent to the twisted IHX–relations above, all other relations were already shown to agree. This is very easy to see in the presence of the interior-twist relations: Together with the (untwisted) IHX–relations, they imply that:

$$0 = \langle I, I - H + X\rangle = \langle I, I\rangle - \langle I, H\rangle + \langle I, X\rangle = 2 \cdot q(I) - \langle I, H\rangle + \langle I, X\rangle$$

This last expression is exactly the difference between the two versions of the twisted IHX–relations. □

Corollary 35 *There is an isomorphism of symmetric quadratic groups*

$$\mathcal{T}_{2n}(m)^c_e \cong \mathcal{T}^{\infty}_{2n}(m)$$

which is the identity on $\mathcal{T}_{2n}(m)$ and takes $q(J)$ to J^{∞} for rooted trees J. The quadratic group structure on $\mathcal{T}^{\infty}_{2n}(m)$ is given by the homomorphisms

$$\mathcal{T}_{2n}(m) \xrightarrow{p} \mathcal{T}^{\infty}_{2n}(m) \xrightarrow{h} \mathcal{T}_{2n}(m)$$

which are uniquely characterized (for unrooted trees t and rooted trees J) by

$$p(t) = t \quad \text{and} \quad h(t) = 2 \cdot t, \quad h(J^{\infty}) = \langle J, J\rangle .$$

Note that Theorem 11 is now a direct consequence of Corollary 30.

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