Knot theory : From Chern-Simons to Goodwillie-Weiss

St. Etienne de Tinée,
Feb. 8, 2018

Peter Teichner,
MPIM, Bonn

Knot table:

Ordered by crossing #

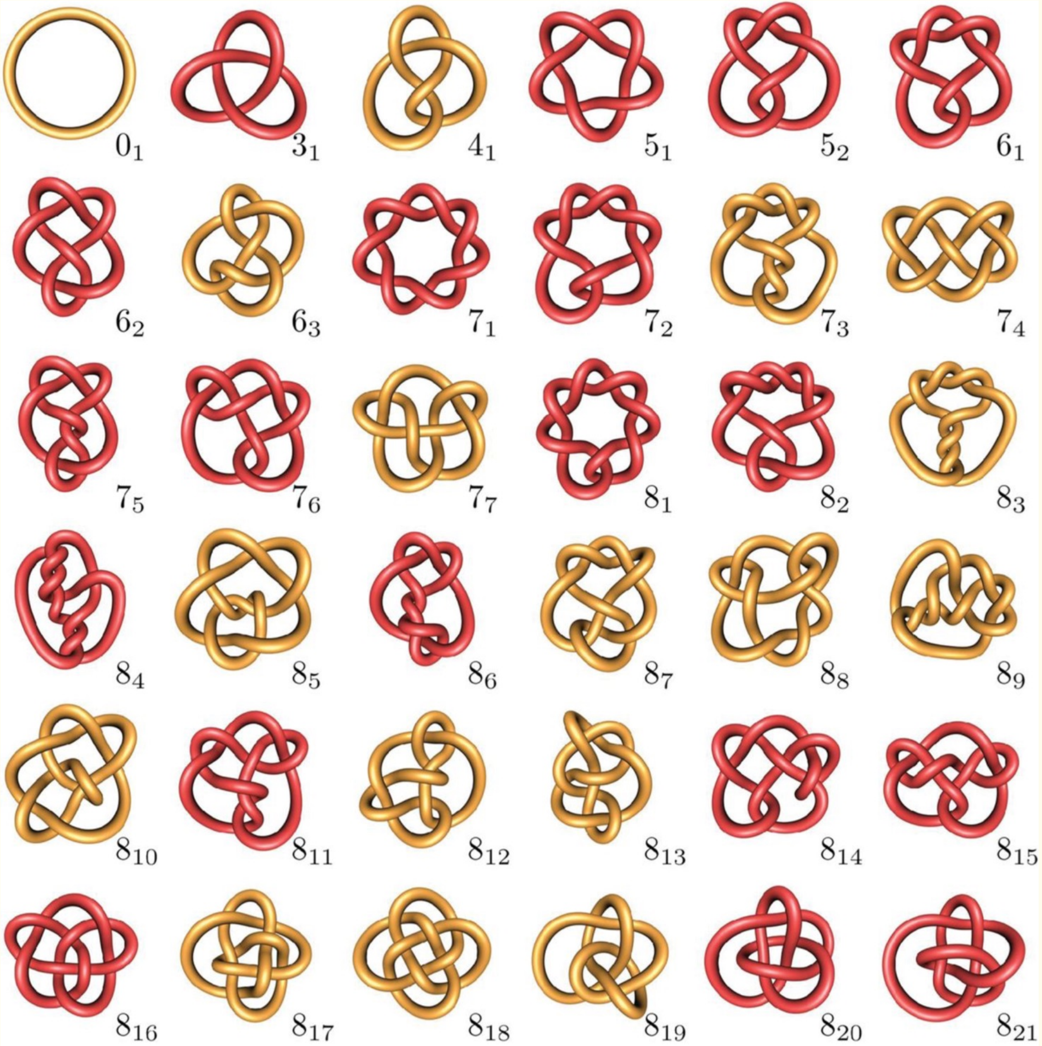
0: unknot

3: trefoil

4: figure 8

⋮

Look for a better organization!



Chern-Simons invariants

Witten, 1989

Vassiliev, 1990

Bar-Natan, thesis 1991

Axelrod-Singer, 1991

Altschuler-Freidel, 1992

Kontsevich, 1992

Bott-Taubes, D. Thurston, 1994

Goussarov, thesis 1994

Habiro, thesis 1999

Lescop, Poirer thesis 2002

Conant-T., 2002

Goodwillie-Weiss calculus

Goodwillie, 1989

Weiss, 1996

Goodwillie-Weiss, 1999

Sinha, thesis 2004

Volic, thesis 2006

Budney-Conant-Scannell-Sinha, 2008

Boavida-Weiss, 2013

Munson-Volic, book 2015

Budney-Conant-Koytcheff-Sinha, 2017

Quantum Chern-Simons knot invariants

Let G be compact Lie, $V \in \mathrm{Rep}(G)$, $\langle , \rangle$ inv. form on $\mathfrak{g}$.

- $$CS_{G,V}(K) = \int_{\mathcal{A}/\mathcal{G}}^{\text{pert.}} \mathrm{Tr}_V\left(\mathrm{hol}_K(A)\right) e^{\frac{iCS(A)}{\hbar}} \, dA \in \mathbb{C}[[\hbar]]$$
- $\mathcal{A} :=$ space of connections on (trivial) G-bundle on S^3 (or $\mathbb{Q}$-homology 3-sphere)
- $\mathcal{G} :=$ gauge group $= C^\infty(S^3, G)$
- $CS : \mathcal{A}/\mathcal{G} \longrightarrow \mathbb{R}/\mathbb{Z}$ usual Chern-Simons action
- $K : S^1 \hookrightarrow S^3$ is a knot and we think of
- $CS_{G,V} :$ Space of knots $\mathcal{K} \longrightarrow \mathbb{C}[[\hbar]]$.

Configuration space integrals

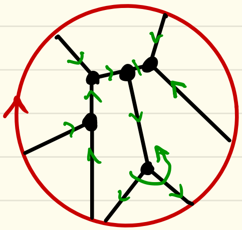
$$CS_{G,V}(K) = \sum_{n\geq 0} \hbar^{n} \cdot \left(\sum_{\substack{\text{Feynman}\\ \text{diagrams } \Gamma \\ \text{of degree } n}} |\mathrm{Aut}\,\Gamma|^{-1} \cdot I_{\Gamma,o}(K) \cdot W_{\Gamma,o}(G,V) \right)$$

analytic part

algebraic part

e.g.

$n=6$



$$\Gamma = (U \sqcup T,\ U \subseteq S^1,\ E)$$ up to isom.

vertices

edges

$$n := \frac{1}{2} \cdot v(\Gamma) = -\chi(\Gamma \cup S^1)$$

$$\underset{\overset{\cap}{\mathbb{Q}}}{I_{\Gamma,o}(K)} := \int_{C_\Gamma(K),\, o_E} \Psi^*(\Omega)$$

integral over configuration space

$$C_\Gamma(K) := \left\{ c : v(\Gamma) \hookrightarrow \mathbb{R}^3 \right\}$$

$$c/_U \simeq K(U) \subseteq K(S^1)$$

$$\begin{tikzcd} C_\Gamma \arrow[d, "\Psi"'] \\ \prod_{e\in E} S^2_e \end{tikzcd}$$

$$\Psi(c) := \prod_{e\in E} \overrightarrow{c(\partial_0 e)\ c(\partial_1 e)}$$

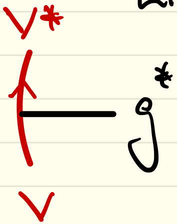
dimension $= \#U + 3 \cdot \#T = 2 \cdot \#E$

$$CS_{G,V}(K)_n = \sum_{\substack{\text{Feynman} \\ \text{diagrams } \Gamma \text{ of deg } n}} |\mathrm{Aut}\,\Gamma|^{-1} \cdot I_{\Gamma,o}(K) \cdot W_{\Gamma,o}(G,V) \in \mathbb{C}$$

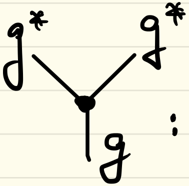
$W_{\Gamma,o}(G,V)$ is the usual Feynman weight, given by contracting the following tensors:

$$\mathrm{Lie}(G) =: \mathfrak{g} \xrightarrow{\rho} \mathrm{End}(V) \qquad\qquad \mathfrak{g}\otimes\mathfrak{g} \xrightarrow{[\,,\,]} \mathfrak{g}$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

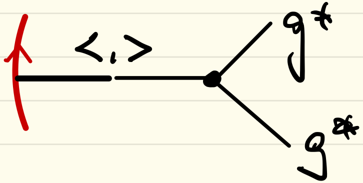


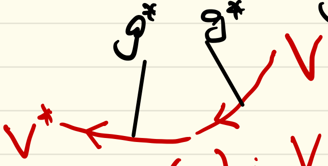
$: \quad \mathfrak{g}^*\otimes V\otimes V^*$



$: \quad \mathfrak{g}^*\otimes\mathfrak{g}^*\otimes\mathfrak{g}$

Uses also the non-deg. $\mathfrak{g}$-invariant pairing on $\mathfrak{g}$ "level":





$(,) : V\otimes V^* \longrightarrow \mathbb{C}$

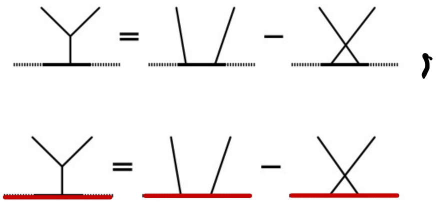
Miracle 1 : $CS_{G,V}(K)_n$ is an isotopy invariant

modulo the "anomaly"

Miracle 2 : $$Z_n(K) := \sum_{\Gamma \text{ of deg } n} |\mathrm{Aut}\,\Gamma|^{-1} \cdot I_{\Gamma,o}(K) \cdot [\Gamma_{,o}]$$

is also an isotopy invariant when considered in

$$\mathcal{A}_n^{\mathbb{Q}} := \left. \begin{array}{c} \mathbb{Q}\text{-linear combinations} \\ \text{of Feynman diagrams } \Gamma_{,o} \\ \text{of degree } n \end{array} \right/ \text{Jacobi-relations}$$



$[a,b] = a \cdot b - b \cdot a$

Goal: Explain a geometric aspect of this version of the "Kontsevich integral" $Z(K)$ and relate that to G-W tower.

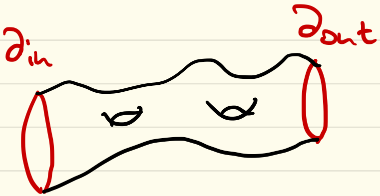
Thm.: [Conant-T., Bar-Natan-Garoufalidis-D. Thurston]

If K and K' cobound an embedded capped grope of degree n in S^3 then

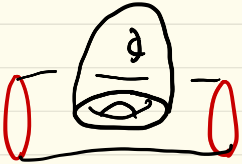
$$Z_{<n}(K) = Z_{<n}(K')$$

Abstract gropes are the following types of 2-complexes:

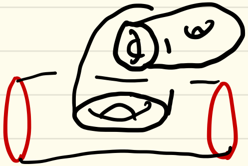
$Y =$



degree = 2



deg = 3



deg = 4

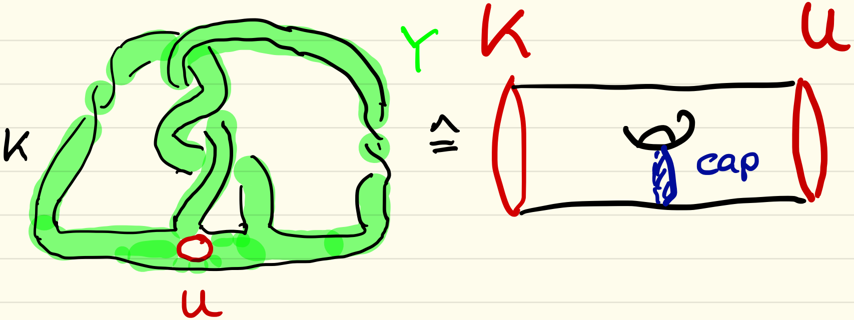
......

If $Y \to X$, it measures $[\partial_{in} Y]^{-1} \cdot [\partial_{out} Y] \in \pi_1(X) / \pi_1(X)_{(n)}$

where the lower central series is $[G, G_{(n-1)}] =: G_{(n)}$.

Def.: A (capped) grope cobordism between $K, K' : S^1 \hookrightarrow \mathbb{R}^3$ is a (capped) embedded grope $Y \hookrightarrow \mathbb{R}^3$ with $\partial Y = K \sqcup K'$. Write $K \overset{(c)}{\underset{n}{\sim}} K'$.

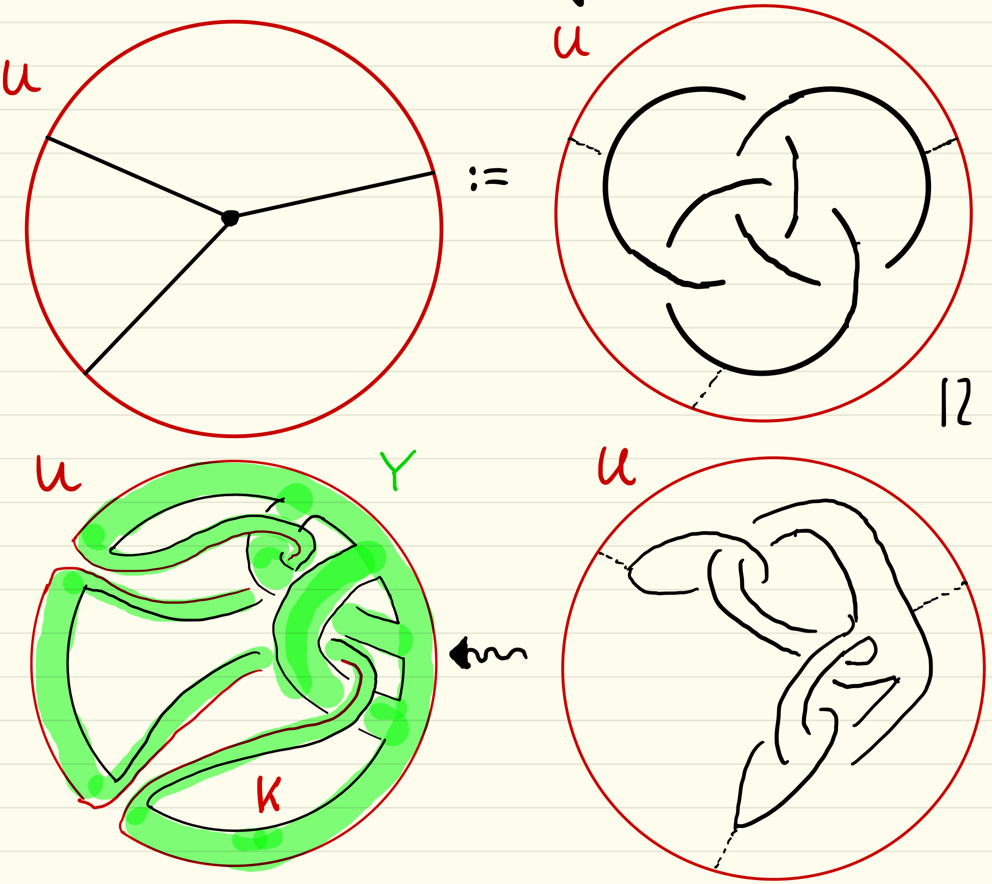
Example: A Seifert surface for K gives a deg 2 grope cobordism from K to the unknot U.



Note that a cap for a band in Y exists ⟺ the band is unknotted and untwisted.

Question: How to construct higher degree grope cobordisms?

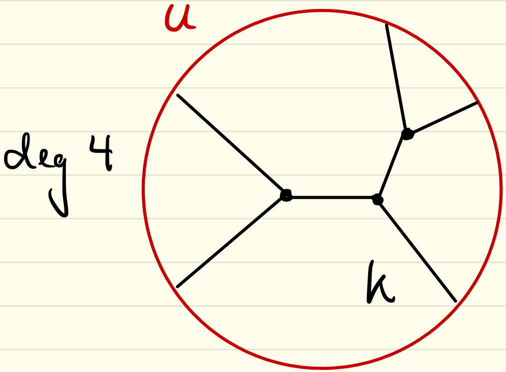
A shortcut: Capped gropes in $\mathbb{R}^3$ from trees in $\mathbb{R}^2$



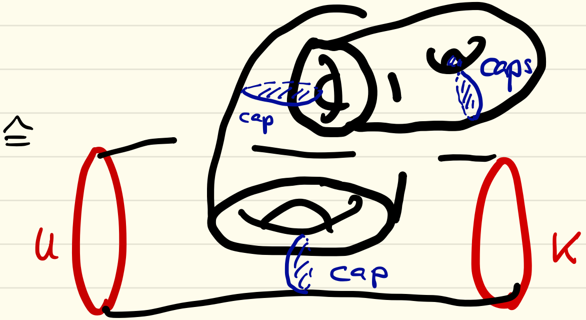
≀ isotopy

all caps exist !

⟹ Capped gropes of higher degree are obtained by induction from trees attached to a knot:



$\triangleq$

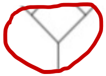
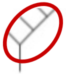
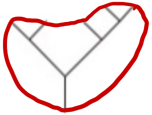
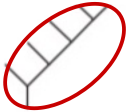
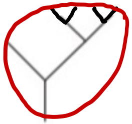


Thm.: $\mathcal{K}/\underset{n}{\sim}$ are finitely generated abelian groups

[Conant-T.] The (capped) grope filtration is (under #).

Def.: $\dots\dots \subseteq G^{(c)}_{n+1} \subseteq G^{(c)}_n := \{K \in \mathcal{K} \mid K \overset{(c)}{\underset{n}{\sim}} U\} \subseteq \dots \subseteq \mathcal{K}$

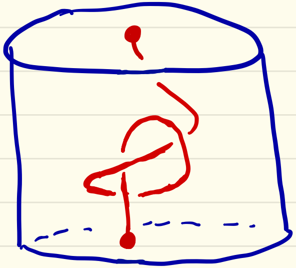
Open problem: $\bigcap_n G^{(c)}_n = \{U\}$ "do Vassiliev inv. detect knots?"

grope degree	type of grope	$\mathcal{K}$/capped gropes	$\mathcal{K}$/grope cobordism
2		$\{0\}$	$\{0\}$
3		$\mathbb{Z}(c_2)$	$\mathbb{Z}/2(\mathrm{Arf})$
4		$\mathbb{Z}(c_3) \oplus \mathbb{Z}(c_2)$	$\mathbb{Z}(c_2)$
4		$\mathbb{Z}(c_3) \oplus \mathbb{Z}(c_2)$	$\mathbb{Z}(c_2)$
5		$\mathbb{Z}(c_4) \oplus \mathbb{Z}(c_4') \oplus$ $\mathbb{Z}(c_3) \oplus \mathbb{Z}(c_2)$	$\mathbb{Z}/2(c_3) \oplus \mathbb{Z}(c_2)$
5		$\mathbb{Z}(c_4) \oplus \mathbb{Z}(c_4') \oplus$ $\mathbb{Z}(c_3) \oplus \mathbb{Z}(c_2)$	$\mathbb{Z}/2(c_3) \oplus \mathbb{Z}(c_2)$
5		?	S-equivalence

Note:

$K \in G_n^c \iff K$ has vanishing Vassiliev inv. of type $< n$.

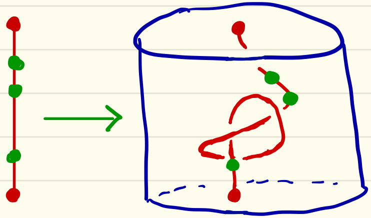
Goodwillie-Weiss tower:



for the space of "long knots"

$$\begin{tikzcd} & \cdots \arrow[d] \\ & T_{n+1} \arrow[d] \\ \mathcal{K} \arrow[r, "ev_n"] \arrow[ruu] \arrow[ru] \arrow[d, phantom, "\parallel"] & T_n = T_n(\mathcal{K}) \arrow[d, phantom, "\cong"] \\ \mathrm{Emb}_\partial(D^1, D^3) \arrow[r, "C_n"] & \mathrm{Strat}(\Delta^n, C_n(D^3)) \end{tikzcd}$$

Idea: Approximate an interval by a finite number n of points! $n=3$:



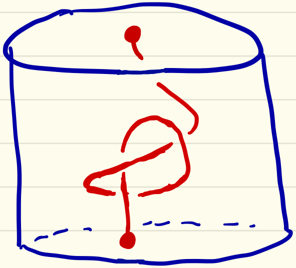
$$\begin{tikzcd} \pi_0(ev_n) : \pi_0(\mathcal{K}) \arrow[r] \arrow[rd] & \pi_0(T_n) \\ & \mathcal{K}/\overset{c}{\underset{n}{\sim}} \arrow[u, "\widetilde{ev}_n"'] \end{tikzcd}$$

$\pi_0(ev_n)$: $\pi_0(\mathcal{K}) \to \pi_0(T_n)$ are knot invariants.

Thm.: They factor through $\mathcal{K}/\overset{c}{\underset{n}{\sim}}$, i.e. through degree n capped grope cobordism, i.e. they are finite type of degree n.

[Budney-Conant-Koytcheff-Sinha 2017]

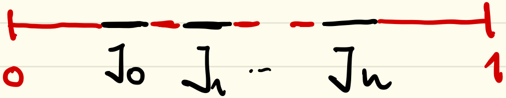
Goodwillie's punctured knots



$$\begin{tikzcd}
 & & \vdots \arrow[d] \\
 & \mathcal{K} \arrow[ur, dashed, "!"] \arrow[r] \arrow[dr, "ev_n"] \arrow[dl, phantom, "=:"] & T_{n+1} \arrow[d] \\
\mathrm{Emb}_\partial(I, D^3) \arrow[d, phantom, "\cong"] & & T_n \arrow[d, phantom, "\rotatebox{90}{$=:$}"] \\
\lim\limits_{S \in P_\gamma[n]} \mathrm{Emb}_\partial(I_S, D^3) \arrow[rr] & & \operatorname{holim}\limits_{S \in P_\gamma[n]} \mathrm{Emb}_\partial(I_S, D^3)
\end{tikzcd}$$

$n \geq 1$:

where



$I_S := I \setminus \bigcup_{s \in S} J_s \ , \ S \subseteq [n]$

Space level refinement:

$$\begin{tikzcd}
\left\{ \begin{array}{c} \text{simple capped} \\ \text{grope cobordisms} \\ \text{of degree } n \end{array} \right\} \arrow[r, "\exists"] \arrow[d, shift right, "\partial_0"'] \arrow[d, shift left, "\partial_1"] & \text{Path-space of } T_n \arrow[d, shift right, "\partial_0"'] \arrow[d, shift left, "\partial_1"] \\
\mathcal{K} \arrow[r, "ev_n"] & T_n
\end{tikzcd}$$

[Danica Kosanović - Yuqing Shi - P.T.]

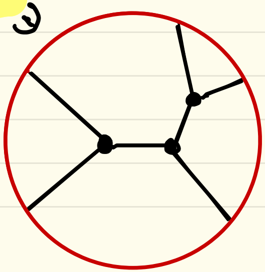
Chern-Simons

$$\begin{tikzcd}
\mathcal{K} \arrow[r, "\log(Z)", "\text{additive}"'] \arrow[d] & \hat{\mathcal{A}}^I \otimes \mathbb{Q} \arrow[d, two heads, "\pi_n"] \\
\mathcal{K}/\sim_{n+1} \arrow[r, "Z_{\leq n}"] & \mathcal{A}^I_{\leq n} \otimes \mathbb{Q} \\
\mathcal{G}_n \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, "Z_n"] & \mathcal{A}^I_n \otimes \mathbb{Q} \arrow[u, phantom, sloped, "\subseteq"] \\
 & \mathcal{A}^I_n \ni \arrow[u] \arrow[ul, two heads, "R_n"]
\end{tikzcd}$$

Goodwillie-Weiss

$$\begin{tikzcd}
\mathcal{K} \arrow[r, "ev_\infty"] \arrow[d, two heads] & \lim(\cdots \to T_{n+1} \to T_n \cdots) \arrow[d, two heads, "\pi_{n+1}"] \arrow[l, bend left, "h"] \\
\mathcal{K}/\sim_{n+1} \arrow[r, "\widetilde{ev}_{n+1}"] & \pi_0(T_{n+1}) \\
\mathcal{G}_n \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, "\widetilde{ev}_{n+1}"] & \mathrm{Ker}(\pi_0 T_{n+1} \to \pi_0 T_n) \arrow[u, phantom, sloped, "\subseteq"] \\
 & E^\infty_{-n-1,n+1} \arrow[u, phantom, sloped, "\cong"] \\
 & E^2_{-n-1,n+1} \arrow[u, two heads, "\text{mod im } d_r,\ r \geq 2"']
\end{tikzcd}$$

$\hat{=}$



$\in \mathcal{A}^I_n \cong E^2_{-n-1,n+1}$ [Conant]

The knot invariant $\widetilde{ev}_n : \mathcal{K}/\underset{n}{\sim} \longrightarrow \pi_0(T_n)$ takes values in an unknown abelian group!

We approach it inductively via $F_{n+1} \to T_{n+1} \to T_n$

Goodwillie-Weiss & Sinha via cosimplicial methods show

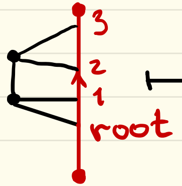
$$F_{n+1} \simeq \Omega^{n+1}\left(\begin{matrix}\text{total fibre}\\ \text{of } (n+1)\text{-cube}\\ S \mapsto C_S(D^3)\end{matrix}\right) \qquad \overset{k}{\vee} S^2 \to C_{k+1}(D^3) \to C_k(D^3)$$

Hilton

$$\Longrightarrow \pi_0 F_{n+1} \cong \pi_{n+1}(\downarrow) \xleftarrow[\text{products } \cong]{\text{Whitehead}} \mathcal{L}_n \cong \mathbb{Z}^{(n-1)!} \text{ where}$$

$$\mathcal{L}_n := \frac{\langle \text{deg } n \text{ trees as before} \rangle}{\text{Jacobi relations}} \subseteq \begin{matrix}\text{Free Lie alg.}\\ \text{on } x_1, \dots, x_n\end{matrix}$$

spanned by basic words, containing all n letters.



$\longmapsto [x_1,[x_2,x_3]]$

$n=3$ = # of caps

Thm. The spectral sequence for GW-tower

[Conant, Scannell-Sinha] $T_n \to T_{n-1} \to \dots \to T_0$ converging to $\pi_*(T_n)$

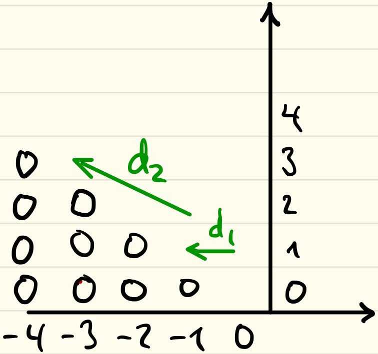
has $E^1_{-n,n} \cong \mathcal{L}_{n-1} \cong \mathbb{Z}^{(n-2)!}$

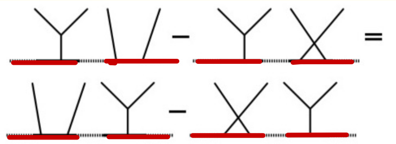
and $E^2_{-n,n} = E^1_{-n,n} / \mathrm{im}\, d_1 \cong \mathcal{L}_{n-1} / STU^2 \cong A^I_{n-1}$

$$E^3_{-n,n} = E^2_{-n,n} / \mathrm{im}\, d_2$$

$$\downarrow \quad \vdots \quad \downarrow$$

$$E^{n+1}_{-n,n} = E^\infty_{-n,n} = \mathrm{Ker}\left(\pi_0(T_n) \longrightarrow \pi_0(T_{n-1}) \right)$$





iso ??

⇕

$d_r = 0$?

$$F_1 \longrightarrow T_1 \longrightarrow T_0, \qquad F_1 \cong \Omega S^2, \quad T_1 \cong \mathrm{Imm}_\partial(D^1, D^3), \quad T_0 = *$$

$$F_2 \longrightarrow T_2 \longrightarrow T_1, \qquad F_2 \cong \Omega^2 S^2, \quad T_2 \cong *, \quad T_1 \cong \Omega S^2$$

$$F_3 \xrightarrow{\sim} T_3 \longrightarrow T_2, \qquad F_3 \cong \Omega^3 \mathrm{fibre}(S^2 \vee S^2 \to S^2 \times S^2), \quad T_2 = *$$

Summary :

capped grope cobordism

$$\begin{tikzcd}[column sep=large]
\pi_0 \mathcal{K} \arrow[r, "Z_{<n}"] \arrow[d, two heads] & \mathcal{A}^I_{<n} \otimes \mathbb{Q} & \\
\mathcal{K}/\sim_n \arrow[ru, "Z_{<n}"] \arrow[r, "\widetilde{ev}_n"'] & \pi_0 T_n \arrow[u, dashed, "(i)", "\tilde{Z}_{<n} := \text{conf. space integral}"'] & \\
\mathbb{G}_{n-1} = \mathcal{G}_{n-1}/\sim_n \arrow[u, phantom, "\cup"] \arrow[r] & E^\infty_{-n,n} \arrow[u, phantom, "\cup"] \arrow[r, phantom, "\cong"] & \mathrm{Ker}(\pi_0 T_n \twoheadrightarrow \pi_0 T_{n-1}) \\
\mathcal{A}^I_{n-1} \arrow[u, two heads, "R_{n-1}"] \arrow[r, phantom, "\overset{\sim}{\cong}"] \arrow[ru, phantom, "(ii)"] & E^2_{-n,n} \arrow[u, two heads, "\text{mod } ind_r,\ r\geq 2"'] &
\end{tikzcd}$$

rational isom. [Conant-T.]

rational iso. [Pedro-Geoffy]

Conjecture [Danica, Yuqing, Peter] : CS & GW are completely compatible, i.e.

(i) The above square commutes, i.p. $\widetilde{ev}_n$ is onto,

(ii) $Z_{<n}$ factors through $\pi_0 T_n$, i.p. $\widetilde{ev}_n \otimes \mathbb{Q}$ is an isom.