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Positivity of topological field theories in dimension at least 5

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Abstract

In this paper we answer a question of Mike Freedman, regarding the efficiency of *positive* topological field theories as invariants of smooth manifolds in dimensions greater than 4. We show that simply connected closed 5-manifolds can be distinguished by such invariants. Using Barden's classification, this follows from our result which says that homology groups and the vanishing of cohomology operations with finite coefficients are detected by positive topological field theories. Moreover, we prove that in the non-simply connected case, as well as in all dimensions $d > 5$, the universal manifold pairing (and in particular, d-dimensional positive topological field theories) are not sufficient to distinguish compact d-manifolds with boundary $S^3 \times S^n$, $n > 1$, and S^{4k-1}, $k > 1$. The latter case is equivalent to the same statement for *closed* $4k$-manifolds.

1. *Introduction*

In [**3**], the authors study the *universal manifold pairing* related to *positive* unitary topological quantum field theories, in short PTFT; see Definition 4. They show that closed smooth oriented manifolds of dimension at most 2 can be detected by PTFT's. Moreover, they prove that in dimension 4 two s-cobordant manifolds, with small 4-balls removed, represent the same vector in the universal vector space $\mathcal{M}_{S^3}$ of the 3-sphere (see below), implying that none of the exotic structures on 4-manifolds can be detected by PTFT. Using every available technique in dimension 3, Calegari, Freedman and Walker recently showed that 3-manifolds are still detected by the universal manifold pairing (leaving open the question of whether they are also detected by PTFT).

This raises the question about dimensions greater than 4, where we give both positive and negative results: We show in Theorem 3 that simply connected 5-manifolds can be detected (as in dimensions at most 3) by PTFT, but that the answer is — as in dimension 4 — negative for general d-manifolds for all $d \geqslant 5$. The precise statement is given in Theorem 2, but we note that there are simply connected examples in dimension at least 6, so that simply connected 5-manifolds are very exceptional in higher dimensions.

We begin with a short summary and notation. Unless stated otherwise, all manifolds are oriented, compact and smooth. For a closed $(d-1)$-manifold S, let $\mathcal{M}_S$ be the $\mathbb{C}$-vector space freely generated by diffeomorphism classes of d-manifolds M with $\partial M = S$. So elements of $\mathcal{M}_S$ are finite sums $x = \sum_i a_i M_i$ with $\partial M_i = S$ and unique coefficients $a_i \in \mathbb{C}$. More precisely, we consider two basis elements M and N of $\mathcal{M}_S$ as equal if and only if there is an orientation-preserving diffeomorphism, whose restriction to the boundary S is the identity map.

Remark 1. A pair (W, φ), where $\varphi : \partial W \longrightarrow S$ is a diffeomorphism, gives a canonical basis element of $\mathcal{M}_S$ as follows: pick a product collar for ∂W and glue a copy of $S \times I$ to W via φ.

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This gives a smooth manifold with boundary *equal to* S. In particular, the diffeomorphism group of S acts on $\mathcal{M}_S$ by this gluing operation.

If S is empty we denote $\mathcal{M}_S$ by $\mathcal{M}$, the $\mathbb{C}$-vector space generated by oriented diffeomorphism classes of closed oriented smooth manifolds of dimension d. There is a hermitian pairing, called the *universal manifold pairing* [3]:

$$\langle\, ,\, \rangle : \mathcal{M}_S \times \mathcal{M}_S \longrightarrow \mathcal{M},$$
$$\left\langle \sum_i a_i M_i, \sum_j b_j N_j \right\rangle := \sum_{i,j} a_i \bar{b}_j (M_i \bigcup_S -N_j).$$

The question raised in that paper is for which dimensions d this hermitian pairing is positive definite, in the sense that $\langle x, x\rangle = 0$ implies $x = 0$.

Our examples in dimension of at least 5 are rather simple counterexamples. We will find a $(d-1)$-manifold S and two d-manifolds W and T with boundary S such that W is not diffeomorphic to T rel. boundary, implying $W - T$ non-zero in $\mathcal{M}_S$, but

$$\langle W - T, W - T\rangle = (W \bigcup_S -W) - (W \bigcup_S -T) + (T \bigcup_S -T) - (T \bigcup_S -W) = 0.$$

Since a closed manifold of the form $W \bigcup_S -W$ always has an orientation-reversing diffeomorphism, the latter is equivalent to the existence of diffeomorphisms

$$W \bigcup_S -W \cong W \bigcup_S -T \cong T \bigcup_S -T.$$

We actually give two classes of examples, one that works only in dimension $d = 4k$ with $k > 1$, where $S = S^{4k-1}$ is a sphere, and another where $S = S^3 \times S^n$ for $n \geqslant 1$ is a product of spheres. The case of a sphere is most interesting because then the classification of compact d-manifolds with boundary S^{d-1} is equivalent to the classification of closed d-manifolds (by gluing in the standard disk D^k).

THEOREM 2. *Let $S = S^{4k-1}$, $k > 1$, or $S = S^3 \times S^n$, $n > 0$. Then there are d-manifolds W and T with boundary S such that*

$$W - T \neq 0, \quad \textit{but} \quad \langle W - T, W - T\rangle = 0.$$

Except for the case $d = 5$, the manifolds W and T may be chosen to be simply connected.

It is an open problem as to whether $S = S^{d-1}$ can be used if d is not divisible by 4. In the smallest unknown (simply connected) case $d = 6$, this is closely related to the algebraic classification of unimodular cubic forms, coming from the triple cup product on $H^2(M^6)$.

In the remaining case of simply connected 5-manifolds, we prove the following result. The new terminology will be explained after the theorem.

THEOREM 3. *For $S = S^4$, the universal manifold pairing is positive on simply connected 5-manifolds. Moreover, such manifolds (and therefore closed simply connected 5-manifolds) can be detected by 5-dimensional PTFT.*

We shall now give a quick review of the terminology. A d-dimensional topological quantum field theory is a symmetric monodial functor from a bordism category $\mathcal{B}_d$ to the category of finite-dimensional vector spaces. Using our convention that all manifolds are oriented, compact and smooth, the objects of $\mathcal{B}_d$ are closed $(d-1)$-manifolds and there are the following two types of morphisms between S_1 and S_2:

(1) orientation-preserving diffeomorphisms $S_1 \longrightarrow S_2$; and
(2) d-dimensional bordisms W with $\partial W = -S_1 \amalg S_2$.

More precisely, the adjective ‘topological’ forces one to use isotopy classes of diffeomorphisms and diffeomorphism classes (rel. boundary) of bordisms. Then there is a well-defined composition of morphisms, given by gluing bordisms (and using Remark 1 to turn a diffeomorphism into a bordism that can be glued on).

Note that, given a bordism W, every boundary component S inherits an orientation from W. One considers S as ‘incoming’ if this orientation disagrees with the one given on the object S in $\mathcal{B}_d$, and otherwise as ‘outgoing’. The gluing operation is compatible with these source and target maps for $\mathcal{B}_d$.

For example, the cylinder $S \times I$ is the identity morphism $\mathrm{id}_S : S \longrightarrow S$ in $\mathcal{B}_d$ (because gluing it to any bordism does not change its diffeomorphism type) but it can also be read as

$$C_S : S \amalg -S \longrightarrow \emptyset,$$

where the disjoint union $\amalg$ is the symmetric monoidal structure on $\mathcal{B}_d$ (whereas the tensor product is used for vector spaces). Then a TQFT E gives linear maps

$$E(C_S) : E(S) \otimes E(-S) \longrightarrow E(\emptyset) \cong \mathbb{C}$$

and it is not hard to see that these pairings $E(C_S)$ are non-degenerate. In fact, using the cylinder also as a bordism $\emptyset \longrightarrow -S \amalg S$, one could derive from the gluing axioms that $E(S)$ must be finite-dimensional without assuming it in the first place!

There are interesting involutions on both categories: on $\mathcal{B}_d$, the involution flips the orientation on both the bordisms and their boundaries: a morphism $W : S_1 \longrightarrow S_2$ leads to a new morphism $-W : -S_1 \longrightarrow -S_2$. For a complex vector space V, one can use the opposite complex structure, usually denoted by $\bar{V}$ (and the identity on linear maps) to define an involution. A *unitary* TQFT preserves these involutions (up to natural isomorphisms). This implies that one has two isomorphisms

$$E(S)^* \cong E(-S) \cong \bar{E}(S),$$

which together give a hermitian pairing on $E(S)$. It is not hard to conclude from the functoriality of E that this pairing must be symmetric, but there is no reason why it should be *positive definite*. However, many of the original examples of TQFT, usually defined with some physical intuition, do indeed lead to Hilbert spaces $E(S)$. Therefore, we provide the following definition.

DEFINITION 4. A PTFT (P for ‘positive’ or ‘physical’) is a unitary topological quantum field theory whose hermitian pairing is positive definite. We removed the Q from the notation because we feel that there is not enough ‘quantum’ theory going on in our discussion.

It is important to summarise the main properties of a PTFT E:

(1) for each object S in $\mathcal{B}_d$, there is a finite-dimensional Hilbert space $E(S)$; and
(2) for each morphism $W : S_1 \longrightarrow S_2$, the image $E(W')$ of $W' := -W : S_2 \longrightarrow S_1$ is the Hilbert space adjoint of $E(W)$.

To connect this with the universal manifold pairing discussed above, note that by evaluating a PTFT E on d-dimensional bordisms, one gets linear maps

$$\mathcal{M}_S \longrightarrow E(S) \quad \text{and in particular,} \quad \mathcal{M} = \mathcal{M}_\emptyset \longrightarrow E_\emptyset = \mathbb{C}.$$

Under these maps, the above universal manifold pairing becomes the (positive definite) inner product on $E(S)$. In particular, if $\langle x, x\rangle = 0$, then the linear combination $x = \sum_i a_i M_i$ of d-manifolds maps to the zero element $\sum_i a_i E(M_i)$ in $E(S)$ because this is a vector of length zero. As a consequence, x is undetectable by E and our Theorem 2 implies that the manifolds W and T cannot be distinguished by any d-dimensional PTFT.

The proof of Theorem 3 follows from Barden's classification [1] via the following general result on d-dimensional PTFT. The particular ones used in this theorem are higher-dimensional versions of Chern–Simons theories with finite Gauge group.

Theorem 5. *PTFT detect homology and cohomology groups additively, and they also determine whether stable cohomology operations with finite coefficients vanish. More precisely, if M is a closed d-manifold, then there are (finitely many) d-dimensional PTFT whose values on M determine the additive homology with finite coefficients and whether stable cohomology operations with finite coefficients vanish.*

It is a consequence of the nature of our counterexamples in Theorem 2 that PTFT cannot detect all cup products in cohomology.

2. *Proof of Theorem* 2

As explained above, we are looking for a closed oriented manifold S of dimension $(d-1)\geqslant 4$ and two d-manifolds W and T with boundary S such that:

(1) W and T are not diffeomorphic rel. boundary; and

(2) $W\bigcup_S -T$, $W\bigcup_S -W$ and $T\bigcup_S -T$ are orientation-preserving diffeomorphic.

We give two classes of examples. The first one, which also motivates the second construction, gives examples in dimension $4k$ for all $k>1$. We begin to describe these manifolds where the boundary $S=S^{4k-1}$ is a sphere. To construct W, consider the positive definite symmetric unimodular form E_8 (even of rank 8) and construct $M(E_8)$, the parallelisable $4k$-dimensional manifold plumbed according to the graph E_8. It is $(2k-1)$-connected with intersection form E_8. Since $k>1$, the boundary of $M(E_8)$ is a homotopy sphere. Moreover, the group of homotopy spheres is finite [4], and hence some boundary-connected sum

$$\sharp^{2r}M(E_8)$$

has boundary diffeomorphic to the standard sphere S^{4k-1}. Note that the boundary of $M(E_8)$ is a generator for the cyclic subgroup of boundary parallelisable homotopy spheres; the order of this subgroup is known [4].

We choose a diffeomorphism φ from this boundary to S^{4k-1}, and attach the cylinder over S^{4k-1} via this diffeomorphism to obtain a manifold W_φ with boundary *equal to* S^{4k-1}. To construct T, we do the same using E_{16} instead of E_8 to plumb the parallelisable manifold $M(E_{16})$. The clue is that $M(E_8)\sharp M(E_8)$ and $M(E_{16})$ have non-isomorphic intersection forms, but same rank and signature 16. Again, we want to consider an appropriate boundary-connected sum of copies of $M(E_{16})$, such that the boundary is S^{4k-1}. Since the homotopy spheres which are boundaries of parallelisable manifolds are determined by the signature of these manifolds [4], and the signature of $M(E_{16})$ is 16, we see that $\sharp^r M(E_{16})$ has boundary diffeomorphic to S^{4k-1}. Again, we choose a diffeomorphism ψ and attach via it the cylinder over S^{4k-1} to get the manifold T_ψ with boundary S^{4k-1}.

Observation. The manifolds W_φ and T_ψ are not diffeomorphic, since their intersection forms are not isomorphic.

We recall Wall's classification of $(2k-1)$-connected stably parallelisable closed manifolds [7]. If X and Y are two such manifolds with isomorphic intersection form, then there is a homotopy sphere Σ such that X is diffeomorphic to $Y\sharp\Sigma$. If X and Y form the boundary of a compact parallelisable manifold, then Σ is the boundary of a compact stably parallelisable manifold, and so Σ is then S^{4k-1} by the h-cobordism theorem. This implies that X and Y are diffeomorphic.

The double $W_\varphi \bigcup_{S^{4k-1}} -W_\varphi$ is a closed stably parallelisable manifold with indefinite intersection form and signature 0. Since indefinite even forms are classified by the signature and rank, the intersection form is isomorphic to that of $\sharp^{16r} S^{2r} \times S^{2r}$. The first manifold is the boundary of the stably parallelisable manifold $W_\varphi \times I$ and the second is the boundary of the stably parallelisable manifold given by the boundary-connected sum of $16r$ copies of $S^{2r} \times D^{2r+1}$. Thus $W_\varphi \bigcup_{S^{4k-1}} -W_\varphi$ is diffeomorphic to $\sharp^{16r} S^{2r} \times S^{2r}$. The same argument implies that $T_\psi \bigcup_{S^{4k-1}} -T_\psi$ is diffeomorphic to $\sharp^{16r} S^{2r} \times S^{2r}$, and so we conclude that

$$W_\varphi \bigcup_{S^{4k-1}} -W_\varphi \cong T_\psi \bigcup_{S^{4k-1}} -T_\psi \cong \sharp^{16r} S^{2r} \times S^{2r}.$$

We stress that these diffeomorphims exist for all choices of φ and ψ.

Applying the same argument to $W_\varphi \bigcup_{S^{4k-1}} -T_\psi$, we conclude that there is a homotopy sphere Σ such that $W_\varphi \bigcup_{S^{4k-1}} -T_\psi$ is diffeomorphic to $\sharp^{16r} S^{2r} \times S^{2r} \sharp \Sigma$. All homotopy spheres of dimension $4k > 4$ are of the form $D^{4k} \bigcup_\rho D^{4k}$ for some diffeomorphism ρ on S^{4k-1}. Thus composing φ with an appropriate diffeomorphism ρ, we conclude that

$$W_{\rho\varphi} \bigcup_{S^{4k-1}} -T_\psi \cong \sharp^{16r} S^{2r} \times S^{2r}.$$

Conclusion. All manifolds $W_{\rho\varphi} \bigcup_{S^{4k-1}} -W_{\rho\varphi}$, $T_\psi \bigcup_{S^{4k-1}} -T_\psi$, and $W_{\rho\varphi} \bigcup_{S^{4k-1}} -T_\psi$ are diffeomorphic to $\sharp^{16r} S^{2r} \times S^{2r}$.

Now we come to the second class of examples. The starting points are again manifolds plumbed according to $E_8 \perp E_8$ and E_{16}, but this time we plumb 4-manifolds. In other words, we start with a 0-handle D^4 and attach sixteen 2-handles $D^2 \times D^2$ according to the linking matrices of these quadratic forms. The boundaries are two (*a priori* distinct) homology 3-spheres. According to Freedman's main theorem [**2**, Theorem 1.4], given any homology 3-sphere Σ, there is a unique contractible *topological* 4-manifold with boundary Σ.

Attaching such a manifold to our homology spheres above, we obtain two closed topological 4-manifolds A and B with intersection forms $E_8 \perp E_8$ and E_{16}, respectively. We remove open discs from the smooth part of these manifolds and denote the result A° and B°. These are topological manifolds with smooth boundary equal to the standard 3-sphere S^3. Although smoothing theory [**5**] does not completely work in dimension 4, part of it works: the obstruction theory for a PL or linear structure on the stable topological tangent bundle. In our situation, this Kirby–Siebenmann obstruction agrees for both cases (PL and smooth) and lies in $\mathbb{Z}/2$. For A and B it is the signature mod 16, and so it vanishes. Similarly, we consider the obstruction for A° and B° (rel. boundary), which is again the signature mod 16, and so it vanishes. There is also an obstruction for uniqueness (rel. boundary) lying in $H^3(-;\mathbb{Z}/2)$. This group vanishes in our situation, and thus in both cases there is a unique reduction of the stable topological tangent bundle to a linear structure.

Now we return to our original problem and Theorem 2. The manifold S is now $S^3 \times S^n, n > 0$, and the manifolds with boundary S are as topological manifolds $A^\circ \times S^n$ and $B^\circ \times S^n$. Since we have a unique reduction of the stable topological tangent bundle to a linear bundle (rel. boundary) on A° and B°, we can take the product structure with the smooth structure on S^n to obtain extensions of the smooth structure of $S^3 \times S^n$ to $A^\circ \times S^n$ and $B^\circ \times S^n$ applying smoothing theory in dimension greater than 4; see [**5**]. We denote these two smooth manifolds with boundary $S^3 \times S^n$ by W and T.

We also consider $A^\circ \bigcup_{S^3} -B^\circ$. By Freedman [**2**], this topological *spin* manifold is up to homeomorphism classified by the intersection form, and so — as in our first class of examples — it is homeomorphic to $\sharp^{16} S^2 \times S^2$. Again, there is a unique linear structure on the stable topological tangent bundle of $A^\circ \bigcup_{S^3} -B^\circ$, and so we obtain a smooth structure on $(A^\circ \bigcup_{S^3} -B^\circ) \times S^n$. By construction, this is diffeomorphic on the one hand to $W \bigcup_{S^3 \times S^n} -T$, and on the other hand (using the agreement of the linear structure on the stable topological tangent bundle) to $(\sharp^{16} S^2 \times S^2) \times S^n$. Now we repeat the same argument with $A^\circ \bigcup_{S^3} -A^\circ$ and $B^\circ \bigcup_{S^3} -B^\circ$

and see that also $W \bigcup_{S^3 \times S^n} -W$ and $T \bigcup_{S^3 \times S^n} -T$ are diffeomorphic to $(\sharp^{16} S^2 \times S^2) \times S^n$, finishing the argument.

It is an exercise left to the reader to show that W and T are not diffeomorphic rel. boundary. In fact, not even their relative cohomology rings are isomorphic. It is clear that these manifolds are simply connected if $n > 1$, that is, in dimensions $d > 5$.

3. *Simply connected* 5*-manifolds*

In this section, we will show that PTFT can distinguish simply connected closed 5-manifolds. This implies Theorem 3 by the discussion in the introduction. We first recall Barden's classification of such manifolds from [**1**].

For any manifold M, we can define an invariant $i(M) \in \{0, 1, \dots, \infty\}$ as the largest integer r such that $w_2(M) \in H^2(M; \mathbb{Z}/2)$ can be lifted to a class in $H^2(M; \mathbb{Z}/2^r)$. By convention, $i(M) := 0$ if and only if $w_2(M) = 0$; that is, M is spin and $i(M) := \infty$ if and only if $w_2(M) \neq 0$ comes from an integral cohomology class.

Theorem 6 (Barden). *Two closed smooth simply connected* 5*-manifolds are diffeomorphic if and only if they have isomorphic second homology and equal* i*-invariants.*

We will show that the invariant $i(M)$ can be detected by the vanishing of certain cohomology operations. This is clear for $i(M) = 0$, which is equivalent to $w_2(M) = 0$, which by the Wu formula is equivalent to $0 = Sq^2 : H^3(M; \mathbb{Z}/2) \longrightarrow H^5(M; \mathbb{Z}/2)$. If $w_2(M) \neq 0$, the non-spin case, we apply the following lemma.

Lemma 7. *For a non-spin closed simply connected* 5*-manifold* M*,* $i(M) > r > 0$ *if and only if the following stable cohomology operation* α_r *vanishes:*

$$\alpha_r : H^2(M; \mathbb{Z}/2^r) \xrightarrow{\beta_r} H^3(M; \mathbb{Z}) \xrightarrow{\mathrm{red}_2} H^3(M; \mathbb{Z}/2) \xrightarrow{Sq^2} H^5(M; \mathbb{Z}/2),$$

where β_r *is the relevant Bockstein,* red_2 *is reduction modulo* 2 *and* Sq^2 *is the second* $\mathbb{Z}/2$*-Steenrod operation.*

Proof. Since M is simply connected, $H^2(M; \mathbb{Z}/2^r) \cong \mathrm{Hom}(H_2(M; \mathbb{Z}), \mathbb{Z}/2^r)$. Applying Poincaré duality and the Wu formula, we identify $w_2(M)$ with the map

$$H^3(M; \mathbb{Z}) \xrightarrow{\mathrm{red}_2} H^3(M; \mathbb{Z}/2) \xrightarrow{Sq^2} H^5(M; \mathbb{Z}/2).$$

Now we apply the Bockstein sequence

$$H^2(M; \mathbb{Z}/2^r) \xrightarrow{\beta_r} H^3(M; \mathbb{Z}) \xrightarrow{2^r} H^3(M; \mathbb{Z})$$

to see that $H^3(M; \mathbb{Z}) \xrightarrow{\mathrm{red}_2} H^3(M; \mathbb{Z}/2) \xrightarrow{Sq^2} H^5(M; \mathbb{Z}/2) \cong \mathbb{Z}/2$ (which corresponds to $w_2(M)$) can be lifted over $\mathbb{Z}/2^{r+1}$ if and only if the precomposition with β_r is zero. □

With this information, all one needs to know to show that PTFT classify simply connected closed 5-manifolds, is Theorem 5 from the introduction. That result follows from a construction going back to Kontsevich, Dijkgraaf–Witten, Segal and Freed–Quinn. We follow the exposition in [**6**] and first introduce the following central notion.

Definition 8. An FH-group is an *H-group* with *finite total homotopy*. A morphism between FH-groups is a product- and unit-preserving continuous map.

So an FH-group is a topological space X with a multiplication $X \times X \longrightarrow X$ that, up to homotopy, is associative and has a unit x_0 and an inverse map $X \longrightarrow X$. Moreover, the finite total homotopy condition means that for all $i \geqslant 0$, the homotopy groups $\pi_i X := \pi_i(X, x_0)$ are finite and non-zero only for finitely many i. Recall that for H-groups, the isomorphism type of each homotopy group is independent of the base point.

Definition 9. The *homotopy order* $\#h(X)$ of an FH-group is the 'alternating product'

$$\#h(X) := \prod_{i=0}^{\infty} |\pi_i X|^{(-1)^i}.$$

This is a rational number, well defined by our assumptions on X.

In the following, we will study spaces $\mathrm{Map}(Y, X)$ of continuous maps, with the compact-open topology, as well as the subspaces $\mathrm{Map}_0(Y, X)$ of maps that preserve a basepoint. Note that if X is an H-space, then so are both types of mapping spaces above, with all structures given pointwise (and units given by constant maps with value x_0). The following lemma is immediate.

Lemma 10. *Let X be an FH-group and C a finite CW-complex.*

(1) *If $f : Y \longrightarrow X$ is a morphism of H-groups, then its homotopy fibre F is an H-group. F is FH (that is, it has finite total homotopy) if and only if Y is FH.*

(2) *The exponential law gives a natural bijection*

$$\pi_n \mathrm{Map}(C, X) \cong [C, \mathrm{Map}_0(S^n, X)].$$

(3) $\mathrm{Map}(C, X)$ *and* $\mathrm{Map}_0(C, X)$ *are FH-groups.*

The main construction in [6] implies the following result.

Proposition 11. *Given an FH-group X and $d \in \mathbb{N}$, there is a d-dimensional PTFT, T_X, whose value on a closed d-manifold M is the homotopy order of the associated mapping space*

$$T_X(M) = \#h(\mathrm{Map}(M, X)).$$

Note that even though our PTFT are defined over the complex numbers, these homotopy orders happen to be rational numbers. Our only contribution to this story is the following simple observation.

Lemma 12. *Let $F \longrightarrow E \xrightarrow{p} B$ be a Serre fibration of FH-groups. Then the above PTFT satisfy the relation*

$$T_F(M) \cdot T_E(M)^{-1} \cdot T_B(M) = \left| \mathrm{coker} \left([M, E] \xrightarrow{[p]} [M, B] \right) \right|$$

In particular, the number on the right-hand side can be detected by PTFT.

Proof. Mapping M into the fibration gives a long exact sequence of groups

$$\cdots \longrightarrow [M, \Omega^n E] \longrightarrow [M, \Omega^n B] \longrightarrow [M, \Omega^{n-1} F] \longrightarrow \cdots \longrightarrow [M, F] \longrightarrow [M, E] \longrightarrow [M, B],$$

where these are free homotopy classes of maps and the identity elements of these sets are the constant maps with value the identity element in the relevant H-groups F, E or B. This identity element is also used when defining the based loop spaces $\Omega F = \Omega_{f_0} F$, and so on.

Part 2 of Lemma 10 above implies $[M, \Omega^n B] \cong \pi_n \mathrm{Map}(M, B)$, and similarly for F and E. Forming the alternating product of all the (finite) orders in the above exact sequence leads to the desired equation. We use here the fact that compact smooth manifolds are finite CW-complexes, so that all orders eventually become trivial. □

As an example, take any FH-group B and use the path-loop fibration with base B. Since both total space and fibre are FH-groups and the total space is contractible, this implies that $|[M, B]|$ is detected by PTFT. A special case would be $B = K(A, n)$, where $n \geqslant 0$ and A is any finite Abelian group. This shows that PTFT can read off the orders of all cohomology groups with finite coefficients A, and by the universal coefficient theorems (and the fact that compact manifolds are finite CW-complexes) the additive homology and cohomology groups of M can also be detected.

To finish the proof of Theorem 5, we need to check that PTFT can determine whether stable cohomology operations with finite coefficients vanish. Such an operation is given by a map of FH-groups

$$\alpha : K(A_1, n_1) \longrightarrow K(A_2, n_2)$$

and by Lemma 10 (1), the homotopy fibre F is again an FH-group. Lemma 12 above shows that PTFT can compute the order of the cokernel of

$$\alpha_* : H^{n_1}(M; A_1) \longrightarrow H^{n_2}(M; A_2),$$

as well as the order of both these cohomology groups. This implies that PTFT can detect whether α_* is trivial or not.

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