Appendix - Clarification of Linear Grope Height Raising

M. Freedman and P. Teichner

Slava Krushkal and Frank Quinn recently brought to our attention misstatements in the proof of our linear grope height raising procedure which we published in 1995 [FT]. This appendix replaces pages 518-522 of that paper with a proof along the same lines but with correct details. The main difference is that we are more careful in which order we add surface stages. This resolves in particular the problem of how to deal with intersections that involve a dual pair of circles on a surface stage: Even though the "key point" in the middle of page 521 is not true as stated (the Borromean rings are not slice after all), the intersections that arise can be dealt with by picking an order and correspondingly decreasing the scale of the relevant lollipops.

We also reformulate the final word length count in terms of coarse geometry, mainly for clarity but also for possible future use.

Since the "warm up" and "warm down" parts of the proof of Theorem 2.1 in [FT] are correct, it suffices to explain the core construction and show that the word length grows linearly. More precisely, we prove the asserted estimate for the word length

$$(*) \qquad \ell(g^{\bullet}_{k+r}) \leq 2r+1$$

in terms of the double point loops of G_k. In the last paragraph on page 522 this assertion is correctly used to finish the proof of Theorem 2.1. We now begin the revision on the top of page 518:

As we start the core construction we have a Capped Grope $G^c := G^c_k$ of height $k \geq 3$. The inductive set up is a Grope G_{h-1} of height $h-1 \geq k$ and an embedding $(G_{h-1}, \gamma) \hookrightarrow (G^c, \gamma)$. One works with the spines, proceeding from g_{h-1} to g_h by adding a finite number of connected surfaces $\Sigma(t)$ to g_{h-1}. To underline the importance of the order in which the surfaces $\Sigma(t)$ are attached, we write

$$g_{h-1} =: g(0) \subset g(1) \subset g(2) \subset \cdots \subset g(n) = g_h$$

where $g(t) := g(t-1) \cup \Sigma(t)$. Even though technically the $g(t)$ are not gropes (since they have heights in between $h-1$ and h), we will still consider them as such. In particular, each $g(t)$ will be thickened to a "Grope" $G(t)$. The surfaces $\Sigma(t)$ are obtained in two steps:

- Step 1 finds surfaces $\Sigma'(t)$ which have (illegal) self-intersections and intersections with grope stages at various heights, but only above

$$Y := \text{base stage} \cup \text{second stage surfaces } \Sigma_1 \cup \{\Sigma_2\} \text{ of } G.$$

  The subspace Y is protected in the construction so that the dual spheres $\{S\}$ will remain geometrically dual to $\{\Sigma_2\}$, the second stages of G, and disjoint from everything else.
- Step 2 only changes the surface $\Sigma'(t)$ to $\Sigma(t)$, removing double points with itself and with earlier stages (and in the process increases the genus of the surface).

Every application of Step 1 involves choosing some obvious surface (often a disk) so, formally, the presence of these obvious surfaces is an inductive hypothesis which must be propagated in passing from g_{h-1} to g_h. The surfaces $\Sigma'(t)$ for Step 1 are of three types:

1. "parallel" copies of the initial caps $g^c \smallsetminus g$,
2. meridional disks to some surface stages of $g(t-1)$, and
3. "parallel" copies of stages of the original Grope G.

Every application of Step 2 is accomplished by a finite number of moves called a *lollipop move* or a *double lollipop move*. The Step 2 algorithm removes all self-intersections and intersections of $\Sigma'(t)$ (in a particular order) to produce the surface $\Sigma(t)$. The caps $g_h^c \smallsetminus g_h$, necessary to define $\ell(g_h)$, are constructed last and in two steps. The preliminary caps cross all grope stages above Y (stages ≥ 3); these are refined to caps disjoint from the grope using the dual spheres $\{S\}$.

We next explain the central move in our grope height raising procedure. Every surface stage Σ in the Grope $G(t-1)$ has a symplectic basis of circles $\alpha_1, \beta_1, ..\alpha_g, \beta_g$ where g is the genus of Σ, along which higher surface stages or caps have been attached. We consider tori $T_{\alpha_i}, i = 1, \ldots, g$ which are ϵ normal circle bundles to Σ in $G(t-1)$ restricted to α_i where ϵ is a small positive number depending on Σ. Notice that all these tori are disjoint. Suppose x is a double point with local sheets $S \subset \Sigma'(t)$ and $S_\beta \subset \Sigma_\beta$, and that the surface stage or cap Σ_β is attached to Σ along β. Symmetrically, if the surface $\Sigma'(t)$ intersects Σ_α then interchange α and β in the next paragraphs.

The *lollipop move* replaces a disk neighborhood S of x with a slightly displaced copy of T_α, made by taking normal ϵ-bundles over a parallel displacement (depending on x) of α in Σ, boundary connected summed to S along a tube which is the normal $\epsilon/10$-bundle of Σ_β in $G(t-1)$ restricted to an arc $\lambda \subset \Sigma_\beta$ from $(T_{\alpha(\text{displaced})}) \cap \Sigma_\beta$ to x. Denote the lollipop by L_α. It is the punctured torus made by attaching the tube (or *stem*) to $T_{\alpha(\text{displaced})}$, see Fig. 2.1 in [FT].

We are now ready to describe the core construction in detail. Let $h-1=k$. The very first application of Step 1 simply attaches one cap of g^c to g. When regarded as a grope stage the self-intersections in the cap are impermissible and thus the cap only gives $\Sigma'(1)$.

We specify that the initial application of Step 2 removes (in some order) all intersections of $\Sigma'(1)$ using lollipop moves. This gives $\Sigma(1)$ and hence $g(1)$. To obtain $\Sigma(2)$ one just repeats Step 1 and Step 2 by starting with the next cap. Note that now the self-intersections of the second cap as well as the intersections with the first cap have to be removed (in some order) by lollipop moves. In the same manner, one constructs all surfaces $\Sigma(t)$ and hence the grope g_{k+1}. Here the scale ϵ of the lollipops is getting rapidly smaller so that they do not intersect the previously constructed surface stages. This is where the order of things is relevant.

In subsequent applications of Step 1 we must specify which surfaces we choose and what the intersections are. Each L_α contains a meridional circle to which we attach the meridional disk (type (2) above) and a longitude ℓ_α (picked out by the standard framing used to thicken g to G) to which we attach a "parallel" copy of the surface stage (type (3)) or cap (type (1)) Σ_α. This surface or cap is only crudely parallel in the sense that we need to glue an annulus A to get from the longitude ℓ_α to $\partial\Sigma_{\alpha(\text{displaced})}$, the attaching circle of a slightly displaced copy of one of the surfaces or caps of G^c. The surface $\Sigma'(t)$ is then defined to be $A\cup\Sigma_{\alpha(\text{displaced})}$. The framing assumption of G implies that for type (3) the surface stage $\Sigma_{\alpha(\text{displaced})}$ will be disjoint from everything constructed previously, i.e. from $g(t-1)$. However, for both types (1) and (3), the annulus A may intersect many $\Sigma(s), s<t$, so that $\Sigma'(t)$ has many intersections with $g(t-1)$. For type (2), $\Sigma'(t)$ is a meridional disk and it will intersect $g(t-1)$ in a single point.

The reader may expect that the next application of Step 2 will use lollipop moves on $\Sigma'(t)$ to remove these intersection points. This is part of the picture, but there is a difficulty. The lollipop moves, if repeated, produce a branch heading inexorably down G: namely resolving (meridian disk) $\cap\Sigma_i$ with a lollipop capped by a (meridian disk) meeting a Σ_{i-1} lead toward the base of G which is Σ_1. There is no way of using a lollipop to remove a point of (meridian disk) $\cap\Sigma_1$. The solution is to use the *double lollipop move* to resolve any intersection of a current top stage meridional disk with a third stage surface Σ_3. This move turns the branch of the growing grope back "upward" to avoid the bottom part Y.

The *double lollipop move* removes an intersection x between a surface $\Sigma'(t)$ and a third story surface Σ_3. This move replaces a small disk neighborhood $S\subset\Sigma'(t)$ of x with L_α/Σ_α. The notation assumes Σ_3 attaches to β (otherwise reverse the labels α and β), L_α is the lollipop made from T_α as describe above, Σ_α is the third story surface attached to α and finally L_α/Σ_α denotes the embedded surface that results by surgering L_α along a parallel copy $\Sigma_{\alpha(\text{displaced})}$ of Σ_α, i.e. $L_\alpha/\Sigma_\alpha = (L_\alpha \smallsetminus \text{nbh. of } \alpha(\text{displaced})) \cup$ two copies of $\Sigma_{\alpha(\text{displaced})}$. Because we have assumed G^c is an untwisted thickening the two copies of $\Sigma_{\alpha(\text{displaced})}$ are disjoint from each other and from the original Σ_α.

Now suppose that we have constructed the grope g_{h-1}. Then the top layer of surfaces has a natural symplectic basis coming from the original grope g and the (meridian, longitude) pair on each lollipop. These bound obvious surfaces $\Sigma'(t)$ of types (1)-(3) as explained above. Applying Step 2 to these surfaces in some chosen order, we remove intersection points by a lollipop move except in the case of intersection with a third stage surface Σ_3 in which case a double lollipop is used. This gives the embedded surfaces $\Sigma(t)$ and hence an embedded grope $(g_h, \gamma) \hookrightarrow (G^c, \gamma)$.

We next check the normal framing. If we assume that each cap has algebraically zero many self-intersections then all surfaces $\Sigma'(t)$ are 0-framed. A lollipop move on a $\pm$-self-intersection changes the relative Euler class by ± 2 (this is best checked in the closed case, $S^2 \times S^2$, where adding the framed dual $0 \times S^2$ to $S^2 \times 0$ gives the diagonal). All other lollipop moves leave the 0-framing unchanged. Thus the passage to $\Sigma(t)$ leaves the relative Euler class trivial so the neighborhood of $g(t)$ agrees with the standard thickening $G(t)$.

To obtain caps $\{\delta\}$ for g_h, we examine the symplectic basis for the top stage of g_h. Some of the curves bound meridian disks to earlier stages of the construction. Some bound "parallel" copies of sub capped gropes of G^c. Contracting, the latter also yield disks. We set $h = k + r$ and

$$g^\bullet_{k+r} := g_{k+r} \cup \{\delta\}$$

The superscript $\bullet$ warns the reader that $g^\bullet_{k+r}$ does not satisfy the definition of a capped grope owing to the cap-grope intersections. These will be removed in the last step, see the last paragraph of page 522 in [FT].

Let us next bound the word length $\ell(g^\bullet_{k+r})$ in terms of the original generators (= double point loops) of the free group $F := \pi_1 G^c$. Recall that we need to prove

$$(*) \qquad \ell(g^\bullet_{k+r}) \leq 2r + 1.$$

For this purpose, we put a *pseudo metric* on the universal covering X of G^c. This is a distance function which still satisfies the triangle inequality but distinct points may have distance zero. Note that pseudo-metrics can be pulled back by arbitrary maps which we will use in the construction as follows. First project X onto the Cayley graph of F such that lifts of the Grope body G map bijectively onto the vertices and lifts of the plumbed squares in the Caps map bijectively onto the centers of the edges. Then take a coarse or pseudo version of the usual path metric on the Cayley graph (in which all edges have length 1) by saying that edge centers have distance 1/2 from all the vertices the edge meets and that all path components of the Cayley graph minus the edge centers have diameter zero. Finally, use the above map to pull this pseudo metric back to X.

For any map $f : Y \to G^c$ which is trivial on π_1, we may then measure the *diameter* of a lift $\tilde{f}(Y)$ in X. For example, if Y is a model capped grope (i.e. with unplumbed caps) such that $f(Y) = g^{\bullet}_{k+r}$ then the diameter of $\tilde{f}(Y)$ is just the word length $\ell(g^{\bullet}_{k+r})$.

If Y happens to be a disk, surface or (capped) grope such that ∂Y maps to G, it is very useful to consider the *radius* of $\tilde{f}(Y)$ around the "point" $\tilde{f}(\partial Y)$. This uses the fact that each lift of G projects onto a vertex in the Cayley graph of F and thus has radius zero itself. For example, if Y is a disk mapping onto a cap of G^c which has one self-intersection, then the radius of $\tilde{f}(Y)$ is 1/2 whereas the diameter is 1.

Let X_r be a lift of g_{k+r} to X and let $X^c_r := \tilde{f}(Y)$ where $f(Y) = g^{\bullet}_{k+r}$ as above. Then the triangle inequality shows that $radius(X^c_r) \leq radius(X_r) + 1/2$ and hence

$$\ell(g^{\bullet}_{k+r}) = diam(X^c_r) \leq 2 \cdot radius(X^c_r) \leq 2 \cdot radius(X_r) + 1.$$

It thus suffices to check that $radius(X_r) \leq r$. This in turn follows by the triangle inequality (applied to the usual tree structure of the grope g_{k+r}) from knowing that the radii of all $S(t)$ are ≤ 1. Here $S(t)$ are lifts to X of the surfaces $\Sigma(t)$ used in the construction of g_{k+r} and the radii are again measured w.r.t. $\partial S(t)$.

We prove that radius $S(t) \leq 1$ by induction on t: Recall that the first surface $\Sigma(1)$ was obtained by applying lollipop moves to the first cap of G^c. Before the lollipop moves, we can lift the (unplumbed) cap to X and as explained above it has radius 1/2 (if the cap is embedded then the radius is zero but we won't consider this easy case). The lollipops then increase this radius to at most 1, independently of how many are used. This follows from the triangle inequality applied to the decomposition of each lollipop into its stem and body (or toral piece). The body has diameter zero since it lies in G whose lift projects to a vertex. The stem has by definition diameter 1/2 since it leads from a plumbed square to the base of the cap.

Now assume by induction that radius $S(s) \leq 1$ for all $s < t$. Let $S'(t)$ be a lift to X of $\Sigma'(t)$. If $\Sigma'(t)$ is of type (2) or (3) then the radius of $S'(t)$ is zero since it lies in a lift of G. For every intersection point of $\Sigma'(t)$ with $g(t-1)$ we add a lollipop or a double lollipop to obtain $\Sigma(t)$. Only the stems of these (double) lollipops will contribute to the radius of $S(t)$ since the bodies lie in G. The induction hypothesis implies that all these stems have diameter ≤ 1 and thus we are done in this case.

Finally, consider the case where $\Sigma'(t)$ has type (1), i.e. is a "parallel" cap. Then its radius is $1/2$ as explained above. For every self-intersection of $\Sigma'(t)$ and every intersection point of $\Sigma'(t)$ with $g(t-1)$ we add a lollipop to obtain $\Sigma(t)$ (note that double lollipops don't occur for caps). Again, only the stems of these lollipops will contribute to the radius of $S(t)$. There are two types of lollipops: One type removes self-intersections and intersections with surface stages of $g(t-1)$ that come from the caps of g^c. As for $\Sigma(1)$ the corresponding lollipop stems have diameter $1/2$ and thus can only increase the radius to 1. The other type of lollipops remove intersections of the annulus $A =$ (collar of $\partial\Sigma'(t)$). This means that, as far as our pseudo metric can measure, the stems of the lollipops start essentially on $\partial\Sigma'(t)$ which is the base point with respect to which we measure the radius. By the induction hypothesis these stems can only bring the radius up to 1. □

Note added in proof: Slava Krushkal has pointed out that in the above proof, the "warm-up" and "warm-down" steps can be replaced by the following easier and shorter argument:

Do the core construction on the originally given Capped Grope of height $k \geq 2$, preserving only the bottom surface Σ_1 instead of the first two stages Y as done above. (No dual spheres need to be constructed.) After the core construction, we have a Capped Grope of height $k+r$ and word length $\leq 2r+1$, with many cap-body intersections but caps are disjoint from the bottom surface Σ_1. Now do symmetric contraction of the bottom surface. This requires taking parallel copies of whatever is attached to it, and reduces the height of the entire Capped Grope by 1. Then push all cap-body intersections down and off the contraction. This at most doubles the estimate on the double point loop length and thus leads to a clean Capped Grope of height $k + (r-1)$ and word length

$$\leq 2(2r+1) = 4(r-1) + 6.$$

Thus linear grope height raising is established.

References

[F] **M.H. Freedman**, *The topology of four-dimensional manifods*, J. Diff. Geom. 17 (1982), 357–453.

[F1] **M.H. Freedman**, *The disk theorem for four-dimensional manifods*, Proc. ICM Warsaw, 1983, pp. 357–453.

[FQ] **M.H. Freedman, F. Quinn**, *Topology of 4-manifolds*, Princeton University Press, 1990.

[FT] **M.H. Freedman, P. Teichner**, *4-manifold topology I: subexponential groups*, Invent. Math. 122 (1995), 509–529.

[K] **V. Krushkal**, *Exponential separation in 4-manifolds*, (preprint).

[Q] **F. Quinn**, *Pseudoisotopies of 4-manifolds*, (preprint).

[S] **R. Stong**, *Four-manifold topology and groups of polynomial growth*, Pacific J. Math. 157 (1993), 145–150.