ELLIPTIC COHOMOLOGY VIA CONFORMAL FIELD THEORY
A CURRENT COURSE AT CAL, DO NOT DISTRIBUTE!

Contents

1. Introduction

Advances in physics have often played an important role in the development of mathematics. In many cases, these advances pre-dated the relevant mathematical theory. For example, the highly effective formalism of differential calculus as introduced by Newton and Leibniz around 1675 did not become a rigorous mathematical theory until Cauchy introduced the notion of limits in the 1820s. Another example is Dirac's delta "function" $\delta(x)$ introduced in the 1920s. It is characterized by the property that the integral over any function f multiplied by δ has the value $f(0)$. Clearly, such a function does not exist, and a precise understanding of δ was not developed until Laurent Schwartz introduced the theory of distibutions for which he was awarded the Fields Medal in 1950. Further examples include the modern physical theories of quantum electrodynamics, quantum chromodynamics, or string theory. None of these theories are, to this day, based on rigorous mathematical foundations. What is missing, from the path integral point of view, are the

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appropriate measures on the spaces of fields. Note that these are well-defined mathematical objects in quantum mechanics: The relevant measure was defined by Wiener, and the path integral formula for the quantum time evolution is due to Feynman and Kac. The main simplification that arises in quantum mechanics is that the spaces of fields have finite dimension only in this case.

In this class, we want to consider recent developments in algebraic topology related to the quantum theories mentioned above. We will need quite a bit of background material, and we will concentrate on the mathematical aspects.

Contents. The outline for the class is as follows:

(i) Classical field theories
   - mechanics, symplectic manifolds
   - Chern-Simons theory
   - σ-models

(ii) Quantization
   - linear quantization: Heisenberg (super) Lie algebras, Fock spaces
   - geometric quantization
   - path integrals

(iii) K-theory via Euclidian field theories
   - Feynman-Kac formula and Wiener measure
   - Dirac operators and index theorems
   - super manifolds and their moduli spaces

(iv) Elliptic cohomology via conformal field theories
   - von Neumann algebras and their bimodules
   - fusion of bimodules
   - elliptic objects

The main references are [Se1] and [ST]. References for specific topics will be given in the corresponding sections.

2. Classical mechanics

Consider a configuration space given by a smooth manifold M. We want to study the time evolution $\gamma : [0,t] \to M$ of the configuration, in the simplest case given by some particle moving in space. In order to speak of the kinetic energy E we endow M with a Riemannian metric g and let

$$E(\gamma(t) := g(\dot{\gamma}(t), \dot{\gamma}(t)) = ||\dot{\gamma}(t)||^2$$

where the mass of the particle is subsumed into the metric g. The potential energy is given by a function $U : M \to \mathbb{R}$, so it only depends on the location of the particle. We describe the three usual formalisms:

Newton's law. According to Newton, the time evolution is described by the equation

$$\ddot{\gamma} = -\nabla U.$$

Here ∇U is the gradient vector field corresponding to dU under the identification $TM \xrightarrow{\cong} T^*M$ given by the metric g, and $\ddot{\gamma} = \nabla_{\dot{\gamma}}\dot{\gamma}$. Newton's equation has a unique solution given any initial condition $(\gamma(0), \dot{\gamma}(0)) \in TM$.

Lagrange's principle of least action. The classical action is a functional $S : PM \to \mathbb{R}$, where $PM := \{\gamma : \mathbb{R} \to M \text{ smooth}\}$, given by the formula

$$S(\gamma) := \int_a^b L(\gamma(t), \dot{\gamma}(t))dt$$

Here the *Lagrangian* L is the difference between the kinetic and the potential energy. The critical points of the functional S are precisely the solutions to Newton's equations.

Hamilton's formalism. Define the *Hamiltonian* $H : T^*M \to \mathbb{R}$ to be the sum of the kinetic and the potential energy. Note that the cotangent bundle T^*M is a symplectic manifold in a canonical way: Using the tautological 1-form α on T^*M one obtains a symplectic form $-d\alpha$ on T^*M.

Let us describe Hamiltion's formalism more generally for any symplectic manifold (X, ω) equipped with a Hamiltonian H. In this situation, the smooth functions on X form a *Poisson algebra*, i.e. we have a Lie bracket $\{f, g\}$ compatible with the algebra structure on $C^\infty(X)$. It is defined as follows: Using the symplectic form we obtain an isomorphism $TX \xrightarrow{\omega} T^*X$, and hence can identify df with a vector field X_f. In other words, X_f is the vector field characterized by the relation

$$i_{X_f}(\omega) = X_f \lrcorner\, \omega = -df.$$

Then the Poisson bracket of two functions f, g is defined by

$$\{f, g\} = X_f(g) = \omega(X_f, X_g) = -X_g(f) = -\{g, f\}$$

In order to describe the time evolution of the system we consider the time evolution of all *observables* (a.k.a. functions) $f : X \to \mathbb{R}$. It is given by the equation

$$\frac{df}{dt}(x) = \{H, f\}.$$

The relation between Lagrange's and Hamilton's formalism. Let M be just a smooth manifold ('configuration space') and $L : TM \to \mathbb{R}$ a smooth map ('Lagrangian').

Theorem 1. *Given M and L there is*

- *a unique function $E_L : TM \to \mathbb{R}$*
- *a unique 1-form α_L on TM*

such that in local coordinates $(q_i, \dot{q}_i)$ *we have*

$$E_L = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \text{ and } \alpha_L = \frac{\partial L}{\partial \dot{q}_i} dq_i$$

Here the $\dot{q}_i$ *are the canonical coordinates on* TM *determined by the coordinates* q_i *on* M*.*

Example 2. If L is the classical Lagrange funtion coming from a metric g and a potential U then

$$L(q_i, \dot{q}_i) = \sum_{i,j} g_{ij}(q)\dot{q}_i\dot{q}_j - U(q)$$

and E_L is the classical Hamiltonian, i.e. the total energy

$$E_L(q_i, \dot{q}_i) = \sum_{i,j} g_{ij}(q)\dot{q}_i\dot{q}_j + U(q).$$

Remark 3. A path $\gamma : \mathbb{R} \to M$ is an extremal point of the functional S defined above if and only if it satisfies "Newton's law"

$$\ddot{\gamma}(t) \lrcorner\, \omega_L = -dE_L(\dot{\gamma}(t))$$

an equality of 1-forms on TM along $\dot{\gamma}$, where $\omega_L := d\alpha_L$.

Definition 4. L *is* non-degenerate *if the matrix*

$$\left(\frac{\partial L}{\partial \dot{q}_i \partial \dot{q}_j}(q, \dot{q}) \right)_{i,j}$$

is invertible for all $(q, \dot{q})$*.*

Lemma 5. *L is non-degenerate if and only if ω_L is a symplectic form. Another equivalent condition is that*

$$(q_1, \ldots, q_n, \frac{\partial L}{\partial \dot{q}_1}, \ldots, \frac{\partial L}{\partial \dot{q}_n})$$

defines a local coordinate system on TM*.*

The *Legendre transform* is the isomorphism $TM \to T^*M$ that is in local coordinates given by the correspondence

$$(q_i, \frac{\partial L}{\partial \dot{q}_i}) \longleftrightarrow (q_i, p_i),$$

where the p_i are the canonical coordinates of the cotangent bundle determined by the q_i. Under this identification α_L is transformed into the tautological 1-form α on T^*M. The function E_L transforms into a function H on T^*M, the 'Hamiltonian'.

Noether's theorem. Let us return to the general situation of a manifold X equipped with an *almost* symplectic structure ω (i.e. the 2-form ω is non-degenerate, but not necessarily closed). As in the symplectic case we obtain a Poisson bracket on $C^\infty(X)$ and for $f \in C^\infty(X)$ an associated vector field X_f on X. Given a 'Hamiltonian' f the classical solutions are given by the flow lines of X_f. Note that by skew-symmetry of the Poisson bracket we have $X_f(f) = 0$, i.e. X_f flows *along* level sets of f. Note that this is quite different (in fact *orthogonal*) to the gradient flow known from Riemannian geometry!

The relation between symmetries and preserved quantities in classical mechanics is given by the following theorem which in our framework is a tautology:

Theorem 6 (Noether)**.** *Let H be a Hamiltion for (X, ω), and let $f \in C^\infty(X)$. Then the condition $\{f, H\} = 0$ is satisfied if and only if f is a preserved quantity (i.e. $X_H(f) = 0$). This is also equivalent to X_f being a symmetry of the system (i.e. that $X_f(H) = 0$).*

Furthermore, the Lie derivative satisfies $\mathcal{L}_{X_f}(\omega) = i_{X_f} d\omega$ by Cartan's formula. This is equal to zero for all f if and only if ω is closed.

Integrability conditions. Let (X, ω) be an almost symplectic manifold. Then the following conditions are equivalent:

- (X, ω) is symplectic, i.e. $d\omega = 0$.
- The Poisson bracket satisfies the Jacobi identity.
- $[X_f, X_g] = X_{\{f,g\}}$ for all f and g. In particular, $\{X_f | f \in C^\infty(X)\} \subset \mathrm{Vect}(X)$ is closed under the Lie bracket.
- ω is integrable, i.e. there are charts in which ω is locally the standard form
$$d(\sum_i p_i dq_i) = \sum_i dp_i \wedge dq_i.$$

The last item is the only hard part of the theorem and it is known as Darboux's theorem.

Let us compare the situation with the case of almost complex manifolds (X, J). Here J is a selfmap of TX such that $J^2 = -\mathrm{id}$. Then the following conditions are equivalent:

- The Nijenhuis tensor of J vanishes.
- $L_J \subset \mathrm{Vect}(X) \otimes \mathbb{C}$ is closed under the Lie bracket. Here L_J is the $(+i)$-eigenspace of $J \otimes \mathrm{id}$.
- J is integrable, i.e. there are J-holomorphic charts making X into a complex manifold. This means that J is locally the standard complex structure on $\mathbb{C}^n$.

The relation between almost symplectic and almost complex manifiolds is given by the following proposition where a hermitian structure on X is defined to be an almost complex structure with a hermitian product on each tangent space. This is equivalent to a compatible almost symplectic structure as explained below.

Proposition 7. *The following conditions are equivalent:*

- *X has an almost symplectic structure.*
- *X has a almost complex structure.*
- *X has a hermitian structure (i.e. X is almost Kähler).*

Why is this true? Equipping X^{2n} with an almost symplectic or almost complex structure corresponds to reducing the structure group of TX from $GL_{2n}(\mathbb{R})$ to $Sp_{2n}(\mathbb{R})$ or $GL_n(\mathbb{C})$, resp. The point is that we can always equip X with a Riemannian metric, reducing its structure group to O_{2n}. However, the interections of the symplectic and the complex general linear group with O_{2n} are equal, namely to the unitary group,

$$Sp_{2n}(\mathbb{R}) \cap O_{2n} = GL_n(\mathbb{C}) \cap O_{2n} = U_n \subset GL_{2n}(\mathbb{R}).$$

If h is a hermitian inner product then its real part is a positive definite inner product g and its imaginary part is a nondegenerate skew form ω that are related by

$$g(v_1, v_2) = \omega(v_1, Jv_2)$$

Remark 8. It is not possible to omit the word 'almost' in the proposition: The corresponding integrability conditions are distinct in all three cases.

Examples 9. We want to explain how to obtain symplectic manifolds as coadjoint orbits (in fact, all these examples are Kähler). Let G be a Lie group and $G \to GL(\mathfrak{g}^*)$ the action dual to the adjoint action of G on $\mathfrak{g}$ ('coadjoint action').

Lemma 10. *For each $\xi \in \mathfrak{g}^*$ there is a unique G-equivariant symplectic structure on the* coadjoint orbit *$\mathcal{O}_\xi := G \cdot \xi \subset \mathfrak{g}^*$ determined by*

$$\phi^*(\omega_\xi) = d\alpha_\xi,$$

where $\phi : G \twoheadrightarrow G/G_\xi \cong \mathcal{O}_\xi$ denotes the quotient map, and α_ξ is the left-invariant 1-form on G determined by $\alpha_\xi(e) = \xi$. Here $e \in G$ the identity element and G_ξ is the stabilizer of ξ.

Proof. Let us check that the form ω_ξ is indeed well-defined and non-degenerate at the identity element. For $v_1, v_2 \in \mathfrak{g}$ we have

$$(d\alpha_\xi)_e(v_1, v_2) = \alpha([v_1, v_2]) = (v_1(\alpha))(v_2),$$

where $v_1(\alpha)$ is the coadjoint action of $v_1 \in \mathfrak{g}$ on $\alpha \in \mathfrak{g}^*$. Hence for fixed v_1 we have

$$(d\alpha_\xi)_e(v_1, v_2) = 0 \text{ for all } v_2 \in \mathfrak{g} \iff v_1(\alpha) = 0 \iff v_1 \in \mathfrak{g}_\xi,$$

where $\mathfrak{g}_\xi$ is the Lie algebra of the stabilizer G_ξ. This shows that ω_ξ is well defined and non-degenerate on the quotient $\mathfrak{g}/\mathfrak{g}_\xi$ and hence it can be extended to a G-equivariant symplectic form on the coadjoint orbit $\mathcal{O}_\xi = G/G_\xi$. □

As a special case, consider $G = U_n$. Using the pairing

$$\mathfrak{u}_n \times \mathfrak{u}_n \to \mathbb{R},\ (x, y) \mapsto \mathrm{Re}(\mathrm{trace}\, xy)$$

we can identify $\mathfrak{u}_n$ with its dual. After conjugation we can assume that our element $\xi \in \mathfrak{u}_n$ is diagonal, i.e.

$$\xi = \begin{pmatrix} ia_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & ia_n \end{pmatrix}$$

with $a_k \in \mathbb{R}$. Clearly, the orbit depends on the stabilizer of the action of U_n at ξ. For example, if all a_k are distinct, then the stabilizer is the maximal torus $T^n = U_1 \times ... \times U_1$. Hence in this case we obtain flag manifolds. In the general case we have diagonal entries a_r with mulitplicity n_r. Hence the corresponding coadjoint orbit is then the quotient of U_n by the stabilizer $U_{n_1} \times ... \times U_{n_r}$. For example, if exactly $n-1$ of the a_k are equal we get $U_n/U_1 \times U_{n-1}$, i.e. complex projective space of dimension $n-1$.

Example 11. Let us consider the case $G = SU_2$ more in detail. Again, we have $\mathfrak{su}_2 \cong \mathfrak{su}_2^*$, so we can think of $\mathfrak{su}_2^*$ as the skew-hermitian matrices with vanishing trace. Up to conjugation, a general element is of the form

$$\xi = \begin{pmatrix} ia_1 & 0 \\ 0 & ia_2 \end{pmatrix},\ \text{where } a_i \in \mathbb{R},\ a_1 + a_2 = 0.$$

If $a_1 \neq 0$ the stabilizer is a circle U_1 so that

$$\mathcal{O}_\xi = \mathbb{CP}^1(a_1) = SU_2/U_1,$$

where the latter notation expresses the dependence of the symplectic structure on a_1. Since the cases $\pm a_1$ are symmetric, we can assume $a_1 > 0$. Since the form ω_ξ is SU_2-equivariant it is determined by its restriction to the tangent space at one point. Hence the only parameter is a scaling factor $a_1 > 0$, and this factor classifies the symplectic manifold $\mathbb{CP}^1(a_1)$. Note that a_1 is the volume of $\mathbb{CP}^1(a_1)$ and that the limit of $a_1 \mapsto 0$ is indeed giving a single point ($= SU_2/SU_2$).

Lemma 12. *The coadjoint G-orbits are in* 1-1*-correspondence with $\mathfrak{g}^*/G$. Moreover,*

$$\mathfrak{g}^*/G = \mathrm{Hom}_{\mathbf{Rings}}(S(\mathfrak{g})^G, \mathbb{R}),$$

where $S(\mathfrak{g})^G$ denotes the G-invariant polynomial functions on $\mathfrak{g}^*$.

For example, in the case of SU_2, $S(\mathfrak{g})^G$ are all even degree polynonials in one variable. The reason why part this lemma is interesting for us is the following difficult

Theorem 13 (Harish-Chandra, Kirilov, Duflo). *There is a bijection between $S(\mathfrak{g})^G$ and the center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$.*

Remark 14. Given an irreducible unitary representation of $\mathfrak{g}$ we get an irreducible *-representation of $U(\mathfrak{g})$ which gives a 'character' in $\mathrm{Hom}_{\mathbf{Rings}}(Z(\mathfrak{g}), \mathbb{C})$. Hence we obtain an embedding of irreducible representations of $\mathfrak{g}$ into the coadjoint G-orbits. Once we get to the concept of geometric quantization, we can say more about the question what it means for the coadjoint orbit to come from an *integrable* representation, namely one that comes from a representation of the group G. These will be the *integral* coadjoint orbits.

Theorem 15. *If X is a symplectic manifold with a transitive Poisson action by a connected Lie group G, then X is a covering of a coadjoint orbit.*

Before we explain the proof, we need to define the notion of a Poisson action. For every symplectic manifold (X, ω) there is an exact sequence

$$0 \longrightarrow H^0_{dR}(X) \longrightarrow C^\infty(X) \longrightarrow \mathfrak{sp}(X, \omega) \longrightarrow H^1_{dR}(X) \longrightarrow 0$$

Here $\mathfrak{sp}(X, \omega)$ is the Lie algebra of symplectic vector fields ξ on X, i.e. ξ's that satisfy $\mathcal{L}_\xi(\omega) = 0$. The arrow from $C^\infty(X)$ to $\mathfrak{sp}(X, \omega)$ is given by associating to f the corresponding Hamiltionian vector field X_f. The map to $H^1_{dR}(X)$ is given by mapping ξ to $\xi \lrcorner \omega$. Exactness at $\mathfrak{sp}(X, \omega)$ follows easily from Cartan's formula. If G acts on (X, w) we can differentiate to obtain a Lie algebra homomorphism

$$\rho : \mathfrak{g} \to \mathfrak{sp}(X, \omega)$$

Definition 16. *We call the action of G*

- Hamiltonian *if ρ maps into the the sub Lie algebra of hamiltonian vector fields on X.*
- Poisson *if in addition ρ lifts to a Lie algebra map to $C^\infty(X)$. The lift is part of the datum of a Poisson action.*

Remark 17. A symplectic G-action is Hamiltonian if any one of the following condition holds:

- $H^1_{dR}(X) = 0$,
- $H^1(\mathfrak{g}) = 0 \iff \mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$,
- $\omega = d\alpha$ and the G-action preserves α. This is true, since in this case we have for $a \in \mathfrak{g}$ that $0 = \mathcal{L}_a(\alpha) = di_a\alpha + i_a d\alpha = i_a d\alpha$. Hence, $i_a(\omega) = i_a(d\alpha) = 0 \in H^1_{dR}$.

Remark 18. A Hamiltonian G-action is Poisson if any one of the following condition holds:

- X is compact,

- $H^2(\mathfrak{g}) = 0$, i.e. all central extensions are trivial,
- $\omega = d\alpha$ and the G-action preserves α. In this case, we can define

$$\mathfrak{g} \to C^\infty(X) \text{ by } a \mapsto \alpha(\rho(a)).$$

Examples 19. (i) Let $G = S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$ act on itself by translation. Here we consider the symplectic form $dx \wedge dy$ on $S^1 \times S^1$. This action is *not* Hamiltonian: For example, the generator $\partial_x \in \mathfrak{g}$ of the action maps to $0 \neq [dy] \in H^1_{dR}$.

(ii) Now consider the translation action of $\mathbb{R}^2$ on itself, where we again look at the form $\omega = dx \wedge dy$. Since $\mathbb{R}^2$ is contracible, this action is clearly Hamiltonian. However, it is not Poisson: Possible lifts for ∂_x and ∂_y are the functions $y + c_1$ and $x + c_2$. Hence we cannot lift $\mathfrak{g} \to \mathfrak{sp}(\mathbb{R}^2, \omega)$ to a Lie homomorphism $\mathfrak{g} \to C^\infty(\mathbb{R}^2)$, since $[\partial_x, \partial_y] = 0$, but $\{x + c_2, y + c_1\} = 1 \neq 0$.

(iii) In order to make the last example work, we introduce the *Heisenberg group* Heis: It is the central extension of $\mathbb{R}^2$ whose Lie algebra has exactly the commutator relations we need in example (ii). More explicitly, Heis is the subgroup of $GL_3(\mathbb{R})$ of upper triangular matrices with all diagonal entries equal to 1. It fits into an exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \text{Heis} \longrightarrow \mathbb{R}^2 \longrightarrow 0,$$

and the first $\mathbb{R}$ in the sequence is the center of Heis. Correspondingly, for the Lie algebra $\mathfrak{heis}$ we have

$$0 \longrightarrow \mathbb{R} =< z > \longrightarrow \mathfrak{heis} \longrightarrow \mathbb{R}^2 =< x, y > \longrightarrow 0,$$

and the generators x, y, z satisfy the commutator relations

$$[x, z] = 0, \ [y, z] = 0, \text{ and } [x, y] = z.$$

From this description it is clear that Heis acts on $\mathbb{R}^2$ with a Poisson lift.

Definition 20. *The* moment map *of a Poisson action is the map*

$$\mu : X \to \mathfrak{g}^*, \ x \mapsto (a \mapsto \lambda(a)(x)),$$

where $\lambda : \mathfrak{g} \to C^\infty(X)$ *is the chosen lift of the Poisson action of* $\mathfrak{g}$. λ *is sometimes called the* comoment map.

Remark 21. If G is connected, it follows from the fact that λ is a Lie homomorphism that μ is G-equivariant.

Proof of Theorem 15. Since G acts transitively on X and since μ is G-equivariant, we see that the image of μ is exactly a single G-orbit $\mathcal{O}_\xi \subset \mathfrak{g}^*$. This implies that the dimension of X is bigger or equal to the dimension of $\mathcal{O}_\xi$. Moreover, the momemt map preserves the symplectic structures and therefore is must be injective on each tangent space. Therefore, it is a submersion with 0-dimensional fibres, and hence a covering. ☐

Exercise 1. Sept. 17

(i) Show that the G-action on coadjoint orbits is a Poisson action.

(ii) Verify that the formula in Remark 18 for the case $\omega = d\alpha$ indeed defines a Poisson action.

(iii) Show that a coadjoint orbit $\mathcal{O}_\xi = G/G_\xi$ is integral, i.e. the corresponding symplectic form comes from integral cohomology, if there is a character $G_\xi \to S^1$ whose derivative is the restriction of ξ to the Lie algebra $\mathfrak{g}_\xi$ of G_ξ.

Symplectic reduction. We end this section on classical mechanics by mentioning a beautiful construction of forming quotients in the symplectic category. Note that naive quotients of symplectic manifolds cannot in general stay symplectic: the dimension might actually become odd. So one needs to find a formalism in which the group is "divided out twice". Consider a Poisson action of G on (X, ω) with moment map by μ. Let $\alpha \in \mathfrak{g}^*$ such that

$$\pi : \mu^{-1}(\alpha) \twoheadrightarrow \mu^{-1}(\alpha)/G_\alpha =: X /\!\!/ G$$

is a submersion of manifolds. Then

Theorem 22. *$X /\!\!/ G$ carries a canonical symplectic structure ω_α that is determined by*

$$\pi^*(\omega_\alpha) = \omega|_{\mu^{-1}(\alpha)}.$$

Examples 23. (i) Let $X = \mathbb{C}^n$ with the canonical symplectic structure. Then the action of $G = S^1$ by scalar multiplication is Poisson: Differentiating the action one finds that the Lie algebra generator $\partial_x \in \mathfrak{s}^1$ maps to the vector field $z \mapsto iz$. This is the Hamiltonian vector field coming from the function

$$f(z) = \frac{1}{2} \sum_j z_j \bar{z}_j.$$

Hence the preimage of an element $\alpha \in \mathbb{R} = (\mathfrak{s}^1)^*$ is the sphere with radius $\sqrt{2\alpha}$ (empty when $\alpha < 0$ and equal to a point for $\alpha = 0$). Hence we obtain symplectic structures on complex projective space,

$$\mathbb{C}^n /\!\!/ S^1 = \mathbb{CP}^{n-1}_\alpha = S^{2n-1}_{\sqrt{2\alpha}}/S^1.$$

(ii) Let G be a Lie group and consider $X = T^*G$ with its canonical symplectic structure and G-action. This action is Poisson and for $\alpha \in \mathfrak{g}^*$ the preimage of α under the moment map $\mu^{-1}(\alpha) \subset T^*G \cong G \times \mathfrak{g}^*$ is the graph of the G-invariant 1-form corresponding to α. Hence

$$T^*G /\!\!/ G = \mu^{-1}(\alpha)/G_\alpha \cong G/G_\alpha = \mathcal{O}_\alpha \subset \mathfrak{g}^*$$

is the coadjoint orbit of α.

3. Quantization

In this section we will make the step from classical to quantum mechanics. It turns out that quite a bit of representation theory comes in, namely of the Heisenberg group. This is a noncompact Liegroup where the interesting representations are all infinite dimensional. From a mathematical point of view, the representation theory of compact groups is slightly easier because the irreducible representations are finite dimensional and the relevant symplectic manifolds are compact.

From the last section we know that there is an injective map that associates to each irreducible unitary representation of a compact Lie group G a coadjoint G-orbit. In this section we want to find out which orbits are integral in the sense that they come from a representation of G. On such orbits we will define a map ('quantization') back to irreducible representations of G which is the content of Borel-Weil theory. The natural generalization of this to nilpotent groups, like the Heisenberg group, is Kirillov theory and will be more relevant for quantum mechanics.

Definition 24. *A symplectic form ω on a manifold X is* integral *if the class $[\omega] \in H^2_{dR}(X) \cong H^2(X;\mathbb{R})$ is integral, i.e. lies in the image of $H^2(X;\mathbb{Z})$. This is the case if and only if for every closed orientable surface Σ and smooth map $f : \Sigma \to X$ we have the condition of* integral periods*:*

$$\int_\Sigma f^*(\omega) \in \mathbb{Z}.$$

Theorem 25. *Let (X, ω) be a symplectic manifold.*

(i) *The form ω is integral if and only if it is the curvature form of a unitary connection on a line bundle $L \to X$.*

(ii) *The line bundle L (with connection) is uniquely determined up to a flat line bundle over X, i.e. if L' is another such bundle, then $L' \cong L \otimes L_{flat}$.*

(iii) *Flat line bundles over X are (up to isomorphism preserving the connection) classified by their holomony, i.e. by an element in*

$$\mathrm{Hom}(\pi_1(X), S^1) \cong \mathrm{Hom}(H_1(X), S^1) = H^1(X;S^1).$$

A line bundle as above is sometimes called a *prequantum* line bundle. We will always consider it as a bundle together with a unitary connection. We obtain the following diagram of exact sequences of Lie algebras respectively Lie groups, connected by the exponential map.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(X;\mathbb{R}) & \longrightarrow & C^\infty(X) & \longrightarrow & \mathfrak{sp}(X,\omega) & \longrightarrow & H^1(X;\mathbb{R}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \exp} & & \downarrow & & \\
0 & \longrightarrow & H^0(X;S^1) & \xrightarrow{\text{central}} & \mathrm{Aut}(L) & \longrightarrow & Sp(X,\omega) & \longrightarrow & H^1(X;S^1) & \longrightarrow & 0,
\end{array}$$

where L is a prequantum line bundle for ω. The group $\mathrm{Aut}(L)$ consists of all automorphisms of L (that preserve the connection) but which are allowed to act by a nontrivial diffeomorphism of the base X. Since the curvature ω of the connection must be preserved by such a diffeomorphism, we obtain the middle map in the second row. The first map is the inclusion of those automorphisms which are the identity on the base, and hence can only act by a scalar that is constant on connected components of X. Finally, the last map takes a diffeomorphism f to the (holonomy of the) flat line bundle $\bar{L} \otimes f^*(L)$.

Example 26. If it happens that there is a *symplectic potential* θ, i.e. a 1-form with $d\theta = \omega$ then on can choose the line bundle L to be trivial and a connection that is determined by the covariant derivative (acting on complex valued functions on X)

$$\nabla_X = X + i\theta(X)$$

Prequantization. Before we get to canonical and geometric quantization we present a nice mathematical formalism that associates to every symplectic manifold (X, ω) a complex Hilbert space H and a Lie algebra homomorphism

$$O : C^\infty(X) \longrightarrow \text{(essentially) skew-adjoint operators on } H$$

such that $O(1) = i\mathbb{1}_H$.

Remark 27. Physicists usually consider *self-adjoint* operators associated to *real valued* functions ('observables') and therefore replace the Lie homomorphism condition by

$$O(f_1)O(f_2) - O(f_2)O(f_1) = -i\hbar O(\{f_1, f_2\}).$$

Note that self-adjoint operators don't form a Lie algebra, that's why the constant i comes in. This is not an important issue because multiplication by i induces a bijection between skew-adjoint and self-adjoint operators. We shall also ignore Planck's constant $\hbar$ in our discussion because it can be subsumed as a factor into the symplectic form. Note however, that the condition $O(1) = i\mathbb{1}_H$ is essential from the physical point of view since it implements Heisenberg's uncertainty principle into the formalism.

Theorem 28. *If (X, ω) is integral, then a natural prequantization (H, O) exists.*

Proof. The first idea that comes to mind is to set $H = L^2(X; \mathbb{C})$, $O(f) = X_f$. Then O is a Lie homomorphism, but does not satisfy the required normalization condition ('Heisenberg's uncertainty principle'). In a second attempt we could introduce a correction term that fixes the normalization: If one takes $O(f) = X_f + im_f$, where m_f is the multiplication operator defined by f, we have $O(1) = i\mathbb{1}_H$, but, unfortunately, this is not a Lie homomorphism any more. Computing the commutators in this case leads to the definition

$$O(f) = X_f + i(m_{\theta(X_f)} + m_f)$$

that satisfies both requirements under the additional assumption that $\omega = d\theta$ for some 1-form θ. Recall from Example 26 that the first two terms are nothing by the covariant derivate ∇_{X_f} acting on sections of the trivial bundle. It is thus natural to define

$$O(f) := \nabla_{X_f} + im_f$$

acting on sections of a prequantum line bundle $\pi : L \to X$. This clearly satisfies $O(1) = i\mathbb{1}_H$ and since commutators can be calculated locally, it is also a Lie homeomorphism.

To explain this a little better, recall that the pullback of ω to L is equal to dA, where A is the curvature of the connection on L. Hence in this case $\pi^*\omega$ is exact and we can apply the above construction to the circle bundle $(S(L), \pi^*\omega)$. More precisely, we don't consider all functions on $S(L)$ but only those that are S^1-equivariant with respect to the to natural S^1 actions on range and domain. These functions can be identified with sections of L and hence we define our Hilbert space to be

$$H := \Gamma_{L^2}(L \xrightarrow{\pi} X)$$

on which the above operators $O(f) = \nabla_{X_f} + im_f$ act in the desired way. □

Example 29. If (X, ω) is linear, i.e. X is a vector space with a skew-form ω, then ω is exact. We can choose θ so that it is determined by ω (and hence natural for the symplectic group) and satisfies

$$(v \lrcorner \theta)(x) = -\omega(v, x), \quad \forall v, x \in X.$$

Here we think of v as a constant vector field on X and of x as a point in X. Conceptually, the above formulas can be interpreted as follows: The skew-form ω on the vector space X can be thought of as a 2-form $\bar{\omega} \in \Omega^2(X)$ on the manifold X that is constant in $x \in X$:

$$\bar{\omega}_x(v_1, v_2) = \omega(v_1, v_2) \quad \forall v_i \in X = T_x X.$$

Alternatively, ω gives a 1-form $\theta \in \Omega^1(X)$ that varies with $x \in X$ according to the formula

$$\theta_x(v) = \omega(x, v) \quad \forall v \in X = T_x X.$$

A geometric way to explain the appearence of the Heisenberg group is to note that the translations $T_x, x \in X$, do not leave θ invariant and hence they don't preserve the unitary connection $d + i\theta$ (on the trivial line bundle over X).

What happens to linear functions $v \in X \cong X^* \subset C^\infty(X)$ under prequantization? We can act on (complex valued) functions on X and use the simple formula

$$O(f) = X_f + i(m_{\theta(X_f)} + m_f)$$

Computation yields that to v we associate the operator

$$\frac{\partial}{\partial v} + 2im_v.$$

on $L^2(X;\mathbb{C})$. However, this is not the answer one expects from quantum mechanics as we shall explain next.

If X is a symplectic vector space, then the constant plus the linear functions

$$\mathbb{R}\cdot 1 \oplus X^* \subset C^\infty(X)$$

form a sub Lie algebra of the Poisson algebra for (X,ω).

Definition 30. *By definition, this is the* Heisenberg Lie algebra $\mathfrak{heis}(X,\omega)$ *associated with* (X,ω)*. It is a central extension of the two trivial Lie algebras* $\mathbb{R}\cdot 1$ *and* $X^*\cong X$ *with the canonical commutation relation*

$$[v,v'] = \omega(v,v')\cdot 1 \quad \forall\ v,v' \in X$$

The corresponding Lie group $\mathrm{Heis}(X,\omega)$ *is also a central extension*

$$1 \longrightarrow \mathbb{R} \longrightarrow \mathrm{Heis}(X,\omega) \longrightarrow (X,+) \longrightarrow 1$$

but in the category of Lie groups. Note that the exponential map is the identity! In the following, we will sometimes also consider a slightly modified Heisenberg group in which X *is centrally extended by a circle* $\mathbb{T}$ *instead of* $\mathbb{R}$*. Hence, in this case we have an exact sequence*

$$1 \longrightarrow \mathbb{T} \longrightarrow \mathrm{Heis}_{\mathbb{T}}(X,\omega) \longrightarrow (X,+) \longrightarrow 0,$$

and the multiplication of two elements in $\mathrm{Heis}_{\mathbb{T}}(X,\omega) = \mathbb{T}\times X$ *is given by*

$$(z_1,v_1)\cdot(z_2,v_2) = (z_1 z_2 e^{i\omega(v_1,v_2)}, v_1+v_2)$$

From the calculations above it follows that the prequantization procedure produces a (unitary) representation of the Heisenberg Lie algebra determined by

$$U(v)(\Psi)(x) = \Psi(x+v)\cdot e^{iw(v,x)}$$

for $x,v\in X$ and $\Psi\in L^2(X;\mathbb{C})$. This is the natural implementation of the translation symmetries of a linear symplectic manifold. From a physical point of view, however, this representation is not satisfying: It is highly reducible even in the case of $X=\mathbb{R}^2$, i.e. where the classical system is the phase space of an elementary particle moving in one real dimension. It is a principle going back to Wigner that 'elementary classical systems', i.e. homogenous symplectic manifolds, should quantize to *irreducible* representations of the symmetry group. Since the translations act transitively on our vector space X, we are really looking for an irreducible representation of the Heisenberg group. This leads us to an attempt of "cutting down the prequantum Hilbert space by half the dimensions".

Canonical quantization. Let again (X, ω) be a symplectic vector space. We choose a splitting of X into position and momentum subspaces (they are *real Lagrangians*)

$$X = M \oplus M^*$$

such that ω vanishes on M and M^* and is the natural evaluation on pairs of vectors from M and M^*. Then the Heisenberg group $\mathrm{Heis}_{\mathbb{T}}(X, \omega)$ acts unitarily on $L^2(M; \mathbb{C})$ as follows:

- a central $z \in \mathbb{T}$ acts by multiplication with the constant function z.
- M acts by translation.
- $\varphi_v \in M^* = \mathrm{Hom}(M, \mathbb{R})$ acts by multiplication: $\varphi_v(\psi) = e^{i\varphi_v}\psi$.

On the Lie algebra level this means that the central element 1 acts as the identity, $v \in M$ acts as $\frac{\partial}{\partial v}$, and $\varphi \in M^*$ acts by multiplication im_φ. One checks the relevant relation

$$[\frac{\partial}{\partial v}, m_\varphi] = \varphi(v) \cdot 1.$$

Remark 31. Unlike for prequantization, this Lie algebra action does not extend to an action of the whole Poisson algebra $C^\infty(X)$. We shall see below that it does extends to quadratic functions $\mathrm{Sym}^{\leq 2}(X)$ which contains quadratic potentials (and the Fourrier transform). However, the action of $\mathrm{Sym}^2(X)$ is not the one predicted by the prequantization rules since second order operators arise.

Theorem 32. *(Stone-von Neumann)*

(i) $L^2(M; \mathbb{C})$ *is an irreducible unitary representation of* $\mathrm{Heis}_{\mathbb{T}}(X, \omega)$.

(ii) *It is the unique irreducible unitary representation of* $\mathrm{Heis}_{\mathbb{T}}(X, \omega)$ *where* $\mathbb{T}$ *acts naturally (i.e. by multiplication).*

(iii) *More generally, every irreducible unitary representation of* $\mathrm{Heis}(X, \omega)$ *is isomorphic to exactly one of the following two families:*

$$H_\lambda, \lambda \in \mathbb{R} \smallsetminus \{0\}, \;\; \textit{or } H_\alpha, \; \alpha \in X^*.$$

The H_α *are exactly the* 1*-dimensional representations (that are trivial on the center). All* H_λ *are infinite-dimensional and determined by the scalar* λ *so that an element* $t \cdot 1$ *in the center acts by multiplication with* $e^{2\pi i t\lambda}$*. The representations that factor through* $\mathrm{Heis}_{\mathbb{T}}(X, \omega)$ *are those with* $\lambda \in \mathbb{Z}$ *and* $L^2(M)$ *corresponds to* $\lambda = 1$.

As a consequence, there is a 1-1-correspondence between unitary irreducible representations of $\mathrm{Heis}(X, \omega)$ and coadjoint $\mathrm{Heis}(X, \omega)$-orbits: Firstly, there are point orbits for each $\alpha \in X^* \subset \mathfrak{heis}^*$. If $\alpha \in \mathfrak{heis}^*$ is off this linear subspace, i.e. if $\alpha(1) \neq 0$, then the orbit is the codimension one affine space parallel to X^* and determined by

$$\lambda := \alpha(1) \neq 0.$$

This result is a special case of Kirrilov's bijection between coadjoint G-orbits and irreducible unitary representation for any 1-connected nilpotent Lie group G.

Remark 33. It follows from the theorem that the linear symplectic automorphism group $\mathrm{Sp}(X, \omega)$ of X acts projectively on each irreducible H_λ, e.g. on $L^2(M)$. This is the usual intertwining argument that goes as follows: Each $g \in \mathrm{Sp}(X, \omega)$ clearly induces an automorphism a_g of $\mathrm{Heis}(X, \omega)$ and therefore one may use a_g to get a new "twisted" action on H_λ. But by the uniqueness part of the theorem, this twisted action must be isomorphic to the untwisted one, i.e. there must be unitary intertwiners $U_g : H_\lambda \to H_\lambda$ satisfying the following identity of operators on H_λ:

$$a_g(h) = U_g \circ h \circ U_g^* \quad \forall\, h \in \mathrm{Heis}(X, \omega), g \in \mathrm{Sp}(X, \omega).$$

By Schur's lemma, the U_g are unique up to phase and hence it follows from the composition property of the a_g that $g \mapsto U_g$ is a projective representation of $\mathrm{Sp}(X, \omega)$. We shall see later it is a 2-fold covering, the *metaplectic group*, that acts without projective anomaly.

Note that by construction we actually constructed a projective representation H_λ of the semidirect product of $\mathrm{Heis}(X, \omega)$ and $\mathrm{Sp}(X, \omega)$. This explains the discussion of Remark 31, namely that we have representations of $\mathrm{Sym}^{\leq 2}(X)$, the Lie algebra of the above semidirect product. To prove this, we need to show that the Lie algebra of $\mathrm{Sp}(X, \omega)$ is isomorphic to the space of quadratic functions $\mathrm{Sym}^2(X)$ on X. This isomorphism is simply given by

$$A \mapsto (x \mapsto \omega(x, Ax)).$$

Remark 34. The appearance of unbounded operators on the Lie algebra level is unavoidable: The commutator relation $PQ - QP = \mathbb{1}$ does not posses bounded solutions.

Example 35. Let us look at the easiest example $(X, \omega) = (\mathbb{R}^2, dx \wedge dy)$. In this case $H = L^2(\mathbb{R}; \mathbb{C})$ and the canonical (self-adjoint) generators act as

$$P = i\partial_x \text{ and } Q = m_x,$$

This is essentially the one-dimensional harmonic oscillator: If we define the *creation and annihilation* operators

$$a := \frac{1}{\sqrt{2}}(P + iQ) \text{ and } a^* := \frac{1}{\sqrt{2}}(P - iQ),$$

then we can write the 'total energy' as

$$E := \frac{1}{2}(-\partial_x^2 + m_{x^2}) = \frac{1}{2}(P^2 + Q^2) = a^*a + \frac{1}{2}.$$

One easily checks that the relation $[a^*, a] = \mathbb{1}$ follows from $[P, Q] = i\mathbb{1}$.

Lemma 36. *Let* $\Omega := e^{-\frac{1}{2}x^2} \in L^2(\mathbb{R})$ *be the 'vacuum vector'. Then*

(i) $a^*\Omega = 0$ *and* $[a^*, a^n] = n \cdot a^{n-1}$,

(ii) $E(a^n\Omega) = (n + \frac{1}{2}) \cdot (a^n\Omega)$, *and these are, up to scalar multiples, all Eigenvectors of* E.

(iii) $a^n\Omega$, $n \in \mathbb{N}$, *is an orthogonal basis for* $L^2(\mathbb{R})$.

(iv) *The standard generators of* $\mathfrak{sl}_2 = \mathfrak{sp}_2$ *act on* $L^2(\mathbb{R})$ *as*

$$\frac{i}{2}P^2, \frac{i}{2}Q^2, \textit{and } \frac{i}{2}(PQ + QP),$$

where $\mathfrak{so}_2 \subset \mathfrak{sl}_2$ *comes from* E. *(This implies* $e^{2\pi i E} = -1$*, so only the double cover of* $SL_2(\mathbb{R})$ *acts; this is the metaplectic group.)*

Proof. We leave the calculations in (i),(ii), (iv) to the reader. To prove the second assertion in (ii) one first needs to verify that, up to scalars, Ω is the only L^2-Eigenvector of E with (minimal) Eigenvalue ('energy') $\frac{1}{2}$. Since Ω is nowhere vanishing, we can write any other Eigenvector as $u\Omega$. Plugging into the Eigenvalue equation leads to

$$\partial_x^2(u) = 2x \cdot \partial_x(u) \quad \text{and hence} \quad \partial_x(u) = ce^{x^2}$$

Unless $c = 0$, it follows that $u(x) > c'e^{x^2}$ and hence $u\Omega$ does not lie in L^2. As a consequence, we also know that Ω spans the line annihilated by a^*. It follows that $a^n\Omega$ are, up to scalars, the only Eigenvectors of E: If v was another Eigenvector, then we can apply powers of a^* to produce new Eigenvectors of smaller energy, one unit per power smaller; this follows directly from the commutation relations. This process either ends at zero (showing that the spectrum of E is indeed $\frac{1}{2} + \mathbb{N}$) or at (a multiple of) Ω in which case we may conclude that v was (a multiple of) $a^n\Omega$ to begin with.
Finally, to prove (iii), we can use the spectral decomposition of $L^2(\mathbb{R})$ with respect to the essentially self-adjoint operator E. □

Remark 37. The lemma implies that we have a subspace

$$L^2(\mathbb{R}) \supset H^{\text{alg}} := \bigoplus_{n\in\mathbb{N}} \text{span}(a^n\Omega) \cong \mathbb{C}[a] \cdot \Omega$$

on which a acts by m_a and a^* acts by ∂_a, satisfying the relevant commutator relations, i.e. giving a representation of the complexified Heisenberg Lie algebra. Moreover, there is a unique inner product on H^{alg} for which Ω has the right length and m_a is the adjoint of ∂_a. Then the Heisenberg *group* acts on the completion and thus this process reduces its representation theory to pure algebra. We shall explain how this can be generalized as soon as we get to complex polarizations.

This algebraic subspace is similar to the subspace of K-finite vectors V^{alg} in a representation V of a non-compact group G, where K is a maximal compact subgroup of G. Here V^{alg} consists of those vectors in V that are contained in a finite dimensional K-invariant subspace. As in our example, the group G usually doesn't preserve V^{alg} but the Lie algebra $\mathfrak{g}$ does. One studies representations V by looking at the action of the pair $(K, \mathfrak{g})$ on V^{alg}.

Bosonic Fock spaces. Now let us try to imitate the above construction for the one-dimensional harmonic oscillator in the case of an arbitrary symplectic vector space (X,ω), possibly infinite dimensional. Consider the complexified Heisenberg Lie algebra, $\mathfrak{heis}_{\mathbb{C}} := \mathfrak{heis}\otimes\mathbb{C}$, and choose a complex Lagrangian $L \subset X_{\mathbb{C}}$ (see definition 40 below). In particular, there is a hermitian inner product on L given by

$$\langle \ell_1, \ell_2 \rangle := i \cdot \omega_{\mathbb{C}}(\bar{\ell}_1, \ell_2)$$

Now define the *bosonic Fock space* as the symmetric algebra on L, thought of as a *quotient* of the tensor algebra $T(L)$, i.e. as the polynomial functions on $\bar{L}$:

$$F_L := \mathrm{Sym}(L) = \bigoplus_{n\in\mathbb{N}_0} \mathrm{Sym}^n(L) =: \mathrm{Pol}(\bar{L}),$$

the last identification being given by sending $a \in L$ to the function $x \mapsto \langle \bar{x}, a\rangle$ on $\bar{L}$. Define an action of $\mathfrak{heis}_{\mathbb{C}}$ on F_L as follows (check that the relevant commutation relations are satisfied!):

- The center acts by multiplication.
- $a \in L$ acts by the *creation operator* m_a, i.e. by multiplication with a. It maps a tensor $a_1 \cdot \dots \cdot a_n$ to $a \cdot a_1 \cdot \dots \cdot a_n$.
- $\bar{a} \in \bar{L}$ acts by the *annihilation operator*, i.e. the derivation ∂_a (of degree -1) determined in degree one by $\partial_a(b) = \omega(\bar{a}, b)$ for $b \in L$.

We put the usual inner product on F_L: The spaces $\mathrm{Sym}^n(L)$ of degree n polynomials are orthogonal and the inner product of $a_1 \dots a_n$ and $b_1 \dots b_n$ is given by the sum over all permutations

$$\frac{1}{n!}\sum_{\sigma\in S_n}\prod_i \langle a_{\sigma(i)}, b_i\rangle$$

It is easy to check that the annihilators are then adjoint to the creators: $m_a^* = \partial_a \quad \forall a \in L$.

Lemma 38. *This action makes the Fock space F_L into an irreducible $\mathfrak{heis}_{\mathbb{C}}$-module.*

Proof. Let $M \subseteq F_L$ be a nontrivial submodule and pick $0 \neq m \in M$. Then m contains a homogenous term of highest degree n. After applying the appropriate n annihilation operators we get a nontrivial multiple of the *vacuum vector* $1 \in \mathrm{Sym}^0(L)$. By applying sequences of creators to 1 we see that $M = F_L$. □

There is also a complexified Heisenberg group $\mathrm{Heis}_{\mathbb{C}}(X,\omega)$; it is a central extension of $X_{\mathbb{C}}$ by $\mathbb{C}^\times$ with the usual commutation relations, e.g. for $a, b \in L$ we have

$$\bar{a}\dot{b} = e^{-\langle a,b\rangle}\dot{b}\bar{a}$$

One gets a representation of this group on $\mathrm{Hol}(\bar{L})$ by letting $a \in L$ act by multiplication

$$(a \cdot f)(\bar{b}) := e^{-\langle a,b\rangle} f(\bar{b})$$

and $\bar{a} \in \bar{L}$ by translations

$$(\bar{a} \cdot f)(\bar{b}) := f(\bar{b} - \bar{a})$$

It is shown in [PS, Prop.9.5.8] that $\mathrm{Heis}_{\mathbb{C}}$ acts unitarily on the subspace $\widehat{F}_L$ of $\mathrm{Hol}(\bar{L})$, the Hilbert space completion of our Fock space F_L. Moreover, this representation is irreducible and hence by Theorem 32 it gives a different description of $L^2(\mathbb{R})$ for the case $X = \mathbb{R}^2$ discussed above.

Remark 39. If X is finite dimensional, Theorem 32 implies that the above representations $\widehat{F}_L$ are independent of the Lagrangian L. However, in infinite dimensions, this no longer holds. In fact, the Siegel-Naimark theorem implies that two such representations are unitarily equivalent if and only if the two Lagrangians differ by trace-class operators. More precisely, we need to replace Lagrangians by complex structures on X as in Lemma 41 before we can take 'differences'. If one is interested in representations of the Heisenberg Lie algebra, instead of the Lie group, then a similar criterion holds with trace class replaced by Hilbert-Schmidt operators.

4. Geometric quantization

We now introduce the method of geometric quantization. The main problem with pre-quantization was that the associated representation was reducible, violating the physicists paradigm that elementary systems, i.e. homogeneous symplectic manifolds, should quantize to irreducible representations. In the case of a linear space (X, ω) the formalism of canonical quantization solved the problem. Now we will generalize it to the non-linear case. We need the following

Definition 40. *Let (X, ω) be a symplectic vector space.*

(i) *A* real Lagrangian *is an isotropic subspace $L \subset X$, i.e. a subspace on which the symplectic form vanishes, and which is maximal with respect to this property. In finite dimensions, the maximality means that L has half the dimension of X and in infinite dimensions one should require a second isotropic subspace L' such that L and L' span X.*

(ii) *A* complex Lagragian *is an isotropic subspace $L \subset X_{\mathbb{C}}$ of the complexification $(X_{\mathbb{C}}, \omega_{\mathbb{C}})$ such that $\bar{L} \oplus L = X_{\mathbb{C}}$ and*

$$i \cdot \omega_{\mathbb{C}}(\bar{\ell}, \ell) > 0$$

for all $0 \neq \ell \in L$.

Note that real Lagrangians can be tensored to give Lagrangians L in $X_{\mathbb{C}}$ satisfying $\bar{L} = L$. This is the 'opposite' property of complex Lagrangians but it allows us to always work in the complexification. There is also a mixed case which we shall not study here.

Lemma 41. *If X is finite dimensional, there are canonical bijections (actually, isomorphisms of complex manifolds) between:*

- *The space of complex Lagrangians of (X, ω),*
- *complex structures $J : X \to X$ compatible with ω in the sense that*

$$\omega(Jx, Jy) = \omega(x, y) \quad \text{and} \quad \omega(Jx, x) > 0, \forall x \neq 0$$

- *symmetric linear maps $A : M_{\mathbb{C}} \to M_{\mathbb{C}}$ whose imaginary part is positive definite. This requires the choice of a real Lagrangian $M \subset X$. The space of such "complex Gaußian's" is also called "Siegel's generalized upper half plane".*
- *symmetric linear maps $B : W \to W^*$ whose operator norm is smaller than 1. This requires the choice of one complex Lagrangian W. The set of such B's is "Siegel's generalized unit disc".*

The last description gives an open, bounded subspace of $\mathbb{C}^{n(n+1)/2}$ if $\dim X = 2n$. This shows most easily that these spaces are contractible.

Proof. The proof is straight forward, for example, the map from complex structures to Lagrangians is given by mapping J to its $(+i)$-Eigenspace. Moreover, a symmetric map $A : M_{\mathbb{C}} \to M_{\mathbb{C}}$ gives a Lagrangian via its graph. □

Definition 42. *A (real respectively complex)* polarization *of a symplectic manifold (X, ω) is an integrable complex distribution $P \subset T_{\mathbb{C}}X$ such that P_x is a (real respectively complex) Lagragian for all $x \in X$.*

Remark 43. Recall that a complex distribution P is integrable if and only if $\bar{\partial}^2 = 0$ where the Dolbeaux operator $\bar{\partial}$ is associated to P via the decomposition of the deRham operator

$$d = \partial + \bar{\partial} : \Omega^0(X; \mathbb{C}) \longrightarrow \Omega^{1,0} \oplus \Omega^{0,1}.$$

Here $\Omega^{1,0}$ is *defined* by P, thought of pointwise as a complex Lagrangian in $T_xX \otimes \mathbb{C}$.

On any symplectic manifold, we can locally always find a *real* symplectic potential, i.e. a 1-form θ_0 with $d\theta_0 = \omega$ but in the Kähler case it is better to work with *complex* potentials. For simplicity, let us work in local coordinates (p, q) on $\mathbb{R}^2$ to explain this point. Here we have $\omega = dp \wedge dq$ and we may choose the real potential to be

$$\theta_0 = 1/2(pdq - qdp).$$

Write $z = p + iq$ for the complex structure and define the *Kähler potential* by

$$K(z) := 1/2(z\bar{z})$$

Then it is easy to check that $\theta := i\partial K = i/2(\bar{z}dz)$ is a symplectic potential:

$$d\theta = d(i\partial K) = i(\partial + \bar{\partial})\partial K = i\partial\bar{\partial}K = 1/2(dz \wedge d\bar{z}) = dp \wedge dq = w.$$

This complex potential θ differs from the above real potential θ_0 by the complex 1-form $(i/2)dK$. Geometrically, this means that the connection $d + i\theta$ (on the trivial line bundle over $\mathbb{R}^2$) is *not* unitary but it can be identified with the unitary connection $d + i\theta_0$ by multiplying with the 'Gaußian' function e^{-K}. This explains the appearence of this function in section 4. We note that on any Kähler manifold, one can locally find a Kähler potential function K satisfying

$$\omega = i\partial\bar{\partial}K$$

and that the corresponding Gaußian appears naturally in the L^2-inner products.

Now let (X, ω) be an integral symplectic manifold, and let $\mathcal{L}$ be the associated prequantum line bundle. Recall that $\mathcal{L}$ comes equipped with a Hermitian metric and a unitary connection ∇ with curvature ω. Furthermore, let P be a complex polarization of (X, ω). Define the *geometric quantization* of the quadruple $(X, \mathcal{L}, \nabla, P)$ by

$$H_P := \{ \, s \in \Gamma_{L^2}(\mathcal{L}) \mid \nabla_{\bar{\xi}}(s) = 0 \text{ for all } \xi \in \Gamma(P) \, \}$$

H_P is the completion of square-integrable C^∞-sections $\Gamma_P(\mathcal{L})$ with respect to a suitable L^2-norm. Consider the subspace of functions

$$C^\infty_P(X) := \{ \, f \in C^\infty(X, \mathbb{C}) \mid \bar{\xi}(f) = 0 \text{ for all } \xi \in \Gamma(P) \, \}.$$

Lemma 44. *The prequantization action of $C^\infty_P(X) \subset C^\infty(X, \mathbb{C})$ on $\Gamma(\mathcal{L})$ preserves the subspace $\Gamma_P(\mathcal{L})$.*

Proof. Assume $\nabla_{\bar{\xi}}(s) = 0$. Then for $f \in C^\infty_P(X)$ we have

$$\nabla_{\bar{\xi}}(O(f)(s)) = \nabla_{\bar{\xi}}(\nabla_{X_f} + im_f)(s) = \nabla_{\bar{\xi}}\nabla_{X_f}(s) + i\nabla_{\bar{\xi}}(f \cdot s).$$

Using the Leibniz rule and the assumptions on f and s one sees that the last term vanishes. The first term can be simplified by using the definition of curvature:

$$\nabla_{\bar{\xi}}\nabla_{X_f}(s) = \nabla_{X_f}\nabla_{\bar{\xi}}(s) + \omega(\bar{\xi}, X_f) \cdot s = 0 - \bar{\xi}(f) \cdot s = 0.$$

All this follows from our assumptions and the definition of X_f. □

From the definition of a complex polarization P it is clear that it makes the symplectic manifold (X, ω) into a Kähler manifold. This is the reason that the quantization procedure above is sometimes also known as *Kähler quantization*.

Remark 45. Using the complex structure on X and the connection ∇ on $\mathcal{L}$ one can make $\mathcal{L}$ into a holomorphic bundle in a canonical way: For any hermitian vector bundle E over X, the complex structure on X gives a splitting

$$\Omega^1(X; E) = \Omega^{1,0}(X; E) \oplus \Omega^{0,1}(X; E)$$

which in turn defines a decomposition of the hermitian connection

$$\nabla = \nabla^{1,0} + \nabla^{0,1} : \Gamma(E) = \Omega^0(X; E) \longrightarrow \Omega^1(X; E).$$

Now the operator $\bar{\partial} := \nabla^{0,1}$ defines a holomorphic structure on E, its kernel being the holomorphic sections. With respect to these holomorphic structures on X and $\mathcal{L}$ we have

$$\Gamma_P(\mathcal{L}) = \Gamma_{\mathrm{hol}}(\mathcal{L}) \text{ and } C_p(X) = \mathrm{Hol}(X).$$

For completeness we recall that, vice versa, a holomorphic structure on E (in the sense of holomorphic coordinate changes $U_\alpha \cap U_\beta \to GL_n(\mathbb{C})$) gives a canonical Dolbeaux operator

$$\bar{\partial} : \Omega^{p,q}(X; E) \oplus \Omega^{p,q+1}(X; E)$$

that is determined on a (local) holomorphic frame $\{e_i\}$ by the formula

$$\bar{\partial}(\sum_i w_i \otimes e_i = \sum_i \bar{\partial}(w_i) \otimes e_i$$

where w_i are (local) scalar (p, q)-forms. If E has in addition a hermitian structure then there is unique connection compatible with the holomorphic and hermitian structures on E.

We shall discuss two important cases of Kähler quantization. First, we will look at the case of a symplectic vector space, after that we will treat integral coadjoint orbits of a compact Lie group G, which is essentially Borel-Weil theory.

Linear quantization. Let (X, ω) be a $2n$-dimensional symplectic vector space. Pick a compatible complex structure J on X. Since we are in the linear case, we have

$$\Gamma_{\mathrm{hol}}(\mathcal{L}) = \mathrm{Hol}(X) \cong \mathrm{Hol}(\bar{L}).$$

We want to complete the space of sections, so we introduce an inner product on $\mathrm{Hol}(\bar{L})$ as follows:

$$\langle \phi_1, \phi_2 \rangle := \int_X \bar{\phi_1} \phi_2 e^{-\frac{1}{2}K} \varepsilon,$$

where ε is the standard Liouville volume form $(n!)^{-1}\omega^n$ on X and K is the positive definite *Kähler potential* as in Remark 43

$$K(\bar{\ell}) := \langle \ell, \ell \rangle.$$

Of course, before we can complete, we have to restrict ourselves to integrable holomorphic sections (containing all polynomial functions on X as a dense subspace). By comparing to section 3 we see that the action of a polynomial $p \in \mathrm{Pol}(L)$ is given by multiplication operators

$$Q(p) = m_p$$

and that $q \in \mathrm{Pol}(L)$ can also be quantized acting as

$$Q(q) = \partial_q.$$

From this it is clear that their action leaves the subspace of polynomials in H_J invariant.

Borel-Weil theory. Let us now look at integral coadjoint orbits $X = \mathcal{O}_\alpha$ for a compact, connected Lie group G and $\alpha \in \mathfrak{g}^*$. All such $\mathcal{O}_\alpha$ are Kähler G-manifolds. One way to see this is that when they are constructed via symplectic reduction the reduction process is actually a Kähler reduction, since one can identify T^*G with the complexification $G_{\mathbb{C}}$. A more direct proof is as follows: Pick a maximal torus in $T \subset G$ and observe that the set of coadjoint orbits can be written as

$$\{\mathcal{O}_\alpha \subset \mathfrak{g}^*\} = \mathfrak{g}^*/G \cong \mathfrak{t}^*/W,$$

where W is the Weyl group of T in G, i.e. $W = N(T)/T$. This means that we may consider $\alpha \in \mathfrak{t}^*$ and in the following we also assume that we are in the generic case where T is the stabilizer of α. The tangent bundle of $\mathcal{O}_\alpha$ can then be written as $G \times_T \mathfrak{g}/\mathfrak{t}$. Hence we can define a polarization of $\mathcal{O}_\alpha$ by giving a complex polarization of $\mathfrak{g}_{\mathbb{C}}/\mathfrak{t}_{\mathbb{C}}$ and translating it under the action of G. We decompose $\mathfrak{g}_{\mathbb{C}}$ under the adjoint action of T to get

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\text{roots } r} \mathfrak{g}_r.$$

Here $\mathfrak{g}_r$ are the root spaces, i.e. the (simultaneous) eigenspaces for the action of the (abelian) group T. They are indexed by roots $r : T \to S^1$ that describe the action of T via

$$t \cdot v = r(t)v \quad \forall t \in T,\ v \in \mathfrak{g}_r.$$

Now the choice of 'positive roots' gives a polarization

$$g_{\mathbb{C}}/\mathfrak{t}_{\mathbb{C}} = \bigoplus_{r>0} \mathfrak{g}_r \oplus \bigoplus_{r<0} \mathfrak{g}_r.$$

Positivity can be defined by a choice of hyperplane in $\mathfrak{t}^*$, missing all (infinitesimal) roots. It remains to check that the constructed polarization $P \subset T_{\mathbb{C}}\mathcal{O}_\alpha$ is integrable. This follows from the Jacobi identity for Lie brackets which implies that

$$[\mathfrak{g}_{r_1}, \mathfrak{g}_{r_2}] \subset \mathfrak{g}_{r_1+r_2}.$$

It follows that $\mathcal{O}_\alpha$ is a Kähler G-manifold.

The next ingredient is the prequantum line bundle $\mathcal{L}$. Here the integrality condition on $\mathcal{O}_\alpha$ comes in: By assumption, the class $[\omega_\alpha] \in H^2_{dR}(\mathcal{O}_\alpha)$ is integral. This condition is equivalent to $\alpha : \mathfrak{t} \to \mathbb{R}$ being the derivative of a homomorphism $a : T \to S^1$. We can write down $\mathcal{L}$ explicitly as the associated bundle

$$\mathcal{L}_a := G \times_{(T,a)} \mathbb{C} \to G/T \cong \mathcal{O}_\alpha.$$

Hence the sections of $\mathcal{L}_a$ can be thought of as functions $f : G \to \mathbb{C}$ that satisfy the following equivariance condition for $t \in T$:

$$f(gt) = a(t)^{-1} f(g).$$

The above decomposition into positive and negative roots induces a holomorphic structure on $\mathcal{L}_a$. One can translate this into saying that a function f as above is 'holomorphic' (at $1 \in G$) if and only if its Lie derivative in the direction of all positive roots vanishes. Clearly, G acts on (holomorphic) sections of $\mathcal{L}_a$ given in the above description by $(h \cdot f)(g) = f(h^{-1}g)$.

Theorem 46 (Borel, Weil)**.** *Let G be a connected, compact Lie group. Then there is a bijection between G-integral coadjoint G-orbits and irreducible (complex) representations of the group G. It is given by Kähler quantization, i.e. it associates with an integral orbit represented by $a : T \to S^1$ the representation $\Gamma_{\mathrm{hol}}(\mathcal{L}_a)$. The inverse is given by looking at the highest weight of a given representation, see below.*

We have to explain the notion of G-integrality. Let (X, ω) be an integral symplectic manifold with prequantum line bundle $\mathcal{L}$. Recall the diagram of Lie algebras and Lie groups relating automorphisms of the line bundle $\mathcal{L}$ with symplectomorphisms of (X, ω). If the action of G on (X, ω) is Poisson, we have, by definition, a lift

$$\mathfrak{g} \longrightarrow C^\infty(\mathcal{O}_\alpha).$$

However, this Lie algebra map does not necessarily come from a homomorphism of Lie groups $G \to \operatorname{Aut} \mathcal{L}$. If it does, we call the action of G on (X, ω) *G-integral*. As we saw in the homework, the action of G on $\mathcal{O}_\alpha$ is always Poisson and the G-integrality is exactly the condition that α comes from a.

Examples 47. The easiest cases are

(i) $G = S^1$. Then $\mathfrak{g}^* = \mathfrak{g}^* = \mathbb{R}$, and each point in $\mathbb{R}$ is an S^1-orbit. $\alpha : \mathbb{R} \to \mathbb{R}$ is S^1-integral if and only if it is the derivative of a group homomorphism $S^1 \to S^1$, hence if and only if α is an integer a. This gives the well known classification of irreducible representations of S^1: they are all 1-dimensional and given by $z \mapsto z^a$ for some $a \in \mathbb{Z}$.

(ii) $G = SU_2$. Recall from example 11 that the different coadjoint orbits are indexed by a non-negative real number a. It turns out that the corresponding coadjoint orbit is SU_2-integral exactly if $a \in \frac{1}{2}\mathbb{N}_0$. The representations corresponding to integers are exactly the ones that descend to SO_3.
To be continued...

5. Path integrals

Geometric quantization as introduced above is, in the case of a general system (X, ω), not very satisfying from a physical point of view, since it does not lead to the quantization of many physically interesting functions. E.g. the Hamiltonian of a system is usually quadratic in the momentum variables, but we did not quantize functions of this type, leaving the generator of time evolution of the system unquantized (except for the case of a linear space, where we were able to quantize quadratic maps in the momentum variables using the Stone-von Neumann theorem). We now want to outline how one can obtain the time evolution operator $e^{it\hat{H}}$ using path integrals for systems given by a Lagranian L.

We start out with the easy case

$$M = \mathbb{R}, \text{ and } L : \mathbb{R}^2 \to \mathbb{R},\ (x, v) \mapsto \|v\|^2.$$

In this case we know how to quantize the energy $H = p^2$, namely, we saw that it acts as $\hat{H} = \partial_x^2$. We will rewrite the time evolution operator $e^{it\hat{H}}$ in terms of a path integral. This representation will generalize to the case of an arbitrary quadratic Lagranian system. The first step is a change of coordinates ('Wick rotation') in order to get rid of the i in the exponential, making the operators involved into Hilbert-Schmidt operators. We need the following

Lemma 48. *If $\hat{H} = \partial_x^2$, the integral kernel of the operator $e^{-t\hat{H}}$ is*

$$P_t(x, y) := \frac{1}{\sqrt{2\pi t}} e^{\frac{1}{2t}|x-y|^2}.$$

This is the well known "heat kernel", describing the distribution of heat. More precisely, if one puts a unit of heat at x and waits for time t, then $P_t(x, y)$ gives the amount of heat at y.

Recall that $k : M \times M \to \mathbb{R}$ is an *integral kernel* for the operator $O : L^2(M) \to L^2(M)$ if for all $f \in L^2(M)$

$$O_k(f)(x) = \int_M k(x, y) f(y) dy,$$

and that the operator O_k is Hilbert-Schmidt if and only if $k \in L^2(M \times M)$.

Now we will express $e^{-t\hat{H}}$ as a path integral. We have for any $n \in \mathbb{N}$:

$$\begin{aligned} e^{-t\hat{H}}(x, y) &= e^{-\frac{t}{n}\hat{H}} \cdot \ldots \cdot e^{-\frac{t}{n}\hat{H}}(x, y) \\ &= \int_{\mathbb{R}} \ldots \int_{\mathbb{R}} P_{\frac{t}{n}}(x, x_1) \cdot \ldots \cdot P_{\frac{t}{n}}(x_{n-1}, y) dx_1 \ldots dx_{n-1} \\ &= \int_{\mathbb{R}^{n-1}} dx_1 \ldots dx_{n-1} \frac{1}{(2\pi t)^{n/2}} e^{-\frac{1}{2}\sum_{i=1}^n \frac{|x_i - x_{i-1}|^2}{t/n}} \\ &= \int_{\mathbb{R}^{n-1}} dx_1 \ldots dx_{n-1} \frac{1}{Z_n(t)} e^{-\frac{1}{2}\int_0^t |\dot{\sigma}(t)|^2 dt}, \end{aligned}$$

where $\sigma : [0, t] \to \mathbb{R}$ is the function that satisfies $\sigma(t) = x_i$ for $t = \frac{it}{n}$ and is linear on all intervals $[\frac{it}{n}, \frac{(i+1)t}{n}]$; here we let $x = x_0$ and $y = x_n$. Hence we can write

$$e^{-t\hat{H}}(x, y) = \int_{\sigma_n} \frac{1}{Z_n(t)} e^{-\frac{1}{2}\int_0^t |\dot{\sigma}_n(t)|^2 dt} d\lambda_{n-1},$$

where σ_n ranges over all piecewise linear paths joining x and y having $(n-1)$ corners. Taking a formal limit $n \to \infty$ we obtain the formula

$$e^{-t\hat{H}}(x,y) = \int_\sigma \frac{1}{Z(t)} e^{-\frac{1}{2}\int_0^t |\dot{\sigma}(t)|^2 dt} d\lambda,$$

where σ now ranges over the space $P_{x,y}$ of all continuous paths joining x to y. The right side is given a precise meaning by the *Wiener measure*. Note that none of the 3 terms in the above integral is well defined but the combination of all three makes good mathematical sense. Note also that two of the three terms are defined for smooth path, however, these have measure zero and hence are not so useful. There are two (dual) ways of understanding the Wiener measure:

- One can consider evaluation maps $e : P_{x,y} \to M^n = \mathbb{R}^n$, given by first partitioning the interval $[0,1]$ by n intermediate points and then evaluating a path on these intermediate points. There are obvious consistency relations among such maps and there is a measure theoretic theorem of Kolmogorov, giving sufficient conditions for a family of measures on $\mathbb{R}^n$ to be the push-forward of a unique measure on $P_{x,y}$. By considerations similar to the above, one can show that the heat measures on $\mathbb{R}^n$ can be used to apply Kolmogorov's theorem for the construction of Wiener measure. Another way to formulate this is to consider *cylinder functions* on $P_{x,y}$ which are just compositions of functions on $\mathbb{R}^n$ with the above evaluation maps. They serve as 'step functions' whose integral is defined first, just by using the finite dimensional measures. Then one needs to check the consistency of this definition to get the integral of cylinder functions over all of $P_{x,y}$.
- A much more direct relation of the above formulas to the actual Wiener measure is given in [AD]. Anderson and Driver show that, for bounded continuous functions on $P_{x,y}$, the finite dimensional approximations by integrals over piecewise linear paths actually converge to a finite number when making the partitions of the interval $[0,t]$ finer and finer. This number turns out to be the integral of the function with respect to Wiener measure.

A similar formula and interpretation for path integrals exists for a general system consisting of a configuration space M and a Lagrangian $L : TM \to \mathbb{R}$ of the form

$$L(x,v) = g(v,v) + f(x)$$

for a Riemannian metric g on M and a potential $f \in C^\infty(M)$. Denote by A_t the action defined on the space of paths $[0,t] \to M$. Then copying the above case yields

$$e^{-t\hat{H}}(x,y) = \int_\sigma e^{A_t(\gamma)} \frac{1}{Z(t)} \mathcal{D}\lambda.$$

where σ ranges over all paths (in a certain class) joining x and y. As above, the right side indeed makes sense as the limit of finite dimensional integrals on M^n if one replaces piecewise linear paths by piecewise geodesics. This is explained very well in [AD].

In later sections, we shall need a further generalization of the path integral giving the kernel of operators acting on sections of certain vector bundles over M, rather then just on functions. An important example is the (square of the) Dirac operator acting on the spinor bundle. To describe such a kernel, one needs an additional integrand in the path-integral which turns out to be the parallel transport in the given bundle. Then a new problem arises because the usual parallel transport is only defined for smooth path which have measure zero. It turns out, however, that there is a subset of continuous paths of measure 1, with respect to the Wiener measure, for which the parallel transport can be defined as the limit over parallel transports over piecewise geodesics. This is the so called *probabilistic parallel transport* and it can be used to define these more general operator kernels via path integrals.

Remark 49. Physicists usually keep the i in the exponent,

$$e^{it\hat{H}}(x,y) = \int_\sigma e^{iA_t(\gamma)} \frac{1}{Z(t)} \mathcal{D}\lambda.$$

and continue to compute with this formula, even though the integral doesn't have a precise (mathematical) meaning. In quantum mechanics,

$$\langle y | e^{it\hat{H}} | x \rangle = e^{it\hat{H}}(x,y)$$

describes the probability amplitude to get from x to y in time t, and the above representation is known as *Feynman's path integral*.

6. Classical field theory

We describe the framework of classical field theory and how quantization leads to a structure known as "quantum field theory".

A *classical field theory* consists of the following components:

- A space-time M, i.e. a semi-Riemannian manifold (M, g). The realistic case is a four-dimensional Lorentz manifold but we shall be very flexible in the following, even allowing M to be 1-dimensional, i.e. to collapse space into a point. We also study the 'Euclidean' case where g is positive definite.
- The 'field content', i.e. a list of *fields* $\Phi(M)$ that appear (usually sections of vector bundles over M, see the examples below).
- An *action functional* $A : \Phi(M) \to \mathbb{R}$ on the space of fields.

We begin by giving a list of examples of *fields* on a space-time M; this should be understood as a 'dictionary' explaining the mathematical meaning of certain physical notions.

We order fields according to their *spin*. They are called *bosonic* if their spin is an integer and *fermionic* if it is a half integer, i.e. an odd multiple of $\frac{1}{2}$.

(i) **spin 0:** One important class of spin 0 fields are *scalar fields*. This includes functions $C^\infty(M;\mathbb{R})$ ('real scalar fields') or, more generally, mapping spaces $C^\infty(M;V)$ into a vector space V ('linear scalar fields'). Other spin 0 fields are smooth maps $M \to X$, where the *target* X is a usually a Riemannian manifold.

(ii) **spin 1:** In this class we have 1-forms on M or 'gauge fields' in $\Omega^1 M/d\Omega^0 M$. Recall that 1-forms can be interpreted as connections on the trivial S^1-bundle, and modding out by the differentials of functions corresponds to dividing by the Gauge group. More generally, one may consider connections (up to equivalence) of G-principal bundles for non-commutative Lie groups G ('non-commutative gauge fields'). Another possible generalization is to replace 1-forms by differential forms of arbitrary degree p to obtain 'p-form fields'. It is not yet completely clear how to define the combination of these two cases, namely 'non-commutative p-form fields', or 'higher non-commutative bundle gerbes'.

(iii) **spin 2:** Typical examples of spin 2 fields are sections of the second symmetric power of the tangent bundle of M, e.g. metrics on M ('gravitational fields').

(iv) **spin $\frac{1}{2}$:** Here we have sections of spinor bundles $S \to M$. By a spinor bundle we mean any vector bundle over M obtained from a $\mathrm{Spin}(r,s)$-principal bundle $P \to M$ by associating a representation of $\mathrm{Spin}(r,s)$ that comes from the Clifford algebra $Cl(r,s)$. Here the group $\mathrm{Spin}(r,s)$ is a double covering of the group $\mathrm{SO}(r,s)$ of isometries of the inner product on $\mathbb{R}^{r+s}$ of signature (r,s). The principal bundle P is a double covering of the $\mathrm{SO}(r,s)$-bundle of orthonormal oriented tangent frames of M and it exists if M has a spin structure.

(v) **spin $\frac{3}{2}$:** Same as for spin $\frac{1}{2}$, but with S replaced by an irreducible part of $S \otimes TM$. These are called 'Rarita-Schwinger fields'.

Next, we explain the names physicists give to some field theories. The spin of a field theory is the highest spin occurring among its fields.

(i) A field theory that only contains (linear) scalar fields is called a *bosonic (linear) σ-model* and has spin 0.

(ii) The fields of a *bosonic gauge theory* usually involve connections and scalar fields. Accordingly, the spin is 1.

(iii) When talking about *gravity*, one has metrics, as well as connections and scalar fields in the game, hence the spin is 2.

In a *supersymmetric field theory* there is a 'supersymmetry' exchanging bosons and fermions, in particular, both of these types of fields must occur. The main examples are

(i) A *supersymmetric σ-model* has scalar fields and spinor fields.

(ii) In *supersymmetric gauge theory* one has spinor fields, connections, and potentially scalar fields.
(iii) Finally, *super gravity* involves Rarita-Schwinger fields, metrics, connections, and scalar fields.

Examples 50. After this truckload of terminology, let us look at some basic examples of classical field theories.

(i) In classical mechanics, the space-time is just $M = \mathbb{R}$, i.e. space is just a point. The (scalar) fields are smooth maps from $\mathbb{R}$ to a configuration space Q. The action A is given by a Lagrangian $\mathcal{L} : TQ \to \mathbb{R}$:

$$A(\phi) = \int_{\mathbb{R}} \mathcal{L}(\phi(t), \dot{\phi}(t))dt.$$

The critical points of A are the classical solutions to the equations of motion.

(ii) (Relativistic) electromagnestism is a spin 1 gauge theory, where $M = \mathbb{R}^{1,3}$. The electromagnetic potential is given by a 1-form $A \in \Omega^1(M)$, the corresponding electromagnetic field is $F = dA$. The action is

$$A : \Omega^2_{closed}(M) \to \mathbb{R},\ A(F) := \int_M F \wedge *F = \int_M |F|^2.$$

The classical solutions are exactly the F's that are closed and co-closed, i.e.

$$dF = 0 = d * F,$$

where $*$ is the Hodge star operator and the first equation follows automatically from the existence of the potential A. These are solutions of Maxwell's equations in the vacuum.

Now we want to explain how the space of classical solutions, i.e. the space of critical points of the action functional A, can (in good cases) be endowed with a symplectic structure. For this, we restrict to the case in which the fields are sections over vector bundles over M and make some mild assumptions on the action A. Denote by $J^r_\Phi(M)$ the *r-jets of* $\Phi(M)$. This space can be described as the total space of the fiber bundle over M whose fiber $J^r_\Phi(m)$ over $m \in M$ are equivalence classes of fields $[\phi]$, where $[\phi_1] = [\phi_1]$ if and only if ϕ_1 and ϕ_2 have the same derivatives up to order r at m. Note that in order for this definition to be meaningful we need connections on the vector bundles involved. These come either from Levi-Civita connections on TM and its associated bundles, or from the connection associated to a Gauge field.

We will from now on assume that the action A is of the form

$$A(\phi) = \int_M \mathcal{L}(\phi_m)|dm|,$$

where $\mathcal{L} : J^r_\Phi(M) \to \mathbb{R}$ and ϕ_m denotes the image of ϕ under the obvious map $\Phi(M) \to J^r_\Phi(m)$. In other words, we assume that the Lagranian $\mathcal{L}$ only depends on the r-jets of $\Phi(M)$ for some r. In the absence of a measure dm on M, we assume that $\mathcal{L}$ is a *density* and hence the above integral is still defined.

Let us first consider the case $M = \mathbb{R}$. Fix a compact interval $[a, b] \subset \mathbb{R}$. Define

$$A_{ab} : \Phi([a,b]) \longrightarrow \mathbb{R},\ \phi \mapsto \int_a^b \mathcal{L}(\phi_m)|dm|.$$

Then

$$dA_{ab}(\phi, \delta\phi) = \int_a^b \frac{\delta L}{\delta \phi}\delta\phi\, dt + \alpha(\phi_b, \delta\phi_b) - \alpha(\phi_a, \delta\phi_a)$$

Hence, restricting A_{ab} to the space X of classical solutions of A we have

$$dA_{ab} = \alpha_b - \alpha_a,$$

where $\alpha_t \in \Omega^1(X)$ for all $t \in \mathbb{R}$. Hence

$$\omega := d\alpha_t$$

is independent of t. Clearly, $d\omega = 0$ but the non-degeneracy of ω is not automatic and requires appropriate additional assumptions. If these are satisfied, ω is the symplectic form on the space of classical solutions X. The Hamiltionian $H : X \to \mathbb{R}$ can be found as follows: Time translation defines a vector field ξ on X. We have

$$i_\xi(\alpha_a) - i_\xi(\alpha_b) = i_\xi(dA_{ab}) = \xi(A_{ab}) = L_b - L_a,$$

where $L_t : X \to \mathbb{R}$ is given by $\phi \mapsto \mathcal{L}(\phi_t)$. Thus, $H : X \to \mathbb{R}$ defined by

$$H(\phi) = i_\xi(\alpha_t) - L_t$$

is independent of the choice of t. In fact, ξ is the Hamiltionian vector field generated by H.

Now let M^{n+1} be any space time. Consider a compact submanifold Σ^{n+1} in M, and set $S^n := \partial\Sigma$. For the restricted action functional

$$A_\Sigma : \Phi(\Sigma) \longrightarrow \mathbb{R},\ \phi \mapsto \int_\Sigma \mathcal{L}(\phi_m)|dm|,$$

we have

$$dA_\Sigma(\phi, \delta\phi) = \int_\Sigma \frac{\delta L}{\delta\phi}\delta\phi|dm| + \int_{\partial\Sigma} \alpha(\phi_x, \delta\phi_x)|dx|.$$

Consequently, on the classical solutions $X \subset \Phi(M)$, we have

$$dA_\Sigma = \alpha_S \in \Omega^1(X)$$

Hence, for a codimension 1 submanifold $S^n \subset M^{n+1}$ we obtain a 1-form α_S on X. As one sees from the above expressions, this 1-form really depends on a (germ of a) neighborhood νS of S in M. The 2-form $\omega_{[S]} := d\alpha_S$ only depends on the homology class $[S] \in H_n(M)$. If

$M = \mathbb{R} \times S$, we obtain a Hamiltonian $H : X \to \mathbb{R}$ just like before (consider $\Sigma = [a, b] \times S \subset M$).

Now we want to go a step further and quantize in order to obtain a QFT. There are two approaches: Geometric quantization of (X, ω, H), or path integrals. We begin by explaining the path integral approach before we get to the mathematically rigorous technique of geometric quantization (which unfortunately only works in very special circumstances).

We 'define' $\mathcal{H}_S := L^2(\Phi(\nu S))$ for $S^n \subset M^{n+1}$, where L^2 only carries a heuristic meaning: In many interesting cases a measure on the space of fields is not known. Furthermore, for a bordism Σ^{n+1} from S_0 to S_1 we have an operator

$$O_\Sigma : \mathcal{H}_{S_0} \longrightarrow \mathcal{H}_{S_1}$$

that is given by the operator kernel

$$O_\Sigma(\phi_0, \phi_1) = \int e^{iA(\phi)} \mathcal{D}\phi,$$

where the integral is taken over $\phi \in \Phi(\Sigma)$ such that $\phi|_{\partial\Sigma} = \phi_1 - \phi_0$. The idea is that in good cases the integrand defines a measure on $\Phi(\Sigma)$. For example, if $\Sigma = [0, t] \times S$, O_Σ is the operator obtained by quantizing time translation by t. Since this is the time evolution in quantum theory, it should be a well defined operator in a theory that describes nature.

7. Axiomatic quantum field theory

We now want to define quantum field theories axiomatically following Atiyah and Segal. We will motivate the definition by extracting the formal properties of the quantum field theories we obtained by quantization from classical field theories. Recall that (formally) we obtained operators

$$\mathcal{O}_\Sigma : \mathcal{H}_{\partial_{in}\Sigma} \longrightarrow \mathcal{H}_{\partial_{out}\Sigma}$$

by the path integral

$$\mathcal{O}_\Sigma(\varphi_{in}, \varphi_{out}) = \int e^{iA_\Sigma(\phi)}\, \mathcal{D}\phi,$$

where ϕ ranges over all fields whose boundary values are given by φ_{in} and φ_{out}. We have the following formal properties:

(i) $\mathcal{H}_{S_1 \amalg S_2} = \mathcal{H}_{S_1} \otimes \mathcal{H}_{S_2}$ and $\mathcal{O}_{\Sigma_1 \amalg \Sigma_2} = \mathcal{O}_{\Sigma_1} \otimes \mathcal{O}_{\Sigma_2}$.

(ii) If $\Sigma = \Sigma_1 \cup_S \Sigma_2$, then $\mathcal{O}_\Sigma = \mathcal{O}_{\Sigma_2} \circ \mathcal{O}_{\Sigma_1}$.

We outline why these identities hold.

(i) We only consider the first equation.

$$\begin{aligned}\mathcal{H}_{S_1 \amalg S_2} &= L^2(\Phi(S_1 \amalg S_2)) \\ &= L^2(\Phi(S_1) \times \Phi(S_2)) \\ &= L^2(\Phi(S_1)) \otimes L^2(\Phi(S_2)) \text{ by 'Fubini's theorem'} \\ &= \mathcal{H}_{S_1} \otimes \mathcal{H}_{S_2}\end{aligned}$$

(ii) Since the action A_Σ is given by integrating the Lagrangian density over Σ, it is clear that $A_\Sigma = A_{\Sigma_1} + A_{\Sigma_2}$. Using this and 'Fubini's theorem' one sees that

$$\int_{\phi \in \Phi(\Sigma)} e^{-A_\Sigma(\phi)} = \mathcal{O}_{\Sigma_2}(\psi, \varphi_{out}) \mathcal{O}_{\Sigma_1}(\varphi_{in}, \psi),$$

where the integral is taken over all ϕ such that

$$\partial_{in}\phi = \varphi_{in}, \partial_{out}\phi = \varphi_{out} \quad \text{and} \quad \phi|_S = \psi.$$

Integrating the left side over all $\psi \in \Phi(S)$ yields $\mathcal{O}_\Sigma(\varphi_{in}, \varphi_{out})$. On the other hand, integrating the right side over all $\psi \in \Phi(S)$ gives the integral kernel

$$(\mathcal{O}_{\Sigma_2} \circ \mathcal{O}_{\Sigma_1})(\varphi_{in}, \varphi_{out}).$$

This shows the second formula.

Now we define QFTs axiomatically. We denote the (Riemannian) cobordism category of dimension $n+1$ by $\mathbf{Cob}_n^{n+1}$. The objects of this category are closed, oriented Riemannian n-manifolds. Morphisms between S_1 and S_2 come in two kinds: We have cobordisms and isometries. By a cobordism, we mean a triple $(\Sigma, [\alpha_{in}], [\alpha_{out}])$, where Σ is an oriented, compact Riemannian $(n+1)$-manifold with a decomposition of its boundary into an incoming and outgoing part, and

$$\alpha_{in} : S_1 \times [0, \varepsilon) \xrightarrow{\cong} \nu(\partial_{in}\Sigma)$$

is an orientation-reversing isometry of a thickening of S_1 onto an open neighbourhood of the incoming boundary part. α_{out} is defined similarly, however, this time the isometry is orientation-preserving. The brackets around the α's indicate that we are only interested in germs of such α's. Furthermore, we consider two bordisms to give the same morphism if they are isomorphic relative boundary. The second part of the morphisms are isometries $S_1 \to S_2$. Composition of cobordisms is given by gluing, cobordisms and isometries are composed by altering the incoming (or outgoing) boundary embedding α by the given isometry.

Let **Hilb** be the category of complex Hilbert spaces and bounded operators between them. Note that both **Cob** and **Hilb** are symmetric monodial categories (with respect to disjoint union and tensor product, respectively). Furthermore, there are certain additional structures on these categories, namely, involutions, anti-involutions, and so-called adjunction transformations. See [ST] for details.

Definition 51. *An* $(n+1)$-dimensional quantum field theory *is a monodial functor*

$$\mathbf{Cob}_n^{n+1} \longrightarrow \mathbf{Hilb}$$

respecting the 'additional structures'.

Note that there are other notions of 'field theories' that are variants of what we called a QFT. For example, a *conformal field theory* is one that only depends on the conformal class of the Riemannian metric. A *toplogical quantum field theory* is one that only depends on the diffeomorphism classes of the manifolds involved.

We should mention that in a similar fashion one can define *supersymmetric quantum field theories.* Roughly speaking, one replaces **Cob** by a bordism category of super manifolds and instead of functors one considers 'super functors', i.e. one makes the target and domain categories into *categories enriched over super manifolds* and looks at enriched functors between them. We will not explain this here, but our definition of the spaces EFT_n in the next section is motivated by this point of view.

8. Super manifolds

We introduce some basic notions of super geometry. Almost all the material is taken from the beautiful article on supersymmetry by Deligne and Morgan, [DM].

Let us begin by explaining briefly what 'super' means in an algebraic context. A *super vector space* or *algebra* is just a vector space or algebra equipped with a $\mathbb{Z}_2$-grading (i.e. a splitting into an 'even' and 'odd' part). The basic rule is

- **Sign rule:** Commuting two odd quantities yields a sign -1.

E.g., a super algebra is (super) *commutative* if for all homogenenous $a, b \in A$ we have

$$ab = (-1)^{|a||b|} ba,$$

where $|.|$ denotes the parity of an element. Examples of commutative super algebras are

- The cohomology ring $H^*(X)$ of a space X
- Exterior algebras $\Lambda^*(\mathbb{R}^q)$, or, more generally, tensor products $\Lambda^*(\mathbb{R}^q) \otimes A$, where A is a commutative algebra (with trivial grading).

The latter example is relevant for the definition of super manifolds: Their rings of functions are obtained by considering usual smooth functions and tensoring them (locally) with an exterior algebra. The generators of $\Lambda^*(\mathbb{R}^q)$ yield so-called odd coordinates; these are useful when one tries to describe physical systems involving Fermions.

Let A be a commutative super algebra. The *derivations* on A are $\mathbb{R}$-linear maps (not necessarily grading preserving) $A \to A$ satisfying the Leibniz rule[1]

$$\mathrm{Der}\, A = \{\, D : A \longrightarrow A \mid D(ab) = Da \cdot b + (-1)^{|D||a|} a \cdot Db \,\}.$$

[1]Whenever we write formulas involving the degree $|.|$ of certain elements, we implicitly assume that these elements are homogenous.

This is a *super Lie algebra* with respect to the bracket operation

$$[D, E] := DE - (-1)^{|D||E|} ED,$$

This means that the bracket is (super) skew symmetric

$$[D, E] + (-1)^{|D||E|} [E, D] = 0$$

and satisfies the (super) *Jacobi identity*

$$[D, [E, F]] + (-1)^{|D|(|E|+|F|)} [E, [F, D]] + (-1)^{|F|(|D|+|E|)} [F, [D, E]] = 0.$$

Note that we cyclically permuted the 3 symbols and put down the signs according to the above rule. Another way to remember the signs in the super Jacobi identity is to say that the map

$$D \mapsto (E \mapsto [D, E])$$

sends the super Lie algebra L to its algebra of derivations $\operatorname{Der} L$ (which is defined by the above sign rule).

Super manifolds. We will define super manifolds as ringed spaces following [DM]. By a morphism we will always mean a map of ringed spaces. The local model for a super manifold of dimension $(p|q)$ is Euclidian space $\mathbb{R}^p$ equipped with the sheaf of commutative super $\mathbb{R}$-algebras $U \mapsto C^\infty(U) \otimes \Lambda^*(\mathbb{R}^q)$. This is usually denoted $\mathbb{R}^{p|q}$.

Definition 52. *A* super manifold M *of dimension* $(p|q)$ *is a pair* $(|M|, \mathcal{O}_M)$ *consisting of a topological space* $|M|$ *together with a sheaf of commutative super* $\mathbb{R}$*-algebras* $\mathcal{O}_M$ *that is locally isomorphic to* $\mathbb{R}^{p|q}$*. Morphisms between super manifolds are defined as continuous maps together with maps of sheafs covering them.*

To every super manifold M there is an associated *reduced manifold*

$$M^{red} := (|M|, \mathcal{O}_M/\text{nil})$$

obtained by dividing out nilpotent functions. By construction, this gives a smooth manifold structure on the underlying topological space $|M|$ and there is an inclusion of super manifolds $M^{red} \hookrightarrow M$.

Other geometric super objects can be defined in a similar way. For example, replacing $\mathbb{R}$ by the complex numbers and C^∞ by analytic functions one obtains *complex (analytic) super manifolds*. Furthermore, there is the notion of *cs manifolds*. These are spaces equipped with sheaves of super $\mathbb{C}$-algebras that locally look like $(\mathbb{R}^p, C^\infty_\mathbb{C} \otimes \Lambda_\mathbb{C}(\mathbb{C}^q))$, i.e. one just replaces smooth real-valued functions by smooth complex-valued functions. The relevance of *cs* manifolds is that they appear naturally as the smooth super manifolds underlying complex analytic super manifolds.

Example 53. Let $E \to M$ be a vector bundle of fiber dimension q over the manifold M^p. Then $(M, \Gamma(\Lambda^* E))$ is a super manifold of dimension (p, q), and denoted by πE. Bachelor's theorem says that every super manifold is isomorphic (but not canonically) to one of this type. This result does not hold in analytic categories, it is important that we consider C^∞ functions.

The following proposition shows that morphisms between super manifolds can be described using coordinates.

Proposition 54. *Let S, M be super manifolds. There is a natural bijection between*

- *morphisms ϕ from S to M, and*
- *super $\mathbb{R}$-algebra homomorphisms $\phi^* : O_M \to O_S$, where $O_X := \mathcal{O}_X(X)$ denotes the algebra of global sections; these are by definition the functions on X.*

In the language of algebraic geometry one may say that 'super manifolds are affine'. If $M \subset \mathbb{R}^{p|q}$ is an open super submanifold (a domain*), maps $S \to M$ are in 1-to-1-correspondence with*

$$\{ (f_1, ..., f_p, \eta_1, ..., \eta_q) \in (O_S^{ev})^p \times (O_S^{odd})^q \mid (f_1(s), ..., f_p(s)) \in |M| \subset \mathbb{R}^p \text{ for all } s \in |S| \}$$

The f_i, η_j are called the coordinates of ϕ and they are given by

$$f_i = \phi^*(x_i) \quad \text{and} \quad \eta_j = \phi^*(\theta_j),$$

where $x_1, ..., x_p, \theta_q, ..., \theta_p$ are coordinates on M.

The proof of the first part is based on the existence of partitions of unity for super manifolds, so it is again false in analytic settings. The second part always holds and is proved in [Lei].

The functor of points approach. Since sheaves are generally difficult to work with, one often thinks of super manifolds in terms of their 'S-points', i.e. instead of M itself one considers the morphism sets $\mathrm{Hom}(S, M)$, where S varies over all super manifolds S. More formally, using the Yoneda lemma we embed the category **Smfds** of super manifolds in the category of functors from **Smfds** to **Sets** by

$$M \mapsto (\ S \mapsto \mathrm{Hom}(S, M)\ ).$$

This embedding identifies super manifolds with representable contravariant functors **Smfds** $\to$ **Sets** and morphisms between super manifolds with natural transformations. Note that the last proposition makes it easy to describe the morphism sets $\mathrm{Hom}(S, M)$. We'd also like to point out that the functor of points approach is very close to the formalism that physicists use to make computations involving odd quantities.

Super Lie groups. According to the functor of points approach, a group object in **Smfds** can be described by giving a representable contravariant functor G : **Smfds** $\to$ **Sets** together with functorial group structures on $G(S)$ for all S.

Examples 55. The most important super Lie groups are as follows.

(i) The additive structure on $\mathbb{R}^{p|q}$ is given by the formula

$$\mathrm{Hom}(S,\mathbb{R}^{p|q})\times\mathrm{Hom}(S,\mathbb{R}^{p|q}) \longrightarrow \mathrm{Hom}(S,\mathbb{R}^{p|q}), (f_1,...,\eta_q),(h_1,...,\psi_q) \mapsto (f_1+h_1,...,\eta_q+\psi_q).$$

(ii) The *super general linear group* $GL(p|q)$ is defined by

$$GL(p|q)(S) := \mathrm{Aut}_{\mathcal{O}_S}(\mathcal{O}_S^{p|q}) \cong \mathrm{Aut}_{O_S}(O_S^{p|q}),$$

where $A^{p|q}$ denotes the A-module freely generated by p even and q odd generators. We need to check that this is representable. We claim that $GL(p|q)(\_)$ is represented by the open super submanifold $G \subset \mathbb{R}^{p^2+q^2|2pq}$ characterized by

$$|G| = \{\, x \in \mathbb{R}^{p^2+q^2} \mid x \in GL_p \times GL_q \,\}.$$

This follows directly from proposition 53 using that a map between super algebras is invertible if and only if it invertible modulo nilpotent elements.

(iii) Using the Berezinian, a super version of the determinant, one can define a super subgroup $SL(p|q) \subset GL(p|q)$.

(iv) On $\mathbb{R}^{1|1}$ one has a 'twisted' super group structure μ defined by

$$\mathrm{Hom}(S,\mathbb{R}^{1|1}) \times \mathrm{Hom}(S,\mathbb{R}^{1|1}) \longrightarrow \mathrm{Hom}(S,\mathbb{R}^{1|1}),\ (f,\eta),(h,\psi) \mapsto (f+h+\eta\psi, \eta+\psi).$$

The relevance of this super group lies in the particular structure of its super Lie algebra: $\mathfrak{g}_{\mathbb{R}^{1|1}}$ is a super Lie algebra *freely* generated by one *odd* generator. This property also explains the appearance of $\mathbb{R}^{1|1}$ in the context of odd ODEs on super manifolds (see below). For us, the multiplication μ (restricted to $\mathbb{R}^{1|1}_{>0}$) will turn out to be important, since it describes the gluing of 'Riemannian' super intervals, see section 9.

Even though we haven't introduced the super Lie algebra of a super Lie group yet, we want to explain how super Lie groups can be understood in terms of super Lie algebras.

Theorem 56. *The following categories are equivalent:*

- *The category of* 1*-connected super Lie groups.*
- *The category of tripels* $(G_0, \mathfrak{g}, a)$*, where* G_0 *is a* 1*-connected Lie group,* $\mathfrak{g}$ *is a super Lie algebra whose even part is the Lie algebra of* G_0*, and* a *is an action of* G_0 *on* $\mathfrak{g}$ *extending the adjoint action of* G_0*.*
- *The category of (finite-dimensional) super Lie algebras over* $\mathbb{R}$

The first equivalence holds even without the assumption on the fundamental group. The second equivalence follows from Lie's theorem. Finite-dimensional simple complex super Lie algebras have been completely classified by Victor Kac in the 70s.

Super vector bundles. What is a (super) vector bundle over a super manifold M? There are two reasonable answers that come to mind:

- A (super) fiber bundle $E \to M$ with structure group $GL(p|q)$.
- A locally free sheaf $\mathcal{E}$ of $\mathcal{O}_M$-modules of dimension $(p|q)$.

The two answers are equivalent. The main point is that coordinate changes between local trivializations are given by the same data in both cases: For a fiber bundle $E \to M$, a change of trivialization over $U \subset M$ is given by a map $\varphi : U \to GL(p|q)$. However, this is nothing but an automorphism of $\mathcal{O}_U^{p|q}$ (recall the definition of $GL(p|q)$ in terms of its S-points) which is exactly the datum giving a change of local trivializations of a locally free sheaf of dimension $(p|q)$.

Let us now look at the basic example of a super vector bundle, the *tangent bundle* of a super manifold $M^{p|q}$. It is the sheaf of $\mathcal{O}_M$-modules TM defined by

$$TM(U) := \operatorname{Der} \mathcal{O}_M(U).$$

TM is locally free of dimension $(p|q)$: If $x_1, ..., \theta_q$ are local coordinates on M, then a local basis is given by $\partial_{x_1}, ..., \partial_{\theta_q}$.

The *cotangent bundle* of M is the sheaf of $\mathcal{O}_M$-modules $\Omega^1 M$ dual to TM. As in the case of usual manifolds on obtains differential forms on M by looking at the exterior algebra of $\Omega^1 M$. Furthermore, a de Rham differential d on $\Omega^* M$ can be defined. The cohomology of this complex is just the usual cohomology $H^*(|M|; \mathbb{R})$.

The super Lie algebra of a super Lie group. Now we can define the super Lie algebra $\mathfrak{g}$ of a super Lie group G. A vector field $\xi \in \Gamma(TG)$ is called *left-invariant* if ξ is related to itself under the left-translation by all $f : S \to G$:

$$S \times G \xrightarrow{f \times \mathrm{id}} G \times G \xrightarrow{\mu} G.$$

Here we interpreted ξ as a vector field on $S \times G$ in the obvious way. The super Lie algebra $\mathfrak{g}$ consists of all left-invariant vector fields on G. Evaluation at $e \in G$ defines an isomorphism $\mathfrak{g} \cong T_e G$, in particular, the vector space dimension of $\mathfrak{g}$ is $(p|q)$.

Example 57. For the twisted super group structure on $\mathbb{R}^{1|1}$, left-translation by a map $f = (f_1, f_2) : S \to \mathbb{R}^{1|1}$ is given by the formula

$$S \times \mathbb{R}^{1|1} \to \mathbb{R}^{1|1},\ (s, t, \theta) \mapsto (f_1(s) + t + f_2(s)\theta, f_2(s) + \theta)$$

Differentiation yields that this maps the vector fields ∂_t and ∂_θ to

$$\partial_t \quad \text{and} \quad -f_2(s)\partial_t + \partial_\theta.$$

Hence ∂_t is a left-invariant vector field. Solving the appropriate linear equation, one sees easily that the second left-invariant vector field is given by

$$D := -\theta\partial_t + \partial_\theta \quad \text{satisfying} \quad D^2 = \frac{1}{2}[D, D] = -\partial_t.$$

Hence we see that the Lie algebra of $\mathbb{R}^{1|1}$ is *freely* generated by one odd generator D.

This is the reason why $\mathbb{R}^{1|1}$ with the twisted super group structure plays a role for odd ODEs on super manifolds: An odd vector field $\xi \in \Gamma_{\text{odd}}(TM)$ determines a unique map from the super Lie algebra of $\mathbb{R}^{1|1}$ to vector fields on TM. This, in turn, generates an action of $\mathbb{R}^{1|1}$ of M. This action $M \times \mathbb{R}^{1|1} \to M$ is the flow of ξ. Hence the flow property for the flow of an odd vector field on a super manifold is expressed in terms of the twisted super group structure of $\mathbb{R}^{1|1}$.

All the subtleties regarding how long the flow is defined only take place in the reduced manifold. An important case of a flow that's always defined is when $[\xi, \xi] = 0$. Then one obtains an action of $\mathbb{R}^{0|1}$ on M. Conversely, any $\mathbb{R}^{0|1}$-action leads to an operator with square zero. An important example is the em odd tangent bundle πTM. By definition, the functions are just differential forms on M. Moreover, πTM turns out to be the super manifold of maps

$$\mathbb{R}^{0|1} \longrightarrow M$$

so it has an obvious action of $\mathbb{R}^{0|1}$ given by translation. This is the most conceptual interpretation of the deRham differential d. Note that $\mathbb{R}^\times$ also acts on the above space of maps, and it turns out that this leads to the grading on differential forms.

Supersymmetric classical mechanics. We briefly describe a supersymmetric variant of classical mechanics in which time $\mathbb{R}$ is replaced by $\mathbb{R}^{1|1}$. This should be thought of as a kind of 'super-time'. We denote coordinates on $\mathbb{R}^{1|1}$ by (s, η). The fields $\Phi(\mathbb{R}^{1|1})$ are defined to be morphisms of super manifolds $F : \mathbb{R}^{1|1} \to X$, where X is some configuration space. Let $D := \partial_\eta - \eta\partial_s$. We define the action functional by

$$A(F) = -\frac{1}{2}\int_{\mathbb{R}^{1|1}} \langle \partial_s F, DF\rangle \, ds d\eta.$$

For $Q = \partial_\eta + \eta\partial_s$ satisfies $[Q, D] = 0$, the vector field Q generates a supersymmetry on $\Phi(\mathbb{R}^{1|1})$. If X is a spin manifold then quantization of this classical field theory gives the L^2-spinors on X. Furthermore, the supersymmetry Q acts as the Dirac operator of X. The quantum 'super-time' evolution is given by $e^{-tD^2+\theta D}$. It should be possible to obtain the integral kernel of this operator as a super path integral analogous to the Feynman-Kac-fomula.

$\mathbb{Z}/2$**-graded vector spaces and super manifolds.** We conclude the section by describing linear infinite-dimensional super manifolds. More precisely, we describe what maps from usual finite-dimensional super manifolds into a $\mathbb{Z}/2$-graded Banach space $B = B_0 \oplus B_1$, considered as a super manifold, are. This exactly amounts to describing the S-points of B, i.e. the functor

$$B : \mathbf{Smfds} \longrightarrow \mathbf{Sets},\ S \mapsto B(S) = \text{" Hom}(S, B)\text{"}$$

Since morphisms are defined locally, it suffices to consider the case $S = U^{p|q}$ of super domains; the general case can be obtained from this by gluing. Let $x_1, ..., \theta_q$ be coordinates on U. A *morphism* $f \in B(U)$ is given by a finite sum

$$\sum_I f_I \theta^I, \text{ where } I \subset \{1, ..., q\},\ \theta^I := \prod_{j \in I} \theta^j, \text{ and the } f_I \text{ are smooth maps } |U| \to B_{|I|}.$$

If $\varphi : U' \to U$ is a map of super domains, the natural transformation $B(\varphi)$ is defined using the formal Taylor expansion as in the case of usual super manifolds. This defines B on super domains.

9. Supersymmetric quantum mechanics and K-theory

In this section we'll begin to explain the relation between the physical topics treated up to now and topology. The punchline is the following: The 'space' of supersymmetric quantum field theories of dimension $0 + 1$ (with $N = 1$ supersymmetry) is a model for the classifying space of K-theory. Let us remark that this is the case of a 0-dimensional, i.e. pointlike, space and a physicist would never call this a field theory because it treats particles rather than fields. From a mathematical point of view, the formalism is exactly the same, regardless of the dimension of space, so we continue our notation. However, we should point out that the theory below would be called "supersymmetric quantum mechanics" in the physics community.

We sum up the situation in the following diagram whose meaning we will explain below.

$$
\begin{array}{ccc}
\left\{\begin{array}{c}\text{closed Riemannian}\\ \text{spin}^c\ n\text{-manifolds}\end{array}\right\} & \xrightarrow{\textit{index of } \mathcal{D}} & \left\{\begin{array}{c}\mathbb{Z} \text{ if } n \text{ is even}\\ 0 \text{ if } n \text{ is odd}\end{array}\right. \\
\Big\downarrow {\scriptstyle \textit{classical susy mechanics}} & & \Big\uparrow \cong \\
\left\{\begin{array}{c}(0+1)\text{-dim. classical super-}\\ \text{symmetric field theories}\end{array}\right\} & & K_n(pt) \\
\Big\downarrow {\scriptstyle \textit{geometric quantization}} & & \Big\uparrow \cong \\
\left\{\begin{array}{c}(0+1)\text{-dim. susy quantum}\\ \text{field theories of degree } n\end{array}\right\} & \xrightarrow{\pi_0} & \pi_0(\mathrm{EFT}_n^{\mathbb{C}})
\end{array}
$$

The left side of the diagram comes from (susy) quantum mechanics as outlined in the previous sections. It turns out that the quantization of the supersymmetric classical mechanical system associated with a Riemannian spin^c n-manifold X can be expressed in terms of spinor bundles and Dirac operators, see [Wi]. The quantization we have in mind is a slightly different one: We want to consider the $\mathbb{C}l_n$-linear spinor bundle $\mathcal{S}_X$ over X (cf. [LM], chapter 2, §7). The Hilbert space of L^2-sections of this bundle is a $\mathbb{C}l_n$-module, and the canonically associated Dirac operator $\mathcal{D}$ on $L^2(\mathcal{S}_X)$ is $\mathbb{C}l_n$-linear. Using $\mathcal{D}$ we obtain a $(0+1)$-dimensional susy quantum field theory, EFT[2], *of degree* n by associating to the super time (t, θ) the operator $e^{-t\mathcal{D}^2+\theta\mathcal{D}}$.

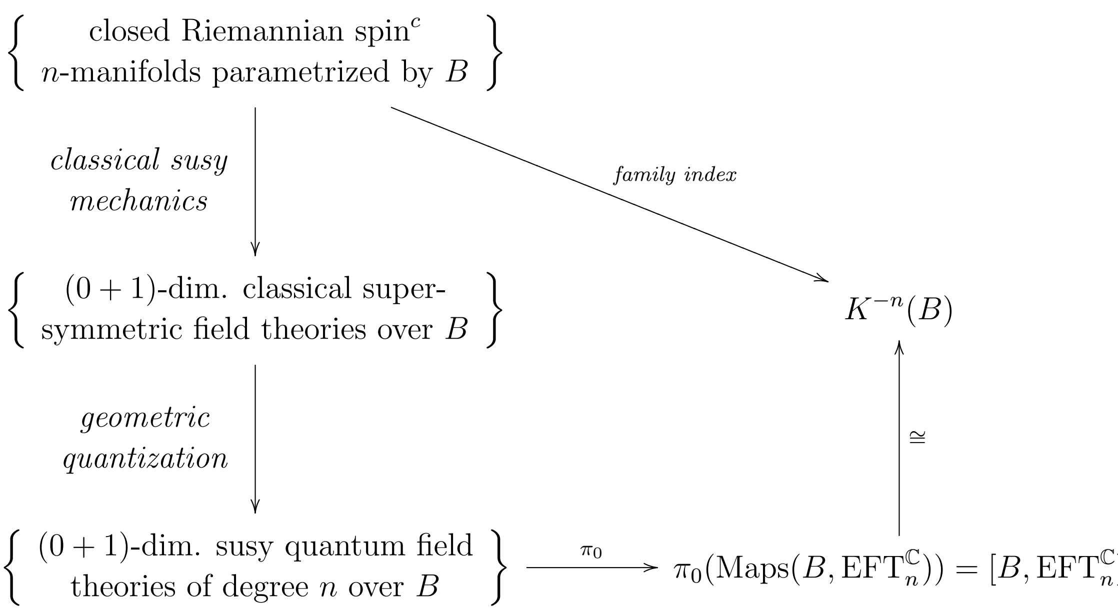
The *degree* n, i.e. the $\mathbb{C}l_n$-action, is important if one wants to ensure that the space of susy EFTs has the same homotopy type as a classifying space for the functor K^{-n}. Hence it would we very desirable to know a classical field theory (i.e. the appropriate Lagrangian) associated with a Riemannian spin^c manifold whose quantization gives the $\mathbb{C}l_n$-linear spinor bundle and Dirac operator.

The diagram indicates that following the arrows counter-clockwise starting on the top left yields the index of the operator $\mathcal{D}$. This is in fact the case, since the index of $\mathcal{D}$ can be computed as the super trace of $e^{-t\mathcal{D}^2}$ according to the Feynman-Kac formula.

[2]EFT is short for Euclidian field theory; the terminology refers to the missing i in the exponent, i.e. to the fact that our operators are not unitary but rather Hilbert-Schmidt operators

We have not yet explained what we mean by the 'space' of $(0+1)$-dim. susy quantum field theories of degree n. This is given a precise meaning on the right side: We will define the topological space $\mathrm{EFT}_n^{\mathbb{C}}$ of $(0+1)$-dimensional Euclidian field theories below.

There is also a real version of the above diagram: Replacing the complex Clifford algebras and Hilbert spaces in the game by their real analogues yields spaces $\mathrm{EFT}_n^{\mathbb{R}}$ that constitute a model for real K-theory. In fact, this is the mathematically more interesting case, and in the following we will pay more attention to it than to the complex variant. Finally, there is a version of the diagram in which everything happens over a parameter space B:



We will now give the definition of the spaces $\mathrm{EFT}_n^{\mathbb{F}}$, and in the next section we will prove that they form an Ω-spectrum representing K-theory. In particular, we shall prove that the vertical arrow on the right of the above diagram is indeed an isomorphism.

Denote by C_n the *Clifford algebra* associated with $\mathbb{R}^n$ equipped with its usual Euclidian inner product; this is the unital $\mathbb{R}$-algebra with n generators $e_1, \ldots, e_n$ satisfying the relations

$$e_i^2 = -1 \text{ for all } i \text{ and } e_i e_j = -e_j e_i \text{ if } i \neq j.$$

In the following, we will fix, for each $n \geq 0$, a separable Hilbert space H_n with an action of C_{n-1} such that each generator e_i acts as a bounded, skew-adjoint operator and such that each irreducible representation of C_{n-1} appears with infinite multiplicity. From this, we obtain a $\mathbb{Z}/2$-graded C_n-module

$$\mathcal{H}_n := H_n \otimes_{C_{n-1}} C_n,$$

where we embed C_{n-1} in C_n using the identification

$$C_{n-1} \xrightarrow{\cong} C_n^{ev},\ e_i \mapsto e_i e_n \text{ for } i = 1, ..., n-1.$$

Definition of EFT_n**.** The discussion in [ST], chapter 3, explains why it is reasonable to define a super symmetric Euclidian field theory of dimension $(0+1|1)$ and degree n as a super semi group homomorphism

$$\phi : \mathbb{R}^{1|1}_{>0} \longrightarrow \mathrm{HS}^{sa}_{C_n}(\mathcal{H}_n).$$

Here $\mathbb{R}^{1|1}_{>0}$ denotes the open sub super manifold of $\mathbb{R}^{1|1}$ defined by the inclusion $\mathbb{R}_{>0} \subset \mathbb{R}$. Note that the twisted super group structure on $\mathbb{R}^{1|1}$ defined in section 7 restricts to a multiplication μ on $\mathbb{R}^{1|1}_{>0}$. Furthermore, we interpret the $\mathbb{Z}/2$-graded real Hilbert space $\mathrm{HS}^{sa}_{C_n}(\mathcal{H}_n)$ (equipped with the Hilbert-Schmidt norm) as an infinite-dimensional super manifold, see the discussion at the end of section 7. Accordingly, a map $\phi : \mathbb{R}^{1|1}_{>0} \to \mathrm{HS}^{sa}_{C_n}(\mathcal{H}_n)$ is given by

$$A(t) + \theta B(t),\ \text{where } A : \mathbb{R}_{>0} \to \mathrm{HS}^{sa,ev}_{C_n}(\mathcal{H}_n) \text{ and } B : \mathbb{R}_{>0} \to \mathrm{HS}^{sa,odd}_{C_n}(\mathcal{H}_n)$$

are smooth maps. $\mathrm{HS}^{sa}_{C_n}(\mathcal{H}_n)$ has a super semi group structure coming from composition. It is defined by

$$\mathrm{HS}^{sa}_{C_n}(\mathcal{H}_n)(U) \times \mathrm{HS}^{sa}_{C_n}(\mathcal{H}_n)(U) \to \mathrm{HS}^{sa}_{C_n}(\mathcal{H}_n)(U),\ (A, B) \mapsto AB$$

for a super domain U. The homomorphism property of ϕ may be expressed as the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Hom}(U, \mathbb{R}^{1|1}_{>0}) \times \mathrm{Hom}(U, \mathbb{R}^{1|1}_{>0}) & \xrightarrow{\mu} & \mathrm{Hom}(U, \mathbb{R}^{1|1}_{>0}) \\ \downarrow{\scriptstyle \phi\times\phi} & & \downarrow{\scriptstyle \phi} \\ \mathrm{HS}^{sa}_{C_n}(\mathcal{H}_n)(U) \times \mathrm{HS}^{sa}_{C_n}(\mathcal{H}_n)(U) & \longrightarrow & \mathrm{HS}^{sa}_{C_n}(\mathcal{H}_n)(U). \end{array}$$

for all U.

Remark 58. The super manifold $\mathbb{R}^{1|1}_{>0}$ appearing here should be thought of as the *moduli super manifold* of 'Euclidian' super intervals. By this, we mean $(1|1)$-dimensional super manifolds (with boundary) equipped with a geometric structure that allows one to associate a 'super length' with such an interval. It turns out that the moduli super manifold of such intervals is $\mathbb{R}^{1|1}_{>0}$ (i.e. an interval is classified by its super length) and that gluing induces the twisted super semi group structure μ on $\mathbb{R}^{1|1}$.

Examples 59. The most important examples of EFTs are as follows.

(i) If $\mathcal{D}$ is the C_n-linear Dirac operator on a spin manifold X, then there is a corresponding field theory given by associating to the super time (t, θ) the operator

$$e^{-t\mathcal{D}^2 + \theta\mathcal{D}} = e^{-t\mathcal{D}^2} + \theta D e^{-t\mathcal{D}^2}.$$

(ii) More generally (and precisely), given any C_n-submodule $V_\infty \subset \mathcal{H}_n$ and any odd, densely defined, self-adjoint operator $\mathcal{D}$ on $V_\infty^\perp$ with compact resolvent, there is a unique super semi group homomorphism into the C^*-algebra of compact operators, self-adjoint and Clifford-linear

$$\phi = A + \theta B : \mathbb{R}^{1|1}_{>0} \longrightarrow K^{sa}_{C_n}(\mathcal{H}_n)$$

defined (using functional calculus) by

$$A(t) = e^{-t\mathcal{D}^2} \text{ and } B(t) = \mathcal{D}e^{-t\mathcal{D}^2} \text{ on } V_\infty^\perp$$

and $A(t) = B(t) = 0$ on V_∞. Checking that this is indeed a super semi group homomorphism is a nice exercise for the reader; the calculation can be found in [ST], page 38. $\mathcal{D}$ defines an EFT if and only if A and B are Hilbert Schmidt for all t. This is the case if the eigenvalues of $\mathcal{D}$ converge to ∞ sufficiently fast. For example, this is true for Dirac operators, see [LM], chapter 3. It is not hard to see that A and B are smooth with respect to the operator norm on $K(\mathcal{H}_n)$. In fact, if A and B are families of Hilbert-Schmidt operators, they are even smooth with respect to the Hilbert-Schmidt norm on $\mathrm{HS}(\mathcal{H}_n)$.

We now define the space $\mathrm{EFT}_n := \mathrm{EFT}_n^{\mathbb{R}}$ to be the space of super semi group homomorphisms $\mathbb{R}^{1|1}_{>0} \longrightarrow \mathrm{HS}^{sa}_{C_n}(\mathcal{H}_n)$. We endow it with the topology of pointwise convergence in A and B. Similarly, one can define spaces $\mathrm{EFT}_n^{\mathbb{C}}$ by replacing $\mathcal{H}_n$ by a complex Hilbert space that is a graded $\mathbb{C}l_n$-module. We can now state the main result:

Theorem 60. *The spaces $\mathrm{EFT}_n^{\mathbb{F}}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, constitute an Ω-spectrum representing real and complex K-theory, resp.*

We will prove this in the next section. In the remainder of this section we will give an interpretation of the spaces EFT_n in terms of configurations on the (compactified) real line indexed by subspaces of the Hilbert space $\mathcal{H}_n$.

Configurations spaces. Let $\mathcal{H}$ be a $\mathbb{Z}/2$-graded Hilbert space and (X, A), $A \subset X$, a pair of spaces equipped with an involution α. Define the *space of configurations* $\mathrm{Conf}(X, A)$ *over* (X, A) *indexed by subspaces of* $\mathcal{H}$ to be the space of maps $c : X \to \mathrm{Proj}(\mathcal{H})$ such that

- $c(x)$ is orthogonal to $c(y)$ if $x \neq y$.
- $\dim c(x) < \infty$ for all $x \in X \smallsetminus A$
- $\{ x \in X \smallsetminus A \mid c(x) \neq 0 \}$ is a discrete subset of $X \smallsetminus A$.
- $\mathcal{H}$ is equal to the Hilbert sum of the $c(x)$, $x \in X$.
- $c(s(x)) = \alpha(c(x))$ for all $x \in X$.

We consider finest topology on this set that allows the following things to happen:

- As long as the nonzero labels $x \in X$ don't collide, the corresponding subspaces $c(x)$ inherit their topology from that of the Graßmannian.

- If two (or more) labels x_i meet, the the associated spaces $c(x_i)$ add.

We will also need the subspace $\mathrm{Conf}^{\mathrm{fin}}(X, A) \subset \mathrm{Conf}(X, A)$ of *finite* configurations, where the word 'discrete' in the definition is replaced by 'finite'. Finally, if C is an $\mathbb{R}$-algebra and $\mathcal{H}$ is a C-module, we can replace subspaces of $\mathcal{H}$ by C-submodules in order to obtain spaces $\mathrm{Conf}_C(X, A)$. The main examples will be Clifford algebras $C = C_n$.

For fixed $\mathcal{H}$ the association $(X, A) \mapsto \mathrm{Conf}(X, A)$ is a functor: Given a continuous map $f : (X, A) \to (Y, B)$, there is an induced continuous map $\mathrm{Conf}(X, A) \to \mathrm{Conf}(Y, B)$. It is clear that such a map preserves the subspace of finite configurations.

We suppressed the Hilbert space $\mathcal{H}$ in our notation for the configuration spaces. In the following, it will be understood that, whenever there is a C_n in the notation, the configurations are indexed by subspaces of $\mathcal{H}_n$.

Proposition 61. *Let $\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$. For all n there is a homeomorphism*

$$\{ \text{ super semi group homomorhisms } \mathbb{R}^{1|1}_{>0} \to K^{sa}_{C_n}(\mathcal{H}_n) \} \cong \mathrm{Conf}_{C_n}(\bar{\mathbb{R}}, \infty),$$

where we endow the one-point compactification $\bar{\mathbb{R}}$ with the involution $s(x) = -x$.

Proof. We use the following technical lemma:

Lemma 62. *Let $A, B : \mathbb{R}_{>0} \to K^{sa}(H)$ be smooth families of self-adjoint compact operators on the Hilbert space H, and assume that the following relations hold for all $s, t > 0$:*

$$A(s+t) = A(s)A(t) \tag{1}$$

$$B(s+t) = A(s)B(t) \tag{2}$$

$$A'(s+t) = -B(s)B(t) \tag{3}$$

Then H decomposes uniquely into orthogonal subspaces H_λ, $\lambda \in \mathbb{R} \cup \{\infty\}$, such that on H_λ

$$A(t) = e^{-t\lambda^2} \text{ and } B(t) = \lambda e^{-t\lambda^2}$$

(where we set $e^{-\infty} = 0$, $\infty \cdot 0 = 0$). For $\lambda \in \mathbb{R}$ the dimension of H_λ is finite; furthermore, the subset of $\mathbb{R}$ consisting of $\lambda \in \mathbb{R}$ with $H_\lambda \neq 0$ is discrete.

Proof. The identities (1) and (2) show that all operators $A(s)$, $B(t)$ commute. We apply the spectral theorem for self-adjoint compact operators to obtain a decomposition of H into simultaneous eigenspaces H_λ of the $A(s)$ and $B(t)$; the label λ takes values in $\mathbb{R} \cup \{\infty\}$ and will be explained presently. We define functions $A_\lambda, B_\lambda : \mathbb{R}_{>0} \to \mathbb{R}$ by

$$A(t)x = A_\lambda(t)x \text{ and } B(t)x = B_\lambda(t)x \text{ for all } x \in H_\lambda$$

Clearly, A_λ and B_λ are smooth and satisfy the same relations (1) - (3) as A and B.

From (1) we see that A_λ is non-negative, and (3) shows $A'_\lambda \leq 0$, i.e. A_λ is decreasing. On the other hand, (1) implies $A_\lambda(\frac{1}{n}) = \sqrt[n]{A_\lambda(1)}$, so that

$$A_\lambda(0) := \lim_{t\to 0} A_\lambda(t) \text{ exists and equals 0 or 1.}$$

In the first case we conclude $A_\lambda \equiv 0$ and thus also $B_\lambda \equiv 0$; the label of the corresponding subspace is $\lambda = \infty$. In the second case, we have $A_\lambda(1) \neq 0$ and using (1) again we compute

$$A'_\lambda(s) = \frac{A_\lambda(1)}{A_\lambda(1)} \lim_{t\to 0} \frac{A_\lambda(s+t) - A_\lambda(s)}{t} = \frac{A_\lambda(s)}{A_\lambda(1)} \lim_{t\to 0} \frac{A_\lambda(1+t) - A_\lambda(1)}{t} = -\lambda^2 A_\lambda(s),$$

where $\lambda^2 := -A'_\lambda(1)/A_\lambda(1)$. Because solutions of ODEs are unique, we must have

$$A_\lambda(t) = e^{-t\lambda^2}.$$

Finally, (3) gives

$$B_\lambda(t) = \sqrt{\lambda^2 e^{-2t\lambda^2}} = \lambda e^{-t\lambda^2},$$

picking the appropriate sign for the label λ. □

Now, given a super semi group homomorphism $\phi = A + \theta B$, we want to exploit the homomorphism property of ϕ for $U = \mathbb{R}^{0|2}$. Let θ, η be the usual odd coordinates on $\mathbb{R}^{0|2}$ and let $s, t \in \mathbb{R}_{>0}$, considered as constant (even) functions on $\mathbb{R}^{0|2}$. We then have

$$\begin{aligned}
\phi(s+t+\eta\theta, \eta+\theta) &= A(s+t+\eta\theta) + (\eta+\theta)B(s+t+\eta\theta)\\
&= A(s+t) + A'(s+t)\eta\theta + (\eta+\theta)(B(s+t) + B'(s+t)\eta\theta)\\
&= A(s+t) + \eta B(s+t) + \theta B(s+t) + \eta\theta A'(s+t)
\end{aligned}$$

which equal to

$$\begin{aligned}
\phi(s,\eta)\phi(t,\theta) &= (A(s) + \eta B(s))(A(t) + \theta B(t)\\
&= A(s)A(t) + \eta B(s)A(t) + \theta A(s)B(t) - \eta\theta B(s)B(t).
\end{aligned}$$

Comparing the coefficients[3] yields exactly the relations in lemma 62 which immediately gives the desired decomposition of $\mathcal{H}_n$. Furthermore, since the operators $B(t)$ are odd, we have

$$\alpha B(t)\alpha = -B(t) \text{ or } \alpha(H_\lambda) = H_{-\lambda},$$

i.e. the have obtained a configuration on $(\bar{\mathbb{R}}, \infty)$. Finally, it is clear that in the C_n-linear case the spaces H_λ are C_n-submodules of $\mathcal{H}_n$, so that we obtain an element in $\mathrm{Conf}_{C_n}(\bar{\mathbb{R}}, \infty)$ associated with ϕ. Conversely, every C_n-linear configuration $\{V_\lambda\}$ defines an odd, self-adjoint operator $\mathcal{D}$ as in example 59 (ii) and thus a super semi group homomorphism $\mathbb{R}^{1|1}_{>0} \to K^{sa}_{C_n}(\mathcal{H}_n)$. It is not hard to check that this bijection is a homeomorphism. □

[3]Just to make the formal aspect of this computation clearer we would like to point out that the considered identity is an equation in the algebra $\mathrm{HS}^{sa}_{C_n}(\mathcal{H}_n)(\mathbb{R}^{0|2}) = \mathrm{HS}^{sa}_{C_n}(\mathcal{H}_n)[\theta, \eta]$.

Let $\mathrm{Conf}_n := \mathrm{Conf}^{\mathrm{fin}}_{C_n}(\bar{\mathbb{R}}, \infty)$.

Corollary 63. *We have a homotopy equivalence*

$$\mathrm{EFT}_n \simeq \mathrm{Conf}_n\,.$$

Proof. Pick $K > 0$ and an increasing continuous map $h_1 : \bar{\mathbb{R}} \xrightarrow{\cong} \bar{\mathbb{R}}$ such that $h|_{(-K,K)}$ is a homeomorphism onto $\mathbb{R}$ and $h|_{(-K,K)^c} = \infty$. Clearly, there is a (linear) homotopy h_t from h_1 to the identity h_0. Applying the functoriality of configuration spaces we see that h_1 is a homotopy equivalence from $\mathrm{Conf}_{C_n}(\bar{\mathbb{R}}, \infty)$ to Conf_n. Now, using the proposition we can interpret EFT_n as a subspace of $\mathrm{Conf}_{C_n}(\bar{\mathbb{R}}, \infty)$. From the construction it is clear that the maps h_t preserve the subspace EFT_n for all times t, and hence the restriction of h_1 to EFT_n yields a homotopy equivalence $\mathrm{EFT}_n \simeq \mathrm{Conf}_n$. □

10. EFTs and spaces of Fredholm operators

The goal of this section is to prove the isomorphisms

$$K^{-n}(B) \cong [B, \mathrm{EFT}^{\mathbb{C}}_n] \text{ and } KO^{-n}(B) \cong [B, \mathrm{EFT}^{\mathbb{R}}_n]$$

that appeared in the diagram in the last section and its real analogue. More precisely, we will show that the spaces $\mathrm{EFT}^{\mathbb{F}}_n$ form an Ω-spectrum representing complex and real K-theory, respectively. This will be accomplished by comparing $\mathrm{EFT}^{\mathbb{F}}_n$ to certain spaces of Fredholm operators $\mathcal{F}^{\mathbb{F}}_n$ introduced by Atiyah and Singer in [AS], where they also show that these spaces form an Ω-spectrum representing K-theory. We begin with some introductory material on Ω-spectra and K-theory. We then turn to the Atiyah-Singer spaces and construct a homotopy equivalence between $\mathcal{F}^{\mathbb{F}}_n$ and $\mathrm{EFT}^{\mathbb{F}}_n$ using the Dold-Thom theory of quasi-fibrations.

Generalized cohomology theories and Ω-spectra. Let h^* be a generalized cohomology theory. Recall that by Brown's representation theorem h^* can be represented by an Ω-spectrum $(E_n, h_n)_{n\in\mathbb{Z}}$, i.e.

$$h^n(X) = [X, E_n] \text{ for all spaces } X \text{ and } n \in \mathbb{Z}.$$

If h^* is multiplicative, the graded ring structure on $h^*(X)$ is given by

$$E_m \wedge E_n \xrightarrow[\mu_{m,n}]{} E_{m+n},$$

the identity element comes from a map $\iota : S^0 \to E_0$, and the suspension map is induced by $\sigma : S^1 \to E_1$ such that under the suspension-loop adjunction

$$S^1 \wedge E_n \xrightarrow[\sigma\wedge\mathrm{id}]{} E_1 \wedge E_n \xrightarrow[\mu_{1,n}]{} E_{n+1} \text{ corresponds to } h_n : E_n \xrightarrow{\simeq} \Omega E_{n+1}$$

Given an Ω-spectrum E we have associated (co)homology theories

$$E^n(X) = [X, E_n] \text{ and } E_n(X) = \pi_n(E \wedge X) = \lim_{n\to\infty} \pi_{n+r}(E_r \wedge X).$$

It follows in particular that $\pi_n(E_0) = \pi_0(E_{-n})$.

K-theory and spectra representing it. The K-group $K^0_{\mathbb{F}}(X)$ associated with a compact space X is the Grothendieck group of the semi group of $\mathbb{F}$-vector bundles over X with respect to Whitney sums. In fact, $K^0_{\mathbb{F}}(X)$ is a commutative ring with multiplication coming from the tensor product of vector bundles. $K^0_{\mathbb{F}}$ is a contravariant functor satisfying the homotopy and exactness axioms of Brown's representation theorem. Hence there exists a classifying space E_0 for $K^0_{\mathbb{F}}$. Note that the functor $K^0_{\mathbb{F}}$ extends to a cohomology theory if and only if there exist 'deloopings' E_n of E_0, i.e. spaces E_n such that $\Omega^n E_n \simeq E_0$. In other words, if E_0 is an infinite loop space. This is, for example, the case if $E_0 = \Omega^k E_0$ for some k; the corresponding cohomology theory is then automatically k-periodic. In the case of K-theory E_0 is of this type as follows from the Bott periodicity theorem:

Theorem 64 (Bott)**.** *For all X we have*

$$\tilde{K}^0_{\mathbb{C}}(\Sigma^2 X) \cong \tilde{K}^0_{\mathbb{C}}(X) \text{ and } \tilde{K}^0_{\mathbb{R}}(\Sigma^8 X) \cong \tilde{K}^0_{\mathbb{R}}(X).$$

Equivalently,

$$E^{\mathbb{C}}_0 \simeq \Omega^2 E^{\mathbb{C}}_0 \text{ and } E^{\mathbb{R}}_0 \simeq \Omega^8 E^{\mathbb{R}}_0.$$

Often, one chooses E_0 to be $\mathbb{Z}$ cross an infinite Graßmannian. In the next section we introduce a more geometric model for E_0, namely the space of Fredholm operators on a Hilbert space H.

Fredhom operators. Recall that a *Fredholm operator* $T : H_1 \to H_2$ is a bounded operator whose kernel and cokernel are finite dimensional. Using the operator norm we make the set of Fredhom operators into a topological space. If $H_1 = H_2 = H$ the space $\mathrm{Fred}(H) \subset B(H)$ is exactly the preimage of the units in the *Calkin algebra* $C(H) := B(H)/K(H)$ of bounded modulo compact operators under the projection $c : B(H) \to C(H)$. In other words,

$$A \text{ is Fredholm } \iff c(A) \in C(H) \text{ is invertible.}$$

The most important invariant of a Fredholm operator T is its *index*

$$\mathrm{index}(T) := \dim(\mathrm{kernel}\, T) - \dim(\mathrm{cokernel}\, T).$$

It turns out that the index is invariant under deformations, i.e. it is a locally constant function on $\mathrm{Fred}(H)$, and that it detects the connected component of $T \in \mathrm{Fred}(H)$, see below.

Elliptic differential operators and Fredholm operators. Interesting examples of Fredholm operators arise from elliptic differential operators on vector bundles. Let $E_i \to X$, $i = 1, 2$, be vector bundles over the Riemannian manifold X.

Definition 65. (i) *A linear map $P : \Gamma(E_1) \to \Gamma(E_2)$ is a* differential operator *if P is* local*, i.e. $P(s)(x)$ depends only on $s|_U$ for any neighborhood U of x. By a result of Petree this is equivalent to saying that P can in local coordinates be written as*

$$P(x) = \sum_{|\alpha| \leq m} A_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha},$$

where A_α is a matrix-valued function on X. We assume that m is minimal in the sense that A_α is non-trival for some α with $|\alpha| = m$. The number m is called the degree *of P.*

(ii) *The* principal symbol *of P is given by a map of vector bundles*

$$\sigma_\xi(P) : E_1 \longrightarrow E_2 \text{ for each } \xi \in \Omega^1(X)$$

that is locally given by

$$\sigma_\xi(P)(x) := \sum_{|\alpha| \leq m} A_\alpha(x) \xi^\alpha(x).$$

P is elliptic *if $\sigma_\xi(P)(x)$ is invertible whenever $\xi(x) \neq 0$.*

Examples 66. (i) Let $E_i = X \times \mathbb{R}$. The *Laplace operator*

$$P = \Delta := d^* d$$

is an elliptic differential operator that is locally given by

$$\Delta(f) = \sum_{i,j} g_{ij} \frac{\partial f}{\partial x_i \partial x_j} + \text{ lower order terms.}$$

The principal symbol is locally given by

$$\sigma_\xi(\Delta)(x) = \sum_{i,j} \xi^i(x) \xi^j(x) = ||\xi(x)||^2$$

which immediately implies the ellipticity of Δ.

(ii) The Dirac operator D on the spinor bundle of a spin manifold is a differential operator of degree 1. Its principal symbol $\sigma_\xi(D)(x)$ is given by Clifford multiplication by ξ; thus D is elliptic.

The *Sobolev s-norm* of a section $\Gamma(E)$ is defined by

$$||u||_s^2 = \sum_{j=0}^{s} \int_X |\nabla ... \nabla u|^2,$$

where the covariant derivative ∇ is applied j-times. From a differential operator one obtains Fredholm operators using the following

Theorem 67. (i) *A differential operator P of degree m extends to a* bounded *linear map*

$$P_s : L^2_s(E_1) \longrightarrow L^2_{s-m}(E_2) \text{ for all } s \geq m.$$

(ii) *If P is elliptic, P_s is a Fredholm operator.*

(iii) *For all $s \geq m$*

$$\operatorname{index}(P_s) = \operatorname{index}(P) := \dim\ker(P) - \dim\ker(P^*).$$

Let us now return to the relation between Fredholm operators and K-theory.

Fredholm operators and the functor $K^0_{\mathbb{F}}$. The connection between the space of Fredholm operators $\mathrm{Fred}(H_{\mathbb{F}})$ on the separably infinite-dimensional Hilbert space $H_{\mathbb{F}}$ over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and the functor $K^0_{\mathbb{F}}$ is given by

Theorem 68 (Atiyah, Palais, Jänich)**.** $\mathrm{Fred}(H_{\mathbb{F}})$ *is a classifying space for the functor $K^0_{\mathbb{F}}$, i.e. for all compact spaces X we have*

$$K^0_{\mathbb{F}}(X) \cong [X, \mathrm{Fred}(H_{\mathbb{F}})].$$

The isomorphism in the theorem is defined as follows: Given a class $\alpha \in [X, \mathrm{Fred}(H_{\mathbb{F}})]$, one can always find a representative f such that the dimensions of the kernel and the cokernel of $f(x)$ are locally constant. This implies that kernel $f(x)$ and cokernel $f(x)$ are vector bundles over X, and we define the image of α to be

$$[\operatorname{kernel} f(x)] - [\operatorname{cokernel} f(x)] \in K^0_{\mathbb{F}}(X).$$

In the case $X = pt$ the theorem gives an isomorphism

$$\pi_0(\mathrm{Fred}(H_{\mathbb{F}})) \cong K^0_{\mathbb{F}}(pt) \cong \mathbb{Z},$$

and from the proof of the theorem it is clear that it is given by sending $[T] \in \pi_0 \mathrm{Fred}(H_{\mathbb{F}})$ to the index of T.

The other spaces E_n in the Ω-spectrum representing K-theory can also be constructed using spaces of Fredholm operators. This will be explained in the next subsection.

The Atiyah-Singer spaces $\mathcal{F}^{\mathbb{F}}_n$. From now on we will restrict our attention to the real case. We will only define the spaces $\mathcal{F}_n := \mathcal{F}^{\mathbb{R}}_n$ and prove the main theorem in this case. The complex case is similar, and the interested reader can certainly work it out after looking up the definition of $\mathcal{F}^{\mathbb{C}}_n$ in [AS].

Let H_n be a real Hilbert space with an action of C_{n-1} as in the last section. Now let

$$\mathcal{F}_n := \{\, T \in \mathrm{Fred}(H_n) \mid T^* = -T \text{ and } Te_i = -e_iT \text{ for } i = 1, ..., n-1 \,\}.$$

If $n \equiv 3\,(4)$ we require the operators $T \in \mathcal{F}_n$ to satisfy the following additional condition: The essential spectrum of the self-adjoint operator $e_1 ... e_{n-1}T$ contains positive *and* negative values ('$e_1 ... e_{n-1}T$ is neither essentially positive nor negative'). The reason we need to impose this condition is that for $n \equiv 3\,(4)$ the space $\mathcal{F}_n(H_n)$, if defined without the additional condition, has three connected components two of which are contractible. However, we are only interested in the third component whose elements can be characterized by the above requirement on the essential spectrum of $e_1 ... e_{n-1}T$. We remind the reader of the definition of the *essential spectrum* of a self-adjoint operator $T \in B(H)$: There is a decomposition of the spectrum

$$\sigma(T) = \{\, \lambda \in \mathbb{C} \mid \lambda I - T \text{ is not invertible} \,\}$$

into two parts,

$$\sigma(T) = \sigma_{\text{discrete}}(T) \amalg \sigma_{\text{ess}}(T),$$

where $\sigma_{\text{discrete}}(T)$ consists of the discrete points in $\sigma(T)$ such that the corresponding eigenspace has finite dimension. A second, equivalent, way to define $\sigma_{\text{ess}}(T)$ is via the equality

$$\sigma_{\text{ess}}(T) = \sigma(c(T)),$$

where $c(T)$ is the image of T in the Calkin algebra (which is a C^*-algebra and hence every element has a well defined spectrum). The main result in [AS] can be formulated as follows:

Theorem 69. *The spaces $\mathcal{F}_n$ constitute an Ω-spectrum representing real K-theory.*

Remark 70. We want to explain how this result together with the classification of Clifford algebras implies Bott periodicity. We begin by some general remarks about Morita equivalence.

Denote by $\mathcal{C}$ the category whose objects are rings and in which the morphism set $\mathcal{C}(R, S)$ is given by isomorphism classes of R-S-bimodules. The composition of two bimodules ${}_RM_S$ and ${}_SN_T$ given by their tensor product over S. The identity morphism $R \to R$ is R considered as a bimodule over itself. Two rings are called *Morita equivalent* if they are isomorphic in $\mathcal{C}$. More explicitly, R and S are Morita equivalent if and only if there are bimodules ${}_RM_S$ and ${}_SN_R$ such that

$${}_RM_S \underset{S}{\otimes} {}_SN_R \cong R \text{ and } {}_SN_R \underset{R}{\otimes} {}_RN_S \cong S.$$

For example, taking M and N to be R^n shows that the ring of $n \times n$-matrices with entries in R is Morita equivalent to R.

Lemma 71. *If R and S are Morita equivalent, then the categories $\mathbf{Mod}_R$ and $\mathbf{Mod}_S$ of R and S left modules are equivalent.*

Proof. We have isomorphisms $M : R \overset{\cong}{\leftrightarrow} S : N$ as above. Define

$$\mathbf{Mod}_R \longrightarrow \mathbf{Mod}_S,\ P \mapsto {}_SN_R \underset{R}{\otimes} {}_RP,$$

and similarly $\mathbf{Mod}_S \to \mathbf{Mod}_R$ by tensoring with M. It is clear that the composition of these two functors is naturally equivalent to the identity functors on $\mathbf{Mod}_R$ and $\mathbf{Mod}_S$. □

Now, using the lemma and $C_{n+8} \cong M_{16}(C_n)$ (see e.g. [LM], chapter 1, §4) we see, in particular, that $\mathcal{F}_{n+8} \cong \mathcal{F}_n$. Here we also used that $H_{n+8} \cong H_n \otimes_{C_n} C_{n+8}$ which follows since each irreducible representation of C_k appears infinitely often in H_k. In a similar fashion one can deduce from the complex version of the Atiyah-Singer theorem that complex K-theory has period 2.

Using the result of Atiyah and Singer we see that our main theorem is implied by the following homotopy equivalence whose proof comprises the remainder of this section. Note that the annoying condition for $n \equiv 3 \mod 4$ does not come up in the definition of our spaces EFT_n.

Theorem 72. *For $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and all $n \geq 0$, there are homotopy equivalences*

$$\mathrm{EFT}_n^{\mathbb{F}} \simeq \mathcal{F}_n^{\mathbb{F}}.$$

As said before, we will only deal with the case $\mathbb{F} = \mathbb{R}$. The first part of the proof consists of showing that the spaces $\mathcal{F}_n$ are homotopy equivalent to certain spaces of configurations. In the second part, we will use Dold-Thom theory to relate these configuration spaces to the configuration spaces that appeared in connection with the spaces EFT_n.

Interpreting the Atiyah-Singer spaces in terms of configurations.

Fact 73. Let $T \in \mathrm{Fred}(\mathcal{H})$. Then

$$\sigma_{ess}(T) \cap (-\varepsilon(T), \varepsilon(T)) = \emptyset \text{ for } \varepsilon(T) = ||c(T)^{-1}||_{C(\mathcal{H})}^{-1}.$$

Here $||.||_{C(\mathcal{H})}$ is the C^*-norm on the Calkin algebra. In other words: The essential spectrum of T has a gap of size at least $\varepsilon(T)$ around 0. Note that $\varepsilon(T)$ depends continuously on T.

Proof. This follows directly from the characterization of the essential spectrum as the spectrum of $c(T)$ in $C(\mathcal{H})$. □

Now we can express $\mathcal{F}_n$ as a configuration space. Let $\tilde{R} := [\infty, \infty]$. We have

$$\begin{aligned} \tilde{\mathcal{F}}_n &\cong \mathrm{Fred}_{C_n}^{odd,sa}(\mathcal{H}_n) \\ &\simeq \mathrm{Conf}_{C_n}^{\mathrm{fin}}(\tilde{\mathbb{R}}, \{\pm\infty\}) \end{aligned}$$

Here $\tilde{\mathcal{F}}_n$ is the same as $\mathcal{F}_n$ if n is not congruent 3 mod 4 and if $n \equiv 3$ mod 4, it denotes the full space of Fredholm operators defined above with the additional condition omitted.

Hence, for all n we have identified $\mathcal{F}_n$ with a connected component $\mathrm{Conf}_n^{\pm\infty}$ of the space $\mathrm{Conf}^{\mathrm{fin}}_{C_n}(\tilde{\mathbb{R}}, \{\pm\infty\})$, and only if $n \equiv 3 \bmod 4$ this component is a proper subspace.

The first map is a homeomorphism and given by sending an operator T to $\hat{T} := T \otimes e_n$. If we decompose $\mathcal{H}_n$ w.r.t. the grading, $\mathcal{H}_n \cong H_n \oplus H_n$, we have

$$\hat{T} \cong \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}.$$

From this it is clear that $\hat{T}$ is odd and self-adjoint.

The second map is the homotopy equivalence given by pushing the spectrum of $\hat{T}$ outside of $[\frac{-\varepsilon(\hat{T})}{2}, \frac{\varepsilon(\hat{T})}{2}]$ to $\pm$infinity. One way to make this precise is to rescale $\hat{T}$ to have norm one and then to apply functional calculus using a function $[-1,1] \to [-1,1]$ that restricts to a homeomorphism $[\frac{-\varepsilon(\hat{T})}{2}, \frac{\varepsilon(\hat{T})}{2}] \xrightarrow{\cong} [-1,1]$ and is equal to ± 1 on $[-1, \frac{-\varepsilon(\hat{T})}{2}]$ and $[\frac{\varepsilon(\hat{T})}{2}, 1]$, resp. This gives a homotopy equivalence to the configurations space $\mathrm{Conf}^{\mathrm{fin}}_{C_n}([-1,1], \{\pm 1\})$ Now use the obvious homeomorphism $([-1,1], \{\pm 1\}) \cong (\tilde{\mathbb{R}} \cup, \{\pm\infty\})$. We'd like to point out that for the continuity of this map the continuous dependence of the size $\varepsilon(\hat{T})$ of the spectral gap is crucial.

The Dold-Thom theory of quasi-fibrations. The next ingredient in the proof is the Dold-Thom theory of quasi-fibrations, see [DT]. The basic notion is

Definition 74. *A map $p : E \twoheadrightarrow B$ is a* quasi-fibration *if for all $b \in B$, $i \in \mathbb{N}$, and $e \in p^{-1}(b)$ p induces an isomorphism*

$$\pi_i(E, p^{-1}(b), e) \xrightarrow{\cong} \pi_i(B, b).$$

From the long exact sequence of homotopy groups for a pair it follows that p is a quasi-fibration exactly if there is a long exact homotopy sequence connecting fibre, total space and base space of p, just like for a fibration. However, p does not need to have any (path) lifting properties as the following example shows.

Example 75. The prototypical example of a quasi-fibration that's not a fibration is the projection of a 'step'

$$(-\infty, 0] \times \{0\} \cup \{0\} \times [0,1] \cup [0, \infty) \times \{1\} \subset \mathbb{R}^2$$

onto the x-axis. Even though all fibers have the same homotopy type (they are contractible), the map doesn't have the lifting property of a fibration, since it is impossible to lift a path that passes through the origin.

The following sufficient condition for a map to be a quasi-fibration is proved in [DT]:

Theorem 76. *The map $p : E \twoheadrightarrow B$ is a* quasi-fibration *if there exists a filtration*

$$F_0 \subset F_1 \subset F_2 \subset \ldots \text{ of } B \text{ such that}$$

(i) *For all i the restriction $p|_{F_i \setminus F_{i-1}}$ is a fibration.*
(ii) *For all i there exists a neighborhood N_i of F_i in F_{i+1} and a homotopy h on N_i such that $h_0 = \mathrm{id}$ and $h_1(N_i) \subset F_i$.*
(iii) *h is covered by $H : p^{-1}(N_i) \times I \to p^{-1}(N_i)$ with $H_0 = \mathrm{id}$ and for all*

$$x \in N_i \text{ we have } H_1(p^{-1}(x)) \subset p^{-1}(h_1(x))$$

Conclusion of the proof of theorem 72 (and thus of theorem 60). We have already shown that

$$\mathrm{EFT}_n \simeq \mathrm{Conf}_n := \mathrm{Conf}^{\mathrm{fin}}_{C_n}(\bar{\mathbb{R}}, \infty) \text{ and } \mathcal{F}_n \simeq \mathrm{Conf}^{\pm\infty}_n \subseteq \mathrm{Conf}^{\mathrm{fin}}_{C_n}(\tilde{\mathbb{R}}, \{\pm\infty\})$$

where $\tilde{\mathbb{R}} := [-\infty, +\infty]$ is the two-point compactification of $\mathbb{R}$ and $\bar{\mathbb{R}}$ is the one-point compactification. Recall that the right hand inclusion is an equality unless $n \equiv 3 \mod 4$. The obvious map $\tilde{\mathbb{R}} \to \bar{\mathbb{R}}$ that is the identity on $\mathbb{R}$ and maps $\pm\infty$ to ∞ induces a map

$$p : \mathrm{Conf}^{\mathrm{fin}}_{C_n}(\tilde{\mathbb{R}}, \{\pm\infty\}) \longrightarrow \mathrm{Conf}_n .$$

We *claim* that, when restricted to $\mathrm{Conf}^{\pm\infty}_n$, p is a quasi-fibration with contractible fiber and hence a homotopy equivalence. This follows from the Dold-Thom theorem and Whitehead's theorem together with the fact that the spaces involved have the homotopy type of CW-complexes. The full map p makes sense without restricting to $\mathrm{Conf}^{\pm\infty}_n$ but if $n \equiv 3 \mod 4$ it is *not* a quasi-fibration as well shall see below (the fibres have distinct homotopy groups).

Let us now prove the claim. We begin by computing the fiber of p over a configuration $c \in \mathrm{Conf}_n$. We have

$$p^{-1}(c) = \text{ space of decompositions of } V_\infty := c(\infty) \text{ as } V_\infty = V \perp \alpha V,$$

where α is the grading involution on $\mathcal{H}_n$. If $\tilde{c} \in p^{-1}(c)$ then we may define $V := \tilde{c}(-\infty)$ and, vice versa, given V an element in $p^{-1}(c)$ is determined by this formula. This implies that $p^{-1}(c)$ is homeomorphic to the space

$$\{ \beta : V_\infty \to V_\infty \; C_n\text{-linear} \mid \beta^2 = \mathrm{id},\ \beta = \beta^*,\ \alpha\beta = -\beta\alpha \}$$

The matrix representation of β with respect to the decomposition $V_\infty = V^{ev}_\infty \oplus V^{odd}_\infty$ is of the form

$$\beta = \begin{pmatrix} 0 & \beta_0^* \\ \beta_0 & 0 \end{pmatrix},$$

where β_0 is orthogonal, C^{ev}_n-linear and with

$$\beta_1 := e \circ \beta_0, \quad e := e_1 \cdot e_2 \cdots e_n \in C^{odd}_n.$$

a *self-adjoint* operator on V^{ev}_∞. Here we assume that $n \equiv 3 \mod 4$, otherwise those subtleties do not appear. We conclude that the fiber $p^{-1}(c)$ is homeomorphic to the space of C^{ev}_n-linear orthogonal (and self-adjoint) involutions

$$\beta_1 : V^{ev}_\infty \longrightarrow V^{ev}_\infty$$

If we intersect this fibre with $\mathrm{Conf}_n^{\pm\infty}$ then we are sure that the ± 1 Eigenspaces of all the β_1 in question are infinite dimensional. Hence (the C_n^{ev}-linear version of) Kuiper's theorem on the contractibility of the orthogonal group of a separable Hilbert space shows that the fibers of p are contractible.

To complete the proof we have to show that p is indeed a quasi-fibration. This follows from theorem 76; we only outline the argument. The filtration F_i is defined by

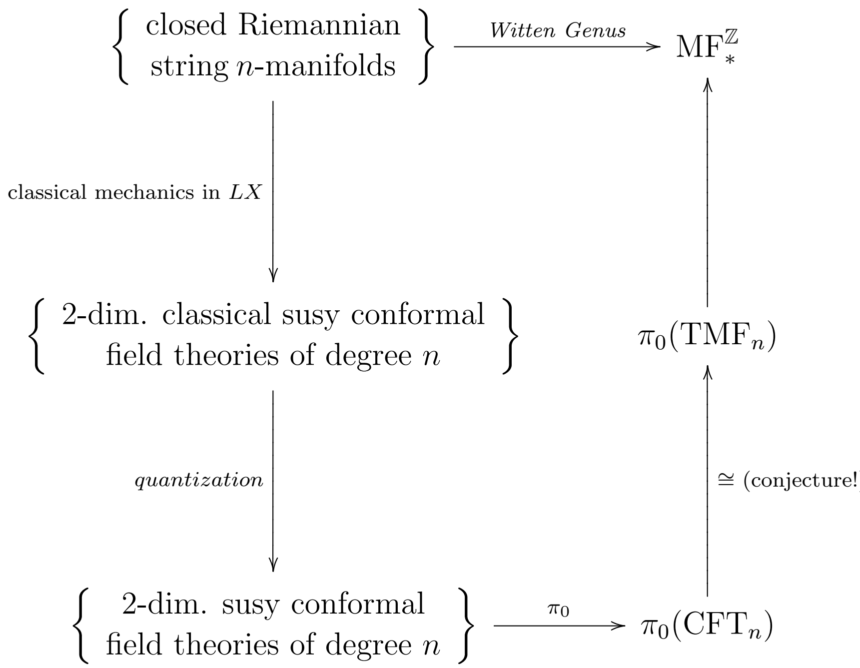
$$F_i := \{ c \in \mathrm{Conf}_n \mid \dim(\oplus_{x\in\mathbb{R}} c(x)) \leq 2i \}.$$

The neighborhoods N_i consist of configurations $c \in F_{i+1}$ such that $c(x) \neq 0$ for exactly one $x \in \mathbb{R}_{>1}$ with $\dim c(x) = 1$. The map H_1 is the inclusion of a smaller unitary group into a bigger one. This completes the proof of theorem 60. □

There is yet another, quite simple, relationship between our spaces EFT_n and the Milnor spaces Ω_n introduced in [Mi] (which also represent K-theory as an Ω-spectrum). It turns out that space of finite rank EFTs of degree n, Conf_n, is actually *homeomorphic* to Ω_{n-1} for $n \geq 1$.

11. Conformal field theories and topological modular forms

We now turn to the 2-dimensional case. The main idea is that there should be a close relationship between the space of susy CFTs of degree n and the $-n^{\text{th}}$ space in the spectrum TMF of topological modular forms, i.e. the universal 'elliptic' cohomology theory constructed by Hopkins and Miller. In this context the index of the Dirac operator is replaced by the Witten genus which should be thought of as the index of the S^1-equivariant Dirac operator on the loop space of a string manifold X. The situation is illustrated by the diagram



Let us explain the components of the diagram more in detail.

String manifolds, the Witten genus, and integral modular forms. The *Witten genus* w is a genus (in the sense of Hirzebruch) with values in the power series ring $\mathbb{R}[[q, \bar{q}]]$ ($\mathbb{Q}$?). It is defined for all smooth manifolds X; however, it is most interesting to consider its properties for manifolds satisfying higher orientability conditions. For example, if X is spin, the coefficients of $w(X)$ are integers. Moreover, if X is *string*, i.e. X is spin and the characteristic class $\frac{p_1}{2}(X)$ is zero, then no $\bar{q}$'s appear. In fact, in this case $w(X)$ is the q-expansion of a modular form. For Witten's interpretation of $w(X)$ as the index of the S^1-equivariant Dirac operator on the loop space LX, see [Wi].

Classical mechanics on LX and conformal field theory. Consider a 2-dimensional space-time M, a target X, and the fields $\Phi(M) := C^\infty(M, X)$. The classical action given by the 'kinetic energy' is

$$(\sigma : M \to X) \mapsto \int_M ||D\sigma||^2 \, \mathrm{vol}_M.$$

This action depends only on the conformal structure on M and hence defines a classical *conformal* field theory.

In the case $M = S^1 \times \mathbb{R}$ we have

$$\Phi(M) = C^\infty(S^1 \times \mathbb{R}, X) = C^\infty(\mathbb{R}, LX),$$

i.e. this is the same as classical mechanics on the (free) loop space LX of X. The classcial solutions are indexed by elements in $T^*(LX)$.

Quantization leads to a conformal field theory. According to the formalism of canonical quantization we would expect

$$\mathcal{H}_{S_1} = L^2(LX) \text{ and that } \mathcal{O}_{S^1\times[0,t]} : \mathcal{H}_{S^1} \longrightarrow \mathcal{H}_{S^1}$$

is given by

$$e^{-tH} = \int_\sigma e^{-A(\sigma)}\mathcal{D}\sigma.$$

However, these things are not defined mathematically. However, as explained in section 8 there is a mathematical notion of a conformal field theory. For more about this and some examples, see the beautiful article by Segal, [Se1].

Topological modular forms. The spectrum TMF was introduced by Hopkins and Miller, see [H]. It has the crucial properties one expects from 'elliptic cohomology': It is the home of the parametrized Witten genus, carries an orientation for string vector bundles, and... However, its construction is purely homotopy theoretic and thus not quite satisfactory from a geometric point of view. The goal of our considerations is to give a geometric description of this theory using conformal field theory. The main reasons to believe that there is a close connection between CFTs and TMF are that to every CFT there is an associated modular form and that the consideration of CFTs leads to an orientation for string vector bundles, see [ST]. We will elaborate on the first point in the next two sections.

12. Modular forms and the moduli space of elliptic curves

We denote by $\mathfrak{h} \subset \mathbb{C}$ the upper half plane.

Definition 77. (i) *A map* $f : \mathfrak{h} \to \mathbb{C}$ *is a* modular form of weight $k \in \mathbb{Z}$ *if*

- *f is holomorphic.*
- *We have*

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau) \text{ for all } \tau \in \mathfrak{h} \text{ and } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{L}_2(\mathbb{Z}).$$

In particular, $f(\tau + 1) = f(\tau)$, i.e. f factors through a map $\bar{f}(q)$,

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{\exp} & \mathring{D}^2 \setminus \{0\} \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \bar{f}} \\ & & \mathbb{C} \end{array}$$

where $\exp(\tau) =: e^{2\pi i\tau} =: q$.

- $\bar{f}$ *is holomorphic at* $q = 0$*, i.e. the Laurent series of* $\bar{f}$ *around zero is of the form*

$$\bar{f}(q) := \sum_{n \geq 0} a_n q^n.$$

(ii) *A* weak modular form *is defined in a similar way, but now we allow that the* q-expansion *is of the form* $\bar{f}(q) := \sum_{n \geq N} a_n q^n$ *for some* $N \in \mathbb{N}$*, i.e.* $\bar{f}$ *may have a pole at* 0*.*

(iii) f *is* integral *if all coefficients* a_n *are integers.*

We will now explain how modular forms can be interpreted as functions on the space of (rank 2) lattices in $\mathbb{C} = \mathbb{R}^2$. Let

$$\mathcal{L} := \{ \text{ space of lattices in } \mathbb{R}^2 \} = GL_2(\mathbb{R})/GL_2(\mathbb{Z}).$$

We will see that $\mathcal{L}$ can be identified with an open dense subset of $\mathbb{C}^2 \setminus \{0\}$.

Definition 78. *A map* $F : \mathcal{L} \to \mathbb{C}$ *is a* lattice function of weight k *if*

- F *is holomorphic.*
- $F(\lambda\Gamma) = \lambda^{-k}F(\Gamma)$ *for all* $\lambda \in \mathbb{C}^\times$ *and lattices* $\Gamma \in \mathcal{L}$*.*
- F *extends to a holomorphic map* $\bar{F} : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}$

We write $\Gamma_{hol}(L^{\otimes k})$ *for this space of* F*'s.*[4]

Lemma 79. *There is an isomorphism of graded rings*

$$\bigoplus_k \Gamma_{hol}(L^{\otimes k}) \longrightarrow \mathrm{MF}_*,\ F \mapsto f(\tau) := F(\tau\mathbb{Z} + \mathbb{Z}).$$

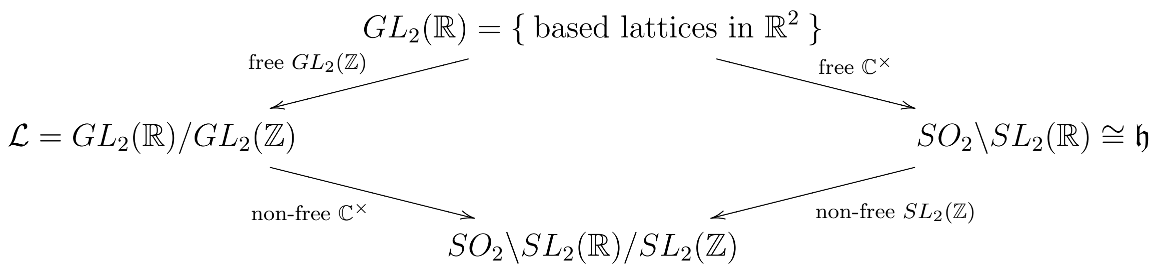
Proof. The transformation properties of the lattice function F and the modular form f correspond to each other; this follows from the computation

$$(c\tau+d)^{-k} f(\frac{a\tau + b}{c\tau + d}) = (c\tau+d)^{-k} F(\frac{a\tau + b}{c\tau + d}\mathbb{Z}+\mathbb{Z}) = F((a\tau+b)\mathbb{Z}+(c\tau+d)\mathbb{Z}) = F(\tau\mathbb{Z}+\mathbb{Z}) = f(\tau)$$

For the other statements we refer to the discussion below. □

Consider the following diagram.

[4]The reason for this notation is that lattice functions can be interpreted as holomorphic sections of the k^{th} power of the (singular) line bundle $L = \mathcal{L} \underset{\mathbb{C}^\times}{\times} \mathbb{C}$ over $\mathcal{L}/\mathbb{C}^\times$.



The diagram illustrates the isomorphism given in the lemma: The spaces $\mathcal{L}$ and $\mathfrak{h}$ are resolutions of the same singular space $SO_2\backslash SL_2(\mathbb{R})/SL_2(\mathbb{Z})$. Thinking of modular forms as sections of certain line bundles over $SO_2\backslash SL_2(\mathbb{R})/SL_2(\mathbb{Z})$ we see that the two descriptions given above are just two sides of the same coin.

We will see soon that the bi-quotient $SO_2\backslash SL_2(\mathbb{R})/SL_2(\mathbb{Z})$ can be interpreted as the moduli space of elliptic curves over $\mathbb{C}$. In fact, one can think of a modular form as an algebraic function on the moduli stack of elliptic curves over $\mathbb{C}$. It is a theorem of Deligne that the integral modular forms correspond precisely to algebraic functions on the moduli stack of elliptic curve over *all* rings.

Examples 80. For $k > 2$ the *Eisenstein series*

$$g_k(\Gamma) := K_k \sum_{\omega \in \Gamma \backslash \{0\}} \frac{1}{\omega^k}$$

defines a lattice function of weight k. From the symmetry of the lattice Γ it follows that for g_k is zero for odd k. The K_k in the formula are constants, we only care to know their value for $k = 4, 6$:

$$K_4 = 60 \text{ and } K_6 = 140.$$

It turns out that MF_* is a polynomial ring in g_4 and g_6, see corollary 87.

Groupoids and moduli of tori, cubics, and curves. We now explain in more detail how the bi-quotient $SO_2\backslash SL_2(\mathbb{R})/SL_2(\mathbb{Z})$ is related to the moduli of elliptic curves over $\mathbb{C}$. Because of the problem of automorphisms, we have to consider groupoids rather than just the space $SO_2\backslash SL_2(\mathbb{R})/SL_2(\mathbb{Z})$. Recall that a groupoid is a category in which all morphisms are invertible.

Definition 81. *Let X be a space with an action of a group G. We define the associated* groupoid (X, G) *by*

$$\mathit{Obj} := X \text{ and } \mathit{Mor} := X \times G$$

The domain and target maps $X \times G \to X$ are given by the projection onto the first factor and the G-action, resp. The identity morphisms are given by $\mathrm{id}_X \times e_G : X \to X \times G$.

Remark 82. If G acts freely on X we have $(X, G) \cong (X/G, \{e\})$.

Examples 83. (i) We define the groupoid of *flat tori and scalings* to be $(\mathcal{L}, \mathbb{C}^\times)$. From the remark and the diagram above we see that

$$(\mathcal{L}, \mathbb{C}^\times) \simeq (\mathfrak{h}, SL_2(\mathbb{Z})).$$

Note that the flat torus associated with an object in $(\mathcal{L}, \mathbb{C}^\times)$ has a canonical base point. This is the reason why we also introduce base points in the subsequent examples.

(ii) The objects of the *groupoid of smooth cubics* (in $\mathbb{CP}^2$) are smooth cubics in $\mathbb{CP}^2$ that go through the point $[0, 1, 0]$. Morphisms are global isomorphisms of $\mathbb{CP}^2$ fixing $[0, 1, 0]$ and taking one cubic to another.

(iii) The groupoid of *elliptic curves over* $\mathbb{C}$ has as objects smooth algebraic curves over $\mathbb{C}$ equipped with a base point, and morphisms are base point preserving isomorphisms.

(iv) Compact Riemann surfaces of genus 1 with a distinguished point and their isomorphisms define the groupoid of *complex curves of genus* 1. As a slight variation we can consider the groupoid of *oriented conformal tori* with a base point. This is isomorphic to the groupoid of genus 1 complex curves, because the structure groups coincide:

$$GL_1(\mathbb{C}) = \mathbb{C}^\times = SO_2 \times \mathbb{R}^\times \subset GL_2(\mathbb{R})$$

Theorem 84. *The four groupoids described in the example are equivalent.*

Proof. We will define maps from each example to the next and from (iv) to (i). It is then not hard to check that composing these four yields a functor equivalent to the identity, no matter at which groupoid one starts. The maps from (ii) to (iii) and (iii) to (iv) are obvious. Let us consider the two remaining interesting cases.

(i) $\to$ (ii): Given $\Gamma \in \mathcal{L}$, the *Weierstraß function*

$$\wp_\Gamma(z) = \frac{1}{z^2} + \sum_{\omega \in \Gamma \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

is a meromorphic function on $\mathbb{C}$ with poles of order 2 exactly at the points of the lattice Γ. $\wp_\Gamma$ is *elliptic*, i.e. it factors through the torus $\mathbb{C}/\Gamma$. This follows easily by first considering its derivative

$$\wp'_\Gamma(z) = \sum_{\omega \in \Gamma} -\frac{2}{(z-\omega)^3}$$

which is (obviously!) an elliptic function. Furthermore, it is not hard to check that $\wp_\Gamma$ and $\wp'_\Gamma$ satisfy the differential equation

$$\wp'_\Gamma(z)^2 = 4\wp_\Gamma(z)^3 - g_4(\Gamma)\wp_\Gamma(z) - g_6(\Gamma).$$

Using the differential equation for $\wp_\Gamma$ one can show that there is a biholomorphism

$$\phi : \mathbb{C}/\Gamma \cong E(g_4, g_6) := \{ [p, q, 1] \in \mathbb{CP}^2 \mid q^2 = 4p^3 - g_4 p - g_6 \} \cup \{ [0, 1, 0] \}$$

given by

$$[z] \mapsto [\wp_\Gamma(z), \wp'_\Gamma(z), 1] \text{ if } z \notin \Gamma \text{ and } [z] \mapsto [0, 1, 0] \text{ if } z \in \Gamma.$$

One way to see that this gives a biholomorphism is to consider the 2-fold branched covers

$$\begin{array}{ccc} \mathbb{C}/\Gamma & \xrightarrow{\ \phi\ } & E(g_4, g_6) \\ & {\scriptstyle \wp_\Gamma} \searrow \quad \swarrow {\scriptstyle pr_1} & \\ & \mathbb{CP}^1 & \end{array}$$

and to check that they have the same branch points (namely e_1, e_2, e_3 and ∞, where the e_i are the roots of the cubic $4p^3 - g_4 p - g_6 = 0$). Hence we associated with each lattice $\Gamma \subset \mathbb{C}$ the smooth cubic $E(g_4, g_6)$ in $\mathbb{CP}^2$ that goes through $[0, 1, 0]$. This gives desired map (i) $\to$ (ii).

(iv) $\to$ (i): Let X be a complex curve of genus one with base point x_0. The Riemann-Roch theorem implies that there exists a non-vanishing holomorphic 1-form θ on X. Define the *period mapping*

$$PM : \pi_1(X, x_0) \longrightarrow \mathbb{C} \text{ by } [\gamma] \mapsto \int_\gamma \theta.$$

This is well defined, since θ is holomorphic and hence closed. Denoting by $\Gamma \subset \mathbb{C}$ the image of the PM, it is clear that we have a biholomorphism

$$X \cong \mathbb{C}/\Gamma \text{ given by } x \mapsto \int_{x_0}^{x} \theta.$$

We would like to point out that in the case $X = E(g_4, g_6)$ one can choose $\theta = \frac{dp}{q}$ and does not need the Riemann-Roch theorem. Furthermore, picking $x_0 = [0, 1, 0]$ we get

$$\int_{[0,1,0]}^{[p,q,1]} \theta = \int_{[0,1,0]}^{[p,q,1]} \frac{dp}{q} = \int_{[0,1,0]}^{[p,q,1]} \frac{dp}{\sqrt{4p^3 - g_4 p - g_6}} = \int_0^y \frac{\wp'(z)}{\sqrt{4\wp(z)^3 - g_4\wp(z) - g_6}} = \int_0^y dz = y$$

which implies that if we start with a lattice in $\mathbb{C}$ and go through the four equivalences, we indeed get the same lattice back. □

Corollary 85. *Let $\Delta := g_4^3 - 27g_6^2$. We have a diffeomorphism*

$$\mathcal{L} \xrightarrow{\cong} \{ (g_4, g_6) \in \mathbb{C}^2 \mid \Delta \neq 0 \},\ \Gamma \mapsto (g_4(\Gamma), g_6(\Gamma)).$$

Proof. The condition $\Delta \neq 0$ says that the cubic in $\mathbb{CP}^2$ given by the coefficients g_4 and g_6 is smooth. To see this, note that in terms of the zeroes e_1, e_2, e_3 of the equation $4p^3 - g_4 p - g_6 = 0$ Δ can be expressed as

$$\Delta = (e_1 - e_2)(e_1 - e_3)(e_2 - e_3)$$

Now the theorem implies that the map $\Gamma \mapsto (g_4, g_6)$ has an inverse. □

Remark 86. The group structure on the quotient $\mathbb{C}/\Gamma$, considered as a cubic in $\mathbb{CP}^2$ as above, can be described as follows: The zero element is $[0, 1, 0]$, and

$$P_1 + P_2 + P_3 = 0 \iff \text{The } P_i \text{ lie on one line in } \mathbb{CP}^2.$$

This leads to an elliptic cohomology theory h s.t.

$$h_*(pt) = \mathbb{Z}[\frac{1}{6}, g_4, g_6],$$

where $g_i \in h_{2i}(pt)$. The theory h is complex oriented and its associated formal group law is the formal group law of the elliptic curve $\mathbb{C}/\Gamma$. There is a map

$$\mathrm{TMF}_*(pt) \longrightarrow h_*(pt)$$

whose image lies in $\mathrm{MF}^{\mathbb{Z}}_{\frac{k}{2}} \cong \mathbb{Z}[g_4, g_6] \subset h^*(pt)$.

The structure of the ring of modular forms. Let us now return to our discussion of modular forms and lattice functions. Consider again the embedding

$$\mathcal{L} \xrightarrow{\cong} \{\, (g_4, g_6) \in \mathbb{C}^2 \mid \Delta \neq 0 \,\} \hookrightarrow \mathbb{C}^2 \setminus \{0\}$$

The $\mathbb{C}^\times$-action on $\mathcal{L}$ corresponds to the $\mathbb{C}^\times$-action on $\mathbb{C}^2 \setminus \{0\}$ given by

$$\lambda(g_4, g_6) = (\lambda^4 g_4, \lambda^6 g_6).$$

We have a commutative diagram

$$\begin{array}{ccccc} \mathcal{L} & \xrightarrow[\cong]{} & \{\, (g_4, g_6) \in \mathbb{C}^2 \mid \Delta \neq 0 \,\} & \longrightarrow & \mathbb{C}^2 \setminus \{0\} \\ \downarrow{\scriptstyle \mathbb{C}^\times} & & & & \downarrow{\scriptstyle \mathbb{C}^\times} \\ \mathcal{L}/\mathbb{C}^\times & \xrightarrow[\cong]{} & \mathfrak{h}/SL_2Z & \xrightarrow{j} & (\mathbb{C}, 2, 3) \end{array}$$

Hence a modular form is the same as a holomorphic function on $\{\, (g_4, g_6) \in \mathbb{C}^2 \mid \Delta \neq 0 \,\}$ that is equivariant w.r.t. the $\mathbb{C}^\times$-action on $\mathbb{C}^2 \setminus \{0\}$. Holomorphicity at $i\infty$ corresponds to a holomorphic extension to $\mathbb{C}^2 \setminus \{0\}$. A holomorphic function on $\mathbb{C}^2 \setminus \{0\}$ that is equivariant w.r.t. the $\mathbb{C}^\times$-action we described above is necessarily a polynomial in g_4 and g_6. Hence we obtain:

Corollary 87. *There is an isomorphism*

$$\mathrm{MF}_* \cong \mathbb{C}[g_4, g_6].$$

Furthermore, since multiplication by a sufficiently high power of the discriminant Δ *makes every weak modular form into an honest one, we have*

$$w\,\mathrm{MF}_* \cong \mathbb{C}[g_4, g_6, \Delta]/(g_4^3 - 27g_6^2 = \Delta).$$

The ring of integral modular forms was computed by Tate:

Theorem 88 (Tate)**.** *The ring of integral modular forms is*

$$\mathrm{MF}_*^{\mathbb{Z}} \cong \mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 = 1728\Delta),$$

where

$$c_4 = 12g_4 \text{ and } c_6 = 216g_6.$$

The q-expansions of c_4, c_6, and Δ are given by

$$c_4(q) = 1 + 240\sum_{n\geq 0}\sigma_3(n)q^n \text{ and } c_6(q) = 1 - 504\sum_{n\geq 1}\sigma_5(n)q^n,$$

where $\sigma_k(n) := \sum_{d|n} d^k$, and

$$\Delta(q) = q\prod_{n\geq 0}(1-q^n)^{24}.$$

In particular, we see that these forms are indeed integral. Tate's theorem implies that

$$\mathrm{MF}_{12*}^{\mathbb{Z}} = \mathbb{Z}[\Delta, c_4^3].$$

Theorem 89 (Hopkins, Mahowald)**.** *Consider the map* $\mathrm{TMF}_*(pt) \to \mathrm{MF}_{*/2}^{\mathbb{Z}}$. *The image of* $\pi_{24*}(\mathrm{TMF})$ *in* $\mathrm{MF}_{12*}^{\mathbb{Z}}$ *is spanned by the monomials*

$$c_4^{3a}\Delta^b\ (a > 1,\ b \geq 0) \text{ and } \frac{24}{(24, b)}\Delta^b\ (b \geq 0).$$

In particular, 24Δ *and* Δ^{24} *lie in the image.*

Corollary 90. *If* M^{24} *is a string manifold, then* $24 \mid \hat{A}(M, TM_{\mathbb{C}})$.

We would like to point out that there is no geometric proof of this theorem yet; it would be nice to have one.

13. CFTs and modular forms

We now explain how conformal field theories and (weak) modular forms are connected.

Theorem 91. *There is a map* $\pi_0 \operatorname{CFT}_{2k} \to w \operatorname{MF}_k^{\mathbb{Z}}$*.*

Remark 92. Introducing the appropriate product structure on CFTs and summing over all k, this will become a ring homomorphism.

We have to define the spaces CFT_n. We will not be too precise about this, and the 2-category aspect of the story (see [ST]) will be ignored completely. However, we will take the super symmetric aspects into account, since they are crucial for obtaining a modular form. We begin with some preparatory material.

Spin structures on 1-dimensional real and complex manifolds. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let X be a 1-dimensional manifold over $\mathbb{F}$. A *spin structure* on X is an $\mathbb{F}$-vector bundle S_X over X together with an isomorphism $\psi_X : S_X \otimes_{\mathbb{F}} S_X \cong T^*X$. In the case $\mathbb{F} = \mathbb{C}$ we assume that S_X and ψ_X are holomorphic. In the case $\mathbb{F} = \mathbb{R}$, we obtain an orientation on X from a spin structure by considering all vectors in T^*X that are squares, i.e. of the form $v^2 := \psi_X(v \otimes v)$.

A spin structure on a complex 1-manifold X with boundary induces a spin structure on ∂X as follows. Define

$$S_{\partial X} = \{\, v \in S_X|_{\partial X} \mid v^2 \text{ annihilates } T(\partial X) \subset TX|_{\partial X} \,\}$$

For example, if $X = D^2 \subset \mathbb{C}$ is the unit disc with its unique spin structure, the spin structure induced on the boundary is given by the Möbius bundle.

$\frac{1}{2}$-forms, Cliffords algebras, and Fock spaces. Now, for X equipped with a spin structure define

$$\Omega^{\frac{1}{2}}(X) := \Gamma(S_X)$$

If $\mathbb{F} = \mathbb{R}$, this has an inner product that is invariant under spin diffeomorphisms of S^1; it is defined by

$$\langle w_1, w_2 \rangle := \int_X \psi_X(w_1(x) \otimes w_2(x)).$$

If $\mathbb{F} = \mathbb{C}$, the holomorphic structure gives a Dolbeault operator

$$\bar{\partial} : \Omega^{\frac{1}{2}}(X) \longrightarrow \Omega^{\frac{3}{2}}(X),$$

and $\ker \bar{\partial}$ is the space of *holomorphic* $\frac{1}{2}$*-forms* on X. The map $\bar{\partial}$ can be identified with the Dirac operator on X, and under this identification the holomorphic $\frac{1}{2}$-forms correspond to harmonic spinors. If X is closed, $\bar{\partial}$ is self-adjoint and ellliptic and hence the space of holomorphic $\frac{1}{2}$-forms is finite-dimensional. In the case $\partial X \neq \emptyset$ the elements in $\ker \bar{\partial}$ restrict to give a Lagrangian subspace in $\Omega^{\frac{1}{2}}(\partial X) \otimes \mathbb{C}$.

We will now associate to each closed spin 1-manifold an algebra $C(X)$ and to each Riemann spin surface Σ a module over $C(\partial\Sigma)$. $C(X)$ is defined to be the Clifford algebra associated with the inner product space

$$(\Omega^{\frac{1}{2}}(X) \otimes \mathbb{C}, \langle .,. \rangle),$$

where $\langle .,. \rangle$ is the $\mathbb{C}$-linear extension of the inner product described above. For a surface Σ we define its associated *Fock space* by

$$F(\Sigma) := \begin{cases} \mathrm{Pf}(\Sigma) = \Lambda^{\mathrm{top}}(\ker \bar\partial) & \text{if } \Sigma \text{ closed} \\ \Lambda^*(\ker \bar\partial) & \text{if } \partial X \neq \emptyset \end{cases}$$

This is a graded irreducible $C(\partial\Sigma)$-module, see e.g. [ST].

Denote by S^1_P and S^1_A the circle equipped with the non-bounding (Periodic) and bounding (Anti-periodic) spin structure. In the following we will, for simplicity, assume that all 1-manifolds occuring are disjoint unions of such standard circles. In order to define CFTs of degree n we fix two graded representations $\mathcal{H}_P$ and $\mathcal{H}_A$ of the algebras $C(S^1_P)^{\otimes n}$ and $C(S^1_A)^{\otimes n}$, respectively.

Definition 93. *A (non-super symmetric)* CFT E of degee n *is a* $C(\partial\Sigma)^{\otimes n}$*-linear assignment*

$$(\Sigma^2, \omega) \mapsto E(\Sigma, \omega) \in \mathcal{H}(\partial\Sigma)$$

that satisfies the usual gluing laws.

Here ω is an element in the n^{th} power of the Fock space $F(\Sigma)^{\otimes n}$. The Hilbert space $\mathcal{H}(\partial\Sigma)$ is an appropriate tensor product of copies of $\mathcal{H}_P$ and $\mathcal{H}_A$ and their duals with multiplicities according to the usual disjoint union axiom. There is an action of $C(\partial\Sigma)^{\otimes n}$ on $\mathcal{H}(\partial\Sigma)$, since $C(\partial\Sigma)^{\otimes n}$ is a tensor product of $C(S^1_P)^{\otimes n}$, $C(S^1_A)^{\otimes n}$, and their opposite algebras with the same multiplicities as for $\mathcal{H}(\partial\Sigma)$. This explains what we mean by $C(\partial\Sigma)^{\otimes n}$-linearity in the definition. We should also point out that giving a vector in $\mathcal{H}(\partial\Sigma)$ is the same as prescribing a Hilbert-Schmidt operator from $\mathcal{H}(\partial_{in}\Sigma)$ to $\mathcal{H}(\partial_{out}\Sigma)$, because of the isomorphism

$$\mathcal{H}(\partial\Sigma) = \mathcal{H}(\partial_{in}\Sigma)^* \otimes \mathcal{H}(\partial_{out}\Sigma) \cong \mathrm{HS}(\mathcal{H}(\partial_{in}\Sigma), \mathcal{H}(\partial_{out}\Sigma)).$$

Remark 94. We want to explain how the same formalism leads to the notion of *degree n* previously introduced in the case of 1-dimensional Euclidian field theories. For simplicity, we will only look at points and intervals. We have the Clifford algebra

$$C(pt) = C(\Omega^{\frac{1}{2}}(pt), \langle .,. \rangle) = C_1$$

and hence $C(pt)^{\otimes n} = C_1^{\otimes n} \cong C_n$. Furthermore,

$$F([0,t]) = \Lambda^*(\text{ harmonic 1/2-forms on } [0,t]\ ) \cong C_1 \quad \text{as } C_1^{op} \otimes C_1\text{-modules},$$

since harmonic 1/2-forms on $[0,t]$ are constant. We thus have $F([0,t])^{\otimes n} \cong C_n$ as an $C_n^{op} \otimes C_n$-module or, equivalently, as a C_n-C_n-bimodule. If we let $Op([0,t]) := E([0,t],1)$ for a EFT E, then,

$$Op([0,t]) = E([0,t], -e_i 1 e_i) = -e_i Op([0,t]) e_i \text{ for all } i,$$

i.e. $Op([0,t])$ is C_n-linear. Here we used the identification $\mathrm{HS}(\mathcal{H}_n) \cong \mathcal{H}_n^* \otimes \mathcal{H}_n$. So it turns out that our new notion of degree coincides with the one introduced earlier.

The modular form associated with a CFT**.** Consider a CFT E of degree n. For a complex torus T we used a non-vanishing 1-form θ and a base-point x_0 to define

$$\Gamma_\theta := \{ \int_\gamma \theta \in \mathbb{C} \mid \gamma \text{ is a loop at } x_0 \}$$

and an isomorphism

$$T \xrightarrow{\cong} \mathbb{C}/\Gamma_\theta, \; x \mapsto \int_{x_0}^x \theta.$$

In other words, the line bundle associated with the $\mathbb{C}^\times$-principle bundle $\mathcal{L} \to \mathcal{M}_{Ell}$ has fiber $\Omega^1_{\mathrm{hol}}(T)^* \cong \Lambda^{top}\Omega^1_{\mathrm{hol}}(T) = \mathrm{Det}(T)$ over T. This implies that the restriction of E to tori defines a (not necessarily holomorphic) lattice function

$$E(\mathrm{tori}) : \mathcal{L} \longrightarrow \mathbb{C}$$

of weight $\frac{n}{2}$. This (and the role of the factor $\frac{1}{2}$) can be seen directly from the identification

$$\mathrm{Pf}(\Sigma)^{\otimes 2} \cong \mathrm{Det}(T).$$

This suggest that we might be able to associate a modular form to every CFT. However, there is no reason why $E(\mathrm{tori})$ should be holomorphic. To ensure this, we need supersymmetry.

Theorem 95. *If E is super symmetric in the sense explained below the map*

$$E(tori) : \mathcal{L} \longrightarrow \mathbb{C}$$

is a weak integral modular form of weight $\frac{n}{2}$.

We will be very brief concerning the meaning of 'supersymmetric' and concentrate on the aspects that imply that $E(\mathrm{tori})$ is holomorphic and integral. Roughly speaking, we replace conformal surfaces by super conformal surfaces and define 'super CFTs' to be (operator valued) functions on the associated super moduli space.

Recall that a complex structure on a surface Σ is just a decomposition

$$T\Sigma \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}$$

and that locally these subbundles are spanned by vector fields ∂_z and $\partial_{\bar{z}}$. Now, a *super conformal structure* on a super manifold of dimension $(2|1)$ is a decomosition

$$T\Sigma^{2|1} \otimes \mathbb{C} = T^{(0,1)|0} \oplus T^{(1,0)|0} \oplus T^{0|1}.$$

Locally, these are spanned by vector fields ∂_z, $\partial_{\bar{z}}$, and $D = \partial_\theta - \theta\partial_{\bar{z}}$. Note that $D^2 = -\partial_{\bar{z}}$.

We remind the reader that in the K-theory case supersymmetry changed the moduli space of intervals $\mathbb{R}_{>0}$ to the super moduli space of super intervals $\mathbb{R}^{1|1}_{>0}$, and the significant feature of this super semi group was the structure of its Lie algebra, which is free in one odd generator. In the 2-dimensional case a similar thing happens. We consider the semi group of 'toy annuli' whose boundary parametrizations are of a particularly simple form. Every conformal annulus A is isomorphic to one of the form $\{\, z \,|\, t \leq |z| \leq 1 \,\} \subset \mathbb{C}$ for some $t \in (0,1)$. We consider the semi group of annnuli whose boundary parametrizations $S^1 \to \{\, z \,|\, t \leq |z| \leq 1 \,\}$ are given by multiplication by a complex number. After multiplication by $z \in S^1$ we can assume that the parametrization of the outgoing boundary is the identity on S^1. Hence such an annulus is characterized by the complex number $q \in D^2 \setminus \{0\}$ that gives the parametrization of the incoming boundary. Gluing of two such annuli A_q and $A_{q'}$ yields $A_{qq'}$ so that the semi group of toy annuli is $D^2 \setminus \{0\}$. It turns out that the corresponding super semi group of super conformal toy annuli is $D^{2|1} \setminus \{0\}$. The super Lie algebra of this is the direct sum of two free super Lie algebras, one with an even and one with an odd generator. This means that in the super version one of the generators of the Lie algebra of the toy annuli group has an odd square root.

Now, from the gluing properties of a CFT one obtains the formula

$$E(T_q) = \mathrm{strace}(A_q),$$

that describes the value of E on the torus obtained from gluing incoming and outgoing boundary components of the annulus A_q together. Recall that in order to evaluate a CFT we also need not only a surface, but also an element in the associated Fock space. For each annulus A_q we have a canonical element in $F(A_q)$, namely the vacuum vector. From this one can construct a canonical element in the Fock space of T_q, see [ST], page 43. If we write $E(T_q)$ it is understood that we use this distinguished element in $F(T_q)$.

Now, every $\mathbb{Z}/2$-equivariant semi group homomorphism $D^2 \to HS^{sa}$ can be written as

$$E(A_q) = q^A \bar{q}^B,$$

where A and B have discrete spectrum and commute. Furthermore, the spectrum of $A - B$ is contained in $\mathbb{Z}$. Taking the super trace we obtain

$$E(T_q) = \sum_{\lambda \in \sigma(A),\, \mu \in \sigma(B)} q^\lambda \bar{q}^\mu \,\mathrm{sdim}\, E_{\lambda,\mu}.$$

If E is super symmetric, then $B = G^2$ for some odd operator G that commutes with A. G gives isomorphisms $E_{\lambda,\mu} \cong E_{\lambda,\bar{\mu}}$, which implies that all contributions in the sum for $\mu \neq 0$ are zero. Thus,

$$E(A_q) = \mathrm{strace}(q^A|_{\ker B}) = \sum_{n\in\mathbb{Z}} q^n \,\mathrm{sdim}\, E_{n,0},$$

where the second equality uses the integrality of $A - B$. It turns out that this series is bounded below (otherwise one runs into a contradiction to $E(A_q)$ being Hilbert-Schmidt) and so we can conclude that $E(tori)$ is an integral weak modular form.

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