# Non-local spin field theories in dimension three

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## 1 The semi characteristic

Our aim is to realize the semi characteristic with $\mathbb{Z}/2$-coefficients

$$s(M) := \dim H_0(M; \mathbb{Z}/2 - \dim H_1(M; \mathbb{Z}/2) \mod 2$$

of a closed 3-manifold $M$ as partition function of a topological field theory with only 1-dimensional vector spaces associated to closed surfaces. It is easy to show that this is not possible on oriented 3-manifolds, but we can do it on Spin-manifolds. This sounds strange since all 3-manifolds have a Spin-structure and the semi characteristic does not depend on the choice of a Spin structure. But working in the category of Spin-manifolds restricts the way one can cut up manifolds and one also the diffeomorphisms on surfaces, to which a field theory attaches a representation. We explain now more precisely what we mean by this.

If $\Sigma$ is a closed Spin-surface, a Spin-structure preserving diffeomorphism (in short, a Spin diffeomorphism) is a pair $(f, h)$, where $f$ is a diffeomorphism on $\Sigma$ and $h$ a homotopy between the original Spin structure and the Spin structure pulled back via $f$. If we consider a Spin structure as a lift of the normal Gauss map to BSpin, then a homotopy of Spin strutures is a fibre homotopy between these lifts. In particular, if $W = W_1 \cup W_2$ is a decomposition of $W$ into two manifolds with same boundary $\Sigma$ and $f : \Sigma \to \Sigma$ is a Spin diffeomorphism, then there is a (unique up to fibre homotopy) Spin structure on the twisted manifold $W_1 \cup_f W_2$ obtained by considering this manifold as $W_1 \cup \Sigma \times [0, 1/2] \cup_f \Sigma \times [1/2, 1] \cup W_2$ and using the homotopy to extend the given Spin structures on $W_i$. In particular we obtain a well defined (up to fibre homotopy) Spin structure on the mapping torus $\Sigma_f$ extending the given Spin-structure on the fibre. There are two such Spin strctures on the mapping torus of a connected surface, which resemble the two choices for a homotopy corresponding to $H^1(\Sigma \times [0,1/2] \cup_f \Sigma \times [1/2,1], \partial; \mathbb{Z}/2)$. A Spin structure on a surface $\Sigma$ is equivalent to a quadratic refinement of the intersection form on $H_1(\Sigma)$ and thus a diffeomorphism preserves the Spin structure, if and only if it commutes with the quadratic refinement, but the homotopy is an additional choice.

For the construction of our field theory and the proof of the axioms we need a few properties of the semi characteristic.

**Lemma 1.** *Let $W$ be a compact Spin 4-manifold with boundary $M$. Then for the mod 2 semi-characteristic $s(M)$ the following formula holds:*

$$s(M) = e(W) \mod 2,$$

*where $e(W)$ is the Euler characteristic.*

*Proof.* Consider the pair sequence with $\mathbb{Z}/2$-coefficients:

$$0 \to H_2(W)/_{\mathrm{im}\, H_2(M)} \to H_2(W,M) \to H_1(M) \to H_1(W) \to \cdots \to H_0(W,M)$$

Since $w_2(W) = 0$ the intersection form on $H_2(M)$ fulfills

$$x \circ x = 0$$

for all $x \in H_2(W)$. Thus the non-degenerate intersection form on $H_2(W)/_{\mathrm{im}\, H_2(M)}$ has even rank. The statement follows from Poincaré Lefschetz duality. □

**Lemma 2.** *Let $M = W \cup W'$ be be the union of two Spin manifolds with the same boundary and $f : \Sigma := \partial W \to \partial W$ be a Spin structure preserving diffeomorphism. Then*

$$s(W \cup W') - s(W \cup_f W') = rank(f_* - 1) \mod 2.$$

*In particular we obtain a formula for the semi characteristic of the mapping tours $\Sigma_f$:*

$$s(\Sigma_f) = rank(f_* - 1) \mod 2$$

*Furthermore it implies that surgery on a 0-sphere changes the semi characteristic by 1.*

*Proof.* There is a well known bordism between $W \cup W'$, $W \cup_f W'$ and the mapping torus $\partial W_f$ obtained by taking $W \times I + W' \times I$ and identifying along $\partial W \times [0, 1/3]$ via id, and along $\partial W \times [2/3, 1]$ via $f \times \mathrm{id}$. Applying Lemma 1 to this bordism we see that

$$s(W \cup W') - s(W \cup_f W') = s(\Sigma_f) + e(W) - e(W') \mod 2.$$

Since $0 = e(W \cup W') = e(W) + e(W') - e(\Sigma)$ and $e(\Sigma)$ is even, the last expression vanishes $\mod 2$. Using the Wang sequence one gets the formula

$$s(\Sigma_f) = \mathrm{rank}(f_* - id).$$

For the last statement consider the disjoint union with $S^3$, which changes the semi-characteristic by 1 and apply the gluing formula to the decomposition of $M + S^3$ into $M - (B^3 + B^3) + S^3 - S^2 \times (0, 1)$, where $B^3$ is the open 3-ball, and $(D^3 + D^3) + S^2 \times [0, 1]$, which after regluing gives the result of the surgery plus two copies of $S^3$. □

This Lemma implies

**Corollary 3.** *The map* $f \mapsto \mathrm{rank}(f_* - \mathrm{id})$ *on Spin-diffeomorphism is a homomorphism, where* $f_*$ *is the induced map with* $\mathbb{Z}/2$*-coefficients.*

*Proof.* Using that $\mathrm{rank}(f_* - \mathrm{id}) = s(\Sigma_f)$ and the standard bordism $W$ between $\Sigma_f$, $\Sigma_g$ and $\Sigma_{fg}$ we know from Lemma 1, that

$$s(\Sigma_f) + s(\Sigma_g) - s(\Sigma fg) = e(W) \mod 2,$$

if the diffeomorphisms preserve the Spin structure. But the standard bordism is homotopy equivalent to the union of $\Sigma_f$ and $\Sigma_g$ glued via the fibre $\Sigma$ and so its Euler characteristic vanishes $\mod 2$. □

Thus we obtain a representation of the Spin self diffeomorphism on $\mathbb{R}$ mapping $f$to $\mathrm{rank}(f_* - \mathrm{id})$. Although we don't need this, the following Lemma might be of some interest since it sheds light on this representation. Namely we investigate those Spin self-diffeomorphisms $f$ of a connected surface $\Sigma$ which preserve a fixed lagrangian $L$. We write $H_1 = L \oplus L^*$, the hyperbolic form, then $f_*$ is in the subgroup $TU(L \oplus L^*)$ /as defined by [**?**])of the isometries $U(L \oplus L^*)$ of the hyperbolic form given by the ismometries which preserve the Lagrangian $L$. Now we stabilize these isometries by adding the identity on hyperbolic planes to obtain the stable subgroup $TU \subset U$. By [**?**]

the subgroup $TU$ is contained in the commutator subgroup $[U, U]$. Since the group of Spin structure preserving diffeomorphisms maps surjectively to the isometries of the intersection form preserving the quadratic refinement, and the map $f \mapsto \text{rank}(f_* - 1) \otimes \mathbb{Z}/2 \mod 2$ is a homomorphism by the Corollary above, the map $U \to \mathbb{Z}/2$ mapping $g$ to $\text{rank}(g-1) \otimes \mathbb{Z}/2 \mod 2$ is a homomorphism. Since $TU$ is contained in the commutator subgroup $[U, U]$, the map $f \mapsto \text{rank}(f_* - 1) \mod 2$ vanishes on those diffeomorphsims which preserve the Spin structure and the Lagrangian. Thus we have proved:

**Proposition 4.** *Let $L$ be a lagrangian in a skew symmetric unimodular bilinear form $\lambda$ with quadratic refinement $q$ . Then for those isometries $g$ of $\lambda$ and $q$ which preserve $L$*

$$\text{rank}(g-1) \otimes \mathbb{Z}/2 = 0 \mod 2.$$

*Proof.* Realize $g$ by a Spin-diffeomorphism on a surface and apply the argument above. □

# 2 Three spin field theories in dimension 3

For a topological field theory we follow Atiyah's definition [**?**]. This means that we associate to each closed Spin surface a vector space. Here we work with the real numbers and associate $V(F) := \mathbb{R}$ to each connected Spin surface $F$. As required by the axioms we associate to the disjoint union the tensor product, which in our situation we identify with $\mathbb{R}$ via the canonical isomorphism $\mathbb{R} \otimes \mathbb{R} \to \mathbb{R}$ mapping $x \otimes y$ to $xy$. A second axiom is that to $-F$ the dual vector space is associated. Since we decided to associate $\mathbb{R}$ to $F$ we have to say how we identify the vector spaces associated to $-F$ with the dual of that ossociated to $\mathbb{R}$. This is equivalent to giving a non-singular pairing $< ..., ... >$ , and our definition for a connected surface $F$ is:

$$V(F) \otimes V(-F) \to \mathbb{R}$$

$$< x, y >_F := (-1)^{e(F)/2 + Arf(F)} xy,$$

where Arf is the Arf invariant of Spin surface. By the tensor product axiom this extends to a pairing on non-connected surfaces, which is again given by the same formula.

The next data required by Atiyah is a functor from the category of surfaces and Spin-diffeomorphisms to the category of vector spaces extending our definition of $V(F)$. Thus we want to define $V(f)$ for a diffeemorphism $f$. For this we first chose a standard model for our surfaces denoted by $\Sigma_k^0$ for surfaces with Arf-invariant 0 and genus $k$ and $\Sigma_k^1$ for surfaces with Arf-invariant 1 and genus $k$. We will do this as follows we take $S^2 := \partial X_0$, where $X_0 = D^3$, for $\Sigma_0^0$ and $S^1 \times S^1 = \partial X_1$, where $X_1 = S^1 \times D^2$, for $\Sigma_1^0$. The surfaces of higher genus are given by the boundary connected sum of $k$ copies: $X_k := \natural X_1 \natural ... \natural X_1$, where we choose once and for ever discs in the boundary along which we perform the boundary connected sum. We define $\Sigma_1^1$ as the torus with the Lie group framing and $\Sigma_k^1 := \Sigma_1^1 \sharp \Sigma_{k-1}^0$. For self diffeomorphisms $f$ of $\Sigma_k^i$ the induced map on $V(\Sigma_k^i)$ is defined as

$$V(f) := rank(f_* - 1),$$

where $f_*$ is the induced map on $H_1(...;\mathbb{Z}/2)$ as considered in the first section. To define it for maps on other surfaces we make a choice. For each connected surface $F$ choose a Spin-diffeomorphism $f(F)$ to $\Sigma_k^i$ for some $k$ and $i$, with the restriction, that if $F = \Sigma_k^i$ we choose the identity map (more generally for $\Sigma_k^i \times \{x\}$ we take the projection) and we define

$$V(f) := \mathrm{id}\,.$$

For a Spin-diffeomorphism $g : F \to F'$ we define

$$V(g) := V(f')gV(f)^{-1}.$$

This obviously gives a functor. To avoid confusion I would like to stress that $-\Sigma_k^i$ is a different surface from $\Sigma_k^i$ and we also there have to choose a diffeomorphism which here we denote by $r(\Sigma_k^i)$, since we can take an appropriate refection preserving the Spin-structure, such that $r^2 = \mathrm{id}$. Even more we can choose $r$ such that it extends to $X_k$ (and $Y_{k,l}$ as we will define below with the help of $X_k$) and do so.

Before we come to the next data for our field theory we would like to describe what we call the canonical manifolds with boundary $\Sigma_k^0$ and $\Sigma_k^1 + \Sigma_l^1$. For $\Sigma_k^0$ we take $X_k$ and for $\Sigma_k^1 + \Sigma_l^1$ we take

$$Y_{k,l} := \Sigma_1^1 \times [0, 1/2] \cup_r \Sigma_1^1 \times [1/2, 1] \natural X_{k-1} \natural X_{l-1}.$$

Using the canonical boundaries we will define the next data required by Atiyah, namely an attachment of an element of $\varphi(W) \in V(\partial W)$ for each Spin diffeomorphism class rel. boundary (meaning that the restriction to the boundary is the identity map). We reduce this to the case where the boundary of $W$ is a disjoint union of the canonical surfaces $\Sigma_k^i$. Namely for each boundary component $F$ of $\partial W$, which is not equal to $\Sigma_k^i$, we replace $W$ by the union of $W$ with the cylinder over $\Sigma_k^i$, glued via $f(F)$. We denote the result of these gluings by $W'$ and define $\varphi(W) := \varphi(W')$, which we still have to define. Thus from now on we only consider manifolds $W$ with boundary a disjoint union of $\Sigma_k^i$ and define $\varphi(W)$ for such manifolds. We have to do it such that Atiyah'2 axioms are fulfilled, in particular the **gluing axiom**, which means that if $\partial W_1 = A + B$ and $\partial W_2 = A + C$, then

$$\varphi(W_1 \cup_{r(A)} W_2) = \varphi(W) \cdot \varphi(W_2),$$

where $r(A)$ is the reflection we have chosen to identify $A$ with $-A$ and where $\cdot$ means: we consider $\varphi(W_1) \otimes \varphi(W_2) \in V(A) \otimes V(B) \otimes V(A) \otimes V(C)$ and map to $V(B) \otimes V(C)$ under the evaluation given by the pairing

$$< ..., ... >_M (\mathrm{id} \times V(r)) : V(A) \otimes V(A) \to V(A) \otimes V(-A) \to K.$$

Now we are ready to define $\varphi(W)$, which we only have to do for Spin-manifolds with boundary a disjoint union of $\Sigma_k^i$. This is done in terms of a construction which is almost canonical. For each component of $\partial W$ equal to $\Sigma_k^0$ we glue $-X_k$ to $W$. Since the Arf invariant of the boundary is zero, all other components come in pairs (but up to a choice of the pairs) $\Sigma_k^1 + \Sigma_l^1$. For each such pair we glue in $-Y_{k,l}$. We denote the resulting manifold by $\hat{W}$ and define:

**Definition 5.**

$$\varphi(W) := s(\hat{W}).$$

To obtain a first feeling for this definition we compute the invariant for our canonical manifolds:

$$\varphi(X_k) = s(X_k \cup (-X_k)) = (-1)^{e(X_k)} = (-1)^{e(\partial X_k)/2} = (-1)^{1-k},$$

where we apply Lemma 1.

$$\varphi(-X_k) = s(-X_k \cup_r (-X_k)) = (-1)^{e(X_k)} = (-1)^{e(\partial X_k)/2} = (-1)^{1-k}.$$

We check the gluing axiom:

$$(-1)^{e(\Sigma_k^0)/2} = s(X_k \cup (-X_k)) = \varphi(X_k \cup (-X_k)),$$

and

$$< \varphi(X_k), \varphi(-X_k) >_{\Sigma_k^0} = (-1)^{e(\Sigma_k^0)/2} \varphi(X_k)^2 = (-1)^{e(\Sigma_k^0)/2},$$

so it is okay. Using Lemma 1 again we obtain:

$$\varphi(Y_{k,l}) = s(Y_{k,l} \cup (-Y_{k,l})) = (-1)^{e(Y_{k,l})} = (-1)^{\partial e(Y_{k,l})/2} = (-1)^{k+l},$$

and

$$\varphi(-Y_{k,l}) = s(-Y_{k,l} \cup_r (-Y_{k,l}) = (-1)^{e(Y_{k,l})} = (-1)^{\partial e(Y_{k,l})/2} = (-1)^{k+l}.$$

Again we check the gluing formula

$$(-1)^{e(\Sigma_k)/2+e(\Sigma_l)/2)} = (-1)^{e(\partial(Y_{k,l})/2} = s(Y_{k,l} \cup (-Y_{k,l}),$$

and

$$< \varphi(Y_{k,l}), \varphi - Y_{k,l}) >_{\partial Y_{k,l}} = (-1)^{e(\Sigma_k)+1} (-1)^{e(\Sigma_l)+1} \varphi(Y_{k,l})^2 = (-1)^{e(\Sigma_k)/2+e(\Sigma_l)/2},$$

and the gluing formula holds.

We have to verify that $\varphi$ is well defined:

**Proposition 6.** *$\varphi(W)$ is well defined*

*Proof.* We have to show that it does not depend on the order we pair the components of $\partial W$ with Arf invariant 1. It is enough to show that for four components $\Sigma_a^1$, $\Sigma_b^1$ , $\Sigma_b^1$ , $\Sigma_d^1$, if we on the one hand pair $\Sigma_a^1 + \Sigma_b^1 = \partial Y_{a,b}$ and $\Sigma_c^1 + \Sigma_d^1 = \partial Y_{c,d}$ and on the other hand pair $\Sigma_a^1 + \Sigma_c^1 = \partial Y_{a,c}$ and $\Sigma_b^1 + \Sigma_d^1 = \partial Y_{b,d}$ and glue these manifolds in, the value of the semi characteristic is the same.

To show this we use the definition of $Y_{a,b}$ and to simplify notation we call the mapping cylinder of a diffeomorphism $f : M \to N$ by $Z(M, f) := M \times [0, 1/2] \cup_f N \times [1/2, 1]$. Then

$$Y_{a,b} + Y_{c,d} = Z(\Sigma_1^1, r) \natural X_{a-1} \natural X_{b-1} + Z(\Sigma_1^1, r) \natural X_{c-1} \natural_{d-1}$$

and

$$Y_{a,c} + Y_{b,d} = Z(\Sigma_1^1, r)\natural X_{a-1}\natural X_{c-1} + Z(\Sigma_1^1, r)\natural X_{b-1}\natural_{d-1}.$$

Now we note that if in the first expression we cut up in the two copies of $Z(\Sigma_1^1, r)$ along $\Sigma_1^1$ and reglue by interchanging the copies, we obtain the second expression (draw a picture). By the gluing formula for the semi characteristic (Lemma 2) the value of the semi characteristic does not change. □

This finishes the definition of our field theory. Finally we have to prove the axioms and begin with the gluing axiom:

**Theorem 7.** *$\varphi(W)$ fulfils the gluing axiom: If $\partial W = F_1 + F_2$ and $\partial W' = -F_2 + F_3$, then*

$$\varphi(W \cup_{F_2} W') =< \varphi(W), \varphi(W') >_{F_2} = \varphi(W)\varphi(W')(-1)^{e(F_2)/2 + Arf(F)},$$

*Proof.* We only have to consider the case where $\partial W$ and $\partial W'$ is the sum of standard surfaces $\Sigma_k^i$. It is enough to consider two cases, where $\partial W = F + \Sigma_k^0$ and $\partial W' = \Sigma_k^0 + F'$ or when $\partial W = F + \Sigma_k^1$ and $\partial W' = \Sigma_k^1 + F'$. Since here $F_2$ has the same orientation we have to glued via $r$ in both cases.

First case: $\partial W = F + \Sigma_k^0$ and $\partial W' = \Sigma_k^0 + F'$. We denote the manifold obtained from $W$ by gluing in all $X_k$ and $Y_{a,b}$ whose boundary is $Y$ and denote the result by $T$, so that $\partial T = \Sigma_k^0$ and do the same for $W'$ to obtain $T'$ with boundary $\Sigma_k^0$. Then

$$\varphi(W + W') = \varphi(W)\varphi(W') = s(T \cup (-X_k))s(T' \cup (-X_k)).$$

And

$$\varphi(W \cup_r W') = s(T \cup_r T').$$

Recall that

$$< ..., ... >_{\Sigma_k} = (-1)^{e(\Sigma_k^0)/2}.$$

Thus we are finished with the first case, if

$$s(T \cup (-X_k))s(T' \cup (-X_k)) = (-1)^{e(\Sigma_k^0)/2} s(T \cup_r T').$$

To see this we cut $T \cup (-X_k) + T' \cup (-X_k)$ along $\Sigma_k$ in each of the pieces and glue after interchanging the copies appropriately to obtain $T \cup_r T' + (-X_k) \cup_r$

$(-X_k)$. Since by the gluing formula for the semi characteristic (Lemma 2) we don't change its value and $s((-X_k) \cup_r (-X_k)) = (-1)^{e(\Sigma_k^0)/2}$, we are done.

Second case: $\partial W = F + \Sigma_k^1$ and $\partial W' = \Sigma_k^1 + F'$. In this case we know that there must be $\Sigma_a^1 \subset \partial W$ and $\Sigma_b^1 \subset \partial W'$ which we use as "partner" for $\Sigma_k^1$. Again we glue into all other boundary components our canonical manifolds $X_r$ resp. $Y_{s,t}$ to obtain in the first case $T$ with boundary $\Sigma_a^1 + \Sigma_k$ and in the second case $T'$ with boundary $\Sigma_k^1 + \Sigma_b^1$. Using this we compute

$$\varphi(W)\varphi(W') = s(T \cup (-Y_{a,k}))s(T' \cup (-Y_{k,b})).$$

We have to show that this is equal to

$$(-1)^{e(\Sigma_k^1)/2+1}\varphi(W \cup_r W') = s(T_1 \cup_r T' \cup Y_{a,b}).$$

But

$$\varphi(W)\varphi(W') = \varphi(W + W') = s(T + T' \cup (-Y_{k,k} \cup -Y_{a,b})),$$

using the fact that the definition of $\varphi$ does not depend on how we group the boundary components (Proposition 6). Then as before we can cut this up, here along $\Sigma_k + \Sigma_k$ and reglue after interchanging the copies. The result is the disjoint union of $T \cup T' \cup (-Y_{a,b})$ and the manifold obtained from $-Y_{a,a}$ by identifying the two boundary components via $r$. This manifold is diffeomorphic to a manifold $\Sigma_1^1 \times S^1$ and the double of $Y_{k-1}$ by making a connected sum twice (on each side two discs are cut out and the boundaries are identified). Since the connected sum changes $(-1)^s$ by the gluing formula by the factor $(-1)$, and this appears twice, and the semi characteristic of $\Sigma_{k-1}^1 \times S^1$ is 0 and $\varphi$ of the double is $(-1)^{e(\Sigma_{k-1})/2} = (-1)^{e(\Sigma_k)/2+1}$, we are done.

□

The other axioms are fulfilled by construction or obvious or easy to prove. By construction $V(\emptyset) = K$, $\varphi(\emptyset) = 1$, and $V(M+N) = V(M) \otimes V(N)$. If we have a Spin-diffeomorphism rel. boundary (meaning that it is the identity on the boundary) from $W$ to $W'$ the vectors $\varphi(W)$ and $\varphi(W')$ agree by construction. Again by construction this implies that if $f : W \to W'$ is a Spin-diffeomorphism, then $V(f)\varphi(W) = \varphi(W')$. We have to show a last axioms:

$$\varphi(M \times I) = \mathrm{id},$$

where we interpret $V(-M)\times V(M)$ via our pairing with $\mathrm{Hom}(V(M), V(M))$. If we glue two cylinders over $M$ the result is diffeomorphic rel. boundary to the cylinder and so by the invariance under diffeomorphisms rel. boundary and the gluing axiom, $\varphi(M \times I)$ is an idempotent, the composition of the corresponding homomorphism is the corresponding homomorphism. Since $\varphi(W)$ is always non-trivial, this implies that the homomorphism is the identity.

We finish this section with another invertible field theory, which is much simpler. It is a field theory whose partition function is trivial, but the field theory is not isomorphic to the trivial theory. The definition is straightforward. We define as before $V(F) := \mathbb{R}$ and the functor as the trivial map, i.e. for a Spin-diffeomorphism $f : F \to F'$ we define $V(f) = \mathrm{id}$. Furthermore, for a connected Spin surface $F$ we define the pairing between $V(F)$ and $V(-F)$ to be

$$< x, y >_F := (-1)^{Arf(F)} xy.$$

Finally we define for a compact Spin 3-manifold $W$ with boundary $\alpha(W)$ to be the number of components with Arf invariant 1 and set

$$\varphi(W) := (-1)^{\alpha(W))/2}.$$

Now, if $\partial W = F_1 + F_2$ and $\partial W' = F_2 + F_3$, and $F_2$ is connected, we compute $\varphi(W \cup_{F_2} W')$. If $Arf(F_2) = 0$, this is $\varphi(W \cup_{F_2} W') = \varphi(W)\varphi(W') =< \varphi(W)\varphi(W') >_{F_2}$. If $Arf(F_2) = 1$, we have $\varphi(W \cup_{F_2} W' = -\varphi(W)\varphi(W') =< \varphi(W), \varphi(W') >_{F_2}$. Thus our field theory fulfills the gluing axiom. As before the other axioms are easy.

Also this field theory has order 2. We obtain a third non-trivial field theory by taking the tensor product of these two theories. We will see in the next sections that these are all isomorphism classes of Spin filed theories in dimension 3. If we pass from the real numbers to the complex numbers, the field theory constructed above is trivia. Namely, for a conncted surface we we map $V(\Sigma)$ to $V(\Sigma)$ by the identity, if $Arf(\Sigma) = 0$ and by $i$, if $Arf(\Sigma) = 1$. It is easy to see that this is an isomorphism to the trivial field theory. Thus over the complex number there is only one non-trivial field theory whose partition function is the semi-characteristic.

# 3 Invertible topological field theories

We summarize in this section the methods developed in [1] that allow us to classify invertible topological field theories. To fix notation for the various flavors of $d$-dimensional field theories, fix a fibration $u : X \to \mathrm{BO}(d)$. For $k \leq d$, an $X$-structure on a smooth manifold $M^k$ (possibly with corners) is a lift over $u$ of the classifying map of $TM \oplus \underline{\mathbb{R}}^{d-k}$. Another way to think about this, is a $d$-dimensional "open collar" around $M$ with an $X$-structure on its tangent bundle. Below we shall take $d = 3$ and $X = \mathrm{BSpin}(3)$ to obtain bordism categories of spin manifolds.

There is a symmetric monoidal $d$-category $X$-$\mathsf{Cob}$ with $k$-morphisms the $X$-manifolds (with corners) of dimension $k$ for $k < d$ and diffeomorphism classes of $d$-manifolds for $k = d$. To make the $k$-morphisms into sets, we should actually take submanifolds of $\mathbb{R}^\infty$. This allows a nice extension to a symmetric monoidal $(\infty, d)$-category $X$-$\mathsf{Bord}$ where the $k$-morphisms are changed only for $k \geq d$: Instead of the discrete set of diffeomorphism classes of compact $X$-manifolds of dimension $d$, we take the *space* of submanifolds. The component of $W^d$ is then homotopy equivalent to $\mathrm{BDiff}(W)$. Here we are using *topological* $d$-categories as a model for $(\infty, d)$-categories which requires some thought to make the composition of $d$-morphisms *strictly* associative.

The *symmetric* monoidal structure allows us to show that the classifying space of $X$-$\mathsf{Bord}$ is an infinite loop space. In fact, there is a canonical connective spectrum $|X\text{-}\mathsf{Bord}|$ whose $n$-the space is the classifying space of the $(\infty, d+n)$-category obtained from $X$-$\mathsf{Bord}$ by introducing a single $k$-morphism for $k = 0, \ldots, n-1$ and turning the actual $k$-morphisms of $X$-$\mathsf{Bord}$ into $k+n$-morphisms for $k = 0, \ldots d$. This is a categorical version of Segal's $\Gamma$-construction.

The following strong version of the main result in [1] was claimed (and used) in [2].

**Theorem 8.** *The connective spectrum $|X\text{-}\mathsf{Bord}|$ is homotopy equivalent to the Thom spectrum* MTX.

Here we use the version of the *Madsen-Tillmann-Weiss spectrum* MTX which has a Thom class in dimension zero (which conflicts with the two versions used in [1] respectively [2]). More precisely, the $n$-th space of MTX is the Thom space of the $n$-dimensional bundle $u_n^*(U_{d,n}^\perp)$ over $X_n$, where $U_{d,n}$ is the universal bundle over the Graßmannian $Gr(d, d+n)$ and the map

$$u_n : X_n \to Gr(d, d+n)$$

is pulled back from $u$ via the inclusion $Gr(d, d+n) \subset Gr(d, \infty) = \mathrm{BO}(d)$.

**Remark 9.** The above construction of a connective spectrum can be reversed. Given a space $Y$ there is an $\infty$-groupoid $\mathsf{C}_Y$ (for any $d \geq 0$, this is a $(\infty, d)$-category with all $k$-morphisms invertible for $k > 0$) whose $k$-morphisms are continuous maps $D^k \to Y$. Its classifying space is homotopy equivalent to $Y$ and hence classical homotopy theory coincides with the homotopy theory of $\infty$-groupoids or $(\infty, 0)$-categories.

If $Y$ is an infinite loop space then all mapping spaces into $Y$, up to homotopy, form abelian groups. It turns out that $\mathsf{C}_Y$ inherits the structure of a symmetric monoidal $\infty$-groupoid with the additional property that all objects are invertible (with respect to the monoidal structure). Such gadgets are called *Picard* $\infty$-groupoids and their homotopy theory is exactly classical stable homotopy theory.

If $Y$ is an infinite loop space, the direct limit of all orthogonal groups $O(d)$ acts on $Y$ up to homotopy. The $O(d)$-action extends to the fully dualizable objects in an $(\infty, d)$-category by Luries cobordism hypothesis. We note that if 1-morphisms are invertible, an object is invertible if and only if it admits left and right duals (which both give an inverse) but in an $(\infty, d)$-category for $d > 0$ it is much harder to have an inverse than duals.

**Corollary 10.** *Let $|\mathsf{C}|$ be the connective spectrum corresponding to a Picard $\infty$-groupoid $\mathsf{C}$. Then there is a bijection*

$$\mathsf{Fun}^{\otimes}(X\text{-}\mathsf{Bord}, \mathsf{C})/_{\simeq} \;\longleftrightarrow\; [\mathrm{MTX}, |\mathsf{C}|]$$

*where the left hand side denotes isomorphism classes of TFTs and the right hand side are homotopy classes of maps between connective spectra. Note that by construction, all TFTs with values in $\mathsf{C}$ are invertible.*

This result follows immediately from the above theorem after observing the adjunction for any symmetric monoidal $(\infty, d)$-category $\mathsf{B}$ and connective spectrum $Y$

$$\mathsf{Fun}^{\otimes}(\mathsf{B}, \mathsf{C}_Y)/_{\simeq} \;\longleftrightarrow\; [|\mathsf{B}|, Y]$$

As an example, we take $Y = \Sigma^{d+1}\,\mathrm{H}\mathbb{Z}$, the shifted Eilenberg-MacLane spectrum. This is a particular delooping of the usual target for invertible TFTs: A $d$-manifold is associated a nonzero number, i.e. an element in $\mathbb{C}^{\times}$. Up to homotopy, this is a $K(\mathbb{Z}, 1)$ whose associated connective spectrum is $\Sigma\,\mathrm{H}\mathbb{Z}$. In order to go down to points, we need to deloop this spectrum $d$ times and the easiest way is a further $d$-fold suspension.

**Corollary 11.** *Let* $\mathsf{C}$ *be the* $(\infty, d)$ *category with associated spectrum* $\Sigma^{d+1}\,\mathrm{H}\mathbb{Z}$*. Then there is a bijection*

$$\mathsf{Fun}^{\otimes}(X\text{-}\mathsf{Bord}, \mathsf{C})/_{\simeq} \;\longleftrightarrow\; H^{d+1}(\mathrm{MTX})$$

*If* $u : X \to \mathrm{BO}(d)$ *factors through* $\mathrm{BSO}(d)$*, i.e. if all manifolds are oriented, then the right hand side is isomorphic to* $H^{d+1}(X)$ *by the Thom isomorphism.*

Taking $X = \mathrm{BSpin}(3)$ we can further compute that $H^4(X) = \mathbb{Z}$, generated by $\frac{p_1}{2}$. Take the generating local field theory and evaluate it on a closed spin surface $\Sigma$. This gives a "characteristic" group homomorphism

$$\mathrm{Diff}^{spin}(\Sigma) \to \mathbb{C}^\times$$

which homotopy class is classified by the Euler class $e \in H^2(\mathrm{BDiff}^{spin}(\Sigma))$ of the corresponding line bundle. It is not hard to check that this class is rationally non-trivial and hence only the trivial local field theory maps to the torsion subgroup

$$\mathrm{Hom}(H_1(\mathrm{BDiff}^{spin}(\Sigma)), \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(\pi_0(\mathrm{Diff}^{spin}(\Sigma)), \mathbb{Q}/\mathbb{Z})$$

This shows that no local field theory can have its characteristic homomorphism be a nontrivial element in the above torsion subgroup group. However, our semi-characteristic field theory has characteristic homomorphism the canonical map onto $\{\pm 1\} \leq \mathbb{Q}/\mathbb{Z}$ for all spin surfaces $\Sigma$. This shows that it cannot be local.

**Remark 12.** What's missing in this argument is the following result which I believe is true: If the vector spaces for $S^2$ and $T^2$ of a local field theory $Z$ are 1-dimensional then $Z$ is invertible. By Freed-Teleman, it suffices to show that $Z(S^1)$ is invertible. If we work with semi-simple categories then the assumption on $T^2$ means that the category $Z(S^1)$ only has one simple object since the number of simple objects is in general the dimension of $Z(T^2)$. This should imply invertability.

This argument may work for general target categories, however, it suffices to show that semi-simple categories represent our particular target $\Sigma^4\,\mathrm{H}\mathbb{Z}$.

The other missing piece in the discussion is a comparison between discrete and continuous field theories. Recall from the beginning of this section the $d$-category $X$-$\mathsf{Cob}$. We can consider it as a topological $d$-category which happens to have discrete spaces of $d$-morphisms. In the language of $(\infty, d)$-categories, this means that there are only identity $k$-morphisms for $k > d$.

Then there is a continuous functor $X\text{-}\mathsf{Bord} \to X\text{-}\mathsf{Cob}$ which forgets the embedding into $\mathbb{R}^\infty$ and only considers the diffeomorphism type of a $d$-manifold, i.e. only considers the connected component of a $d$-morphism.

In fact this construction works for any topological $d$-cagetory $\mathsf{B}$: The space $\mathsf{B}_d(X, Y)$ of d-morphisms maps to the discrete space of its connected components $\pi_0(\mathsf{B}_d(X, Y))$. The latter gives rise to a new $d$-category $\pi_0(\mathsf{B})$ (which can be considered a topological $d$-category with discrete spaces of $d$-morphisms). By construction, it comes equipped with a functor $\mathsf{B} \to \pi_0(\mathsf{B})$ and we see that $X\text{-}\mathsf{Cob} = \pi_0(X\text{-}\mathsf{Bord})$.

In addition to the functor $Y \mapsto \pi_0(Y)$ from topological spaces to sets, there is another functor, namely $Y \mapsto Y_\delta$ (same set as $Y$ but with discrete topology). In this case, the canonical map goes the other way, namely the identity on $Y$ is continuous as a map $Y_\delta \to Y$.

Given a topological $d$-category $\mathsf{C}$, we can hence construct a new (discrete) $d$-category $\mathsf{C}_\delta$ together with a continuous functor $\mathsf{C}_\delta \to \mathsf{C}$. Combining these two constructions for topological $d$-categories $\mathsf{B}, \mathsf{C}$, we get a functor

$$\mathsf{Fun}(\pi_0(\mathsf{B}), \mathsf{C}_\delta) \to \mathsf{Fun}(\mathsf{B}, \mathsf{C})$$

where the functors on the right hand side are assumed to be continuous whereas the left hand side has no topological information. Adding symmetric monoidal structures and restricting to $\mathsf{B} = X\text{-}\mathsf{Bord}$, we arrive at continuous symmetric monoidal functors $X\text{-}\mathsf{Bord} \to \mathsf{C}$ which be call *continuous* topological field theories. Symmetric monoidal functors $X\text{-}\mathsf{Cob} \to \mathsf{C}_\delta$ are called *discrete* TFTs. Note that both versions are *local* since they go down to points and that the above gives a canonical functor from discrete to continuous field theories.

It is the invertible continuous field theories that are computed via [1]. This means that we actually show a stronger result, namely that the discrete field theory coming from the semi-characteristic is not localizable, even when considered as a continuous field theory. In fact, the above discussion can be complemented by noting that our original construction gives a nontrivial homomorphism

$$\alpha : \pi_0(\mathrm{Diff}^{spin}(\Sigma)) \twoheadrightarrow \{\pm 1\} \hookrightarrow \mathbb{C}^\times_\delta$$

which can be turned into a continuous homomorphism by the canonical maps

$$\mathrm{Diff}^{spin}(\Sigma) \twoheadrightarrow \pi_0(\mathrm{Diff}^{spin}(\Sigma)) \xrightarrow{\alpha} \mathbb{C}^\times_\delta \longrightarrow \mathbb{C}^\times$$

When computing the Euler class $e \in H^2(\mathrm{BDiff}^{spin}(\Sigma))$ of the corresponding flat line bundle, we saw that $e$ is a non-trivial torsion class. This is actually true for any non-trivial homomorphism $\alpha : \pi_0(\mathrm{Diff}^{spin}(\Sigma)) \to \mathbb{Q}/\mathbb{Z}$ since $\mathbb{Q}/\mathbb{Z}$ is the torsion subgroup of $\mathbb{C}^\times_\delta$, possibly leading to more non-localizable field theories.

**Remark 13.** Recall Milnor's examples of flat line bundles with non-torsion Euler classes to see that the $\mathbb{Q}/\mathbb{Z}$ is important in the above discussion.

It is possible to compute all "partially extended" field theories from the above formalism as follows. If we consider the empty set $\emptyset$ as the only $k$-manifold for $k = 0, \dots, b-1$ for some $0 \le b \le d$ ($b$ is then the *bottom* dimension of possibly non-empty closed $b$-manifolds) then we obtain a topological $d$-subcategory of $X$-$\mathsf{Bord}$, denoted by $X\text{-}\mathsf{Bord}^d_b$. This symbolizes that there are interesting manifolds of dimensions $b$ through $d$. Recall that the $X$-structure is given by a fibration $X \to \mathrm{BO}(d)$ so the top dimension $d$ is implicit in $X$, that's why we haven't added it explicitly into the notation. Note that in the new notation $X\text{-}\mathsf{Bord} = X\text{-}\mathsf{Bord}^d_0$.

Taking the classifying space (respectively spectrum), we see that

$$|X\text{-}\mathsf{Bord}^d_b| \simeq |X\text{-}\mathsf{Bord}|_b := |X\text{-}\mathsf{Bord}|\langle b-1\rangle$$

is the $(b-1)$-connected cover of the connective spectrum $|X\text{-}\mathsf{Bord}|$. We remark that there seems to be a shift by one between Eilenberg, 1944, and Whitehead, 1952 on how to denote connected covers. For reasons visible in the above homotopy equivalence, we introduce $Y_b$ for $Y\langle b-1\rangle$ (for spectra or spaces $Y$).

**Corollary 14.** *Let* $\mathsf{C}$ *be the* $(\infty, d)$ *category with associated spectrum* $\Sigma^{d+1}\,\mathrm{H}\mathbb{Z}$*. Then there is a bijection*

$$\mathsf{Fun}^{\otimes}(X\text{-}\mathsf{Bord}^d_b, \mathsf{C})/_{\simeq} \;\longleftrightarrow\; H^{d+1}(\mathrm{MTX}_b)$$

*where* $MTX_b$ *is the* $(b-1)$*-connected cover of the connective spectrum* MTX*.*

Local field theories correspond to exactly those cohomology classes that come from $H^{d+1}(\mathrm{MTX})$ under the natural map $\mathrm{MTX}_b \to \mathrm{MTX}$. Let's compute this for $d = 3$ and $X = \mathrm{BSpin}(3)$ and start with the case $b = 1$. Denoting MTX in this case by $M$, there is a fibration of spectra

$$M_1 \to M \to \mathrm{H}\mathbb{Z}$$

since $M_1$ is connected whereas $\pi_0(M) = \Omega_0^{spin} \cong \mathbb{Z}$. Since $H^k(\mathrm{H}\mathbb{Z})$ is torsion for $k > 0$ and $H^5(M) = H^5(\mathrm{BSpin}(3)) = 0$, we get a short exact sequence

$$0 \to \mathbb{Z} \cong H^4(M) \to H^4(M_1) \to H^5(\mathrm{H}\mathbb{Z}) \to 0$$

Looking at tables, we see that $H^5(\mathrm{H}\mathbb{Z}) \cong \mathbb{Z}/6$. The remaining question is whether the above sequence splits, i.e. how divisible $p_1$ becomes in $H^4(M_1)$.

Now to the most interesting case, namely $X$-$\mathsf{Bord}_2^3$, classified by the 1-connected cover $M_2 \to M \to \mathbb{S}[1]$. The latter is the first Postnikov section of the sphere spectrum $\mathbb{S}$ which ends at $\pi_1(\mathbb{S}) \cong \mathbb{Z}/2$. A similar discussion as for $M_1$ leads to a short exact sequence

$$0 \to \mathbb{Z} \cong H^4(M) \to H^4(M_2) \to H^5(\mathbb{S}[1]) \to 0$$

and known information about the Steenrod algebra gives $H^5(\mathbb{S}[1]) \cong \mathbb{Z}/6$. What must happen is that our field theory gives a (2- or 4-) torsion element in $H^4(M_2)$ which maps non-trivially to $H^5(\mathbb{S}[1])$. Since the map $\mathbb{S}[1] \to \mathrm{H}\mathbb{Z}$ induces multiplication by 2 on $H^5$ (basically because $Sq^2Sq^3 \neq 0$), it follows that our field theory does not come from $H^4(M_1)$, i.e. that it can't even be defined on the circle:

**Theorem 15.** *The discrete field theory coming from the semi-characteristic does not extend to a continuous functor on $X$-$\mathsf{Bord}_1^3$. In particular, it can't be extended to a discrete field theory on $X$-$\mathsf{Cob}_1^3$, either. (This still assumes that any such extension would automatically be invertible).*

# 4 Invertible field theory for other targets

It is very interesting to use other target categories then those corresponding to $\Sigma^4\,\mathrm{H}\mathbb{Z}$, for example the Anderson dual $A$ of the sphere spectrum. It is more naturally associated to spin manifolds since it does involve $\mathbb{Z}/2$-gradings. In the above computation, the Thom isomorphism is not available because the Anderson dual is not a ring spectrum, so the group of local field theories could be much larger. Another possible target would be the 4-th stage ko[4] of the Postnikov tower of the connective $K$-theory spectrum ko. Here the Thom isomorphism is available for spin bundles because this is true for ko and hence for ko[4] (check this!).

Let's start again with the case of $X$-manifolds where $X = \mathrm{BSpin}(3)$ and let's stop distinguishing in notation between connective spectra and symmetric monoidal $(\infty, 0)$-categories.

By the Thom isomorphism, we see that

$$\mathsf{Fun}(X\text{-}\mathsf{Bord}, \mathrm{ko}[4])/_{\simeq} \cong [\mathrm{MTX}, \mathrm{ko}[4]] \cong [\Sigma^\infty \mathrm{BSpin}(3), \mathrm{ko}[4]]$$

Even though BSpin(3) is 3-connected, its infinite suspension spectrum isn't because

$$\pi_0(\Sigma^\infty \mathrm{BSpin}(3)) = \lim_n \pi_n(\Sigma^n \mathrm{BSpin}(3)) \cong \mathbb{Z}$$

In fact the Atiyah-Hirzebruch spectral sequence shows that the two maps $\Sigma^4 \mathrm{H}\mathbb{Z} \to \mathrm{ko}[4]$ (3-connected cover) and $p_0 : \mathrm{ko}[4] \to \mathrm{H}\mathbb{Z}$ (projection to bottom Postnikov section) give a short exact sequence of abelian groups

$$0 \to H^4(\mathrm{BSpin}(3)) \to [\Sigma^\infty \mathrm{BSpin}(3), \mathrm{ko}[4]] \xrightarrow{p_0^*} H^0(\mathrm{BSpin}(3)) \to 0$$

and hence there are $\mathbb{Z} \times \mathbb{Z}$ many local field theories with target ko[4].

**Example 16.** The composition with the Atiyah orientation

$$\mathrm{MTX} \to \mathrm{MSpin} \to \mathrm{ko} \to \mathrm{ko}[4]$$

is a generators $\alpha \in [\mathrm{MTX}, \mathrm{ko}[4]]$ which maps onto a generator via $p_0^*$. In particular, the corresponding twist does not come from $\Sigma^4 \mathrm{H}\mathbb{Z}$ (the gradings on lines and algebras are important!).

The following computes invertible local field theories with target $A$. Up to homotopy, the characterizing property of $A$ gives a unique map $a : \mathrm{ko} \to A$ inducing an isomorphism on $\pi_4 \cong \mathbb{Z}$. We believe that it induces an isomorphism on $\pi_1 \cong \pi_2 \cong \mathbb{Z}/2$ and an epimorphism $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/24$ on $\pi_0$. Therefore, there is a fibration $\mathrm{H}\mathbb{Z} \to \mathrm{ko}[4] \to A_0$ inducing an exact sequence

$$[\mathrm{MTX}, \mathrm{H}\mathbb{Z}] \to [\mathrm{MTX}, \mathrm{ko}[4]] \to [\mathrm{MTX}, A] \to [\mathrm{MTX}, \Sigma\, \mathrm{H}\mathbb{Z}] \cong H^1 \mathrm{BSpin}(3) = 0$$

Since $[\mathrm{MTX}, \mathrm{H}\mathbb{Z}] \cong \mathbb{Z}$ we can figure out the left-most map after composition with $p_0$. By definition, this composition $\mathrm{H}\mathbb{Z} \to \mathrm{ko}[4] \to \mathrm{H}\mathbb{Z}$ is multiplication by 24, implying the following result.

**Corollary 17.** *There is an short exact sequence*

$$0 \to \mathsf{Fun}(X\text{-}\mathsf{Bord}, \Sigma^4 \mathrm{H}\mathbb{Z})/_{\simeq} \to \mathsf{Fun}(X\text{-}\mathsf{Bord}, A)/_{\simeq} \to \mathbb{Z}/24 \to 0$$

*where the composition of $\alpha$ from Example 16 with $a : \mathrm{ko}[4] \to A$ maps to the generator on the right hand side.*

**Lemma 18.** *There is a group isomorphism* $\mathsf{Fun}(X\text{-}\mathsf{Bord}, A)/_{\simeq} \cong \mathbb{Z} \times \mathbb{Z}/2$.

*Proof.* For any spectrum $E$, we have a short exact sequence

$$0 \to \mathrm{Ext}(\pi_3(E), \mathbb{Z}) \to [E, A] \to \mathrm{Hom}(\pi_4(E), \mathbb{Z}) \to 0$$

and by our corollary above it suffices to show that $\pi_3(\mathrm{MTX}) \cong \mathbb{Z}/2$. Via Pontrjagin-Thom, this group is isomorphic to closed spin 3-manifolds, modulo bordism by spin 4-manifolds with vanishing Euler characteristic. This comes from the fact that the tangent bundle is classifies in BSpin(3).

Any spin 3-manifold is spin zero bordant and by adding copies of $S^2 \times S^2$ respectively $T^4$ we can assume that the Euler characteristic is either zero or one. On the other hand, a closed 4-manifold has even Euler characteristic if it spin since then the intersection form is even and non-degenerate. This shows that the Euler characteristic of any bounding spin 4-manifold gives an isomorphism $\pi_3(\mathrm{MTX}) \cong \mathbb{Z}/2$. □

It is very surprising to observe that the above short exact sequence seems to imply that the 2-connective cover $E_3 \to E$ induces an isomorphism $[E_3, A] \cong [E, A]$. In particular, any field theory with target $A$ can be localized, even if it's just defined on (the space of) all closed 3-manifolds alone!

# 5 2-dimensional topological field theories

To related these twists with our 2|0-EFTs, we have to consider a different case, namely $X = \mathrm{BSpin}(2)$. In this case, the rational homotopy groups of MTX are concentrated in all even dimensions. For example, $\pi_2(\mathrm{MTX}) = \mathbb{Z} \times \mathbb{Z}/2$ via Euler characteristic and Arf invariant, the proof being similar to that of Lemma 18 above.

**Conjecture 19.** The finite abelian group

$$\mathrm{Tors}\,\mathsf{Fun}(X\text{-}\mathsf{Bord}, A)/_{\simeq} \cong \mathrm{Tors}[\mathrm{MTX}, A] \cong \mathrm{Ext}(\pi_3(\mathrm{MTX}), \mathbb{Z}) \cong \pi_3(\mathrm{MTX})$$

contains $\mathbb{Z}/24$ as a subgroup.

Via Pontrjagin-Thom, this group is isomorphic to closed spin 3-manifolds with a prefered oriented plane-bundle, modulo bordism by spin 4-manifolds with tangent bundles that split off a 2-dimensional trivial bundle (whose complement extends the plane-bundles on the boundary). The hope is that Rohlin's theorem would somehow contribute to this 24.

There is another group $\mathbb{Z}/24$ that contributes to 2|0-EFT twists: It comes from the fact that only genus one surfaces are flat and hence the generating field theory restricts to something of order 24 in that subcategory: this comes from the fact that $H^2(\mathcal{M}_{T^2}) \cong \mathbb{Z}/24$.

Let $ET$ denote the group of 2|0-EFT twists with target $A$, then we can evaluate on the moduli stack of (non-zero bordant) spin tori to get a homomorphism

$$ET \longrightarrow H^2(\mathcal{M}_{T^2}) \cong \mathbb{Z}/24$$

whereas restriction to flat surfaces leads to a map the other way:

$$\mathbb{Z}/24 \leq \mathsf{Fun}(X\text{-}\mathsf{Bord}, A)/_{\simeq} \longrightarrow ET$$

This leaves the desired possibility that $ET \cong \mathbb{Z}/24^2$ if we can show that the following composition is zero:

$$\mathrm{Tors}\,\mathsf{Fun}(X\text{-}\mathsf{Bord}, A)/_{\simeq} \longrightarrow ET \longrightarrow H^2(\mathcal{M}_{T^2})$$

Identifying moduli of conformal spin tori with $\mathrm{BDiff}^{spin}(T^2)$, this composition is the evaluation of a topological field theory on diffeomorphisms of spin tori. It extends to all genera and hence it factors through the stable group $H^2(\mathrm{BDiff}^{spin}(\Sigma))$ where $\Sigma$ has large genus. Extending a diffeomorphism (that's the identity on a disk) by the identity from genus one to higher genus induces a map $\mathrm{BDiff}^{spin}(T^2) \to \mathrm{BDiff}^{spin}(\Sigma)$. Hence our composition factors as

$$\mathbb{Z}/24 \leq \mathsf{Fun}(X\text{-}\mathsf{Bord}, A)/_{\simeq} \longrightarrow H^2(\mathrm{BDiff}^{spin}(\Sigma)) \longrightarrow H^2(\mathrm{BDiff}^{spin}(T^2))$$

Since the group in the middle is torsionfree (or has a $\mathbb{Z}/2$, check that!) the $\mathbb{Z}/24$ on the left must die ...

# References


[1] Galatius, Madsen, Weiss, Tillmann

[2] Lurie *TFT*