

THE ROUND HANDLE PROBLEM

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ABSTRACT. We present the Round Handle Problem (RHP), proposed by Freedman and Krushkal. It asks whether a collection of links, which contains the Generalised Borromean Rings (GBRs), are slice in a 4-manifold R constructed from adding round handles to the four ball. A negative answer would contradict the union of the surgery conjecture and the s -cobordism conjecture for 4-manifolds with free fundamental group.

1. STATEMENT OF THE RHP

These notes give an alternative proof of the connection of the Round Handle Problem to the topological surgery and s -cobordism conjectures (these will all be recalled below). The Round Handle Problem (RHP) was formulated in [FK16, Section 5.1]. We give a shorter and easier argument that the above mentioned conjectures imply a positive answer to the RHP.

Let $L = L_1 \sqcup \cdots \sqcup L_m$ be an oriented ordered link in S^3 with vanishing pairwise linking numbers. We will be particularly concerned with the Generalised Borromean Rings (GBRs). By definition these are the collection of links arising from iterated Bing doubling starting with a Hopf link.

Write $X_L := S^3 \setminus \nu L$ for the exterior of L . Let $\mu_i \subset X_L$ be an oriented meridian of the i th component of L , and let $\lambda_i \subset X_L$ be a zero-framed oriented longitude. Make μ_i small enough that $\text{lk}(\mu_i, \lambda_i) = 0$ (of course $\text{lk}(\mu_i, L_i) = 1$ and $\text{lk}(\lambda_i, L_i) = 0$).

A *Round handle* H is a copy of $S^1 \times D^2 \times D^1$. The *attaching region* is $S^1 \times D^2 \times S^0 \subset \partial(S^1 \times D^2 \times D^1) \cong S^1 \times S^2$.

Definition 1.1. Given an m -component link L , construct a manifold $R(L)$ by attaching m round handles $\{H_i\}_{i=1}^m$ to D^4 as follows. For the i th round handle, glue $S^1 \times D^2 \times \{-1\}$ to $\nu\mu_i \subset X_L \subset S^3 = \partial D^4$, and glue $S^1 \times D^2 \times \{1\}$ to $\nu\lambda_i$. In both cases use the zero-framing for the identification of $\nu\mu_i$ and $\nu\lambda_i$ with $S^1 \times D^2$. Note that the link L lies in $\partial R(L)$.

The key question will be whether L is slice in $R(L)$.

Definition 1.2 (Round Handle Slice). A link L is *Round Handle Slice (RHS)* if $L \subset \partial R(L)$ is slice in $R(L)$, that is if L is the boundary of a disjoint union of locally flat embedded discs in $R(L)$.

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Theorem 1.3. *Suppose that the topological surgery and s -cobordism conjectures hold. Then for any link L with pairwise linking numbers all zero, L is round handle slice.*

Problem 1.4. The Round Handle Problem is to determine whether all pairwise linking number zero links are slice in $R(L)$.

A negative answer for one such link would contradict the logical union of the topological surgery conjecture and the s -cobordism conjecture. It is suggested by Freedman and Krushkal, but by no means compulsory, to focus on the links arising as GBRs. It is also suggested that one might try to adapt Milnor's invariants to provide obstructions. The primary purpose of this problem, like the AB slice problem, is to provide a way to get obstructions to surgery and s -cobordism.

We briefly recall the statements of these conjectures and their relation to the disc embedding problem.

Conjecture 1.5 (Topological surgery conjecture). *Any degree one normal map $(M, \partial M) \rightarrow (X, \partial X)$ from a 4-manifold M to a 4-dimensional Poincaré pair $(X, \partial X)$, that is a $\mathbb{Z}[\pi_1(X)]$ -homology equivalence on the boundary, is topologically normally bordant rel. boundary to a homotopy equivalence if and only if the surgery obstruction in $L_4(\mathbb{Z}[\pi_1(X)])$ vanishes.*

Conjecture 1.6 (s -cobordism conjecture). *Every compact topological 5-dimensional s -cobordism $(W; M_0, M_1)$, that is a product on the boundary, is homeomorphic to a product $W \cong M_0 \times I \cong M_1 \times I$, extending the given product structure on the boundary.*

In Section 3 we will explain why the union of these two conjectures is equivalent to the disc embedding conjecture below. In the statement of this conjecture below we use the equivariant intersection form $\lambda: \pi_2(M) \times \pi_2(M) \rightarrow \mathbb{Z}[\pi_1(M)]$. Also note that the transverse spheres are required to be *framed*, which means that they have trivialised normal bundles.

Conjecture 1.7 (Disc embedding conjecture). *Let $f_i: (D^2, S^1) \looparrowright (M, \partial M)$ be a collection of immersed discs in a 4-manifold M , and suppose that there are framed immersed spheres $g_i: S^2 \looparrowright M$ such that $\lambda(g_i, g_j) = 0$, and the g_i are transverse spheres, so $\lambda(f_i, g_j) = \delta_{ij}$. Then the circles $f_i(S^1)$ bound disjointly embedded discs in M with transverse spheres, inducing the same framing on $f_i(S^1)$ as the f_i .*

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2. PROOF OF THEOREM 1.3

The proof of Theorem 1.3 involves the construction of an s -cobordism rel. boundary from the manifold $R(L)$, henceforth abbreviated to R , to another 4-manifold R' , in which L is slice.

We begin with a Kirby diagram for R , shown in Figure 1. First we will explain the figure, then we will explain why this is a diagram for R . The diagram does not show the literal Kirby diagram for R . Rather, the curve labelled d specifies a solid torus, as the complement of a regular neighbourhood of this curve. Inside the solid torus two dotted circles, corresponding to 1-handles, and two zero-framed circles, corresponding to 2-handles, can be seen. Embed a copy of this solid torus into νL_i for each $i = 1, \dots, m$ using the zero framing. The diagram also shows the link L parallel to the core of the solid torus.

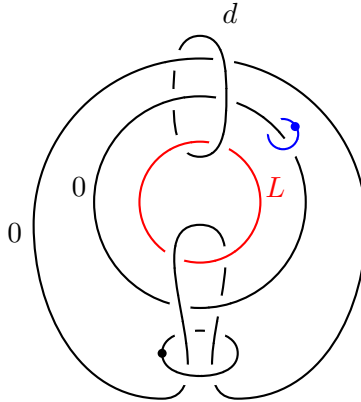
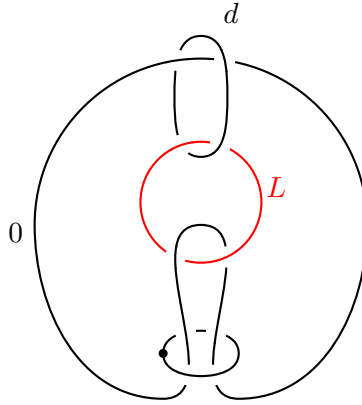
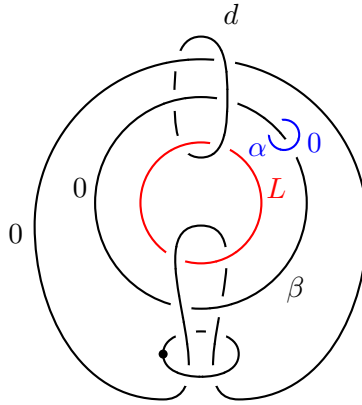


FIGURE 1. A handle diagram for R .

Now we explain why Figure 1 is a diagram for the 4-manifold R . Observe that one of the 1-handles and one of the 2-handles are in cancelling position. Cancel this pair, to obtain the handle diagram shown in Figure 2. This diagram can be seen with a little thought to be a diagram for R . A round handle can be constructed from a 1-handle and a 2-handle whose boundary goes around one attaching circle of the round handle (a meridian of L), traverses the 1-handle, goes around the other attaching circle (a zero-framed longitude of the same component of L), and then traverses the 1-handle in the other direction. Ignoring the link L , we see that R is diffeomorphic to the zero-trace of L with m 1-handles added.

Next, Figure 3 shows a Kirby diagram, with the same convention as above, for a 4-manifold that we call R_M . Here M stands for “middle,” since this manifold will lie in the middle of an s -cobordism that we are about to construct.

The diagram for R_M is very similar to the diagram for R from Figure 1; in order to get from the diagram for R_M to that for R , change the zero-framed

FIGURE 2. The diagram for R from Figure 1 after cancellations.FIGURE 3. A handle diagram for R_M .

2-handle whose attaching curve is labelled α in Figure 3 to a 1-handle. That is, perform surgery on the 2-sphere obtained from the core of the 2-handle union a disc bounded by the attaching circle in D^4 . Note that, by virtue of the cores of the 2-handle labelled β in Figure 3, L is slice in R_M . To see this, just observe that L can be passed through the attaching region of the α 2-handles in Figure 3. On the other hand, in Figure 1, L cannot be passed through a dotted circle corresponding to a 1-handle, so this argument cannot be used to show that L is slice in R in Figure 1. If one passes the link past the attaching region of a 2-handle, one cannot later use the core of that 2-handle to construct an embedded slice disc. So likewise one cannot use Figure 2 to see that L is slice in R .

On the other hand, there are also immersed 2-spheres in R_M obtained from the union of the cores of the β 2-handles with immersed discs in D^4 bounded by the β attaching curves. The linking number zero hypothesis implies that the algebraic intersection numbers between these 2-spheres vanish.

These β spheres have framed dual spheres arising from the round handle 2-handles; namely the 2-handles that also appear in Figure 2. These 2-handles are algebraically dual to the β spheres because they have attaching curves that link the β curves once. By the disc embedding conjecture, we can therefore find framed embedded spheres representing the regular homotopy classes of the β 2-spheres. We actually apply disc embedding to Whitney discs pairing up double points of the given immersed discs, in the complement of the slice discs for L in R_M . Construct the necessary transverse spheres from Clifford tori for the double points, with caps given by normal discs to the β spheres tubed into the dual spheres. Use the caps to symmetrically contract [FQ90, Section 2.3] the tori to immersed spheres. See [FQ90, Corollary 5.2B] for more details. Whitney moves across the embedded discs resulting from Conjecture 1.7 give a regular homotopy to the desired framed embedded spheres.

Perform surgery on R_M using these framed embedded spheres, and define R' to be the 4-manifold obtained as result of these surgeries. Note that L is still slice in R' , since our initial immersed discs and their duals lie in the complement of the slice discs, and we apply disc embedding in this complement. The disc embedding conjecture has no hypothesis on the fundamental group, so we do not need to control the fundamental group here.

Lemma 2.1. *The 4-manifolds R and R' are s -cobordant rel. boundaries.*

Proof. Start with R_M . The trace of surgeries on the α spheres gives a cobordism to R . The trace of surgeries on the β spheres gives a cobordism to R' . The union of the two cobordisms along R_M is an s -cobordism from R to R' , since algebraically the intersection numbers $\alpha_i \cdot \beta_j = \delta_{ij}$. \square

Note that we used duals to the β spheres twice, once to apply surgery and once to prove that we have an s -cobordism. However we use *different* duals. For the surgery we use the duals arising from the round handle 2-handles. For the s -cobordism, we use the α spheres.

Then since R and R' are s -cobordant, the s -cobordism Conjecture 1.6 implies that they are homeomorphic. The homeomorphism is an identity on the boundary, so preserves the link L . Thus the image of the slice discs for L in R' under the homeomorphism $f: R' \rightarrow R$ are slice discs for L in R . It follows that L is Round Handle Slice as desired. This completes the proof of Theorem 1.3.

3. DISC EMBEDDING IS EQUIVALENT TO SURGERY AND s -COBORDISM

In this section we briefly argue that the disc embedding Conjecture 1.7 is equivalent to the surgery and s -cobordism Conjectures 1.5 and 1.6 respectively.

We will argue that the following are equivalent: (i) surgery and s cobordism; (ii) disc embedding; (iii) height 1.5 capped gropes contain embedded

discs with the same boundary; (iv) certain links $L \cup m$, to be described below, are slice with standard slice discs for L .

- (1) The fact that the disc embedding conjecture implies surgery and s -cobordism follows from inspection of the high dimensional proof: the proof of topological surgery in dimension four and the five dimensional topological s -cobordism theorem can be reduced to precisely the need to find embedded discs in the presence of transverse spheres. See for example [LÖ2] for an exposition of the high dimensional theory. The s -cobordism theorem requires extra argument to find the transverse spheres, which can be found in [FQ90, Chapter 7].
- (2) The disc embedding conjecture is equivalent to the statement that every height 1.5 capped grope has a neighbourhood that contains an embedded disc with the same framed boundary. For one direction, if disc embedding holds, then we can use it to find a disc in a neighbourhood of a height 1.5 capped grope, as follows. The caps on the height 1 side are immersed discs, and parallel copies of the symmetric contraction of the height 1.5 side, together with annuli in neighbourhoods of the boundary circles, give transverse spheres that have the right algebraic intersection data. See [FQ90, Section 2.6] for the construction of transverse gropes within a grope neighbourhood, which are then symmetrically contracted [FQ90, Section 2.3] to yield transverse spheres. Apply disc embedding to find embedded discs with framed boundary the same as the height 1 caps' framed boundary. These correctly framed embedded discs can be used to asymmetrically contract the first stage of the height 1.5 grope to an embedded disc. On the other hand, a collection of discs with transverse spheres as in Conjecture 1.7 gives rise to a height 1.5 capped grope with the same boundary, as shown in [FQ90, Section 5.1]. Thus if every height 1.5 capped grope has a neighbourhood that contains an embedded disc, then disc embedding holds.
- (3) A Kirby diagram for a capped grope consists of an unlink L , in the form of a link obtained from the unknot by iterated ramified Bing doubling, followed by a single operation of ramified Whitehead doubling. Place a dot on every component to denote that they correspond to 1-handles; a neighbourhood of a capped grope is diffeomorphic to a boundary connected sum of copies of $S^1 \times D^3$. The boundary circle of the grope is represented by a meridian m to the original unknot. One can think of performing the ramified Bing and Whitehead doubling on one component of the Hopf link. A grope contains an embedded disc with the same framed boundary if and only if this link $L \cup m$ is slice with standard slice discs for all the dotted components. The desired embedded disc is the slice disc for m . See [FQ90, Proposition 12.3A] for further details.

- (4) Let $L \cup m$ be any link from the family constructed in the previous item, using iterated ramified Bing and Whitehead doubling on one component of the Hopf link. The zero surgery on $L \cup m$ bounds a spin 4-manifold over a wedge of circles since the Arf invariants of the components vanish. By the topological surgery conjecture, this can be improved, via a normal bordism rel. boundary, to be homotopy equivalent to the wedge of circles. Attach a 2-handle to fill in the surgery torus $D^2 \times S^1$ of m . The remaining 4-manifold is homeomorphic to a boundary connected sum of copies of $S^1 \times D^3$, by the s -cobordism conjecture. Therefore it is homeomorphic to the exterior of standard slice discs for L in D^4 . We have no control over the remaining slice disc, whose boundary is the link component m . Thus surgery and s -cobordism imply that the link $L \cup m$ is slice with standard slice discs for L . More details are given in [FQ90, Section 11.7C] and the preceding sections of Chapter 11. By the previous two items, this implies the disc embedding conjecture.

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