

Grope cobordism and feynman diagrams

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Abstract. We explain how the usual algebras of Feynman diagrams behave under the grope degree introduced in [CT]. We show that the Kontsevich integral rationally classifies grope cobordisms of knots in 3-space when the “class” is used to organize gropes. This implies that the grope cobordism equivalence relations are highly nontrivial in dimension 3. We also show that the class is not a useful organizing complexity in 4 dimensions since only the Arf invariant survives. In contrast, measuring gropes according to “height” does lead to very interesting 4-dimensional information [COT]. Finally, several low degree calculations are explained, in particular we show that S-equivalence is the same relation as grope cobordism based on the smallest tree with an internal vertex.

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1. Introduction

In [CT] we introduced the notion of a *grope cobordism* between two knots in 3-space, which places Vassiliev theory in a natural topological context. Gropes are certain 2-complexes built out of several surface stages, whose complexity can be measured by either the *class* (corresponding to nilpotent groups) or the *height* (corresponding to solvable groups). The analogy to group theory arises by observing that a continuous map ϕ of a circle (into some target space) represents a commutator in the fundamental group if and only if it extends to a map of a surface (which is the simplest possible grope, of class 2 and height 1). Similarly, ϕ represents an element in the k -th term of the *lower central series* (respectively *derived series*) of the fundamental group if and only if it extends to a continuous map of a grope of class k (respectively height k).

In knot theory, one replaces continuous maps of a circle by smooth embeddings of a circle into 3-space. Accordingly, one should study *embeddings* of gropes into 3-space. More precisely, one obtains two sequences of new geometric equivalence relations on the set of knot types by calling two knots equivalent if they cobound an *embedded grope* (of a specified class or height).

It is the purpose of this paper to show that the invariants associated to grope cobordism are extremely interesting. Let \mathcal{K} be the abelian monoid of knot types, i. e. isotopy classes of oriented knots in 3-space (under connected sum). We proved in [CT] that the quotients $\mathcal{K}/G_k := \mathcal{K}$ modulo grope cobordism of class k in 3-space, are in fact finitely generated abelian groups. In Section 3.1 we start the investigation of these groups by showing that there is an epimorphism

$$\mathcal{B}_{<k}^g \twoheadrightarrow \mathcal{K}/G_k,$$

where $\mathcal{B}_{<k}^g$ is the usual (primitive) diagram space known from the theory of finite type invariants but *graded by the grope degree*. More precisely, $\mathcal{B}_{<k}^g$ is the abelian group generated by connected uni-trivalent graphs of grope degree i , $1 < i < k$, with at least one univalent vertex and a cyclic ordering at each trivalent vertex. The relations are the usual IHX and AS relations. The grope degree is the Vassiliev degree (i.e. half the number of vertices) *plus* the first Betti number of the graph. Observe that both relations preserve this new degree.

We then show in Section 3.1 that as in the usual theory of finite type invariants, the above map has an inverse, the *Kontsevich integral*, after tensoring with the

rational numbers. See Definition 3.4 for the definition of the Kontsevich integral $Z_{<k}^g$:

Theorem 1.1. $Z_{<k}^g$ induces an isomorphism of \mathbb{Q} -vector spaces

$$\mathcal{K}/G_k \otimes \mathbb{Q} \cong \mathcal{B}_{<k}^g \otimes \mathbb{Q}$$

This result was inspired by the recent discovery of Garoufalidis and Rozansky [GR] that the Kontsevich integral not only preserves the Vassiliev filtration but also the “loop filtration”, where one grades diagrams by the first Betti number (and correspondingly the clasper surgeries are reorganized). We decided to give an independent proof of Theorem 1.1, by using properties of the Kontsevich integral explained in [Aa] as well as a result from [CT] which says that grope cobordisms can be refined into simple clasper surgeries. In Conjecture 4.5, we take a guess at what the groups \mathcal{K}/G_k could be integrally. There is an analogue of this Theorem 1.1 which says that *capped* grope cobordism is rationally computed by $\mathcal{B}_{<k}^v$, which is the same diagram space as above but graded by the Vassiliev degree. This latter result follows by using work of Habiro, and was announced in [H2].

If one uses class as an organizational tool for *grope concordance*, i.e. for gropes embedded in $S^3 \times [0, 1]$ and with boundary in $S^3 \times \{0, 1\}$, then the theory collapses:

Proposition 3.8. *For each $k \geq 3$, two knots $K_i \subset S^3 \times \{i\}$, are class k grope concordant if and only if their Arf invariants agree.*

It should be mentioned that Schneiderman has independently given a beautiful geometric argument for the above fact: He directly constructs a *weak Whitney tower of class k* in 4-space, cobounding two knots with equal Arf invariants. One can then turn this weak Whitney tower into a grope concordance of class k .

It turns out that in order to derive interesting information about knot concordance (i.e. 4-dimensional knot theory), one needs to imitate the *derived series* of a group geometrically. This can be done by restricting attention to gropes which grow symmetrically from the root. Such gropes have a *height h* and the class k can be calculated as $k = 2^h$, exactly as for group commutators. It was shown by Cochran, Orr and Teichner in [COT] that *symmetric* grope concordance filters the knot concordance group in such a way as to yield all known concordance invariants in the first few steps, leaving a huge tower of concordance invariants yet to be discovered. The first new graded quotient (above the one leading to Casson-Gordon invariants) was shown to be nontrivial in [COT] by using certain von Neumann signatures associated to solvable quotients of the knot group. It is now known [CoT] that all the graded quotients are nontrivial.

Challenge. *Try to understand the equivalence relation of symmetric grope cobordism in 3-space. In particular, determine the role of the von Neumann signatures.*

We will show that very interesting things happen even at very small heights. At height $h = 1.5$ one gets isomorphism of Blanchfield forms (i.e. S-equivalence) for grope cobordisms in 3-space, whereas the 4-dimensional analogue gives *cobordism* of Blanchfield forms. So in this setting the “kernel” from dimensions 3 to 4 is given by connected sums $K \# K^!$, where K is any oriented knot and $K^!$ is its reversed mirror image, the concordance inverse.

The reader might be irritated about the occurrence of the non-integral height $h = 1.5$ but that’s just a special case of the following equivalence relations on knots: Fix a rooted tree type \mathcal{T} and consider only grope cobordisms of type \mathcal{T} . The notation \mathcal{T} (respectively $c\mathcal{T}$) in the following table refers to the equivalence relation given by grope cobordisms (respectively capped grope cobordism) in 3-space using gropes of tree-type \mathcal{T} , as explained in [CT]. One can also study grope cobordism in $S^3 \times [0, 1]$ which is denoted by \mathcal{T}^4 below. Note that in dimension 4 there is no difference between capped and uncapped grope cobordism because intersections and self-intersections of the caps can always be pushed down into the bottom stage. The following table summarizes our calculations in Section 4.

Theorem 1.2. *For the smallest rooted tree types \mathcal{T} , the grope cobordism (respectively grope concordance) relations are given by the following table:*

Tree Type \mathcal{T}	$\mathcal{K}/c\mathcal{T}$	\mathcal{K}/\mathcal{T}	$\mathcal{K}/\mathcal{T}^4$
	{0}	{0}	{0}
	$\mathbb{Z}(c_2)$	$\mathbb{Z}/2(\text{Arf})$	$\mathbb{Z}/2(\text{Arf})$
	$\mathbb{Z}(c_3) \oplus \mathbb{Z}(c_2)$	$\mathbb{Z}(c_2)$	$\mathbb{Z}/2(\text{Arf})$
	$\mathbb{Z}(c_3) \oplus \mathbb{Z}(c_2)$	$\mathbb{Z}(c_2)$	$\mathbb{Z}/2(\text{Arf})$
	$\mathbb{Z}(c_4) \oplus \mathbb{Z}(c'_4) \oplus \mathbb{Z}(c_3) \oplus \mathbb{Z}(c_2)$	$\mathbb{Z}/2(c_3) \oplus \mathbb{Z}(c_2)$	$\mathbb{Z}/2(\text{Arf})$
	$\mathbb{Z}(c_4) \oplus \mathbb{Z}(c'_4) \oplus \mathbb{Z}(c_3) \oplus \mathbb{Z}(c_2)$	$\mathbb{Z}/2(c_3) \oplus \mathbb{Z}(c_2)$	$\mathbb{Z}/2(\text{Arf})$
	?	S-equivalence or Bl-forms	cobordism of Bl-forms

Here the c_i are generating degree i Vassiliev invariants, and *Bl* is the Blanchfield form.

Corollary 1.3. c_3 modulo two is an S-equivalence invariant.

Challenge. Understand the monoids \mathcal{K}/\mathcal{T} for more tree types \mathcal{T} .

Our paper ends with Section 5 where we have collected several results that are relevant in the context of grope cobordism. Recall that if a knot K cobounds a

grope with the unknot U , then K and U might very well be linked in a nontrivial way. Thus it is a much stronger condition on K to assume that it is the boundary of a grope. For example, if K bounds a grope of class 3 in S^3 then the Alexander polynomial vanishes. The following statement generalizes this vanishing result by using the “null-filtration” of [GR], explained in Section 5.1.

Proposition 5.2. *If a knot K bounds an embedded grope of class k in a 3-manifold M , then the pair (M, K) is $(k - 3)$ -null equivalent to (M, U) .*

The converse is not true for $k = 3$: In $M = S^3$, knots which are null equivalent to the unknot ($k = 3$ above), are exactly knots with trivial Alexander polynomial. On the other hand, knots which bound a grope of class 3 in S^3 have a Seifert surface such that the rank of the Seifert form equals the genus of the surface. It is shown in [GT] that this *minimal Seifert rank* condition is much stronger than having trivial Alexander polynomial.

We investigate the behavior of orientation reversal:

Proposition 5.3. *Let ρ be the map reversing a knot’s orientation. Then for every knot K in the k -th term G_k of the grope filtration of \mathcal{K} , one has*

$$K \equiv (-1)^k \rho(K) \pmod{G_{k+1}}.$$

Our final result implies that the main theorem of [CT] can now be phrased as follows: Grope cobordisms of class k in S^3 are in 1-1 correspondence with sequences of simple clasper surgeries of grope degree *exactly* k . In [CT] we had to allow simple clasper surgeries of grope degree $\geq k$.

Theorem 5.4. *A simple clasper surgery of grope degree $(k + 1)$ may be realized by a sequence of simple clasper surgeries of grope degree k .*

2. Gropes, clasps and diagrams

2.1. Basic notions

For the reader’s convenience we recall some basic notions and results from [CT]. In that paper we introduced *rooted clasps* which are connected clasps in the sense of Habiro [H1, H2], where one leaf (the *root*) is an unknotted meridian of a knot. Given a knot K and a disjoint rooted clasper C in a 3-manifold M , one can construct a grope cobordism $G(K, C)$ between K and K_C in M , where K_C is the surgery of K along C . The “root” of the clasper is a *simple leaf* (i.e. a leaf with an embedded cap intersecting K once) which makes sure that the ambient 3-manifold M is unchanged. *Caps* of a clasper C are disjointly embedded disks with interiors disjoint from C , which bound some of the leaves of C . The grope cobordism $G(K, C)$ is *capped* if all the leaves of the clasper have caps, which are allowed to intersect the knot. (If there is a single cap disjoint from K then K_C is isotopic

to K .) Finally, in a *simple* clasper, all the leaves have caps and the knot intersects each cap exactly once. Thus this notion only makes sense for the pair (K, C) .

The class of $G(K, C)$ is given by the *grope degree* of the clasper. Recall that a clasper C has an underlying uni-trivalent graph Γ which is obtained by removing the leaves and collapsing to the spine. The grope degree of C is defined as the Vassiliev degree plus the first Betti number of Γ . The Vassiliev degree is one half the number of vertices of Γ .

The construction of $G(K, C)$ depends on a choice of $b_1(\Gamma)$ many “cuts”, which turn Γ into a rooted tree, giving the precise grope type of $G(K, C)$. Note that each cut increases the Vassiliev degree by one, but leaves the grope degree unchanged, as it should. Note also that a cut introduces a pair of Hopf-linked leaves into the clasper, and hence the resulting grope cobordism cannot have two disjoint caps at the corresponding tips. This explains why capped grope cobordism corresponds to the Vassiliev degree: cuts are not allowed (since caps must be disjointly embedded), and for trees the two degrees agree.

Let \mathcal{K} denote the monoid of oriented knot types in 3-space (with respect to connected sum), and let \mathcal{K}/G_k be \mathcal{K} modulo the equivalence relation of grope cobordism of class k . It turns out that this is an abelian group. By Theorem 2 of [CT], a grope cobordism of class k corresponds to a sequence of simple clasper surgeries of grope degree $\geq k$. We will show in Theorem 5.4 that a grope cobordism of class k also corresponds to a sequence of simple clasper surgeries of grope degree *exactly equal to* k .

Let \mathcal{K}/G_k^{cap} denote the abelian group of oriented knot types modulo capped grope cobordism of class k in 3-space. Capped grope cobordism of class k coincides with Vassiliev degree k simple clasper surgeries, and with Vassiliev $(k - 1)$ -equivalence. By letting G_k be the subset of knots which are class k grope cobordant to the unknot (and similarly G_k^{cap} for capped grope cobordisms) we can form the associated graded quotients

$$\mathcal{G}_k := G_k / G_{k+1} \quad \text{respectively} \quad \mathcal{G}_k^{cap} := G_k^{cap} / G_{k+1}^{cap}$$

These graded quotients will be related in Lemmas 2.1 and 3.2 to certain Feynman diagrams of a fixed degree.

Let \mathcal{T} be a rooted tree type. Then \mathcal{K}/\mathcal{T} is defined as the monoid of knots modulo \mathcal{T} grope cobordism. Define $\mathcal{K}/\mathcal{T}^{cap}$ to be the monoid of knots modulo capped \mathcal{T} grope cobordism. A capped \mathcal{T} grope cobordism corresponds to a sequence of simple tree clasper surgeries of type \mathcal{T} . As a consequence, $\mathcal{K}/\mathcal{T}^{cap}$ depends only on the unrooted tree type.

2.2. Feynman diagrams

Let $\tilde{\mathcal{A}}_k$ denote the free abelian group generated by connected trivalent graphs with $2k$ vertices and one distinguished (oriented) cycle, such that each trivalent vertex

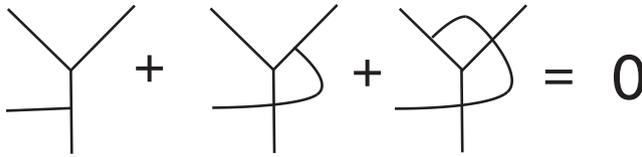


Fig. 1. The IHX relation. Note that the 4-valent vertices are not present

has a cyclic orientation.¹ The distinguished cycle is often called “the outer circle,” and the rest is sometimes called “the dashed part.”

Define \mathcal{A}_k to be the quotient of $\tilde{\mathcal{A}}_k$ by the usual IHX and AS relations. The AS (antisymmetry) relation is a relation of the form $G_1 + G_2 = 0$, where the G_i differ by a cyclic orientation at a given vertex. The IHX relation is pictured in Figure 1. If the distinguished cycle runs through the part of the diagram involved in an IHX relation, the relation is called an STU relation. As proven in [BN], STU relations generate all IHX relations. Then $\mathcal{A} = \bigoplus_k \mathcal{A}_k$ is the well-known algebra of Feynman diagrams. (Feynman diagrams enjoy a plethora of other names, including web diagrams, Jacobi diagrams, and Chinese character diagrams.) The algebra structure is given by “connected sum”, which turns out to be well-defined. Moreover, \mathcal{A} is a graded Hopf algebra as explained in [BN]. The *primitive* elements $\text{Prim}(\mathcal{A})$ are generated by diagrams which stay connected when removing the outer circle. However, such diagrams are *not* closed under the STU relation, so it is convenient to also consider the group $\mathcal{A}^I := \mathcal{A}_{>0} / \mathcal{A}_{>0}^2$ of *indecomposable* elements. Since \mathcal{A} is a commutative and cocommutative Hopf algebra, a famous theorem of Milnor and Moore implies that, say over \mathbb{Q} , \mathcal{A} is a polynomial algebra in the primitive elements. In particular,

$$\text{Prim}(\mathcal{A}) \otimes \mathbb{Q} \cong \mathcal{A}^I \otimes \mathbb{Q}.$$

There is an analogous abelian group of diagrams \mathcal{B} defined just like \mathcal{A} but without an outer circle. Here the subspace of connected diagrams is closed under IHX and AS relations, so that the above problem for primitives does not occur. There is an averaging map $\chi : \mathcal{B} \rightarrow \mathcal{A}$ which puts back the outer circle in all possible ways. It was shown in [BN] that this is rationally an isomorphism.

Since we will only be interested in connected diagrams, we *define* \mathcal{B}_k^v to be the abelian group spanned by connected uni-trivalent graphs with $2k$ vertices (such that each trivalent vertex has a cyclic orientation, and with at least one univalent vertex) modulo the AS and IHX relations. Since we are dealing with unframed knots, we just *set* the group $\mathcal{B}_1^v := 0$. (Note that \mathcal{B}_0^v is spanned by the empty graph.) Define the graded abelian group

$$\mathcal{B}^v := \bigoplus_k \mathcal{B}_k^v.$$

¹ Since trivalent vertices that lie on the distinguished cycle can be canonically oriented, the convention in much of the literature is to not orient these vertices.

The superscript ‘ v ’ indicates that we are using the Vassiliev degree, and it also serves to distinguish from the usual group \mathcal{B} (which would also contain non-connected diagrams as well as “struts”). As a consequence of these definitions, one has an averaging isomorphism of graded vector spaces

$$\chi : \mathcal{B}^v \otimes \mathbb{Q} \xrightarrow{\cong} \mathcal{A}^I \otimes \mathbb{Q}. \tag{1}$$

We shall show that these groups, modulo terms of degree $> k$, are isomorphic to $\mathcal{K}/G_k^{cap} \otimes \mathbb{Q}$.

2.3. Maps relating capped gropes and Feynman diagrams

There is a well-known map that sends a chord diagram to an alternating sum over crossing changes on the unknot guided by the chords. This map can in fact be extended to all diagrams, as in the next lemma, whose proof is found in the next section.

Lemma 2.1. *For each $k > 1$, there is an epimorphism*

$$\Phi_k : \mathcal{A}_k^I \longrightarrow \mathcal{G}_k^{cap}$$

defined by sending a diagram to the alternating sum over clasper surgeries corresponding to each connected component of “the dashed part.”

The (unframed) Kontsevich integral is a map on isotopy classes of oriented knots

$$Z : \mathcal{K} \longrightarrow \widehat{\mathcal{A}} := \prod_k \mathcal{A}_k \otimes \mathbb{Q}$$

which sends connected sums of two knots to their *product* in $\widehat{\mathcal{A}}$. The image of Z lies inside the group-like elements of the complete Hopf algebra $\widehat{\mathcal{A}}$. Thus we may compose it with the *logarithm* in this complete Hopf algebra to end up in the subspace of primitive elements. Then $\log Z$ takes connected sum to addition! Now decompose $\log Z$ according to Vassiliev degree and denote the degree k part by

$$Z_k^v : \mathcal{K} \longrightarrow \text{Prim}(\mathcal{A}_k) \otimes \mathbb{Q} \cong \mathcal{A}_k^I \otimes \mathbb{Q}.$$

Then Z_k^v factors through simple clasper surgeries of degree $k + 1$ by [H2], and hence through $\mathcal{K}/G_{k+1}^{cap}$ by [CT]. Restricting to knots in G_k^{cap} we get a homomorphism

$$Z_k^v : \mathcal{G}_k^{cap} \longrightarrow \mathcal{A}_k^I \otimes \mathbb{Q}.$$

The definition of Φ_k given in the next section, and in particular the fact that it extends the original definition on chord diagrams, implies by the universality of the Kontsevich integral the following

Lemma 2.2. $Z_k^v \circ \Phi_k = Id$.

This shows that Φ_k is an isomorphism modulo torsion because \mathcal{A}_k are finitely generated.

2.4. Claspers and Feynman diagrams

It is the goal of this section to show that the map Φ_k extends to all diagrams in a natural way. This will lead to Lemma 2.13 which will be used in the proof of Theorem 4.2. This material is well-known to the experts and was essentially announced by Habiro in [H2]. A complication we face is that we include all capped claspers, not just those which are trees. Since Habiro usually restricts to trees, we need to include some extra arguments for the general case.

We begin with some definitions which will be used in this section.

Definitions 2.3. • A CC-set is a union of finitely many disjoint capped claspers on a knot K .

- The degree of a CC-set is the minimum of the degrees of each connected component.
- A CC-scheme is a collection $\{C_1, C_2, \dots, C_l\}$ where each C_i is a CC-set.
- The degree of a CC-scheme is the sum of the degrees of each C_i .
- A CC-scheme is called simple if each CC-set is a single simple clasper.
- Let $\mathcal{S} = \{C_1, C_2, \dots, C_l\}$ be a CC-scheme on a knot K . Define $[K; \mathcal{S}] = \sum_{\sigma \subset \mathcal{S}} (-1)^{l+|\sigma|} K_\sigma \in \mathbb{Z}[\mathcal{K}]$, where K_σ is the knot modified by each CC-set in σ .

We list a couple of useful facts about brackets.

Lemma 2.4. Let C_1, C_2 be disjoint CC-sets on K and denote by $C_1 \cup C_2$ the CC-set which is the union of C_1 and C_2 . If \mathcal{S} is a CC-scheme on K , disjoint from $C_1 \cup C_2$, then

$$[K; \{C_1 \cup C_2\} \cup \mathcal{S}] = [K; \{C_1\} \cup \mathcal{S}] + [K_{C_1}; \{C_2\} \cup \mathcal{S}],$$

See [G], Lemma 5.2 mutatis mutandis.

Lemma 2.5. Consider the central part of Figure 2, with two disjoint capped claspers a, b , disjoint from a CC-scheme \mathcal{S} . Then

$$[K; \{a\} \cup \{b\} \cup \mathcal{S}] = [K; \{a \cup b\} \cup \mathcal{S}]$$

Proof. If $\sigma \subset \mathcal{S}$ then $K_{\{a\} \cup \sigma} = K_\sigma = K_{\{b\} \cup \sigma}$. This implies $[K_a; \mathcal{S}] = [K_b; \mathcal{S}] = [K; \mathcal{S}]$ and hence

$$\begin{aligned} [K; \{a\} \cup \{b\} \cup \mathcal{S}] &= [K; \mathcal{S}] - [K_b; \mathcal{S}] - [K_a; \mathcal{S}] + [K_{a \cup b}; \mathcal{S}] \\ &= -[K; \mathcal{S}] + [K_{a \cup b}; \mathcal{S}] \\ &= [K; \{a \cup b\} \cup \mathcal{S}] \quad \square \end{aligned}$$

The linear span of all knot types, $\mathbb{Z}[\mathcal{K}]$, can be filtered by defining $\mathcal{F}_k^v \subset \mathbb{Z}[\mathcal{K}]$ to be the linear span of all brackets $[K; \mathcal{S}]$ where \mathcal{S} is of degree k .

Lemma 2.6. If C is a clasper on a knot disjoint from a CC-scheme \mathcal{S} , then $[K; \mathcal{S}] - [K_C; \mathcal{S}] \in \mathcal{F}_{|\mathcal{S}|+|C|}^v$.

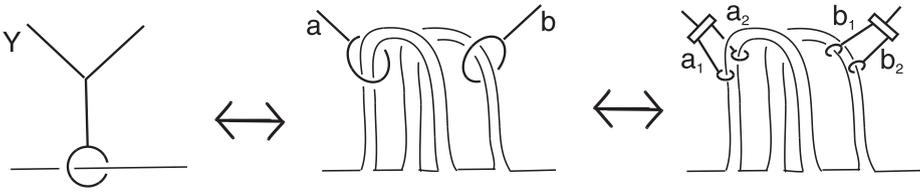


Fig. 2. A clasper identity

Proof. This is just the simple calculation $[K; \mathcal{S} \cup \{C\}] = [K; \mathcal{S}] - [K_C; \mathcal{S}]$. \square

The following proposition explains the superscript “ v .”

Proposition 2.7. $\mathcal{F}_k^v \subset \mathbb{Z}[K]$ agrees with the usual Vassiliev filtration of $\mathbb{Z}[K]$.

Proof. One can take the usual Vassiliev filtration to be defined by brackets where each CC-set is a single simple degree 1 clasper. Thus \mathcal{F}_k^v contains the usual Vassiliev filtration.

The converse follow from Theorem 6.7 (3) of [H2], with one complication. Namely, that Theorem is phrased only for tree claspers, but it can be easily enhanced to work for all capped claspers. The proof works by applying the move from Lemma 2.5, thereby breaking a clasper into two claspers of lower degree. Iterating, one reduces all of the trees to degree one capped claspers. This fails for graphs if in Figure 2, the two claspers a and b are the same clasper. In this case we’ve still made progress since there are a reduced number of loops. To be precise, induct on the number of internal vertices. (Compare the proof of Lemma 2.8.) \square

Lemma 2.8. On a knot K , let $\mathcal{S}_1 = \{C_1^1, \dots, C_l^1\}$ and $\mathcal{S}_2 = \{C_1^2, \dots, C_l^2\}$ be two CC-schemes of degree k , where each CC-set consists of a single clasper. Suppose the CC-schemes differ only by a homotopy of an edge of one of the claspers. Then

$$[K; \mathcal{S}_1] - [K; \mathcal{S}_2] \in \mathcal{F}_{k+1}^v.$$

Proof. A homotopy of an edge can be realized by a degree one clasper, one leaf of which links the edge and the other leaf being embedded arbitrarily away from the caps. Using the zip construction, this can be realized by degree one claspers E where one leaf links the edge, and the other leaf either: is a meridian to an edge, is a meridian to the knot, or is a trivial 1-framed leaf. This last case is covered by Corollary 2.11. We prove the other two cases by induction on the number of trivalent vertices. The base case is when all the claspers C_j^i are of degree one (eyeglasses). Then the leaves of the clasper E can be slid off the end of the (up to two) eyeglasses that it links, introducing two new intersections with the knot for each one. Let the slid clasper be called E' . Then we have argued

$$[K; \mathcal{S}_2] = [K_{E'}; \mathcal{S}_1].$$

By Lemma 2.6 we are done. Now suppose that a pair C_j^1, C_j^2 has a trivalent vertex. Apply Figure 2 to a leg of C_j^i , denoting the resulting claspers by a^i, b^i . (If E happens to link the leg we are expanding, push it off so that it links the knot instead.) If a^i and b^i are different, then Lemma 2.5 implies that, defining $S'_i = \{C_1^i, C_2^i, \dots, C_{j-1}^i, a^i, b^i, C_{j+1}^i, \dots, C_l^i\}$,

$$[K; S_i] = [K; S'_i]$$

and since S'_1, S'_2 differ by the homotopy coming from E , $[K; S'_1] = [K; S'_2] \text{ mod } \mathcal{F}_{k+1}^v$ by induction. If $a^i = b^i$, life is even simpler: define

$$S'' = \{C_1^i, C_2^i, \dots, C_{j-1}^i, a^i, C_{j+1}^i, \dots, C_l^i\}.$$

Then $[K; S] = [K; S'']$ and we are done by induction as above. □

Lemma 2.9. *Let C be a tree clasper on a knot K . There is a clasper \tilde{C} in a regular neighborhood of C such that $K_{C \cup \tilde{C}} = K$. Moreover, the leaves of C and \tilde{C} are parallel (in the given framing of C) and one may arrange the edges of \tilde{C} which go through a regular neighborhood of a leaf of C to be parallel to the leaves of C .*

In Figure 3(a), the above process is depicted in the vicinity of a Hopf-pair of C . (The two degree 1 claspers labelled E are not relevant now.)

Proof. The clasper \tilde{C} is constructed using Figure 27 of [H2] and the version of the zip construction in [CT, Sec.4]. □

Lemma 2.10. *Let C and C' be two simple claspers of degree k on a knot K , which differ by a single half-twist. Then $K_C + K_{C'} - 2K \in \mathcal{F}_{k+1}^v$.*

Proof. Insert Hopf-pairs of leaves to make C a tree clasper and let \tilde{C} be an inverse to C , as in Lemma 2.9. For each (non-root) cap of C and \tilde{C} there is a degree 1 clasper (or eyeglass), surgery on which pushes the knot out of the cap. Choose one of the leaves in each Hopf pair of C . As this can be regarded as an additional cap of the tree clasper, there is an eyeglass which pushes everything out of this leaf, just like above. See Figure 3 for an example of these eyeglasses E_i . Let \mathcal{E}_1 denote the CC-scheme whose elements are the (at least k) eyeglasses which push things out of C , and let \mathcal{E}_2 be the same, only for \tilde{C} . The claspers in $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$ are capped, when considered on the knot $K = K_{C \cup \tilde{C}}$. Note that $K_\sigma = K$ if σ has nontrivial intersection with \mathcal{E}_1 and with \mathcal{E}_2 . If $\emptyset \neq \sigma \subset \mathcal{E}_1$, then $K_\sigma = K_{\tilde{C}}$. If $\emptyset \neq \sigma \subset \mathcal{E}_2$ then $K_\sigma = K_C$. This analysis implies that

$$-[K; \mathcal{E}] = K_C + K_{\tilde{C}} - 2K.$$

The left-hand side is in $\mathcal{F}_{2k}^v \subset \mathcal{F}_{k+1}^v$.

This is almost what we want, except that \tilde{C} has edges that wind around itself and may wind through the Hopf pairs of leaves. Let \mathcal{E}_3 be the CC-scheme consisting

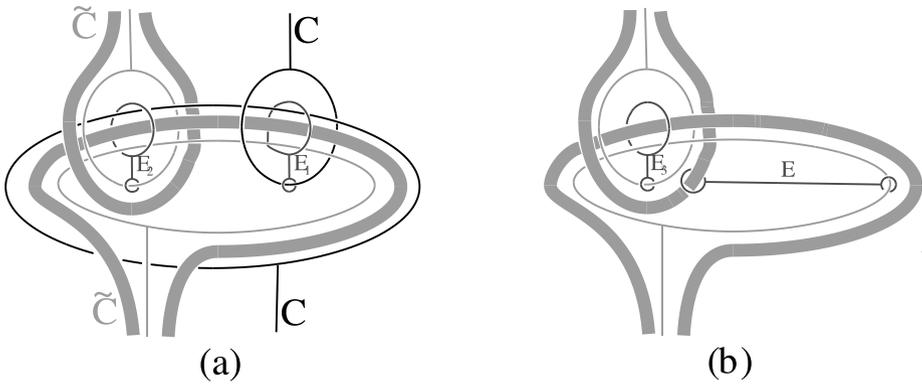


Fig. 3. From the proof of Lemma 2.10. The thick lines are where edges of \tilde{C} might be

of k degree 1 claspers on $K_{\tilde{C}}$, which push things out of the leaves. There is an additional clasper, denoted by E which pushes the edges out of the other disk in each Hopf pair. It is a union of eyeglasses, one for each leaf, see Figure 3(b). Now

$$[K_{\tilde{C}}; \mathcal{E} \cup \{E\}] = K_{\tilde{C}} - K_{\bar{C}},$$

where \bar{C} is formed from \tilde{C} by applying E . Symmetrically we can push the edges out of the other leaves in each Hopf pair, modulo \mathcal{F}_{k+1}^v to get the clasper C' in the statement of the lemma. \square

Corollary 2.11. *Let C_1 and C_2 be two claspers on a knot K which differ by a full twist along an edge. Then $K_{C_1} - K_{C_2} \in \mathcal{F}_{k+1}^v$.*

Proof. Let C be a clasper that differs from both C_1 and from C_2 by a half-twist. Then Lemma 2.10 implies that $K_{C_1} = 2K - K_C = K_{C_2} \pmod{\mathcal{F}_{k+1}^v}$. \square

Let \mathcal{CF}_k be the set of pairs (K, \mathcal{S}) where $K \in \mathcal{K}$ and \mathcal{S} is a simple CC-scheme of degree k on the knot K . These are considered up to isotopy. The bracket defines a map $\mathcal{CF}_k \rightarrow \mathbb{Z}[\mathcal{K}]$. One can also define a map

$$\Psi : \mathcal{CF}_k \rightarrow \mathcal{A}_k$$

as follows: The unoriented trivalent graph, D , associated to (K, \mathcal{S}) is gotten by collapsing simple leaves to points and forgetting the embedding. The distinguished cycle is the knot. The orientation of the Feynman diagram is more subtle. Recall that each clasper C_i is a certain thickening of a graph. Hence C_i may not be orientable (even though the thickened leaves are because they have integral framing, and hence a full number of twists). However, each C_i may be made orientable by replacing l_i bands by half-twisted bands, where l_i is the dimension of the 2-torsion subgroup of $H_1(C_i)$. Do this, and let $l = \sum l_i$ be the total number

of half twisted bands glued in to every C_i . Now choose orientations of each new thickening C_i . These give rise to cyclic orders of the in-coming edges at each node. Denote the induced orientation of D by or . The choice of orientation also determines a normal direction. Let m be the number of leaves where the knot pierces contrary to the distinguished normal direction. Now $\Psi(K, \mathcal{S}) = (-1)^{m+l}(D, or)$. Ψ is well defined. It does not depend on the choice of orientations, because the parity of the number of leaves of each C_i is the same as the parity of the number of nodes. The key thing to notice about this orientation convention is that it switches under the introduction of a half-twist in any edge of a clasper in \mathcal{S} .

Now we can define the homomorphism

$$\Phi_k : \mathcal{A}_k^I \rightarrow \mathcal{G}_k^{cap}$$

from Lemma 2.1 as follows: For an oriented diagram $D \in \tilde{\mathcal{A}}_k$, define $\Phi_k(D) := cs_k([U; \mathcal{S}])$, for some framed embedding \mathcal{S} of the diagram D on the unknot U , so that $\Psi(U, \mathcal{S}) = D$. (Note that framed embedded diagrams can be regarded as simple CC-schemes). Here $cs_k : \mathbb{Z}[\mathcal{K}] \rightarrow \mathcal{K}/\mathcal{G}_{k+1}^{cap}$ is the map given by sending addition to connected sum.

By definition, Φ_k vanishes on diagrams of degree $> k$ and, by Proposition 2.7 lands in $\mathcal{G}_k^{cap} = \mathcal{F}_k^v/\mathcal{F}_{k+1}^v$. This, together with Lemma 2.8 and Corollary 2.11, implies that, extending linearly, we get a well defined map:

$$\Phi_k : \tilde{\mathcal{A}}_k \rightarrow \mathcal{G}_k^{cap}$$

This means that the choice of the framed embedding of the diagram D is irrelevant. Clearly $\Phi_k = 0$ on separated diagrams, so it will factor through \mathcal{A}_k^I . Thus to get a map on \mathcal{A}_k , we only need to show that Φ_k vanishes on the STU- and AS-relations.

Lemma 2.12. Φ_k vanishes on all STU and AS relations.

Proof. First we show it vanishes on AS relations. Suppose D_1 and D_2 are two diagrams that differ by a cyclic order at one vertex. Let C_1 be a clasper representing D_1 : $\Phi_k(D_1) = U_{C_1}$. Let C_2 be the same, except for three half-twists on the edges incident to the vertex. By our orientation conventions, $\Phi_k(D_2) = U_{C_2}$. Applying Lemma 2.10 three times, with $K = U$, we see that

$$\Phi_k(D_1 + D_2) = U_{C_1} \# U_{C_2} = 0 \in \mathcal{G}_k^{cap},$$

as desired. Next we consider the STU relation. Let $\tilde{\Phi}_k : \tilde{\mathcal{A}}_k \rightarrow \mathbb{Z}[\mathcal{K}]$ be the lift of Φ_k discussed above, i. e. before applying the summation maps cs_k . Consider an STU relation $D_s = D_t - D_u$, where D_s has one more trivalent vertex than D_t and D_u . Let Y be the component of the dashed part of D_s which has the additional vertex. Then

$$\tilde{\Phi}_k(D_s) = [U; \{Y, C_2, \dots, C_l\}]$$

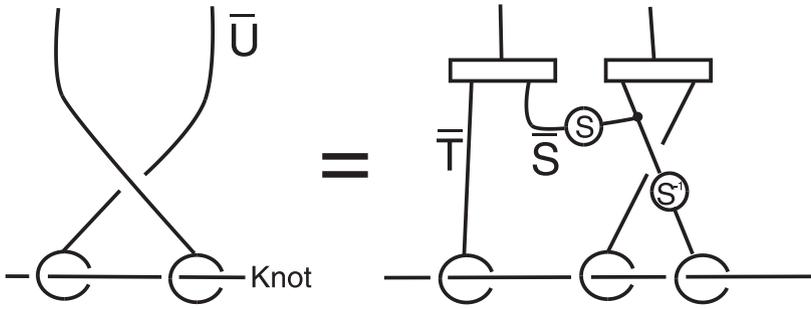


Fig. 4. A clasper identity

where the clasper representing Y is also called Y , and is pictured in Figure 2. The claspers a and b from Figure 2 can be further subdivided into claspers a_1, a_2, b_1, b_2 using the zip construction, assuming that a and b are not the same clasper. By the previous lemmas we have

$$\begin{aligned}
 & [U; \{a \cup b, C_2, \dots, C_l\}] \\
 &= [U; \{a, b, C_2, \dots, C_l\}] \\
 &= [U; \{a_1 \cup a_2, b_1 \cup b_2, C_2, \dots, C_l\}] \\
 &= [U; \{a_1, b_1, C_2, \dots, C_l\}] + [U_{b_1}; \{a_1, b_2, C_2, \dots, C_l\}] \\
 &\quad + [U_{a_1}; \{a_2, b_1, C_2, \dots, C_l\}] + [U_{a_1 \cup b_1}; \{a_2, b_2, C_2, \dots, C_l\}] \\
 &\equiv [U; \{a_1, b_1, C_2, \dots, C_l\}] + [U; \{a_1, b_2, C_2, \dots, C_l\}] \\
 &\quad + [U; \{a_2, b_1, C_2, \dots, C_l\}] + [U; \{a_2, b_2, C_2, \dots, C_l\}] \pmod{\mathcal{F}_{k+1}^v(\mathbb{Z}\mathcal{K})} \\
 &= -\tilde{\Phi}_k(D_t) + \tilde{\Phi}_k(D_t) - \tilde{\Phi}_k(D_u) - \tilde{\Phi}_k(D_t) = \tilde{\Phi}_k(D_t) - \tilde{\Phi}_k(D_u)
 \end{aligned}$$

If, in Figure 2, the two claspers a and b are really two ends of the same clasper, we use the construction of Proposition 4.6 of [H2] instead. See Figure 4 which is a clasper identity of [H2].

Let D_s, D_t, D_u be the three diagrams in an STU relation such that the corresponding a and b claspers are part of the same clasper. Then

$$\begin{aligned}
 \tilde{\Phi}_k(D_u) &= [U; \{\bar{U}, C_2, \dots, C_l\}] = [U; \{\bar{T}, \bar{S}\}, C_2, \dots, C_l] \\
 &= [U; \{\bar{T}, C_2, \dots, C_l\}] + [U_{\bar{T}}; \{\bar{S}, C_2, \dots, C_l\}] \\
 &\equiv [U; \{\bar{T}, C_2, \dots, C_l\}] + [U; \{\bar{S}, C_2, \dots, C_l\}] \\
 &= \tilde{\Phi}_k(D_t) + \tilde{\Phi}_k(D_s)
 \end{aligned}$$

□

The proof of Lemma 2.1 is now complete.

Lemma 2.13. *Let C be a simple clasper surgery of Vassiliev degree k on a knot K . Let D be the diagram obtained by thinking of the clasper as the dashed part, and of the knot as the outer circle. Then $\Phi_k(\pm D) = K_C \# K^{-1} \in \mathcal{G}_k^{cap}$.*

Remark 2.14. Here is a sketch of a proof that Lemma 2.13 is true rationally. The proof that it is true integrally is given below. One can think of the Kontsevich integral of $K_C - K$ as the Aarhus integral of the difference of two links $K \cup L_C - K$, where L_C is the link associated to the simple clasper. Now, it is not hard to show that the lowest degree term of the Aarhus integral is exactly the diagram corresponding to the graph type of the clasper. (By arguments analogous to those in section 3.3 of [GR]. In the absence of a knot K , this is the statement that the Aarhus/LMO invariant is universal with respect to Goussarov’s Y-filtration.) Now the result follows by Lemma 2.2.

Proof of 2.13. Let \tilde{C} be a clasper on the unknot with diagram D . Put in half-twists so that it has the same sign as C . By definition, $\Phi_k(D) = U_{\tilde{C}} = U_{\tilde{C}}\#K\#K^{-1}$. By Lemma 2.8, we can move \tilde{C} into a position corresponding to the clasper C on the knot K . Using Corollary 2.11, we can add an even number of half-twists until the framings agree. □

3. The grope degree

We first introduce a second degree on the graphs Γ generating \mathcal{B}^v :

Definition 3.1. *The grope degree of Γ is*

$$g(\Gamma) := b_1(\Gamma) + v(\Gamma).$$

Let \mathcal{B}_k^g be the grope degree k part of the group \mathcal{B}^v .

We note that the grope degree is preserved by the IHX and AS relations, and hence really gives a new *graded* abelian group

$$\mathcal{B}^g := \bigoplus_k \mathcal{B}_k^g.$$

3.1. Feynman diagrams and the grope degree

In a similar fashion to Φ_k , we define a surjective map

$$\Phi_k^g : \mathcal{B}_k^g \rightarrow \mathcal{G}_k$$

on connected Feynman diagrams of grope degree k as follows. For a connected diagram $D \in \tilde{\mathcal{B}}_k$, let $\tilde{D} \in \tilde{\mathcal{A}}_k$ be a diagram formed by attaching the univalent vertices of D to the outer circle in some order. Let $(U, C) \in \mathcal{CF}_k^v$ satisfy $\Psi(U, C) = \tilde{D}$, where C is a single clasper of grope degree k , and U is the unknot. Now define

$$\Phi_k^g(D) := U_C \in \mathcal{K}/G_{k+1}.$$

In the previous section we used various lemmas of Habiro [H2] (for moving claspers around modulo higher Vassiliev degree) in order to show that Φ_k is well

defined. Section 3.3 contains the relevant lemmas in the case of grope degree. In fact, we need to show that Φ_k^g does not depend on the order in which we attached the univalent vertices, and that AS and IHX get killed. The independence of order follows from Lemma 3.9(a), and the AS and IHX relations follow from 3.11(a) and 3.11(c). We have thus proven the following analogue of Lemma 2.1.

Lemma 3.2. *For each $k > 1$, there is an epimorphism*

$$\Phi_k^g : \mathcal{B}_k^g \longrightarrow \mathcal{G}_k.$$

In order to show that Φ_k^g is rationally an isomorphism, we want to study the behavior of the Kontsevich integral with respect to the grope degree.

Recall that there is an isomorphism $\hat{A} \cong \hat{B}$ of graded *algebras* given by the composition $\partial_\Omega \sigma$, where $\sigma : \hat{A} \rightarrow \hat{B}$ is the inverse of the averaging map χ from equation 1, and ∂_Ω is the “wheeling” automorphism of \hat{B} [T]. Here \hat{A} has the multiplication given by connected sum, whereas on \hat{B} the multiplication is just disjoint union. It follows that

$$\log_{\hat{B}}(\partial_\Omega \sigma) = (\partial_\Omega \sigma) \log_{\hat{A}}. \tag{2}$$

Following [GR], define the *Euler degree* of a diagram in \mathcal{B} to be the number of *internal* trivalent vertices, which by definition are trivalent vertices not adjacent to univalent vertices. It is called Euler degree because for uni-trivalent graphs Γ one has

$$e(\Gamma) = 2(b_1(\Gamma) - b_0(\Gamma)).$$

Decompose the composition $\log_{\hat{B}}(\partial_\Omega \sigma)Z$ according to Euler degree to obtain

$$Z_k^e : \mathcal{K} \rightarrow \mathcal{B}_k^e$$

as the Euler degree k part of the “Kontsevich integral”. The following proposition follows by work of Garoufalidis and Rozansky [GR] using [Aa], but for the sake of completeness we provide an argument using only [Aa].

Proposition 3.3. *Let C be a simple clasper of Euler degree n on a knot K . Then $Z_m^e(K) - Z_m^e(K_C) = 0$ for all $m < n$.*

Proof. We factor through the Aarhus integral, A , of pairs (M, K) , [Aa]. When normalized appropriately, $A(S^3, K) = \sigma Z(K)$. (In the definition of the Aarhus integral we apply σ to the knot K as well as to the surgery link components.) We will assume familiarity with the Aarhus integral in the following proof.

Break C into a union of Y ’s. If a Y has no leaf that links the knot, it will be called *internal* since the corresponding trivalent vertex is internal. Each Y has an associated 6 component link. Let the three components in the Borromean rings be called the “B” components, and the three other components be called the “L” components. Let the link corresponding to the union of all Y ’s be called L_C . Fix

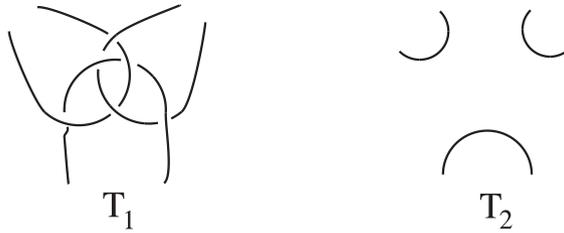


Fig. 5. The tangles T_1 and T_2 . The difference $\mu := Z(T_1) - Z(T_2)$ is comprised of terms with at least one trivalent vertex

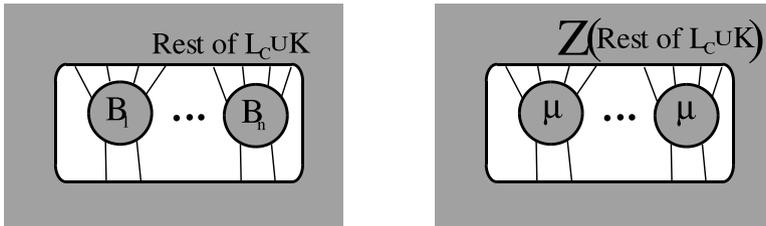


Fig. 6. The Kontsevich integral of $[K \cup L_C; S]$ is computed by gluing the Kontsevich integral at each ball B_i to the Kontsevich integral of the exterior

balls which meet the link at the Borromean rings corresponding to *internal* Y 's as on the left hand side of Figure 5. For each ball, there is an associated move that replaces the tangle on the left of Figure 5 with the trivial one on the right. Let S be the set of such moves, one for each internal vertex of C . Consider the alternating sum $[K \cup L_C; S]$. Then the (alternating sum of) Kontsevich integrals $Z[K \cup L_C; S]$ can be computed as the Kontsevich integral of the difference of tangles in each ball, called μ , glued to the Kontsevich integral of the exterior to the balls. See Figure 6.

Now we use the fact that each summand of μ always contains trivalent vertices. This follows since the degree 1 part is given by linking numbers which are zero. Therefore, since $\sigma Z(T_i)$ is of the form $\exp(\text{struts}) \exp(\text{rest})$, there is no strut part to μ . (We need to apply σ in order to have an algebra structure.) Therefore, we have shown that each term of $Z[K \cup L_C; S]$ contains at least n *special* vertices, which by definition are trivalent vertices only adjacent to other trivalent vertices, or to univalent vertices which lie on internal Borromean rings components. This also holds for the LMMO normalization $\check{Z}[K \cup L_C; S]$ which only differs by factors of the Kontsevich integral of the unknot v .

When we apply σ to $\check{Z}[K \cup L_C; S]$, the number of special vertices can only increase. This can be seen by examining the definition of σ [BN], which is iteratively defined by operations which involve removing the skeleton and tacking things on to the created univalent vertices.

Let $S(L_C)$ denote the surgery along L_C . Then the Aarhus integral is defined as

$$A[(S(L_C), K); S] := \left(\int^{FG} \sigma \check{Z}[K \cup L_C; S] \right) \cdot (\text{Kirby I-move normalizations})$$

The formal Gaussian integration is with respect to the negative inverse of the linking matrix of the link L_C , and this linking matrix is the same in each summand. Notice that the linking matrix and its inverse are of the forms

$$\Lambda = \begin{pmatrix} 0 & I \\ I & A \end{pmatrix}, \Lambda^{-1} = \begin{pmatrix} -A & I \\ I & 0 \end{pmatrix},$$

where the first row and column refers to B-components and the second to L-components. We claim that special vertices descend to internal vertices after applying \int^{FG} . If not, then there is a special vertex adjacent to a univalent vertex labeled by a B component, b , and a strut with one endpoint labeled by K and the other labeled by some component x , which are glued together along a strut labeled by b and x , with coefficient coming from $-\Lambda^{-1}$. By consideration of linking number, the only K - x struts are when x is an L component that links the knot, and hence is not part of an internal Y. Therefore the $b - x$ strut is between a B component and an L component in different Y's, which therefore don't link. Thus, by examining Λ^{-1} we see that the weight is zero.

Therefore we have argued that all summands of $A[(S(L_C), K); S]$ have at least n internal vertices. Multiplying by the normalizations from the Kirby I-move can only increase this number.

Notice that for any nonempty $s \subset S$ we have $(S(L_C), K)_s = (S^3, K)$. Therefore, we have $A[(S(L_C), K); S] = A((S^3, K)_C) - A(S^3, K)$. The right hand side of this last equation is $\sigma Z(K_C) - \sigma Z(K)$, which has Euler degree $\geq n$, since we argued that the left-hand side of the equation has that property. Notice that the wheeling isomorphism can never decrease Euler degree, since it involves attaching wheels to diagrams. Thus $\partial_\Omega \sigma Z(K_C) - \partial_\Omega \sigma Z(K)$ is of Euler degree $\geq n$.

Finally we must take the logarithm. Write $a = \partial_\Omega \sigma Z(K_C) - 1$ and $b = \partial_\Omega \sigma Z(K) - 1$. We are interested in

$$\log(a + 1) - \log(b + 1) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{a^k - b^k}{k}.$$

Notice that this is divisible by $a - b = (a + 1) - (b + 1)$ which we calculated was of Euler degree $\geq n$. Since the Euler degree adds under disjoint union (i.e. multiplication) it follows that the whole expression is of Euler degree $\geq n$. \square

Definition 3.4. Let $Z_k^s : \mathcal{K} \rightarrow \mathcal{B}_k^s$ be the grope degree k part of

$$\log_{\beta}(\partial_\Omega \sigma)Z = (\partial_\Omega \sigma) \log_{\hat{A}} Z.$$

Corollary 3.5. Z_k^g vanishes on G_{k+1} .

Proof. Let K be a knot and C be a simple clasper of grope degree $(k + 1)$. We need to show that

$$z := \log_{\widehat{\mathcal{B}}}(\partial_\Omega \sigma)Z(K) - \log_{\widehat{\mathcal{B}}}(\partial_\Omega \sigma)Z(K_C) \in \widehat{\mathcal{B}}$$

has no terms of grope degree $\leq k$. Write $k + 1 = v + b_1 = v + \frac{e}{2} + 1$ in terms of the Vassiliev degree and the first Betti number. Then the Euler-degree of C is $2(b_1 - 1)$ implying by the previous proposition that z starts with terms of Euler-degree $2(b_1 - 1)$. Similarly, by the usual properties of the Kontsevich integral, we know that z starts with terms of Vassiliev degree v (this also covers the case $b_1 = 0$). Hence our claim follows. \square

Lemma 3.6. $Z_k^g \circ \Phi_k^g = Id$.

Proof. Similarly to the above proof, let U be the unknot and C be a simple clasper of grope degree k . Now we need to show that the \mathbb{Z} -linear combination

$$z := \log Z(U_C) \in \widehat{\mathcal{B}}$$

of diagrams D_i starts (in the grope filtration) with the diagram D underlying the clasper C . Writing $k = v + b_1 = v(D) + b_1(D)$, we conclude as in the above argument that

$$v(D_i) \geq v \text{ and } b_1(D_i) \geq b_1.$$

This implies as before that $g(D_i) \geq g(D) = k$ but also that the grope degree part of z consists of the linear combination of those D_i for which $v(D_i) = v$ and $b_1(D_i) = b_1$. By the usual universality of the Kontsevich integral and Lemma 2.13, the first property alone shows that exactly one $D_i = D$ with coefficient $+1$. \square

Corollary 3.7. Z_k^g induces an isomorphism

$$\mathcal{G}_k \otimes \mathbb{Q} \cong \mathcal{B}_k^g \otimes \mathbb{Q}.$$

Proof. The fact that the map is well-defined is the content of Corollary 3.5. By Lemma 3.2 the map Φ_k^g is an epimorphism. Now Lemma 3.6 implies that it is injective modulo torsion, and hence a rational isomorphism. Therefore its rational inverse $\log Z_k^g$ is also an isomorphism. \square

This result clearly implies Theorem 1.1 from the introduction.

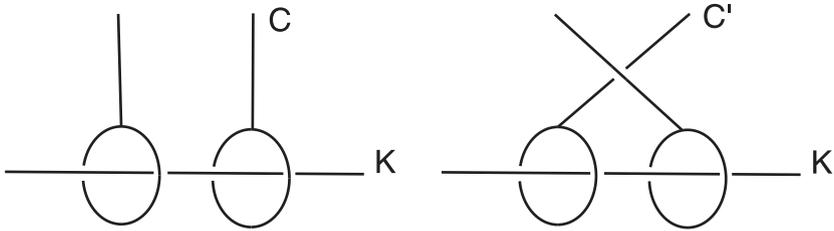


Fig. 7. Interchanging the order of leaves

3.2. 4-dimensional grope cobordism: Grope concordance

Proposition 3.8. *For each $k \geq 3$, two knots are class k grope concordant if and only if their Arf invariants agree.*

Proof. It was shown in [COT] that a grope concordance of class ≥ 3 preserves the Arf invariant. So pick $k \geq 3$ and suppose that K_1 and K_2 have the same Arf invariant. We shall construct a grope concordance of class k as follows:

Using Lemma 4.3 one can perform clasper surgeries on claspers *with loops* until the two knots share Vassiliev invariants up to order k . Hence by [CT, Thm.4], the two knots are concordant to knots K'_1 and K'_2 that share Vassiliev invariants up to order k . By [CT, Thm.1] K'_1 and K'_2 are then related by a (capped) grope cobordism of class k in 3-space. This grope cobordism can be glued to the two concordances to obtain a grope concordance of class k between K_1 and K_2 . \square

3.3. Clasper moves and the grope degree

It is the purpose of this section to prove some lemmas on the behavior of claspers with respect to the grope degree.

Lemma 3.9. *Let C be a rooted tree clasper of degree c on a knot K .*

- Suppose two leaves of C hit K as on the left in Figure 7. Let C' be obtained from C by interchanging the order of the leaves as on the right of Figure 7. Then $K_C = K_{C'} \pmod{G_{c+1}}$.*
- Suppose C' is a rooted clasper obtained from C by homotoping one of the edges. Then $K_C = K_{C'} \pmod{G_{c+1}}$. Indeed, when C is a tree clasper, $K_C = K_{C'} \pmod{T}$, where T is formed from the tree type of C by adding a hair to the edge that is homotoped.*
- Suppose L is a leaf that bounds a disk, and that the leaf has trivial linking number with K . Let C' be the clasper which has these intersections pushed off of the disk. Then $K_C = K_{C'} \pmod{G_{c+1}}$.*

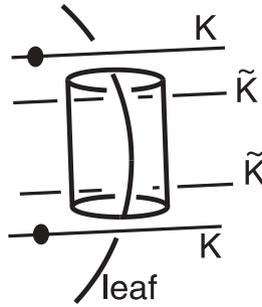


Fig. 8. The proof of 3.9(c)

Proof. [H2, Prop.4.4] proves (a) when the two leaves belong to different simple clasps. The hypothesis of “simple” is not used in the proof. (His Figure 29 is still valid, and the zip construction proceeds without a hitch.) In order to use this fact, insert some Hopf-linked tips into some edges of C , in order to break C into two clasps, each containing exactly one of the leaves to be interchanged. This proves (a).

To prove part (b), let E be a degree 1 clasper with one leaf L linking an edge of C as a meridian, and the other leaf embedded arbitrarily, so that surgery on E realizes the homotopy of the edge. Insert a Hopf-pair in the edge of C . There are two cases: either this disconnects C into two clasps C' and C'' or it doesn't. Now L bounds a disk that hits one of the Hopf-pair in two points. Add a tube to get rid of the intersections. The resulting surface bounding L has a symplectic basis bounding disks each of which hits one of the Hopf-pair in one point. In the disconnected case, in the complement of K_C , these curves therefore bound gropes of the same tree type as C' and C'' respectively. Therefore surgery on E is the same as surgery on a clasper formed by gluing the C^1 and C^2 trees onto the tips of a “Y”. This is exactly the tree type T . In the connected case, just use one of the symplectic basis elements.

To prove part (c), consider two intersections of K with the disk of opposite sign. Let \tilde{K} in $S^3 \setminus C$ be a parallel to K with the two intersection points pushed off of the disk. Now Figure 8 shows a genus one surface cobounding K and \tilde{K} in S^3_C . This cobounding surface has a cap which is pierced by the leaf once. So by Theorem 11 of [CT], K and \tilde{K} cobound a class $c + 1$ grope in S^3_C , which says that K_C and K'_C cobound a class $c + 1$ grope in S^3 . Iterate this procedure until all intersections are removed. □

The following theorem is well-known to both Goussarov and Habiro. Indeed we thank Habiro for mentioning the statement to us in an email. See [G2] for a similar statement. We will give a nice grope proof in [CST].

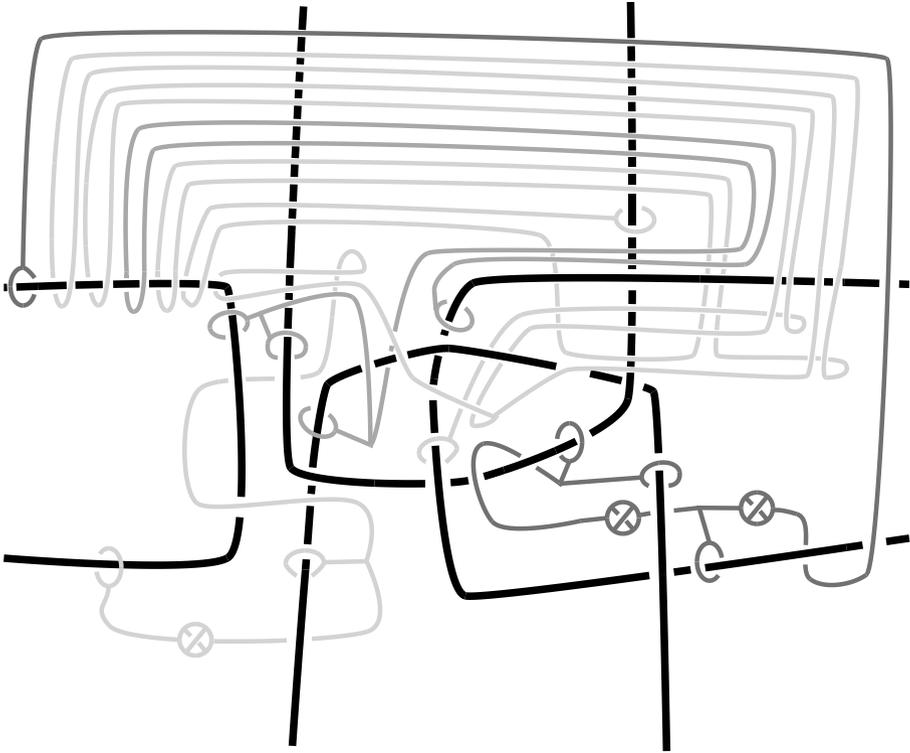


Fig. 9. The topological IHX relation

Theorem 3.10. *Suppose three claspers C_i represent the three terms in an IHX relation. Given an embedding of C_1 into a 3-manifold, there are embeddings of C_2 and C_3 inside a regular neighborhood of C_1 , such that the leaves are parallel copies of the leaves of C_1 , and the edges avoid any caps that C_1 may have. Moreover, surgery on $C_1 \cup C_2 \cup C_3$ is diffeomorphic (rel boundary) to doing no surgery at all.*

Sketch of proof. A picture of the relation for the case of Vassiliev degree 3 trees is given in Figure 3.10. The precise details of the picture are not relevant for our application in this paper, only the existence of such a picture is needed. As mentioned above, this existence is well known to the experts but unfortunately not present in the literature, yet. That's why we explain our own picture which is motivated from certain configurations of Whitney disks in four dimensions, see [CST]. This is not the right place to explain this motivation.

Figure 3.10 gives three claspers in a genus 4 handlebody, where three of the arcs that have been drilled out of a 3-ball have been isotoped into a Borromean rings configuration. (This comes from the way in which the picture was derived in

[CST].) We denote by R the fourth arc (which is on the left of our figure). Three clasps are shown of the form I, H, and X. In this picture notice that there are symbols indicating half-twists. One can verify by direct, if tedious, computation that surgery on these three clasps moves the arc R to an isotopic position. To see this, one applies the “second Morse cancellation” of Figure 15 of [CT] to the leaves of the three clasps which are meridians to R in Figure 3.10. After doing this there are three Y clasps which have one leaf each that links R in two parallel strands. One can then apply Figure 15 of [CT] again, which still works on sets of parallel strands. The result is that R has been moved to a new (complicated) position which is isotopic to its original position.

We should also check that the handlebody is indeed a regular neighborhood of one of the clasps, which may not be obvious at first glance. It is actually a regular neighborhood of the clasper which should be denoted C_1 and which has an edge that appears right-most in Figure 3.10 and has two pictured half-twists. Ignoring the other two clasps, one can slide the arcs around inside the ball so that they are standard and such that we don't mess up the clasper C_1 . The resulting picture demonstrates that the handlebody is a regular neighborhood of C_1 .

Finally, we need to go from the degree 3 picture to the case of arbitrary degree. Given a clasper C_1 of arbitrary degree > 3 . Introduce Hopf pairs of leaves into the clasper C_1 so that C_1 is equivalent to a degree 3 tree clasper, T , union the rest of the clasper which we denote T' . (T' could have up to 4 connected components.) This new tree should be positioned to correspond to the piece of the clasper affected by the IHX relation. Insert the degree 3 picture into a regular neighborhood of T , calling the three degree 3 clasps \bar{C}_1 , \bar{C}_2 , and \bar{C}_3 . In a regular neighborhood of each leaf of T , there are many parallel edges of the \bar{C}_2 and \bar{C}_3 clasps. These edges can be slid over T' using Corollary 33 of [CT]. Now each of the leaves of T' which link with T now link with exactly three leaves each: one from each of the \bar{C}_i . Using the zip construction of [CT] on each component of T' we can find three daughter clasps in a regular neighborhood of each component of T' , such that each of the cut leaves now links with one of the \bar{C}_i . Cancelling the resulting clean Hopf pairs, we get three clasps one of which is the original C_1 , and the other two are the desired C_2 and C_3 . □

Lemma 3.11. *Let U be the unknot.*

- (a) *Suppose that C and C' are two simple clasps of grope degree c on U which differ only by a half twist along any edge. Then $U_C \# U_{C'} \equiv U \pmod{G_{c+1}}$.*
- (b) *G_c / G_{c+1} is generated under connected sum by knots which are simple clasper surgeries of grope degree c on U .*
- (c) *Suppose three clasps C_i of grope degree c on U differ according to the IHX relation, see Figure 1. Then*

$$U_{C_1} \# U_{C_2} \# U_{C_3} \in G_{c+1}.$$

(d) Suppose two claspers C', C'' of grope degree c , on a knot K , differ by a full twist along an edge. Then $K_{C'} = K_{C''} \pmod{G_{c+1}}$.

Proof. Part (a): First, insert Hopf-linked pairs of tips to make C, C' trees. Use Lemma 2.9 to find \tilde{C} , such that $U_{C \cup \tilde{C}} = U$. We need to disentangle \tilde{C} . Those leaves that were meridians to U on C , are still meridians on C' . For every tree clasper D of degree exceeding 1 on a knot K , a meridian to each (non-root) leaf and edge links K_D algebraically trivially. Hence the pushed off Hopf-linked tips on \tilde{C} link U_C algebraically trivially, and so all intersections can be pushed out by Lemma 3.9(c). The pushed off Hopf-linked tips of \tilde{C} , are now Hopf-linked in the same way as C . Denote the new clasper \tilde{C}' . Let B be a ball meeting U_C in a standard unknotted arc away from C . Slide the leaves of \tilde{C}' into B . By Lemma 3.9(b), we can pull \tilde{C}' into B modulo G_{c+1} . Let this new clasper be called C' . We have just demonstrated that

$$U \equiv (U_C)\tilde{C} \equiv U_C \# U_{C'} \pmod{G_{c+1}}$$

and that C' is of the required form.

Part (b): By Theorem 2 of [CT], if $K \in G_c$, then there are knots K_i and simple claspers C_i of grope degree $\geq c$, such that $K_0 = U, K_i = (K_{i-1})_{C_i}, K_N = K$. Modulo G_{c+1} , we can discard all C_i except those of degree c . As in part a, slide the leaves of C_N into a ball on K_{N-1} , and then pull C_N into the ball by Lemma 3.9(b). Call the new clasper C'_N . Then $K_N \equiv K_{N-1} \# U_{C'_N} \pmod{G_{c+1}}$. Inductively, we are done.

Part (c): By Theorem 3.10, we can find three tree claspers differing by the IHX relation inside a regular neighborhood of each other, such that surgery on all three is null-isotopic. As noted previously, we can pull them apart modulo higher grope degree. After pulling them apart Hopf pairs of leaves can be blown down into edges.

Part (d): There is a clasper \tilde{C} that differs from both C' and C'' by a single half-twist. Then the proof of part (a) implies that $K_{C'} = (U_{\tilde{C}})^{-1} \pmod{G_{c+1}}$, and similarly $K_{C''} = (U_{\tilde{C}})^{-1} \pmod{G_{c+1}}$. □

4. Low degree calculations

All the 4-dimensional results in Table 1.2 are contained in [COT], so we work purely in dimension 3. We begin by making a general observation which turns out to be very useful for all our calculations.

Proposition 4.1.

$$K_1 \equiv K_2 \in \mathcal{K}/G_k^{cap} \text{ implies } K_1 \equiv K_2 \in \mathcal{K}/G_k \text{ implies } K_1 \equiv K_2 \in \mathcal{K}/G_{\lfloor \frac{k}{2} \rfloor + 1}^{cap}$$

Remark. The second implication is a slight improvement over [C2], although its proof depends on Theorem 2 of [CT], which in turn depends on [C2].

Proof. The first implication is obvious: a capped grope is also an uncapped grope. The second implication arises as follows. A simple clasper of grope degree k has minimal Vassiliev degree when the first Betti number is maximized. Suppose $k = 2t + 1$. Then the number of leaves of the clasper when the edges are cut to make a tree is $2t + 2$. Note that the grope degree is unchanged under performing such cuts. At most $2t$ of these edges could have been paired together to form t loops. The Vassiliev degree is then $k - t = \lfloor \frac{k}{2} \rfloor + 1$.

If $k = 2t$, then there are $2t + 1$ leaves of the associated tree clasper, $2t$ of which can be paired to make a tree. In that case, we would have a clasper of degree t , with a single leaf hitting the knot. Since the corresponding Feynman diagram is trivial modulo STU, the map Φ indicates that it must be trivial modulo G_{t+1}^{cap} . \square

Moreover, the preceding argument proves that the only type $\lfloor k/2 \rfloor + 1$ invariant values that can be attained by a G_k -trivial knot are those corresponding to linear combinations of connected Feynman diagrams with $\lfloor (k - 1)/2 \rfloor$ loops, using Lemma 2.13.

4.1. The groups \mathcal{K}/G_k for $k \leq 5$

Theorem 4.2. *We have the following calculations:*

$$\begin{aligned} \mathcal{K}/G_1 &= \{0\} & \mathcal{K}/G_2 &= \{0\} \\ \mathcal{K}/G_3 &\cong \mathbb{Z}/2(c_2) & \mathcal{K}/G_4 &\cong \mathbb{Z}(c_2) \\ \mathcal{K}/G_5 &\cong \mathbb{Z}(c_2) \oplus \mathbb{Z}/2(c_3) \end{aligned}$$

Here c_2 and c_3 denote some choice of the degree 2 and 3 Vassiliev invariants. The proof uses the following well-known calculations of the indecomposable elements \mathcal{A}^l , see for instance [G] or [BN]. The last statement is due to [Ng].

Lemma 4.3. $\mathcal{A}_2^l \cong \mathbb{Z}$, with generator  and $\mathcal{A}_3^l \cong \mathbb{Z}$ with generator . In general, \mathcal{A}_k^l is generated by connected diagrams with at least 1 loop for $k \geq 3$.

Definition 4.4. Let $\mathcal{A}_k^l[m]$ denote \mathcal{A}_k^l modulo diagrams with m -loops.

Proof of 4.2. Suppose \mathcal{A}_m^l is torsion free for $m \leq k$. We have seen that the maps $\Phi_m : \mathcal{A}_m^l \rightarrow G_m^{cap}$ from Lemma 2.1 are then isomorphisms for $m \leq k$. Hence

$$\mathcal{K}/G_k^{cap} \cong \bigoplus_{m < k} \mathcal{A}_m^l$$

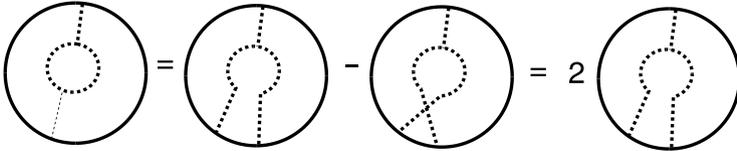
By Proposition 4.1, \mathcal{K}/G_k is a quotient group of \mathcal{K}/G_k^{cap} by simple clasper moves of grope degree k . A simple clasper move represents a diagram in \mathcal{A}_l^l for some l , and by definition the number of loops is $n - l$. Hence, by Lemma 2.13, the degree l part changes according to the corresponding Feynman diagram. Thus we have a relation of the form

$$(0, \dots, 0, \alpha, *, \dots, *) \in \mathcal{A}_2^l \oplus \mathcal{A}_3^l \oplus \dots \oplus \mathcal{A}_l^l \oplus \dots \oplus \mathcal{A}_k^l$$

where α lies in the subspace of \mathcal{A}_l^I generated by diagrams with $n - l$ loops. These constitute all of the relations, but notice that we have no control over the $*$'s, so this fact will only be useful in (very) low degrees.

Now \mathcal{K}/G_2 is a quotient of $\mathcal{K}/G_2^{cap} = \{0\}$, and is therefore trivial. That \mathcal{K}/G_2 is trivial also follows from the statement that all knots cobound a surface with the unknot. For instance, one may take a punctured Seifert surface.

By the above remarks \mathcal{K}/G_3 is a quotient of \mathcal{A}_2^I , by the subspace of diagrams with one loop. The 1-loop subspace is generated by the following diagram, which, as shown, is equal to twice the generator.



Since the Arf invariant is the mod 2 reduction of c_2 , we have proven what we needed to. For the next degree, note that \mathcal{K}/G_4 is a quotient of $\mathcal{A}_2^I \oplus \mathcal{A}_3^I$ by relations of the form:

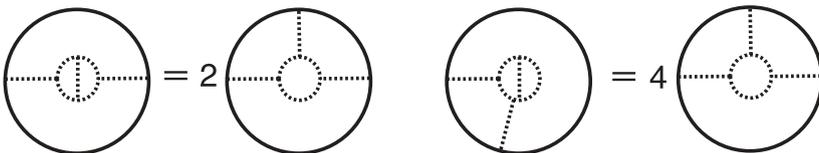
$$(2\text{-loop}, *) \text{ and } (0, 1\text{-loop})$$

By Lemma 4.3, the relations of the second type kill off \mathcal{A}_3^I , and we are left with $\mathcal{A}_2^I[2]$. The 2 loop subspace of \mathcal{A}_2^I is generated by a diagram with one foot on the solid circle, which is trivial modulo STU . Hence $\mathcal{A}_2^I[2] = \mathcal{A}_2^I$.

Now \mathcal{K}/G_5 is a quotient of $\mathcal{A}_2^I \oplus \mathcal{A}_3^I \oplus \mathcal{A}_4^I$, by relations of the form

$$(0, 2\text{-loop}, *) \text{ and } (0, 0, 1\text{-loop})$$

Again, by 4.3 the second type of relation kills off \mathcal{A}_4^I , and we are left with $\mathcal{A}_2^I \oplus \mathcal{A}_3^I[2]$. By [Ng]'s arguments the 2-loop subspace is generated by diagrams of the form  attached to the outer loop by some permutation. For instance we do not need to separately consider the diagrams . Up to sign, there is only one diagram of each of the good types, and they can each be represented as follows.



Since twice the generator is realized, but the generator is not realized, $c_3 \text{ mod } 2$ is all that survives. □

The previous calculations are quite suggestive of the following conjecture.

Conjecture 4.5. *Suppose \mathcal{A}^l is torsion free up to degree k . Then*

$$\mathcal{K}/G_k \cong \mathcal{A}_{k-1}[1] \oplus \mathcal{A}_{k-2}[2] \oplus \mathcal{A}_{k-3}[3] \oplus \cdots \oplus \mathcal{A}_2[k-2].$$

We remark that this is true rationally, without the hypothesis, see Corollary 3.7. We end this section with a generalization of one of our results above to knots in integral homology spheres.

Proposition 4.6. *Two knots in an integral homology sphere have the same Arf invariant iff they are class 3 grope cobordant.*

Proof. Let K be a knot in an integral homology sphere M . It bounds a surface. By Matveev’s result [M], there is a collection of Vassiliev degree 2 claspers which turn M into S^3 . We may assume that the claspers are disjoint from the surface since we can perturb them by isotopy. One can then find inverse claspers in a regular neighborhood, so that we are in the situation of a knot in S^3 together with some claspers disjoint from a Seifert surface, such that surgery on these claspers takes us to the original pair (M, K) . Now by Lemma 3.9(c), K is class 3 cobordant to a knot in $D^3 \subset M$. Now we can use the result for S^3 . □

4.2. Tree types of class 4

In Theorem 4.2, we analyzed the equivalence relation given by grope cobordisms of a fixed class, up to class 5. In general, when one refines these equivalence relations to be of a fixed tree type, one gets a different answer. For instance, S-equivalence is generated by a specific tree type of class 5 (Theorem 4.9). Degree 4 is the first place that the rooted tree type is not unique, but we prove in this section that class 4 cobordism is generated by either of the two rooted tree types:



Proposition 4.7. $\mathcal{K}/\Upsilon \cong \mathcal{K}/\Upsilon' \cong \mathcal{K}/G_4$. *That is, grope cobordism of class four is generated by either of the two rooted tree types.*

Proof. Let the two tree types be called T_1 and T_2 . We have the following commutative diagram.

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & Ker & \longrightarrow & \mathcal{K}/G_4^{cap} & \longrightarrow & \mathcal{K}/G^4 & \longrightarrow & 0 \\
 \uparrow = & & \uparrow_{onto} & & \uparrow = & & \uparrow_{onto} & & \uparrow = \\
 0 & \longrightarrow & Ker_i & \longrightarrow & \mathcal{K}/G_4^{cap} & \longrightarrow & \mathcal{K}/T_i & \longrightarrow & 0
 \end{array}$$

As we saw in the proof of Theorem 4.2, the Ker is generated by a wheel with three legs attached to the outer circle. After cutting this, the root can be chosen so that it is either of the two tree types. That implies the map $Ker_i \rightarrow Ker$ is onto. By the five lemma the map $\mathcal{K}/T_i \rightarrow \mathcal{K}/G_4$ is an isomorphism. □

4.3. Tree types of class 5

In this section we prove that grope cobordism of class 5 is generated by \mathcal{Y} -equivalence, or by \mathcal{Y}^{cap} -equivalence. In [CT], it was proven that “half gropes” generate equivalence by (capped) gropes of a given class. Here a half grope has a tree type representing a right-normed commutator. This implies the first isomorphism in the following lemma.

Lemma 4.8. $\mathcal{K}/\mathcal{Y}^{\text{cap}} \cong \mathcal{K}/G_5 \cong \mathcal{K}/\mathcal{Y}$

Proof. As in the proof of Lemma 4.7, it suffices to show that the kernel of $\mathcal{K}/G_5^{\text{cap}} \rightarrow \mathcal{K}/G_5$ is generated by both of the tree types. This kernel is generated by a circle with four legs, something that can be thought of as either tree type, as well as twice the generator corresponding to c_3 . This is realized by a theta with two legs which can be cut apart to be either of the two tree types. \square

4.4. S -equivalence

Let \mathcal{S}_5 denote the following tree type:



This is the simplest tree with an internal vertex.

Theorem 4.9. *Two knots are S -equivalent iff they are \mathcal{S}_5 -equivalent.*

Remark 4.10. In [COT] it is proven that the corresponding move in 4 dimensions gives Blanchfield forms up to cobordism. Thus the kernel of going from 3 to 4 dimensions consists of adding the relation $K + K^! = 0$ where $K^!$ is the mirror image and is an inverse in the knot concordance group.

Proof. That \mathcal{S}_5 preserves S -equivalence follows by a construction of Murakami and Ohtsuki [MO, p.6], applied to a disk leaf of the \mathcal{S}_5 move. For the converse we use a result of Naik and Stanford [NS] (see also Murakami and Nakanishi [MN]), that the *doubled delta move* generates S -equivalence. In fact, [NS] proves the stronger result that the doubled delta move applied to bands of some Seifert surface generates S -equivalence. If this move is applied to three bands, no two of which are dual, it is easy to construct an \mathcal{S}_5 grope cobounding the knots before and after the doubled delta move. We construct this grope by constructing disjoint surfaces bounded by the three leaves of the doubled delta move. These surfaces are constructed by tubing into the dual band. See Figure 11 If two or three leaves link the same band, we can construct the surfaces by nesting the tubes.

Now that we have these 3 surfaces, we get an \mathcal{S}_5 grope by the discussion of grope-clasper duality given in [CT]. A meridian of an innermost tube provides the root. We are left with the following two cases:

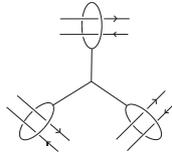


Fig. 10. Stanford and Naik’s doubled delta move

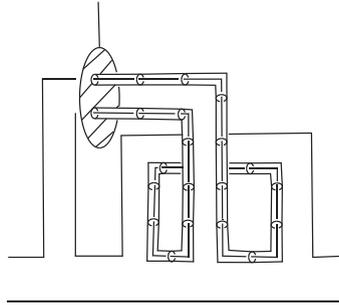


Fig. 11. Tubing into the dual band

- (a) Two leaves link one band, and the other leaf links the dual band.
- (b) Two leaves link dual bands, and the third leaf links some other band.

Note that in both cases we can construct disjoint surfaces on two of the leaves. The argument that S -equivalence is generated by the S_5 grope is given by diagram 12. We now give the explanation of the figure. First, the two leaves which bound surfaces are the leaves at the bottom of frame 1. When constructing these surfaces we can assume that they run parallel to only one of the strands going through the top leaf. Let this strand be the top strand of the picture. Going to frame 2 is Habiro’s move 8. Going to frame 3 follows easily from Habiro’s move 11. In frame 4 we have added the two surfaces to the tips of one of the clasps. Rather than drawing the details, we have represented things schematically. The heavy line at the top of the picture reminds us that pieces of the new clasper run through here. In frame 5 we have slid one of the leaves along the knot until it is opposite the other leaf. Because of the fact that the strands were oriented oppositely, the leaves face each other as shown. Going to frame 6 is where we use the S_5 move. We pull the edge of the clasper which is incident to the leaf we just slid into the shown position. Then by Lemma 3.9(b), the knots in pictures 5 and 6 cobound an S_5 grope. To get to frame 7 we forget the fact that there are surfaces bounding two of the leaves. (Basically the inverse of $3 \rightarrow 4$.) In frame 8 we do Habiro’s move 11 in reverse. But then we have a clasper with trivial leaf, which does not alter the knot as in frame 9. □

Note that Corollary 1.3 follows immediately, because by Theorem 4.2, $c_3 \pmod 2$ is an invariant of all class 5 grope cobordisms.

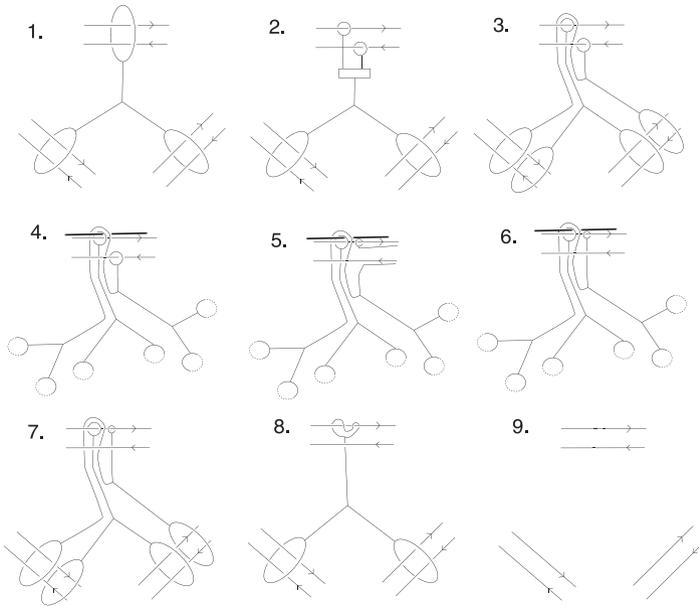


Fig. 12. The proof that the S_5 grope generates S -equivalence

This corollary is somewhat surprising. [MO] have proven that the only rational finite type S -equivalence invariants are the coefficients of the Alexander-Conway polynomial. Recall that $c_3 \pmod 2$ is not an Alexander-Conway coefficient. Recently, Ted Stanford [S] discovered that $c_3 \pmod 2$ can be expressed as a polynomial in the Conway coefficients c_2 and c_4 . Hence all known finite type invariants of S -equivalence come from the Alexander polynomial.

We have classified the behavior of all capped and uncapped class 5 trees but the following.

Problem 4.11. *What is $\mathcal{K}/(\mathcal{S}_5)^{cap}$?*

Clearly a $(\mathcal{S}_5)^{cap}$ move must preserve S -equivalence and also type 4 Vassiliev invariants. Conjecturally this completely characterizes the move.

We close this section with a natural conjecture based on our low-degree calculations.

Conjecture 4.12. *\mathcal{K}/T only depends on the unrooted tree type of T .*

5. Miscellaneous results

5.1. Null filtration of knots which bound a grope

In the first author’s Ph.D. thesis [C] (see also [C2]), the question of a knot *bounding* a grope is considered. This is much stronger than cobounding a grope with

the unknot. In particular, if a knot bounds a grope of class at least three, the knot has trivial Alexander polynomial, whereas this is certainly not the case for the cobounding situation. The central result in [C] is the following:

Theorem 5.1. *If a knot K bounds an embedded grope of class k into S^3 , then Vassiliev invariants up to degree $\lceil k/2 \rceil$ vanish on K .*

In fact the bounding of a grope is an extremely restrictive condition. This can be very well expressed in terms of the *null filtration* of [GR]. It is obtained by the usual alternating sum formalism by declaring a null clasper surgery on a Y to have degree one. Here the word “null” expresses the condition that the leaves of the clasper must have trivial linking numbers with the knot. It follows that to be null equivalent to the unknot (the case $k = 3$ below) is the same as having trivial Alexander polynomial, at least in a homology sphere.

Proposition 5.2. *If a knot K bounds an embedded grope of class k in a 3-manifold M , then the pair (M, K) is $(k - 3)$ -null equivalent to the unknot in M .*

Proof. A knot bounding a grope of class k can be obtained from the unknot U by surgery on a rooted tree clasper of Vassiliev degree k , where U is a meridian to the root. In particular, the other leaves do not link U .

Break the clasper into a union of $(k - 1)$ Y 's. Surger the knot along the Y which contains the root. This leads to a union of $(k - 2)$ Y 's, which the surgered knot links trivially. Now consider the alternating sum, surgering over all subsets of the Y 's. It is easy to see that surgery on any proper subset does not change the knot, because there will be leaves which bound embedded disks. Hence the alternating sum reduces to $(M, U) - (M, K)$ and since we did $(k - 2)$ Y -surgeries, these two knots are $(k - 3)$ -null equivalent. \square

5.2. Grope cobordism and orientation reversal

Proposition 5.3. *Let ρ be the map reversing a knot's orientation. Then for every knot K in the k -th term G_k of the grope filtration of \mathcal{K} , one has*

$$K \equiv (-1)^k \rho(K) \pmod{G_{k+1}}.$$

One can filter the primitive Feynman diagrams by grope degree, and in the associated graded group, it is straightforward to show that $D = (-1)^{|D|} \rho(D)$. Conjecturally the graded pieces of this group are isomorphic to \mathcal{G}_k , in which case we'd be done. However, we can still mimic the Feynman diagram computation geometrically.

Proof of Proposition 5.3. By Lemma 3.11(b) and Lemma 3.9(b), it suffices to show the result for $K = U_C$, where C is a simple clasper of degree k which hits the unknot U as on the left of Figure 13. Here B is a ball which contains most of

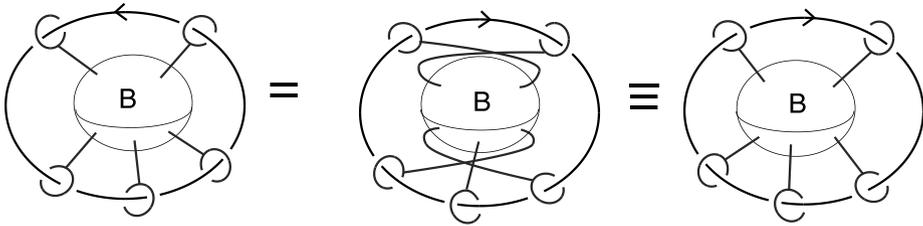


Fig. 13. The proof of Proposition 5.3

the clasper, excluding the leaves. One rotates the outer unknot about the vertical axis lying in the page, while keeping the ball B fixed. By Lemma 3.9, we can reorder the leaves as on the right of the picture. Each edge incident to a leaf has picked up a half twist. Note that the parity of the number of leaves of a simple clasper matches the parity of the grope degree k . By Lemma 3.11(a), we have shown that $U_C = (-1)^k \rho(U_C) \in G_k/G_{k+1}$. \square

5.3. Simple clasper surgeries and the grope degree

In this section, we restrict attention to knots and claspers in 3-space. Consider the statement,

“A simple clasper surgery of degree k may be realized by a sequence of simple clasper surgeries of degree $(k - 1)$.”

When the degree k is the Vassiliev degree, this follows from Habiro’s work: By his main theorem any simple clasper surgery of Vassiliev degree k may be realized by a sequence of simple *tree* clasper surgeries of Vassiliev degree k . For a simple tree claspers, one breaks the clasper into a union of a degree $(k - 1)$ tree clasper and a Y , with two leaves linking as a Hopf pair (exactly one of the leaves belongs to the Y). Surgering the knot along the Y , we get a capped tree clasper of Vassiliev degree $(k - 1)$, which can then be refined via the zip construction into a sequence of simple tree clasper surgeries of Vassiliev degree $(k - 1)$.

It is the purpose of this section to demonstrate that the above statement holds also for the grope degree.

Theorem 5.4. *A simple clasper surgery of grope degree k may be realized by a sequence of simple clasper surgeries of grope degree $(k - 1)$.*

We make the following preliminary definition for the purpose of this section.

Definition 5.5. *A clasper is said to be admissible if it is simple and the graph type is obtained from a connected trivalent graph with no separating edges by adding a positive number of legs to the edges. We also call the trivalent graph admissible.*

Let \mathcal{S} denote the set of finite sequences of connected uni-trivalent graphs with at least one univalent vertex. Consider the partial order generated by the relation

$$(G_1, \dots, G_i, \dots, G_N) < (G_1, \dots, G_i^1, G_i^2, G_{i+1}, \dots, G_N),$$

where G_i^j are the two other terms in an IHX relation involving G_i . This partial order gives a convenient language to state the following lemma.

Lemma 5.6. *For all $s \in \mathcal{S}$, there is an $s_0 \in \mathcal{S}$ such that $s < s_0$ and $s_0 = (G_i)$ is a sequence of admissible graphs G_i .*

Proof. Straightforward. □

The point is that these elementary relations can be realized topologically by embedded claspers, as the next proposition makes clear.

Proposition 5.7. *Let G_1, G_2, G_3 be uni-trivalent graphs related by an IHX relation, and let C_1 be a simple clasper of type G_1 , embedded in the complement of a knot K . Then C_1 may be realized by a sequence of claspers one of which is of type G_2 , one of which is of type G_3 , and the rest of which have increased grope degree and (at least) the same number of simple caps.*

Proof. First, we must convert some edges of C_1 to Hopf pairs to obtain a tree clasper \hat{C}_1 of tree type \hat{G}_1 . This gives rise to induced tree types \hat{G}_2, \hat{G}_3 . There is an inverse \bar{C}_1 to the clasper \hat{C}_1 inside a regular neighborhood of \hat{C}_1 . The leaves of the inverse are parallels of the original, but the edges may wander around the regular neighborhood in a complicated way. The edges do however avoid any caps that \hat{C}_1 may have. (See Lemma 2.9.) If K_1 is the knot after surgery on C_1 (equivalently on \hat{C}_1) then \bar{C}_1 sits on the knot K_1 , and surgery on it produces the original knot. The knot K_1 will wander through the Hopf pairs of \bar{C}_1 , but will link these leaves trivially. By Lemma 3.9(c), the surgery \bar{C}_1 can be realized modulo higher grope degree, by surgery along a clasper \tilde{C}_1 obtained from \bar{C}_1 by pushing the knot out of the Hopf pairs.

Now we use the topological IHX relation, Theorem 3.10. There is a union of two tree claspers \hat{C}_2 and \hat{C}_3 in a regular neighborhood of \tilde{C}_1 which are of type \hat{G}_2 and \hat{G}_3 respectively, such that surgery on $\tilde{C}_1 \cup \hat{C}_2 \cup \hat{C}_3$ is null isotopic. The leaves of \hat{C}_2 and \hat{C}_3 are parallels of the corresponding leaves of \tilde{C}_1 . The edges of \hat{C}_2 and \hat{C}_3 may run through the regular neighborhood of C_1 , but avoid any caps that C_1 may have.

Where there were Hopf pairs of leaves on C_1 , all the leaves of the three claspers link. However, these leaves of \hat{C}_3 have trivial linking number with $(K_1)_{\tilde{C}_1 \cup \hat{C}_2}$. Thus, by Lemma 3.9(c), $K_1 = ((K_1)_{\tilde{C}_1 \cup \hat{C}_2})_{\hat{C}_3}$ is equivalent modulo higher grope degree to $((K_1)_{\tilde{C}_1 \cup \hat{C}_2})_{\tilde{C}_3}$ where \tilde{C}_3 is a clasper obtained from \hat{C}_3 by pushing strands of the knot out of the Hopf pairs of leaves. Similarly, modulo higher grope degree \hat{C}_2 can be realized by a clasper \tilde{C}_2 , where the knot has been pushed out of the

Hopf pairs. Thus K_1 is equivalent modulo higher grope degree to $((((K_1)_{\tilde{C}_1})_{\tilde{C}_2})_{\tilde{C}_3})$. However $(K_1)_{\tilde{C}_1}$ is equivalent modulo higher grope degree to K . Hence $(K_{\tilde{C}_2})_{\tilde{C}_3}$ is equivalent modulo higher grope degree to K_1 . The claspers \tilde{C}_2, \tilde{C}_3 have clean Hopf pairs, and when these are converted to edges, the resulting simple claspers have graph type G_2 and G_3 respectively. \square

Recall that a cap is an embedded disk bounding a leaf of a clasper C , with interior disjoint from C . If there are several caps, they are assumed to be embedded *disjointly*. A cap is simple (with respect to a knot K) if it has a single intersection with K .

Definitions 5.8. *Let C be a clasper having some caps.*

- (a) *Let $c_1(C)$ be the number of simple caps.*
- (b) *Let $g(C)$ be the grope degree of the clasper C . After breaking some edges into Hopf pairs of leaves to make C a tree clasper, $g(C)$ is the number of leaves minus one.*

Lemma 5.9. *Consider a surgery on a clasper C .*

- (a) *It may be realized by a sequence of surgeries on claspers C_i which are of the following two possible forms.*
 - (i) $g(C_i) = g(C)$, C_i is admissible, $c_1(C_i) \geq c_1(C)$
 - (ii) $g(C_i) = g(C) + 1$, $c_1(C_i) \geq c_1(C)$.
- (b) *If C is admissible then the clasper surgery may be realized by a sequence of surgeries on claspers C_i of the following two possible forms.*
 - (i) $g(C_i) = g(C) - 1$, $c_1(C_i) > c_1(C)$
 - (ii) $g(C_i) = g(C)$, $c_1(C_i) > c_1(C)$.

Proof. Part (a): By Theorem 20 of [CT], we may realize C by a sequence of *simple* clasper surgeries of the same grope degree and at least the same number of simple caps together with some surgeries of higher grope degree which have at least the same number of simple caps. (One must check that [CT] Lemma 19 does not decrease the number of simple caps. Some caps may be destroyed when adding the nested tubes at one stage of the proof. However, instead of adding the nested tubes, push the disk over the cap as we did for the root leaf.)

Hence it suffices to prove (a) for simple claspers. For a simple clasper C , it is straightforward to show that there is a sequence of admissible graph types s_0 such that $s_0 \geq (G)$, where G is the graph type of C . Then we have $(G) \leq s_1 \leq s_2 \leq \dots \leq s_n \leq s_0$, where each sequence is related to the next by a replacement of a single graph by a pair related by IHX.

Proposition 5.7 implies that an elementary relation $s_i \leq s_{i+1}$ can be realized geometrically modulo higher grope degree. More precisely, if s_i represents a sequence of clasper surgeries between a knot K_0 and K_1 , then there is a sequence of clasper surgeries of the form s_{i+1} which also go between K_0 and K_1 . Therefore, we have

that C can be implemented, modulo higher degree, by a sequence of claspers of the form s_0 , which are admissible. This concludes the proof of part (a).

Part (b): Since the clasper is admissible, every univalent vertex is part of a Y , such that the two other vertices of the Y are trivalent, and there is a path connecting them which doesn't hit the Y 's interior. Surger the knot along the Y . This gives a connected clasper which has two new capped leaves having two intersections with the knot each. Refining this clasper using Theorem 20 of [CT], we will get some claspers of higher grope degree and some simple claspers where some number of edges have been cut. All of the latter type have increased c_1 , since we took a single simple cap of the original and converted it to two simple caps in all the daughters. Cutting of edges will add even more simple caps. \square

Lemma 5.10. (a) *A clasper surgery of grope degree $(2k - 1)$ is realizable by a sequence of simple tree clasper surgeries of grope degree $(k - 1)$.*
 (b) *A clasper surgery which has $k + 1$ caps is realizable by a sequence of simple tree clasper surgeries of grope degree $(k - 1)$.*

Proof. The proofs of both of these facts use the main theorem of Habiro [H2], which is that if two knots have the same degree $(k - 1)$ invariants, then they are related by a sequence of simple tree clasper surgeries of Vassiliev = grope degree k . The fact that grope degree $(2k - 1)$ clasper surgeries preserve type $k - 1$ invariants is Theorem 3 of [C2]. For part (b), if a rooted clasper C has $k + 1$ caps, it must have k non-root caps. Then there are k groups of crossing changes on K_C which correspond to pushing the knot out of the caps. Since surgery on a clasper which has a cap that does not intersect the knot is trivial, doing any collection of these k groups of crossings will yield the knot K . Hence K and K_C are $k - 1$ -equivalent. \square

Proof of Theorem 5.4. Define a complexity function, ordered lexicographically as the triplet (c_1, g, a) where c_1 is the number of simple capped leaves, g is the grope degree, and $a(G)$ is zero unless G is admissible, in which case it is one. If $g \geq 2k - 1$ or $c_1 \geq k + 1$ then by the previous lemma the surgery is realizable by simple grope degree $k - 1$ surgeries and we are done.

We prove the following statement by contradiction: "Every clasper surgery of grope degree $\geq k - 1$ is realizable by a sequence of simple clasper surgeries of grope degree $(k - 1)$."

Assume C is a counterexample. Then, as noted above, C lies inside the range $c_1 \leq k$ and $k - 1 \leq g \leq 2k - 2$. Hence it makes sense to take C to have maximal complexity (c_1, g, a) . There are two cases: either $g(C) = k - 1$ or $g(C) \geq k$. In the former case, C cannot be admissible, since it would then be simple and hence not a counterexample. Therefore, $a = 0$, in which case Lemma 5.9(a) says that C can be realized by claspers with higher complexity. Since C is a counterexample, one of these daughters must also be a counterexample. But this contradicts that C was of maximal complexity.

So we are left with the case that $g(C) \geq k$. If $a = 0$, then as above Lemma 5.9(a) furnishes a contradiction. Otherwise, $a = 1$ and so Lemma 5.9(b) says that C can be realized by claspers of degree $\geq k - 1$, and of higher complexity. As before, this contradicts the maximality of C . \square

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