Invariants of knots and 3-manifolds: Survey on 3-manifolds

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### Tentative plan of the course

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- No talks on May 1, May 10, May 22, May 24, May 31, July 5.

- On demand there can be a discussion session at the end of the Thursday lecture.
We give an introduction and survey about 3-manifolds.

We cover the following topics:
- Review of surfaces
- Prime decomposition and the Kneser Conjecture
- Jaco-Shalen-Johannsen splitting
- Thurston’s Geometrization Conjecture
- Fibering 3-manifolds
- Fundamental groups of 3-manifolds
Some basic facts surfaces

- **Surface** will mean compact, connected, orientable 2-dimensional manifold possibly with boundary.

- Every surface has a preferred structure of a PL-manifold or smooth manifold which is unique up to PL-homeomorphism or diffeomorphism.

- Every surface is homeomorphic to the standard model $F^d_g$, which is obtained from $S^2$ by deleting the interior of $d$ embedded $D^2$ and taking the connected sum with $g$-copies of $S^1 \times S^1$.

- The standard models $F^d_g$ and $F^{d'}_{g'}$ are homeomorphic if and only if $g = g'$ and $d = d'$ holds.

- Any homotopy equivalence of closed surfaces is homotopic to a homeomorphism.
The following assertions for two closed surfaces \( M \) and \( N \) are equivalent:

- \( M \) and \( N \) are homeomorphic;
- \( \pi_1(M) \cong \pi_1(N) \);
- \( H_1(M) \cong H_1(N) \);
- \( \chi(M) = \chi(N) \).

A closed surface admits a complete Riemannian metric with constant sectional curvature 1, 0 or \(-1\) depending on whether its genus \( g \) is 0,1 or \( \geq 2 \). For \(-1\) there are infinitely many such structures on a given surface of genus \( \geq 2 \).

A closed surface is either simply connected or aspherical.

A simply connected closed surface is homeomorphic to \( S^2 \).

A closed surface carries a non-trivial \( S^1 \)-action if and only if it is \( S^2 \) or \( T^2 \).
The fundamental group of a compact surface $F^d_g$ is explicitly known.

The fundamental group of a compact surface $F^d_g$ has the following properties

- It is either trivial, $\mathbb{Z}^2$, a finitely generated one-relator group, or a finitely generated free group;
- It is residually finite;
- Its abelianization is a finitely generated free abelian group;
- It has a solvable word problem, conjugacy problem and isomorphism problem.

**Question**

*Which of these properties carry over to 3-manifolds?*
3-manifold will mean compact, connected, orientable 3-dimensional manifold possibly with boundary.

Every 3-manifold has a preferred structure of a PL-manifold or smooth manifold which is unique up to PL-homeomorphism or diffeomorphism.

This is not true in general for closed manifolds of dimension $\geq 4$. 
Prime decomposition and the Kneser Conjecture

- Recall the connected sum of compact, connected, orientable \( n \)-dimensional manifolds \( M_0 \# M_1 \) and the fact that \( M \# S^n \) is homeomorphic to \( M \).

**Definition (prime)**

A 3-manifold \( M \) is called prime if for any decomposition as a connected sum \( M_0 \# M_1 \) one of the summands \( M_0 \) or \( M_1 \) is homeomorphic to \( S^3 \).

**Theorem (Prime decomposition)**

*Every 3-manifold \( M \), which is not homeomorphic to \( S^3 \), possesses a prime decomposition*

\[
M \cong M_1 \# M_2 \# \cdots \# M_r
\]

where each \( M_i \) is prime and not homeomorphic to \( S^3 \). This decomposition is unique up to permutation of the summands and homeomorphism.
**Definition (incompressible)**

Given a 3-manifold $M$, a compact connected orientable surface $F$ which is properly embedded in $M$, i.e., $\partial M \cap F = \partial F$, or embedded in $\partial M$, is called **incompressible** if the following holds:

- The inclusion $F \to M$ induces an injection on the fundamental groups;
- $F$ is not homeomorphic to $S^2$;
- If $F = D^2$, we do not have $F \subseteq \partial M$ and there is no embedded $D^3 \subseteq M$ with $\partial D^3 \subseteq D^2 \cup \partial M$.

One says that $\partial M$ is **incompressible in** $M$ if and only if $\partial M$ is empty or any component $C$ of $\partial M$ is incompressible in the sense above.

- $\partial M \subseteq M$ is incompressible if for every component $C$ the inclusion induces an injection $\pi_1(C) \to \pi_1(M)$ and $C$ is not homeomorphic to $S^2$. 
Theorem (The Kneser Conjecture is true)

Let $M$ be a compact 3-manifold with incompressible boundary. Suppose that there are groups $G_0$ and $G_1$ together with an isomorphism $\alpha : G_0 * G_1 \xrightarrow{\cong} \pi_1(M)$.

Then there are 3-manifolds $M_0$ and $M_1$ coming with isomorphisms $u_i : G_i \xrightarrow{\cong} \pi_1(M_i)$ and a homeomorphism $h : M_0 \# M_1 \xrightarrow{\cong} M$

such that the following diagram of group isomorphisms commutes up to inner automorphisms

$$
\begin{array}{ccc}
\pi_1(M_0) * \pi_1(M_1) & \xrightarrow{\cong} & \pi_1(M_0 \# M_1) \\
\uparrow u_0 * u_1 & & \downarrow \pi_1(h) \\
G_0 * G_1 & \xrightarrow{\cong} & \pi_1(M)
\end{array}
$$
**Definition (irreducible)**

A 3-manifold is called **irreducible** if every embedded two-sphere $S^2 \subseteq M$ bounds an embedded disk $D^3 \subseteq M$.

**Theorem**

A prime 3-manifold $M$ is either homeomorphic to $S^1 \times S^2$ or is irreducible.

**Theorem (Knot complement)**

The complement of a non-trivial knot in $S^3$ is an irreducible 3-manifold with incompressible toroidal boundary.
The Sphere and the Loop Theorem

**Theorem (Sphere Theorem)**

Let $M$ be a 3-manifold. Let $N \subseteq \pi_2(M)$ be a $\pi_1(M)$-invariant subgroup of $\pi_2(M)$ with $\pi_2(M) \setminus N \neq \emptyset$.

Then there exists an embedding $g : S^2 \to M$ such that $[g] \in \pi_2(M) \setminus N$.

- Notice that $[g] \neq 0$.
- However, the Sphere Theorem does not say that one can realize a given element $u \in \pi_2(M) \setminus N$ to be $u = [g]$.

**Corollary**

An irreducible 3-manifold is aspherical if and only if it is homeomorphic to $D^3$ or its fundamental group is infinite.
Theorem (Loop Theorem)

Let $M$ be a 3-manifold and let $F \subseteq \partial M$ be an embedded connected surface. Let $N \subseteq \pi_1(F)$ be a normal subgroup such that
\[
\ker(\pi_1(F) \to \pi_1(M)) \setminus N \neq \emptyset.
\]
Then there exists a proper embedding $(D^2, S^1) \to (M, F)$ such that $[g|_{S^1}]$ is contained in $\ker(\pi_1(F) \to \pi_1(M)) \setminus N$

- Notice that $[g] \neq 0$.
- However, the Loop Theorem does not say that one can realize a given element $u \in \ker(\pi_1(F) \to \pi_1(M)) \setminus N$ to be $u = [g]$. 
Haken manifolds

**Definition (Haken manifold)**
An irreducible 3-manifold is **Haken** if it contains an incompressible embedded surface.

**Lemma**
*If the first Betti number \( b_1(M) \) is non-zero, which is implied if \( \partial M \) contains a surface other than \( S^2 \), and \( M \) is irreducible, then \( M \) is Haken.*

A lot of conjectures for 3-manifolds could be proved for Haken manifolds first using an inductive procedure which is based on cutting a Haken manifold into pieces of smaller complexity using the incompressible surface.
**Conjecture (Waldhausen’s Virtually Haken Conjecture)**

*Every irreducible 3-manifold with infinite fundamental group has a finite covering which is a Haken manifold.*

**Theorem (Agol, [1])**

*The Virtually Haken Conjecture is true.*

- **Agol** shows that there is a finite covering with non-trivial first Betti number.
We use the definition of Seifert manifold given in the survey article by Scott [8], which we recommend as a reference on Seifert manifolds besides the book of Hempel [4].

**Lemma**

If a 3-manifold $M$ has infinite fundamental group and empty or incompressible boundary, then it is Seifert if and only if it admits a finite covering $\overline{M}$ which is the total space of a $S^1$-principal bundle over a compact orientable surface.

**Theorem (Gabai [3])**

An irreducible 3-manifold $M$ with infinite fundamental group $\pi$ is Seifert if and only if $\pi$ contains a normal infinite cyclic subgroup.
Definition (Hyperbolic)

A compact manifold (possible with boundary) is called hyperbolic if its interior admits a complete Riemannian metric whose sectional curvature is constant $-1$.

Lemma

Let $M$ be a hyperbolic 3-manifold. Then its interior has finite volume if and only if $\partial M$ is empty or a disjoint union of incompressible tori.
A **geometry** on a 3-manifold $M$ is a complete locally homogeneous Riemannian metric on its interior.

- Locally homogeneous means that for any two points there exist open neighbourhoods which are isometrically diffeomorphic.

- The universal cover of the interior has a complete homogeneous Riemannian metric, meaning that the isometry group acts transitively. This action is automatically proper.

- **Thurston** has shown that there are precisely eight maximal simply connected 3-dimensional geometries having compact quotients, which often come from left invariant Riemannian metrics on connected Lie groups.
\[ S^3, \quad \text{Isom}(S^3) = O(4); \]
\[ \mathbb{R}^3, \quad 1 \to \mathbb{R}^3 \to \text{Isom}(\mathbb{R}^3) \to O(3) \to 1; \]
\[ S^2 \times \mathbb{R}, \quad \text{Isom}(S^2 \times \mathbb{R}) = \text{Isom}(S^2) \times \text{Isom}(\mathbb{R}); \]
\[ H^2 \times \mathbb{R}, \quad \text{Isom}(H^2 \times \mathbb{R}) = \text{Isom}(H^2) \times \text{Isom}(\mathbb{R}); \]
\[ \widetilde{\text{SL}_2(\mathbb{R})}, \quad 1 \to \mathbb{R} \to \text{Isom}(\widetilde{\text{SL}_2(\mathbb{R})}) \to \text{PSL}_2(\mathbb{R}) \to 1; \]
\[ \text{Nil} := \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\}, \quad 1 \to \mathbb{R} \to \text{Isom}(\text{Nil}) \to \text{Isom}(\mathbb{R}^2) \to 1; \]
\[ \text{Sol} := \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}; \quad 1 \to \text{Sol} \to \text{Isom}(\text{Sol}) \to D_{2.4} \to 1; \]
\[ \mathbb{H}^3, \quad \text{Isom}(\mathbb{H}^3) = \text{PSL}_2(\mathbb{C}). \]
A geometry on a 3-manifold $M$ modelled on $S^3$, $IR^3$ or $H^3$ is the same as a complete Riemannian metric on the interior of constant section curvature with value 1, 0 or $-1$.

If a closed 3-manifold admits a geometric structure modelled on one of these eight geometries, then the geometry involved is unique.

The geometric structure on a fixed 3-manifold is in general not unique. For instance, one can scale the standard flat Riemannian metric on the torus $T^3$ by a real number and just gets a new geometry with different volume which of course still is a $R^3$-geometry.
**Theorem (Mostow Rigidity)**

Let $M$ and $N$ be two hyperbolic $n$-manifolds with finite volume for $n \geq 3$. Then for any isomorphism $\alpha : \pi_1(M) \xrightarrow{\cong} \pi_1(N)$ there exists an isometric diffeomorphism $f : M \to N$ such that up to inner automorphism $\pi_1(f) = \alpha$ holds.

- This is not true in dimension 2, see Teichmüller space.
A 3-manifold is a Seifert manifold if and only if it carries one of the geometries $S^2 \times \mathbb{R}$, $\mathbb{R}^3$, $H^2 \times \mathbb{R}$, $S^3$, $\text{Nil}$, or $\text{SL}_2(\mathbb{R})$. In terms of the Euler class $e$ of the Seifert bundle and the Euler characteristic $\chi$ of the base orbifold, the geometric structure of a closed Seifert manifold $M$ is determined as follows:

<table>
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<tr>
<th>$e = 0$</th>
<th>$\chi &gt; 0$</th>
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<th>$\chi &lt; 0$</th>
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<td>$S^2 \times \mathbb{R}$</td>
<td>$\mathbb{R}^3$</td>
<td>$H^2 \times \mathbb{R}$</td>
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<tr>
<td>$e \neq 0$</td>
<td>$S^3$</td>
<td>$\text{Nil}$</td>
<td>$\text{SL}_2(\mathbb{R})$</td>
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Let $M$ be a prime 3-manifold with empty boundary or incompressible boundary. Then it is a Seifert manifold if and only if it is finitely covered by the total space $\overline{M}$ of an principal $S^1$-bundle $S^1 \to \overline{M} \to F$ over a surface $F$.

Moreover, $e(M) = 0$ if and only if this $S^1$-principal bundle is trivial, and the Euler characteristic $\chi$ of the base orbifold of $M$ is negative, zero or positive according to the same condition for $\chi(F)$.

The boundary of a Seifert manifold is incompressible unless $M$ is homeomorphic to $S^1 \times D^2$.

A Seifert manifold is prime unless it is $\mathbb{RP}^3 \# \mathbb{RP}^3$.

Let $M$ be a Seifert manifold with finite fundamental group. Then $M$ is closed and carries a $S^3$-geometry.
A 3-manifold admits an $S^1$-foliation if and only if it is a Seifert manifold.

Every $S^1$-action on a hyperbolic closed 3-manifold is trivial.

A 3-manifold carries a Sol-structure if and only if it is finitely covered by the total space $E$ of a locally trivial fiber bundle $T^2 \to E \to S^1$ with hyperbolic glueing map $T^2 \to T^2$, where hyperbolic is equivalent to the condition that the absolute value of the trace of the automorphism of $H_1(T^2)$ is greater or equal to 3.
Theorem (Jaco-Shalen [5], Johannson [6])

Let $M$ be an irreducible 3-manifold $M$ with incompressible boundary.

1. There is a finite family of disjoint, pairwise-nonisotopic incompressible tori in $M$ which are not isotopic to boundary components and which split $M$ into pieces that are Seifert manifolds or are geometrically atoroidal, i.e., they admit no embedded incompressible torus (except possibly parallel to the boundary).

2. A minimal family of such tori is unique up to isotopy.
Definition (Toral splitting or JSJ-decomposition)

We will say that the minimal family of such tori gives a toral splitting or a JSJ-decomposition.

We call the toral splitting a geometric toral splitting if the geometrically atoroidal pieces which do not admit a Seifert structure are hyperbolic.
Thurston’s Geometrization Conjecture

Conjecture (Thurston’s Geometrization Conjecture)

- An irreducible 3-manifold with infinite fundamental group has a geometric toral splitting;

- For a closed 3-manifold with finite fundamental group, its universal covering is homeomorphic to $S^3$, the fundamental group of $M$ is a subgroup of $SO(4)$ and the action of it on the universal covering is conjugated by a homeomorphism to the restriction of the obvious $SO(4)$-action on $S^3$.

Theorem (Perelmann, see Morgan-Tian [7])

Thurston’s Geometrization Conjecture is true.
Thurston’s Geometrization Conjecture implies the 3-dimensional Poincaré Conjecture.

Thurston’s Geometrization Conjecture implies:
- The fundamental group of a 3-manifold $M$ is residually finite, Hopfian and has a solvable word, conjugacy and membership problem.
- If $M$ is closed, $\pi_1(M)$ has a solvable isomorphism problem.
- Every closed 3-manifold has a solvable homeomorphism problem.

Thanks to the proof of the Geometrization Conjecture, there is a complete list of those finite groups which occur as fundamental groups of closed 3-manifolds. They all are subgroups of $SO(4)$.

Recall that, for every $n \geq 4$ and any finitely presented group $G$, there exists a closed $n$-dimensional smooth manifold $M$ with $\pi_1(M) \cong G$. 
Thurston’s Geometrization Conjecture implies the **Borel Conjecture** in dimension 3 stating that every homotopy equivalence of aspherical closed 3-manifolds is homotopic to a homeomorphism.

There are irreducible 3-manifolds with finite fundamental group which are homotopy equivalent but not homeomorphic, namely the lens spaces $L(7; 1, 1)$ and $L(7; 1, 2)$.

Thurston’s Geometrization Conjecture is needed in the proof of the **Full Farrell-Jones Conjecture** for the fundamental group of a (not necessarily compact) 3-manifold (possibly with boundary).
Thurston’s Geometrization Conjecture is needed in the complete calculation of the $L^2$-invariants of the universal covering of a 3-manifold.

These calculations and calculations of other invariants follow the following pattern:

- Use the prime decomposition to reduce it to irreducible manifolds.
- Use the Thurston Geometrization Conjecture and glueing formulas to reduce it to Seifert manifolds or hyperbolic manifolds.
- Treat Seifert manifolds with topological methods.
- Treat hyperbolic manifolds with analytic methods.
Theorem (Stallings [9])

The following assertions are equivalent for an irreducible 3-manifold $M$ and an exact sequence $1 \to K \to \pi_1(M) \to \mathbb{Z} \to 1$:

- $K$ is finitely generated;
- $K$ is the fundamental group of a surface $F$;
- There is a locally trivial fiber bundle $F \to M \to S^1$ with a surface $F$ as fiber such that the induced sequence

$$1 \to \pi_1(F) \to \pi_1(E) \to \pi_1(S^1) \to 1$$

can be identified with the given sequence.
Conjecture (Thurston’s Virtual Fibering Conjecture)

Let $M$ be a closed hyperbolic 3-manifold. Then a finite covering of $M$ fibers over $S^1$, i.e., is the total space of a surface bundle over $S^1$.

- A locally compact surface bundle $F \to E \to S^1$ is the same as a selfhomeomorphism of the surface $F$ by the mapping torus construction.
- Two surface homeomorphisms are isotopic if and only if they induce the same automorphism on $\pi_1(F)$ up to inner automorphisms.
- Therefore mapping class groups play an important role for 3-manifolds.
Theorem (Agol, [1])

The Virtually Fibering Conjecture is true.

Definition (Graph manifold)

An irreducible 3-manifold is called graph manifold if its JSJ-splitting contains no hyperbolic pieces.

- There are aspherical closed graph manifolds which do not virtually fiber over $S^1$.
- There are closed graph manifolds, which are aspherical, but do not admit a Riemannian metric of non-positive sectional curvature.
- Agol proved the Virtually Fibering Conjecture for any irreducible manifold with infinite fundamental group and empty or incompressible toral boundary which is not a closed graph manifold.
Actually, Agol, based on work of Wise, showed much more, namely that the fundamental group of a hyperbolic 3-manifold is virtually compact special. This implies in particular that they occur as subgroups of RAAG-s (right Artin angled groups) and that they are linear over $\mathbb{Z}$ and LERF (locally extended residually finite). For the definition of these notions and much more information we refer for instance to Aschenbrenner-Friedl-Wilton [2].
On the fundamental groups of 3-manifolds

- The fundamental group plays a dominant role for 3-manifolds what we want to illustrate by many examples and theorems.

- A 3-manifold is prime if and only if $\pi_1(M)$ is prime in the sense that $\pi_1(M) \cong G_0 \ast G_1$ implies that $G_0$ or $G_1$ are trivial.

- A 3-manifold is irreducible if and only if $\pi_1(M)$ is prime and $\pi_1(M)$ is not infinite cyclic.

- A 3-manifold is aspherical if and only if its fundamental group is infinite, prime and not cyclic.

- A 3-manifold has infinite cyclic fundamental group if and only if it is homeomorphic to $S^1 \times S^2$. 
Let $M$ and $N$ be two prime closed 3-manifolds whose fundamental groups are infinite. Then:

- $M$ and $N$ are homeomorphic if and only if $\pi_1(M)$ and $\pi_1(N)$ are isomorphic.
- Any isomorphism $\pi_1(M) \xrightarrow{\cong} \pi_1(N)$ is induced by a homeomorphism.

Let $M$ be a closed irreducible 3-manifold with infinite fundamental group. Then $M$ is hyperbolic if and only if $\pi_1(M)$ does not contain $\mathbb{Z} \oplus \mathbb{Z}$ as subgroup.

Let $M$ be a closed irreducible 3-manifold with infinite fundamental group. Then $M$ is a Seifert manifold if and only if $\pi_1(M)$ contains a normal infinite subgroup.
A closed Seifert 3-manifold carries precisely one geometry and one can read off from $\pi_1(M)$ which one it is:

- $S^3$
  
  $\pi_1(M)$ is finite.

- $\mathbb{R}^3$
  
  $\pi_1(M)$ contains $\mathbb{Z}^3$ as subgroup of finite index.

- $S^2 \times \mathbb{R}$
  
  $\pi_1(M)$ is virtually cyclic.

- $H^2 \times \mathbb{R}$
  
  $\pi_1(M)$ contains a subgroup of finite index which is isomorphic to $\mathbb{Z} \times \pi_1(F)$ for some closed surface $F$ of genus 2.

- $\tilde{SL}_2(\mathbb{R})$
  
  $\pi_1(M)$ contains a subgroup of finite index $G$ which can be written as a non-trivial central extension $1 \to \mathbb{Z} \to G \to \pi_1(F) \to 1$ for a surface $F$ of genus $\geq 2$.

- Nil
  
  $\pi_1(M)$ contains a subgroup of finite index $G$ which can be written as a non-trivial central extension $1 \to \mathbb{Z} \to G \to \mathbb{Z}^2 \to 1$. 

Definition (deficiency)

The deficiency of a finite presentation \( \langle g_1, \ldots, g_m \mid r_1 \ldots, r_n \rangle \) of a group \( G \) is defined to be \( m - n \).

The deficiency of a finitely presented group is defined to be the supremum of the deficiencies of all its finite presentations.

Lemma

Let \( M \) be an irreducible 3-manifold. If its boundary is empty, its deficiency is 0. If its boundary is non-empty, its deficiency is \( 1 - \chi(M) \).
We have already mentioned the following facts:

- The fundamental group of a 3-manifold is residually finite, Hopfian and has a solvable word and conjugacy problem.
- If $M$ is closed, $\pi_1(M)$ has a solvable isomorphism problem.
- There is a complete list of those finite groups which occur as fundamental groups of closed 3-manifolds. They all are subgroups of $SO(4)$.
- The fundamental group of a hyperbolic 3-manifold is virtually compact special and linear over $\mathbb{Z}$. 
Some open problems

Definition (Poincaré duality group)

A Poincaré duality group $G$ of dimension $n$ is a finitely presented group satisfying:

- $G$ is of type FP;
- $H^i(G; \mathbb{Z}G) \cong \begin{cases} 0 & i \neq n; \\ \mathbb{Z} & i = n. \end{cases}$

Conjecture (Wall)

Every Poincaré duality group is the fundamental group of an aspherical closed manifold.
Conjecture (Cannon’s Conjecture in the torsionfree case)

A torsionfree hyperbolic group $G$ has $S^2$ as boundary if and only if it is the fundamental group of a closed hyperbolic 3-manifold.

Conjecture (Bergeron-Venkatesh)

Suppose that $M$ is a closed hyperbolic 3-manifold. Let

$$\pi_1(M) = G_0 \supseteq G_1 \supseteq G_2 \supseteq$$

be a nested sequence of normal subgroups $G_i$ of finite index of $\pi_1(M)$ with $\bigcap_i G_i = \{1\}$. Let $M_i \to M$ be the finite covering associated to $G_i \subseteq \pi_1(M)$. Then

$$\lim_{i \to \infty} \frac{\ln(\|\text{tors}(H_1(G_i))\|)}{[G : G_i]} = \frac{1}{6\pi} \cdot \text{vol}(M).$$
Let $M$ be an aspherical 3-manifold with empty or toroidal boundary with fundamental group $G = \pi_1(M)$, which does not admit a non-positively curved metric.

1. Is $G$ linear over $\mathbb{C}$?

2. Is $G$ linear over $\mathbb{Z}$?

3. If $G$ is not solvable, does it have a subgroup of finite index which is for every prime $p$ residually finite of $p$-power?

4. Is $G$ virtually bi-orderable?

5. Does $G$ satisfy the Atiyah Conjecture about the integrality of the $L^2$-Betti numbers of universal coverings of closed Riemann manifolds of any dimension and fundamental group $G$?

6. Is the group ring $\mathbb{Z}G$ a domain?
Questions

- Does the isomorphism problem has a solution for the fundamental groups of (not necessarily closed) 3-manifolds?
- Does the homeomorphism problem has a solution for (not necessarily closed) 3-manifolds?
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