$L^2$-Betti numbers

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We introduce $L^2$-Betti numbers.

We present their basic properties and tools for their computation.

We compute the $L^2$-Betti numbers of all 3-manifolds.

We discuss the Atiyah Conjecture and the Singer Conjecture.
Basic motivation

- Given an invariant for finite $CW$-complexes, one can get much more sophisticated versions by passing to the universal covering and defining an analogue taking the action of the fundamental group into account.

Examples:

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We want to apply this principle to (classical) Betti numbers

\[ b_n(X) := \dim_{\mathbb{C}}(H_n(X; \mathbb{C})). \]

Here are two naive attempts which fail:

- \( \dim_{\mathbb{C}}(H_n(\tilde{X}; \mathbb{C})) \)
- \( \dim_{\mathbb{C}[\pi]}(H_n(\tilde{X}; \mathbb{C})) \),

where \( \dim_{\mathbb{C}[\pi]}(M) \) for a \( \mathbb{C}[\pi] \)-module could be chosen for instance as \( \dim_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{C}G} M) \).

The problem is that \( \mathbb{C}[\pi] \) is in general not Noetherian and \( \dim_{\mathbb{C}[\pi]}(M) \) is in general not additive under exact sequences.

We will use the following successful approach which is essentially due to Atiyah [1].

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L^2-Betti numbers
Throughout these lectures let $G$ be a discrete group.

Given a ring $R$ and a group $G$, denote by $RG$ or $R[G]$ the group ring.

Elements are formal sums $\sum_{g \in G} r_g \cdot g$, where $r_g \in R$ and only finitely many of the coefficients $r_g$ are non-zero.

Addition is given by adding the coefficients.

Multiplication is given by the expression $g \cdot h := g \cdot h$ for $g, h \in G$ (with two different meanings of $\cdot$).

In general $RG$ is a very complicated ring.
Denote by $L^2(G)$ the Hilbert space of (formal) sums $\sum_{g \in G} \lambda_g \cdot g$ such that $\lambda_g \in \mathbb{C}$ and $\sum_{g \in G} |\lambda_g|^2 < \infty$.

**Definition**

Define the group von Neumann algebra

$$\mathcal{N}(G) := \mathcal{B}(L^2(G), L^2(G))^G = \mathbb{C}G^{\text{weak}}$$

to be the algebra of bounded $G$-equivariant operators $L^2(G) \to L^2(G)$. The von Neumann trace is defined by

$$\text{tr}_{\mathcal{N}(G)} : \mathcal{N}(G) \to \mathbb{C}, \quad f \mapsto \langle f(e), e \rangle_{L^2(G)}.$$

**Example (Finite $G$)**

If $G$ is finite, then $\mathbb{C}G = L^2(G) = \mathcal{N}(G)$. The trace $\text{tr}_{\mathcal{N}(G)}$ assigns to $\sum_{g \in G} \lambda_g \cdot g$ the coefficient $\lambda_e$. 
Example \((G = \mathbb{Z}^n)\)

Let \(G = \mathbb{Z}^n\). Let \(L^2(T^n)\) be the Hilbert space of \(L^2\)-integrable functions \(T^n \to \mathbb{C}\). Fourier transform yields an isometric \(\mathbb{Z}^n\)-equivariant isomorphism

\[
L^2(\mathbb{Z}^n) \xrightarrow{\sim} L^2(T^n).
\]

Let \(L^\infty(T^n)\) be the Banach space of essentially bounded measurable functions \(f : T^n \to \mathbb{C}\). We obtain an isomorphism

\[
L^\infty(T^n) \xrightarrow{\sim} \mathcal{N}(\mathbb{Z}^n), \quad f \mapsto M_f
\]

where \(M_f : L^2(T^n) \to L^2(T^n)\) is the bounded \(\mathbb{Z}^n\)-operator \(g \mapsto g \cdot f\).

Under this identification the trace becomes

\[
\text{tr}_{\mathcal{N}(\mathbb{Z}^n)} : L^\infty(T^n) \to \mathbb{C}, \quad f \mapsto \int_{T^n} f d\mu.
\]
### Definition (Finitely generated Hilbert module)

A finitely generated Hilbert $\mathcal{N}(G)$-module $V$ is a Hilbert space $V$ together with a linear isometric $G$-action such that there exists an isometric linear $G$-embedding of $V$ into $L^2(G)^n$ for some $n \geq 0$.

A map of finitely generated Hilbert $\mathcal{N}(G)$-modules $f : V \to W$ is a bounded $G$-equivariant operator.

### Definition (von Neumann dimension)

Let $V$ be a finitely generated Hilbert $\mathcal{N}(G)$-module. Choose a $G$-equivariant projection $p : L^2(G)^n \to L^2(G)^n$ with $\text{im}(p) \cong \mathcal{N}(G) V$.

Define the von Neumann dimension of $V$ by

$$\dim_{\mathcal{N}(G)}(V) := \text{tr}_{\mathcal{N}(G)}(p) := \sum_{i=1}^{n} \text{tr}_{\mathcal{N}(G)}(p_{i,i}) \in \mathbb{R}_{\geq 0}.$$
Example (Finite $G$)
For finite $G$ a finitely generated Hilbert $\mathcal{N}(G)$-module $V$ is the same as a unitary finite dimensional $G$-representation and

$$\dim_{\mathcal{N}(G)}(V) = \frac{1}{|G|} \cdot \dim_{\mathbb{C}}(V).$$

Example ($G = \mathbb{Z}^n$)
Let $G$ be $\mathbb{Z}^n$. Let $X \subset T^n$ be any measurable set with characteristic function $\chi_X \in L^\infty(T^n)$. Let $M_{\chi_X} : L^2(T^n) \to L^2(T^n)$ be the $\mathbb{Z}^n$-equivariant unitary projection given by multiplication with $\chi_X$. Its image $V$ is a Hilbert $\mathcal{N}(\mathbb{Z}^n)$-module with

$$\dim_{\mathcal{N}(\mathbb{Z}^n)}(V) = \text{vol}(X).$$

In particular each $r \in \mathbb{R}_{\geq 0}$ occurs as $r = \dim_{\mathcal{N}(\mathbb{Z}^n)}(V)$. 
Definition *(Weakly exact)*

A sequence of Hilbert $\mathcal{N}(G)$-modules $U \xrightarrow{i} V \xrightarrow{p} W$ is **weakly exact at $V$** if the kernel $\ker(p)$ of $p$ and the closure $\overline{\text{im}(i)}$ of the image $\text{im}(i)$ of $i$ agree.

A map of Hilbert $\mathcal{N}(G)$-modules $f : V \to W$ is a **weak isomorphism** if it is injective and has dense image.

**Example**

The morphism of $\mathcal{N}(\mathbb{Z})$-Hilbert modules

$$M_{z-1} : L^2(\mathbb{Z}) = L^2(S^1) \to L^2(\mathbb{Z}) = L^2(S^1), \quad u(z) \mapsto (z - 1) \cdot u(z)$$

is a weak isomorphism, but not an isomorphism.
Theorem (Main properties of the von Neumann dimension)

1. **Faithfulness**
   
   We have for a finitely generated Hilbert $\mathcal{N}(G)$-module $V$
   
   \[ V = 0 \iff \dim_{\mathcal{N}(G)}(V) = 0; \]

2. **Additivity**
   
   If $0 \to U \to V \to W \to 0$ is a weakly exact sequence of finitely generated Hilbert $\mathcal{N}(G)$-modules, then
   
   \[ \dim_{\mathcal{N}(G)}(U) + \dim_{\mathcal{N}(G)}(W) = \dim_{\mathcal{N}(G)}(V); \]

3. **Cofinality**
   
   Let $\{V_i \mid i \in I\}$ be a directed system of Hilbert $\mathcal{N}(G)$-submodules of $V$, directed by inclusion. Then
   
   \[ \dim_{\mathcal{N}(G)} \left( \bigcup_{i \in I} V_i \right) = \sup \{ \dim_{\mathcal{N}(G)}(V_i) \mid i \in I \}. \]
**Definition** (*$L^2$*-homology and *$L^2$*-Betti numbers)

Let $X$ be a connected $CW$-complex of finite type. Let $\tilde{X}$ be its universal covering and $\pi = \pi_1(M)$. Denote by $C_\bullet(\tilde{X})$ its cellular $\mathbb{Z}\pi$-chain complex. Define its cellular *$L^2$*-chain complex to be the Hilbert $\mathcal{N}(\pi)$-chain complex

$$C^{(2)}_\bullet(\tilde{X}) := L^2(\pi) \otimes_{\mathbb{Z}\pi} C_\bullet(\tilde{X}) = C_\bullet(\tilde{X}).$$

Define its *$n$*-th *$L^2$*-homology to be the finitely generated Hilbert $\mathcal{N}(G)$-module

$$H^{(2)}_n(\tilde{X}) := \ker(c^{(2)}_n) / \text{im}(c^{(2)}_{n+1}).$$

Define its *$n$*-th *$L^2$*-Betti number

$$b^{(2)}_n(\tilde{X}) := \dim_{\mathcal{N}(\pi)}(H^{(2)}_n(\tilde{X})) \in \mathbb{R}^{\geq 0}.$$
Theorem (Main properties of $L^2$-Betti numbers)

Let $X$ and $Y$ be connected CW-complexes of finite type.

- **Homotopy invariance**
  If $X$ and $Y$ are homotopy equivalent, then
  
  $$b_n^{(2)}(\tilde{X}) = b_n^{(2)}(\tilde{Y});$$

- **Euler-Poincaré formula**
  We have
  
  $$\chi(X) = \sum_{n \geq 0} (-1)^n \cdot b_n^{(2)}(\tilde{X});$$

- **Poincaré duality**
  Let $M$ be a closed manifold of dimension $d$. Then
  
  $$b_n^{(2)}(\tilde{M}) = b_{d-n}^{(2)}(\tilde{M});$$
Theorem (Continued)

- **Künneth formula**

\[ b_n^{(2)}(X \times Y) = \sum_{p+q=n} b_p^{(2)}(X) \cdot b_q^{(2)}(Y); \]

- **Zero-th $L^2$-Betti number**

We have

\[ b_0^{(2)}(\widetilde{X}) = \frac{1}{|\pi|}; \]

- **Finite coverings**

If $X \to Y$ is a finite covering with $d$ sheets, then

\[ b_n^{(2)}(\widetilde{X}) = d \cdot b_n^{(2)}(\widetilde{Y}). \]
Example (Finite $\pi$)

If $\pi$ is finite then

$$b_n^{(2)}(\tilde{X}) = \frac{b_n(\tilde{X})}{|\pi|}.$$ 

Example ($S^1$)

Consider the $\mathbb{Z}$-$CW$-complex $\tilde{S^1}$. We get for $C_*^{(2)}(\tilde{S^1})$

$$\ldots \rightarrow 0 \rightarrow L^2(\mathbb{Z}) \xrightarrow{M_{z^{-1}}} L^2(\mathbb{Z}) \rightarrow 0 \rightarrow \ldots$$

and hence $H_n^{(2)}(\tilde{S^1}) = 0$ and $b_n^{(2)}(\tilde{S^1}) = 0$ for all $n \geq 0$. 
Example \((\pi = \mathbb{Z}^d)\)

Let \(X\) be a connected \(CW\)-complex of finite type with fundamental group \(\mathbb{Z}^d\). Let \(\mathbb{C}[\mathbb{Z}^d]^{(0)}\) be the quotient field of the commutative integral domain \(\mathbb{C}[\mathbb{Z}^d]\). Then

\[
b_n^{(2)}(\tilde{X}) = \dim_{\mathbb{C}[\mathbb{Z}^d]^{(0)}} \left( \mathbb{C}[\mathbb{Z}^d]^{(0)} \otimes_{\mathbb{Z}[\mathbb{Z}^d]} H_n(\tilde{X}) \right)
\]

Obviously this implies

\[
b_n^{(2)}(\tilde{X}) \in \mathbb{Z}.
\]
For a discrete group $G$ we can consider more generally any free finite $G$-$CW$-complex $\overline{X}$ which is the same as a $G$-covering $\overline{X} \to X$ over a finite $CW$-complex $X$. (Actually proper finite $G$-$CW$-complex suffices.)

The universal covering $p: \tilde{X} \to X$ over a connected finite $CW$-complex is a special case for $G = \pi_1(X)$.

Then one can apply the same construction to the finite free $\mathbb{Z}G$-chain complex $C_*(\overline{X})$. Thus we obtain the finitely generated Hilbert $\mathcal{N}(G)$-module

$$H_n^{(2)}(\overline{X}; \mathcal{N}(G)) := H_n^{(2)}(L^2(G) \otimes_{\mathbb{Z}G} C_*(\overline{X})), $$

and define

$$b_n^{(2)}(\overline{X}; \mathcal{N}(G)) := \dim_{\mathcal{N}(G)}(H_n^{(2)}(\overline{X}; \mathcal{N}(G))) \in \mathbb{R}_{\geq 0}. $$
Let $i: H \to G$ be an injective group homomorphism and $C_\ast$ be a finite free $\mathbb{Z}H$-chain complex.

Then $i_\ast C_\ast := \mathbb{Z}G \otimes_{\mathbb{Z}H} C_\ast$ is a finite free $\mathbb{Z}G$-chain complex.

We have the following formula

$$
\dim_{\mathcal{N}(G)}(H_n^{(2)}(L^2(G) \otimes_{\mathbb{Z}G} i_\ast C_\ast)) = \dim_{\mathcal{N}(H)}(H_n^{(2)}(L^2(H) \otimes_{\mathbb{Z}H} C_\ast)).
$$

**Lemma**

If $\overline{X}$ is a finite free $H$-CW-complex, then we get

$$
b_n^{(2)}(i_\ast \overline{X}; \mathcal{N}(G)) = b_n^{(2)}(\overline{X}; \mathcal{N}(H)).
$$
The corresponding statement is wrong if we drop the condition that $i$ is injective.

An example comes from $p: \mathbb{Z} \to \{1\}$ and $\tilde{X} = \tilde{S}^1$ since then $p_*\tilde{S}^1 = S^1$ and we have for $n = 0, 1$

$$b^{(2)}_n(\tilde{S}^1; \mathcal{N}(\mathbb{Z})) = b^{(2)}_n(\tilde{S}^1) = 0,$$

and

$$b^{(2)}_n(p_*\tilde{S}^1; \mathcal{N}(\{1\})) = b_n(S^1) = 1.$$
The $L^2$-Mayer Vietoris sequence

**Lemma**

Let $0 \to C^{(2)}_\ast \xrightarrow{i^{(2)}_*} D^{(2)}_\ast \xrightarrow{p^{(2)}_*} E^{(2)}_\ast \to 0$ be a weakly exact sequence of finite Hilbert $\mathcal{N}(G)$-chain complexes.

Then there is a long weakly exact sequence of finitely generated Hilbert $\mathcal{N}(G)$-modules

\[
\ldots \xrightarrow{\delta_{n+1}^{(2)}} H_n^{(2)}(C^{(2)}_\ast) \xrightarrow{H_n^{(2)}(i^{(2)}_\ast)} H_n^{(2)}(D^{(2)}_\ast) \xrightarrow{H_n^{(2)}(p^{(2)}_\ast)} H_n^{(2)}(E^{(2)}_\ast) \\
\xrightarrow{\delta_n^{(2)}} H_{n-1}^{(2)}(C^{(2)}_\ast) \xrightarrow{H_{n-1}^{(2)}(i^{(2)}_\ast)} H_{n-1}^{(2)}(D^{(2)}_\ast) \xrightarrow{H_{n-1}^{(2)}(p^{(2)}_\ast)} H_{n-1}^{(2)}(E^{(2)}_\ast) \xrightarrow{\delta_{n-1}^{(2)}} \ldots .
\]
Lemma

Let

\[
\begin{array}{ccc}
\overline{X}_0 & \longrightarrow & \overline{X}_1 \\
\downarrow & & \downarrow \\
\overline{X}_2 & \longrightarrow & \overline{X} \\
\end{array}
\]

be a cellular G-pushout of finite free G-CW-complexes, i.e., a G-pushout, where the upper arrow is an inclusion of a pair of free finite G-CW-complexes and the left vertical arrow is cellular.

Then we obtain a long weakly exact sequence of finitely generated Hilbert \( \mathcal{N}(G) \)-modules

\[
\cdots \rightarrow H_n^{(2)}(\overline{X}_0; \mathcal{N}(G)) \rightarrow H_n^{(2)}(\overline{X}_1; \mathcal{N}(G)) \oplus H_n^{(2)}(\overline{X}_2; \mathcal{N}(G)) \\
\rightarrow H_n^{(2)}(\overline{X}; \mathcal{N}(G)) \rightarrow H_{n-1}^{(2)}(\overline{X}_0; \mathcal{N}(G)) \\
\rightarrow H_{n-1}^{(2)}(\overline{X}_1; \mathcal{N}(G)) \oplus H_{n-1}^{(2)}(\overline{X}_2; \mathcal{N}(G)) \rightarrow H_{n-1}^{(2)}(\overline{X}; \mathcal{N}(G)) \rightarrow \cdots .
\]
Proof.

- From the cellular $G$-pushout we obtain an exact sequence of $\mathbb{Z}G$-chain complexes

$$0 \to C_*(\overline{X}_0) \to C_*(\overline{X}_1) \oplus C_*(\overline{X}_2) \to C_*(\overline{X}) \to 0.$$ 

- It induces an exact sequence of finite Hilbert $\mathcal{N}(G)$-chain complexes

$$0 \to L^2(G) \otimes_{\mathbb{Z}G} C_*(\overline{X}_0) \to L^2(G) \otimes_{\mathbb{Z}G} C_*(\overline{X}_1) \oplus L^2(G) \otimes_{\mathbb{Z}G} C_*(\overline{X}_2) \to L^2(G) \otimes_{\mathbb{Z}G} C_*(\overline{X}) \to 0.$$ 

- Now apply the previous result.
Definition ($L^2$-acyclic)

A finite (not necessarily connected) $CW$-complex $X$ is called $L^2$-acyclic, if $b_n^{(2)}(\tilde{C}) = 0$ holds for every $C \in \pi_0(X)$ and $n \in \mathbb{Z}$.

If $X$ is a finite (not necessarily connected) $CW$-complex, we define

$$b_n^{(2)}(\tilde{X}) := \sum_{C \in \pi_0(X)} b_n^{(2)}(\tilde{C}) \in \mathbb{R}_{\geq 0}.$$
Definition ($\pi_1$-injective)

A map $X \to Y$ is called $\pi_1$-injective, if for every choice of base point in $X$ the induced map on the fundamental groups is injective.

Consider a cellular pushout of finite $CW$-complexes

$$
\begin{array}{ccc}
X_0 & \longrightarrow & X_1 \\
\downarrow & & \downarrow \\
X_2 & \longrightarrow & X
\end{array}
$$

such that each of the maps $X_i \to X$ is $\pi_1$-injective.
Lemma

We get under the assumptions above for any $n \in \mathbb{Z}$

- If $X_0$ is $L^2$-acyclic, then
  \[ b_n^{(2)}(\tilde{X}) = b_n^{(2)}(\tilde{X}_1) + b_n^{(2)}(\tilde{X}_2). \]

- If $X_0, X_1$ and $X_2$ are $L^2$-cyclic, then $X$ is $L^2$-acyclic.
Proof.

Without loss of generality we can assume that $X$ is connected.

By pulling back the universal covering $\tilde{X} \to X$ to $X_i$, we obtain a cellular $\pi = \pi_1(X)$-pushout

$$
\begin{array}{ccc}
X_0 & \longrightarrow & X_1 \\
\downarrow & & \downarrow \\
X_2 & \longrightarrow & \tilde{X}
\end{array}
$$

Notice that $\overline{X}_i$ is in general not the universal covering of $X_i$. 
Proof continued.

Because of the associated long exact $L^2$-sequence and the weak exactness of the von Neumann dimension, it suffices to show for $n \in \mathbb{Z}$ and $i = 1, 2$

\[
H_n^{(2)}(\overline{X}_0; \mathcal{N}(\pi)) = 0; \\
b_n^{(2)}(\overline{X}_i; \mathcal{N}(\pi)) = b_n^{(2)}(\tilde{X}_i).
\]

This follows from $\pi_1$-injectivity, the lemma above about $L^2$-Betti numbers and induction, the assumption that $X_0$ is $L^2$-acyclic, and the faithfulness of the von Neumann dimension.
Some computations and results

Example (Finite self coverings)

We get for a connected $CW$-complex $X$ of finite type, for which there is a selfcovering $X \to X$ with $d$-sheets for some integer $d \geq 2$,

$$b_n^{(2)}(\tilde{X}) = 0 \quad \text{for } n \geq 0.$$ 

This implies for each connected $CW$-complex $Y$ of finite type that $S^1 \times Y$ is $L^2$-acyclic.
Example ($L^2$-Betti number of surfaces)

- Let $F_g$ be the orientable closed surface of genus $g \geq 1$.
- Then $|\pi_1(F_g)| = \infty$ and hence $b_0^{(2)}(\widetilde{F_g}) = 0$.
- By Poincaré duality $b_2^{(2)}(\widetilde{F_g}) = 0$.
- Since $\dim(F_g) = 2$, we get $b_n^{(2)}(\widetilde{F_g}) = 0$ for $n \geq 3$.
- The Euler-Poincaré formula shows

\[
\begin{align*}
b_1^{(2)}(\widetilde{F_g}) &= -\chi(F_g) = 2g - 2; \\
b_n^{(2)}(\widetilde{F_0}) &= 0 \text{ for } n \neq 1.
\end{align*}
\]
Theorem ($S^1$-actions, Lück)

Let $M$ be a connected compact manifold with $S^1$-action. Suppose that for one (and hence all) $x \in X$ the map $S^1 \to M, \ z \mapsto zx$ is $\pi_1$-injective.

Then $M$ is $L^2$-acyclic.

Proof.

Each of the $S^1$-orbits $S^1/H$ in $M$ satisfies $S^1/H \cong S^1$. Now use induction over the number of cells $S^1/H_i \times D^n$ and a previous result using $\pi_1$-injectivity and the vanishing of the $L^2$-Betti numbers of spaces of the shape $S^1 \times X$. 

Theorem ($S^1$-actions on aspherical manifolds, Lück)

Let $M$ be an aspherical closed manifold with non-trivial $S^1$-action. Then

1. The action has no fixed points;
2. The map $S^1 \to M$, $z \mapsto zx$ is $\pi_1$-injective for $x \in M$;
3. $b_n^{(2)}(\tilde{M}) = 0$ for $n \geq 0$ and $\chi(M) = 0$.

Proof.

The hard part is to show that the second assertion holds, since $M$ is aspherical. Then the first assertion is obvious and the third assertion follows from the previous theorem.
Theorem ($L^2$-Hodge - de Rham Theorem, Dodziuk [2])

Let $M$ be a closed Riemannian manifold. Put

$$\mathcal{H}^n_{(2)}(\tilde{M}) = \{ \tilde{\omega} \in \Omega^n(\tilde{M}) \mid \tilde{\Delta}_n(\tilde{\omega}) = 0, \|\tilde{\omega}\|_{L^2} < \infty \}$$

Then integration defines an isomorphism of finitely generated Hilbert $\mathcal{N}(\pi)$-modules

$$\mathcal{H}^n_{(2)}(\tilde{M}) \xrightarrow{\cong} H^n_{(2)}(\tilde{M}).$$

Corollary ($L^2$-Betti numbers and heat kernels)

$$b_n^{(2)}(\tilde{M}) = \lim_{t \to \infty} \int_{\mathcal{F}} \text{tr}_R(e^{-t\tilde{\Delta}_n}(\tilde{x}, \tilde{y})) \, d\text{vol}.$$ 

where $e^{-t\tilde{\Delta}_n}(\tilde{x}, \tilde{y})$ is the heat kernel on $\tilde{M}$ and $\mathcal{F}$ is a fundamental domain for the $\pi$-action.
Theorem (hyperbolic manifolds, Dodziuk [3])

Let $M$ be a hyperbolic closed Riemannian manifold of dimension $d$. Then:

$$b_n^{(2)}(\tilde{M}) = \begin{cases} 
0 & \text{if } 2n \neq d; \\
> 0 & \text{if } 2n = d.
\end{cases}$$

Proof.

A direct computation shows that $\mathcal{H}^{(2)}_p(\mathbb{H}^d)$ is not zero if and only if $2n = d$. Notice that $M$ is hyperbolic if and only if $\tilde{M}$ is isometrically diffeomorphic to the standard hyperbolic space $\mathbb{H}^d$. 

\[\square\]
Corollary

Let $M$ be a hyperbolic closed manifold of dimension $d$. Then

1. If $d = 2m$ is even, then

\[ (-1)^m \cdot \chi(M) > 0; \]

2. $M$ carries no non-trivial $S^1$-action.

Proof.

(1) We get from the Euler-Poincaré formula and the last result

\[ (-1)^m \cdot \chi(M) = b^{(2)}_m(\tilde{M}) > 0. \]

(2) We give the proof only for $d = 2m$ even. Then $b^{(2)}_m(\tilde{M}) > 0$. Since $\tilde{M} = \mathbb{H}^d$ is contractible, $M$ is aspherical. Now apply a previous result about $S^1$-actions.
Theorem (3-manifolds, Lott-Lück [7])

Let the 3-manifold $M$ be the connected sum $M_1 \# \ldots \# M_r$ of (compact connected orientable) prime 3-manifolds $M_j$. Assume that $\pi_1(M)$ is infinite. Then

$$b_1^{(2)}(\tilde{M}) = (r - 1) - \sum_{j=1}^{r} \frac{1}{|\pi_1(M_j)|} - \chi(M)$$

$$+ \left| \left\{ C \in \pi_0(\partial M) \mid C \cong S^2 \right\} \right| ;$$

$$b_2^{(2)}(\tilde{M}) = (r - 1) - \sum_{j=1}^{r} \frac{1}{|\pi_1(M_j)|}$$

$$+ \left| \left\{ C \in \pi_0(\partial M) \mid C \cong S^2 \right\} \right| ;$$

$$b_n^{(2)}(\tilde{M}) = 0 \quad \text{for } n \neq 1, 2.$$
Proof.

- We have already explained why a closed hyperbolic 3-manifold is $L^2$-acyclic.

- One of the hard parts of the proof is to show that this is also true for any hyperbolic 3-manifold with incompressible toral boundary.

- Recall that these have finite volume.

- One has to introduce appropriate boundary conditions and Sobolev theory to write down the relevant analytic $L^2$-deRham complexes and $L^2$-Laplace operators.

- A key ingredient is the decomposition of such a manifold into its core and a finite number of cusps.
Proof continued.

This can be used to write the $L^2$-Betti number as an integral over a fundamental domain $\mathcal{F}$ of finite volume, where the integrand is given by data depending on $\mathbb{H}^3$ only:

$$b_n^{(2)}(\tilde{M}) = \lim_{t \to \infty} \int_{\mathcal{F}} \text{tr}_\mathbb{R}(e^{-t\tilde{\Delta}_n}(\tilde{x}, \tilde{x})) \, d\text{vol}.$$ 

Since $\mathbb{H}^3$ has a lot of symmetries, the integrand does not depend on $\tilde{x}$ and is a constant $C_n$ depending only on $\mathbb{H}^3$.

Hence we get

$$b_n^{(2)}(\tilde{M}) = C_n \cdot \text{vol}(M).$$

From the closed case we deduce $C_n = 0$. 

Proof continued.

- Next we show that any Seifert manifold with infinite fundamental group is $L^2$-acyclic.

- This follows from the fact that such a manifold is finitely covered by the total space of an $S^1$-bundle $S^1 \to E \to F$ over a surface with injective $\pi_1(S^1) \to \pi_1(E)$ using previous results.

- In the next step one shows that any irreducible 3-manifold $M$ with incompressible or empty boundary and infinite fundamental group is $L^2$-acyclic.

- Recall that by the Thurston Geometrization Conjecture we can find a family of incompressible tori which decompose $M$ into hyperbolic and Seifert pieces. The tori and all these pieces are $L^2$-acyclic.

- Now the claim follows from the $L^2$-Mayer Vietoris sequence.
Proof continued.

In the next step one shows that any irreducible 3-manifold $M$ with incompressible boundary and infinite fundamental group satisfies

$$b_1^{(2)}(\tilde{M}) = -\chi(M) \text{ and } b_n^{(2)}(\tilde{M}) = 0 \text{ for } n \neq 1.$$  

This follows by considering $N = M \cup_{\partial M} M$ using the $L^2$-Mayer-Vietoris sequence, the already proved fact that $N$ is $L^2$-acyclic and the previous computation of the $L^2$-Betti numbers for surfaces.

In the next step one shows that any irreducible 3-manifold $M$ with infinite fundamental group satisfies $b_1^{(2)}(\tilde{M}) = -\chi(M)$ and $b_n^{(2)}(\tilde{M}) = 0$ for $n \neq 1.$
Proof continued.

- This is reduced by an iterated application of the Loop Theorem to the case where the boundary is incompressible. Namely, using the Loop Theorem one gets an embedded disk $D^2 \subset M$ along which one can decompose $M$ as $M_1 \cup_{D^2} M_2$ or as $M_1 \cup_{S^0 \times D^2} D^1 \times D^2$ depending on whether $D^2$ is separating or not.

- Since the only prime 3-manifold that is not irreducible is $S^1 \times S^2$, and every manifold $M$ with finite fundamental group satisfies the result by a direct inspection of the Betti numbers of its universal covering, the claim is proved for all prime 3-manifolds.

- Finally one uses the $L^2$-Mayer Vietoris sequence to prove the claim in general using the prime decomposition.
Corollary

Let $M$ be a 3-manifold. Then $M$ is $L^2$-acyclic if and only if one of the following cases occur:

- $M$ is an irreducible 3-manifold with infinite fundamental group whose boundary is empty or toral.
- $M$ is $S^1 \times S^2$ or $\mathbb{RP}^3 \# \mathbb{RP}^3$.

Corollary

Let $M$ be a compact $n$-manifold such that $n \leq 3$ and its fundamental group is torsionfree.

Then all its $L^2$-Betti numbers are integers.
Theorem (mapping tori, Lück [9])

Let $f : X \to X$ be a cellular selfhomotopy equivalence of a connected CW-complex $X$ of finite type. Let $T_f$ be the mapping torus. Then

$$b_n^{(2)}(\tilde{T}_f) = 0 \quad \text{for } n \geq 0.$$ 

Proof.

- As $T_{fd} \to T_f$ is up to homotopy a $d$-sheeted covering, we get

$$b_n^{(2)}(\tilde{T}_f) = \frac{b_n^{(2)}(\tilde{T}_{fd})}{d}.$$
Proof continued.

- If $\beta_n(X)$ is the number of $n$-cells, then there is up to homotopy equivalence a $CW$-structure on $T_{fd}$ with $\beta_n(T_{fd}) = \beta_n(X) + \beta_{n-1}(X)$. We have

$$b_n^{(2)}(\widetilde{T}_{fd}) = \dim_{\mathcal{N}(G)} \left( H_n^{(2)}(C_n^{(2)}(\widetilde{T}_{fd})) \right) \leq \dim_{\mathcal{N}(G)} \left( C_n^{(2)}(\widetilde{T}_{fd}) \right) = \beta_n(T_{fd}).$$

- This implies for all $d \geq 1$

$$b_n^{(2)}(\widetilde{T}_f) \leq \frac{\beta_n(X) + \beta_{n-1}(X)}{d}.$$

- Taking the limit for $d \to \infty$ yields the claim.
Let $M$ be an irreducible manifold $M$ with infinite fundamental group and empty or incompressible toral boundary which is not a closed graph manifold.

Agol proved the Virtually Fibering Conjecture for such $M$.

This implies by the result above that $M$ is $L^2$-acyclic.
Conjecture (Atiyah Conjecture for torsionfree finitely presented groups)

Let $G$ be a torsionfree finitely presented group. We say that $G$ satisfies the Atiyah Conjecture if for any closed Riemannian manifold $M$ with $\pi_1(M) \cong G$ we have for every $n \geq 0$

$$b_n^{(2)}(\tilde{M}) \in \mathbb{Z}.$$ 

- All computations presented above support the Atiyah Conjecture.
The fundamental square is given by the following inclusions of rings

\[
\begin{array}{c}
\mathbb{Z}G \\ \downarrow \\
D(G) \\ \downarrow \\
\mathcal{U}(G)
\end{array} \quad \begin{array}{c}
\mathcal{N}(G) \\
\downarrow \\
\mathcal{U}(G)
\end{array}
\]

\( \mathcal{U}(G) \) is the algebra of affiliated operators. Algebraically it is just the Ore localization of \( \mathcal{N}(G) \) with respect to the multiplicatively closed subset of non-zero divisors.

\( D(G) \) is the division closure of \( \mathbb{Z}G \) in \( \mathcal{U}(G) \), i.e., the smallest subring of \( \mathcal{U}(G) \) containing \( \mathbb{Z}G \) such that every element in \( D(G) \), which is a unit in \( \mathcal{U}(G) \), is already a unit in \( D(G) \) itself.
If $G$ is finite, its is given by

\[
\begin{array}{ccc}
\mathbb{Z}G & \longrightarrow & \mathbb{C}G \\
\downarrow & & \downarrow \text{id} \\
\mathbb{Q}G & \longrightarrow & \mathbb{C}G
\end{array}
\]

If $G = \mathbb{Z}$, it is given by

\[
\begin{array}{ccc}
\mathbb{Z}[\mathbb{Z}] & \longrightarrow & L^\infty(S^1) \\
\downarrow & & \downarrow \\
\mathbb{Q}[\mathbb{Z}]^{(0)} & \longrightarrow & L(S^1)
\end{array}
\]
If $G$ is elementary amenable torsionfree, then $\mathcal{D}(G)$ can be identified with the Ore localization of $\mathbb{Z}G$ with respect to the multiplicatively closed subset of non-zero elements.

In general the Ore localization does not exist and in these cases $\mathcal{D}(G)$ is the right replacement.
Conjecture (Atiyah Conjecture for torsionfree groups)

Let $G$ be a torsionfree group. It satisfies the Atiyah Conjecture if $\mathcal{D}(G)$ is a skew-field.

- A torsionfree group $G$ satisfies the Atiyah Conjecture if and only if for any matrix $A \in M_{m,n}(\mathbb{Z}G)$ the von Neumann dimension
  \[
  \dim_{\mathcal{N}(G)}(\ker(r_A : \mathcal{N}(G)^m \to \mathcal{N}(G)^n))
  \]
  is an integer. In this case this dimension agrees with
  \[
  \dim_{\mathcal{D}(G)}(\ker(r_A : \mathcal{D}(G)^m \to \mathcal{D}(G)^n)).
  \]
- The general version above is equivalent to the one stated before if $G$ is finitely presented.
The Atiyah Conjecture implies the Zero-divisor Conjecture due to Kaplansky saying that for any torsionfree group and field of characteristic zero $F$ the group ring $FG$ has no non-trivial zero-divisors.

There is also a version of the Atiyah Conjecture for groups with a bound on the order of its finite subgroups.

However, there exist closed Riemannian manifolds whose universal coverings have an $L^2$-Betti number which is irrational, see Austin, Grabowski [4].
Theorem (Linnell [6], Schick [11])

1. Let $\mathcal{C}$ be the smallest class of groups which contains all free groups, is closed under extensions with elementary amenable groups as quotients and directed unions. Then every torsionfree group $G$ which belongs to $\mathcal{C}$ satisfies the Atiyah Conjecture.

2. If $G$ is residually torsionfree elementary amenable, then it satisfies the Atiyah Conjecture.
Strategy to prove the Atiyah Conjecture

1. Show that $K_0(\mathbb{C}) \to K_0(\mathbb{C}G)$ is surjective
   (This is implied by the Farrell-Jones Conjecture)

2. Show that $K_0(\mathbb{C}G) \to K_0(\mathcal{D}(G))$ is surjective.

3. Show that $\mathcal{D}(G)$ is semisimple.
In general there are no relations between the Betti numbers $b_n(X)$ and the $L^2$-Betti numbers $b^{(2)}_n(\tilde{X})$ for a connected CW-complex $X$ of finite type except for the Euler Poincaré formula

$$\chi(X) = \sum_{n \geq 0} (-1)^n \cdot b^{(2)}_n(\tilde{X}) = \sum_{n \geq 0} (-1)^n \cdot b_n(X).$$
Given an integer $l \geq 1$ and a sequence $r_1, r_2, \ldots, r_l$ of non-negative rational numbers, we can construct a group $G$ such that $BG$ is of finite type and

\[
b^{(2)}_n(BG) = r_n \quad \text{for } 1 \leq n \leq l;
b^{(2)}_n(BG) = 0 \quad \text{for } l + 1 \leq n;
b_n(BG) = 0 \quad \text{for } n \geq 1.
\]

For any sequence $s_1, s_2, \ldots$ of non-negative integers there is a $CW$-complex $X$ of finite type such that for $n \geq 1$

\[
b_n(X) = s_n;
b^{(2)}_n(\tilde{X}) = 0.
\]
Theorem (Approximation Theorem, Lück [8])

Let $X$ be a connected CW-complex of finite type. Suppose that $\pi$ is residually finite, i.e., there is a nested sequence

$$\pi = G_0 \supset G_1 \supset G_2 \supset \ldots$$

of normal subgroups of finite index with $\bigcap_{i \geq 1} G_i = \{1\}$. Let $X_i$ be the finite $[\pi : G_i]$-sheeted covering of $X$ associated to $G_i$.

Then for any such sequence $(G_i)_{i \geq 1}$

$$b_n^2(\widetilde{X}) = \lim_{i \to \infty} \frac{b_n(X_i)}{[G : G_i]}.$$
Ordinary Betti numbers are not multiplicative under finite coverings, whereas the $L^2$-Betti numbers are. With the expression

$$\lim_{i \to \infty} \frac{b_n(X_i)}{[G : G_i]}$$

we try to force the Betti numbers to be multiplicative by a limit process.

The theorem above says that $L^2$-Betti numbers are asymptotic Betti numbers. It was conjectured by Gromov.
Let $G$ be a finitely presented group. Define its **deficiency**

$$\text{defi}(G) := \max\{g(P) - r(P)\}$$

where $P$ runs over all presentations $P$ of $G$ and $g(P)$ is the number of generators and $r(P)$ is the number of relations of a presentation $P$. 
Example

- The free group $F_g$ has the obvious presentation $\langle s_1, s_2, \ldots s_g \mid \emptyset \rangle$ and its deficiency is realized by this presentation, namely $\text{defi}(F_g) = g$.

- If $G$ is a finite group, $\text{defi}(G) \leq 0$.

- The deficiency of a cyclic group $\mathbb{Z}/n$ is 0, the obvious presentation $\langle s \mid s^n \rangle$ realizes the deficiency.

- The deficiency of $\mathbb{Z}/n \times \mathbb{Z}/n$ is $-1$, the obvious presentation $\langle s, t \mid s^n, t^n, [s, t] \rangle$ realizes the deficiency.
Example *(deficiency and free products)*

The deficiency is not additive under free products by the following example due to Hog-Lustig-Metzler. The group

$$(\mathbb{Z}/2 \times \mathbb{Z}/2) \ast (\mathbb{Z}/3 \times \mathbb{Z}/3)$$

has the obvious presentation

$$\langle s_0, t_0, s_1, t_1 \mid s_0^2 = t_0^2 = [s_0, t_0] = s_1^3 = t_1^3 = [s_1, t_1] = 1 \rangle$$

One may think that its deficiency is $-2$. However, it turns out that its deficiency is $-1$ realized by the following presentation

$$\langle s_0, t_0, s_1, t_1 \mid s_0^2 = 1, [s_0, t_0] = t_0^2, s_1^3 = 1, [s_1, t_1] = t_1^3, t_0^2 = t_1^3 \rangle.$$
Lemma

Let $G$ be a finitely presented group. Then

$$\text{defi}(G) \leq 1 - |G|^{-1} + b_1^{(2)}(G) - b_2^{(2)}(G).$$

Proof.

We have to show for any presentation $P$ that

$$g(P) - r(P) \leq 1 - b_0^{(2)}(G) + b_1^{(2)}(G) - b_2^{(2)}(G).$$

Let $X$ be a $CW$-complex realizing $P$. Then

$$\chi(X) = 1 - g(P) + r(P) = b_0^{(2)}(\tilde{X}) + b_1^{(2)}(\tilde{X}) - b_2^{(2)}(\tilde{X}).$$

Since the classifying map $X \to BG$ is 2-connected, we get

$$b_n^{(2)}(\tilde{X}) = b_n^{(2)}(G) \quad \text{for } n = 0, 1;$$

$$b_2^{(2)}(\tilde{X}) \geq b_2^{(2)}(G).$$
Theorem (Deficiency and extensions, Lück)

Let \( 1 \to H \overset{i}{\to} G \overset{q}{\to} K \to 1 \) be an exact sequence of infinite groups. Suppose that \( G \) is finitely presented and \( H \) is finitely generated. Then:

1. \( b_1^{(2)}(G) = 0 \);
2. \( \text{defi}(G) \leq 1 \);
3. Let \( M \) be a closed oriented 4-manifold with \( G \) as fundamental group. Then
   \[
   |\text{sign}(M)| \leq \chi(M).
   \]
The Singer Conjecture

Conjecture (Singer Conjecture)

*If M is an aspherical closed manifold, then*

\[ b_n^{(2)}(\tilde{M}) = 0 \quad \text{if } 2n \neq \dim(M). \]

*If M is a closed Riemannian manifold with negative sectional curvature, then*

\[ b_n^{(2)}(\tilde{M}) \begin{cases} = 0 & \text{if } 2n \neq \dim(M); \\ > 0 & \text{if } 2n = \dim(M). \end{cases} \]
The computations presented above do support the Singer Conjecture.

Under certain negative pinching conditions the Singer Conjecture has been proved by Ballmann-Brüning, Donnelly-Xavier, Jost-Xin.

The Singer Conjecture gives also evidence for the Atiyah Conjecture.
Because of the Euler-Poincaré formula

\[ \chi(M) = \sum_{n \geq 0} (-1)^n \cdot b_n^2(\tilde{M}) \]

the Singer Conjecture implies the following conjecture provided that \( M \) has non-positive sectional curvature.

**Conjecture (Hopf Conjecture)**

*If \( M \) is a closed Riemannian manifold of even dimension with sectional curvature \( \sec(M) \), then*

\[
\begin{align*}
(-1)^{\dim(M)/2} \cdot \chi(M) &> 0 \quad \text{if} \quad \sec(M) < 0; \\
(-1)^{\dim(M)/2} \cdot \chi(M) &\geq 0 \quad \text{if} \quad \sec(M) \leq 0; \\
\chi(M) &= 0 \quad \text{if} \quad \sec(M) = 0; \\
\chi(M) &\geq 0 \quad \text{if} \quad \sec(M) \geq 0; \\
\chi(M) &> 0 \quad \text{if} \quad \sec(M) > 0.
\end{align*}
\]
Definition (Kähler hyperbolic manifold)

A Kähler hyperbolic manifold is a closed connected Kähler manifold $M$ whose fundamental form $\omega$ is $\tilde{d}$(bounded), i.e. its lift $\tilde{\omega} \in \Omega^2(\tilde{M})$ to the universal covering can be written as $d(\eta)$ holds for some bounded 1-form $\eta \in \Omega^1(\tilde{M})$.

Theorem (Gromov [5])

Let $M$ be a closed Kähler hyperbolic manifold of complex dimension $c$. Then

$$b_n^{(2)}(\tilde{M}) = 0 \quad \text{if } n \neq c;$$
$$b_n^{(2)}(\tilde{M}) > 0;$$
$$(-1)^m \cdot \chi(M) > 0;$$
Let $M$ be a closed Kähler manifold. It is Kähler hyperbolic if it admits some Riemannian metric with negative sectional curvature, or, if, generally $\pi_1(M)$ is word-hyperbolic and $\pi_2(M)$ is trivial.

A consequence of the theorem above is that any Kähler hyperbolic manifold is a projective algebraic variety.
M. F. Atiyah.
Elliptic operators, discrete groups and von Neumann algebras.

J. Dodziuk.
de Rham-Hodge theory for $L^2$-cohomology of infinite coverings.

J. Dodziuk.
$L^2$ harmonic forms on rotationally symmetric Riemannian manifolds.

Ł. Grabowski.
On Turing dynamical systems and the Atiyah problem.

M. Gromov.
Kähler hyperbolicity and $L_2$-Hodge theory.
P. A. Linnell.
Division rings and group von Neumann algebras.

J. Lott and W. Lück.
$L^2$-topological invariants of 3-manifolds.

W. Lück.
Approximating $L^2$-invariants by their finite-dimensional analogues.

W. Lück.
$L^2$-Betti numbers of mapping tori and groups.

W. Lück.
T. Schick.
Integrality of $L^2$-Betti numbers.