Pulling Apart 2–Spheres in 4–Manifolds

Rob Schneiderman and Peter Teichner

Received: November 5, 2012
Revised: May 1, 2014

Communicated by Ursula Hamenstädt

Abstract. An obstruction theory for representing homotopy classes of surfaces in 4–manifolds by immersions with pairwise disjoint images is developed, using the theory of *non-repeating* Whitney towers. The accompanying higher-order intersection invariants provide a geometric generalization of Milnor's link-homotopy invariants, and can give the complete obstruction to pulling apart 2–spheres in certain families of 4–manifolds. It is also shown that in an arbitrary simply connected 4–manifold any number of parallel copies of an immersed 2–sphere with vanishing self-intersection number can be pulled apart, and that this is not always possible in the non-simply connected setting. The order 1 intersection invariant is shown to be the complete obstruction to pulling apart 2–spheres in any 4–manifold after taking connected sums with finitely many copies of $S^2 \times S^2$; and the order 2 intersection indeterminacies for quadruples of immersed 2–spheres in a simply-connected 4–manifold are shown to lead to interesting number theoretic questions.

2010 Mathematics Subject Classification: Primary 57M99; Secondary 57M25.
Keywords and Phrases: 2–sphere, 4–manifold, disjoint immersion, homotopy invariant, non-repeating Whitney tower.

1 Introduction

We study the question of whether a map $A : \Sigma \to X$ is homotopic to a map A' such that $A'(\Sigma_i)$ are pairwise *disjoint* subsets of X, where $\Sigma = \amalg_i \Sigma_i$ is the decomposition into connected components. In this case, we will say that A *can be pulled apart.*

This question arises as a precursor to the embedding problem – whether or not A is homotopic to an embedding. It also arises in the study of configuration spaces $X^{(m)}$ of m *distinct ordered* points in X, where elements of $\pi_n X^{(m)}$ are represented by m disjoint maps of n–spheres to X, and one might ask whether or not a given element of the m-fold product $\prod^m \pi_n X$ lies in the image of the map $\pi_n X^{(m)} \to \prod^m \pi_n X$ induced by the canonical projections $p_1, \ldots, p_m : X^{(m)} \to X$.

For example, let $\Sigma = \amalg_i S^n$ be a disjoint union of n–spheres and X be a connected $2n$–manifold. For $n \geq 2$, there are Wall's well known intersection numbers $\lambda(A_i, A_j) \in \mathbb{Z}[\pi_1 X]$, where $A_i : S^n \to X$ are the components of A [33]. These are obstructions for representing A by an embedding, and the main geometric reason for the success of surgery theory is that, for $n \geq 3$, they are (almost) complete obstructions: The only missing ingredient is Wall's self-intersection invariant μ, a quadratic refinement of λ. However, for the question of making the $A_i(S^n)$ disjoint, it is necessary and sufficient that $\lambda(A_i, A_j) = 0$ for $i \neq j$. We abbreviate this condition on intersection numbers by writing $\lambda_0(A) = 0$.

As expected, the condition $\lambda_0(A) = 0$ is not sufficient for pulling apart A if $n = 2$, but this failure is surprisingly subtle: Given only *two* maps $A_1, A_2 : S^2 \to X^4$ with $\lambda(A_1, A_2) = 0$, one can pull them apart by a clever sequence of finger moves and Whitney moves, see [19] and Section 1.1 below. However, this is *not* true any more for three (or more) 2–spheres in a 4–manifold. In [30] we defined an additional invariant $\lambda_1(A)$ which takes values in a quotient of $\mathbb{Z}[\pi_1 X \times \pi_1 X]$ and was shown to be the complete obstruction to pulling apart a *triple* $A = A_1, A_2, A_3 : S^2 \to X$ of 2–spheres mapped into an arbitrary 4–manifold X with vanishing $\lambda_0(A)$. For trivial $\pi_1 X$ the analogous obstruction was defined earlier in [25, 34].

In this paper, we extend this work to an arbitrary number of 2–spheres (and other surfaces – see Remark 16) in 4–manifolds. The idea is to apply a variation of the theory of *Whitney towers* as developed in [3, 4, 5, 6, 29, 30, 31] to address the problem. Before we introduce the relevant material on Whitney towers, we mention a couple of new results that can be stated without prerequisites.

Throughout this paper the letter m will usually denote the number of surface components to be pulled apart, and from now on the letters Σ and X will be used to denote surfaces and 4–manifolds, respectively. The distinction between a map of a surface and its image in X will frequently be disregarded in the interest of brevity.

Pulling apart parallel 2–spheres

The following theorem is discussed and proven in Section 6:

Theorem 1 *If X is a simply connected 4–manifold and $A : \amalg^m S^2 \to X$ consists of m copies of* the same *map $A_0 : S^2 \to X$ of a 2–sphere with trivial normal Euler number, then A can be pulled apart if and only if $[A_0] \in H_2(X; \mathbb{Z})$ has vanishing homological self-intersection number $[A_0] \cdot [A_0] = 0 \in \mathbb{Z}$.*

Note that each transverse self-intersection of A_0 gives rise to $m^2 - m$ intersections among the m parallel copies A, not counting self-intersections, see Figure 1. As a consequence, there cannot be a simple argument to pull A apart. In fact, the analogous statement fails for non-simply connected 4–manifolds, see Example 6.2.

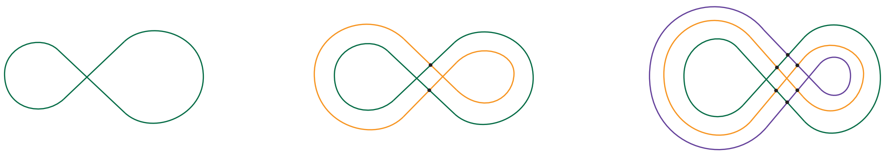

Figure 1: One self-intersection leads to $m^2 - m$ intersections among m copies.

Stably pulling apart 2–spheres

We say that surfaces $A : \Sigma \to X$ can be *stably* pulled apart if A can be pulled apart after taking the connected sum of X with finitely many copies of $S^2 \times S^2$. The invariants λ_0 and λ_1 are unchanged by this stabilization, and in this setting they give the complete obstruction to pulling apart m maps of 2–spheres:

Theorem 2 *$A : \amalg^m S^2 \to X$ can be stably pulled apart if and only if $\lambda_0(A) = 0 = \lambda_1(A)$.*

This result also holds when the stabilizing factors $S^2 \times S^2$ are replaced by any simply-connected closed 4-manifolds (other than S^4). It also holds when components of A are maps of disks. The invariant λ_1 is described precisely in sections 2 and 8; and the proof of Theorem 2 is given in section 7.2. (Note that X is not required to be simply connected in Theorem 2.) We remark that the stronger invariant $\tau_1(A)$ of [30], together with Wall's self-intersection invariant $\tau_0(A)$, is the complete obstruction to *stably embedding* A, see [29, Cor.1].

Remark 3 *The question of pulling apart surfaces in 4–manifolds is independent of category. More precisely, any connected 4–manifold can be given a smooth structure away from one point [12] and any continuous map can be approximated arbitrarily closely by a smooth map. As a consequence, we can work in the smooth category and as a first step, we can always turn a map $A : \Sigma^2 \to X^4$ into a generic immersion. We will also assume that surfaces are* properly *immersed, i.e. $A(\partial\Sigma) \subset \partial X$ with the interior of Σ mapping to the interior of X, and that homotopies fix the boundary.*

1.1 Pulling apart *two* 2–spheres in a 4–manifold

To motivate the introduction of Whitney towers into the problem, it is important to understand the basic case of pulling apart *two* maps of 2–spheres

$A_1, A_2 : S^2 \to X$. Wall's intersection pairing associates a sign and an element of $\pi_1 X$ to each transverse intersection point between the surfaces, and the vanishing of $\lambda(A_1, A_2)$ implies that all of these intersections can be paired by *Whitney disks*. As illustrated in Figures 2 and 3, these Whitney disks can be used to pull apart A_1 and A_2 by first pushing any intersection points between A_2 and the interior of a Whitney disk $W_{(1,2)}$ down into A_2, and then using the Whitney disks to guide Whitney moves on A_1 to eliminate all intersections between A_1 and A_2 (details in [19]).

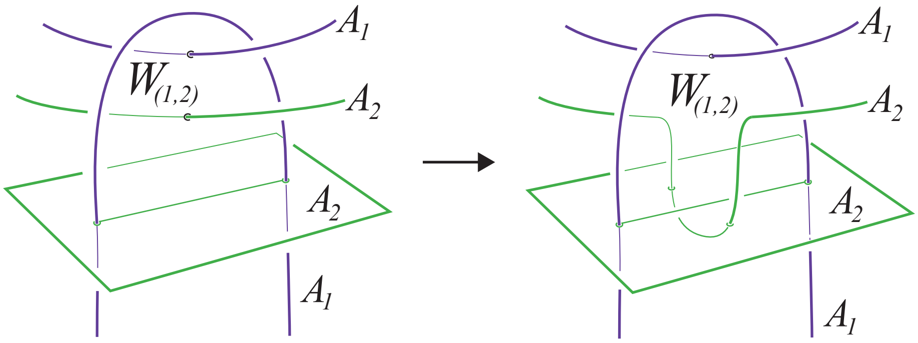

Figure 2: Pushing an intersection between A_2 and the interior of a Whitney disk $W_{(1,2)}$ down into A_2 only creates (two) self-intersections in A_2.

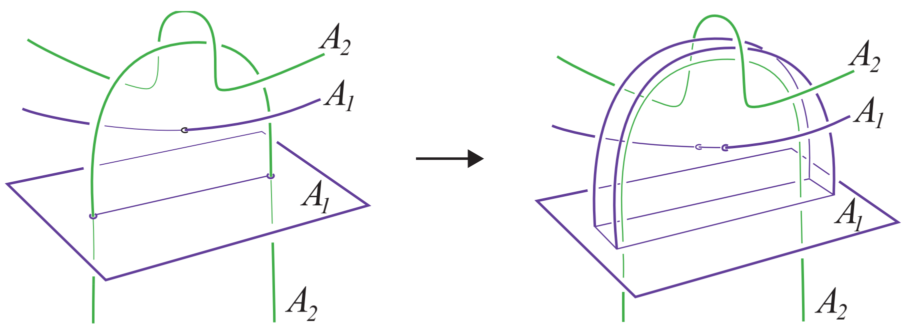

Figure 3: A Whitney move guided by the Whitney disk of Figure 2. The intersection between A_1 and the interior of the Whitney disk becomes a pair of self-intersections of A_1 after the Whitney move.

1.2 Pulling apart *three or more* 2–spheres

Note that for a triple of spheres one cannot use the method of figures 2 and 3 to eliminate an intersection point between one sphere and a Whitney disk that pairs intersections between the *other* two spheres. Such "higher-order" intersections were used in [30] to define the invariant $\lambda_1(A)$ discussed above. In this case, the procedure for separating the surfaces involves constructing "second order" Whitney disks which pair the intersections between surfaces and Whitney disks. The existence of these second order Whitney disks allows for an analogous pushing-down procedure which only creates self-intersections and cleans up the Whitney disks enough to pull apart the surfaces by an ambient homotopy.

Building on these ideas, we will describe an obstruction theory in terms of *non-repeating Whitney towers* $\mathcal{W}$ built on properly immersed surfaces in X, and *non-repeating intersection invariants* $\lambda_n(\mathcal{W})$ taking values in quotients of the group ring of $(n+1)$ products of $\pi_1 X$. The *order* n of the non-repeating Whitney tower $\mathcal{W}$ determines how many of the underlying surfaces at the bottom of the tower can be made pairwise disjoint by a homotopy, and the vanishing of $\lambda_n(\mathcal{W})$ is sufficient to find an order $n+1$ non-repeating Whitney tower.

Non-repeating Whitney towers are special cases of the Whitney towers defined in [31] (see also [3, 5, 6, 7, 27, 28, 30]). An introduction to these notions is sketched here with details given in Section 2. We work in the smooth oriented category, with orientations usually suppressed.

1.3 Whitney towers and non-repeating Whitney towers

Consider $A : \Sigma = \amalg_i \Sigma_i^2 \to X^4$ where the surface components Σ_i are spheres or disks (and see Remark 3 for initial clean-ups on A). To begin our obstruction theory, we say that A forms a *Whitney tower of order 0*, and define the *order* of each properly immersed connected surface $A_i : \Sigma_i \to X$ to be zero.

If all the singularities (transverse intersections) of A can be paired by Whitney disks then we get a *Whitney tower of order 1* which is the union of these *order* 1 *Whitney disks* and the order 0 Whitney tower.

If we only have Whitney disks pairing the intersections between *distinct* order 0 surfaces $A_i : \Sigma_i \to X$ of A, then we get an *order* 1 *non-repeating Whitney tower.*

If it exists, an *order* 2 *Whitney tower* also includes Whitney disks (of *order* 2) pairing all the intersections between the order 1 Whitney disks and the order 0 surfaces. An *order* 2 *non-repeating Whitney tower* only requires second order Whitney disks for intersections between an A_i and Whitney disks pairing A_j and A_k, where i, j and k are distinct. As explained in Section 2, all of this generalizes to higher order, including the distinction between non-repeating and repeating intersection points, however things get more subtle as different "types" of intersections of the same order can appear (parametrized by isomorphism classes of unitrivalent trees).

An *order n Whitney tower* has Whitney disks pairing up all intersections of order less than n, and an *order n non-repeating Whitney tower* is only required to have Whitney disks pairing all non-repeating intersections of order less than n (sections 2.1 and 2.4). So "order n non-repeating" is a weaker condition than

"order n".
The underlying order 0 surfaces A in a Whitney tower $\mathcal{W}$ are said to *support* $\mathcal{W}$, and we say that A *admits* an order n Whitney tower if A is homotopic (rel boundary) to A' supporting $\mathcal{W}$ of order n.

1.4 Pulling apart surfaces in 4–manifolds

As a first step towards determining whether or not $A : \Sigma \to X$ can be pulled apart, we have the following translation of the problem into the language of Whitney towers. This is the main tool in our theory:

Theorem 4 *Let m be the number of components of Σ. Then $A : \Sigma \to X$ can be pulled apart if and only if A admits a non-repeating Whitney tower of order $m - 1$.*

The existence of a non-repeating Whitney tower of sufficient order encodes "pushing down" homotopies and Whitney moves which lead to disjointness, as will be seen in the proof of Theorem 4 given in Section 3. It will be clear from the proof of Theorem 4 that for $1 < n < m$ the existence of a non-repeating Whitney tower of order n implies that any $n + 1$ of the order 0 surfaces $A_i : \Sigma_i \to X$ can be pulled apart.

1.5 Higher-order intersection invariants

An immediate advantage of this point of view is that the higher-order intersection theory of [31] can be applied inductively to increase the order of a Whitney tower or, in some cases, detect obstructions to doing so. The main idea is that to each unpaired intersection point p in a Whitney tower $\mathcal{W}$ on A one can associate a decorated unitrivalent tree t_p which bifurcates down from p through the Whitney disks to the order 0 surfaces A_i (Figure 4, also Figure 12). The *order* of p is the number of trivalent vertices in t_p. The univalent vertices of t_p are labeled by the elements $i \in \{1, \ldots, m\}$ from the set indexing the A_i. The edges of t_p are decorated with elements of the fundamental group $\pi := \pi_1 X$ of the ambient 4–manifold X. Orientations of A and X determine vertex-orientations and a sign $\mathrm{sign}(p) \in \{\pm\}$ for t_p, and the *order n intersection invariant $\tau_n(\mathcal{W})$* of an order n Whitney tower $\mathcal{W}$ is defined as the sum

$$\tau_n(\mathcal{W}) := \sum \mathrm{sign}(p) \cdot t_p \in \mathcal{T}_n(\pi, m)$$

over all order n intersection points p in $\mathcal{W}$. Here $\mathcal{T}_n(\pi, m)$ is a free abelian group generated by order n decorated trees modulo relations which include the usual antisymmetry (AS) and Jacobi (IHX) relations of finite type theory (Figure 5). Restricting to non-repeating intersection points in an order n non-repeating Whitney tower $\mathcal{W}$, yields the analogous order n *non-repeating intersection invariant* $\lambda_n(\mathcal{W})$:

$$\lambda_n(\mathcal{W}) := \sum \mathrm{sign}(p) \cdot t_p \in \Lambda_n(\pi, m)$$

which takes values in the subgroup $\Lambda_n(\pi, m) < \mathcal{T}_n(\pi, m)$ generated by order n trees whose univalent vertices have *distinct* labels. We refer to Definition 18 for more precise statements. In the following we shall sometimes suppress the number m of components of Σ and just write $\Lambda_n(\pi)$.

REMARK 5 *We will show in Lemma 19 that $\Lambda_n(\pi, m)$ is isomorphic to the direct sum of $\binom{m}{n+2}n!$-many copies of the integral group ring $\mathbb{Z}[\pi^{(n+1)}]$ of the $(n+1)$-fold cartesian product $\pi^{(n+1)} = \pi \times \pi \times \cdots \times \pi$. Note that $\Lambda_n(\pi, m)$ is trivial for $n \geq m-1$ since an order n unitrivalent tree has $n+2$ univalent vertices. For π left-orderable, $\mathcal{T}_1(\pi, m)$ is computed in [29, Sec.2.3.1]. For π trivial, $\mathcal{T}_n(m) := \mathcal{T}_n(1, m)$ is computed in [7] for all n, and in [5] the torsion subgroup of $\mathcal{T}_n(m)$ (which is only 2-torsion) is shown to correspond to obstructions to "untwisting" Whitney disks in* twisted Whitney towers *in the 4–ball. The absence of torsion in $\Lambda_n(\pi, m)$ corresponds to the fact that such obstructions are not relevant in the non-repeating setting since a* boundary-twisting *operation [12, Sec.1.3] can be used to eliminate non-trivially twisted Whitney disks at the cost of only creating repeating intersections.*

In the case $n = 0$, our notation $\lambda_0(A) \in \Lambda_0(\pi)$ just describes Wall's Hermitian intersection pairing $\lambda(A_i, A_j) \in \mathbb{Z}[\pi]$ (see section 2.6).
For $n = 1$, we showed in [30] that if $\lambda_0(A) = 0$ then taking $\lambda_1(A) := \lambda_1(\mathcal{W})$ in an appropriate quotient of $\Lambda_1(\pi)$ defines a homotopy invariant of A (independent of the choice of non-repeating Whitney tower $\mathcal{W}$). See sections 2.7 and 8.2.
The main open problem in this intersection theory is to determine for $n \geq 2$ the largest quotient of $\Lambda_n(\pi)$ for which $\lambda_n(\mathcal{W})$ only depends on the homotopy class of $A : \Sigma \to X$. Even for $n = 1$, this quotient will generally depend on A, unlike Wall's invariants λ_0.

1.6 THE GEOMETRIC OBSTRUCTION THEORY

In Theorem 2 of [31] it was shown that the vanishing of $\tau_n(\mathcal{W}) \in \mathcal{T}_n(\pi)$ implies that A admits an order $n+1$ Whitney tower. The proof of this result uses controlled geometric realizations of the relations in $\mathcal{T}_n(\pi)$, and the exact same constructions (which are all homogeneous in the univalent labels – see Section 4 of [31]) give the analogous result in the non-repeating setting:

THEOREM 6 *If $A : \Sigma \to X$ admits a non-repeating Whitney tower $\mathcal{W}$ of order n with $\lambda_n(\mathcal{W}) = 0 \in \Lambda_n(\pi)$, then A admits an order $(n+1)$ non-repeating Whitney tower.* □

Combining Theorem 6 with Theorem 4 above yields the following result, which was announced in [31, Thm.3]:

COROLLARY 7 *If Σ has m components and $A : \Sigma \to X$ admits a non-repeating Whitney tower $\mathcal{W}$ of order $m-2$ such that $\lambda_{m-2}(\mathcal{W}) = 0 \in \Lambda_{m-2}(\pi)$, then A can be pulled apart.* □

Thus, the problem of deciding whether or not any given A can be pulled apart can be attacked inductively by determining the extent to which $\lambda_n(\mathcal{W})$ only depends on the homotopy class of A.

The next two subsections describe settings where $\lambda_n(\mathcal{W}) \in \Lambda_n(\pi)$ does indeed tell the whole story. For Whitney towers in simply connected 4–manifolds, we drop π from the notation, writing $\Lambda_n(m)$, or just Λ_n if the number of order 0 surfaces is understood.

1.7 PULLING APART DISKS IN THE 4–BALL

A *link-homotopy* of an m-component link $L = L_1 \cup L_2 \cup \cdots \cup L_m$ in the 3–sphere is a homotopy of L which preserves disjointness of the link components, i.e. during the homotopy only self-intersections of the L_i are allowed. In order to study "linking modulo knotting", Milnor [23] introduced the equivalence relation of link-homotopy and defined his (non-repeating) μ-invariants, showing in particular that a link is link-homotopically trivial if and only if it has all vanishing μ-invariants. In the setting of link-homotopy, Milnor's algebraically defined μ-invariants are intimately connected to non-repeating intersection invariants as implied by the following result, proved in Section 4 using a new notion of *Whitney tower-grope duality* (Proposition 25).

THEOREM 8 *Let L be an m-component link in S^3 bounding $D : \amalg^m D^2 \to B^4$. If D admits an order n non-repeating Whitney tower $\mathcal{W}$ then $\lambda_n(\mathcal{W}) \in \Lambda_n(m)$ does not depend on the choice of $\mathcal{W}$. In fact, $\lambda_n(\mathcal{W})$ contains the same information as all non-repeating Milnor invariants of length $n+2$ and it is therefore a link-homotopy invariant of L.*

We refer to Theorem 24 for a precise statement on how Milnor's invariants are related to $\lambda_n(L) := \lambda_n(\mathcal{W}) \in \Lambda_n(m)$. Together with Corollary 7 we get the following result:

COROLLARY 9 *An m-component link L is link-homotopically trivial if and only if $\lambda_n(L)$ vanishes for all $n = 0, 1, 2, \ldots, m-1$.* $\square$

This recovers Milnor's characterization of links which are link-homotopically trivial [23], and uses the fact that L bounding disjointly immersed disks into B^4 is equivalent to L being link-homotopically trivial [13, 14].

REMARK 10 *A precise description of the relationship between general (repeating) Whitney towers on D and Milnor's μ-invariants (with repeating indices [24]) for L is given in [6]. Our current discussion is both easier and harder at the same time: We only make a statement about non-repeating Milnor invariants, a subset of all Milnor invariants, but as an input we only use a non-repeating Whitney tower, an object containing less information then a Whitney tower.*

1.8 Pulling apart 2–spheres in special 4–manifolds

The relationship between Whitney towers and Milnor's link invariants can be used to describe some more general settings where the non-repeating intersection invariant $\lambda_n(\mathcal{W}) \in \Lambda_n$ of a non-repeating Whitney tower gives homotopy invariants of the underlying order 0 surfaces. Denote by X_L the 4–manifold which is gotten by attaching 0-framed 2–handles to the 4–ball along a link L in the 3–sphere. The following theorem is proved in Section 5:

Theorem 11 *If a link L bounds an order n Whitney tower on disks in the 4–ball, then:*

(i) Any map $A : \amalg^m S^2 \to X_L$ of 2–spheres into X_L admits an order n Whitney tower.

(ii) For any order n non-repeating Whitney tower $\mathcal{W}$ supported by $A : \amalg^m S^2 \to X_L$, the non-repeating intersection invariant $\lambda_n(A) := \lambda_n(\mathcal{W}) \in \Lambda_n(m)$ is independent of the choice of $\mathcal{W}$.

Note that the number m of 2–spheres need not be equal to the number of components of the link L. Using the realization techniques for Whitney towers in the 4–ball described in [5, Sec.3], examples of such A realizing any value in $\Lambda_n(m)$ can be constructed.

Corollary 12 *If a link L bounds an order n Whitney tower on disks in the 4–ball, then:*

(i) $A : \amalg^m S^2 \to X_L$ admits an order $n+1$ non-repeating Whitney tower if and only if $\lambda_n(A) = 0 \in \Lambda_n(m)$.

(ii) In the case $m = n+2$, we have that $A : \amalg^m S^2 \to X_L$ can be pulled apart if and only if $\lambda_{m-2}(A) = 0 \in \Lambda_{m-2}(m)$. □

The "if" parts of the statements in Corollary 12 follow from Theorem 6 and Corollary 7 above. The "only if" statements follow from the fact that the second statement of Theorem 11 implies that $\lambda_n(A) := \lambda_n(\mathcal{W}) \in \Lambda_n(m)$ only depends on the homotopy class of A, as explained in Proposition 14 below. In this setting, Kojima [20] had identified (via Massey products) the first non-vanishing Milnor invariant $\mu_L(123\cdots m)$ of an m-component link L as an obstruction to pulling apart the collection of m 2–spheres determined by L in X_L.

1.9 Indeterminacies from lower-order intersections.

The sufficiency results of Theorem 6 and Corollary 7 show that the groups $\Lambda_n(\pi)$ provide upper bounds on the invariants needed for a complete obstruction theoretic answer to the question of whether or not $A : \Sigma \to X$ can be pulled apart. And as illustrated by Theorem 8 and Theorem 11 above, there are settings in which $\lambda_n(\mathcal{W}) \in \Lambda_n$ only depends on the homotopy class (rel boundary) of A, sometimes giving the complete obstruction to pulling A apart.

In general however, more relations are needed in the target group to account for indeterminacies in the choices of possible Whitney towers on a given A. In particular, for Whitney towers in a 4–manifold X with non-trivial second homotopy group $\pi_2 X$, there can be indeterminacies which correspond to tubing the interiors of Whitney disks into immersed 2–spheres. Such INT *intersection relations* are, in principle, inductively manageable in the sense that they are determined by strictly lower-order intersection invariants on generators of $\pi_2 X$. For instance, the INT_1 relations in the target groups of the order 1 invariants τ_1 and λ_1 of [25, 30] are determined by the order 0 intersection form on $\pi_2 X$. However, as we describe in Section 8, higher-order INT relations can be non-linear, and if one wants the resulting target to carry exactly the obstruction to the existence of a higher-order tower then interesting subtleties already arise in the order 2 setting.

It is interesting to note that these INT indeterminacies are generalizations of the Milnor-invariant indeterminacies in that they may involve intersections between 2–spheres *other* than the A_i. The Milnor link-homotopy invariant indeterminacies come from sub-links because there are no other essential 2-spheres in X_L. For instance, the proof of Theorem 2 exploits the hyperbolic summands of the stabilized intersection form on π_2. We pause here to note another positive consequence of the intersection indeterminacies before returning to further discussion of the well-definedness of the invariants.

1.9.1 Casson's separation lemma

The next theorem shows that in the presence of algebraic duals for the order 0 surfaces A_i, all our higher-order obstructions vanish. This recovers the following result of Casson (proved algebraically in the simply-connected setting [2]) and Quinn (proved using transverse spheres [10, 26]):

Theorem 13 *If $\lambda(A_i, A_j) = 0$ for all $i \neq j$, and there exist 2–spheres $B_i : S^2 \to X$ such that $\lambda(A_i, B_j) = \delta_{ij}$ for all i, then A_i can be pulled apart.*

Here λ denotes Wall's intersection pairing with values in $\mathbb{Z}[\pi]$, and $\delta_{ij} \in \{0, 1\}$ is the Kronecker delta. Note that there are no restrictions on intersections among the dual spheres B_i. Theorem 13 is proved in section 7.1.

1.9.2 Homotopy invariance of higher-order intersection invariants

Our proposed program for pulling apart 2–spheres in 4–manifolds involves refining Theorem 6 by formulating (and computing) the relations $\mathrm{INT}_n(A) \subset \Lambda_n(\pi)$ so that $\lambda_n(A) := \lambda_n(\mathcal{W}) \in \Lambda_n(\pi)/\mathrm{INT}_n(A)$ is a homotopy invariant of A (independent of the choice of order n non-repeating Whitney tower $\mathcal{W}$) which represents the complete obstruction to the existence of an order $n+1$ non-repeating tower supported by A. Via Theorem 4 this would provide a procedure to determine whether or not A can be pulled apart. The following observation clarifies what needs to be shown:

Proposition 14 *If for a* fixed *immersion A the value of $\lambda_n(\mathcal{W}) \in \Lambda_n(\pi)/\mathrm{INT}_n(A)$ does not depend the choice of order n non-repeating Whitney tower $\mathcal{W}$ supported by A, then $\lambda_n(A) := \lambda_n(\mathcal{W}) \in \Lambda_n(\pi)/\mathrm{INT}_n(A)$ only depends on the homotopy class of A.* □

To see why this is true, observe that, up to isotopy, any generic regular homotopy from A to A' can be realized as a sequence of finitely many finger moves followed by finitely many Whitney moves. Since any Whitney move has a finger move as an "inverse", there exists A'' which differs from each of A and A' by only finger moves (up to isotopy). But a finger move is supported near an arc, which can be assumed to be disjoint from the Whitney disks in a Whitney tower, and the pair of intersections created by a finger move admit a local Whitney disk; so any Whitney tower on A or A' gives rise to a Whitney tower on A'' with the same intersection invariant.

Thus, the problem is to find $\mathrm{INT}_n(A)$ relations which give independence of the choice of $\mathcal{W}$ for a fixed immersion A, and can be realized geometrically so that $\lambda_n(\mathcal{W}) \in \mathrm{INT}_n(A)$ implies that A bounds an order $n+1$ non-repeating Whitney tower. We conjecture that all these needed relations do indeed correspond to lower-order intersections involving 2–spheres, and hence deserve to be called "intersection" relations. Although such $\mathrm{INT}_n(A)$ relations are completely understood for $n = 1$ (see 8.2 below), a precise formulation for the $n = 2$ case already presents interesting subtleties. We remark that for maps of higher genus surfaces there can also be indeterminacies (due to choices of boundary arcs of Whitney disks) which do not come from 2–spheres; see [29] for the order 1 invariants of immersed annuli.

Useful necessary and sufficient conditions for pulling apart four or more 2–spheres in an arbitrary 4–manifold are not currently known. In Section 8 we examine the intersection indeterminacies for the relevant order 2 non-repeating intersection invariant λ_2 in the simply connected setting, and show how they can be computed as the image in $\Lambda_2(4) \cong \mathbb{Z}^2$ of a map whose non-linear part is determined by certain Diophantine quadratic equations which are coupled by the intersection form on $\pi_2 X$ (see section 8.3.6). Carrying out this computation in general raises interesting number theoretic questions, and has motivated work of Konyagin and Nathanson in [21].

We'd like to pose the following *challenge*: Formulate the $\mathrm{INT}_n(A)$ relations for $n \geq 2$ which make the following conjecture precise and true:

Conjecture 15 *$A : \amalg^m S^2 \to X$ can be pulled apart if and only if $\lambda_n(A) := \lambda_n(\mathcal{W})$ vanishes in $\Lambda_n(\pi)/\mathrm{INT}_n(A)$ for $n = 0, 1, 2, 3, \ldots, m-2$.*

2 Whitney towers

This section contains a summary of relevant Whitney tower notions and notations as described in more detail in [3, 5, 6, 27, 28, 29, 30, 31]. Recall our blurring of the distinction between a map $A : \Sigma \to X$ and its image, which leads us to speak of A as a "collection" of immersed connected surfaces in X.

REMARK 16 *Although this paper focuses on pulling apart A in the case where the components Σ_i of Σ are spheres and/or disks, much of the discussion is also relevant to the* π_1-null *setting; i.e. the Σ_i are compact connected surfaces of arbitrary genus and the component maps $A_i : \Sigma_i \to X$ induce trivial maps $\pi_1\Sigma_i \to \pi_1 X$ on fundamental groups.*

2.1 WHITNEY TOWERS

The following formalizes the discussion from the introduction by inductively defining Whitney towers of order n for each non-negative integer n.

DEFINITION 17

- *A* surface of order 0 *in a 4–manifold X is a properly immersed connected compact surface (boundary embedded in the boundary of X and interior immersed in the interior of X). A* Whitney tower of order 0 *in X is a collection of order 0 surfaces.*
- *The* order of a (transverse) intersection point *between a surface of order n_1 and a surface of order n_2 is $n_1 + n_2$.*
- *The* order of a Whitney disk *is $n + 1$ if it pairs intersection points of order n.*
- *For $n \geq 0$, a* Whitney tower of order $n+1$ *is a Whitney tower $\mathcal{W}$ of order n together with Whitney disks pairing all order n intersection points of $\mathcal{W}$. These order $n + 1$ Whitney disks are allowed to intersect each other as well as lower-order surfaces.*

The Whitney disks in a Whitney tower are required to be framed *[5, 12, 30] and have disjointly embedded boundaries. Each order* 0 *surface in a Whitney tower is also required to be framed, in the sense that its normal bundle in X has trivial (relative) Euler number. Interior intersections are assumed to be transverse. A Whitney tower is* oriented *if all its surfaces (order* 0 *surfaces* and *Whitney disks) are oriented. Orientations and framings on any boundary components of order* 0 *surfaces are required to be compatible with those of the order* 0 *surfaces. A* based *Whitney tower includes a chosen basepoint on each surface (including Whitney disks) together with a* whisker *(arc) for each surface connecting the chosen basepoints to the basepoint of X.*

We will assume our Whitney towers are based and oriented, although whiskers and orientations will usually be suppressed from notation. The collection A of order 0 surfaces in a Whitney tower $\mathcal{W}$ is said to *support* $\mathcal{W}$, and we also say that $\mathcal{W}$ is a Whitney tower *on* A. A collection A of order 0 surfaces is said to *admit* an order n Whitney tower if A is homotopic (rel boundary) to A' supporting an order n Whitney tower.

2.2 Trees for Whitney disks and intersection points.

In this paper, a *tree* will always refer to a finite oriented unitrivalent tree, where the *(vertex) orientation* of a tree is given by cyclic orderings of the adjacent edges around each trivalent vertex. The *order* of a tree is the number of trivalent vertices. Univalent vertices will usually be labeled from the set $\{1, 2, 3, \ldots, m\}$ indexing the order 0 surfaces, and we consider trees up to isomorphisms preserving these labelings. A tree is *non-repeating* if its univalent labels are distinct. When X is not simply connected, edges will be oriented and labeled with elements of $\pi_1 X$. A *root* of a tree is a chosen univalent vertex (usually left un-labeled).

We start by considering the case where X is simply connected:

Formal non-associative bracketings of elements from the index set are used as subscripts to index surfaces in a Whitney tower $\mathcal{W} \subset X$, writing A_i for an order 0 surface (dropping the brackets around the singleton i), $W_{(i,j)}$ for an order 1 Whitney disk that pairs intersections between A_i and A_j, and $W_{((i,j),k)}$ for an order 2 Whitney disk pairing intersections between $W_{(i,j)}$ and A_k, and so on, with the ordering of the bracket components determined by an orientation convention described below (2.3). When writing $W_{(I,J)}$ for a Whitney disk pairing intersections between W_I and W_J, the understanding is that if a bracket I is just a singleton i then the surface $W_I = W_i$ is just the order zero surface A_i. Note that both Whitney disks and order 0 surfaces are referred to as "surfaces in $\mathcal{W}$".

Via the usual correspondence between non-associative brackets and rooted trees, this indexing gives a correspondence between surfaces in $\mathcal{W}$ and rooted trees: To a Whitney disk $W_{(I,J)}$ we associate the rooted tree corresponding to the bracket (I, J). We use the same notation for rooted trees and brackets, so the bracket operation corresponds to the *rooted product* of trees which glues together the root vertices of I and J to a single vertex and sprouts a new rooted edge from this vertex. With this notation the order of a Whitney disk W_K is equal to the order of (the rooted tree) K.

The rooted tree (I, J) associated to $W_{(I,J)}$ can be considered to be a subset of $\mathcal{W}$, with its root edge (including the root edge's trivalent vertex) sitting in the interior of $W_{(I,J)}$, and its other edges bifurcating down through lower-order Whitney disks. The unrooted tree t_p associated to any intersection point $p \in W_{(I,J)} \cap W_K$ is the *inner product* $t_p = \langle\, (I, J), K \,\rangle$ gotten by identifying the roots of the trees (I, J) and K to a single non-vertex point. Note that t_p also can be considered as a subset of $\mathcal{W}$, with the edge of t_p containing p a sheet-changing path connecting the basepoints of $W_{(I,J)}$ and W_K (see Figure 4).

If X is not simply connected, then the edges of the just-described trees are decorated by elements of $\pi_1 X$ as follows: Considering the trees as subsets of $\mathcal{W}$, each edge of a tree is a sheet-changing path connecting basepoints of adjacent surfaces of $\mathcal{W}$. Choosing orientations of these sheet-changing paths determines elements of $\pi_1 X$ (using the whiskers on the surfaces) which are attached as labels on the correspondingly oriented tree edges.

Note that the notation for trees is slightly different in the older papers [27, 31], where the rooted tree associated to a bracket I is denoted $t(I)$, and the rooted and inner products are denoted by $*$ and $\cdot$ respectively. The notation of this paper agrees with the more recent papers [4, 5, 6, 7, 8, 29].

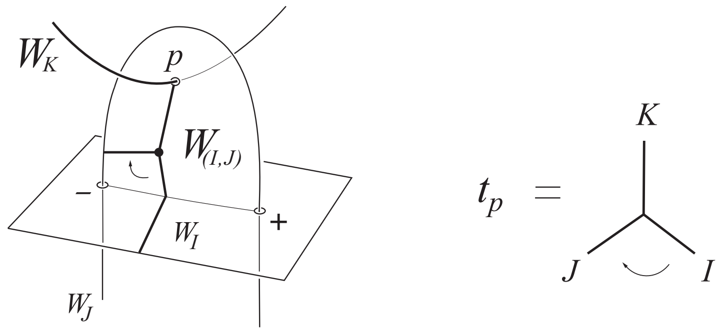

Figure 4: A local picture of the tree $t_p = \langle (I,J),K\rangle$ associated to $p \in W_{(I,J)} \cap W_K$ near a trivalent vertex adjacent to the edge of t_p passing through an unpaired intersection point p in a Whitney tower $\mathcal{W}$. On the left t_p is pictured as a subset of $\mathcal{W}$, and on the right as an abstract labeled vertex-oriented tree. In a non-simply connected 4–manifold X the edges of t_p would also be oriented and labeled by elements of $\pi_1 X$ (as in Figure 5 below).

2.3 Orientation conventions

Thinking of the tree I associated to a Whitney disk W_I as a subset of $\mathcal{W}$, it can be arranged that the trivalent orientations of I are induced by the orientations of the corresponding Whitney disks: Note that the pair of edges which pass from a trivalent vertex down into the lower-order surfaces paired by a Whitney disk determine a "corner" of the Whitney disk which does not contain the other edge of the trivalent vertex. If this corner contains the *negative* intersection point paired by the Whitney disk, then the vertex orientation and the Whitney disk orientation agree. Our figures are drawn to satisfy this convention.
This "negative corner" convention (also used in [5, 6]), which differs from the positive corner convention used in [3, 31], turns out to be compatible with the usual commutator conventions, for instance in the setting of Milnor invariants (see Figure 13).

2.4 Non-repeating Whitney towers

Whitney disks and intersection points are called *non-repeating* if their associated trees are non-repeating. This means that the univalent vertices are labeled by *distinct* indices (corresponding to distinct order 0 surfaces, i.e. distinct connected components of A). A Whitney tower $\mathcal{W}$ is an order n *non-repeating*

Whitney tower if all non-repeating intersections of order (strictly) less than n are paired by Whitney disks. In particular, if $\mathcal{W}$ is an order n Whitney tower then $\mathcal{W}$ is also an order n non-repeating Whitney tower. In a non-repeating Whitney tower repeating intersections of any order are not required to be paired by Whitney disks.

2.5 Intersection invariants

For a group π, denote by $\mathcal{T}_n(m,\pi)$ the abelian group generated by order n (decorated) trees modulo the relations illustrated in Figure 5.

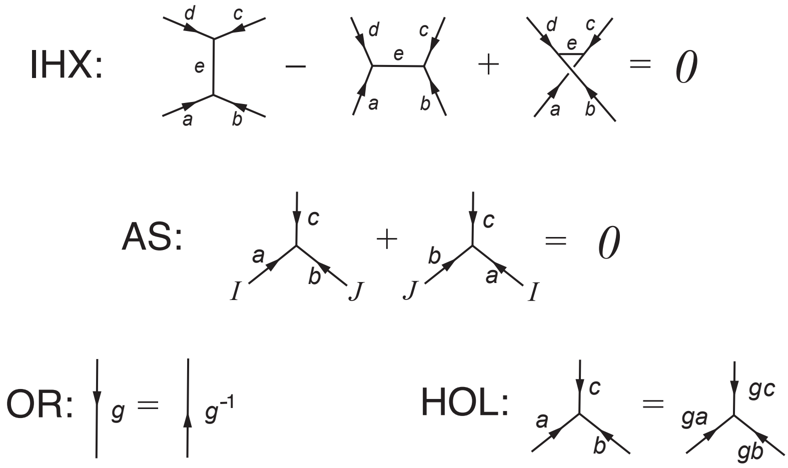

Figure 5: The relations in $\mathcal{T}_n(\pi,m)$: IHX (Jacobi), AS (antisymmetry), OR (orientation), HOL (holonomy). These are 'local' pictures, meaning that the unlabeled univalent vertices extend to fixed decorated subtrees in each equation. For instance, in the right-hand term of the HOL relation only the three visible edge decorations are multiplied by the element g, corresponding to a change of whisker on a Whitney disk at the indicated trivalent vertex. All vertex-orientations are induced from a fixed orientation of the plane; in particular, the two terms in the AS relation only differ by the orientation at the indicated trivalent vertex, where the two edges extending to the subtrees I and J have been interchanged.

Note that when π is the trivial group, the edge decorations (orientations and π-labels) disappear, and the relations reduce to the usual AS antisymmetry and IHX Jacobi relations of finite type theory (compare also the decorated graphs of [15]). All the relations are homogeneous in the univalent labels, and restricting the generating trees to be non-repeating order n trees defines the subgroup $\Lambda_n(m,\pi) < \mathcal{T}_n(m,\pi)$. (See sections 2.1 and 3 of [31] for explanations of these relations.)

Definition 18 *For an order n (oriented) Whitney tower $\mathcal{W}$ in X, the* order n intersection invariant $\tau_n(\mathcal{W})$ *is defined by summing the signed trees $\pm t_p$ over all order n intersections $p \in \mathcal{W}$:*

$$\tau_n(\mathcal{W}) := \sum \operatorname{sign}(p) \cdot t_p \in \mathcal{T}_n(\pi).$$

Here $\pi = \pi_1 X$; and $\operatorname{sign}(p) = \pm$, for $p \in W_I \cap W_J$, is the usual sign of an intersection between the oriented Whitney disks W_I and W_J.
If $\mathcal{W}$ is an order n non-repeating Whitney tower, the order n non-repeating intersection invariant $\lambda_n(\mathcal{W})$ *is analogously defined by*

$$\lambda_n(\mathcal{W}) := \sum \operatorname{sign}(p) \cdot t_p \in \Lambda_n(\pi)$$

where the sum is over all order n non-repeating intersections $p \in \mathcal{W}$.

2.6 Order 0 intersection invariants

The order 0 intersection invariants τ_0 and λ_0 for $A : \amalg^m S^2 \to X$ carry the same information as Wall's [33] Hermitian intersection form μ, λ: The generators in $\tau_0(A) \in \mathcal{T}_0(\pi, m)$ with both vertices labeled by the same index i correspond to Wall's self-intersection invariant $\mu(A_i)$. For $\mu(A_i)$ to be a *homotopy* (not just regular homotopy) invariant, one must also mod out by a *framing relation* which kills order 0 trees labeled by the trivial element in π (see [5] for higher-order framing relations). Wall's homotopy invariant Hermitian intersection pairing $\lambda(A_i, A_j) \in \mathbb{Z}[\pi]$ for $i \neq j$ corresponds to $\lambda_0(A) \in \Lambda_0(\pi, m)$.
The vanishing of these invariants corresponds to the order 0 intersections coming in canceling pairs (after perhaps a homotopy of A), so A admits an order 1 Whitney tower if and only if $\tau_0(A) = 0 \in \mathcal{T}_0(\pi, m)$, and admits an order 1 non-repeating Whitney tower if and only if $\lambda_0(A) = 0 \in \Lambda_0(\pi, m)$.

2.7 Order 1 intersection invariants

It was shown in [30], and for $\pi_1 X = 1$ and $m = 3$ in [25, 34], that for $A : \amalg^m S^2 \to X$ admitting an order 1 Whitney tower (resp. non-repeating Whitney tower) $\mathcal{W}$, the order 1 intersection invariant $\tau_1(A) := \tau_1(\mathcal{W})$ (resp. order 1 non-repeating intersection invariant $\lambda_1(A) := \lambda_1(\mathcal{W})$) is a homotopy invariant of A, if taken in an appropriate quotient of $\mathcal{T}_1(\pi, m)$ (resp. $\Lambda_1(\pi, m)$). The relations defining this quotient are determined by order 0 intersections between the A_i and immersed 2–spheres in X. These are the order 1 intersection relations INT_1 which are described in [30] (in slightly different notation) and below in Section 8 (for λ_1). As remarked in the introduction, for τ_1 there are also framing relations, but there are no framing relations for λ_n (for all n) because Whitney disks can always be framed by the boundary-twisting operation [12, Sec.1.3] which creates only repeating intersections.

From [30], we have that A admits an order 2 Whitney tower (resp. order 2 non-repeating Whitney tower) if and only if $\tau_1(A)$ (resp. $\lambda_1(A)$) vanishes. In particular, $\lambda_1(A_1, A_2, A_3) \in \Lambda_1(\pi, 3)/\mathrm{INT}_1$ is the complete obstruction to pulling apart three order 0 surfaces with vanishing $\lambda_0(A_1, A_2, A_3)$.

2.8 Order n intersection invariants

As was shown in Theorem 2 of [31], for A admitting a Whitney tower $\mathcal{W}$ of order n, if $\tau_n(\mathcal{W}) = 0 \in \mathcal{T}_n(\pi)$ then A admits a Whitney tower of order $n+1$. The proof of this result proceeds by geometrically realizing the relations in the target group of the intersection invariant in a controlled manner, so that one can convert "algebraic cancellation" of pairs of trees to "geometric cancellation" of pairs of points (paired by next-order Whitney disks). The exact same arguments work restricting to the non-repeating case to prove Theorem 6 of the introduction: For A admitting a non-repeating Whitney tower $\mathcal{W}$ of order n, if $\lambda_n(\mathcal{W}) = 0 \in \Lambda_n(\pi)$ then A admits a non-repeating Whitney tower of order $n+1$. Beyond this "sufficiency" result, it is not known for $n \geq 2$ what additional relations $\mathrm{INT}_n \subset \Lambda_n(\pi)$ would also make the vanishing of $\lambda_n(\mathcal{W})$ in the quotient a necessary condition for A to admit a non-repeating Whitney tower of order $n+1$, as discussed in 1.9.2 of the introduction.

2.9 The groups Λ_n

The groups $\Lambda_n(\pi, m)$ provide upper bounds for the order n non-repeating obstruction theory, and hence by Corollary 7 also for the obstructions to pulling apart surfaces. The following result describes the structure of $\Lambda_n(\pi, m)$:

Lemma 19 *$\Lambda_n(\pi, m)$ is isomorphic (as an additive abelian group) to the $\binom{m}{n+2}n!$-fold direct sum of the integral group ring $\mathbb{Z}[\pi^{n+1}]$ of the $(n+1)$-fold cartesian product $\pi^{n+1} = \pi \times \pi \times \cdots \times \pi$.*

Proof: First consider the case where π is trivial. Since the relations in $\Lambda_n(m)$ are all homogenous in the univalent labels, $\Lambda_n(m)$ is the direct sum of subgroups $\Lambda_n(n+2)$ over the $\binom{m}{n+2}$ choices of $n+2$ of the m labels. (As noted in the introduction, $\Lambda_n(\pi, m)$ is trivial for $n \geq m-1$ since an order n unitrivalent tree has $n+2$ univalent vertices.) We will show that each of these subgroups has a basis given by the $n!$ distinct *simple* non-repeating trees shown in Figure 6 (ignoring the edge decorations for the moment), where an order n tree is simple if it contains a geodesic of edge-length $n+1$.
For a given choice of $n+2$ labels, placing a root at the minimal-labeled vertex of each order n tree gives an isomorphism from $\Lambda_n(n+2)$ to the subgroup of non-repeating length $n+1$ brackets in the free Lie algebra (over $\mathbb{Z}$) on the other labels (with AS and IHX relations going to skew-symmetry relations and Jacobi identities). This "reduced" free Lie algebra (see also 4.1 below) is known to have rank $n!$, as explicitly described in [22, Thm.5.11] (also implicitly contained in [23, Sec.4–5]), so the trees in Figure 6 are linearly independent if they span.

To see that the trees in Figure 6 form a spanning set, first observe that for a given choice of $n+2$ labels, each order n non-repeating tree t has a distinguished geodesic edge path T_t from the minimal-label univalent vertex to the maximal-label univalent vertex. For an orientation-inducing embedding of t in the plane, it can be arranged that all the sub-trees of t emanating from T_t lie on a preferred side of T_t by applying AS relations at the trivalent vertices of T_t as needed. Then, by repeatedly applying IHX relations (replacing the left-most I-tree by the difference of the H-tree and X-tree in the IHX relation of Figure 5) at trivalent vertices of distinguished geodesics to reduce the order of the emanating sub-trees one eventually gets a linear combination of simple non-repeating trees as in Figure 6 which is uniquely determined by t. (To see how the IHX relation reduces the order of subtrees emanating from a distinguished geodesic, observe that if the central edge of the I-tree in an IHX relation is the first edge of such a subtree, then the corresponding emanating subtrees in the H-tree and X-tree both have order decreased by one.)

In the case of non-trivial π, the group elements decorating the edges of the simple trees can always be (uniquely) normalized to the trivial element on all but $n+1$ of the edges as shown in Figure 6 (by applying HOL relations from Figure 5 and working from the minimal towards the maximal vertex label). $\square$

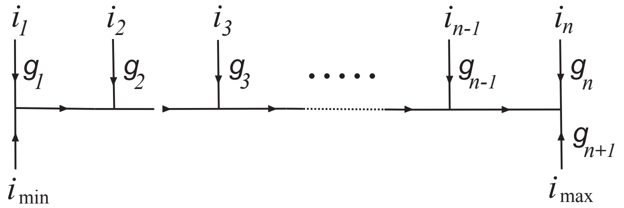

Figure 6: A *simple* order n tree with minimal- and maximal-labeled vertices connected by a length $n+1$ geodesic; $1 \leq i_{\min} < i_{\max} \leq m$, and $i_{\min} < i_k < i_{\max}$ for $1 \leq k \leq n$. Vertex orientations are induced by the planar embedding. By the HOL relations, all but $n+1$ of the edge decorations can be set to the trivial element in π (indicated by the 'empty-labeled' edges in the figure).

2.10 Some properties of Whitney towers

For future reference, we note here some elementary properties of Whitney towers and their intersection invariants.

Let $A : \Sigma \to X$ support an order n Whitney tower $\mathcal{W} \subset X$, where Σ has m connected components Σ_i. We will consider the effects on $\tau_n(\mathcal{W})$ of changing the order 0 surfaces $A_i : \Sigma_i \to X$ of A by the operations of re-indexing, including parallel copies, taking internal sums, switching orientations, and deletions; all of which preserve the property that A supports an order n Whitney tower. We will focus on the case where X is simply connected, which will be used in Section 5. (Analogous properties hold in the non-simply connected setting, although when taking internal sums (2.10.3) some care would be needed in keeping track of the effect on the edge decorations due to choices of arcs guiding the sums.)

2.10.1 Re-indexing order 0 surfaces

For $A : \Sigma \to X$ the natural indexing of the order 0 surfaces of $\mathcal{W}$ is by $\pi_0\Sigma$. In practice, we fix an identification of $\pi_0\Sigma$ with the label set $\{1, 2, \ldots, m\}$, and the effect of changing this identification is given by the corresponding permutation of the univalent labels on all the trees representing $\tau_n(\mathcal{W})$.

2.10.2 Parallel Whitney towers

Suppose A is extended to A' by including a parallel copy A_{m+1} of the last order 0 surface A_m of A. Recall from Definition 17 that order 0 surfaces have trivial (relative) normal Euler numbers, so each self-intersection of A_m will give rise to a single self-intersection of A_{m+1} and a pair of intersections between A_{m+1} and A_m; and each intersection between A_m and any A_i, for $i \neq m$, will give rise to a single intersection between A_{m+1} and A_i; and no other intersections in A' will be created. By the splitting procedure of [31, Lem.13] (also [27, Lem.3.5]) it can be arranged that all Whitney disks in $\mathcal{W}$ are embedded and contained in standard 4–ball thickenings of their trees. Since the Whitney disks are all framed, $\mathcal{W}$ can be extended to an order n Whitney tower $\mathcal{W}'$ on A' by including parallel copies of the Whitney disks in $\mathcal{W}$ as illustrated by Figure 7. This new Whitney tower $\mathcal{W}'$ can be constructed in an arbitrarily small neighborhood of $\mathcal{W}$, and the intersection invariants are related in the following way.

Define $\delta\colon \mathcal{T}_n(m) \to \mathcal{T}_n(m+1)$ to be the homomorphism induced by the map which sends a generator t having r-many m-labeled univalent vertices to the 2^r-term sum over all choices of replacing the label m by the label $(m+1)$. Then $\tau_n(\mathcal{W}') = \delta(\tau_n(\mathcal{W}))$. (In the non-simply connected setting, group elements decorating the edges would be preserved by taking parallel whiskers.)

Via re-indexing, the effect of including a parallel copy of any ith order 0 surface can be described by analogous relabeling maps δ_i, and iterating this procedure constructs an order n Whitney tower near $\mathcal{W}$ on any number of parallel copies of any order 0 surfaces of A, with the resulting change in $\tau_n(\mathcal{W})$ described by compositions of the δ_i maps.

2.10.3 Internal sums

Suppose A' is formed from A by taking the ambient connected sum of A_{m-1} with A_m in X (or by joining ∂A_{m-1} to ∂A_m with a band in ∂X), so that A' has $m-1$ components. Since it may be assumed that the interior of the arc guiding the sum is disjoint from $\mathcal{W}$, it is clear that A' bounds an order n Whitney tower $\mathcal{W}'$ all of whose Whitney disks and singularities are identical to $\mathcal{W}$. Then $\tau_n(\mathcal{W}') = \sigma(\tau_n(\mathcal{W})) \in \mathcal{T}_n(m-1)$, where the map $\sigma : \mathcal{T}_n(m) \to \mathcal{T}_n(m-1)$ is induced by the relabeling map on generators which changes all m-labeled

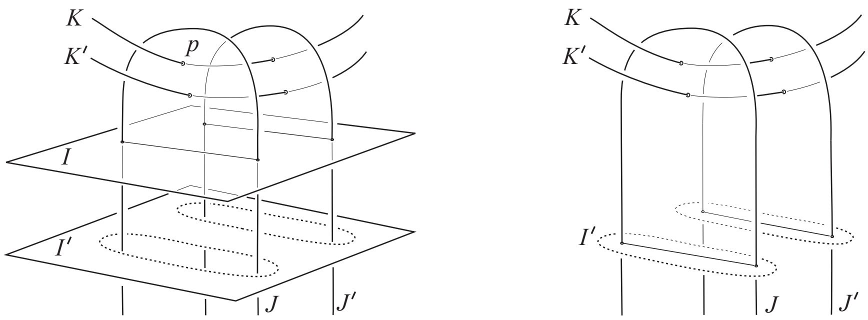

Figure 7: Extending a Whitney tower using parallel Whitney disks: An unpaired intersection point $p \in W_{(I,J)} \cap W_K$ in a Whitney tower on A, where each of I, J and K contain exactly one occurrence of the index m, gives rise to eight unpaired intersections in four Whitney disks in the new Whitney tower on A', formed from A by including a parallel copy of A_m. The dotted oval loops (left) bound neighborhoods of the Whitney arcs in the opaque I' sheet which have been perturbed into a nearby time coordinate along with the two corresponding translucent Whitney disks (right).

univalent vertices to $(m-1)$-labeled univalent vertices. (In the non-simply connected setting, group elements decorating all edges would be preserved if the guiding arc together with the whiskers on A_{m-1} and A_m formed a null-homotopic loop.)

Via re-indexing, the effect of summing any A_i with any A_j $(j \neq i)$ is described by the analogous map σ_{ij}, and for iterated internal sums the resulting intersection invariant is described by compositions of the σ_{ij} maps.

2.10.4 Switching order 0 surface orientations

As explained in [31, Sec.3], the orientation of A determines the vertex-orientations of the trees representing $\tau_n(\mathcal{W})$ up to AS relations, via our above convention (2.3). The effect on $\tau_n(\mathcal{W})$ of switching the orientation of an order 0 surface A_i of A is described as follows.

Define $s_i\colon \mathcal{T}_n(m) \to \mathcal{T}_n(m)$ to be the automorphism induced by the map which sends a generator t to $(-1)^{i(t)}t$, where again $i(t)$ denotes the multiplicity of the univalent label i in t. Then if $\mathcal{W}'$ is a reorientation of $\mathcal{W}$ which is compatible with a reversal of orientation of A_i, then we have $\tau_n(\mathcal{W}') = s_i(\tau_n(\mathcal{W}))$.

The effect on the intersection invariant of reorienting any number of order 0 surfaces of A is described by compositions of the s_i maps.

2.10.5 Deleting order 0 surfaces

The result A' of deleting the last order 0 surface A_m of A supports an order n Whitney tower $\mathcal{W}'$ formed by deleting those Whitney disks from $\mathcal{W}$ which involve A_m; that is, deleting any Whitney disk whose tree has at least one univalent vertex labeled by m. We have $\tau_n(\mathcal{W}') = e(\tau_n(\mathcal{W}))$, where the homomorphism $e\colon \mathcal{T}_n(m) \to \mathcal{T}_n(m-1)$ is induced by the map which sends a generator t to zero if m appears as a label in t, and is the identity otherwise. Via re-indexing, the effect of deleting any A_i can be described by analogous maps e_i, and the change in $\tau_n(\mathcal{W})$ due to multiple deletions of order 0 surfaces is described by compositions of the e_i.

2.10.6 Canceling parallels

We note here the following easily-checked lemma, which will be used in Section 5:

Lemma 20 *The composition $\sigma_{ji'} \circ \sigma_{i'i''} \circ s_{i''} \circ \delta_{i'} \circ \delta_i$ is the identity map on $\mathcal{T}_n(m)$.* □

Lemma 20 describes the effect on the intersection invariant that corresponds to including two parallel copies A_i' and A_i'' of A_i, switching the orientation on A_i'', then recombining A_i' and A_i'' by an internal sum into a single i'th component, and then internal summing this combined i'th component into any jth component of A. (Note that applying the analogous sequence of operations to a link obviously preserves the isotopy class of the link.)

3 Proof of Theorem 4

We want to show that m connected surfaces $A_i : \Sigma_i \to X$ can be pulled apart if and only if they admit an order $m-1$ non-repeating Whitney tower.

Proof: The "only if" direction follows by definition, since disjoint order 0 surfaces form a non-repeating Whitney tower of any order. So let $\mathcal{W}$ be a non-repeating Whitney tower of order $m-1$ on $A_1, A_2, \ldots, A_m$. If $\mathcal{W}$ contains no Whitney disks, then the A_i are pairwise disjoint. In case $\mathcal{W}$ does contain Whitney disks, we will describe how to use finger moves and Whitney moves to eliminate the Whitney disks of $\mathcal{W}$ while preserving the non-repeating order $m-1$.

First note that $\mathcal{W}$ contains no unpaired non-repeating intersections: All non-repeating intersections of order $< m-1$ are paired by definition; and since trees of order $\geq m-1$ have $\geq m+1$ univalent vertices, all intersections of order greater than or equal to $m-1$ in any Whitney tower on m order 0 surfaces must be repeating intersections.

Now consider a Whitney disk $W_{(I,J)}$ in $\mathcal{W}$ of maximal order. If $W_{(I,J)}$ is *clean* (the interior of $W_{(I,J)}$ contains no singularities) then do the $W_{(I,J)}$-Whitney move on either W_I or W_J. This eliminates $W_{(I,J)}$ (and the corresponding canceling pair of intersections between W_I and W_J) while creating no new intersections, hence preserves the order of the resulting non-repeating Whitney tower which we continue to denote by $\mathcal{W}$.

If any maximal order Whitney disk $W_{(I,J)}$ in $\mathcal{W}$ is not clean, then the singularities in the interior of $W_{(I,J)}$ are exactly a finite number of unpaired intersection points, all of which are repeating. (Since $W_{(I,J)}$ is of maximal order, the interior of $W_{(I,J)}$ contains no Whitney arcs; and $\mathcal{W}$ contains no unpaired non-repeating intersections, as noted above.) So, for any $p \in W_{(I,J)} \cap W_K$, at least one of (I,K) or (J,K) is a repeating bracket. Assuming that (I,K), say, is repeating, push p off of $W_{(I,J)}$ down into W_I by a finger move (Figure 8). This creates only a pair of repeating intersections between W_I and W_K. After pushing down all intersections in the interior of $W_{(I,J)}$ by finger moves in this way, do the clean $W_{(I,J)}$-Whitney move on either W_I or W_J. Repeating this procedure on all maximal order Whitney disks eventually yields the desired order $m-1$ non-repeating Whitney tower (with no Whitney disks) on order 0 surfaces A_i'. The A_i' are regularly homotopic to the A_i; the pushing-down finger moves will have created pairs of self-intersections in the pairwise disjointly immersed A_i'. □

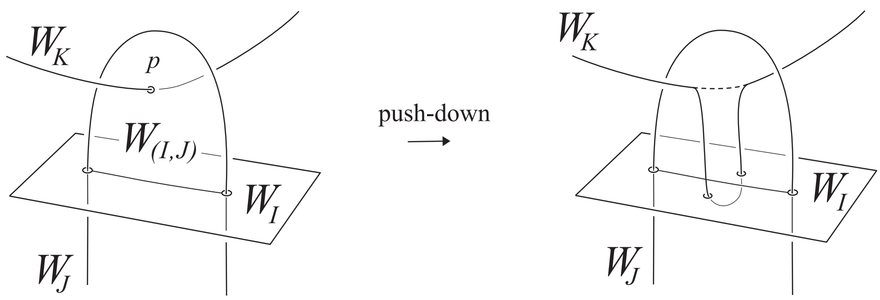

Figure 8:

4 Proof of Theorem 8

Consider a link $L = L_1 \cup L_2 \cup \cdots \cup L_m \subset S^3$ that bounds an order n non-repeating Whitney tower $\mathcal{W}$ on immersed disks in the 4–ball. We will prove Theorem 8 by relating $\lambda_n(\mathcal{W})$ to Milnor's length $n+2$ link-homotopy μ-invariants of L in Theorem 24, showing in particular that $\lambda_n(L) := \lambda_n(\mathcal{W}) \in \Lambda_n(m)$ only depends on the link-homotopy class of L (and not on the Whitney tower $\mathcal{W}$).

The essential idea is that $\mathcal{W}$ can be used to compute the link longitudes as iterated commutators in Milnor's nilpotent quotients of the fundamental group of the link complement. The proof uses a new result, *Whitney tower-grope duality*, which describes certain class $n+2$ gropes that live in the complement of an order n Whitney tower in any 4–manifold (Proposition 25). After fixing notation for the first-non-vanishing Milnor invariants of L in section 4.1, we give the explicit identification of them with $\lambda_n(\mathcal{W})$ in Theorem 24 of section 4.2.

4.1 Milnor's link-homotopy μ-invariants

This subsection briefly reviews and fixes notation for the first non-vanishing non-repeating μ-invariants of a link. See any of [1, 16, 17, 23] for details.

For a group G *normally* generated by elements $g_1, g_2, \ldots, g_m$, the *Milnor group* of G (with respect to the g_i) is the quotient of G by the subgroup normally generated by all commutators between g_i and $g_i^h := hg_ih^{-1}$, so we kill the elements

$$[g_i, g_i^h] = g_i g_i^h g_i^{-1} g_i^{-h}$$

for $1 \leq i \leq m$, and all $h \in G$. One can prove (e.g. [16, Lem.1.3]) by induction on m that this quotient is nilpotent and (therefore) generated by $g_1, \ldots, g_m$.

The *Milnor group* $\mathcal{M}(L)$ of an m-component link L is the Milnor group of the fundamental group of the link complement $\pi_1(S^3 \smallsetminus L)$ with respect to a generating set of meridional elements. Specifically, $\mathcal{M}(L)$ has a presentation

$$\mathcal{M}(L) = \langle x_1, x_2, \ldots, x_m \,|\, [\ell_i, x_i],\, [x_j, x_j^h] \rangle$$

where each x_i is represented by a meridian (one for each component), and the ℓ_i are words in the x_i determined by the link longitudes. The Milnor group $\mathcal{M}(L)$ is the largest common quotient of the fundamental groups of all links which are link homotopic to L. Since $\mathcal{M}(L)$ only depends on the conjugacy classes of the meridional generators x_i, it only depends on the link L (and no base-points are necessary).

A presentation for the Milnor group of the unlink (or any link-homotopically trivial link) corresponds to the case where all $\ell_i = 1$, and Milnor's μ-invariants (with non-repeating indices) compare $\mathcal{M}(L)$ with this *free Milnor group* $\mathcal{M}(m)$ by examining each longitudinal element in terms of the generators corresponding to the *other* components. Specifically, mapping $x_i^{\pm 1}$ to $\pm X_i$ induces a canonical isomorphism

$$\mathcal{M}(m)_{(n)}/\mathcal{M}(m)_{(n+1)} \cong \mathsf{RL}_n(m)$$

from the lower central series quotients to the *reduced* free Lie algebra $\mathsf{RL}(m) = \oplus_{n=1}^m \mathsf{RL}_n(m)$, which is the quotient of the free $\mathbb{Z}$-Lie algebra on the X_i by the relations which set an iterated Lie bracket equal to zero if it contains more than one occurrence of a generator. This isomorphism takes a product of length n commutators in distinct x_i to a sum of length n Lie brackets in distinct X_i. In particular, $\mathsf{RL}_n(m) = 0$ for $n > m$.

Let $\mathcal{M}^i(L)$ denote the quotient of $\mathcal{M}(L)$ by the relation $x_i = 1$. If the element in $\mathcal{M}^i(L)$ determined by the longitude ℓ_i lies in the $(n+1)$th lower central subgroup $\mathcal{M}^i(L)_{(n+1)}$ for each i, then we have isomorphisms:

$$\mathcal{M}(L)_{(n+1)}/\mathcal{M}(L)_{(n+2)} \cong \mathcal{M}(m)_{(n+1)}/\mathcal{M}(m)_{(n+2)} \cong \mathsf{RL}_{(n+1)}(m).$$

Via the usual identification of non-associative bracketings and binary trees, $\mathsf{RL}_{(n+1)}(m)$ can be identified with the abelian group on order n *rooted* non-repeating trees modulo IHX and antisymmetry relations as in Figure 5 (with π trivial). This identification explains the subscripts in the following definition:

Definition 21 *The elements* $\mu_n^i(L) \in \mathsf{RL}^i_{(n+1)}(m)$ *determined by the longitudes* ℓ_i *are the* non-repeating Milnor-invariants *of* order n. *Here* $\mathsf{RL}^i(m)$ *is the reduced free Lie algebra on the* $m-1$ *generators* X_j, *for* $j \neq i$.

This definition of non-repeating μ-invariants was originally given by Milnor [23]. He later expressed the elements $\mu_n^i(L)$ in terms of integers $\mu_L(i, k_1, \ldots, k_{n+1})$, which are the coefficients of $X_{k_1} \cdots X_{k_{n+1}}$ in the Magnus expansion of ℓ_i. We note that our *order* n corresponds to the originally used *length* $n+2$ (of entries in μ_L).
By construction, these non-repeating μ-invariants depend only on the link-homotopy class of the link L. We have only defined order n μ-invariants assuming that the lower-order μ-invariants vanish, which will turn out to be guaranteed by the existence of an order n non-repeating Whitney tower.

4.2 Mapping from trees to Lie brackets

For each i, define a map

$$\eta_n^i : \Lambda_n(m) \to \mathsf{RL}^i_{(n+1)}(m)$$

by sending a tree t which has an i-labeled univalent vertex v_i to the iterated bracketing determined by t with a root at v_i. Trees without an i-labeled vertex are sent to zero. For example, if t is an order 1 Y-tree with univalent labels $1, 2, 3$, and cyclic vertex orientation $(1, 2, 3)$, then $\eta_1^1(t) = [X_2, X_3]$, and $\eta_1^3(t) = [X_1, X_2]$, and $\eta_1^2(t) = [X_3, X_1]$. Note that the IHX and AS relations in $\Lambda_n(m)$ go to the Jacobi and skew-symmetry relations in $\mathsf{RL}^i_{(n+1)}(m)$, so the maps η_n^i are well-defined.

Lemma 22

$$\sum_{i=1}^{m} \eta_n^i : \Lambda_n(m) \longrightarrow \oplus_{i=1}^m \mathsf{RL}^i_{(n+1)}(m)$$

is a monomorphism.

Proof: Putting an i-label in place of the root in a tree corresponding to a Lie bracket in $\mathsf{RL}^i_{(n+1)}(m)$ gives a left inverse to η_n^i. In fact, for the top degree $n+2 = m$, this is an inverse because every index i appears exactly once in a tree t of order $n = m-2$. For arbitrary n, it is easy to check that composing the sum of these left inverse maps with $\sum_{i=1}^m \eta_n^i$ is just multiplication by $n+2$ on $\Lambda_n(m)$. Since $\Lambda_n(m)$ is torsion-free by Lemma 19, it follows that $\sum_{i=1}^m \eta_n^i$ is injective. $\square$

Remark 23 *The monomorphism* $\sum_{i=1}^m \eta_n^i$ *fits into the bottom row of a commutative diagram:*

$$\begin{tikzcd}
\mathcal{T}_n(m) \arrow[r, tail, "\eta_n"] \arrow[d, two heads] & \oplus_{i=1}^m \mathsf{L}_{(n+1)}(m) \arrow[d, two heads] \\
\Lambda_n(m) \arrow[r, tail] & \oplus_{i=1}^m \mathsf{RL}^i_{(n+1)}(m)
\end{tikzcd}$$

Here the upper row is relevant for repeating *Milnor invariants as explained in [4, 5]. The injectivity of the top horizontal map η_n, defined by Jerry Levine, is much harder to show and is the central result of [7] (implying that $\mathcal{T}_n(m)$ has at most 2-torsion). The two vertical projections simply set trees with repeating labels to zero.*

The maps η^i_n correspond to tree-preserving geometric constructions which desingularize an order n Whitney tower to a collection of class $n+1$ *gropes*, as described in detail in [27], and sketched in section 4.3 below. Gropes are 2-complexes built by gluing together compact orientable surfaces, and this correspondence will be used in the proof of the following theorem:

Theorem 24 *If a link $L \subset S^3$ bounds a non-repeating Whitney tower $\mathcal{W}$ of order n on immersed disks $D = \amalg^m D^2 \to B^4$, then for each i the longitude ℓ_i lies in $\mathcal{M}^i(L)_{(n+1)}$, and*

$$\eta^i_n(\lambda_n(\mathcal{W})) = \mu^i_n(L) \ \in \mathsf{RL}^i_{(n+1)}(m)$$

Since the sum of the η^i_n is injective, this will prove Theorem 8: The intersection invariant $\lambda_n(\mathcal{W}) \in \Lambda_n(m)$ does not depend on the Whitney tower $\mathcal{W}$ and is a link homotopy invariant of L, denoted by $\lambda_n(L)$.
For L bounding an honest order n Whitney tower, one can deduce this theorem from the main result in [6, Thm.5] (and the diagram in Remark 23 above); but here we only have a non-repeating order n Whitney tower as an input.
Proof: We start by giving an outline of the argument, introducing some notation that will be clarified during the proof:

(i) First the Whitney tower will be cleaned up, including the elimination of all repeating intersections of positive order and all repeating Whitney disks, to arrive at an order n non-repeating Whitney tower $\mathcal{W}$ bounded by L such that all unpaired intersection points of positive order have non-repeating trees (so the only repeating intersections are self-intersections in the order 0 disks D_j).

(ii) Then the preferred order 0 disk D_i (and all Whitney disks involving D_i) will be resolved to a grope G_i of class $n+1$ bounded by L_i, such that G_i is in the complement $B^4 \smallsetminus \mathcal{W}^i$, where $\mathcal{W}^i$ is the result of deleting D_i and the Whitney disks used to construct G_i from $\mathcal{W}$. The grope G_i will display the longitude ℓ_i in $\pi_1(B^4 \smallsetminus \mathcal{W}^i)$ as a product of $(n+1)$-fold commutators of meridians to the order 0 surfaces $D^i := \cup_{j \neq i} D_j$ of $\mathcal{W}^i$ corresponding to putting roots at all i-labeled vertices of the trees representing $\lambda_n(\mathcal{W})$. This is the same formula as in the definition of the map η^i_n, so it only remains to show that $\mu^i_n(L)$ can be computed in $\pi_1(B^4 \smallsetminus \mathcal{W}^i)$.

(iii) This last step is accomplished by using *Whitney tower-grope duality* (Proposition 25) and Dwyer's theorem [9] to show that the inclusion $S^3 \smallsetminus \partial D^i \to B^4 \smallsetminus \mathcal{W}^i$ induces an isomorphism on the Milnor groups modulo the $(n+2)$th terms of the lower central series.

Step (i): Let $\mathcal{W}$ be an order n non-repeating Whitney tower on $D \to B^4$ bounded by $L \subset S^3$. As described in [27, Lem.3.5] (or [31, Lem.13]), $\mathcal{W}$ can be *split*, so that each Whitney disk of $\mathcal{W}$ is embedded, and the interior of each Whitney disk contains either a single unpaired intersection p or a single boundary arc of a higher-order Whitney disk, and no other singularities. This splitting process does not change the trees representing $\lambda_n(\mathcal{W})$, and results in each tree t_p associated to an order n intersection p being contained in a 4-ball thickening of t_p, with all these 4–balls pairwise disjoint. Splitting simplifies combinatorics, and facilitates the use of local coordinates for describing constructions. Also, split Whitney towers correspond to *dyadic* gropes (whose upper stages are all genus one), and dyadic gropes are parametrized by trivalent (rooted) trees.

We continue to denote the split order n non-repeating Whitney tower by $\mathcal{W}$, and will keep this notation despite future modifications. In the following, further splitting may be performed without mention.

If $\mathcal{W}$ contains any repeating intersections of positive order, then by following the pushing-down procedure described in the proof of Theorem 4 given in section 3, all these repeating intersections can be pushed-down until they create (many) pairs of self-intersections in the order 0 disks. Then all repeating Whitney disks are clean, and by doing Whitney moves guided by these clean Whitney disks it can be arranged that $\mathcal{W}$ contains no repeating Whitney disks and no repeating intersections of positive order.

Step (ii): Consider now the component L_i bounding D_i. We want to convert D_i into a class $n+1$ grope displaying the longitude ℓ_i as a product of $(n+1)$-fold iterated commutators in meridians to the $D_{j\neq i}$ using the tree-preserving Whitney tower-to-grope construction of [27, Thm.5]. This construction is sketched roughly below in section 4.3, and a simple case is illustrated in Figure 12. Actually, the resulting grope G_i comes with *caps*, which in this setting are embedded normal disks to the other D_j which are bounded by essential circles called *tips* on G_i. For our purposes the caps only serve to show that these tips are meridians to the D_j. The trees associated to gropes are rooted trees, with the root vertex corresponding to the bottom stage surface, and the other univalent vertices corresponding to the tips (or to the caps). Since $\mathcal{W}$ was split, the upper surface stages of G_i will all be genus one, so the collection of order n unitrivalent trees $t(G_i)$ associated to G_i will contain one tree for each dyadic branch of upper stages, with each trivalent vertex of a tree corresponding to a genus one surface in a branch. In this setting the class of G_i is equal to $n+1$, the number of non-root univalent vertices in each tree in the collection $t(G_i)$ (see e.g. [27, Sec.2.3]).

Applying the construction of [27, Thm.5] to D_i converts D_i and all the Whitney disks of $\mathcal{W}$ corresponding to trivalent vertices in trees containing an i-label into a class $n+1$ grope G_i. This grope G_i (without the extra caps provided by [27, Thm.5]) is disjoint from $\mathcal{W}^i \subset \mathcal{W}$, where the order n non-repeating Whitney tower $\mathcal{W}^i$ consists of the order 0 immersed disks $D^i := \cup_{j\neq i} D_j$ together with the Whitney disks of $\mathcal{W}$ whose trees do not have an i-labeled vertex. In the present setting, any self-intersections of D_i will give rise to self-intersections in the bottom stage surface of G_i (which is bounded by L_i), but all higher stages of G_i will be embedded.

At the level of trees, this construction of G_i corresponds to replacing each i-labeled vertex on a tree representing $\lambda_n(\mathcal{W})$ with a root [27, Thm.5(v)], which is the same formula as the map on generators defining η^i_n (signs and orientations are checked in [6, Lem.31] and [6, Sec.4.2] in the setting of repeating Milnor invariants; see also sketch in section 4.3 below). So we have shown that, as an element in $\pi_1(B^4 \smallsetminus \mathcal{W}^i)$, the ith longitude ℓ_i is represented by the iterated commutators in meridians to the D_j that correspond to the image of $\eta^i_n(\lambda_n(\mathcal{W}))$ if the inclusion $S^3 \smallsetminus \partial D^i \to B^4 \smallsetminus \mathcal{W}^i$ induces an isomorphism on the quotients of the Milnor groups by the $(n+2)$th terms of the lower central series.

Step (iii): To finish the proof of Theorem 24 we will use Dwyer's Theorem [9] and a new notion of Whitney tower-grope duality to check that the inclusion $S^3 \smallsetminus \partial D^i \to B^4 \smallsetminus \mathcal{W}^i$ does indeed induce the desired isomorphism on the quotients of the Milnor groups by the $(n+2)$th terms of the lower central series. It is easy to check that the inclusion induces an isomorphism on first homology, so by [9] the kernel of the induced map on π_1 is generated by the attaching maps of the 2-cells of surfaces generating the (integral) second homology group $H_2(B^4 \smallsetminus \mathcal{W}^i)$. The order 0 self-intersections $D_j \cap D_j$ only contribute Milnor relations, coming from the attaching maps of the 2-cells of the Clifford tori around the self-intersections. If the D_j were pairwise disjoint, then by introducing (more) self-intersections as needed (by finger moves realizing the Milnor relations, see e.g. [18, XII.2]), it could be arranged that $\pi_1(B^4 \smallsetminus D^i)$ was in fact isomorphic to the free Milnor group. Since the D_j will generally intersect each other, we have to use the fact that $\mathcal{W}^i$ is a non-repeating Whitney tower of order n to show that any new relations coming from (higher-order) intersections are trivial modulo $(n+2)$-fold commutators. Since $H_2(B^4 \smallsetminus \mathcal{W}^i)$ is Alexander dual to $H_1(\mathcal{W}^i, \partial D^i)$, the proof of Theorem 24 is completed by applying the following general duality result to $\mathcal{W}^i \subset B^4$, which shows that the other generating surfaces extend to class $n+2$ gropes. $\square$

Proposition 25 (Whitney tower-grope duality) *If $\mathcal{V}$ is a split Whitney tower on $A : \Sigma = \amalg_j \Sigma^2_j \to X$, where each order 0 surface A_j is a sphere $S^2 \to X$ or a disk $(D^2, \partial D^2) \to (X, \partial X)$, then there exist dyadic gropes $G_k \subset X \smallsetminus \mathcal{V}$ such that the G_k are geometrically dual to a generating set for the relative first homology group $H_1(\mathcal{V}, \partial A)$. Furthermore, the tree $t(G_k)$ associated to each G_k is obtained by attaching a rooted edge to the interior of an edge of a tree t_p associated to an unpaired intersection p of $\mathcal{V}$.*

Here *geometrically dual* means that the bottom stage surface of each G_k bounds a 3–manifold which intersects exactly one generating curve of $H_1(\mathcal{V}, \partial A)$ transversely in a single point, and is disjoint from the other generators. In particular, there are as many gropes G_k as free generators of $H_1(\mathcal{V}, \partial A)$. Note that it follows from the last sentence of the proposition that if $\mathcal{V}$ is order n, then each G_k is class $n+2$.

Proof: Since the A_j are simply-connected, the group $H_1(\mathcal{V}, \partial A)$ is generated by sheet-changing curves in $\mathcal{V}$ which pass once through a transverse intersection (and avoid all other transverse intersections in $\mathcal{V}$). Such curves either pass through an unpaired intersection or a paired intersection. First we consider a sheet-changing curve through an unpaired intersection $p \in W_I \cap W_J$ (so $t_p = \langle I, J\rangle$). The Clifford torus T around p is geometrically dual to the curve, and the dual pair of circles in T represent meridians to W_I and W_J, respectively (recall our convention that if, say, $J = j$ is order 0, Then $W_J = A_j$ is an order 0 surface). The next lemma shows that the circles on T bound branches of the desired grope $G_{(I,J)}$, with $t(G_{(I,J)}) = (I, J)$.

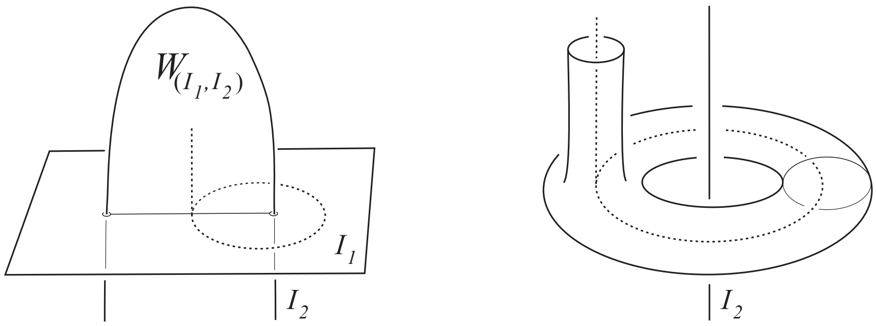

Figure 9: The normal circle bundle T_I to W_{I_1} and $W_{(I_1,I_2)}$ over the dotted circle and arc on the left is shown on the right.

LEMMA 26 *Any meridian to a Whitney disk $W_{(I_1,I_2)}$ in a Whitney tower $\mathcal{V} \subset X$ bounds a grope $G_{(I_1,I_2)} \subset X \smallsetminus \mathcal{V}$ such that $t(G_{(I_1,I_2)}) = (I_1, I_2)$.*

Proof: As illustrated in Figure 9, such a meridian bounds a punctured Clifford torus T_I around one of the intersections paired by $W_{(I_1,I_2)}$. Each of a symplectic pair of circles on T_I is a meridian to one of the Whitney disks W_{I_i} paired by $W_{(I_1,I_2)}$, so iterating this construction until reaching meridians to order 0 surfaces yields the desired grope $G_{(I_1,I_2)}$ with bottom stage T_I. $\square$

Now we consider the sheet-changing curves through intersection points that are paired by Whitney disks. Let $W_{(I,J)}$ be a Whitney disk, and consider the boundary γ of a neighborhood of a boundary arc of $W_{(I,J)}$ in one of the sheets paired by $W_{(I,J)}$, as illustrated in the left-hand side of Figure 10. We call such a loop γ an *oval* of the Whitney disk. Clearly, an oval intersects once with a sheet-changing curve that passes once through one of the two intersections paired by $W_{(I,J)}$. So the normal circle bundle to the sheet over an oval is geometrically dual to such a sheet-changing curve. The following lemma completes the proof of Proposition 25. $\square$

LEMMA 27 *Let $W_{(I,J)}$ be a Whitney disk in a split Whitney tower $\mathcal{V}$ such that $W_{(I,J)}$ contains a trivalent vertex of a tree $t_p = \langle (I,J), K\rangle$ associated to an*

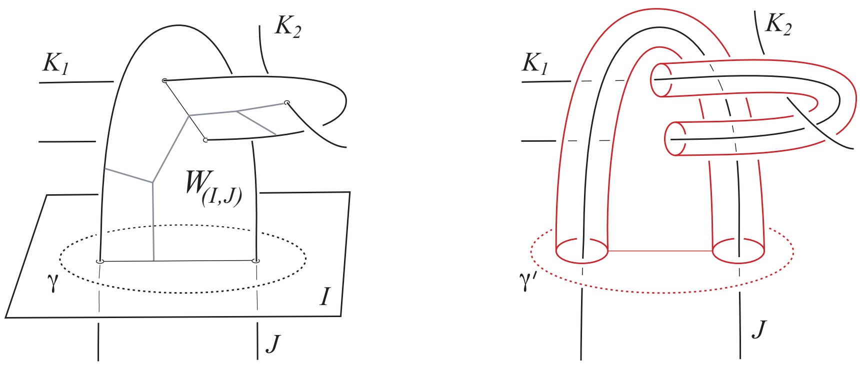

Figure 10: Left: A (dotted) oval $\gamma \subset W_I$. Right: A (dotted) parallel γ' to the oval bounds a grope in a nearby 'time' coordinate. Not shown is the dual branch of the grope attached to the meridian of W_J (as per Lemma 26).

unpaired intersection point $p \in \mathcal{V}$. If $\gamma \subset W_I$ is an oval of $W_{(I,J)} \subset \mathcal{V}$; then the normal circle bundle T to W_I over γ is the bottom stage of a dyadic grope $G \subset (X \smallsetminus \mathcal{V})$, such that $t(G) = (I,(J,K))$.

Proof: The torus T contains a symplectic pair of circles, one of which is a meridian to W_I, while the other is a parallel γ' of γ. By Lemma 26, the meridian to W_I bounds a grope G_I with $t(G_I) = I$, so we need to check that γ' bounds a grope $G_{(J,K)}$ with tree (J,K).
As shown in Figure 10, γ' bounds a grope whose bottom stage contains a symplectic pair of circles, one of which is a meridian to W_J; while the other is either parallel to an oval in $W_{(I,J)}$ around the boundary arc of a higher-order Whitney disk $W_{((I,J),K_1)}$ for $K = (K_1, K_2)$ (as shown in the figure), or is a meridian to W_K if $W_{(I,J)}$ contains the unpaired intersection $p = W_{(I,J)} \cap W_K$ (since $\mathcal{V}$ is split, these are the only two possible types of singularities in $W_{(I,J)}$). By Lemma 26, the meridian to W_J bounds a grope G_J; and inductively the oval-parallel circle, or again by Lemma 26 the meridian to W_K, bounds a grope G_K; so the grope bounded by γ' does indeed have tree (J,K). □

4.3 The Whitney tower-to-grope construction

This subsection briefly sketches the Whitney tower-to-grope construction used above in Step (ii) of the proof of Theorem 24. In [6] this procedure of converting Whitney towers to capped gropes in order to read off commutators determined by link longitudes is covered in detail in the setting of repeating Milnor invariants. The analogous computation of repeating Milnor invariants from capped gropes described there is trickier in that meridians to a given link component L_i can also contribute to the same longitude ℓ_i. Hence the computation of ℓ_i uses a push-off G_i' of the grope body G_i, and there may be intersections

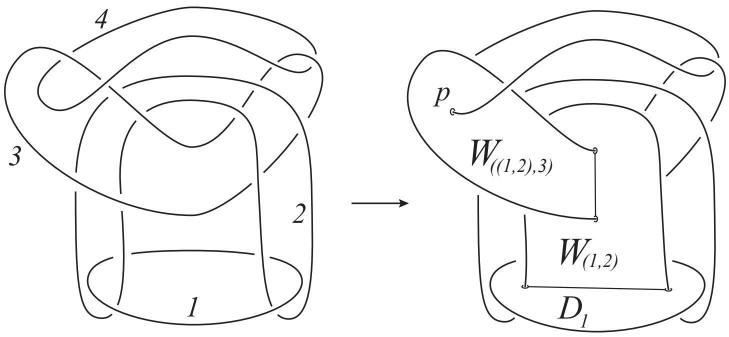

Figure 11: Moving radially into B^4 from left to right, a link $L \subset S^3$ bounds an order 2 (non-repeating) Whitney tower $\mathcal{W}$: The order 0 disk D_1 consists of a collar on L_1 together with the indicated embedded disk on the right. The other three order 0 disks in $\mathcal{W}$ consist of collars on the other link components which extend further into B^4 and are capped off by disjointly embedded disks. The order 1 Whitney disk $W_{(1,2)}$ pairs $D_1 \cap D_2$, and the order 2 Whitney disk $W_{((1,2),3)}$ pairs $W_{(1,2)} \cap D_3$, with $p = W_{((1,2),3)} \cap D_4$ the only unpaired intersection point in $\mathcal{W}$. See Figure 12 for the tree-preserving resolution of $\mathcal{W}$ to a grope.

between the bottom stage of G_i' and caps on G_i^c which correspond to repeating indices on the associated tree.

Here in the non-repeating setting, ℓ_i can be computed as described above directly from the body G_i of the capped grope G_i^c, by throwing away the caps and just remembering how the tips of G_i are meridians to the other components corresponding to the univalent labels on $t(G_i^c)$. See Figures 11 and 12 for the local model near a tree.

A typical 0-surgery which converts a Whitney disk $W_{(I,J)}$ into a cap $c_{(I,J)}$ is illustrated in Figure 13, which also shows how the signed tree is preserved. The sign associated to the capped grope is the product of the signs coming from the intersections of the caps with the bottom stages, which corresponds to the sign of the un-paired intersection point in the Whitney tower; (surgering along the other boundary arc of the Whitney disk, and the other sign cases are checked similarly). If either of the J- and K-labeled sheets is a Whitney disk, then the corresponding cap will be surgered after a Whitney move which turns the single cap-Whitney disk intersection into a cancelling pair of intersections between the cap and a surface sheet that was paired by the Whitney disk, as described in Section 4.2 of [27] (with orientations checked in Lemma 14, Figures 10 and 11 of [31]).

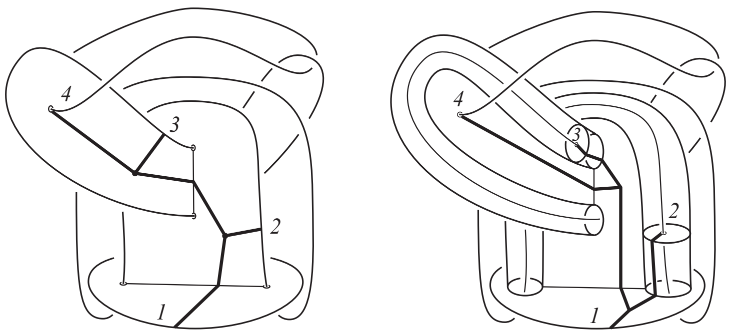

Figure 12: Both sides of this figure correspond to the slice of B^4 shown in the right-hand side of Figure 11. The tree $t_p = \langle((1,2),3),4\rangle \subset \mathcal{W}$ is shown on the left. Replacing this left-hand side by the right-hand side illustrates the construction of a class 3 (capped) grope G_1^c bounded by L_1, shown (partly translucent) on the right, gotten by surgering D_1 and $W_{(1,2)}$. Each of the disks D_2, D_3 and D_4 has a single intersection with a cap of G_1^c, with G_1 displaying the longitude of L_1 as the triple commutator $[x_2,[x_3,x_4]]$ in $\pi_1(B^4 \smallsetminus \mathcal{W}^1)$, where $\mathcal{W}^1 = D_2 \cup D_3 \cup D_4$. This simple case illustrates the local picture of the general computation of $\eta_n^i(\lambda_n(\mathcal{W}))$: For a more complicated $L = \partial\mathcal{W}$ this construction would be carried out in a 4–ball neighborhood of each tree containing an i-labeled vertex, and $\mathcal{W}^i$ would consist of other Whitney disks as well as the $D_{j\neq i}$.

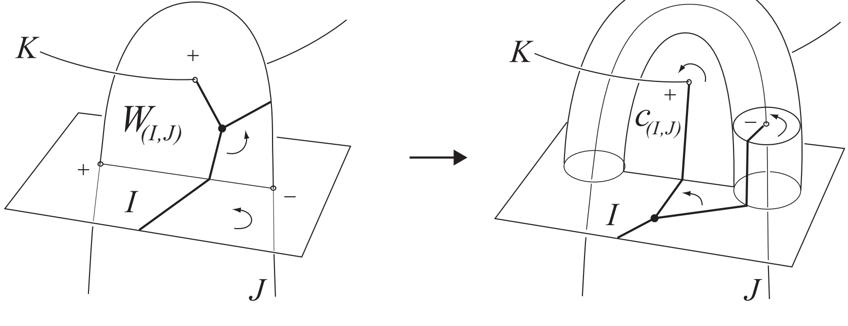

Figure 13: Resolving a Whitney tower to a capped grope preserves the associated oriented trees. The boundary of the I-labeled sheet represents the commutator $[x_J, x_K]$, up to conjugation, of the meridians x_J and x_K to the J- and K-labeled sheets.

5 Proof of Theorem 11

Let $L \subset S^3$ bound an order n Whitney tower in B^4, and let X_L be the 4–manifold gotten by attaching 0-framed 2–handles to L. We need to show:

(i) Any map $A : \amalg^m S^2 \to X_L$ of 2–spheres into X_L admits an order n Whitney tower.

(ii) For any order n non-repeating Whitney tower $\mathcal{W}$ supported by $A : \amalg^m S^2 \to X_L$, the non-repeating intersection invariant $\lambda_n(A) := \lambda_n(\mathcal{W}) \in \Lambda_n(m)$ is independent of the choice of $\mathcal{W}$.

The first statement of Theorem 11 follows from the observations in section 2.10: Any $A : \amalg^m S^2 \to X_L$ is homotopic to the union of band sums of parallel copies of cores of the 2–handles of X_L with 0-framed immersed disks bounded by a link L' formed from L by the operations of adding parallel components, switching orientations, taking internal band sums and deleting components. Any order n Whitney tower on immersed disks in the 4–ball bounded by L can be modified to give an order n Whitney tower on immersed disks bounded by L' as described in subsection 2.10. Then the union of the Whitney tower bounded by L' with the 2–handle cores forms an order n Whitney tower supported by A.

To prove the second statement of Theorem 11 we will use the following consequence of Theorem 8: If $\mathcal{V}$ is any order n non-repeating Whitney tower on a collection of m immersed 2–spheres in the 4–sphere, then the order n non-repeating intersection invariant $\lambda_n(\mathcal{V})$ must vanish in $\Lambda_n(m)$. Otherwise, the 2–spheres supporting $\mathcal{V}$ could be tubed into disjointly embedded 2–disks in the 4–ball bounded by an unlink U in the 3–sphere to create an order n Whitney tower $\mathcal{W}_U$ in $B^4 = B^4 \# S^4$ with $\lambda_n(U) = \lambda_n(\mathcal{W}_U) = \lambda_n(\mathcal{V}) \neq 0 \in \Lambda_n(m)$.

We start with the case where L is an m-component link, and $A = \amalg_{i=1}^m A_i : S^2 \to X_L$ is such that each A_i goes geometrically once over the 2–handle of X_L attached to the ith component L_i of L, and is disjoint from all other 2–handles. We assume that the orientations of A and L are compatible. In this case, the union of an order n Whitney tower $\mathcal{W}_L$ bounded by L with the cores of the 2–handles forms an order n Whitney tower $\mathcal{W}$ on A, with $\lambda_n(\mathcal{W}) = \lambda_n(L) := \lambda_n(\mathcal{W}_L) \in \Lambda_n(m)$. If $\mathcal{W}'$ is any other order n non-repeating Whitney tower on A', with A' homotopic to A, then we will show that $\lambda_n(\mathcal{W}') = \lambda_n(\mathcal{W}) \in \Lambda_n(m)$ by exhibiting the difference $\lambda_n(\mathcal{W}) - \lambda_n(\mathcal{W}')$ as $\lambda_n(\mathcal{V})$, where $\mathcal{V}$ is an order n non-repeating Whitney tower on a collection of immersed 2–spheres in the 4–sphere.

To start the construction, let $\mathcal{W}' \subset X_L = B \cup H_1 \cup H_2 \cup \cdots \cup H_m$ be an order n non-repeating Whitney tower on A', with A' homotopic to A. Here B is the 4–ball, and the H_i are the 0-framed 2–handles. Any singularities of $\mathcal{W}'$ which are contained in the H_i can be pushed off by radial ambient isotopies, so that $\mathcal{W}'$ may be assumed to only intersect the H_i in disjointly embedded disks which are parallel copies of the handle cores. These embedded disks lie in the order 0 2–spheres and the interiors of Whitney disks of $\mathcal{W}'$. It also may be assumed that the trees representing $\lambda_n(\mathcal{W}')$ are disjoint from all the H_i.

The intersection $\mathcal{W}' \cap \partial B$ is a link L' in S^3, such that L' is related to L by adding some parallel copies of components and switching some orientations. Note that since each A_i goes over H_i algebraically once, L' contains L as a sublink. Write L' as the union $L' = L^0 \cup L^1$ of two links where the components of L^0 bound handle core disks in the order 0 2–spheres of $\mathcal{W}'$, and the components of L^1 bound handle core disks in the Whitney disks (surfaces of order at least 1) of $\mathcal{W}'$. For each i, the components of L^0 which are parallel to $L_i \subset L$ must come in oppositely oriented pairs except for one component which is oriented the same as L_i. The components of L^1 can be arbitrary parallels of components of L.

Now delete the H_i from X_L, and form S^4 by gluing another 4–ball B^+ to B along their 3–sphere boundaries. Since L bounds the order n Whitney tower $\mathcal{W}_L$ in B^+, an order n Whitney tower $\mathcal{W}^+ \subset B^+$ bounded by L' can be constructed using parallel order 0 disks and Whitney disks of $\mathcal{W}_L$ as in section 2.10 above. The union of $\mathcal{W}^+$ together with $\mathring{\mathcal{W}}' := \mathcal{W}' \cap B$ is an order n non-repeating Whitney tower $\mathcal{V} := \mathcal{W}^+ \cup \mathring{\mathcal{W}}'$ on m immersed 2–spheres in S^4. Figure 14 gives a schematic illustration of $\mathcal{V}$.

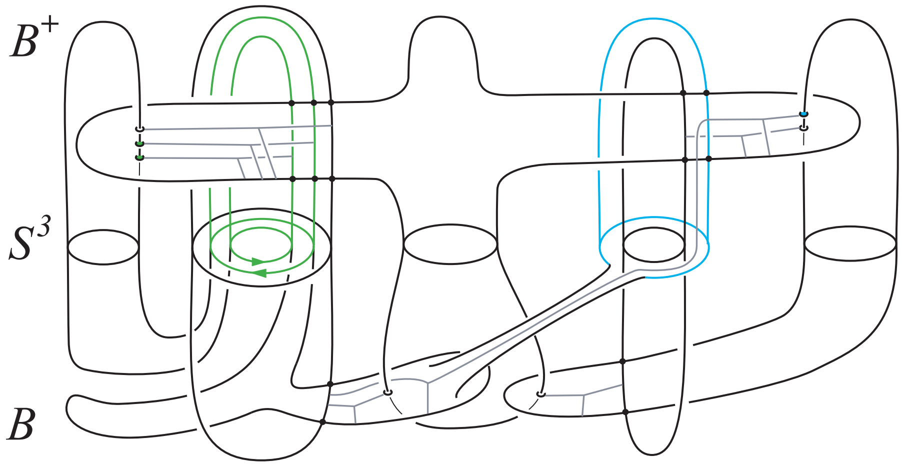

Figure 14: The non-repeating Whitney tower $\mathcal{V} = \mathcal{W}^+ \cup \mathring{\mathcal{W}}' \subset S^4 = B \cup_{S^3} B^+$: The links $L \subset L' \subset S^3$ are shown in the horizontal middle part of the figure. The components of L are black; the component of $L^1 \subset L'$ is blue, and an oppositely oriented pair of components in $L^0 \subset L'$ are shown in green. The lower part of the figure shows $\mathring{\mathcal{W}}' \subset B$, and the upper part shows $\mathcal{W}^+ \subset B^+$. The tree shown involving the blue L^1-component passes down through L^1 into a Whitney disk of $\mathring{\mathcal{W}}'$ and then down into a pair of order 0 disks in $\mathring{\mathcal{W}}'$, so the tree is of order greater than n. The pair of trees each having a univalent vertex on a green order 0 disk in $\mathcal{W}^+$ have opposite signs due to the opposite orientations on the green disks.

We will check that $\lambda_n(\mathcal{V}) = \lambda_n(\mathcal{W}_L) - \lambda_n(\mathcal{W}') \in \Lambda_n(m)$, which will complete the proof in this case by the opening observation that $\lambda_n(\mathcal{V})$ vanishes. We take the orientation of $(B^+, \partial B^+)$ in S^4 to be the standard orientation of (B^4, S^3), and that of $(B, \partial B)$ to be the opposite. Since $\mathring{\mathcal{W}}' \subset B$ contains all the trees representing $\lambda_n(\mathcal{W}')$, these trees contribute the term $-\lambda_n(\mathcal{W}')$ to $\lambda_n(\mathcal{V})$.

Consider next the trees corresponding to intersections in $\mathcal{W}^+$ involving components of L^1 (i.e. the trees that intersect at least one order 0 disk of $\mathcal{W}^+$ bounded by a component of L^1). In $\mathcal{V}$ these trees are subtrees of trees (for the same intersections) which pass down through L^1 into the Whitney disks of $\mathring{\mathcal{W}}'$ until reaching the order 0 disks in $\mathring{\mathcal{W}}'$. Any such tree is of order strictly greater than n, since it contains an order n proper subtree (the part of the tree in $\mathcal{W}^+$) – see Figure 14. Such higher-order trees do not contribute to $\lambda_n(\mathcal{V})$.

Consider now the remaining trees in $\mathcal{V}$ which only involve the components of L^0. These trees represent $\lambda_n(L^0) = \lambda_n(\mathcal{W}^0)$, where $\mathcal{W}^0 \subset \mathcal{W}^+$ is the order n Whitney tower in B^+ bounded by $L^0 \subset \partial B^+$; but we claim that in $\mathcal{V}$ these trees contribute exactly $\lambda_n(L)$, which completes the proof in this case. To see the claim, recall that L^0 consists of L together with oppositely oriented pairs of parallel components of L. Denote by $+L_i^j$, $-L_i^j$ such a pair which is parallel to the ith component L_i of L, and which bounded oppositely oriented handle-cores $+H_i$ and $-H_i$ in the jth component A'_j of A'. The univalent labels on trees representing $\lambda_n(L^0) = \lambda_n(\mathcal{W}^0)$ which correspond to $+L_i^j$ and $-L_i^j$ when considered as trees in $\mathcal{V}$ are labeled by the same label j. Such re-labelings correspond exactly to the operations of Lemma 20 in section 2.10.6, which implies that all trees involving such pairs of components contribute trivially to $\lambda_n(\mathcal{V})$, verifying the claim.

The proof of Theorem 11 in the general case follows the argument just given with essentially only notational differences: An arbitrary A is represented by the union of a linear combination of cores of the H_i with immersed disks in B^4 bounded by a link L_A formed from L by the operations of adding parallel components, switching orientations, taking internal sums and deleting components. Since L bounds an order n Whitney tower, so does L_A by section 2.10.6. Hence A supports an order n Whitney tower $\mathcal{W}$ with $\lambda_n(\mathcal{W}) = \lambda_n(L_A) \in \Lambda_n(m)$. One shows that $\lambda_n(\mathcal{W}') = \lambda_n(L_A)$ for any non-repeating $\mathcal{W}'$ on A' homotopic to A by proceeding as above with L_A taking the place of L.

6 Pulling apart parallel 2–spheres

In this section we prove Theorem 1 of the introduction, which states that for a map $A_0 : S^2 \to X$ of a 2-sphere in a simply connected 4–manifold X with vanishing normal Euler number, the homological self-intersection number $[A_0] \cdot [A_0]$ vanishes if and only if any number of parallel copies of A_0 can be pulled apart.

Note that since the Euler number $e(A)$ of the normal bundle of a map $A : S^2 \to X$ of a 2-sphere in a 4–manifold X can be changed by ± 2 by performing a cusp homotopy of A, the condition $e(A) = 0$ can be arranged if and only if the second Stiefel-Whitney class $\omega_2 \in H^2(X; \mathbb{Z}_2)$ vanishes on $[A]$ (see e.g. [12, Sec.1.3A]). On the other hand, if $\omega_2([A]) \neq 0$, then $[A] \cdot [A]$ is odd and hence not even two copies of A can be pulled apart.

The proof of Theorem 1 includes a geometric proof that boundary links in the 3–sphere are link-homotopically trivial (Proposition 28 below). We also give an example (6.2) illustrating that the "only if" direction of Theorem 1 is not generally true in *non*-simply connected 4–manifolds.

6.1 Proof of Theorem 1.

We drop the subscript from notation and consider a map $A : S^2 \to X$ with vanishing normal Euler number $e(A) = 0$ and X simply connected. From the relationship $[A] \cdot [A] = e(A) + \lambda(A, A')$, and the hypothesis that $e(A) = 0$, we have that $[A] \cdot [A]$ is equal to the Wall pairing $\lambda(A, A')$ which counts signed intersections between A and any transverse parallel copy A' (a generic normal section). (Since X is simply connected, the Wall pairing is just the usual algebraic intersection number in $\mathbb{Z}$.) So the "if" direction of Theorem 1 is clear, since $\lambda(A, A')$ obstructs pulling apart any two copies of A.

For the other direction, start by observing that the vanishing of $[A] \cdot [A] = \lambda(A, A')$ implies that A supports an order 1 Whitney tower $\mathcal{W}$: The intersections between the parallels A and A' correspond in pairs to self-intersections of A, so $\lambda(A, A')$ is equal to twice the sum of signed self-intersections of A. These self-intersections must come in oppositely signed pairs, which admit Whitney disks since X is simply connected.

First consider the case where A also has vanishing order 1 intersection invariant (section 2.7): If A is characteristic, then $\tau_1(A) := \tau(\mathcal{W}) = 0 \in \mathcal{T}_1(1)/\mathrm{INT}_1(A) \cong \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$; or if A is not characteristic, then $\mathcal{T}_1(1)/\mathrm{INT}_1(A)$ is trivial (see [30, Sec.1]). By Theorem 2 of [30] the vanishing of $\tau_1(A)$ implies that A admits an order 2 Whitney tower, so by Lemma 3 of [28] for *any* $m \in \{3, 4, 5, \ldots\}$, A admits a Whitney tower $\mathcal{W}$ of order m. (The fact that A is *connected* and X is *simply connected* is crucial here, since under these hypotheses Lemma 3 of [28] shows that higher-order Whitney towers can be built using a Whitney disk boundary-twisting construction.) Now, taking parallel copies of the Whitney disks in $\mathcal{W}$ yields an order m Whitney tower on $m + 1$ parallel copies of A, as observed above in 2.10.2. In particular, we get an order m non-repeating Whitney tower so, by Theorem 4, the $m + 1$ parallel copies of A can be pulled apart.

Consider now the case where $\tau_1(A) = \tau_1(\mathcal{W})$ is the non-trivial element in $\mathbb{Z}_2$. We will first isolate (to a neighborhood of a point) the obstruction to building an order 2 Whitney tower, and then combine the previous argument away from this point with an application of Milnor's Theorem 4 of [23] (which we will also prove geometrically in Proposition 28 below).

As illustrated in Figure 15, a trefoil knot in the 3–sphere bounds an immersed 2–disk in the 4–ball which supports an order 1 Whitney tower containing exactly one Whitney disk whose interior contains a single order 1 intersection point. It follows that the square knot, which is the connected sum of a right- and a left-handed trefoil knot, bounds an immersed disk D in the 4–ball which supports a Whitney tower $\mathcal{V}$ containing exactly two first order Whitney disks, each of which contains a single order 1 intersection point with D. Being a well-known slice knot, the square knot also bounds an embedded 2–disk D' in the 4–ball,

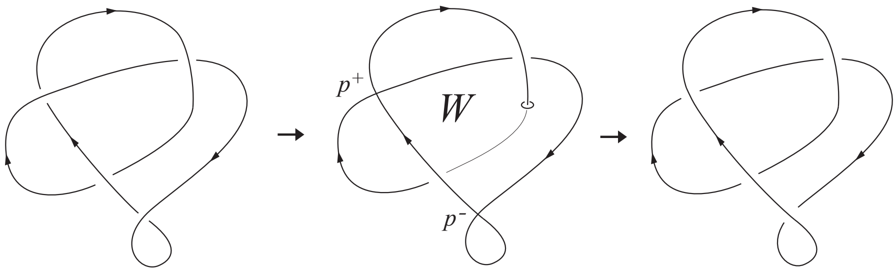

Figure 15: Moving into B^4 from left to right: A trefoil knot in S^3 (left) bounds an immersed disk having a single pair of self-intersections $p^{\pm}$ admitting a Whitney disk W containing a single order 1 intersection point (center). An unknotted 'slice' of the immersed disk is shown on the right. The rest of the immersed disk is described by a null isotopy further into B^4 (not shown) of this unknot.

and by gluing together two 4–balls along their boundary 3–spheres we get an immersed 2–sphere $S = D \cup D'$ in the 4–sphere having the square knot as an "equator" and supporting the obvious order 1 Whitney tower $\mathcal{W}_S$ consisting of S together with the two Whitney disks from $\mathcal{V}$ pairing the intersections in $D \subset S$.

Now take $\mathcal{W}_S$ in a (small) 4–ball neighborhood of a point in X (away from A), and tube (connected sum) A into S. This does not change the (regular) homotopy class of A, so we will still denote this sum by A. Note that by construction there is a (smaller) 4–ball B^4 such that the intersection of the boundary ∂B^4 of B^4 with A is a trefoil knot (one of the trefoils in the connected sum decomposition of the square knot), and B^4 contains one of the two Whitney disks of $\mathcal{W}_S$. Denote by X° the result of removing from X the interior of B^4, and denote by A° the intersection of A with X° (so A° is just A minus a small open disk). Since the order 1 intersection point in the Whitney disk of $\mathcal{W}_S$ which is *not* contained in B^4 now cancels the obstruction $\tau_1(\mathcal{W}) \in \mathbb{Z}_2$, we have that A° admits an order 2 Whitney tower in X°, and hence again by Lemma 3 of [28], A° admits a Whitney tower of any order in X°. As before, it follows that parallel copies of A° can be pulled apart by using parallel (non-repeating) copies of the Whitney disks in a high order Whitney tower on A°. The parallel copies of A° restrict on their boundaries to a link of 0-parallel trefoil knots in ∂B^4, and the proof of Theorem 1 is completed by the following lemma which implies that these trefoil knots bound disjointly immersed 2–disks in B^4. □

Proposition 28 *If the components L_i of a link $L = \cup L_i \subset S^3$ are the boundaries of disjointly embedded orientable surfaces $F_i \subset S^3$ in the 3–sphere, then the L_i bound disjointly immersed 2–disks in the 4–ball.*

This proposition first appeared as Milnor's Theorem 4 of [23], and is a special case of the general results of [32] which are proved using symmetric surgery.

Proof: Choose a symplectic basis of simple closed curves on each F_i bounding properly immersed 2–disks into the 4–ball. We shall refer to these disks as *caps*. These caps may intersect each other, but the interiors of these caps lie in the interior of B^4 and so are disjoint from $\cup_i F_i \subset S^3$. The proof proceeds inductively by using half of these caps to surger each F_i to an immersed disk F_i^0, while using the other half of the caps to construct Whitney disks which guide Whitney moves to achieve disjointness.
We start with F_1. Let D_{1r} and D_{1r}^* denote the caps bounded by the symplectic circles in F_1, with ∂D_{1r} geometrically dual to ∂D_{1r}^* in F_1.
Step 1: Using finger moves, remove any interior intersections between the D_{1r}^* and any D_{1s} by pushing the D_{1s} down into F_1 (Figure 16).

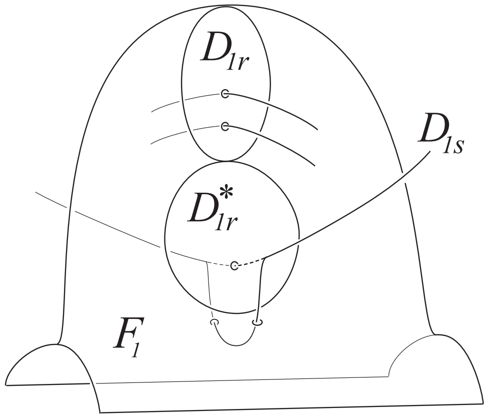

Figure 16:

Step 2: Surger F_1 along the D_{1r} (Figure 17). The result is a properly immersed 2–disk F_1^0 in the 4–ball bounded by L_1 in S^3. The self-intersections in F_1^0 come from intersections and self-intersections in the surgery disks D_{1r}, and any intersections between the D_{1r} and F_1 created in Step 1, as well as any intersections created by taking parallel copies of the D_{1r} during surgery. We don't care about any of these self-intersections in F_1^0, but we do want to eliminate all intersections between F_1^0 and any of the disks D_j on the other F_j, $j \geq 2$. These intersections between F_1^0 and the disks on the other F_j all occur in cancelling pairs, with each such pair coming from an intersection between a D_{1r} and a D_j. Each of these cancelling pairs admits a Whitney disk W_{1r}^* constructed by adding a thin band to (a parallel copy of) the dual disk D_{1r}^* as illustrated in Figure 17. Note that by Step 1 the interiors of the D_{1r}^* are disjoint from F_1^0, hence the interiors of the W_{1r}^* are also disjoint from F_1^0. The interiors of the W_{1r}^* may intersect the D_j, but we don't care about these intersections.
Step 3: Do the W_{1r}^* Whitney moves on the D_j. This eliminates all intersections between F_1^0 and the disks D_j on all the other F_j ($j \geq 2$). Note that any interior

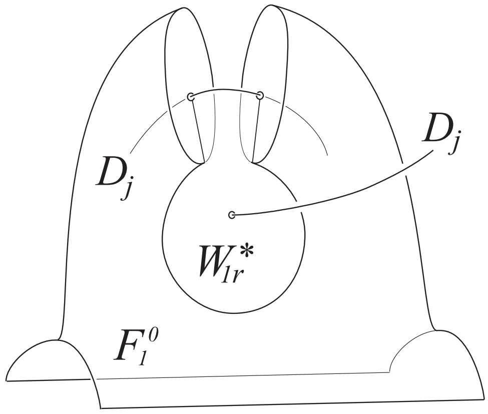

Figure 17:

intersections the W^*_{1r} may have had with the D_j only lead to more intersections among the D_j, so these three steps may be iterated, starting next by applying Step 1 to F_2. □

6.2 Example

If $\pi_1 X$ is non-trivial, then the conclusion of Theorem 1 may not hold, as we now illustrate. Let X be a 4–manifold with $\pi_1 X \cong \mathbb{Z}$, such that $\pi_2 X$ has trivial order 0 intersection form; and let A_1 be an immersed 0-framed 2–sphere admitting an order 1 Whitney tower $\mathcal{W}$ in X with a single order 1 intersection point p such that $\tau_1(A_1) = t_p \in \mathcal{T}_1(\mathbb{Z}, 1)$ is represented by the single Y-tree $Y(e, g, h) = t_p$ having one edge labeled by the trivial group element e, and the other edges labeled by non-trivial elements $g \neq h$, all edges oriented towards the trivalent vertex. Such examples are given in [30], and can be easily constructed by banding together Borromean rings in the boundary of $B^3 \times S^1$ and attaching a 0-framed two handle.

If A_2 and A_3 are parallel copies of A_1, then the order 1 non-repeating intersection invariant $\lambda_1(A_1, A_2, A_3)$ takes values in $\Lambda_1(\mathbb{Z}, 3)$ (since the vanishing of the order 0 intersections means that all INT_1 relations are trivial). By normalizing the group element decorating the edge adjacent to the 1-label to the trivial element using the HOL relations, $\Lambda_1(\mathbb{Z}, 3)$ is isomorphic to $\mathbb{Z}[\mathbb{Z} \times \mathbb{Z}]$. Using six parallel copies of the Whitney disk in $\mathcal{W}$, we can compute that $\lambda_1(A_1, A_2, A_3)$ is represented by the sum of six Y-trees $Y(e, g, h)$, where the univalent vertex labels vary over the permutations of $\{1, 2, 3\}$ (see [30, Thm.3.(iii)]). This element corresponds to the element

$$(g,h)-(h,g)+(hg^{-1},g^{-1})-(g^{-1},hg^{-1})+(gh^{-1},h^{-1})-(h^{-1},gh^{-1}) \in \mathbb{Z}[\mathbb{Z}\times\mathbb{Z}]$$

which is non-zero if (and only if) g and h are distinct non-trivial elements of $\mathbb{Z}$. Since $\lambda_1(A_1, A_2, A_3) \in \Lambda_1(\mathbb{Z}, 3)$ is a well-defined homotopy invariant [30, Thm.3], the A_i can not be pulled apart whenever g and h are distinct and non-trivial.

7 Dual spheres and stablizations

This section contains proofs of Theorem 2 and Theorem 13, both of which involve using low-order intersections to kill higher-order obstructions.

7.1 Proof of Theorem 13

We need to show that surfaces A_i with pairwise vanishing Wall intersections can be pulled apart if they have algebraic duals $B_i : S^2 \to X$.
Proof: Wall's intersection pairing $\lambda(A_i, A_j) \in \mathbb{Z}[\pi]$ is defined when the $A_k : \Sigma_k \to X$ are maps of simply connected surfaces Σ_k, or more generally when the A_k are π_1-null (Remark 16). The pairwise vanishing of Wall's invariant gives an order 1 non-repeating Whitney tower on the A_i (2.6). Assuming inductively for $1 \leq n < m-2$ the existence of an order n non-repeating Whitney tower $\mathcal{W}$ on the A_i, it is enough to show that it can be arranged that $\lambda_n(\mathcal{W}) = 0 \in \Lambda_n(\pi, m)$, which allows us to find an order $n+1$ non-repeating Whitney tower by Theorem 6, and then to apply Theorem 4 when $n = m-2$. By performing finger moves to realize the rooted product, any order n Whitney tower $\mathcal{W} \subset X$ can be modified (in a neighborhood of a 1-complex) to have an additional clean order n Whitney disk W_J whose decorated tree corresponds to any given bracket J, with edges labeled by any given elements of $\pi := \pi_1 X$. If J is non-repeating and does not contain the label i, then tubing the 2–sphere B_i into W_J will change $\lambda_n(\mathcal{W})$ exactly by adding the order n generator $\pm\langle J, i\rangle_g$, where the sign can be chosen by the choice of orientation on B_i, and the element $g \in \pi$ decorating the i-labeled edge is determined by the choice of arc guiding the tubing (together with the whiskers on W_J and A_i): Since W_J is order n, any intersections between B_i and other Whitney disks in $\mathcal{W}$ will only contribute intersections of order strictly greater than n; and since $\lambda(A_j, B_i) = \delta_{ij} \in \mathbb{Z}[\pi]$, any other intersections between A_j and B_i contribute only canceling pairs of order n intersections. For $1 \leq n$, any generator of $\Lambda_n(\pi, m)$ can be realized as $\langle J, i\rangle_g$, so the just-described tubing procedure can be used to modify $\mathcal{W}$ until $\lambda_n(\mathcal{W}) = 0 \in \Lambda_n(\pi, m)$. □

7.2 Proof of Theorem 2

We need to show that $\lambda_1(A) = 0 \in \Lambda_1(\pi, m)/\mathrm{INT}_1(A)$ if and only if $A : \amalg^m S^2 \to X$ can be pulled apart stably.
Note that the vanishing of the homotopy invariant $\lambda_0(A)$ is implied by $\lambda_1(A)$ being defined.
Proof: The "if" direction is immediate since $\lambda_1(A)$ only depends on the homotopy class of A (by [30]), and any 2–spheres carried by the stabilization con-

tribute trivially to $\mathrm{INT}_1(A)$. (See section 8.2 below for details on the $\mathrm{INT}_1(A)$ relations in the cases $m = 3, 4$ and X simply connected.)

For the "only if" direction, observe first that the vanishing of $\lambda_1(A)$ gives an order 2 non-repeating tower supported by A (by Theorem 6). Assuming inductively for $2 \leq n < m - 2$ the existence of an order n non-repeating Whitney tower $\mathcal{W}$ on A, it is enough to show that it can be arranged that $\lambda_n(\mathcal{W}) = 0 \in \Lambda_n(\pi, m)$, which allows us to find an order $n+1$ non-repeating Whitney tower by Theorem 6, and then to apply Theorem 4 when $n = m - 2$.

For $n \geq 2$, any generator of $\Lambda_n(\pi, m)$ can be written as $\langle I, J\rangle_g$ where I and J are both of order greater than or equal to 1, and $g \in \pi$ decorates the edge where the roots of I and J are joined. As in the above proof of Theorem 13, any order n non-repeating Whitney tower $\mathcal{W}$ on A can be modified to have new clean Whitney disks W_I and W_J, without affecting $\lambda_n(\mathcal{W})$. Stabilizing the ambient 4–manifold by $S^2 \times S^2$ and tubing W_I and W_J into a pair of dual 2–spheres coming from the stablization creates an intersection realizing the generator $\langle I, J\rangle_g$, where the element $g \in \pi$ is determined by the choices of whiskers on W_I and W_J and the tubes into the dual spheres. By realizing generators in this way it can be arranged that $\lambda_n(\mathcal{W}) = 0$.

By Poincaré duality, the same holds for any closed simply connected 4–manifold other than S^4. For instance, for stablization by $\mathbb{CP}^2$ (or $\overline{\mathbb{CP}}^2$), where the dual 2–spheres are copies of $\mathbb{CP}^1$, the framings on W_I and W_J can be recovered by boundary-twisting [12, Sec.1.3], which only creates repeating intersections. □

We remark that some control on the number of stablizations needed in Theorem 2 can be obtained in terms of m when X is simply connected (so that $\Lambda_n(m)$ is finitely generated). For instance, a single stablization realizes k times a generator by tubing W_I or W_J into k copies (tubed together) of one of the dual spheres.

8 Second order intersection indeterminacies

It is an open problem to give necessary and sufficient algebraic conditions for determining whether or not an arbitrary quadruple $A : \amalg^4 S^2 \to X$ of 2–spheres in a 4–manifold can be pulled apart. The vanishing of $\lambda_0(A)$ and $\lambda_1(A)$ is of course necessary, and is equivalent to A admitting an order 2 non-repeating Whitney tower. As explained in the introduction (1.9), refining the sufficiency statement provided by Corollary 7 requires the introduction of *intersection relations* $\mathrm{INT}_2(A)$ in the target of $\lambda_2(A)$ which correspond to order 0 and order 1 intersections involving 2-spheres which can be tubed into the Whitney disks of any Whitney tower $\mathcal{W}$ supported by A.

With an eye towards stimulating future work, the goals of this section are to present some relevant details, describe some partial results, and introduce a related number theoretic problem, while formulating order 2 intersection relations which make the following conjecture precise:

Conjecture 29 *If a quadruple of immersed* 2*–spheres* $A : \amalg^4 S^2 \to X$ *in a simply connected* 4*–manifold* X *admits an order* 2 *non-repeating Whitney tower* $\mathcal{W}$*, then* A *can be pulled apart if and only if* $\lambda_2(A) := \lambda_2(\mathcal{W})$ *vanishes in* $\Lambda_2(4)/\mathrm{INT}_2(A)$*.*

This section is somewhat technical, so we begin by providing an outline: After quickly recalling in 8.1 the lack of indeterminacies in the order 0 non-repeating invariant λ_0, the intersection relations INT_1 in the target of the order 1 non-repeating invariant λ_1 are examined in detail for triples and then quadruples of 2–spheres, including notation and examples intended to clarify and motivate the introduction of the intersection relations $\mathrm{INT}_2(A)$ in the target of the order 2 non-repeating invariant λ_2. These $\mathrm{INT}_2(A)$ relations, which are determined by λ_0 and λ_1 on $\pi_2 X$, are discussed throughout section 8.3. Section 8.3.1 observes that if A has any non-trivial order 0 intersections with any other 2–spheres in X, then the target $\Lambda_2(4)/\mathrm{INT}_2(A)$ of $\lambda_2(A)$ must be finite; and presents two related results, Proposition 30 and Proposition 31, which give sufficient conditions for pulling apart A in the setting where $\lambda_2(A)$ is defined. Section 8.3.3 describes the INT_2 relations as the image in $\Lambda_2 \cong \mathbb{Z} \oplus \mathbb{Z}$ of a linear map determined by λ_1 on $\pi_2 X$ in the setting where λ_0 vanishes on $\pi_2 X$, as motivation for the discussion in section 8.3.5 on how non-trivial values of λ_0 away from A can affect the INT_2 relations. Section 8.3.6 shows how the INT_2 relations can be computed as the image of a map whose non-linear part is determined by Diophantine quadratic equations coupled by the order 0 intersection form λ_0 on $\pi_2 X$, leading naturally to some relevant number theoretic questions.
Throughout the rest of this section we assume that the ambient 4–manifold X is simply connected. For brevity we suppress the domains of the components of A from notation and consider collections $A = A_1, \ldots, A_m \looparrowright X$ of immersed 2–spheres.

8.1 Order 0 intersection invariants

Recall (2.6) that the order 0 non-repeating intersection invariant $\lambda_0(A_1, \ldots, A_m) = \sum \mathrm{sign}(p) \cdot i \,\text{—}\, j \in \Lambda_0(m)$ on 2–spheres immersed in a simply connected 4–manifold X carries exactly the same information as the integral homological intersection form on $H_2(X)$, with the sum of the coefficients of the $i \,\text{—}\, j$ corresponding to the usual homological intersection number $[A_i] \cdot [A_j] \in \mathbb{Z}$. There are no intersection indeterminacies in this order 0 setting, and $A_1, \ldots, A_m$ admits an order 1 non-repeating Whitney tower if and only if $\lambda_0(A_1, \ldots, A_m)$ vanishes in $\Lambda_0(m)$ (which is isomorphic to a direct sum of $\binom{m}{2}$ copies of $\mathbb{Z}$, one for each (unordered) pair of distinct indices i, j).

8.2 Order 1 intersection relations.

The order 1 intersection relations INT_1 are described by order 0 intersections λ_0. These INT_1 relations are examined here in detail for triples and quadruples of 2–spheres, as notational and motivational preparation for describing the order 2 intersection relations.

8.2.1 ORDER 1 TRIPLE INTERSECTIONS

For a triple of immersed 2–spheres $A_1, A_2, A_3 \looparrowright X$ with $\lambda_0(A_1, A_2, A_3) = 0$, the order 1 non-repeating intersection invariant $\lambda_1(A_1, A_2, A_3)$ is a sum of order 1 Y-trees in $\Lambda_1(3) \cong \mathbb{Z}$ modulo the $\mathrm{INT}_1(A_1, A_2, A_3)$ intersection relations:

$$ {}^{j}_{i} \succ\!\!-\lambda_0(S_{(i,j)}, A_k) = 0 $$

where $S_{(i,j)}$ ranges over $\pi_2 X$, and (i, j) ranges over the three choices of pairs from $\{1, 2, 3\}$. (Here the notation ${}^{j}_{i} \succ\!\!-\lambda_0(S_{(i,j)}, A_k)$ indicates the sum of trees gotten by attaching the root of (i, j) to the (i, j)-labeled univalent vertices in $\lambda_0(S_{(i,j)}, A_k)$ corresponding to $S_{(i,j)}$.) Geometrically, these relations correspond to tubing any Whitney disk $W_{(i,j)}$ into any 2–sphere $S_{(i,j)}$. Via the identification $\Lambda_1(3) \cong \mathbb{Z}$, the quotient $\Lambda_1(3)/\mathrm{INT}_1(A_1, A_2, A_3)$ is isomorphic to $\mathbb{Z}_d = \mathbb{Z}/d\mathbb{Z}$, where d is the greatest common divisor of all the $\lambda_0(S_{(i,j)}, A_k)$. This invariant $\lambda_1(A_1, A_2, A_3) \in \Lambda_1(3)/\mathrm{INT}_1(A_1, A_2, A_3)$ is the *Matsumoto triple* [25] which vanishes if and only if A_1, A_2, A_3 admit an order 2 non-repeating Whitney tower (and hence can be pulled apart [34]).
EXAMPLES: In the 4–manifold X_L gotten by attaching 0-framed 2-handles to the Borromean rings $L = L_1 \cup L_2 \cup L_3 \subset S^3 = \partial B^4$, all INT_1 relations are trivial, and the triple A_1, A_2, A_3 of 2–spheres determined (up to homotopy) by the link components can not be pulled apart since $\lambda_1(A_1, A_2, A_3)$ is equal to $(\pm)$ the generator ${}^{2}_{1} \succ\!\!- 3$ of $\Lambda_1(3) \cong \mathbb{Z}$.
If X_L is changed to X'_L by attaching another 2-handle along a meridional circle to L_3, then $\mathrm{INT}_1(A_1, A_2, A_3) = \Lambda_1(3)$ since ${}^{2}_{1} \succ\!\!-\lambda_0(S_{(1,2)}, A_3) = (\pm){}^{2}_{1} \succ\!\!- 3$, where $S_{(1,2)}$ is the new 2–sphere which is dual to A_3. Now $A_1, A_2, A_3 \looparrowright X'_L$ can be pulled apart since $\lambda_1(A_1, A_2, A_3)$ takes values in the trivial group.

8.2.2 COMPUTING THE $\mathrm{INT}_1(A_1, A_2, A_3)$ INTERSECTION RELATIONS

Since each element of $\pi_2 X$ can affect the non-repeating order 1 indeterminacies in three independent ways (by tubing 2–spheres into Whitney disks $W_{(1,2)}$, $W_{(1,3)}$, and $W_{(2,3)}$) the $\mathrm{INT}_1(A_1, A_2, A_3)$ relations can be computed as the image of a linear map $\mathbb{Z}^r \oplus \mathbb{Z}^r \oplus \mathbb{Z}^r \longrightarrow \mathbb{Z}$, with r the rank of the $\mathbb{Z}$-module $\pi_2 X$ modulo torsion. Specifically, let S^α be a basis for $\pi_2 X$ (mod torsion), and define integers $a^\alpha_{ij} := \lambda_0(S^\alpha_{(i,j)}, A_k)$ for $S^\alpha_{(i,j)}$ ranging over the basis, and i, j, k distinct. Then, identifying $\Lambda_1(3) \cong \mathbb{Z}$, the $\mathrm{INT}_1(A_1, A_2, A_3)$ intersection relations can be described as

$$ \sum_\alpha (x^\alpha_{12} a^\alpha_{12} + x^\alpha_{31} a^\alpha_{31} + x^\alpha_{23} a^\alpha_{23}) = 0 $$

with the coefficients x^α_{ij} ranging (independently) over $\mathbb{Z}$.

Using integer vector notation, this map can be written concisely as:

$$(x_{12}, x_{31}, x_{23}) \longmapsto x_{12} \cdot a_{12} + x_{31} \cdot a_{31} + x_{23} \cdot a_{23}.$$

with "$\cdot$" denoting the dot product in $\mathbb{Z}^r$.

8.2.3 Order 1 quadruple intersections

For a collection A of *four* immersed 2–spheres $A = A_1, A_2, A_3, A_4 \looparrowright X$ with vanishing $\lambda_0(A)$, the order 1 non-repeating intersection invariant $\lambda_1(A)$ takes values in $\Lambda_1(4)/\mathrm{INT}_1(A)$, with the $\mathrm{INT}_1(A)$ relations given by

$$\overset{j}{\underset{i}{}} \succ\!\!- \lambda_0(S_{(i,j)}, A_k, A_l) = 0$$

where $S_{(i,j)}$ ranges over $\pi_2 X$, and (i,j) ranges over the six choices of distinct pairs from $\{1,2,3,4\}$. Each such relation corresponds to tubing the 2–sphere $S_{(i,j)}$ into a Whitney disk $W_{(i,j)}$. Here $\Lambda_1(4) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, and each generator of the rank r $\mathbb{Z}$-module $\pi_2 X$ modulo torsion gives six relations, so the target group $\Lambda_1(4)/\mathrm{INT}_1(A)$ of $\lambda_1(A)$ is the quotient of Z^4 by the image of a linear map from $\mathbb{Z}^{6r}$. The invariant $\lambda_1(A)$ vanishes in $\Lambda_1(4)/\mathrm{INT}_1(A)$ if and only if A admits an order 2 non-repeating Whitney tower.

Example: Note that each of the four copies of $\mathbb{Z}$ in $\Lambda_1(4)$ corresponds to a target of a Matsumoto triple (a choice of three distinct indices), but the vanishing of the all the triples is not sufficient to get an order 2 non-repeating Whitney tower on the A because of "cross-terms" in the INT_1 relations; the simplest example is the following:

Consider a five component link $L = L_1 \cup \cdots \cup L_5 \subset S^3 = \partial B^4$ such that $L_1 \cup L_2 \cup L_3$ forms a Borromean rings which is split from the component L_4, and L_5 is a band sum of (positive) meridians to L_3 and L_4. In the 4–manifold gotten by attaching 0-framed 2-handles to B^4 along L, let A_i denote the immersed 2–sphere determined (up to homotopy) by the core of the 2-handle attached to L_i.

Now any three of the quadruple A_1, A_2, A_3, A_4 will have vanishing first order triple $\lambda_1(A_i, A_j, A_k)$ in $\Lambda_1(3)/\mathrm{INT}_1(A_i, A_j, A_k)$ for any choice of distinct i, j, k: Since A_5 is dual to A_3, the generator $\overset{2}{\underset{1}{}} \succ\!\!- 3$ of $\Lambda_1(3)$ is killed by $\mathrm{INT}_1(A_1, A_2, A_3)$; and for the other choices of $1 \leq i < j < k \leq 4$ it is clear that $\lambda_1(A_i, A_j, A_k)$ vanishes since $L_i \cup L_j \cup L_k$ is a split link (so A_i, A_j, A_k can be pulled apart).

But the first order quadruple $\lambda_1(A_1, A_2, A_3, A_4) = \overset{2}{\underset{1}{}} \succ\!\!- 3 = -\overset{2}{\underset{1}{}} \succ\!\!- 4$ is non-zero in $\Lambda_1(4)/\mathrm{INT}_1(A_1, A_2, A_3, A_4) \cong \mathbb{Z}^3$, where the only non-trivial $\mathrm{INT}_1(A_1, A_2, A_3, A_4)$ relation is

$$\overset{2}{\underset{1}{}} \succ\!\!- \lambda_0(A_5, A_3, A_4) = \overset{2}{\underset{1}{}} \succ\!\!- 3 + \overset{2}{\underset{1}{}} \succ\!\!- 4 = 0.$$

Geometrically, any order 1 intersection in a Whitney tower on $A_1 \cup A_2 \cup A_3$ can be killed by tubing A_5 into a Whitney disk pairing intersection between A_1 and A_2 to create a canceling order 1 intersection, but this also creates an order 1 intersection between the Whitney disk and A_4.

8.2.4 COMPUTING THE $\mathrm{INT}_1(A_1, A_2, A_3, A_4)$ RELATIONS

Choose a basis S^α for $\pi_2 X$ (mod torsion), and define integers $a^\alpha_{ij,k} := \lambda_0(S^\alpha_{(i,j)}, A_k)$. Then each element of the subgroup $\mathrm{INT}_1(A_1, A_2, A_3, A_4) < \Lambda_1(4)$ can be written

$$\begin{aligned} {}^{j}_{i}\succ\!\!- \lambda_0(\textstyle\sum_\alpha x^\alpha_{ij} S^\alpha_{(i,j)}, A_k, A_l) &= (\textstyle\sum_\alpha x^\alpha_{ij} a^\alpha_{ij,k})\, {}^{j}_{i}\succ\!\!-k + (\textstyle\sum_\alpha x^\alpha_{ij} a^\alpha_{ij,l})\, {}^{j}_{i}\succ\!\!-l \\ &= (x_{ij} \cdot a_{ij,k})\, {}^{j}_{i}\succ\!\!-k + (x_{ij} \cdot a_{ij,l})\, {}^{j}_{i}\succ\!\!-l \end{aligned}$$

where the coefficients in the last expression are dot products of vectors in $\mathbb{Z}^r$, with i, j, k, l distinct, and r the rank of $\pi_2 X$ (mod torsion). Using the basis

$$\{\, {}^{1}_{2}\succ\!\!-3\,,\ {}^{1}_{2}\succ\!\!-4\,,\ {}^{1}_{3}\succ\!\!-4\,,\ {}^{2}_{3}\succ\!\!-4 \,\}$$

for $\Lambda_1(4)$, the subgroup $\mathrm{INT}_1(A_1, A_2, A_3, A_4)$ is the image of the linear map $\mathbb{Z}^{6r} \longrightarrow \mathbb{Z}^4$:

$$\begin{pmatrix} x_{12} \\ x_{13} \\ x_{41} \\ x_{23} \\ x_{24} \\ x_{34} \end{pmatrix} \longmapsto \begin{pmatrix} a_{12,3} & -a_{13,2} & 0 & a_{23,1} & 0 & 0 \\ a_{12,4} & 0 & a_{41,2} & 0 & a_{24,1} & 0 \\ 0 & a_{13,4} & a_{41,3} & 0 & 0 & a_{34,1} \\ 0 & 0 & 0 & a_{23,4} & -a_{24,3} & a_{34,2} \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{13} \\ x_{41} \\ x_{23} \\ x_{24} \\ x_{34} \end{pmatrix}$$

where the multiplication of entries is the vector dot product in $\mathbb{Z}^r$.

8.3 ORDER 2 INTERSECTION RELATIONS.

Now consider a quadruple of immersed 2–spheres $A = A_1, A_2, A_3, A_4 \looparrowright X$ in a simply connected 4–manifold X, such that $\lambda_1(A) = 0 \in \Lambda_1(4)/\mathrm{INT}_1(A)$, so that A supports an order 2 non-repeating Whitney tower $\mathcal{W} \subset X$.
Recall that we want to describe order 2 intersection relations $\mathrm{INT}_2(A)$ which account for changes in the choice of Whitney tower on A and define the target of $\lambda_2(A) \in \Lambda_2(4)/\mathrm{INT}_2(A)$. Note that $\Lambda_2(4)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, generated, for instance, by the elements

$$t_1 := {}^{2}_{1}\succ\!\!-\!\!\prec^{3}_{4} \quad \text{and} \quad t_2 := {}^{3}_{1}\succ\!\!-\!\!\prec^{2}_{4},$$

with the IHX relation giving:

$${}^{4}_{1}\succ\!\!-\!\!\prec^{2}_{3} = t_1 + t_2.$$

We will mostly be concerned with the case that A is in the radical of λ_0 on $\pi_2 X$, so that for each $i \in \{1, 2, 3, 4\}$ the order 0 pairing $\lambda_0(S, A_i)$ vanishes for any immersed 2–sphere S, but first we make some quick general observations related to Theorems 2 and 13 above.

8.3.1 Tubing order 2 Whitney disks into spheres

Let i, j, k, l be distinct indices from $\{1, 2, 3, 4\}$. As already observed in the proof of Theorem 13, $\mathcal{W}$ can be modified to have an additional clean order 2 Whitney disk $W_{((i,j),k)}$ without creating any new unpaired intersections. If $S_{((i,j),k)}$ is any immersed 2–sphere, then tubing $W_{((i,j),k)}$ into $S_{((i,j),k)}$ preserves the order of the Whitney tower and changes $\lambda(\mathcal{W})$ by $a_{ijk} \cdot \langle((i,j),k),l\rangle$, where $a_{ijk} = \lambda_0(S_{((i,j),k)}, A_l) \in \mathbb{Z}$ (since any intersections between the new Whitney disk $W_{((i,j),k)} \# S_{((i,j),k)}$ and A_i, A_j, A_k are repeating intersections).
Letting $S_{((i,j),k)}$ vary over a basis S^α for $\pi_2 X$ (mod torsion) for distinct triples i, j, k in $\{1, 2, 3, 4\}$, this construction generates a subgroup of $\Lambda_2(4)$ isomorphic to $d\mathbb{Z} \oplus d\mathbb{Z}$, where d is the greatest common divisor of $\lambda_0(S^\alpha, A_i)$ over all S^α and i. In particular, if these order 0 intersections are relatively prime, then the target $\Lambda_2(4)/\mathrm{INT}_2(A)$ of $\lambda_2(A)$ is trivial:

Proposition 30 *If $A = A_1, A_2, A_3, A_4$ admits an order 2 non-repeating Whitney tower and if $\gcd(\{\lambda_0(S^\alpha, A_i)\}_{\alpha,i}) = 1$, then A can be pulled apart.* □

8.3.2 Tubing order 1 Whitney disks into spheres

Again as in the proof of Theorem 13, for any choice of distinct indices $\mathcal{W}$ can be modified to have two additional clean order 1 Whitney disks $W_{(i,j)}$ and $W_{(k,l)}$. Tubing either of these Whitney disks into an arbitrary 2–sphere might create unpaired order 1 non-repeating intersections (between A and the 2–sphere) and hence not preserve the order of $\mathcal{W}$, however tubing into 2–spheres created by stabilization does indeed preserve the order (since intersections among order 1 Whitney disks are of order 2). In fact, a single stablization is all that is needed to kill any obstruction to pulling apart the quadruple A_1, A_2, A_3, A_4:

Proposition 31 *If $A = A_1, A_2, A_3, A_4$ admit an order 2 non-repeating Whitney tower, then A can be pulled apart in the connected sum of X with a* single *$S^2 \times S^2$ (or a* single *$\mathbb{CP}^2$, or a* single *$\overline{\mathbb{CP}}^2$).*

Proof: $S^2 \times S^2$, or $\mathbb{CP}^2$, or $\overline{\mathbb{CP}}^2$. We will show how to change $\lambda_2(\mathcal{W})$ by any integral linear combination $a_1t_1 + a_2t_2$ of the above generators t_1, t_2 of $\Lambda_2(4)$: To create a_1t_1, first modify $\mathcal{W}$ to have two additional clean Whitney disks $W_{(1,2)}$ and $W_{(3,4)}$, then tube $W_{(1,2)}$ into S, and tube $W_{(3,4)}$ into $|a_1|$-many copies of S' (where the sign of a_1 corresponds to the orientations of the copies of S'). Note that in case of stabilization by $\mathbb{CP}^2$ or $\overline{\mathbb{CP}}^2$, the extra intersections coming from taking $|a_1|$ copies of $\mathbb{CP}^1$ are all repeating intersections, so that $\lambda_2(\mathcal{W})$ is indeed only changed by a_1t_1. Now, to further create a_2t_2 proceed in the same way starting with two additional clean Whitney disks $W_{(1,3)}$ and $W_{(2,4)}$, which are tubed into a parallel copy of the same S and $|a_2|$-many copies of S'. This will also create intersections with the previous copies of S and S', but these extra intersections will all be repeating intersections. (Any Whitney disks tubed into copies of $\mathbb{CP}^1$ can be framed as in the proof of Theorem 2, see section 7.2.) □

REMARK 32 *By Poincaré duality the statement of Proposition 31 holds for a single stabilization by taking the connected sum of X with any simply connected closed 4–manifold other than S^4.*

From the observations just before Proposition 30, the existence of any non-trivial order 0 intersections between any A_i and any 2–spheres in X implies that the obstruction to pulling apart the A_i lives in a finite quotient of $\Lambda_2(4)$. Returning to our goal of defining the INT_2 relations which clarify Conjecture 15, we will consider settings where the target for $\lambda_2(A)$ is potentially infinite.

8.3.3 LINEAR INT_2 RELATIONS

Assume first that all order 0 non-repeating intersections λ_0 on $\pi_2 X$ vanish.
Let i, j, k, l denote distinct indices in $\{1, 2, 3, 4\}$.
Suppose that $W_{(i,j)}$ is an order 1 Whitney disk in $\mathcal{W}$, and that $W'_{(i,j)}$ is a different choice of order 1 Whitney disk with the same boundary as $W_{(i,j)}$ such that all intersections $W'_{(i,j)} \cap A_k$ and $W'_{(i,j)} \cap A_l$ are paired by order 2 Whitney disks. Then replacing $W_{(i,j)}$ by $W'_{(i,j)}$, and replacing the order 2 Whitney disks supported by $W_{(i,j)}$ with those supported by $W'_{(i,j)}$, changes $\mathcal{W}$ to another order 2 non-repeating Whitney tower $\mathcal{W}'$ on A. The union of $W_{(i,j)}$ with $W'_{(i,j)}$ along their common boundary is a 2–sphere $S_{(i,j)} = W_{(i,j)} \cup W'_{(i,j)}$ with $\lambda_0(S_{(i,j)}, A_k, A_l) = 0 \in \Lambda_0((i,j), k, l)$ as pictured (schematically) in Figure 18.

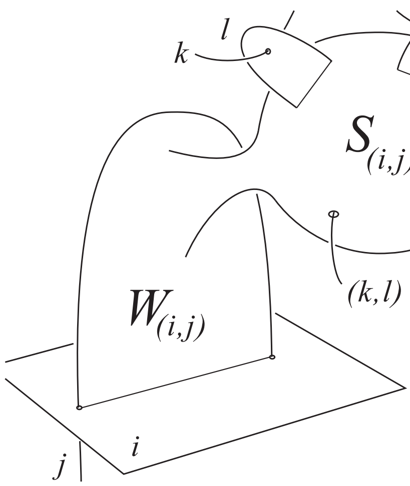

Figure 18: Changing the interior of $W_{(i,j)}$ to $W'_{(i,j)}$ corresponds to tubing $W_{(i,j)}$ into a 2–sphere $S_{(i,j)}$. Only intersections which contribute to the difference $\lambda_2(\mathcal{W}) - \lambda_2(\mathcal{W}') \in \Lambda_2(4)$ are shown.

Via the map $\Lambda_1((i,j), k, l) \to \Lambda_2(4)$ induced by sending $(i,j) \prec^k_l$ to ${}^j_i \succ\!\!\prec^k_l$,

the corresponding change $\lambda_2(\mathcal{W})-\lambda_2(\mathcal{W}')\in\Lambda_2(4)$ is equal to the image of the order 1 non-repeating intersection invariant $\lambda_1(S_{(i,j)},A_k,A_l)$, which is defined in $\Lambda_1((i,j),k,l)$ since the vanishing of λ_0 means that all INT_1 relations are trivial.
Similarly changing the interiors of any number of the order 1 Whitney disks in $\mathcal{W}$ leads to the following definition, which makes Conjecture 29 precise in this setting:

Definition 33 *For a quadruple of 2–spheres $A=A_1,A_2,A_3,A_4 \looparrowright X$ with $\lambda_1(A)=0$, with X simply connected and λ_0 vanishing on π_2X, define the order 2 intersection relations $\mathrm{INT}_2(A)<\Lambda_2(4)$ to be the subgroup generated by*

$$ {}^{j}_{i} \succ\!\!- \lambda_1(S_{(i,j)},A_k,A_l)$$

over all choices of distinct i,j,k,l and all representatives $S_{(i,j)}$ of π_2X.

This definition of $\mathrm{INT}_2(A)$ describes all possible changes in the order 2 intersections due to choices of Whitney disks for fixed choices of boundaries of order 1 Whitney disks (up to isotopy), so by Proposition 14 what remains to be done to confirm Conjecture 29 in this case is to show that $\lambda_2(\mathcal{W})\in\Lambda_2(4)/\mathrm{INT}_2(A)$ is independent of the choice of order 1 Whitney disk boundaries.

8.3.4 Computing the linear INT_2 relations

In this setting (where λ_0 vanishes on π_2X), the subgroup $\mathrm{INT}_2(A)$ can be computed as follows:
For a basis S^α for the rank r $\mathbb{Z}$-module π_2X (mod torsion), and integers a^α_{ij} defined by

$$\lambda_1(S^\alpha_{(i,j)},A_k,A_l)=a^\alpha_{ij}\ (i,j) -\!\!\prec {}^{k}_{l},$$

the $\mathrm{INT}_2(A)$ relations are described as the image of the linear map $\mathbb{Z}^{6r}\to\mathbb{Z}^2$ given in the basis $\{t_1,t_2\}=\{\,{}^2_1 \succ\!\!-\!\!\prec {}^3_4\,,\ {}^3_1 \succ\!\!-\!\!\prec {}^2_4\}$ by:

$$\begin{pmatrix}x_{12}\\x_{34}\\x_{13}\\x_{24}\\x_{14}\\x_{23}\end{pmatrix}\mapsto\begin{pmatrix}a_{12}&a_{34}&0&0&a_{14}&a_{23}\\0&0&a_{13}&a_{24}&a_{14}&a_{23}\end{pmatrix}\begin{pmatrix}x_{12}\\x_{34}\\x_{13}\\x_{24}\\x_{14}\\x_{23}\end{pmatrix}$$

where the multiplication of entries is vector inner product.
Examples in this setting realizing any coefficient matrix can be constructed by attaching 2-handles to B^4 along 0-framed links in S^3 with vanishing linking matrix.

8.3.5 QUADRATIC INT_2 RELATIONS

Now assume that the A_i represent elements in the radical of λ_0 on $\pi_2 X$, but that λ_0 may otherwise be non-trivial.

We continue to investigate changes in order 2 intersections due to choices of interiors of Whitney disks in $\mathcal{W}$ supported by A. Changing the interior of a Whitney disk $W_{(i,j)}$ to $W'_{(i,j)}$ along their common boundary again leads to a 2–sphere $S_{(i,j)} = W_{(i,j)} \cup W'_{(i,j)}$ whose order 1 intersections with A_k, A_l determine $\lambda_2(\mathcal{W}) - \lambda_2(\mathcal{W}') \in \Lambda_2(4)$, but the order 1 invariant $\lambda_1(S_{(i,j)}, A_k, A_l)$ that we want to use to measure this change may now itself have indeterminacies coming from non-trivial order 0 intersections between $S_{(i,j)}$ and any 2–spheres in X.

Specifically, $\lambda_1(S_{(i,j)}, A_k, A_l)$ takes values in $\Lambda_1((i,j),k,l)$ modulo $\mathrm{INT}_1(S_{(i,j)}, A_k, A_l)$, where the $\mathrm{INT}_1(S_{(i,j)}, A_k, A_l)$ relations are:

$$ {}^{k}_{(i,j)} >\!\!- \lambda_0(S_{((i,j),k)}, A_l) = 0 \qquad (1)$$

$$ {}^{(i,j)}_{l} >\!\!- \lambda_0(S_{(l,(i,j))}, A_k) = 0 \qquad (2)$$

$$ {}^{l}_{k} >\!\!- \lambda_0(S_{(k,l)}, S_{(i,j)}) = 0. \qquad (3)$$

Note that the first two relations are empty by our assumption that the A_i have vanishing order 0 intersections with all 2–spheres. The third relation corresponds to indeterminacies in $\lambda_1(S_{(i,j)}, A_k, A_l)$ due to the choice of interiors of order 1 Whitney disks pairing $A_k \cap A_l$, so computing with the order 1 Whitney disks $W_{(k,l)}$ in $\mathcal{W}$ determines a lift $\lambda_1^{\mathcal{W}}(S_{(i,j)}, A_k, A_l) \in \Lambda_1((i,j),k,l)$. Mapping $(i,j) -\!\!<^{k}_{l}$ to ${}^{j}_{i} >\!\!-\!\!<^{k}_{l}$, we have:

$$\lambda_2(\mathcal{W}) - \lambda_2(\mathcal{W}') = {}^{j}_{i} >\!\!- \lambda_1^{\mathcal{W}}(S_{(i,j)}, A_k, A_l) \in \Lambda_2(4).$$

Now consider changing both $W_{(i,j)}$ to $W'_{(i,j)}$, and some $W_{(k,l)}$ to $W'_{(k,l)}$ in as illustrated in Figure 19 (recall that i,j,k,l are distinct). The resulting change $\Delta^{\mathcal{W}}(S_{(i,j)}, S_{(k,l)}) := \lambda_2(\mathcal{W}) - \lambda_2(\mathcal{W}') \in \Lambda_2(4)$ can be expressed as $\Delta^{\mathcal{W}}(S_{(i,j)}, S_{(k,l)}) =$

$${}^{j}_{i} >\!\!- \lambda_1^{\mathcal{W}}(S_{(i,j)}, A_k, A_l) + {}^{j}_{i} >\!\!- \lambda_0(S_{(i,j)}, S_{(k,l)}) -\!\!<^{k}_{l} + \lambda_1^{\mathcal{W}}(A_i, A_j, S_{(k,l)}) -\!\!<^{k}_{l}$$

Here the 2–sphere $S_{(k,l)}$ determined by $W_{(k,l)}$ and $W'_{(k,l)}$ contributes the right-hand term $\lambda_1^{\mathcal{W}}(A_i, A_j, S_{(k,l)})$ just as discussed above for $S_{(i,j)}$, but now there is also a "cross-term" coming from order 0 intersections between $S_{(i,j)}$ and $S_{(k,l)}$. As in the previous paragraph $\lambda_1^{\mathcal{W}}(S_{(i,j)}, A_k, A_l)$ and $\lambda_1^{\mathcal{W}}(A_i, A_k, S_{(k,l)})$ are lifts of the corresponding order 1 invariants. The three homotopy invariants $\lambda_1(S_{(i,j)}, A_k, A_l)$, $\lambda_1(A_i, A_j, S_{(k,l)})$, and $\lambda_0(S_{(i,j)}, S_{(k,l)})$ are independent, so the given expression for $\Delta^{\mathcal{W}}(S_{(i,j)}, S_{(k,l)})$ only depends on $\mathcal{W}$ and the homotopy classes of $S_{(i,j)}$ and $S_{(k,l)}$.

Observe that, since the intersection invariants sum over contributions from the Whitney disks, this entire discussion applies word for word to changing *all* the

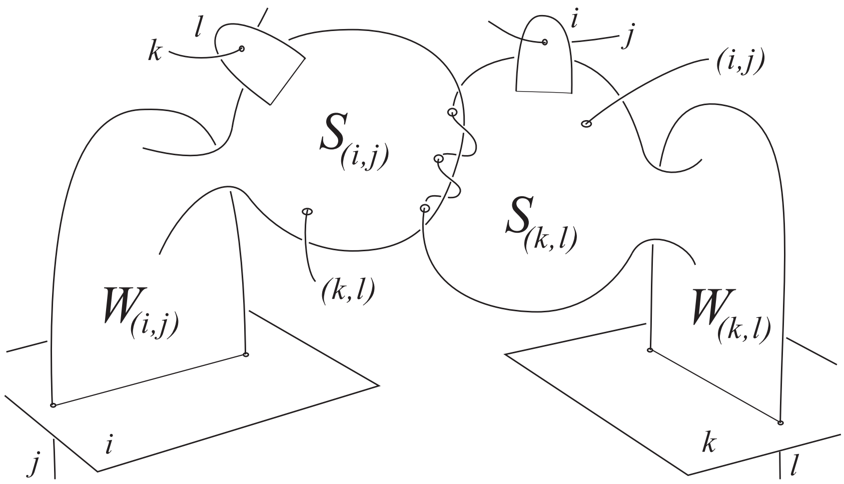

Figure 19: A schematic illustration of how order 0 intersections between $S_{(i,j)}$ and $S_{(k,l)}$ can contribute order 2 indeterminacies. (Only relevant intersections are shown.)

first order Whitney disks $W_{(i,j)}$ on A_i and A_j, and *all* the first order Whitney disks $W_{(k,l)}$ on A_k and A_l; with the 2–spheres $S_{(i,j)}$ and $S_{(k,l)}$ interpreted as sums (geometrically: unions) of the 2–spheres determined by each pair of Whitney disks.

Definition 34 *For a quadruple of 2–spheres* $A = A_1, A_2, A_3, A_4 \looparrowright X$ *with* X *simply connected and* A *in the radical of* λ_0 *on* $\pi_2 X$*, define the order* 2 *intersection relations* $\mathrm{INT}_2^{\mathcal{W}}(A) < \Lambda_2(4)$ *to be the sub*set

$$\mathrm{INT}_2^{\mathcal{W}} := \bigcup\{-\Delta^{\mathcal{W}}(S_{(1,2)}, S_{(3,4)}) - \Delta^{\mathcal{W}}(S_{(1,3)}, S_{(2,4)}) - \Delta^{\mathcal{W}}(S_{(1,4)}, S_{(2,3)})\} \subset \Lambda_2(4).$$

where (i,j)*,* (k,l) *vary over the pair-choices* $(1,2),(3,4)$*, and* $(1,3),(2,4)$ *and* $(1,4),(2,3)$*; and where* $S_{(i,j)}$ *and* $S_{(k,l)}$ *vary over all (homotopy classes of)* 2*–spheres in* X*.*

Note that, as defined, $\mathrm{INT}_2^{\mathcal{W}}$ is only a subset of $\Lambda_2(4)$.
Since the above construction can be carried out simultaneously for the three pair-choices, it follows that if $\lambda_2(\mathcal{W}) \in \mathrm{INT}_2^{\mathcal{W}}$, then it can be arranged that the A_i support $\mathcal{W}'$ with $\lambda_2(\mathcal{W}') = 0 \in \Lambda_2(4)$, so the A_i can be pulled apart.
Since $\mathrm{INT}_2^{\mathcal{W}}$ always contains the zero element of $\Lambda_2(4)$, the statement of Conjecture 29 makes sense, with $\mathrm{INT}_2^{\mathcal{W}}$ taking the place of $\mathrm{INT}_2(A)$. It would be desirable to have a formulation of the general INT_2 relations just in terms of A, rather than $\mathcal{W}$.

In the case that all order 0 intersections vanish on $\pi_2 X$, then $\mathrm{INT}_2^{\mathcal{W}}$ reduces to the subgroup $\mathrm{INT}_2(A) < \Lambda_2(4)$ of Definition 33.

8.3.6 Computing the quadratic INT_2 relations

In this setting, $\mathrm{INT}_2^{\mathcal{W}}$ can be computed as follows:
For a basis S^α for the rank r $\mathbb{Z}$-module $\pi_2 X$ (mod torsion), let $Q = q^{\alpha\beta} = \lambda_0(S^\alpha, S^\beta)$ denote the intersection matrix. For integers a^α_{ij} defined by

$$\lambda_1^{\mathcal{W}}(S^\alpha_{(i,j)}, A_k, A_l) = a^\alpha_{ij}\ (i,j) -\!\!<^k_l$$

we have the formula

$$\begin{aligned}\Delta^{\mathcal{W}}\left(\textstyle\sum_\alpha x^\alpha_{ij} S^\alpha_{(i,j)}, \sum_\beta x^\beta_{kl} S^\beta_{(k,l)}\right) &= \\ = \textstyle\sum_\alpha x^\alpha_{ij} a^\alpha_{ij} + \sum_\beta x^\beta_{kl} a^\beta_{kl} + \sum_\alpha \sum_\beta x^\alpha_{ij} x^\beta_{kl} q^{\alpha\beta} \\ = \qquad x_{ij}\cdot a_{ij} + x_{kl}\cdot a_{kl} + x_{ij} Q x^T_{kl}\end{aligned}$$

where the x_{uv} and a_{uv} are vectors in $\mathbb{Z}^r$.
Using the basis $\{t_1, t_2\} = \{\, {}^2_1 >\!\!-\!\!<^3_4\,,\ {}^3_1 >\!\!-\!\!<^2_4 \}$ for $\Lambda_2(4)$, computing $\mathrm{INT}_2^{\mathcal{W}}$ amounts to determining the image of the map $\mathbb{Z}^{6r} \to \mathbb{Z}^2$:

$$\begin{pmatrix} x_{12}\\ x_{34}\\ x_{13}\\ x_{24}\\ x_{14}\\ x_{23}\end{pmatrix} \mapsto \begin{pmatrix} a_{12} & a_{34} & 0 & 0 & a_{14} & a_{23}\\ 0 & 0 & a_{13} & a_{24} & a_{14} & a_{23}\end{pmatrix}\begin{pmatrix} x_{12}\\ x_{34}\\ x_{13}\\ x_{24}\\ x_{14}\\ x_{23}\end{pmatrix} + \begin{pmatrix} x_{12}Qx^T_{34} + x_{14}Qx^T_{23}\\ x_{13}Qx^T_{24} + x_{14}Qx^T_{23}\end{pmatrix}$$

where the multiplication of entries is vector inner product.
For example, in the easiest case where just a single 2–sphere generator S has non-trivial self-intersection number $\lambda_0(S, S') = q \neq 0 \in \mathbb{Z}$, we have that $\mathrm{INT}_2^{\mathcal{W}}$ is the image of the map $\mathbb{Z}^6 \to \mathbb{Z}^2$ given by:

$$\begin{pmatrix} x_{12}\\ x_{34}\\ x_{13}\\ x_{24}\\ x_{14}\\ x_{23}\end{pmatrix} \mapsto \begin{pmatrix} a_{12} & a_{34} & 0 & 0 & a_{14} & a_{23}\\ 0 & 0 & a_{13} & a_{24} & a_{14} & a_{23}\end{pmatrix}\begin{pmatrix} x_{12}\\ x_{34}\\ x_{13}\\ x_{24}\\ x_{14}\\ x_{23}\end{pmatrix} + q\begin{pmatrix} x_{12}x_{34} + x_{14}x_{23}\\ x_{13}x_{24} + x_{14}x_{23}\end{pmatrix}$$

Examples in this setting realizing any coefficient matrix can be constructed by attaching 2-handles to B^4 along links in S^3, and the following questions arise: Is this image always a subgroup of $\mathbb{Z} \oplus \mathbb{Z}$? Konyagin and Nathanson have shown in [21, Thm.3] that the image always projects to subgroups in each $\mathbb{Z}$-summand. And under what conditions will the image be all of $\mathbb{Z} \oplus \mathbb{Z}$? This would imply that the A_i can be pulled apart. What about analogous questions in the general case where the equations are coupled by the intersection matrix Q?

References

[1] D Bar-Natan, *Vassiliev homotopy string link invariants*, J. Knot Theory Ramifications 4 (1995) 13–32.

[2] A Casson, *Three Lectures on new infinite constructions in 4–dimensional manifolds*, Notes prepared by L Guillou, Prepublications Orsay 81T06 (1974).

[3] J Conant, R Schneiderman, P Teichner, *Jacobi identities in low-dimensional topology*, Compositio Mathematica 143 Part 3 (2007) 780–810.

[4] J Conant, R Schneiderman, P Teichner, *Higher-order intersections in low-dimensional topology*, Proc. Natl. Acad. Sci. USA vol. 108, no. 20, (2011) 8131–8138.

[5] J Conant, R Schneiderman, P Teichner, *Whitney tower concordance of classical links*, Geom. Topol. (2012) 16 (2012) 1419–1479.

[6] J Conant, R Schneiderman, P Teichner, *Milnor invariants and twisted Whitney towers*, J. Topology, Volume 7, Issue 1, (2013) 187–224.

[7] J Conant, R Schneiderman, P Teichner, *Tree homology and a conjecture of Levine*, Geom. Topol. 16 (2012) 555–600.

[8] J Conant, R Schneiderman, P Teichner, *Universal quadratic forms and Whitney tower intersection invariants*, Geom. Topol. Monographs 18 (2012) 35–60.

[9] W Dwyer, *Homology, Massey products and maps between groups* , J. Pure Appl. Alg. 6 (1975) 177–190.

[10] R Edwards, *The solution of the four-dimensional annulus conjecture (after Frank Quinn)*, Four-manifold Theory, Contemporary Mathematics AMS Vol 35 (1982) 211–264.

[11] M Freedman, *The topology of four-dimensional manifolds*, J. Diff. Geom. 17 (1982) 357–453.

[12] M Freedman, F Quinn, *The topology of 4–manifolds*, Princeton Math. Series 39 Princeton, NJ, (1990).

[13] C H Giffen *Link concordance implies link homotopy*, Math. Scand. 45 (1979) 243–254.

[14] D Goldsmith *Concordance implies homotopy for classical links in M^3*, Comment. Math. Helvetici. 54 (1979) 347–355.

[15] S Garoufalidis, J Levine, *Homology surgery and invariants of 3–manifolds*, Geom. Topol. Vol. 5 (2001) 75–108.

[16] N Habegger, X Lin, *The classification of links up to homotopy*, Journal of the AMS Vol. 3 (1990) 389-419.

[17] N Habegger, G Masbaum, *The Kontsevich integral and Milnor's invariants*, Topology 39 (2000) 1253-1289.

[18] R Kirby, *The Topology of 4-manifolds*, Lecture Notes in Math. 1374 Springer–Verlag, Berlin–Heidelberg–New York (1989).

[19] K Kobayashi, *On a homotopy version of* 4*–dimensional Whitney's lemma*, Math. Sem. Notes 5 (1977) 109–116.

[20] S Kojima, *Milnor's* $\overline{\mu}$*–invariants, Massey products and Whitney's trick in* 4*–dimensions*, Topol. Appl. 16 (1983) 43–60.

[21] S Konyagin, M Nathanson, *Sums of products of congruence classes and of arithmetic progressions*, Int. J. Number Theory Vol. 5 Issue 4 (2009) 625–634.

[22] W Magnus, A Karass, D Solitar, *Combinatoral group theory*, Dover Publications, Inc. (1976).

[23] J Milnor, *Link groups*, Annals of Math. 59 (1954) 177–195.

[24] J Milnor, *Isotopy of links*, Algebraic geometry and topology Princeton Univ. Press (1957).

[25] Y Matsumoto, *Secondary intersectional properties of* 4*–manifolds and Whitney's trick*, Proceedings of Symposia in Pure mathematics Vol. 32 Part 2 (1978) 99–107.

[26] F Quinn, *Ends of maps, III: dimensions 4 and 5*, J. Diff. Geom. 17 (1982) 503–521.

[27] R Schneiderman, *Whitney towers and Gropes in 4–manifolds*, Trans. Amer. Math. Soc. 358 (2006), 4251-4278.

[28] R Schneiderman, *Simple Whitney towers, half-gropes and the Arf invariant of a knot*, Pacific J. Math. Vol. 222 No. 1 Nov (2005).

[29] R Schneiderman, *Stable concordance of knots in* 3*–manifolds*, Alg. and Geom. Topology 10 (2010) 37–432.

[30] R Schneiderman, P Teichner, *Higher order intersection numbers of* 2*–spheres in 4–manifolds*, Alg. and Geom. Topology 1 (2001) 1–29.

[31] R Schneiderman, P Teichner, *Whitney towers and the Kontsevich integral*, Proceedings of a conference in honor of Andrew Casson, UT Austin 2003, Geometry and Topology Monograph Series Vol. 7 (2004), 101–134.

[32] P Teichner, *Symmetric surgery and boundary link maps*, Math. Ann. 312 (1998) 717–735.

[33] T Wall, *Surgery on Compact Manifolds*, London Math. Soc. Monographs 1, Academic Press 1970 or Second Edition, edited by A. Ranicki, Math. Surveys and Monographs 69 A.M.S.

[34] M Yamasaki, *Whitney's trick for three* 2*–dimensional homology classes of* 4*–manifolds*, Proc. Amer. Math. Soc. 75 (1979) 365–371.

Rob Schneiderman
Department of Mathematics
and Computer Science,
Lehman College,
City University of New York,
New York NY 10468 USA
robert.schneiderman@
lehman.cuny.edu

Peter Teichner
Max-Planck Institut
für Mathematik
Vivatsgasse 7, 53111 Bonn
Germany
teichner@mac.com