Topic 2*: Link homotopy in 3-space

* first blackboard talk since 2.5 years !

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MPIM

lecture hall

Pete

Recall :

$$\begin{tikzcd}
\mathrm{Embeddings}(X,Y) \arrow[r] & \mathrm{Link\ maps}(X,Y) \arrow[r] & \mathrm{Maps}(X,Y) \\
{[0,1]} \arrow[u, "\text{isotopy}", "\text{smooth path}"'] & {[0,1]} \arrow[u, "\text{link homotopy}", "\text{path}"'] & {[0,1]} \arrow[u, "\text{homotopy}", "\text{path}"']
\end{tikzcd}$$

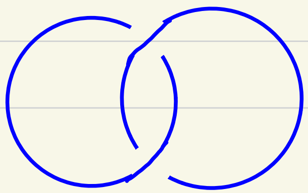
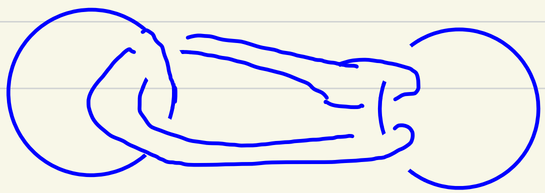
$$LM^4_{2,2} := \left\{ S^2 \sqcup S^2 \xrightarrow{\text{link map}} \mathbb{R}^4 \right\} \Big/ \text{link homotopy} \cong \bigoplus^{\infty} \mathbb{Z} \quad (\text{Topic 1a, 1b})$$

upcoming

$$LM^3_{1,1} := LM\left[S^1 \sqcup S^1, \mathbb{R}^3 \right] \cong \mathbb{Z} \text{ via linking number.}$$

What about $LM\left[\coprod^n S^1, \mathbb{R}^3 \right]$? (link homotopically)

Definition: L is almost trivial if $L \setminus \ell_k$ is trivial $\forall k = 1, \dots, n$.

e.g. "Hopf"  or  "Bor"

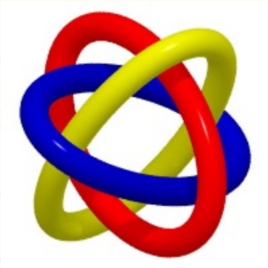
Milnor [1950s]: $\{\text{almost trivial links}\} \subseteq LM[\coprod^n S^1, \mathbb{R}^3]$

$$\cong \Big\downarrow \mu$$

$$\mathbb{Z}^{(n-2)!}$$

$$\mu = \bigoplus_{\sigma \in S_{n-2}} \mu(\sigma(1), \dots, \sigma(n-2), n-1, n)$$

(non-repeating) **Milnor invariants**

Bor = 

e.g. $n=2$, $\mu = \mu(12)$ = linking number $\in \mathbb{Z}$

$n=3$, $\mu = \mu(123)$ = Milnor's triple invariant $\in \mathbb{Z}$:

$\mu_L(ij) = 0 \ \forall i \neq j \overset{\text{tubes}}{\iff} \exists$ Seifert surfaces F_1, F_2, F_3 for L s.t. $F_i \subseteq \mathbb{R}^3 \setminus (\ell_j \sqcup \ell_k)$. Then $\mu_L(123) = F_1 \pitchfork F_2 \pitchfork F_3$

$n=4$, $\mu = \mu(1234) \oplus \mu(2134) \in \mathbb{Z}^2$ .... ?

Goal of next few lectures: **Geometric understanding** of this μ & repeating partners like $\mu(1122)$ etc.

$n=3$: For $L = (\underbrace{\ell_1, \ell_2}, \ell_3)$, $[L] \in LM[3]$ only depends on

$$\ell_3 : S^1 \hookrightarrow \mathbb{R}^3 \setminus \bigcirc \; \bigcirc \qquad [\ell_3] \in \pi_1\left(\mathbb{R}^3 \setminus (\ell_1 \sqcup \ell_2)\right)$$

$$\cong$$

free group $F := F(m_1, m_2)$

$\big\downarrow$ forget

free abelian group $\mathbb{Z} \times \mathbb{Z} \cong F/F_2$

$$\mu_L(13) = 0 = \mu_L(23) \iff \ell_3 \in F' = F_2 = \text{commutator subgroup}$$

Surprise: $[L]$ actually only depends on $[\ell_3] \in F_2/F_3 \le F/F_3 =$ **free nilpotent group** of class 3.

$$[\ell_3] \mapsto \mu_L(123), \qquad F_2/F_3 \cong \mathbb{Z}$$

Moreover, $\mu_L(123) \in \mathbb{Z}$ is another incarnation of the triple invariant.

Reason: Self-intersections of ℓ_1 and ℓ_2 introduce relations $[[m_1, m_2], m_1], [[m_1, m_2], m_2] \in F_3$.

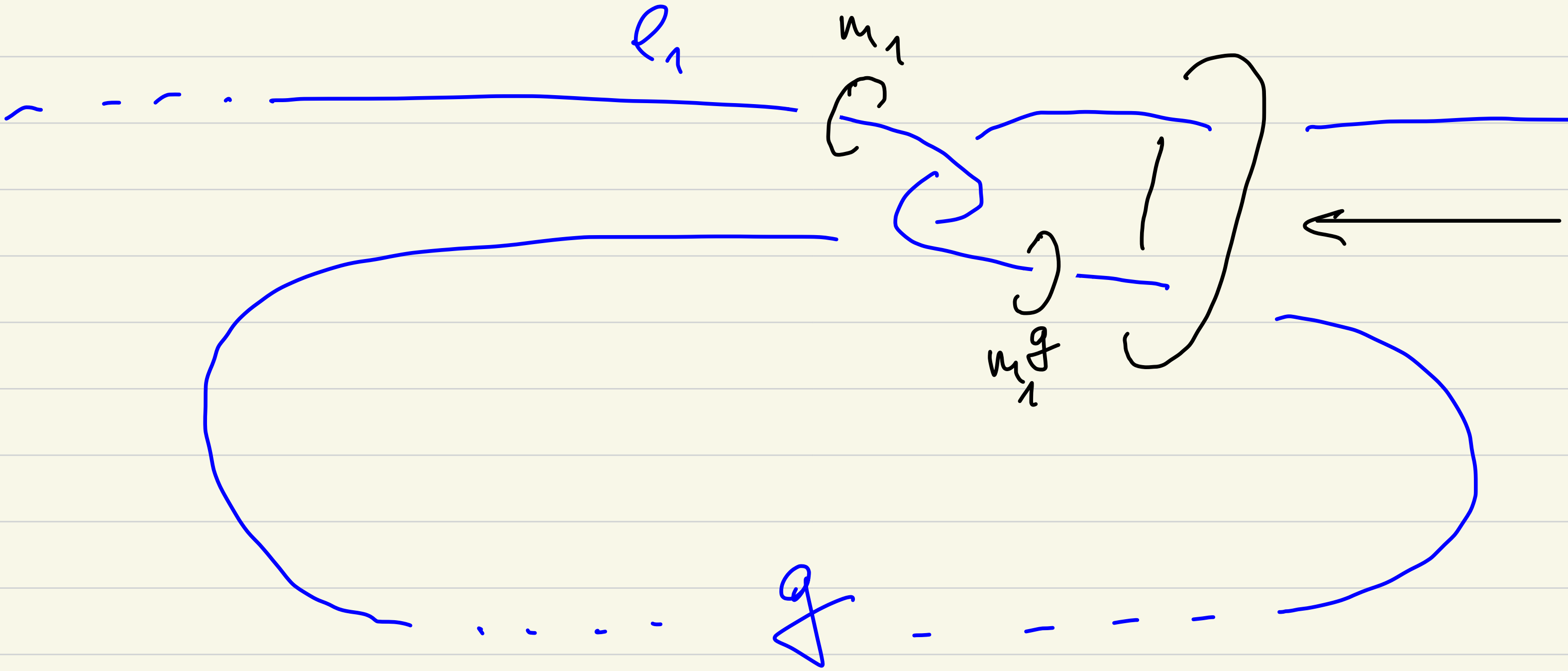
$$[m_1, m_2] = m_1 m_2 m_1^{-1} m_2^{-1}$$
$$= m_1 \cdot \left(m_1^{-1}\right)^{m_2}$$
$$= m_2^{m_1} \cdot m_2^{-1}$$

$$\Updownarrow$$

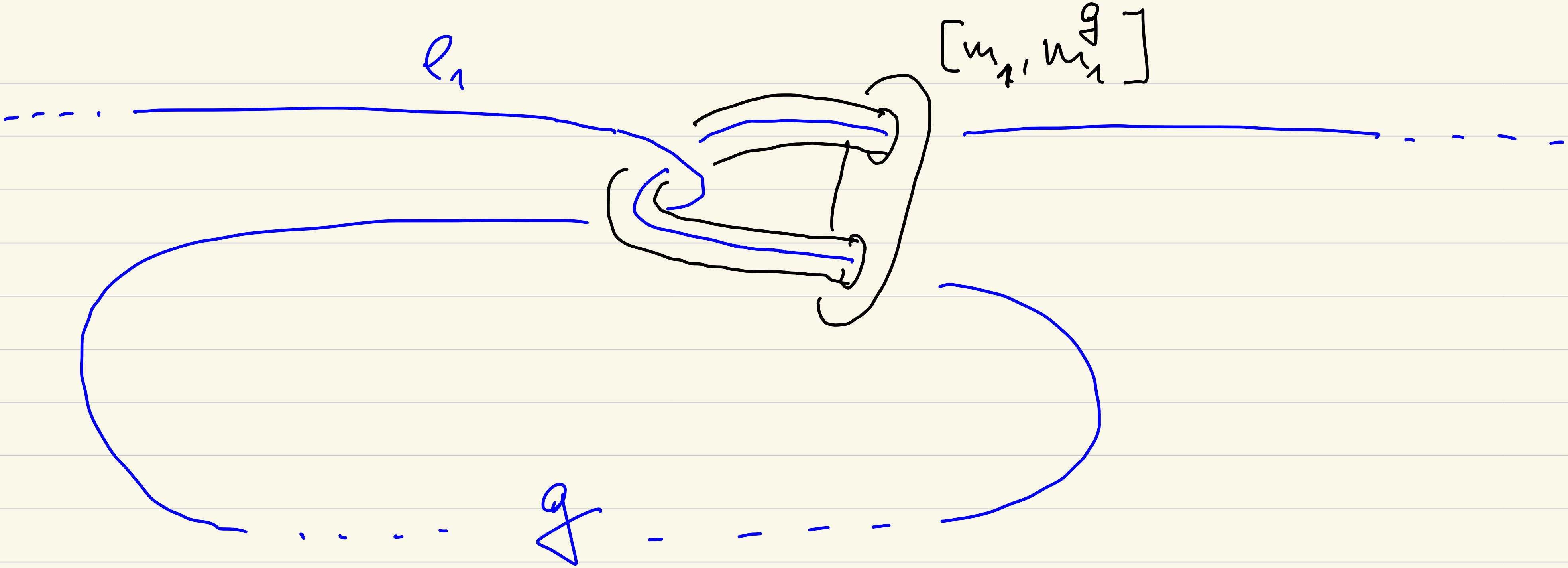
$$\left[m_1^{-m_2}, m_1\right], \quad \left[m_2^{m_1}, m_2\right] = 1$$

$$\Updownarrow$$

all conjugates of m_i commute with m_i, $i = 1, 2$.



slips off after self-intersection of l_1 !



Def.: The **Milnor group of L** is the quotient

$$M(L) := \pi_1(\mathbb{R}^3 \setminus L) \Big/ [m_i, m_i^g] = 1 \quad \forall i \ \ \forall g \in \pi_1$$

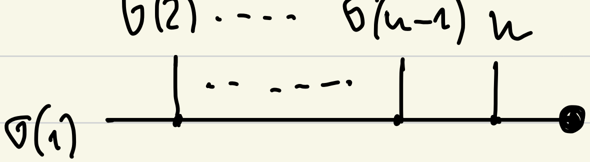
Thm.: (i) $M(L)$ only depends on $[L] \in LM[\coprod^n S^1, \mathbb{R}^3]$

(ii) $M(L)$ is nilpotent of class $n+1$, i.e. $M(L)_{n+1} = \{1\}$

(iii) $[L]$ trivial $\Rightarrow$ $\mathrm{Ker}\left(M(L) \longrightarrow \prod_{k=1}^{n} M(L \setminus \ell_k)\right) \cong \mathbb{Z}^{(n-1)!}$

where the generator of $\mathbb{Z}_\sigma$ in $\mathbb{Z}^{(n-1)!} = \bigoplus_{\sigma \in S_{n-1}} \mathbb{Z}_\sigma$

is sent to $[m_{\sigma(1)}, [m_{\sigma(2)}, [\,..\,[m_{\sigma(n-1)}, m_n]\cdots] \in M(L)_n$

$\hat{=}$ 

"right normed" commutator.

Proof: (i) follows from our pictures, together with the fact that any homotopy can be perturbed to become "generic", i.e. a finite sequence of isotopies and finger moves.

(ii) We'll use that $\pi_1(\mathbb{R}^3 \setminus L)$ is normally generated by one meridian m_i per component. The relation imply that for $n=1$, the group $M(L)$ is abelian, i.e. that commutators $M(L)_2 = \{1\}$.

Ind. on $n-1 \mapsto n$: It suffices to show that

class goes up by at most 1.

$\mathrm{Ker}\left(M(L) \longrightarrow \prod_{k=1}^{n} M(L \setminus \ell_k)\right)$ lies in the center of $M(L)$,

that any element $k \in \mathrm{Ker}$ commutes with all m_ℓ^g. Now m_ℓ normally generates (kernel of $M(L) \to M(L \setminus \ell_\ell)$) $\supseteq$ Ker ∎