KNOT CONCORDANCE AND VON NEUMANN ρ-INVARIANTS

TIM D. COCHRAN and PETER TEICHNER

Abstract

We present new results, announced in [T], on the classical knot concordance group $\mathscr{C}$. We establish the nontriviality at all levels of the (n)-solvable *filtration*

$$\cdots \subseteq \mathscr{F}_n \subseteq \cdots \subseteq \mathscr{F}_1 \subseteq \mathscr{F}_0 \subseteq \mathscr{C}$$

introduced in [COT1]. Recall that this filtration is significant due to its intimate connection to tower constructions arising in work of A. Casson and M. Freedman on the topological classification problem for 4-manifolds and due to the fact that all previously known concordance invariants are reflected in the first few terms in the filtration. In [COT1], nontriviality at the first new level $n = 3$ was established. Here, we prove the nontriviality of the filtration for all n, *hence giving the ultimate justification to the theory.*

A broad range of techniques is employed in our proof, including cut-and-paste topology and analytical estimates. We use the Cheeger-Gromov estimate *for von Neumann ρ-invariants, a deep analytic result. We also introduce a number of new algebraic arguments involving noncommutative localization and Blanchfield forms. We have attempted to make this article accessible to readers with only passing knowledge of [COT1].*

Contents

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1. Introduction

A *knot* is a smoothly embedded circle in 3-space. In [FM], Fox and Milnor introduced the notion of a *slice knot* as a knot that is the intersection of 3-space with a 2-sphere smoothly embedded in 4-space. They showed that an isolated piecewise linear singularity of a surface in a 4-manifold can be resolved smoothly if and only if the link of the singularity is a slice knot.

The question of which knots are slice knots lies at the heart of the classification of 4-dimensional manifolds since it is an attempt to bound a knot in a 3-manifold N by an embedded disk in a 4-manifold M (with boundary $\partial M = N$). The knot-slice problem is the simplest case $(M, N) = (D^4, S^3)$. The classification of higher-dimensional manifolds can largely be reduced to algebra via the techniques of *surgery* and the *s-cobordism theorem*. A key step in this reduction is representing a k-dimensional homotopy class of a $2k$-dimensional manifold M^{2k} by an *embedded* k-sphere. A crucial tool in completing this program is the *Whitney trick*, which allows for the *geometric* elimination of a pair of self-intersection points of an immersed sphere $S^k \looparrowright M^{2k}$ if this pair cancels *algebraically*, using certain embedded 2-disks called *Whitney disks*. However, when $k = 2$, the Whitney trick, and hence the entire program, fails because the immersed Whitney disk itself has potentially essential singularities; the problem of embedding the Whitney disks is just as difficult as the original problem. Thus it was that in the early 1980s, the question of which knots are slice knots became the central problem in topological 4-manifolds. Moreover, the failure of the Whitney trick suggested a hierarchy of algebraic *obstructions*. More precisely, the singularities of the Whitney disks can be viewed as "second-order" algebraic obstructions to the original program. If *these* in turn vanish, then the pairs of intersections of the first Whitney disks have their own (second-order) Whitney circles and disks whose self-intersections could be viewed as "third-order" obstructions, leading to the notion of a *Whitney tower* of height n, considered by A. Casson and M. Freedman.

However, the actual existence of an infinite sequence of nontrivial algebraic obstructions to finding Whitney towers of height n has remained unconfirmed until this article. Indeed, the following result is a direct consequence of Proposition 1.3 and Theorem 1.4.

THEOREM 1.1
For any positive integer n, there exists an immersion $D^2 \looparrowright B^4$ whose homotopy class (relative its boundary knot) can be represented by a smooth Whitney tower of height n but cannot be represented by any smooth or even topological Whitney tower of height $n + 1$.

It is fascinating that it took (for us, at least!) techniques of von Neumann algebras and other deep analytical results to achieve this topological result.

The study of slice knots is facilitated by a natural group structure. Kervaire and Milnor [KM] observed that the connected sum operation gives the set of all knots, modulo slice knots, the structure of an abelian group, now called the *smooth-knot concordance group*. More than 45 years after their work, and despite its connection to the classification problem, the concordance group remains far from being understood. Using (locally flat) topological embeddings, one gets the *topological* knot concordance group $\mathscr{C}$, which is a quotient of its smooth partner but is also very much unknown. This article gives new information about both of these groups using techniques of noncommutative algebra and analysis.

In the late 1960s, Levine [L] defined an epimorphism from $\mathscr{C}$ to an *algebraic concordance group* that he showed was isomorphic to $\mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$, given by the Arf invariant, certain discriminants, and twisted signatures associated to the infinite cyclic cover of the knot complement. In the early 1970s, Casson and Gordon [CG] defined new invariants via dihedral covers; these were used to show that the kernel of Levine's map has infinite rank (see [Li] for a more detailed history).

In [COT1], Kent Orr and the authors used arbitrary solvable covers of the knot complement to exhibit a new filtration of $\mathscr{C}$,

$$\cdots \subseteq \mathscr{F}_n \subseteq \cdots \subseteq \mathscr{F}_1 \subseteq \mathscr{F}_{0.5} \subseteq \mathscr{F}_0 \subseteq \mathscr{C}.$$

This filtration is significant for several reasons. Most importantly, it successfully mirrors the constructions of towers of Whitney disks as discussed above. Similarly, it mirrors the closely related approach using *Gropes*, utilized by Freedman and Quinn in their foundational book on 4-manifolds (see [FQ]). It also corresponds well to the surgery-theoretic approach of Cappell and Shaneson [CS]. The precise connections between the filtration and these other notions were formulated and proven in [COT1, Theorems 8.11, 8.12, 8.4, 8.8]. In addition, it was shown there that the filtration "contains," in its associated graded quotients of low degree, all of the previously known concordance invariants. It was also confirmed that there is indeed *new* information in the filtration. In particular, it was shown in [COT2] that $\mathscr{F}_2/\mathscr{F}_{2.5}$ contains an infinite rank summand of concordance classes of knots not detectable by previously known invariants. However, the crucial question of whether or not the filtration was *nontrivial* for larger n was left open.

The primary result of this article is to prove nontriviality of the filtration for all n. To explain our results in more detail, recall that for each positive half-integer n, the subgroup $\mathscr{F}_n$ consists of all (n)-*solvable* knots. $\mathscr{F}_0$ consists of the knots with Arf invariant zero, and $\mathscr{F}_{0.5}$ consists of the algebraically slice knots (i.e., knots in the kernel of Levine's map). In general, (n)-solvability is defined using intersection forms

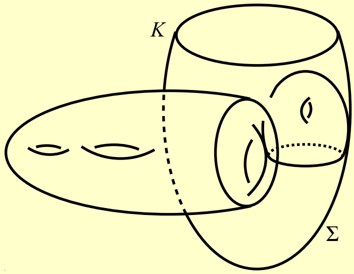

Figure 1.2. A grope of height 2

in (n)-solvable covering spaces of certain 4-manifolds. (The definition is reviewed in Section 4.) There is a much easier (for an introduction!), closely related filtration that is geometrically more intuitive, namely, a filtration defined using *Gropes*. Denote by $\mathcal{G}_n$ the subgroup of $\mathcal{C}$ consisting of all knots that bound a Grope of height n in D^4. A Grope is a 2-complex that approximates the 2-disk. The precise definitions are reviewed in Section 3, but a grope of height 2 is shown in Figure 1.2, and the general concept is quite simple: Gropes of larger and larger height are, on one hand, a geometric version of the derived series of a group; on the other hand, they form a better and better approximation to the desired slice disk for a given knot. There is always a (Seifert) surface Σ bounding the knot, and if some basis of curves on Σ bounds disjointly embedded disks, then Σ can be turned (or surgered) into a slice disk. However, the question now iterates, namely, the curves on Σ may bound merely surfaces (rather than disks), and hence we find a grope of height 2 as in Figure 1.2. In this article, we use exclusively *symmetric* gropes, corresponding to the derived series of a group (leading to solvable groups) (see [T] for other notions of gropes and their applications). For example, the much-simpler theory of the lower central series (leading to nilpotent groups) gives, in its incarnation via half-gropes, a geometric interpretation of Vassiliev invariants and the Kontsevich integral of knots (see [CT]).

The filtrations of $\mathcal{C}$ by solvability and by Gropes are not the same, but the following result from [COT1] shows that they are closely correlated, and so the reader may (in this introduction) safely consider only the Grope filtration.

PROPOSITION 1.3 ([COT1, Theorem 8.11])
If a knot K in S^3 bounds a Grope or Whitney tower of height $(n+2)$ in D^4, then K is (n)-solvable. In particular, $\mathcal{G}_{n+2} \subseteq \mathcal{F}_n$.

The main result of this article—and, in some sense, the ultimate justification for the entire theory—is as follows.

THEOREM 1.4

For any $n \in \mathbb{N}_0$, the quotient groups $\mathcal{F}_n/\mathcal{F}_{n.5}$ contain elements of infinite order. Similarly, the groups $\mathcal{G}_{n+2}/\mathcal{G}_{n+2.5}$ contain elements of infinite order. In fact, the groups $\mathcal{G}_{n+2}$ contain knots that have infinite order modulo $\mathcal{F}_{n.5}$.

We note that our construction of examples is all done in the smooth category, so that we actually also prove the corresponding statements about the smooth-knot concordance group; that is, our examples bound smooth $(n+2)$-Gropes but do not bound any smoothly (or even topologically) embedded $(n+2.5)$-Grope.

For $n = 0$, examples of our theorem can be detected by the Levine-Tristram signature obstructions (see [COT1, Remark 1.13]); for $n = 1$, one can use Casson-Gordon invariants (which vanish on $\mathcal{F}_{1.5}$ by [COT1, Theorem 9.11]); and for $n = 2$, this is the main result of [COT1, Theorem 6.4]. The results for $n > 2$ are new.

The proof of this theorem turns out to be difficult—and, actually, impossible (for us)—without introducing a new tool from analysis. The main problem is the higher-order obstruction-theoretic nature of all known invariants, a problem that already arises in using Casson-Gordon invariants. In a nutshell, there *are* higher-order signature invariants that obstruct the existence of a Grope of height $n.5$ *extending* a fixed given Grope of height n. But, in order to show that a knot does not bound a Grope of height $n.5$, one then has to show that this signature is nontrivial *for all possible* Gropes of height n bounding the knot. This turns out to be a formidable task. It was resolved by Casson and Gordon [CG] and in [COT1, Section 6] by finding knots for which, roughly speaking, there is a *unique* Grope of height n, at least as far as the relevant algebra can see. This works for $n = 1, 2$, but we have been unable to find such knots for $n > 2$. In this article, we use analytic tools to resolve the issue along with a deep new result using the noncommutative localization techniques of [COT1] and ideas from [COT2] to construct the relevant knots with the little more care necessary to make sure that they lie in $\mathcal{G}_{n+2}$ rather than just in $\mathcal{F}_n$. The new analytic methods show that these knots are not $(n.5)$-solvable.

To see why analysis can play a role, recall that the signature that turns out to be relevant for $n > 1$ is in fact a (real-valued) von Neumann signature associated to a certain intersection form on a 4-manifold constructed from a Grope of height n. This 4-manifold has boundary M_K, the zero surgery on the given knot K. The main information about the Grope is encoded by the homomorphism

$$\phi : \pi_1 M_K \longrightarrow \pi$$

induced by the inclusion of the boundary into the 4-manifold. Here, π is the quotient of the 4-manifold group by all $(n+1)$-fold commutators. Since the Grope is variable, we do not have any information on the group π; in particular, we do not know any interesting representations, except for the canonical one on $\ell^2(\pi)$. That is why von Neumann algebras enter the story.

By the von Neumann index theorem, the difference between the von Neumann signature and the usual (untwisted) signature of this 4-manifold is equal to the invariant $\rho(M_K, \phi)$ of the boundary. This von Neumann ρ-invariant is the difference between the von Neumann η-invariant and the untwisted η-invariant. It is a real-valued topological invariant depending only on the covering determined by ϕ and is explained in detail in Section 2.

Thus the main question of how to control the von Neumann signatures of the 4-manifolds associated to all possible Gropes of height n with boundary K translates into the question of how to control the invariants $\rho(M_K, \phi)$ for all possible ϕ. Analysis enters prominently because of the following estimate of Cheeger and Gromov [CG1] for the von Neumann ρ-invariants

$$\exists\, C_M > 0 \text{ such that } |\rho(M, \phi)| < C_M \,\forall \phi, \tag{1.5}$$

where M is a closed oriented manifold of odd dimension. Thus, for any fixed 3-manifold, the set of all possible ρ-invariants is bounded in absolute value. In addition to [CG1], Ramachandran's article [R] is a good source to learn about the estimate in more generality (see our Section 2).

It remains to be seen whether our filtration can be used to fully understand the topological knot concordance group $\mathscr{C}$. The main open question may be whether a knot in the intersection of all $\mathscr{F}_n$ must be topologically slice. The other main remaining open questions about our filtrations are the following.

CONJECTURE

For any $n \in \mathbb{N}_0$, the quotient groups $\mathscr{F}_n/\mathscr{F}_{n.5}$ have infinite rank. Moreover, for $n > 0$, the groups $\mathscr{F}_{n.5}/\mathscr{F}_{n+1}$ are nontrivial.

The first part of the conjecture is true for $n \leq 2$: $n = 0$ is verified using twist knots and the Seifert form obstructions; for $n = 1$, this can be established by using examples due to Casson and Gordon [CG]; and $n = 2$ is the main result of [COT2].

Outline of the proof of Theorem 1.4

We start with the zero surgery M on a fibered genus 2 ribbon knot R (which bounds an embedded disk in B^4 and is thus (n)-solvable for all n). We prove the existence of a certain collection of circles $\eta_1, \ldots, \eta_m$ in the nth derived subgroup of $\pi_1 M$ which forms a trivial link in S^3. We then modify R (we call this a *genetic infection*) using a certain auxiliary knot J (called the *infection knot*) along the circles $\eta_1, \ldots, \eta_m$ (called *axes*).

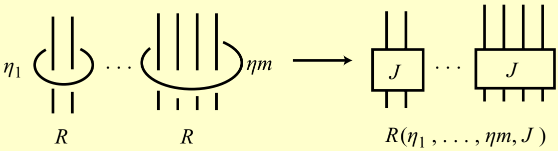

Figure 1.6. Genetic infection of R by J along η_i

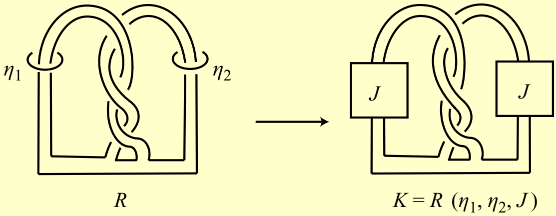

Figure 1.7. An example for $n = 1$

This language, introduced in [COT2], just expresses a standard construction, where the η_i bound disjointly embedded disks in S^3 and the knot J is tied into all strands of R passing through one of these disks, as illustrated in general in Figure 1.6 and in a particular case in Figure 1.7, where $n = 1$, $m = 2$.

With a good choice of an infection knot J, we can make sure that all resulting knots $K = R(\eta_1, \ldots, \eta_m, J)$ bound Gropes of height $(n + 2)$ (see Theorem 3.8).

If the von Neumann signature corresponding to $\pi_1(M_K) \to \mathbb{Z}$ of J is bigger than C_M (the "infection is strong enough"), then the Cheeger-Gromov estimate (1.5) shows that K is not $(n.5)$-solvable, provided that for all (relevant) homomorphisms ϕ, one of the axes η_i is mapped nontrivially. The precise statement is given in Theorem 4.2.

The last condition on the axes η_i turns out to be more subtle than expected, complicated by the fact that $\pi_1 M^{(n)}/\pi_1 M^{(n+1)}$ may not be finitely generated for $n > 1$. Fortunately, it turns out that we can use the higher-order Blanchfield forms from [COT1] to carefully construct axes η_i with the desired property (see Theorem 4.3).

Our article is organized as follows. In Section 2, we give a survey of the analytical results surrounding the von Neumann ρ-invariant; no originality is claimed. In Theorem 3.8, we construct a large class of knots that bound Gropes of height $(n + 2)$ in the 4-ball. Theorems 4.2 and 4.3 together imply that many of these knots are not $(n.5)$-solvable. Therefore, our main result, Theorem 1.4, follows. Section 5 reviews from [COT1] some algebraic topological results, higher-order Alexander modules and their Blanchfield forms, used for Theorem 6.3, which is at the heart of the proof of Theorem 4.3.

This article uses many notions and ideas from [COT1]. However, we have made a conscious effort to include here all relevant definitions, and we have stated all results needed explicitly and with precise references. We hope that, as a consequence, most readers are able to enjoy this article independently of [COT1]. Those who want to see the proofs of the stated results can find them easily in [COT1].

2. Von Neumann ρ-invariants

For the convenience of the reader, this section gives a short survey of the more analytic aspects of the invariant that is used in the rest of the article. Throughout this section, we work with the signature operator on 4-dimensional manifolds (and their boundary), even though everything works as well on $4k$-manifolds. One needs to replace $p_1/3$ by Hirzebruch's L-polynomial (and the 1-forms in (2.4) by $(2k-1)$-forms on the boundary). In fact, most of our discussion applies to all Dirac-type operators in any dimension instead of just the signature operator.

The signature theorem for manifolds with boundary

Let W be a compact oriented Riemannian 4-manifold with boundary M, and assume that the metric is a product near the boundary (or at least that the first two normal derivatives of the metric vanish on M). The Atiyah-Patodi-Singer index theorem for the case of the signature operator implies the signature theorem (see [APS, Theorem 4.14])

$$\sigma(W) = \int_W \frac{p_1(W)}{3} - \eta(M), \tag{2.1}$$

where $\sigma(W)$ is the signature of the intersection form on $H_2(W)$, $p_1(W)$ is the first Pontrjagin form of the tangent bundle (which depends on the metric), and $\eta(M)$ is a spectral invariant of the boundary. It is the value at $s = 0$ of the η-function

$$\eta(s) = \sum_{\lambda \neq 0} (\operatorname{sign} \lambda)|\lambda|^{-s}. \tag{2.2}$$

Here, λ runs through the nonzero eigenvalues of the signature operator D on M. More precisely, D is the self-adjoint operator on *even* differential forms on M defined by $\pm(*d - d*)$, where the Hodge $*$-operator depends on the metric. The η-function is defined by analytic continuation, and it turns out that it is holomorphic for $\operatorname{Re}(s) > -1/2$. In fact, one can get the explicit formula

$$\eta(2s) = \frac{1}{\Gamma(s+1/2)} \int_0^\infty t^{s-1/2} \operatorname{trace}(De^{-tD^2})\, dt, \tag{2.3}$$

where the trace-class operator De^{-tD^2} is defined by functional calculus and the gamma function is given by

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t}\, dt.$$

Using Hodge theory, [APS, Proposition 4.20] shows that for the purpose of calculating the η-function, one may restrict to the operator $d*$ acting on the space $d\Omega^1(M)$. As pointed out by [APS], this translation leads to the following suggestive interpretation of $\eta(M)$. Define a quadratic form Q by

$$Q(\alpha) := \int_M \alpha \wedge d\alpha, \quad \alpha \in \Omega^1(M), \tag{2.4}$$

and observe that Q has radical ker(d) and hence gives a form on $d\Omega^1(M)$. Moreover, for eigenvectors $d\alpha$ of $d*$, the corresponding eigenvalue has opposite sign as $Q(\alpha)$. Hence one can formally interpret the correction term $-\eta(M)$ in (2.1) as the "signature" of the quadratic form $-Q$. Note that Q does *not* depend on the metric, but since Q is defined on the infinite-dimensional space, the metric is used to give the proper meaning to its signature.

Even though signature theorem (2.1) is what we need in this article, it may be good to remind the reader of the relation to the index of the signature operator on W. In fact, the index theorem for the signature operator D_W on W reads as (see [APS, Theorem 4.3])

$$\text{index}(D_W) = \int_W \frac{p_1(W)}{3} - \frac{h + \eta(0)}{2}, \tag{2.5}$$

where h is the dimension of the space of harmonic forms on M and $\eta(0)$ is the value at $s = 0$ of the η-function for the signature operator on *all* forms on M. Since this operator preserves the parity of forms and commutes with the Hodge $*$-operator, the above η-function is just *twice* the η-function (on even forms) considered in the signature theorem. This explains the disappearance of the factor $1/2$ from (2.5). Moreover, by considering L^2-solutions on W with an infinite cylinder attached, [APS, Theorem 4.8] shows that

$$\text{index}(D_W) = \sigma(W) - h,$$

which explains how (2.1) is derived from (2.5).

In [CG2], Cheeger and Gromov looked for geometric conditions under which the integral term in the signature theorem is a topological invariant. They proved the following beautiful result: assume that the open 4-manifold X admits a *complete* metric with finite volume whose sectional curvature satisfies

$$|K(X)| \le 1.$$

Assume, furthermore, that X has *bounded covering geometry* in the sense that there is a normal covering $\tilde{X}$ (with covering group Γ) such that the injectivity radius of the pullback metric on $\tilde{X}$ is ≥ 1. (This condition gets weaker with the covering getting larger.) Then [CG2, Theorem 6.1] says that one has a topological invariant (in fact, a proper homotopy invariant)

$$\int_X \frac{p_1(X)}{3} = \sigma_\Gamma(X), \tag{2.6}$$

where $\sigma_\Gamma(X)$ is the *von Neumann* or L^2-signature of the Γ-cover $\tilde{X}$ to which we turn in the next section. We point out that the main step in the proof of this theorem is an estimate for the η-invariant of 3-manifolds (with similarly bounded geometry; cf. Theorem 2.9). It implies what we called the *Cheeger-Gromov estimate* in our introduction.

The L^2-signature theorem

Returning to a compact 4-manifold W with product metric near the boundary, we can study twisted signatures given by bundles with connection over W. If the bundle is flat, then these signatures have again a homological interpretation. (And the integral term in the signature theorem is unchanged.) A flat bundle is given by a representation of the fundamental group $\pi_1 W$. However, in the application that we have in mind, there are no such preferred representations mainly because $\pi_1 W$ is an unknown group. All we are given is a homomorphism $\pi_1 W \to \Gamma$, where Γ is usually a solvable group. Fortunately, there is a highly nontrivial *canonical* representation of any group, namely, on $\ell^2(\Gamma)$. It turns out that one can twist the signature operator with this representation and then calculate its index using von Neumann's Γ-dimension. We used these real numbers in [COT1] to prove our main results, and Cochran, Orr, and Teichner gave a survey in [COT1, Section 5], similar to the current one, including all relevant references. Since then, Lück and Schick have written [LS], where they prove that all known definitions of von Neumann signatures agree.

There is an L^2-signature theorem, analogous to (2.1), that can again be derived from an L^2-index theorem, proven for general Dirac-type operators in [R]. A more direct argument for signatures is given in [LS, Theorem 3.10], where the authors use Vaillant's thesis to translate the right-hand side of (2.7) to the L^2-signature of the intersection pairing on L^2-harmonic 2-forms on W with an infinite cylinder attached. Lück and Schick then translate this signature into a purely homological setting, obtaining the following result. Given a compact oriented 4-manifold W together with a homomorphism $\pi_1 W \to \Gamma$, one has

$$\sigma_\Gamma(W) = \int_W \frac{p_1(W)}{3} - \eta_\Gamma(M). \tag{2.7}$$

Here, the left-hand side is the von Neumann signature of the intersection form on $H_2(W; \mathcal{N}\Gamma)$, where $\mathcal{N}\Gamma$ is the von Neumann algebra of the group Γ. The von Neumann η-invariant can also be defined in a straightforward way, using (2.3) rather than (2.2):

$$\eta_\Gamma(M) := \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2}\, \mathrm{trace}_\Gamma(\tilde{D} e^{-t\tilde{D}^2})\, dt. \tag{2.8}$$

Here, $\tilde{D}$ is the signature operator on the even forms of the induced Γ-cover $\tilde{M}$ of M. If $k_t(x, y)$ denotes the smooth kernel of the operator $\tilde{D} e^{-t\tilde{D}^2}$, then the above Γ-trace is given by the integral

$$\int_{\mathcal{F}} \mathrm{trace}_x\, k_t(x, x)\, dx,$$

where $\mathcal{F}$ is a fundamental domain for the Γ-action on $\tilde{M}$.

As before, one can alternatively use the lift of $d*$ to calculate this η_Γ-invariant. A key step in the proof of (2.7) is the following result of [CG1, Theorem 4.10].

THEOREM 2.9
There is a constant C, depending only on the local geometry of M, such that for all Γ-covers of M,

$$|\eta_\Gamma(M)| \le C \cdot \mathrm{volume}(M).$$

This is also proven in [R, Theorem 3.1.1] for general Dirac-type operators, but there seems to be a problem for s near t in the fourth line of [R, (3.1.10)]. As Bruce Driver [D] pointed out to us, this problem can be fixed by using an L^1-estimate instead of the L^∞-estimate in [R, (3.1.10)]. For the signature operator $d*$, one can also carefully read [CG1, pages 23–24], inserting the symbols $*d$ into the decisive definition [CG1, (4.15)] of the von Neumann η-invariant. Note that Cheeger and Gromov use the operator $*d$ on coexact 1-forms which is conjugate, under Hodge-$*$, to $d*$ on $d\Omega^1$ used above.

The von Neumann ρ-invariant
Subtracting expressions (2.1) and (2.7), one gets the equation

$$\sigma_\Gamma(W) - \sigma(W) = \eta(M) - \eta_\Gamma(M), \tag{2.10}$$

which allows the following beautiful interpretation. The left-hand side is independent of the metric, whereas the right-hand side does not depend on the zero bordism W for M. As a consequence, the expression (2.10) must be a topological invariant of M! This argument works as long as, for a given 3-manifold M, one can find a metric (easy) and a zero bordism (also easy, except if it has to be over the group Γ). In fact, it suffices to find the zero bordism over a group into which Γ embeds, and this can always be done. Alternatively, one applies (2.10) to the product $M \times I$, equipped with a metric inducing a path of metrics on M. Since the signature terms vanish on the product, one concludes that the right-hand side is indeed independent of the metric.

Definition 2.11
Let M be a closed oriented 3-manifold, and fix a homomorphism $\phi : \pi_1 M \to \Gamma$. Define the von Neumann ρ-invariant

$$\rho_\Gamma(M, \phi) := \eta(M) - \eta_\Gamma(M)$$

with respect to any metric on M. If the group (or the homomorphism) is clear from the context, we suppress it from the notation, as we already did for η_Γ.

It is extremely interesting to find a combinatorial interpretation of this von Neumann ρ-invariant. Since it does not depend on the choice of a metric, a definition along the lines of the quadratic form Q in (2.4) may not be out of reach. All our calculations of ρ_Γ are based on (2.10); in fact, by choosing our examples of knot carefully, we manage to reduce the Γ-signature calculations to the case $\Gamma = \mathbb{Z}$. There it boils down to an integral, over the circle, of all twisted signatures, one for each $U(1)$-representation of $\mathbb{Z}$ (cf. Lemma 4.5).

The Cheeger-Gromov estimate from Theorem 2.9 now clearly implies the following innocent-looking estimate. It is crucial for our purposes.

THEOREM 2.12
For any closed oriented 3-manifold M, there is a constant C_M such that for all groups Γ and all homomorphisms $\phi : \pi_1 M \to \Gamma$,

$$|\rho_\Gamma(M, \phi)| < C_M.$$

We noticed long ago that such an estimate is extremely helpful for understanding our filtration of the knot concordance group. We made several unsuccessful attempts at proving this estimate by using (2.10), that is, the interpretation in terms of signature defects. Our lesson is that estimates are best approached with analytic tools.

3. Gropes of height $(n+2)$ in D^4

In this section, we review the definition of a *Grope* and then describe, for each positive integer n, large families of knots in S^3 which bound embedded Gropes of height $(n+2)$ in D^4. In Section 4, we show that among these are knots that *do not* bound any Grope of height $(n+2.5)$ in D^4.

We now review the definition of a *Grope* (see [FQ], [FT]). More precisely, we only define *symmetric* gropes (and Gropes) since these are used exclusively. We therefore suppress the adjective *symmetric* (for a survey of other notions and applications, see [T]).

In the following, when we refer to a *surface*, we mean a compact, connected, oriented surface with one boundary component. Recall that each connected component of an abstract grope G is built up from a connected *first-stage surface* G^1 by gluing *second-stage* surfaces to each circle in a symplectic basis $\{a_j, b_j\}$, $1 \leqq j \leqq 2g$, for G^1. This growth process continues, so that, in general, the k-stage surface $(G^k)_{a_j}$ is glued to a_j and $(G^k)_{b_j}$ is glued to b_j, where $\{a_j, b_j\}$ is a symplectic basis of circles for one of the $(k-1)$-stage surfaces. One does this for each $(k-1)$-stage surface. If a grope has n stages in all, then we say that it has *height* n. A grope of height $n+0.5$ is a grope of height n together with further surfaces attached to only *half* of a symplectic basis for each one of the nth-stage surfaces. A grope of height zero is understood to be merely a circle (no surfaces). The union of all of the circles in symplectic bases for the n-stage surfaces is called the set of *tips* of the grope. No surfaces are attached to the tips.

The reader may have observed that gropes are related to the derived series of a group as follows. Let $A^{(i)}$ denote the ith *derived group* of a group A, inductively defined by $A^{(0)} := A$ and $A^{(i+1)} := [A^{(i)}, A^{(i)}]$. A group A is (n)-*solvable* if $A^{(n+1)} = 1$ (zero-solvable corresponds to abelian), and A is *solvable* if such a finite n exists. The connection between gropes and the derived series is then given by the following statement (see [FT, Part II, Lemma 2.1]). A loop $\gamma : S^1 \to X$ in a space X extends to a continuous map of a height n grope $G \to X$ if and only if γ represents an element of the nth term of the derived series of the fundamental group of X. In particular, for γ to bound an *embedded* such grope is a strictly stronger statement. In this article, when we say that a circle *bounds a grope* in a space X, we mean that it bounds an *embedded* copy of the abstract grope. The difference between this geometric condition and its algebraic counterpart is a central underlying theme of this section.

If a grope G is embedded in a 4-dimensional manifold, we usually are able to take arbitrarily many disjoint parallel copies; hence we require the following framing condition. A neighborhood of G is diffeomorphic to the product of $\mathbb{R}$ with a neighborhood of a standard embedding of G into $\mathbb{R}^3$. Another way of expressing this is to say that the relative Euler number of each surface stage vanishes. This relative invariant is defined because the boundary circle of each surface stage, except the bottom, inherits

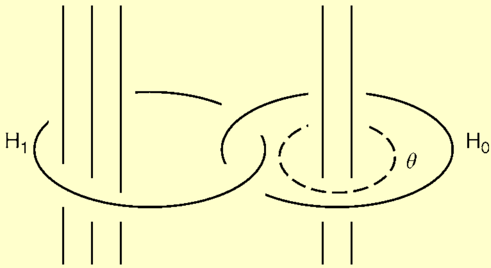

Figure 3.2. A Hopf link in $S^3 - R$

a framing from its embedding into the previous stage. In [FT], neighborhoods of such framed gropes in 4-manifolds are called *Gropes*, and we retain this convention (without explicitly distinguishing the neighborhood and its spine). Note that a grope embedded in a 3-manifold is automatically framed, so we can be sloppy with the capitalization. Finally, if one removes a disk from the bottom surface of a grope (away from the attaching circles of the next surfaces), one obtains an *annular* grope that has two boundary circles.

Definition 3.1
Two links L_0 and L_1 in S^3 are called *height n Grope concordant* if they cobound a disjoint union of annular height n Gropes in $S^3 \times I$, where each Grope has one boundary component in L_0 and one in L_1. If a link is height n Grope concordant to the unlink, then we say that it is *height n Grope slice*. If the links lie in $S^3 \setminus R$ for some other link R, and the annular Grope also lies in $(S^3 \setminus R) \times I$, then we refer to it as a Grope concordance rel R.

It is easy to see that the quotient group $\mathscr{C}/\mathscr{G}_n$ from the introduction is the same as the quotient group given by knots modulo the equivalence relation of height n Grope concordance.

Suppose that R is a knot in S^3 and that $H = \{H_0, H_1\}$ is a zero framed ordered Hopf link in S^3 which misses R, such as is shown in Figure 3.2. Since zero framed surgery on a Hopf link in S^3, denoted $(S^3)_H$, is well known to be homeomorphic to S^3 (see [K]), the image of the knot R under this homeomorphism is a new knot K (i.e., $(S^3, R)_H \approxeq (S^3, K)$). We say that K is the *result of surgery on H* (see [K]). Let θ denote the dashed circle shown in Figure 3.2 which is a parallel of H_0 but does not link H_1.

By sliding the left-hand strands of R over H_0 and then canceling the Hopf pair, one sees that the effect of one Hopf surgery is as in Figure 3.3.

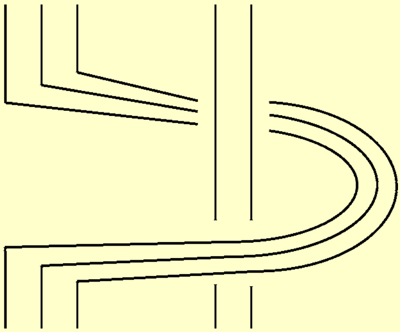

Figure 3.3. The effect of a Hopf surgery

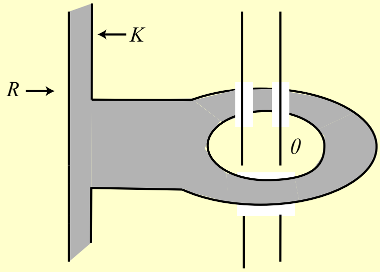

Figure 3.5. A punctured annulus cobounding K and R

The following proposition discusses the effect of repeated application of the above Hopf surgery. In particular, we assume that several *separated* Hopf links are given; that is, the Hopf links together with their bounding 2-disks are embedded disjointly.

PROPOSITION 3.4
Suppose that R is a knot, and suppose that K is the result of surgery on separated Hopf links H^i, $1 \le i \le m$, in $S^3 \setminus R$. Suppose that the link $\theta_1, \dots, \theta_m$ (see Figure 3.2) is height n Grope slice rel R. *Then K and R are height n Grope concordant.*

Proof
We first assume that there is only one Hopf link $H = \{H_0, H_1\}$ and later indicate the modifications necessary in the general case. First, observe that in S^3, K and R cobound an embedded punctured annulus. If there is only one strand of R going through H_1 in Figure 3.2, this annulus has one puncture, namely, θ, and is shown in Figure 3.5. If there are r strands, just take r parallel copies of this figure, noticing that the relevant

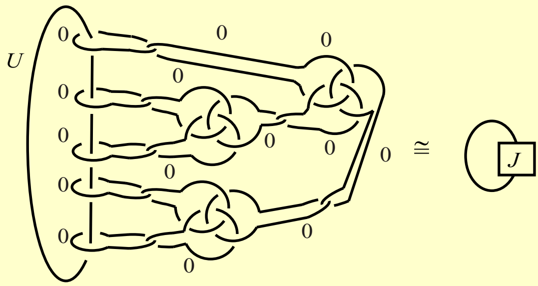

Figure 3.6. The knot J as surgery on an unknot U

part is planar. Thus the result is an embedded annulus with r punctures, each a zero framed copy of θ.

We can extend this annulus by $R \times I$ to get a punctured annulus A in $S^3 \times I$, leading from K down to R. By assumption, there is a height n Grope G that bounds θ in $(S^3 \smallsetminus R) \times I$. Since G is framed, we can get r disjoint parallel copies and glue them into the punctures of A. The result is a height n Grope concordance between K and R.

In the general case that there is more than one Hopf link, the same argument works: first, one constructs a punctured annulus A between R and K, embedded in S^3 except for the portion $R \times I$. The punctures of A are now parallels of the link $\theta_1, \dots, \theta_m$, but by assumption they can be filled by disjoint Gropes that miss the interior of A because they lie in the complement of $R \times I$. □

Now we want to describe a particular large class of knots that bound $(n+2)$-Gropes in D^4 by virtue of satisfying Proposition 3.4. We fix a very specific knot J, as defined implicitly by Figure 3.6, which is obtained from the trivial knot U by performing zero framed surgery on the 14-component link shown in Figure 3.6.

The surgery point of view which defines J has enormous flexibility for our purposes. In fact, the picture for J is *redrawn* five times in this article, depending exactly on the aspect of knot theory which we study. For those familiar with the language of claspers (see [H]), J can also be described as the knot obtained by performing clasper surgery on the unknot along the height 2 clasper, as shown in Figure 3.7. (We have used a convention wherein an edge corresponds to a left-handed Hopf clasp.) It was shown in [CT] that J cobounds with the unknot a grope of height 2, embedded in S^3. Here, we do not want to use this 3-dimensional notion and prefer to give a direct construction of a certain height $(n+2)$-Grope concordance. Note, however, that the shift by 2 is a direct consequence of this special feature of J.

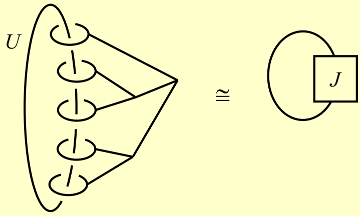

Figure 3.7. A clasper construction of J

Suppose that R is a slice knot. Given any η_i, $1 \leq i \leq m$, disjointly embedded circles forming a trivial link in S^3 such that $[\eta_i] \in \pi_1(S^3 \smallsetminus R)^{(n)}$, we can consider the knot $R(\eta_1, \ldots, \eta_m, J)$ obtained from R by genetic infection along each η_i by the auxiliary knot J (see Figure 1.6).

In Section 4, we show that for many R, there exist certain choices of the *homotopy* classes $[\eta_i]$ such that for any sufficiently large positive integer N, the infected knot $R(\eta_1, \ldots, \eta_m, \#_{j=1}^N J)$ *does not* bound any Grope of height $(n + 2.5)$ in D^4. When we say this, we mean for *any* choice of representatives η_i of $[\eta_i]$. In the rest of Section 3, we prove that $R(\eta_1, \ldots, \eta_m, \#_{j=1}^N J)$ *does* bound a Grope of height $(n + 2)$ in D^4. However, we cannot prove this in all cases. The (*isotopy* classes) η_i need to be chosen carefully to have the stronger *geometric* property that they are height n Grope slice, rel R (see Definition 3.1). We demonstrate that this can be arranged by choosing the isotopy class of each η_i carefully within its homotopy class. We then use Proposition 3.4, resulting in the following theorem.

THEOREM 3.8
Suppose that R is a slice knot, J is the knot of Figure 3.6, N is a positive integer, and $[\eta_i] \in \pi_1(S^3 \smallsetminus R)^{(n)}$, $1 \leq i \leq m$. There exist representatives $\eta_1, \ldots, \eta_m$ disjointly embedded and forming a trivial link in S^3 such that $R(\eta_1, \ldots, \eta_m, \#_{j=1}^N J)$ bounds a Grope of height $(n + 2)$ in D^4, that is, is a height $(n + 2)$ Grope slice.

Proof
We need the very general Lemma 3.9. Here, a *capped Grope* is a Grope equipped with a set of 2-disks (called *caps*) whose boundaries are the (full set of) tips of the grope. In Lemma 3.9, our capped Gropes are embedded in S^3 (although the caps may intersect the knot R). Our only use for the caps is that they are a good way to make sure that the collection of boundary circles η_i of a disjoint union of such capped Gropes is a trivial link in S^3. This follows since “one-half” of the caps can be used to ambiently “surger” the Gropes, producing disjointly embedded disks with boundary η_i, each of

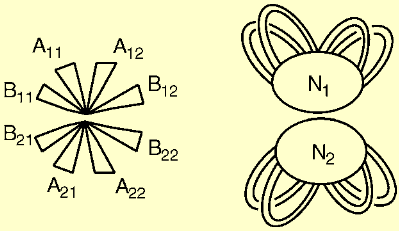

Figure 3.10. The inductive step

which lies in a small regular neighborhood of its corresponding capped Grope. The technique for the following result was used in [FT, Part II, Lemma 2.8].

LEMMA 3.9
Suppose that R is a knot, and suppose that $[\eta_i] \in \pi_1(S^3 \smallsetminus R)^{(n)}$, $1 \leq i \leq m$. Then there exist height n capped Gropes G_i, disjointly embedded in S^3, disjoint from R except for the caps and such that, for each i, ∂G_i is in the homotopy class of $[\eta_i]$.

Proof
The proof is by induction on n. Suppose that $n = 0$. By general position, we can choose disjoint embedded representatives η_i of the $[\eta_i]$. Moreover, by further "crossing changes," we may suppose that η_i forms a trivial link in S^3. Setting $G_i = \eta_i$ completes the case $n = 0$ since a height zero Grope is merely a circle. The set of caps is the set of disks that the η_i bound. Now, suppose the theorem is true for $n - 1$. Suppose that $[\eta_i] = \prod_j [a_{ij}, b_{ij}]$ for some a_{ij}, b_{ij} in $\pi_1(S^3 \smallsetminus K)^{(n-1)}$. Using the induction hypothesis applied to each of the a_{ij} and b_{ij}, choose embedded height $(n-1)$ capped Gropes H_{ij} and L_{ij}, pairwise disjoint except for the base point, so that $[\partial H_{ij}] = a_{ij}$, $[\partial L_{ij}] = b_{ij}$ for all i and all j. Let $A_{ij} = \partial H_{ij}$, and let $B_{ij} = \partial L_{ij}$. We can alter these Gropes to assume that for each fixed i, the A_{i1}, B_{i1}, A_{i2}, $B_{i2}, \ldots$ share a common point (as illustrated for the case i, $j \leq 2$ on the left-hand side of Figure 3.10) with the H_{ij} coming up straight out of the plane of this page and the L_{ij} going down straight below the plane of this page.

Then we can "thicken" this wedge slightly (as shown on the right-hand side of Figure 3.10), avoiding all the caps for H_{ij} and L_{ij}, to form a new embedded surface N_i whose boundary has the homotopy class of $[\eta_i]$. Then $N_i \cup \left(\bigcup_j H_{ij} \cup_j L_{ij}\right)$ forms a height n capped Grope G_i, and these are disjoint for different i. □

Remark 3.11
In Lemma 3.9, S^3 may be replaced by any orientable 3-manifold X as long as the η_i are trivial in $\pi_1(X)$.

Continuing with the proof of Theorem 3.8, by Lemma 3.9 there exist embedded representatives η_i, $1 \leq i \leq m$, forming a trivial link in S^3 which bound disjointly embedded height n Gropes in $S^3 \smallsetminus R$ (with caps that intersect R). We get height n Gropes $G_1, \ldots, G_m$ in $(S^3 \smallsetminus R) \times I$ by pushing slightly into the I-direction, leaving only the boundary in S^3.

For each i, form $5N$ parallel pushoffs of η_i, denoted η_{ijk}, $1 \leq j \leq N$, $1 \leq k \leq 5$. Now, for each fixed i and j, connect the five circles by four short arcs, connecting η_{ij1} to η_{ij2}, η_{ij2} to η_{ij3}, and so forth. For each fixed i and j, *inside a regular neighborhood of the short arcs*, connect the five circles $\eta_{ij1}, \ldots, \eta_{ij5}$ by a collection of arcs to form a copy of the embedded tree, as shown in the left-hand side of Figure 3.7. Then for varying i, j, using the mN trees as guides, replace each tree with a copy L_{ij} of the 14-component link of Figure 3.6. These trees, and hence the links, are pairwise disjoint. Let K denote the result of zero framed surgery on $\coprod_{i,j} L_{ij}$.

First, we claim that K is isotopic to $R(\eta_1, \ldots, \eta_m, \#_{j=1}^N J)$. Since, for fixed (i, j), $\eta_{ij1}, \ldots, \eta_{ij5}$ are parallel, there is a 3-ball B_{ij} such that $(B_{ij}, B_{ij} \cap R)$ is a trivial tangle. Moreover, we may assume that this ball contains L_{ij}. In this way, one sees that the effect of the ijth-surgery is locally the same as the effect of the 14-component link of Figure 3.6 on a trivial tangle. This effect is clearly to tie all the parallel strands of the trivial tangle into the knot J. Moreover, since, for each fixed i, the N circles η_{ij1}, $1 \leq j \leq N$, are also parallel, modifying each by J is the same as infecting a single one, say, η_{i11}, by $\#_{j=1}^N J$. Thus K is the result of genetic modification of R along the original m-circles $\{\eta_i\}$ via $\#_{j=1}^N J$.

Second, we claim that K satisfies the hypotheses of Proposition 3.4 for $(n+2)$, and so K and R are height $(n+2)$ Grope concordant. Since R was assumed slice, the verification of this claim finishes the proof of the theorem. To apply Proposition 3.4, we must establish that K can be viewed as the result of surgery on a disjoint union of Hopf links in the exterior of some slice knot and show that the appropriate circles θ associated to each Hopf link are height $(n+2)$ Grope slice rel R. We focus on one link L_{ij} and order its components $(H_0, \ldots, H_{13})$ as shown in Figure 3.12. Ignore the shading for now.

The pictured link is merely a copy of that shown in Figure 3.6, except that the three copies of the Borromean rings have been altered by isotopy to arrive at the embedding shown in Figure 3.12. We can now perform four handle cancellations, namely, the four pairs on the left read from the bottom as

$$H_{10} - H_5, \qquad H_{11} - H_6, \qquad H_{12} - H_7, \qquad H_{13} - H_8.$$

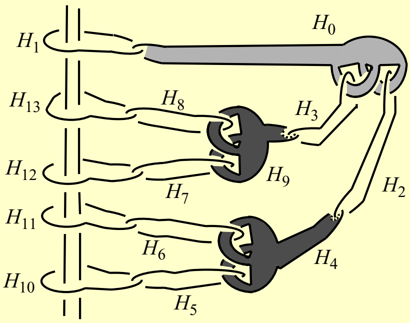

Figure 3.12. The link L_{ij}

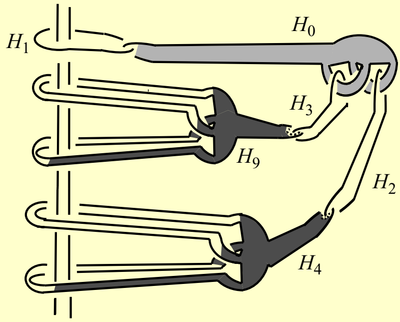

Figure 3.13. The link N_{ij}

What remains is a 6-component link N_{ij}, shown in Figure 3.13.

Let $L := N_{ij} - \{H_0, H_1\}$. Since L consists of two separated Hopf links, surgery along L yields S^3 again. Furthermore, the knot R can be disentangled from these two Hopf links by sliding over H_2 and H_3. Therefore, the image of R in S^3_L remains a slice knot R'. Thus we have shown that K is the image of a slice knot R' in a manifold S^3_L, homeomorphic to S^3, after surgery on a 2-component link (H_0, H_1). Using handle slides over H_4 and H_9, one sees that this link is indeed a Hopf link in S^3_L.

We now describe a height 2 Grope embedded in S^3 whose boundary is θ. Recall from the notation of Figure 3.2 that θ is merely a parallel of H_0, which is short-circuited to avoid linking H_1. The (punctured) torus shown lightly shaded in Figure 3.13 is the first stage of the Grope. The obvious symplectic basis for this torus consists of two circles that are (isotopic to) meridians of H_2 and H_3, shown as dashed in Figure 3.13. The annuli that go between the symplectic basis and the dashed circles are part of the Grope and are often called *pushing annuli*. These dashed meridians in turn bound punctured tori, which are shown darkly shaded (partially) in Figure 3.13. This gives the embedded Grope G of height 2, each of whose surfaces has genus 1, together with four pushing annuli identifying the tips of the Grope with $\{\eta_2, \eta_3, \eta_4, \eta_5\}$. Recall that we are fixing i, j, and hence, there are really mN such Gropes G_{ij} with boundary θ_{ij} embedded disjointly in S^3_L.

The Gropes G_{ij} constitute the first two stages of the height $(n+2)$ Gropes whose boundary is θ_{ij} which we need to exhibit in $(S^3_L \smallsetminus R) \times I$ to verify the hypothesis of Proposition 3.4. The higher stages of this Grope are just $4N$ parallel copies of the height n Gropes G_i bounding η_i which exist by assumption. These are attached to the tips of G_{ij}. This finishes the verification that K satisfies the hypotheses of Proposition 3.4 for $(n+2)$, and so K bounds a Grope in D^4 of height equal to $(n+2)$. This concludes the proof of Theorem 3.8. □

We note in passing that the argument used in the verification that the knot K satisfied the hypotheses of Proposition 3.4 generalizes to establish Corollary 3.14 (which we make no use of in this article). We state it in the language of claspers and the trees that underlie them (see [CT] for definitions). It has generalizations to nonsymmetric trees, but we suppress this for simplicity. Suppose that (T, r) is a uni-trivalent tree T equipped with a distinguished univalent vertex r called the *root*. Observe that each rooted (tree) clasper has an underlying rooted tree (cf. Figure 3.7).

COROLLARY 3.14
Suppose that K is the result of clasper surgery on a knot R along $\coprod_{i=1}^{\ell} C_{(T_i,r_i)}$ (a disjoint union of claspers whose root leaves bound disjoint disks in S^3). Suppose that each (T_i, r_i) is a rooted symmetric tree of height m and that the nonroot leaves of $C_{(T_i,r_i)}$ form a link that is height n Grope slice rel R. *Then K and R are height $(m+n)$ Grope concordant.*

4. Knots that are not $(n.5)$-solvable

In this section, we complete the proof of our main theorem, Theorem 1.4. First, we review the necessary definitions.

For a CW-complex W, we define $W^{(n)}$ to be the regular covering corresponding to the subgroup $\pi_1(W)^{(n)}$. If W is an oriented 4-manifold, then there is an intersection form

$$\lambda_n : H_2(W^{(n)}) \times H_2(W^{(n)}) \longrightarrow \mathbb{Z}[\pi_1(W)/\pi_1(W)^{(n)}].$$

Similarly, there is a self-intersection form μ_n. For $n \in \mathbb{N}_0$, an *n-Lagrangian* is a submodule $L \subset H_2(W^{(n)})$ on which λ_n and μ_n vanish and which maps onto a Lagrangian of λ_0.

Definition 4.1 ([COT1, Sections 7, 8])
A knot is called (n)-*solvable* if M (the zero surgery on the knot K) bounds a spin 4-manifold W, so that the inclusion map induces an isomorphism on first homology (a 4-manifold satisfying only this is called an H_1-*bordism* for M) and so that W admits two *dual* n-Lagrangians. This means that the intersection form λ_n pairs the two Lagrangians nonsingularly and that their images together freely generate $H_2(W)$. Then M is also called (n)-solvable, and W is called an (n)-*solution* for M and K. A knot is called $(n.5)$-*solvable*, $n \in \mathbb{N}_0$, if M bounds a spin 4-manifold W, so that the inclusion map induces an isomorphism on first homology and so that W admits an $(n+1)$-Lagrangian and a dual n-Lagrangian in the above sense. The W is called an $(n.5)$-*solution* for K and M.

Recall also the *rational derived series* $G_r^{(n)}$ of a group G wherein $G_r^{(0)} \equiv G$ and $G_r^{(n+1)} \equiv \{g \mid g^k \in [G_r^{(n)}, G_r^{(n)}]$ for some positive integer $k\}$. The terms of the rational derived series are slightly larger than the terms of the derived series, but they have the key technical advantage that their successive quotients are torsion-free abelian groups (see [Ha, Section 3]).

The following are the major theorems of Section 4. Combined with Theorem 3.8, they give large families of knots that are (n)-solvable but not $(n.5)$-solvable.

THEOREM 4.2
Let R be an (n)-solvable knot ($n \geq 1$), and let M be the zero framed surgery on R. Suppose that there exists a collection of homotopy classes

$$[\eta_i] \in \pi_1(M)^{(n)}, \quad 1 \leq i \leq m,$$

which has the following property. For any *(n)-solution W of M, there exists* some *i such that $j_*(\eta_i) \notin \pi_1(W)_r^{(n+1)}$, where $j_* : \pi_1(M) \longrightarrow \pi_1(W)$.*

Then for any oriented trivial link $\{\eta_1, \ldots, \eta_m\}$ in $S^3 \smallsetminus R$ which represents the $[\eta_i]$, and for any m-tuple $\{J_1, \ldots, J_m\}$ of Arf invariant zero knots for which $\rho_{\mathbb{Z}}(J_i) > C_M$ (the Cheeger-Gromov constant of M), the knot

$$K = R(\eta_1, \ldots, \eta_m, J_1, \ldots, J_m)$$

formed by genetic infection is (n)-solvable but not $(n.5)$-solvable. Moreover, K is of infinite order in $\mathscr{F}_n/\mathscr{F}_{n.5}$.

THEOREM 4.3
Suppose that R is a genus 2 fibered knot that is (n)-solvable (e.g., a genus 2 fibered ribbon knot). Then there exists a collection of homotopy classes satisfying the hypotheses of Theorem 4.2.

Proof of Theorem 4.2
Let V' be an (n)-solution for M. Let N be the zero surgery on K. Using V', we show that N (and hence, K) is (n)-solvable. Moreover, by assuming the existence of an $(n.5)$-solution V for N, we derive a contradiction.

There is a standard cobordism C between M and N which can be described as follows. For each $1 \le i \le m$, choose a spin 4-manifold W_i whose boundary is M_{J_i}, the zero surgery on J_i, so that $\pi_1(W_i) \cong \mathbb{Z}$ generated by a meridian of J_i and so that the intersection form on $H_2(W_i)$ is a direct sum of hyperbolic forms. Such a W_i is a zero solution of M_{J_i}. It exists whenever the Arf invariant of J_i is zero. Now form C from $M \times [0, 1]$ and $\coprod_{i=1}^m W_i$ by identifying (for each i) the solid torus in $\partial W_i \equiv (S^3 \setminus J_i) \cup (S^1 \times D^2)$ with a regular neighborhood of $\eta_i \times \{1\}$ in such a way that a meridian of J_i is glued to a longitude of η_i and a longitude of J_i is glued to a meridian of η_i. Then $\partial_+ C \cong N$ and $\partial_- C \cong M$. Also observe that C can be assumed to be spin by changing the spin structure on W_i if necessary.

Let $W = C \cup V$, and let $W' = C \cup V'$. We claim that W is an (n)-solution for M and that W' is an (n)-solution for N. Clearly, W is an H_1-bordism for M, W' is an H_1-bordism for N, and

$$H_1(M) \cong H_1(C) \cong H_1(N) \cong H_1(V) \cong H_1(W) \cong H_1(W') \cong \mathbb{Z},$$

all induced by inclusion. Thus V and $N = \partial V$ have two distinct spin structures, and changing the spin structure on V changes that induced on ∂V (and similarly for M and V'). Hence, spin structures on the manifolds V and V' can be chosen to agree with those induced on N and M by the spin manifold C. Thus W and W' are spin. It is easy to see that $H_2(C)$ is isomorphic to $\bigoplus_{i=1}^m H_2(W_i) \oplus H_2(M)$ by examining the Mayer-Vietoris sequence for $C \cong (M \times [0, 1]) \cup \left(\coprod_{i=1}^m W_i\right)$. By a similar sequence for $W \cong V \cup C$, one sees that

$$H_2(W) \cong \big(H_2(C) \oplus H_2(V)\big)/i_*\big(H_2(N)\big).$$

Since $N \longrightarrow V$ induces an isomorphism on H_1, one easily sees that it induces the zero map on H_2. Moreover, a generator of $H_2(N)$ under the map

$$H_2(N) \longrightarrow H_2(C) \cong \bigoplus_{i=1}^m H_2(W_i) \oplus H_2(M)$$

goes to a generator of $H_2(M)$ since it is represented by a capped-off Seifert surface for R which can be chosen to miss the η_i (since $n \ge 1$). It follows that

$$H_2(W) \cong \bigoplus_{i=1}^m H_2(W_i) \oplus H_2(V).$$

Similarly, $H_2(W') \cong \bigoplus_{i=1}^m H_2(W_i) \oplus H_2(V')$. Since V is an $(n.5)$-solution for N, it is an (n)-solution for N, so there exists an n-Lagrangian with n-duals. This may be thought of as collections $\mathscr{L}$ and $\mathscr{D}$ of based immersed surfaces that lift to the $\pi_1(V)^{(n)}$ cover of V (called n-surfaces in [COT1, Sections 7, 8]), that together constitute a basis of $H_2(V;\mathbb{Z})$, and that satisfy

$$\lambda_n(\ell_i, \ell_j) = \mu_n(\ell_i) = 0, \qquad \lambda_n(\ell_i, d_j) = \delta_{ij}$$

for the intersection and self-intersection forms with $\mathbb{Z}[\pi_1(V)/\pi_1(V)^{(n)}]$-coefficients. These same collections are certainly n-surfaces in W since $\pi_1(V)^{(n)}$ maps into $\pi_1(W)^{(n)}$ under the inclusion. Since the equivariant intersection form can be calculated from the collections $\mathscr{L}$ and $\mathscr{D}$ (or by naturality), these also retain the above intersection properties with $\mathbb{Z}[\pi_1(W)/\pi_1(W)^{(n)}]$-coefficients. Now consider collections of zero Lagrangians $\mathscr{L}_i$ and zero duals $\mathscr{D}_i$ for the zero solutions W_i. Since the map

$$\pi_1(W_i) \longrightarrow \pi_1(C) \longrightarrow \pi_1(W)/\pi_1(W)^{(n)}$$

is zero (since $\pi_1(W_i)$ is generated by η_i), these zero surfaces are n-surfaces in W. Since the union of all of these collections is a basis for $H_2(W)$, it constitutes an n-Lagrangian and n-duals for W. Thus W is an (n)-solution for M. Similarly, W' is an (n)-solution for N. In particular, K is (n)-solvable.

Now, let $\Gamma = \pi_1(W)/\pi_1(W)^{(n+1)}_r$. It is straightforward to verify that Γ is an n-solvable poly-(torsion-free-abelian) group (PTFA) (see [Ha, Section 3]). Let $\psi : \pi_1(W) \longrightarrow \Gamma$ be the projection. Let ϕ and ϕ' denote the induced maps on $\pi_1(M)$ and $\pi_1(N)$, respectively.

PROPOSITION 4.4
The ρ-invariants of M and N are related by

$$\rho_\Gamma(M, \phi) - \rho_\Gamma(N, \phi') = \sum_{i=1}^m \epsilon_i \rho_{\mathbb{Z}}(J_i),$$

where $\epsilon_i = 0$ or 1 according to whether $\phi(\eta_i) = e$ or not.

Proof
By [COT2, Proposition 3.2],

$$\rho_\Gamma(M, \phi) - \rho_\Gamma(N, \phi') = \sum_{i=1}^m \rho_\Gamma(M_{J_i}, \psi_{|\pi_1(M_{J_i})}).$$

Since $\psi_{|\pi_1(M_{J_i})}$ factors through $\mathbb{Z}$ generated by η_i, its image is zero if $\epsilon_i = 0$ and is $\mathbb{Z}$ if $\epsilon_i = 1$. In the former case, $\rho_\Gamma = 0$ since $\sigma^{(2)}_\Gamma$ is then the ordinary signature. In the latter case, $\rho_\Gamma = \rho_{\mathbb{Z}}$ (replacing Γ by the image of $\psi_{|\pi_1(M_{J_i})}$) by [COT1, Proposition 5.13]. But $\rho_{\mathbb{Z}}(M_{J_i}, \psi_{|\pi_1(M_{J_i})})$ is just what we have called $\rho_{\mathbb{Z}}(J_i)$. □

Moreover, since V is assumed to be an $(n.5)$-solution for N and ϕ' extends to $\pi_1(V)$, $\rho_\Gamma(N, \phi') = 0$ by [COT1, Theorem 4.2]. Finally, by hypothesis, there exists some i such that $\phi(\eta_i) \neq e$. Thus, by Proposition 4.4, $|\rho_\Gamma(M, \phi)| > C_M$, a contradiction. Therefore, no $(n.5)$-solution exists for N; that is to say, K is not $(n.5)$-solvable.

Now suppose that $K = K(R, \eta, J)$ are of finite order $k > 0$ in $\mathscr{F}_n/\mathscr{F}_{n.5}$. Let $N_\#$ denote the zero framed surgery on $\#_{j=1}^k K$, and let $V_\#$ denote an $(n.5)$-solution for $N_\#$. We arrive at a contradiction, implying that K is of infinite order. There is a standard cobordism D from $\coprod_{j=1}^k N_j$ (where N_j is the jth copy of N) to $N_\#$ which is obtained from $\coprod_{j=1}^k N_j \times [0, 1]$ by adding $(k-1)$ 1-handles and then $(k-1)$ 2-handles. After just adding the 1-handles, one has a cobordism from $\coprod_{j=1}^k N_j$ to the zero framed surgery on the split link consisting of the disjoint union of K_j. Next, add zero framed 2-handles to loops, the jth, $2 \leq j \leq k$, of which goes once around the meridian of K_1 and once around the meridian of K_j. Using Kirby's calculus of framed links, one sees that the resulting 3-manifold is homeomorphic to $N_\#$ (see [COT2, page 113]). Let W denote the 4-manifold

$$W := V_\# \cup_{N_\#} D \cup \Big(\coprod_{j=1}^k C_j \Big),$$

where C_j is the jth copy of the standard cobordism C (described at the start of Theorem 4.2) between M and N. Let $\Gamma = \pi_1(W)/\pi_1(W)_r^{(n+1)}$, and let $\psi : \pi_1(W) \to \Gamma$ be the projection. Let ϕ_j, ϕ'_j, and ϕ denote the restrictions of ψ to $\pi_1(M_j)$, $\pi_1(N_j)$, and $\pi_1(N_\#)$, respectively. Since D is a homology cobordism with coefficients in $\mathscr{K}(\Gamma)$, the quotient field of $\mathbb{Z}\Gamma$ (see [COT2, Lemma 4.2]), $\sum_{j=1}^k \rho(N_j, \phi'_j) = \rho(N_\#, \phi_\#)$. Since $V_\#$ is an $(n.5)$-solution for $N_\#$ over which $\phi_\#$ extends, $\rho(N_\#, \phi_\#) = 0$ by [COT1, Theorem 4.2]. Thus,

$$\sum_{j=1}^k \rho(M_j, \phi_j) = \sum_{j=1}^k \sum_{i=1}^m \epsilon_{ji} \rho_{\mathbb{Z}}(J_i),$$

where $\epsilon_{ji} = 0$ if $\phi_j(\eta_{ji}) = e$ and $\epsilon_{ji} = 1$ if $\phi_j(\eta_{ji}) \neq e$. We assert that for each j, there exists some i such that $\phi_j(\eta_{ji}) \neq e$. Thus $\left| \sum_{j=1}^k \rho(M_j, \phi_j) \right| > kC_M$, which is a contradiction since $|\rho(M, \phi_j)| < C_M$. To establish the assertion, fix $j = j_0$, and consider the 4-manifold

$$\overline{W} = W \cup \Big(\bigcup_{\substack{j=1 \\ j \neq j_0}}^k -W_j \Big),$$

where W_j is a copy of the (n)-solution for M discussed above. Then $\partial\overline{W} = M_{j_0}$. By an argument as above, one sees that $\overline{W}$ is an (n)-solution for M_{j_0}. By the assumption in Theorem 4.2, there exists some i (depending on j_0) such that $j_*(\eta_{j_0 i}) \notin \pi_1(\overline{W})_r^{(n+1)}$, where $j : M_{j_0} \to \overline{W}$ is the inclusion. Since this inclusion factors $M_{j_0} \xrightarrow{i_*} W \longrightarrow \overline{W}$, it follows that $i_*(\eta_{j_0 i}) \notin \pi_1(W_r^{(n+1)})$. Hence $\phi_{j_0}(\eta_{j_0 i}) \neq e$ in Γ. This establishes the assertion and completes the proof of Theorem 4.2. □

We can now give the proof of Theorem 1.4.

Proof of Theorem 1.4

First, we show that $\mathscr{F}_n/\mathscr{F}_{n.5}$ contains an element of infinite order. We may assume that $n \geq 1$ since this result was known previously for $n = 0$, 1, and 2. There exist genus 2 fibered ribbon knots, for example, the connected sum of two figure-eight knots. Hence, by Theorem 4.3, there exist an (n)-solvable knot R and classes $\{[\eta_i]\}$ satisfying the hypotheses of Theorem 4.2. Certainly, there exist representatives of these classes which form a trivial link since we can alter any collection by crossing changes to achieve this. There also exist Arf invariant zero knots with $\rho_{\mathbb{Z}} > C_M$, for example, the connected sum of a suitably large *even* number of left-handed trefoil knots. Then Theorem 4.2 implies that the knot K formed by genetic infection of R along any such collection $\{\eta_i\}$ using any such $\{J_i\}$ is of infinite order in $\mathscr{F}_n/\mathscr{F}_{n.5}$.

Now, in order to prove the other statements of Theorem 1.4, we just need to redo the proof of the previous paragraph and be a little more careful in choosing the infection knots. In fact, the shift by two for the inclusion $\mathscr{G}_{n+2} \leq \mathscr{F}_n$ is related to the fact that the knot J from Figure 3.6 by construction cobounds, with the unknot, a Grope of height 2 in S^3, yet it also satisfies the essential nontriviality condition that the integral, over the circle, of the Levine-Tristram signature function of J is *nonzero*.

Let R be a genus 2 fibered ribbon knot. Let M be the zero surgery on R, and let N be an even integer greater than C_M, where C_M is the *Gromov-Cheeger constant* for M. Given n, by Theorem 4.3 there exist homotopy classes $[\eta_i] \in \pi_1(S^3 \smallsetminus R)^{(n)}$, $1 \leq i \leq m$, with the property that for any (n)-solution W of M, there exists some i such that $[\eta_i]$ is not mapped to zero under the homomorphism $\pi_1(M) \to \pi_1(W) \to \pi_1(W)/\pi_1(W)_r^{(n+1)}$. Let J be the knot shown in Figure 3.6. Applying Theorem 3.8, we get representatives $\eta_1, \ldots, \eta_m$ disjointly embedded, forming a trivial link in S^3 such that $K = R(\eta_1, \ldots, \eta_m, \#_{j=1}^N J)$ bounds a Grope of height $(n+2)$ in D^4. Hence, $R(\eta_1, \ldots, \eta_m, \#_{j=1}^N J) \in \mathscr{G}_{n+2}$. Yet by Theorem 4.2, $\#^\ell K$ is not $(n.5)$-solvable for any $\ell \geq 1$ since, by Lemma 4.5, $\rho_{\mathbb{Z}}(\#_{j=1}^N J) = N\rho_{\mathbb{Z}}(J)$ is strictly more than C_M. Note that the Arf invariant of J is zero since our 4-manifold W is spin. Thus $\#^\ell K \notin \mathscr{F}_{n.5}$, and it follows a fortiori that $\#^\ell K$ does not bound any $(n+2.5)$-Grope in D^4.

LEMMA 4.5
The von Neumann signature of J associated to the infinite cyclic cover, denoted $\rho_{\mathbb{Z}}(J)$, satisfies

$$\rho_{\mathbb{Z}}(J) = \frac{4}{3}.$$

In fact, J has the same Levine-Tristram signatures as the left-handed trefoil knot.

Proof
Recall that we can compute $\rho_{\mathbb{Z}}(J)$ from any $\mathbb{Z}$-bordism W for M_J as follows. Let M be a matrix for the intersection form on $H_2(W; \mathbb{Z}[\mathbb{Z}])$. Then the integral over the unit circle of the signature of $M(\omega)$ gives the von Neumann signature of W associated to the map to $\mathbb{Z}$, $\sigma_{\mathbb{Z}}(W)$, and by definition, $\rho_{\mathbb{Z}}(J) = \sigma_{\mathbb{Z}}(W) - \sigma(W)$. Recall that J is obtained from the unknot U by zero framed surgery on the 14-component link shown in Figure 3.6 (see also Figure 3.12 for the labeling of the components). This picture also encodes an assortment of possible 4-manifolds W, each with boundary the zero surgery M_J. This is seen as follows. Consider labeling some of the components of the 14-component link by dots, and label the remainder by zeros. Also label the unknot U by a dot, subject to the restriction that the collection A, of all components labeled with dots constitutes a trivial link in S^3, hence bounds a collection of disjointly embedded 2-disks in the 4-ball. Let V be the exterior of this collection of disks in the 4-ball. Now interpret the remaining circles with zero labels as zero framed 2-handles attached to $S^3 - A$, which is part of the boundary of V. The result is a 4-manifold W.

Since V is diffeomorphic to the manifold obtained from a 4-ball by *adding* 1-handles (one for each dotted component), one can think of the choices above as deciding which components of the 14-component link represent 1-handles and which represent 2-handles. Also, since V has the same *boundary* as the manifold obtained by adding zero framed 2-handles along the trivial link represented by A, the boundary of *any* such W is the same as doing zero framed surgery on *all* the components, namely, zero framed surgery on J.

The best decision for our purposes is as shown in Figure 4.6. Note that the Borromean rings formed by $\{H_0, H_2, H_3\}$ have been altered by an isotopy to yield a different embedding than that shown in Figure 3.12. This allows for six $(1-2)$-handle cancellations, so that the diagram can be simplified by handle slides to leave only one 1-handle (corresponding to U) and two 2-handles (corresponding to H_2 and H_4). The resulting handle decomposition is shown in Figure 4.7.

It follows that $\pi_1 W \cong \mathbb{Z}$, generated by a meridian, and that $\pi_2 W$ is free on the two 2-handles H_2, H_4 in the sense that the core of a 2-handle leads to a 2-sphere in W if the attaching circle is null homotopic in the 4-ball (minus the disk corresponding to the 1-handle). In fact, the circle labeled H_4 bounds an immersed 2-disk that can be

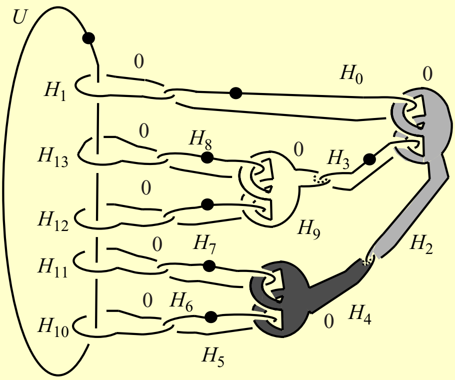

Figure 4.6. A 4-manifold with boundary M_J

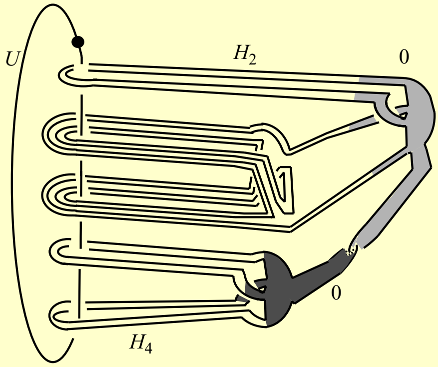

Figure 4.7. The 4-manifold W

seen as follows. Note that H_4 bounds the (punctured) torus shown darkly shaded in Figure 4.7. But the two circles representing a symplectic basis for this torus are freely homotopic meridians of U. Thus the circle labeled H_4 in Figure 4.7 can be altered by a local homotopy (no intersections with U) until it is undone from U. Similarly, H_2 winds around U in a way that is precisely a "doubling" of the way H_4 wraps around U in Figure 4.7. It follows that H_2 is homotopic to a circle that is undone from U. This implies

$$W \simeq S^1 \vee S^2 \vee S^2.$$

Examining the intersections created during these homotopies (in the group ring $\mathbb{Z}[\mathbb{Z}]$) gives the intersection form λ on $\pi_2 W$. For example, since the two homotopies above are disjoint, there are immersed 2-spheres representing $[H_2]$ and $[H_4]$ which intersect in precisely one point. Therefore $\lambda([H_2], [H_4]) = 1$. It turns out that the intersection matrix is given by

$$\lambda = \begin{pmatrix} -t - t^{-1} + 2 & 1 \\ 1 & -t - t^{-1} + 2 \end{pmatrix}.$$

Since the determinant $(t + t^{-1} - 2)^2 - 1$ of λ factors as $(t + t^{-1} - 1)(t - 3 + t^{-1})$, its only zeros on the unit circle are the two primitive 6th roots of unity. These are exactly the zeros of the first factor, which is the Alexander polynomial of the trefoil knot. Since the signature function on the unit circle is constant except at these values, it suffices to check that the signature of λ is $+2$ when substituting $t = -1$. Note that if both diagonal entries of λ are changed in sign, then the analysis is the same except that J has the signature of the right-handed trefoil knot. Since all we really need is that the integral of the signature function is *nonzero*, this also suffices. □

This completes the proof of Theorem 1.4 modulo the proof of Theorem 4.3. □

5. Higher-order Alexander modules and Blanchfield duality

We have completed the proofs of our main theorems modulo the proof of Theorem 4.3, which is a deep result concerning the nontriviality of the inclusion maps $\pi_1(M_R) \to \pi_1(W)$, where W is an arbitrary (n)-solution for the knot R. To underscore the nature and difficulty of the task, note that it is a direct consequence of Theorem 4.3 that if R is a genus 2 fibered ribbon knot and W is the exterior in B^4 of *any* slice disk for R, then $\pi_1(W)$ *is not a solvable group*! For such a W is an (n)-solution for every n, and if $\pi_1(W)^{(n)} = 0$, then for any $\eta_i \in \pi_1(M_R)^{(n)}$, $j_*(\eta_i) = 0$. To prove Theorem 4.3, we need the full power of the duality results of [COT1, Section 2], which we here review for the convenience of the reader.

In [COT1], it was observed that the classical Alexander module and the classical Blanchfield linking form have higher-order generalizations that can be defined using noncommutative algebra. These modules reflect the highly nonsolvable nature of the fundamental group of the knot exterior. Using these modules as measures of nonsolvability, it was shown that if a knot is a slice knot with slice disk Δ, then $\pi_1(B^4 - \Delta)$ is not arbitrary but rather is constrained by $\pi_1(S^3 - K)$. It is Poincaré duality with twisted coefficients which provides connections between the higher-order modules of $B^4 - \Delta$ and those of $S^3 - K$. These constraints are expressed through higher-order Blanchfield duality. We also obtained similar results if K is an (n)-solvable knot.

More specifically, recall that the classical Alexander module and Blanchfield form are associated to the infinite cyclic cover $X_{\mathbb{Z}}$ of the exterior X of a knot K in S^3, being merely $H_1(X_{\mathbb{Z}}, \mathbb{Z})$ viewed as a module over $\mathbb{Z}[t^{\pm 1}]$. More generally, consider any regular covering space X_Γ that is obtained by taking *iterated* covering spaces, each with torsion-free abelian covering group. The group of covering translations Γ of such a cover is PTFA and hence solvable and torsion-free. We call $H_1(X_\Gamma, \mathbb{Z})$, viewed as a module over $\mathbb{Z}\Gamma$, a *higher-order Alexander module*. Even in the classical case, the ring $\mathbb{Z}[t^{\pm 1}]$ can be simplified by localizing the coefficient group $\mathbb{Z}$ to get $\mathbb{Q}$ and considering the *classical rational Alexander module* $\mathscr{A}_\mathbb{Z}$ a module over the principal ideal domain (PID) $\mathbb{Q}[t^{\pm 1}]$. Similarly, after appropriate localization of the ring $\mathbb{Z}\Gamma$, the higher-order modules become finitely generated torsion modules $\mathscr{A}_\Gamma$ over a noncommutative PID $\mathbb{K}[t^{\pm 1}]$ (see [COT1, Proposition 2.11, Corollary 3.3]). Briefly, this is seen as follows. There is a split exact sequence

$$1 \to \Gamma' \to \Gamma \to \mathbb{Z} \to 1,$$

where Γ' is the commutator subgroup of Γ. It follows that $\mathbb{Z}\Gamma$ can be identified with a twisted Laurent polynomial ring $\mathbb{Z}\Gamma'[t^{\pm 1}]$. Because Γ is torsion-free and solvable, $\mathbb{Z}\Gamma$ and $\mathbb{Z}\Gamma'$ are right Ore domains and thus admit classical right quotient fields $\mathscr{K}$ and $\mathbb{K}$. Let $\mathscr{R}$ denote the ring obtained from $\mathbb{Z}\Gamma$ by inverting the nonzero elements of $\mathbb{Z}\Gamma'$, so that

$$\mathbb{Z}\Gamma \subset \mathscr{R} \subset \mathscr{K}.$$

The localized ring $\mathscr{R}$ is then identified with the twisted Laurent polynomial ring $\mathbb{K}[t^{\pm 1}]$ with coefficients in the skew field $\mathbb{K}$. Thus $\mathscr{R}$ is seen to be a (noncommutative) PID by considering the obvious degree function on the polynomial ring. There is a classification theorem for modules over a noncommutative PID analogous to that for a commutative PID. Consequently, the finitely generated modules

$$\mathscr{A}_\Gamma \cong \bigoplus_i \big(\mathscr{R}/p_i(t)\mathscr{R}\big)$$

then have a measurable "size," namely, their ranks over $\mathbb{K}$ (which are directly interpretable as the degrees of "higher-order Alexander polynomials" of the knot K). These ranks are an important measure of nontriviality in the proofs of Section 6. Henceforth, we restrict our attention to these (*localized*) higher-order Alexander modules.

Slightly more generally, if X is any compact space and $\phi : \pi_1(X) \to \Gamma$ is any homomorphism where Γ is a PTFA group such that $\Gamma/\Gamma' \cong \mathbb{Z}$, then we may associate a (localized) higher-order Alexander module $\mathscr{A}_\Gamma(X)$ as above. Viewing ϕ as a system of twisted coefficients on X, note that $\mathscr{A}_\Gamma(X)$ is merely $H_1(X, \mathscr{R}) \cong H_1(X, \mathbb{Z}\Gamma) \otimes \mathscr{R}$. We apply this to two spaces: first to M, the 3-manifold obtained by zero framed surgery on K, and second to a 4-manifold W whose boundary is M. We see in Theorem 5.2 that if W is the complement of a slice disk for K or, more generally, if W is an (n)-solution for M, then the Alexander modules of M and W are strongly related.

Moreover, we can define a higher-order linking form

$$B\ell_\Gamma : \mathscr{A}_\Gamma(M) := H_1(M;\mathscr{R}) \to \mathrm{Hom}_{\mathscr{R}}\left(\mathscr{A}_\Gamma(M), \mathscr{K}/\mathscr{R}\right) =: \mathscr{A}_\Gamma(M)^\#$$

as the composition of the Poincaré duality isomorphism to $H^2(M;\mathscr{R})$, the inverse of the Bockstein to $H^1(M;\mathscr{K}/\mathscr{R})$, and the usual Kronecker evaluation map to $\mathscr{A}_\Gamma(M)^\#$. In the case where $\phi : \pi_1(M) \to \Gamma$ is the abelianization map to $\mathbb{Z}$, we recover the classical $\mathscr{A}_\mathbb{Z}$ and $B\ell_\mathbb{Z}$.

THEOREM 5.1 *([COT1, Theorem 2.13])*
There is a nonsingular Hermitian linking form

$$B\ell_\Gamma : H_1(M;\mathscr{R}) \to H_1(M;\mathscr{R})^\#$$

defined on the higher-order Alexander module $\mathscr{A}_\Gamma(M) = H_1(M;\mathscr{R})$.

THEOREM 5.2
Suppose that M is (n)-solvable via W, and suppose that $\phi : \pi_1(W) \to \Gamma$ is a nontrivial coefficient system, where $\Gamma^{(n)} = 0$. Then the linking form $B\ell_\Gamma(M)$ is hyperbolic, and in fact, the kernel $\mathscr{P}$ of

$$j_* : H_1(M;\mathscr{R}) \to H_1(W;\mathscr{R})$$

is self-annihilating. ($\mathscr{P} = \mathscr{P}^\perp$ with respect to $B\ell_\Gamma$.)

It follows easily that the induced map $B\ell_\Gamma : \mathscr{P} \to (\mathscr{A}_\Gamma(M)/\mathscr{P})^\#$ is an isomorphism (see [COT1, Lemma 2.14]) and thus, by examining the exact sequence

$$0 \to \mathscr{P} \to \mathscr{A}_\Gamma(M) \to \mathscr{A}_\Gamma(M)/\mathscr{P} \to 0,$$

that $\mathrm{rank}_\mathbb{K}(\mathscr{P}) = \mathrm{rank}_\mathbb{K}(\mathscr{A}_\Gamma(M)/\mathscr{P}) = (1/2)\,\mathrm{rank}_\mathbb{K}(\mathscr{A}_\Gamma(M))$. Therefore we obtain the following crucial consequence of these theorems: the “size” of the *image* of $j_* : H_1(M;\mathscr{R}) \to H_1(W;\mathscr{R})$, as measured by the rank over $\mathbb{K}$, is *precisely one-half* of the “size” of $\mathscr{A}_\Gamma(M)$. In particular, as long as the latter is nontrivial, the image is nontrivial. But the “size” of the higher-order Alexander modules is easy to establish for fibered knots and turns out to be bounded below by $d-2$, where $d = \mathrm{rank}_\mathbb{Q}(\mathscr{A}_\mathbb{Z}(M))$ is the degree of the classical Alexander polynomial of K (whence the restriction in Theorem 4.3 to knots with $d = 4$). In [C, Corollary 4.8], this bound was indeed implicitly established for *any* knot. Indeed, in [Ha], such bounds were established explicitly for any closed 3-manifold using the Alexander norm. But in the fibered case, the situation is much simpler.

The nontriviality that results from this sequence of ideas is the key element in the proof of Theorem 4.3 in Section 6, where more details are given.

6. Proof of Theorem 4.3

Suppose that R is a genus 2 fibered knot in S^3, and suppose that M is the result of zero framed surgery on R. Suppose also that W is an (n)-solution for M. The principle behind Theorem 4.3 is as follows. If Γ is some coefficient system for W such that $\Gamma^{(n)} = 1$, then one may expect $H_1(M;\mathbb{Z}\Gamma)$ to be finitely generated and generated by nth-order commutators of $\pi_1(M)$. Neither is generally true. Certainly, any nth-order commutator of $\pi_1(M)$ represents an element of $H_1(M;\mathbb{Z}\Gamma)$, and certainly, $H_1(M;\mathbb{Z}\Gamma)$ is finitely generated *after* localizing the coefficient system to $\mathcal{R}$. In this section, we show that not only can we choose a finite number of nth-order commutators of $\pi_1(M)$ which span $H_1(M;\mathcal{R})$ for any particular W and Γ, but we show that such a finite set can be chosen *independently* of W and Γ. By the techniques of Section 5, roughly one-half of these must survive in $H_1(W;\mathcal{R})$. This then translates quickly into the π_1-statement that some of the nth-order commutators of $\pi_1(M)$ must survive in $\pi_1(W)/\pi_1(W)^{(n+1)}$, giving our desired injectivity result. This result is quite delicate and somewhat mysterious.

Note that M fibers over S^1 with fiber a closed genus 2 surface Σ. Let S denote $\pi_1(\Sigma)$, which can be identified (under inclusion) with $\pi_1(M)^{(1)}$. The inclusion $j : M \longrightarrow W$ induces a map $j : S \longrightarrow \pi_1(W)^{(1)}$. Let $G = \pi_1(W)^{(1)} = \pi_1(W)^{(1)}_r$. From this data, provided by the (n)-solution W, we abstract certain algebraic properties and call this an *algebraic* (n)-*solution*. For Definition 6.1, we let $F = F\langle x_1, x_2, x_3, x_4\rangle$ be the free group and consider $F \twoheadrightarrow S$ in the standard way with $\{x_i\}$ corresponding to a symplectic basis of $H_1(S)$ so that the kernel is normally generated by $[x_1, x_2][x_3, x_4]$. We adopt the shorthand $G_k = G/G_r^{(k)}$.

Definition 6.1

A homomorphism $r : S \to G$ is called an *algebraic* (n)-*solution* $(n \geq 1)$ if the following hold:

(1) $r_* : H_1(S;\mathbb{Q}) \longrightarrow H_1(G;\mathbb{Q})$ has 2-dimensional image, and after possibly reordering $\{x_1, x_2\}$ and $\{x_3, x_4\}$, $r_*(x_1)$ and $r_*(x_3)$ are nontrivial;

(2) for each $0 \leq k \leq n-1$, the following composition is nontrivial even after tensoring with the quotient field $\mathbb{K}(G_k)$ of $\mathbb{Z}G_k$:

$$H_1(S;\mathbb{Z}G_k) \xrightarrow{r_*} H_1(G;\mathbb{Z}G_k) \cong G_r^{(k)}/[G_r^{(k)}, G_r^{(k)}] \twoheadrightarrow G_r^{(k)}/G_r^{(k+1)}.$$

We note that if $r : S \longrightarrow G$ is an algebraic (n)-solution, then for any $k < n$, it is an algebraic k-solution.

PROPOSITION 6.2
The map $j : S \longrightarrow G$ (induced by the (n)-solution W above) is an algebraic (n)-solution.

We postpone the proof of this proposition.

THEOREM 6.3
For each $n \geq 0$, there is a finite set $\mathscr{P}_n$ of pairs of elements of $F^{(n)}$ with the following property. For any algebraic (n)-solution $r : S \longrightarrow G$ (no condition for $n = 0$), at least one such pair (which is called a special pair *for r) maps to a $\mathbb{Z}G_n$-linearly independent set under the composition*

$$F^{(n)} \longrightarrow S^{(n)}/S^{(n+1)} \cong H_1(S; \mathbb{Z}S_n) \xrightarrow{r_*} H_1(S; \mathbb{Z}G_n).$$

Assuming this theorem and Proposition 6.2, we finish the proof of Theorem 4.3. Apply Theorem 6.3 to find a finite set $\mathscr{P}_{n-1}$ of pairs of elements of $F^{(n-1)}$. Since $S^{(n-1)} = \pi_1(M)^{(n)}$, the union of the elements of $\mathscr{P}_{n-1}$ is a finite set $\{\alpha_1, \ldots, \alpha_m\}$ of elements of $\pi_1(M)^{(n)}$, as required by Theorem 4.3. Suppose that W is an (n)-solution for M. Then by Proposition 6.2, the induced map $j : S \longrightarrow G$ is an algebraic (n)-solution. Suppose that $n \geq 2$. Since j is also an algebraic $(n-1)$-solution, by Theorem 6.3 at least one pair $(y, z) \in \mathscr{P}_{n-1}$ spans a 2-dimensional subspace (over $\mathbb{K}(G_{n-1})$) of $H_1(S; \mathbb{K}(G_{n-1}))$. But we claim that if $n \geq 2$, $H_1(S; \mathbb{K}(G_{n-1}))$ has rank 2, so this subspace is all of $H_1(S; \mathbb{K}(G_{n-1}))$. To establish this claim, observe that the Euler characteristic of Σ can be computed using homology with coefficients in the skew field $\mathbb{K}(G_{n-1})$ (cf. [C, page 357]). Thus $-2 = \chi(\Sigma) = b_0 - b_1 + b_2$, where b_i is the rank of $H_i(\Sigma; \mathbb{K}(G_{n-1})) = H_i(S; \mathbb{K}(G_{n-1}))$. The coefficient system $S \to G/G_r^{(1)}$ is nontrivial by Definition 6.1(1). Hence $b_0 = 0$ (see [COT1, Proposition 2.9]), and consequently, b_2 is also zero by Poincaré duality. Thus $b_1 = 2$, as claimed. Then, applying Definition 6.1(2) of an algebraic (n)-solution (with $k = n - 1$), we see that at least one of $\{y, z\}$ maps nontrivially under

$$F^{(n-1)} \longrightarrow S^{(n-1)} \xrightarrow{j} G_r^{(n-1)}/G_r^{(n)}.$$

If $n = 1$, at least one of $\{x_1, x_2\}$ maps nontrivially by Definition 6.1(1) of an algebraic 1-solution. Hence, in any case, at least one α_i has the property that $j_*(\alpha_i) \notin G_r^{(n)} = \pi_1(W)_r^{(n+1)}$. Since each α_i actually comes from $F^{(n-1)}$, and since $\pi_1(M - \Sigma) \cong S$, we can represent the α_i by simple closed curves in the complement of Σ and hence in the exterior of a Seifert surface (in S^3) for the knot R. This is the collection $\{\eta_i\}$ required by Theorem 4.3, whose proof is thus completed. □

Proof of Proposition 6.2

Since M is (n)-solvable via W, Theorem 6.4 applies with $\Gamma = \mathbb{Z}$ and $n = 1$ to show that $j_* : H_1(M; \mathbb{Q}[t, t^{-1}]) \longrightarrow H_1(W; \mathbb{Q}[t, t^{-1}])$ has rank $r/2$ over $\mathbb{Q}$, where $r = \operatorname{rank}_{\mathbb{Q}} H_1(M_\infty; \mathbb{Q})$. But for a fibered knot, $H_1(M; \mathbb{Q}[t, t^{-1}]) \cong H_1(M_\infty; \mathbb{Q})$ is equal to $H_1(S; \mathbb{Q})$; on the other hand, $H_1(W; \mathbb{Q}[t, t^{-1}])$ is given by

$$\pi_1(W)^{(1)}/\pi_1(W)^{(2)} \otimes \mathbb{Q} \cong G/G^{(1)} \otimes \mathbb{Q} \cong H_1(G; \mathbb{Q}).$$

For a genus 2 fibered knot, $r = 4$, and so j_* has 2-dimensional image, as required by Definition 6.1(1). Since the inclusion $M \xrightarrow{j} W$ is an isomorphism on H_1, there is a map $f : W \longrightarrow S^1$ such that f^{-1} (regular value) is an embedded 3-manifold Y whose boundary is Σ. We have a factorization of j_* as

$$H_1(\Sigma; \mathbb{Q}) \xrightarrow{i_*} H_1(Y; \mathbb{Q}) \xrightarrow{k_*} H_1(G; \mathbb{Q}).$$

By the usual Poincaré duality argument, the kernel of i_* is a Lagrangian of the intersection form on $H_1(\Sigma; \mathbb{Q})$. In particular, in our setting, it has dimension 2. This also implies that the image of i_* has rank 2, so kernel $j_* =$ kernel i_*. Suppose that both x_1 and x_2 lay in kernel j_* and hence in kernel i_*. This is a contradiction since $x_1 \cdot x_2 \neq 0$ in $H_1(\Sigma; \mathbb{Q})$. For if $\langle\ ,\ \rangle$ is the intersection form $H_2(Y, \partial Y) \otimes H_1(Y) \longrightarrow \mathbb{Q}$ and $\partial_* : H_2(Y, \partial Y; \mathbb{Q}) \longrightarrow H_1(\partial Y; \mathbb{Q})$, then $\langle z, i_* x_2 \rangle = \partial_* z \cdot x_2$. So if x_1 lay in kernel i_*, then x_1 would be of the form $\partial_* z$ for some z, implying $x_1 \cdot x_2 = 0$. Similarly, at least one of $\{x_3, x_4\}$ has nonzero image. This completes the verification of Definition 6.1(1).

Now suppose that $k \leq n - 1$. Let $\Gamma = \pi_1(W)/\pi_1(W)_r^{(k+1)}$ so that Γ is PTFA and $\Gamma^{(n)} = \{e\}$. Then $\Gamma^{(1)} = G/G_r^{(k)} \equiv G_k$. By Theorem 6.4, the map $j_* : H_1(M; \mathbb{Z}\Gamma^{(1)}[t^{\pm 1}]) \longrightarrow H_1(W; \mathbb{Z}\Gamma^{(1)}[t^{\pm 1}])$ has rank at least 1 as a map of $\mathbb{Z}\Gamma^{(1)}$-modules, that is, $\mathbb{Z}G_k$-modules. But as $\mathbb{Z}G_k$ modules, this map is identical to $j_* : H_1(M_\infty; \mathbb{Z}G_k) \longrightarrow H_1(W_\infty; \mathbb{Z}G_k)$, where W_∞ is the infinite cyclic cover of W. But $H_1(M_\infty; \mathbb{Z}G_k) = H_1(S; \mathbb{Z}G_k)$, and $H_1(W_\infty; \mathbb{Z}G_k) = H_1(G; \mathbb{Z}G_k)$, and we have shown that j_* is nontrivial even after tensoring with $\mathbb{K}(G_k)$. Since the kernel of $G_r^{(k)}/[G_r^{(k)}, G_r^{(k)}] \longrightarrow G_r^{(k)}/G_r^{(k+1)}$ is $\mathbb{Z}$-torsion, it is an isomorphism after tensoring with $\mathbb{K}(G_k)$ (which contains $\mathbb{Q}$). Thus j is an algebraic (n)-solution. □

THEOREM 6.4

Let M be zero surgery on a knot. Suppose that M is (n)-solvable via W, and suppose that $\psi : \pi_1(W) \longrightarrow \Gamma$ induces an isomorphism upon abelianization, where Γ is PTFA group and $\Gamma^{(n)} = \{e\}$. Let $r = \operatorname{rank}_{\mathbb{Q}} H_1(M_\infty; \mathbb{Q})$, where M_∞ is the infinite cyclic cover of M. Then

$$j_* : H_1(M; \mathbb{K}[t^{\pm 1}]) \longrightarrow H_1(W; \mathbb{K}[t^{\pm 1}])$$

has rank at least $(r-2)/2$ *if* $n>1$ *and has rank* $r/2$ *if* $n=1$ *as a map of* $\mathbb{K}$ *vector spaces, where* $\mathbb{K}$ *is the quotient field of* $\mathbb{Z}\Gamma^{(1)}$.

Proof
Let $\mathscr{R}=\mathbb{K}[t^{\pm 1}]$. By Theorem 5.1, there exists a nonsingular linking form $\mathscr{B}\ell : H_1(M;\mathscr{R}) \longrightarrow H_1(M;\mathscr{R})^{\#}$. Let $\mathscr{A}=H_1(M;\mathscr{R})$. By Theorem 5.2, $P=\text{kernel}(j_*)$ is an $\mathscr{R}$-submodule of $\mathscr{A}$ which is self-annihilating with respect to $\mathscr{B}\ell$. It follows that the map $h : P \longrightarrow (\mathscr{A}/P)^{\#}$ given by $p \mapsto \mathscr{B}\ell(p,\cdot)$ is an isomorphism (see [COT1, Lemma 2.14]). Note that $\mathscr{A}/P$ is isomorphic to the image of j_*. We claim that the rank over $\mathbb{K}$ of a finitely generated right $\mathscr{R}$-module $\mathscr{M}$ is equal to the $\mathbb{K}$-rank of the right $\mathscr{R}$-module $\overline{\text{Hom}_{\mathscr{R}}(\mathscr{M},\mathscr{K}/\mathscr{R})} \equiv \mathscr{M}^{\#}$. Since $\mathscr{R}$ is a noncommutative PID (see [J, Chapter 3], [C, Proposition 4.5]), any finitely generated $\mathscr{R}$-module is a direct sum of cyclic modules (see [Co, Theorem 2.4, page 494]). Hence our claim can be seen by examining the case of a cyclic module $\mathscr{M}=\mathscr{R}/p(t)\mathscr{R}$ and verifying that, in this case, $\mathscr{M}^{\#} \cong (\overline{\mathscr{R}/\mathscr{R}p(t)}) \cong \mathscr{R}/\bar{p}(t)\mathscr{R}$, where $\bar{p}(t)$ is the result of applying the involution from the group ring $\mathbb{Z}\Gamma$. One also verifies that (just as in the commutative case) the $\mathbb{K}$-rank of such a cyclic module is the degree of the Laurent polynomial $p(t)$. Since the degree of $\bar{p}(t)$ equals the degree of $p(t)$, we are done. Hence $\text{rank}_{\mathbb{K}}(P)=\text{rank}_{\mathbb{K}}(\text{image } j_*)$, and so this rank is at least $\text{rank}_{\mathbb{K}}(\mathscr{A})/2$. It remains only to show that $\text{rank}_{\mathbb{K}}(\mathscr{A})$ is at least $r-2$ if $n>1$ and is r if $n=1$.

By hypothesis, M is zero framed surgery on a knot R in S^3. Then r is interpretable as the degree of the Alexander polynomial of R. If $n=1$, then $\mathbb{K}=\mathbb{Q}$ and $\mathscr{A}$ is the classical Alexander module, which is well known to have $\mathbb{Q}$-rank r. If $n>1$, by [C, Corollary 4.8], $\text{rank}_{\mathbb{K}}\, H_1(S^3 \smallsetminus R;\mathbb{K}[t^{\pm 1}]) \geq r-1$. Since $\mathscr{A}$ depends only on $\pi_1(M)$ and the latter is obtained from $\pi_1(S^3 \smallsetminus R)$ by killing the longitude, $\mathscr{A}$ is obtained from $H_1(S^3 \smallsetminus R;\mathbb{K}[t^{\pm 1}])$ by killing the $\mathbb{K}[t^{\pm 1}]$-submodule generated by the longitude ℓ. If $n=1$, the longitude is trivial. Since ℓ commutes with the meridian of R,

$$(t-1)_*[\ell]=0 \in H_1(S^3 \smallsetminus R;\mathbb{K}[t^{\pm 1}]),$$

implying that this submodule is a quotient of $\mathbb{K}[t^{\pm 1}]/(t-1)\mathbb{K}[t^{\pm 1}] \cong \mathbb{K}$. Hence, if $n>1$, $\text{rank}_{\mathbb{K}}\, \mathscr{A} \geq r-2$. □

Proof of Theorem 6.3
Set $\mathscr{P}_0=\{(x_1,x_2)\}$, and set

$$\mathscr{P}_1=\Big\{([x_i,x_j],[x_i,x_k]) \,\Big|\, i,j,k \text{ distinct}\Big\}.$$

Supposing that $\mathcal{P}_k$ has been defined, define $\mathcal{P}_{k+1}$ as follows. For each $(y, z) \in \mathcal{P}_k$, include the following twelve pairs in $\mathcal{P}_{k+1}$: $([y, y^{x_i}], [z, z^{x_i}])$, $([y, z], [z, z^{x_i}])$, $([y, y^{x_i}], [y, z])$ for $1 \leq i \leq 4$ and $y^x \equiv x^{-1}yx$.

Now, we fix n and show that $\mathcal{P}_n$ satisfies the conditions of the theorem. Fix an algebraic (n)-solution $r : S \longrightarrow G$. We must show that there exists a special pair in $\mathcal{P}_n$ corresponding to r. This is true for $n = 0$ since $H_1(S; \mathbb{Z}G_0) \cong H_1(S; \mathbb{Z})$, so we assume that $n \geq 1$. Now we need some preliminary definitions.

Let F be the free group on $\{x_1, \ldots, x_4\}$. Its classifying space has a standard cell structure as a wedge of four circles W. Our convention is to consider its universal cover $\widetilde{W}$ as a right F-space as follows. Choose a preimage of the zero cell as base point denoted $*$. For each element $w \in F \equiv \pi_1(W)$, lift w^{-1} to a path $(\tilde{w}^{-1})$ beginning at $*$. There is a unique deck translation $\Phi(w)$ of $\widetilde{W}$ which sends $*$ to the end point of this lift. Then w acts on $\widetilde{W}$ by $\Phi(w)$. This is the conjugate action of the usual left action, as in [M]. Taking the induced cell structure on $\widetilde{W}$ and tensoring with an arbitrary left $\mathbb{Z}F$-module A gives an exact sequence

$$0 \longrightarrow H_1(F; A) \xrightarrow{d} A^4 \longrightarrow A \longrightarrow H_0(F; A) \longrightarrow 0. \tag{6.5}$$

Specifically, consider $A = \mathbb{Z}G$, where $\mathbb{Z}F$ acts by left multiplication via a homomorphism $\phi : F \longrightarrow G$. From the interpretation of $H_1(F; \mathbb{Z}G)$ as H_1 of a G-cover of W, one sees that an element g of $\ker(\phi)$ can be considered as an element of $H_1(F; \mathbb{Z}G)$. We claim that the composition $\ker(\phi) \longrightarrow H_1(F; \mathbb{Z}G) \xrightarrow{d} (\mathbb{Z}G)^4$ can be calculated using the free differential calculus $\partial = (\partial_1, \ldots, \partial_4)$, where $\partial_i : F \longrightarrow \mathbb{Z}F$. Specifically, we assert that the diagram (6.6) commutes. Henceforth, we abbreviate maps of the form $(r, r, r, r) : (\mathbb{Z}F_n)^4 \longrightarrow (\mathbb{Z}G_n)^4$ by r:

$$\begin{array}{ccc} F & \xrightarrow{\quad\quad \partial \quad\quad} & (\mathbb{Z}F)^4 \\ \uparrow & & \downarrow \phi \\ \ker(\phi) \longrightarrow & H_1(F; \mathbb{Z}G) \xrightarrow{d} & (\mathbb{Z}G)^4 \end{array} \tag{6.6}$$

where $\partial_i(x_j) = \delta_{ij}$, $\partial_i(e) = 0$, and $\partial_i(gh) = \partial_i g + (\partial_i h)g^{-1}$ for each $1 \leq i \leq 4$. This can be seen as follows. Let e_1, e_2, e_3, e_4 denote lifts of the 1-cells of W to $\widetilde{W}$ which emanate from $*$ and are oriented compatibly with $x_1, \ldots, x_4$. For any word $g \in F \equiv \pi_1(W)$, ∂g is (by definition) the $\mathbb{Z}F$-linear combination of the e_i which describes the 1-chain determined by lifting a path representing g to a path $\tilde{g}$ in $\widetilde{W}$ starting at $*$. If $g \in \ker(\phi)$, then g lifts to a loop in the ϕ-cover of W (and $H_1(F; \mathbb{Z}G)$ is the first homology of this cover), and the 1-cycle it represents can be obtained by pushing down $\tilde{g}$ from the universal cover. In other words, the 1-chain $\tilde{g}$ in $C_1(\widetilde{W}) \otimes_{\mathbb{Z}F} \mathbb{Z}G$ is obtained from the 1-chain in $C_1(\widetilde{W})$ by applying ϕ in each coordinate. It only remains to justify the formula for ∂_i. Note that the usual formula for the standard left action is

$$d_i(gh) = d_i(g) + gd(h).$$

Our formula is obtained by setting $\partial_i = \bar{d}_i$, where $\bar{}$ is the involution on the group ring. Alternatively, this can be verified explicitly by induction on the length of h. Suppose first that the length of h is 1. If $h \neq x_i$, then the formula is clearly true, so consider $\partial_i(gx_i)$. The path $\tilde{g}\tilde{x}_i$, viewed as a 1-chain, is obviously equal to the 1-chain given by $\tilde{g}$ (whose ith coordinate is $\partial_i g$) plus a certain translate of e_i. Since the path from $*$ to the initial point of $\tilde{x}_i$ is $\tilde{g}$, this is the image of e_i under the action of g^{-1} (in our convention). Hence $\partial_i(gx_i) = \partial_i g + g^{-1}$, as claimed. Now it is a simple matter to verify the inductive step by expressing $h = h' \cdot h''$, where h' and h'' are of lesser length. This is left to the reader.

Note that $r\pi_k : F \to F_k \to G_k$ is the same as $\pi_k r : F \to S \to G \to G_k$.

LEMMA 6.7

Given an algebraic (n)*-solution* $(n \geq 1)$ $r : S \longrightarrow G$, *after reordering* $\{x_1, x_2\}$ *and* $\{x_3, x_4\}$ *so that* $r_*(x_1)$ *and* $r_*(x_3)$ *are nontrivial as per Definition 6.1(1), for each* k, $1 \leq k \leq n$, *there is at least one pair* $(y, z) \in \mathscr{P}_k$ *with the following* good *properties:*

(1) $\partial_4 y = \partial_4 z = 0$;

(2) *the vectors* $(r\pi_k\partial_2 y, r\pi_k\partial_3 y)$ *and* $(r\pi_k\partial_2 z, r\pi_k\partial_3 z)$ *(i.e., the vectors consisting of the second and third coordinates of the images of* y *and* z *under the composition* $F^{(k)} \xrightarrow{\pi_k\partial} (\mathbb{Z}F_k)^4 \xrightarrow{r} (\mathbb{Z}G_k)^4$*) are right linearly independent over* $\mathbb{Z}G_k$. *(Note that property (1) ensures that the fourth coordinates are zero).*

Proof that Lemma 6.7 $\Longrightarrow$ *Theorem 6.3*

The set $\mathscr{P}_n$ was defined in the proof of Theorem 6.3. Given an algebraic (n)-solution $r : S \longrightarrow G$, reorder $\{x_1, x_2\}$ and $\{x_3, x_4\}$ so that $r_*(x_1)$ and $r_*(x_3)$ are nontrivial, as is possible by Definition 6.1(1). Then Lemma 6.7 provides a pair $(y, z) \in \mathscr{P}_n$ which has the listed *good* properties with respect to r. We verify that (y, z) is a *special pair* with respect to r. Consider the diagram below; recall $G_n = G/G_r^{(n)}$:

$$\begin{array}{ccccc}
F^{(n)} & \longrightarrow & H_1(F;\mathbb{Z}F_n) & \xrightarrow{d} & (\mathbb{Z}F_n)^4 \\
\Big\downarrow r^{(n)} & & \Big\downarrow r_* & & \Big\downarrow (r_n)^4 \\
 & & H_1(F;\mathbb{Z}G_n) & \xrightarrow{d'} & (\mathbb{Z}G_n)^4 \\
 & & \Big\downarrow i_* & & \Big\downarrow \\
S^{(n)} & \longrightarrow & H_1(S;\mathbb{Z}G_n) & \xrightarrow{d''} & (\mathbb{Z}G_n)^4/\langle d'(T)\rangle
\end{array} \qquad (6.8)$$

The horizontal composition on top is $\pi_n \partial$. The right-top square commutes by naturality of the sequence (6.5). Let $T \in F$ denote the single relation so that $S = F/\langle T\rangle$. Since T is in the kernel of $F \longrightarrow S \longrightarrow G_n$, it represents an element of $H_1(F;\mathbb{Z}G_n)$, and it generates the kernel of the epimorphism i_*. Since d and d' are monomorphisms, d'' is a monomorphism. Hence, to show that (y,z) is special, it suffices to show that the set of three 4-tuples $\{r\pi_n\partial(y), r\pi_n\partial(z), d'T\}$ is $\mathbb{Z}G_n$-linearly independent in $(\mathbb{Z}G_n)^4$. This follows immediately from the *good* properties of y and z once we verify that the fourth coordinate of $d'T$ is nonzero. After our possible reordering, the standard relation T is either $g[x_3,x_4]$ or $g[x_4,x_3]$, where g is either $[x_1,x_2]$ or $[x_2,x_1]$. If d stands for any one of the ∂_i, then one calculates that $d(g^{-1}) = -(dg)g$ and that

$$d([g,h]) = dg + (dh)g^{-1} - (dg)gh^{-1}g^{-1} - (dh)hgh^{-1}g^{-1}.$$

Using these, one calculates that $\partial_4(g[x_3,x_4]) = (x_3^{-1} - [x_4,x_3])g^{-1}$ and that $\partial_4(g[x_4,x_3]) = (1 - x_4x_3^{-1}x_4^{-1})g^{-1}$. The fourth coordinate of $d'T$ is $r\pi_n\partial_4(T)$, by diagram (6.6). If this vanished in $\mathbb{Z}G_n$, then its image $r\pi_1\partial_4(T)$ would certainly vanish in $\mathbb{Z}G_1 = \mathbb{Z}[G/G_r^{(1)}]$ (recall that $n \geq 1$). But $r\pi_1\partial_4(T)$ is either $r(x_3^{-1}) - 1$ or $1 - r(x_3^{-1})$, which can only vanish if $r(x_3)$ is trivial in $G/G_r^{(1)} = H_1(G;\mathbb{Z})/\text{torsion}$, that is, $r_*(x_3) = 0$ in $H_1(G;\mathbb{Q})$, contradicting our choice of x_3. Thus the fourth coordinate of $d'T$ is nontrivial. This completes the verification that the lemma implies the theorem. □

Proof of Lemma 6.7

The integer $n \geq 1$ is fixed throughout. By definition of an algebraic (n)-solution, we may reorder so that $r_*(x_1)$ and $r_*(x_3)$ are nontrivial in $H_1(G;\mathbb{Q}) = G_1 \otimes \mathbb{Q}$. This implies that $r\pi_1(x_1)$ and $r\pi_1(x_3)$ are nontrivial in G_1. We prove the lemma by induction on k. We begin with $k=1$. Consider the pair

$$(y,z) = ([x_1,x_2],[x_1,x_3]) \in \mathcal{P}_1.$$

We claim that (y,z) has the *good* properties. Certainly, $\partial_4 y = \partial_4 z = 0$ since x_4 does not appear in the words y and z. Similarly, the third coordinate of the image of y and the second coordinate of the image of z are zero. Hence, to establish the second good property, it suffices to show that $r\pi_1\partial_2 y \neq 0 \neq r\pi_1\partial_3 z$. Since $\partial_2 y = x_1^{-1} - [x_2,x_1]$ and $\partial_3 z = x_1^{-1} - [x_3,x_1]$, this follows since $r\pi_1(x_1^{-1})$ and $r\pi_1(x_3^{-1})$ are nontrivial in G_1. Therefore the base of the induction ($k=1$) is established.

Now suppose that the conclusions of the lemma have been established for $1,\dots,k$, where $k<n$. We establish them for $k+1$. Let $(y,z) \in \mathcal{P}_k$ be a pair that has the good properties. This means that $\partial_4 y = \partial_4 z = 0$ and that the vectors $(r\pi_k\partial_2 y, r\pi_k\partial_3 y)$ and $(r\pi_k\partial_2 z, r\pi_k\partial_3 z)$ are (right) linearly independent over $\mathbb{Z}G_k$. Consider the following

three elements of $\mathscr{P}_{k+1}$,

$$([y, y^{x_1}], [z, z^{x_1}]), \qquad ([y, z], [z, z^{x_1}]), \qquad ([y, y^{x_1}], [y, z]),$$

where $y^x = x^{-1}yx$. We show that at least one of these pairs (y_{k+1}, z_{k+1}) has the good properties, finishing the inductive proof of Lemma 6.7.

First, note that in all cases, $\partial_4 y_{k+1} = \partial_4 z_{k+1} = 0$ since $\partial_4 y = \partial_4 z = 0$. For the remainder of this proof, we write x for x_1, suppressing the subscript. We need to show that there is at least one of the three pairs (y_{k+1}, z_{k+1}) such that the vectors $(r\pi_{k+1}\partial_2 y_{k+1}, r\pi_{k+1}\partial_3 y_{k+1})$ and $(r\pi_{k+1}\partial_2 z_{k+1}, r\pi_{k+1}\partial_3 z_{k+1})$ are $\mathbb{Z}G_{k+1}$-linearly independent.

Case 1. Suppose that both $r\pi_{k+1}(y)$ and $r\pi_{k+1}(z)$ are nontrivial in G_{k+1}.

In this case, we show that the pair $(y_{k+1}, z_{k+1}) = ([y, y^x], [z, z^x])$ satisfies property (2). Let d be either ∂_2 or ∂_3. One has $dx = 0$ and $dy^x = (dy)x$. Using this and our previous computations of $d([g, h])$, one computes that $d([y, y^x]) = (dy)p$, where

$$p = 1 + xy^{-1} - (y^x)^{-1}[y^x, y] - x[y^x, y].$$

Similarly, $d([z, z^x]) = (dz)q$, where $q = 1 + xz^{-1} - (z^x)^{-1}[z^x, z] - x[z^x, z]$. We must show that the vectors $\big(r\pi_{k+1}((\partial_2 y)p), r\pi_{k+1}((\partial_3 y)p)\big)$ and $\big(r\pi_{k+1}((\partial_2 z)q), r\pi_{k+1}((\partial_3 z)q)\big)$ are $\mathbb{Z}G_{k+1}$-linearly independent. Note that the first vector is a right multiple of $v_{k+1} = (r\pi_{k+1}\partial_2 y, r\pi_{k+1}\partial_3 y)$ by $r\pi_{k+1}p$ and that the second is a right multiple of $w_{k+1} = (r\pi_{k+1}\partial_2 z, r\pi_{k+1}\partial_3 z)$ by $r\pi_{k+1}q$. The right factor $r\pi_{k+1}p$ is seen to be nontrivial in $\mathbb{Z}G_{k+1}$ as follows. First, observe that $[y, y^x] \in F^{(k+1)}$, so $r\pi_{k+1}([y, y^x]) = 1$. Then note that

$$r\pi_{k+1}p = r\pi_{k+1}\big(1 + xy^{-1} - (y^x)^{-1} - x\big)$$

is a linear combination of four group elements e, $r\pi_{k+1}(xy^{-1})$, $r\pi_{k+1}((y^x)^{-1})$, and $r\pi_{k+1}(x)$ in G_{k+1}. For $r\pi_{k+1}p$ to vanish in $\mathbb{Z}G_{k+1}$, the elements must pair up in a precise way and, in particular, such that $r\pi_{k+1}(x) = r\pi_{k+1}(xy^{-1})$ in G_{k+1}. This is a contradiction since $r\pi_{k+1}(y) \neq e$ by hypothesis. No other pairing is possible because the projections of the four elements to G_1 are e, $r\pi_1(x)$, e, and $r\pi_1(x)$, and we have already noted that $r\pi_1(x)$ is nontrivial in G_1. An entirely similar argument using the nontriviality of $r\pi_{k+1}(z)$ shows that the right factor $r\pi_{k+1}q$ is nontrivial. Since these right factors are nontrivial and $\mathbb{Z}G_{k+1}$ has no zero divisors, the linear independence of $\{v_{k+1}, w_{k+1}\}$ is sufficient to imply the linear independence of the original set of vectors. Recall that our hypothesis on (y, z) ensures that the set

$$\{v_k, w_k\} = \big\{(r\pi_k\partial_2 y, r\pi_k\partial_3 y),\ (r\pi_k\partial_2 z, r\pi_k\partial_3 z)\big\}$$

is linearly independent in $\mathbb{Z}G_k$. Note that v_k and w_k are the images of v_{k+1} and w_{k+1} under the canonical projection $(\mathbb{Z}G_{k+1})^2 \longrightarrow (\mathbb{Z}G_k)^2$. We assert that the linear independence of $\{v_k, w_k\}$ implies the linear independence of $\{v_{k+1}, w_{k+1}\}$ since the kernel of $G_{k+1} \longrightarrow G_k$ is a torsion-free abelian group (details follow). This completes the verification, under the assumptions of Case 1, that at least one pair (y_{k+1}, z_{k+1}) has *good* properties.

To establish our assertion, consider more generally an arbitrary endomorphism f of free right $\mathbb{Z}G_{k+1}$ modules $f : (\mathbb{Z}G_{k+1})^2 \longrightarrow (\mathbb{Z}G_{k+1})^2$ given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = (ax+by, cx+dy)$ for $a, b, c, d, x, y \in \mathbb{Z}G_{k+1}$. Then f induces $\bar{f} : (\mathbb{Z}G_k)^2 \longrightarrow (\mathbb{Z}G_k)^2$ given by the matrix $\left(\begin{smallmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{smallmatrix}\right)$, where $\bar{a}$ is the projection of a. Suppose that $\bar{f}$ is injective. Let $H = G_r^{(k)}/G_r^{(k+1)}$, and note that $\mathbb{Z}G_{k+1} \otimes_{\mathbb{Z}[H]} \mathbb{Z} \cong \mathbb{Z}G_k$, where $a \otimes 1 \mapsto \bar{a}$, as $\mathbb{Z}G_{k+1} - \mathbb{Z}$ bimodules. Moreover (under this identification), f descends to

$$f \otimes \mathrm{id} : (\mathbb{Z}G_k)^2 \longrightarrow (\mathbb{Z}G_k)^2,$$

sending $(\bar{z}, \bar{w})$ (for $z, w \in \mathbb{Z}G_{k+1}$) to $(\bar{a}\bar{z} + \bar{c}\bar{w}, \bar{b}\bar{z} + \bar{d}\bar{w})$, thus agreeing with $\bar{f}$. Since H is torsion-free abelian, a theorem of Strebel [S, Section 1, page 305] ensures that the injectivity of $f \otimes \mathrm{id} = \bar{f}$ implies the injectivity of f. An application of this general fact with $\{(a, c), (b, d)\} = \{v_{k+1}, w_{k+1}\}$ shows the latter set is right linearly independent.

Case 2. Suppose that $r\pi_{k+1}(y) = e$ and that $r\pi_{k+1}(z) \neq e$ in G_{k+1}.

In this case, we claim that the pair $(y_{k+1}, z_{k+1}) = ([y, z], [z, z^x])$ satisfies the *good* property (2). We see that $dy_{k+1} = dy(1 - z^{-1}[z, y]) + dz(y^{-1} - [z, y])$, where $d = \partial_2$ or $d = \partial_3$. Thus

$$r\pi_{k+1} dy_{k+1} = (r\pi_{k+1} dy)\big(1 - r\pi_{k+1}(z^{-1})\big).$$

Therefore, one of the vectors, $(r\pi_{k+1}\partial_2 y_{k+1}, r\pi_{k+1}\partial_3 y_{k+1})$, is a right multiple of the vector $v_{k+1} = (r\pi_{k+1}\partial_2 y, r\pi_{k+1}\partial_3 y)$ by a nonzero divisor $1 - r\pi_{k+1}(z^{-1})$. Hence, just as in Case 1, we can abandon the former and consider v_{k+1}. Recall also that $dz_{k+1} = (dz)q$, as in Case 1. One checks that $r\pi_{k+1} q \neq 0$ in $\mathbb{Z}G_{k+1}$, just as in Case 1, using the fact that $r\pi_{k+1}(z) \neq e$. Thus we can reduce to considering the vector $w_{k+1} = (r\pi_{k+1}\partial_2 z, r\pi_{k+1}\partial_3 z)$ as in Case 1. We finish the proof of Case 2 just as in Case 1, using the fact that our hypothesis guarantees that the vectors called w_k and v_k are linearly independent in $\mathbb{Z}G_k$.

Case 3. Suppose that $r\pi_{k+1}(y) \neq e$ and that $r\pi_{k+1}(z) = e$ in G_{k+1}.

In this case, we claim that the pair $(y_{k+1}, z_{k+1}) = ([y, y^x], [y, z])$ satisfies the *good* property (2). Following Case 2, $r\pi_{k+1} dz_{k+1}$ is equal to $(r\pi_{k+1} dz)(r\pi_{k+1}(y^{-1}) - 1)$, where $d = \partial_2$ or ∂_3. Moreover, $dy_{k+1} = (dy)p$. One finishes just as in Case 2.

Case 4. Suppose that $r\pi_{k+1}(y) = r\pi_{k+1}(z) = e$ in G_{k+1}.

We claim that this case cannot occur, for recall that, by the inductive hypothesis, the pair $(y, z) \in P_k$ has the *good* properties for the algebraic (n)-solution r, where $1 \leq k \leq n - 1$. But r is also an algebraic k-solution since $k < n$ and, by the proof of Lemma 6.7 $\Longrightarrow$ Theorem 6.3, the pair (y, z) is a *special pair* for r. Thus, under the composition

$$F^{(k)} \longrightarrow S^{(k)} \longrightarrow H_1(S; \mathbb{Z}S_k) \xrightarrow{r_*} H_1(S; \mathbb{Z}, G_k),$$

the set $\{y, z\}$ maps to a linearly independent set. Since $H_1(S; \mathbb{Z}G_k)$ has rank 2 (as we showed in the paragraph following the statement of Theorem 6.3), this set is a basis of $H_1(S; \mathbb{K}(G_k))$. Since r is also an algebraic (n)-solution and $k \leq n - 1$, by Definition 6.1(2), the composition of the above with the map

$$H_1(S; \mathbb{Z}G_k) \xrightarrow{r_*} H_1(G; \mathbb{Z}G_k) \cong G_r^{(k)}/G_r^{(k+1)}$$

is nontrivial when restricted to $\{y, z\}$. On the other hand, this combined map $F^{(k)} \longrightarrow G_r^{(k)}/G_r^{(k+1)}$ is clearly given by

$$y \mapsto r\pi_{k+1}(y) \qquad \text{and} \qquad z \mapsto r\pi_{k+1}(y),$$

so it is not possible that both $r\pi_{k+1}(y)$ and $r\pi_{k+1}(z)$ lie in $G_r^{(k+1)}$. Therefore Case 4 is not possible.

This completes the proof that one of the three new pairs (y_{k+1}, z_{k+1}) satisfies the *good* properties and hence concludes the inductive step of our proof of Lemma 6.7, as well as Theorem 6.3. □

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Cochran
Department of Mathematics, Rice University, Houston, Texas 77005-1892, USA; cochran@math.rice.edu

Teichner
Department of Mathematics, University of California, Berkeley, Berkeley, California 94720-3840, USA; teichner@math.berkeley.edu