4-MANIFOLDS

WS 18/19

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Lectures 8 - 15

(November)

Class 8 — Nov 6th

Recall:

$$\mathcal{D} := \left\{ \Delta : D^2 \sqcup D^2 \hookrightarrow D^4 : \begin{array}{l}\Delta_1 \text{ embedded} \\ \Delta_1 \cap \Delta_2 = \emptyset\end{array} \right\}$$

$$\big\downarrow \partial$$

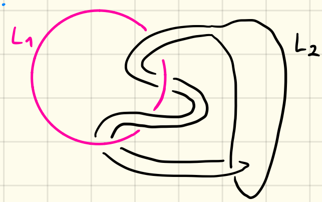
$$\{L \in \mathcal{L} \mid L_1 \text{ is unknot}\} \subseteq \operatorname{im}(\partial) \subseteq \mathcal{L} := \{ L : S^1 \sqcup S^1 \hookrightarrow S^3 : lk(L) = 0 \}$$

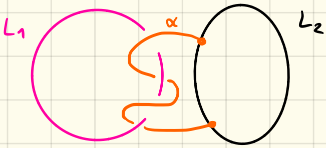
Def. The Kirk invariant of $L \in \operatorname{im}(\partial)$ is:

= "the secondary linking"

$$\hat{\mu}_2(\Delta_2) \in t\cdot\mathbb{Z}[t] \cong \frac{\mathbb{Z}[t, t^{-1}]}{g \sim g^{-1},\, 1} \longleftarrow \frac{\mathbb{Z}\pi_1(D^4 - \Delta_1)}{g \sim g^{-1},\, 1}$$

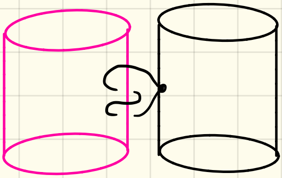
example.

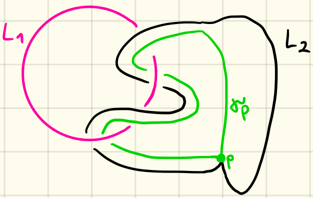




pick an arc α

discs:





Calculate $\mathcal{K}(L) = \pm t^n$

where

$$n = |lk(L_1, \gamma)| \in \mathbb{N}$$

Here $\mathcal{K}(L) = -t^2$.

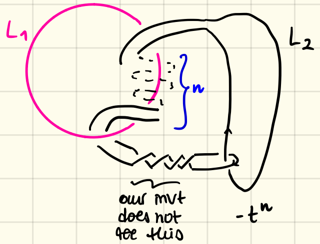
The choice of loop γ_p is irrelevant because: $lk(L_1, L_2) = 0 \Rightarrow \Delta_1 \cap \Delta_2 = \emptyset$

so picking the other loop in L_2 gives the same result.

Corollary. $\forall p \in \mathbb{Z}[t]$ $\exists L = (\text{unknot}, L_2)$ s.t. $\mathcal{K}(L) = p$.

proof.

$$p(t) = \sum_{n=1}^{N} a_n t^n \qquad \text{for } \pm t^n \text{ do as above.}$$

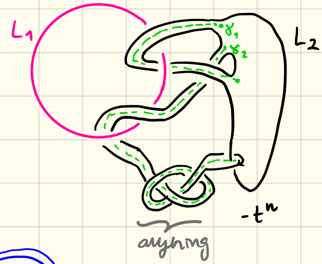


Need $|a_n|$ many arcs that all link L_1 n times.

"We are undoing L_2 in the complement of L_1 and counting the linking numbers of $\underline{L_1}$ with double-point loops $\underline{\gamma_p}$."

Different choices for $\gamma_1, \dots, \gamma_r$ – the guiding arcs for finger moves give links with same $\mathcal{K}(L)$.

e.g.



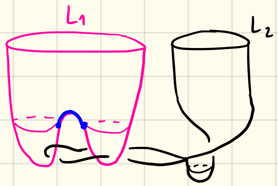
link mvt does not see this difference!

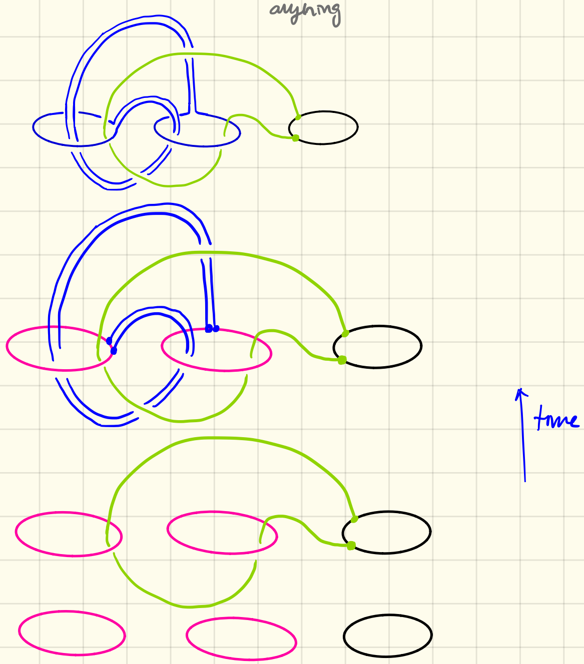
so L_2 can even not be slice

and $\mathcal{K}(L)$ could vanish.

$\therefore$ $\mathcal{K}(L)$ does not detect slice $\square$

HW.





note Δ_1 is not a undisk.

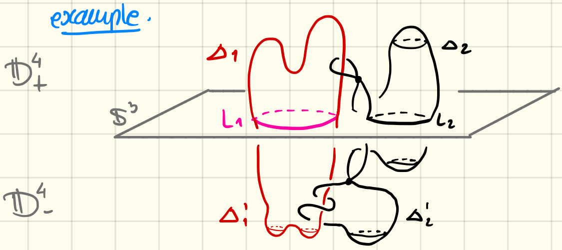
blue = saddle

green = guiding arc for finger move

Remark. Can interchange saddles and finger moves.

proof that Kirk's invt is well-defined: i.e. $\mathcal{K}(L)$ does not depend on Δ.

example.



$\hat{\mu}_{\mathbb{Z}}(R_2) = \pm t \pm t^2 \neq 0$
This is impossible!

Rem. if Δ_1 is undisk, we would have only one minimum, so get $\pi_1(S^4 \setminus \Delta_1) \cong \mathbb{Z}$

Claim. $\hat{\mu}_{\mathbb{Z}}(\Delta_2) - \hat{\mu}_{\mathbb{Z}}(\Delta_2') = 0$

To prove this, use $R := \Delta \cup \Delta' : S^2 \sqcup S^2 \looparrowright S^4$ with R_1 embedded, $R_1 \cap R_2 = \emptyset$

We define $\hat{\mu}_{\mathbb{Z}}(R_2) \in t\mathbb{Z}[t]$ s.t. $\hat{\mu}_{\mathbb{Z}}(\underset{\Delta_2 \cup \Delta_2'}{\overset{\parallel}{R_2}}) = \hat{\mu}_{\mathbb{Z}}(\Delta_2) - \hat{\mu}_{\mathbb{Z}}(\Delta_2')$

Main Claim. $\hat{\mu}_{\mathbb{Z}}(R_2) = 0.$

(do not draw whiskers since $\mathbb{Z} = \langle t \rangle$ abelian & changing whisker is conjugation.)

Def. of $\hat{\mu}_{\mathbb{Z}}(R_2)$:

$$[R_2] \in \pi_2(S^4 \setminus R_1) \Rightarrow \hat{\mu}(R_2) \in \frac{\mathbb{Z}[\pi_1(S^4 \setminus R_1)]}{g \sim g^{-1}, 1} \xrightarrow{\;\star\;} t\mathbb{Z}[t] \ni \hat{\mu}_{\mathbb{Z}}(R_2)$$

To define $\star$ we need a gp homomorphism: $\pi_1(S^4 \setminus R_1) \longrightarrow \mathbb{Z}$

We pick the following composition:

$$\pi_1(S^4 \setminus R_1) \xrightarrow{ab.} H_1(S^4 \setminus R_1) \underset{A.D.}{\xrightarrow{\cong}} H^2(R_1) \cong \mathbb{Z}$$

Claim. This map is precisely the linking number: $\underset{\pi_1(S^4 \setminus R_1)}{lk(\overset{}{\underset{\cap}{\gamma}}, R_1)} \in \mathbb{Z}$

Denote $M := S^4 \setminus R_1$

Then we have:

$$\begin{tikzcd}
\text{univ. cover} & \tilde{\tilde{M}} \arrow[d, "p"] & \\
\mathbb{Z}\text{-cover} & \tilde{M} \arrow[d, "q"] & \\
& M & S^2 \arrow[l, "R_2"] \arrow[u, bend right, "\tilde{R}_2"'] \arrow[uu, bend right, "\tilde{\tilde{R}}_2"']
\end{tikzcd}$$

We used univ. cover for def of λ

Now we use the $\mathbb{Z}$-cover instead, call it $\lambda_{\mathbb{Z}}$.

Lemma. (i) $\hat{\mu}_{\mathbb{Z}}(R_2)$ is determined by $\lambda_{\mathbb{Z}}$.

(ii) $\lambda_{\mathbb{Z}}(R_2, R_2)$ only depends on the image of R_2 under

$$\begin{array}{ccccc} \pi_2 M & \xrightarrow[\text{Hurew. iso.}]{h} & H_2(\tilde{\tilde{M}}) & \xrightarrow{p_*} & H_2(\tilde{M}) \\ \rotatebox{90}{$\in$} & & & & \rotatebox{90}{$\in$} \\ \tilde{\tilde{R}}_2 & & & & \tilde{R}_2 \end{array}$$

proof of Lem. (i). Recall: $\lambda(R_2, R_2) = \mu(R_2) + \overline{\mu(R_2)} + e(\nu R_2)\cdot 1$

so get: $\underset{\displaystyle t\mathbb{Z}[t^{\pm1}]}{\underset{\cap}{\lambda_{\mathbb{Z}}(R_1, R_2)}} = \underset{\displaystyle \sum_{n=1}^{N} a_n t^n}{\underset{\|}{\mu_{\mathbb{Z}}(R_2)}} + \underset{\displaystyle \sum_{n=1}^{N} a_n t^{-n}}{\underset{\|}{\overline{\mu_{\mathbb{Z}}(R_2)}}}$

(ii) General fact about $\lambda_{\mathbb{Z}}$, proven exactly the same way as for λ. (see Ani's class).

$\square$

Now we use the Gysin sequence (or Bockstein seq.):

$$\begin{array}{ccccccc} \cdots \longrightarrow H_2(\tilde{M}) & \xrightarrow{\cdot(t-1)} & H_2(\tilde{M}) & \xrightarrow{q_*} & H_2(M) & \longrightarrow \cdots \\ & & \rotatebox{90}{$\in$} & & \| & \\ & & \tilde{R}_2 & & H_2(S^4 \setminus R_1) & \\ & & & & \| & \\ & & & & H^1(R_1) = 0 & \end{array}$$

since R_1 is a sphere!

$\Rightarrow$ so $\tilde{R}_2$ is in image of $\cdot(t-1)$.

$\Rightarrow$ $\forall n \in \mathbb{N}\ \exists a_n \in H_2(\tilde{M})$ $\tilde{R}_2 = (t-1)^n a_n$

It follows that: $\lambda_{\mathbb{Z}}(\tilde{R}_2, \tilde{R}_2) = \lambda_{\mathbb{Z}}((t-1)^n a_n, (t-1)^n a_n)$

$$= (t-1)^n (t^{-1}-1)^n \lambda_{\mathbb{Z}}(a_n, a_n)$$

$\Rightarrow \quad (t-1)^n \mid \lambda_{\mathbb{Z}}(\tilde{R}_2, \tilde{R}_2)$ for all n in $\mathbb{Z}[t,t^{-1}]$

$\lightning$ since unique factoriz. domain.

$\Rightarrow \quad \lambda_{\mathbb{Z}}(\tilde{R}_2, \tilde{R}_2) = 0$ $\square$

Recall: **Class 9** **Nov 8th**

M^{2n} connected mfld, $n \geq 1$

$$\begin{array}{ccc} \hat{\mu}: \pi_n M^{2n} & \longrightarrow & \widehat{\mathbb{Z}[\pi_1 M]} \\ \cong & & \parallel \\ \dfrac{\left\{ \begin{array}{c} S^n \looparrowright M^{2n} \\ + \text{whisker} \end{array} \right\}}{\substack{\text{isotopy,} \\ \text{finger/Whitney move} \\ \text{cusp move}}} & \xrightarrow[\text{count}]{\text{self-inter.}} & \dfrac{\mathbb{Z}[\pi_1 M]}{\langle g - g^{-1}, 1 \rangle} \end{array}$$

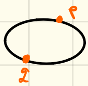
$\hat{\mu}(f)$ is (the first) obstruction for f being homotopic to an embedding.

Whitney trick Theorem. For $n \geq 3$ we have: $\hat{\mu}(f)$ is the only obstruction

i.e.

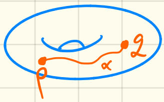
$$\hat{\mu}(f) = 0 \quad \Rightarrow \quad f \simeq \text{embedding}$$

Question: $n = 1$?



$\looparrowright M^2$ Only $\hat{\mu}(f) \in \mathbb{Z}[\pi_1 M / \langle f_* S^1 \rangle]$ well-defined. Is it also the only obstr?

example.



$\looparrowright M^4$ $f(p) = f(q)$

pick α. $f(\alpha)$ is a loop! + whisker $\Rightarrow$ get an elt $[g_p] \in \pi_1 M$.

But another choice of an arc α' would matter: $\alpha\alpha' \neq 0 \in \pi_1 T^2$. ?

Fact: $(a,b) \in \mathbb{Z}^2 = \pi_1 T^2$ is represented by an embedding $S^1 \hookrightarrow T^2$
(HW6) $\iff$ a and b coprime.
$\iff$ $(a,b) \in H_1(T^2)$ primitive class.

Note: In dim $=4$: $\hat{\mu}(f)$ can be zero even if $f \not\simeq$ emb. (See Claim on next p.)
We will see some "higher obstructions".

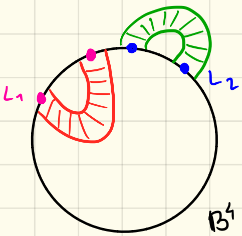
Remark. Whitney: $\pi_1 M = 1$ then any $N^n \looparrowright M^{2n}$
can be homotoped to $N^n \hookrightarrow M^{2n}$.

Remark. In case: $\pi_1 M = \{1\}$
Is $\hat{\mu}$ a homomorphism? we saw in HW4:
$$\mu(f_1+f_2) - \mu(f_1) - \mu(f_2) = \lambda(f_1,f_2)$$
<u>NO</u>: it is *quadratic*!

Lemma. There exists a map:

$$\begin{tikzcd}
\left\{\begin{array}{l} L: S^1 \sqcup S^1 \hookrightarrow S^3 \\ L_i \text{ unknots} \\ lk\, L = 0 \end{array}\right\} \arrow[r, "L \mapsto M_L"] \arrow[dr, "\mathcal{K}"', "L \mapsto \hat{\mu}_{\mathbb{Z}}(\Delta_2)"] & \left\{\begin{array}{l} \text{conn. or. 4-mflds } M \\ \text{together with} \\ M \simeq S^1 \vee S^2 \end{array}\right\} \arrow[d, "\hat{\mu}\ (\text{same gener. of } \pi_2(M) \cong \mathbb{Z}[\mathbb{Z}])"] \arrow[dl, phantom, "\circlearrowright"] \\
 & +\mathbb{Z}[t] \cong \widehat{\mathbb{Z}[\mathbb{Z}]}
\end{tikzcd}$$

proof.
Given L construct
$$M_L := B^4 \setminus \mathring{\nu}(\text{undisk on } L_1) \cup_{L_2 \times D^2} (D^2 \times D^2)$$
$\nwarrow$ 2-handle



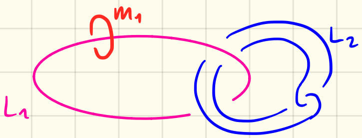
Note:
$$M_L \cong \underbrace{(S^3 \setminus \mathring{\nu} L_1)}_{= S^1 \times D^2 \simeq S^1_{m_1}} \times I \cup_{L_2 \times D^2} \text{2-handle.}$$

$M_L \simeq S^1_{m_1} \cup_{L_2}$ 2-cell

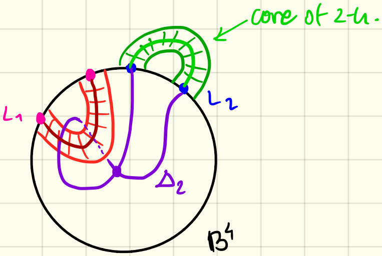
need orient of L_1

We claim: $\quad M_L \xrightarrow[h]{\simeq} S^1 \vee S^2$

because the boundary of the 2-cell is null-htpic in $S^1_{m_1}$ since $lk(L) = 0$.



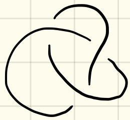
Note: h does not depend on choice of Δ_2



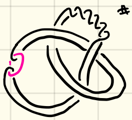
Now our diagram in the statement of the Lemma commutes, because d.p. of $\Delta_2 \cup$ core are precisely d.p. of Δ_2. $\Rightarrow$ $\hat{\mu}_{\mathbb{Z}}(\Delta_2) = \hat{\mu}(\Delta_2 \cup \text{core})$.

$$\hat{\mu}_{\mathbb{Z}}(\Delta_2) = \hat{\mu}(\Delta_2)$$

$\square$

Def. Bing double of a knot $K =$ 

is the link

$B(K) =$ 

twists s.t. two blam strands unlinked. $= -$ writhe of K.

Addendum

HW5 (1.d)) : Bing double of a knot is always a "boundary link".
i.e. the components L_i bound disjoint Seifert surfaces F_i.

Claim. Let $L = B(\text{trefoil})$. Then $\hat{\mu}(L) \equiv 0$ on $\pi_2 M_L$. (by HW5.1)
but: the generator of $\pi_2 M_L$ is not represented by an embedding !

Moreover:
Bing(trefoil) is not slice with $\Delta_1 =$ undisk.
(can prove using L^2-signatures).

$$\begin{tikzcd}
\{\text{Knots}\} \arrow[r, "\text{Bing double}"] \arrow[d, "\sigma_{L^2}"] \arrow[drr, dashed, "\text{zero map!}"'] & \left\{\begin{array}{l} L : S^1 \sqcup S^1 \hookrightarrow S^3 \\ L_1 \text{ unknot} \\ lk(L)=0 \end{array}\right\} \arrow[r, "L \mapsto M_L"] \arrow[dl, "\widetilde{\sigma}_{L^2}"] \arrow[dr, "\varkappa"] & \left\{\begin{array}{l} \text{conn. or. 4-mflds} \\ \text{with } M^4 \simeq S^1 \vee S^2 \end{array}\right\} \arrow[d, "\hat{\mu}", "\exists\,\text{thm}"'] \\
\mathbb{R} & & t\mathbb{Z}[t]
\end{tikzcd}$$

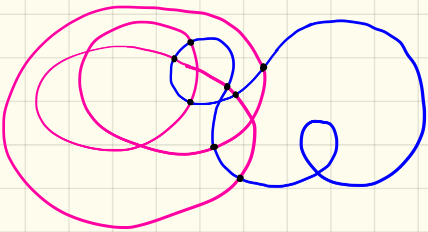
L^2-signatures $\leftarrow \sigma_{L^2}$ (cf. class last year)

Rem. Try to prove Kirk invt of B(trefoil) is zero directly: shrink the trefoil comp. in the complement of the unknot by doing some crossing changes and tracing what gp. elts are.

Let $n \geq 2$, M^{2n} connected.

Recall:

$$\lambda : \pi_n M^{2n} \times \pi_n M^{2n} \longrightarrow \mathbb{Z}[\pi_1 M]$$



λ is Hermitian form

$\mathbb{Z}[\pi_1 M]$ - bilinear form.

Then λ is an obstruction to making f_1 and f_2 homotopic to maps with disjoint images.

Theorem. λ is the only obstruction.

? Also in dimension 4 i.e. $n=2$.

proof. next time! ("the immersed Wh. move").

For higher n again Whitney trick. We actually get:

$$\left.\begin{array}{l} f_1 \dots f_k \in \pi_n M^{2n} \\ \hat{\mu}(f_i) = 0 \\ \lambda(f_i, f_j) = 0 \ \forall i \neq j \end{array}\right\} \text{ Then: they are homotopic to embeddings } f_1' \dots f_k'.$$

Class 10. Nov 13 Tue

Main invariants of a connected 4-manifold M are:

$$\star \qquad (\pi_1 M,\ \pi_2 M,\ \lambda_M,\ \hat{\mu}_M)$$

quadratic refinement

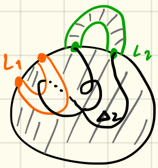
first obstruction to realizing a class in $\pi_2 M$ by an embedding. (We'll see secondary obstructions!)

hermitian form

Theorem.
The obstruction to realizing $[f_1], [f_2] \in \pi_2 M$ by maps with disjoint images.

Recall our examples

$$\underset{(L_1, L_2)}{M_L} \simeq S^1 \vee S^2$$



$$\cong S^1 \times D^3 \underset{(L_2, n)}{\cup} h^2$$

$$\pi_2 M_L \cong \mathbb{Z}[t^{\pm}]$$

then:

$$t\mathbb{Z}[t] \ni \hat{\mu}\left(\begin{matrix}\pi_1 M_L\text{-generator of}\\ \pi_2 M_L\end{matrix}\right) = \mathcal{K}(L) \quad \text{can be any polynomial!}$$

$\star$ is preserved under homotopy equivalences $(M, \partial M) \simeq (M', \partial M')$ which are homeomorphisms on the boundary.

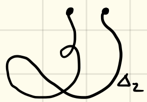
In fact (HW6.1b): $\mathcal{K}(L)$ can be read off from ∂M_L !

Later (Am): $M_L \underset{\partial M_L}{\cup} M_L \cong S^1 \times S^3 \# S^2 \times S^2$ (for framing of h^2 n = even)

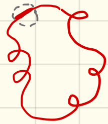
Remark. μ, λ are also defined for $(D^2, \partial D^2) \hookrightarrow (M^4, \partial M)$
We need not close Δ_2 above to a sphere:

? note: this generic $\Rightarrow \partial D^2 \hookrightarrow \partial M$

$$\Delta_2 : \underset{S^1}{\underset{\|}{(D^2, \partial)}} \hookrightarrow \underset{S^1 \times D^2}{\underset{\cup\!\shortmid}{(S^1 \times D^3,\ S^1 \times S^2)}}$$



So can count double points as before to get $\lambda, \hat{\mu}$.
They do not change under homotopies of $(\mathbb{D}^2, \partial) \longrightarrow (M, \partial)$ as long as they restrict to isotopies on ∂.



Note: can calculate $\lambda, \hat{\mu}$ for a sphere by removing a small disc and calculating $\lambda, \hat{\mu}$ on the disc = remaining complement

Theorem. Given $f: (\mathbb{D}^2 \sqcup \mathbb{D}^2, \partial) \xrightarrow{(f_1, f_2)} (M^4, \partial)$ we have:

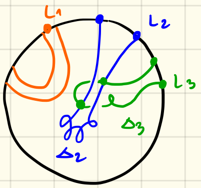
$$\boxed{\lambda([f_1],[f_2]) = 0} \iff \boxed{f \underset{\text{rel}\,\partial}{\simeq} f' \text{ s.t. } \operatorname{im} f_1' \cap \operatorname{im} f_2' = \emptyset}$$

Corollary 1. $lk(L_1, L_2) = 0 \iff \exists \Delta' : (\mathbb{D}^2 \sqcup \mathbb{D}^2, \partial) \looparrowright (\mathbb{D}^4, \partial)$
with $\partial \Delta' = L$ and $\Delta_1' \cap \Delta_2' = \emptyset$

since: $lk(L_1, L_2) = \# \Delta_1 \cap \Delta_2 \in \mathbb{Z}$
$= \lambda(\Delta_1, \Delta_2)$

Q: More Elementary way to prove Cor 1?

Corollary 2. (HW4.2 b))



For $L = (L_1, L_2, L_3)$ s.t. L_1 unknot, $lk(L_1, L_2) = lk(L_1, L_3) = 0$
calculate the Kirk invariant for (L_2, L_3) as $\lambda(\Delta_2, \Delta_3)$
← for $\mathbb{D}^4$-complement
Then
$\mathcal{K}(L_2, L_3) = 0$ iff $\exists \Delta_2', \Delta_3'$ disjoint from Δ_1 and $\Delta_2' \cap \Delta_3' = \emptyset$.

Geometric Cancellation Theorem (will imply Thm.)

Let
$f: (\mathbb{D}^2 \sqcup \mathbb{D}^2, \partial) \looparrowright (M, \partial M)$ s.t. $\exists$ inters. points $p_+, p_- \in f_1 \pitchfork f_2$ with $g_{p_+} = g_{p_-}$ in $\pi_1 M$.

Then there is a homotopy $f \underset{\text{rel}\,\partial}{\simeq} f'$ such that:

$$f_1' \pitchfork f_2' = f_1 \pitchfork f_2 \setminus \{p_+, p_-\} \quad \text{and:}$$

and the homotopy consists of:

- isotopies rel ∂
- self-finger moves on f_i
- one Whitney move cancelling p^+ and p^-.

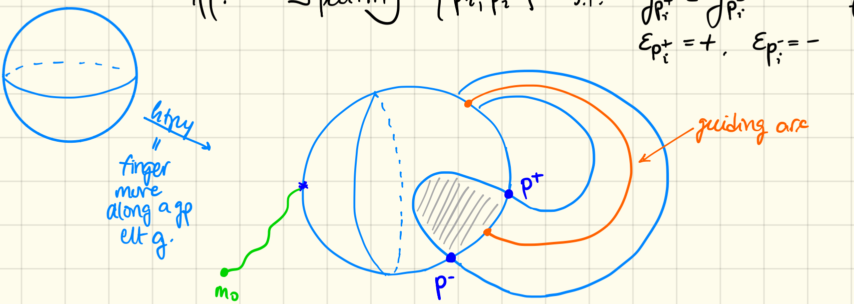
proof of Thm follows from this lemma:

Note: $$\lambda(f_1,f_2) = \sum_{p \in f_1 \pitchfork f_2} \underbrace{\varepsilon_p}_{\{\pm 1\}} \cdot \underbrace{g_p}_{\in \pi_1 M} = 0 \quad \in \mathbb{Z}[\pi_1 M]$$

iff. $\exists$ pairing $\{p_i^-, p_i^+\}$ s.t. $g_{p_i^+} = g_{p_i^-}$, $\varepsilon_{p_i^+} = +$, $\varepsilon_{p_i^-} = -$ for all $i = 1..n$

$\square$

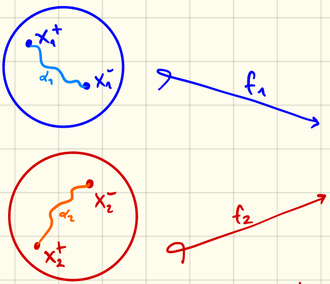
example.



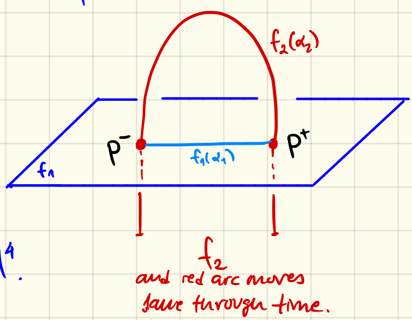
Have that: $g_{p^+} = g_{p^-} = g$. ?

proof of Lemma.

this proof is due to A. Casson 1970's.



$M^4 \ni p^\pm = f_1(x_1^\pm) = f_2(x_2^\pm)$



Pick embedded arcs $x_1^+ \overset{\alpha_1}{\leadsto} x_1^-$ and $x_2^+ \overset{\alpha_2}{\leadsto} x_2^-$ that are away from other double points.

then

$f_1(\alpha_1) \cup f_2(\alpha_2)$ is an embedded circle in M^4.

Moreover:

it is null-homotopic in M iff $g_{p^+} = g_{p^-}$.

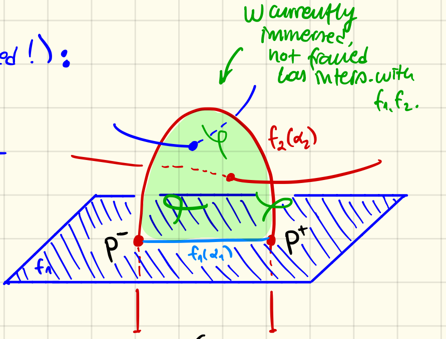
It follows that $\exists\, W \hookrightarrow M^4$ with $\partial W = f_1(\alpha_1) \cup f_2(\alpha_2)$

CASE EASY (Embedded):

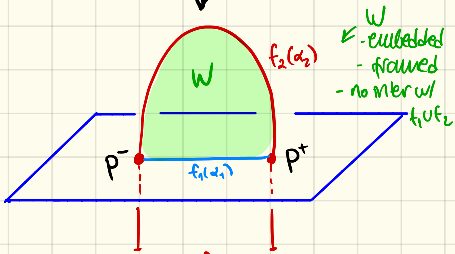
Assume W can be chosen such that (∂W fixed!):

1) W is embedded : $W \hookrightarrow M$
2) W is framed : $t(W) = 0 \in \mathbb{Z}$
3) $W(\mathring{D}^2) \cap (f_1 \cup f_2) = \emptyset$.

Then the Whitney move does the job.



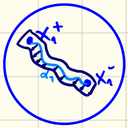
we will later reduce to the case E



f_2 and red arc moves same through time to form its rectangle

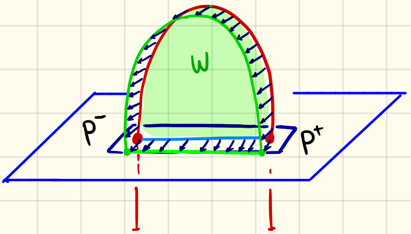
proof. Note: $\nu(W) \cong \mathbb{D}^4$

with:



$\nu W \cap f_1$ rectangle around α_1

$\nu W \cap f_2$ rectangle around α_2.



$*$

$z_1 :=$ unit vector field along ∂W

which is:
- normal to W
- non-vanishing
- tangent to f_1
- normal to f_2

Choice of such z_1 gives $\nu(W) = \mathbb{D}^4 \cong \mathbb{D}^3 \times I$

It gives an associated invariant called the twist number of W:

$$t(W) := \langle e(\nu W, z_1), [W, \partial] \rangle \in \mathbb{Z}$$

Having that $t(W) = 0$ tells us that we can indeed draw ∂W as $*$ and find W exactly as embedded disc there:

$W \subseteq \mathbb{D}^3 =$ normal bundle of W along z_1.

Now do the Whitney move and note: 3) $\Rightarrow$ no new intersections between f_1 & f_2.

$\square$ of case E.

General Case: Reduce to E in 3 steps:

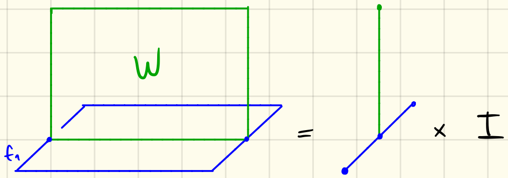
1) Arrange that W is framed
2) Arrange that W is embedded

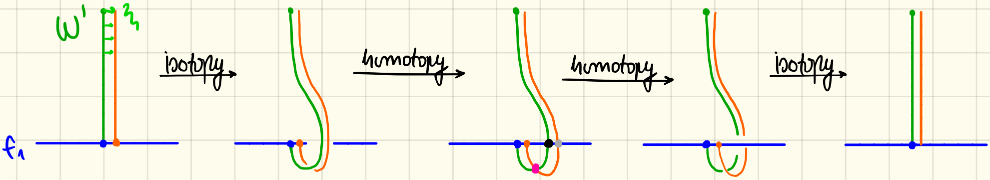
} $\rightarrow$ does not change $\underline{f}$.

3) Arrange that $\emptyset = W(\mathring{D}) \cap (f_1 \cup f_2)$

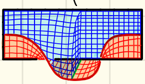
need self-finger moves on f_1 / on f_2

1): Near a point in $\partial W \cap f_1$ we replace a small rectangle in W by the boundary twist and obtain W':





another picture:



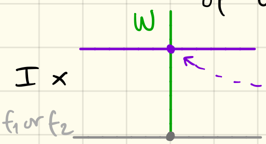
Disc in $\mathbb{R}^3$ with a line of self-intersection (green). Perturb interior of blue into the past, and interior of red into the future to get an immersed disc with 1 d.p.

black pt is new intersection of W and f_1

pink pt is intersection of W with the normal push-off so that:

$$t(W') = t(W) \pm 1.$$

2&3): Another trick: pushing down intersections of W with $(f_1 \cup f_2)$:

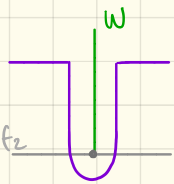


$= W$ or f_1 or f_2

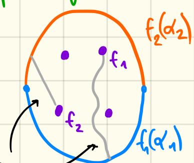
Want to get rid of this!

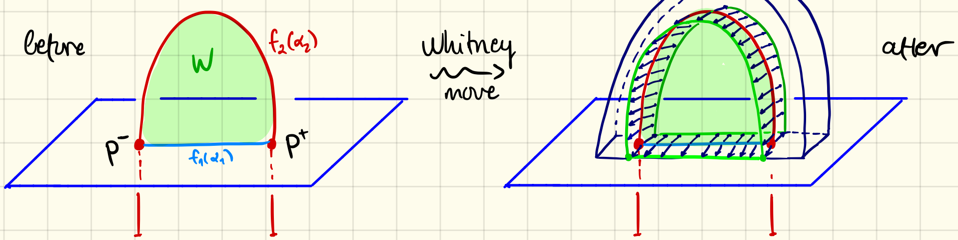
$\rightsquigarrow$

finger move along an arc in W across ∂W



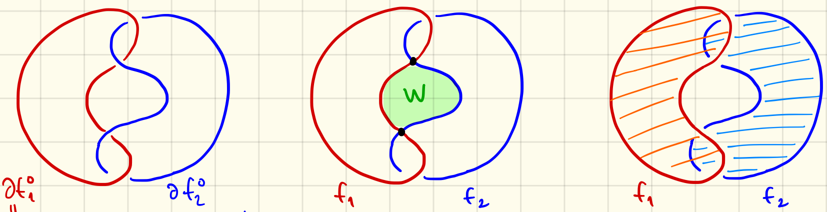
The preimage in $W = D^2$:





Symmetric version :

before

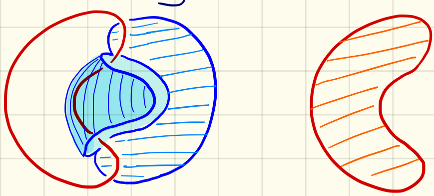


$\partial f_1^\circ = \partial(\nu W \cap f_1)$ $\partial f_2^\circ = \partial(\nu W \cap f_2)$

middle level of null-homotopy of $(\partial f_1, \partial f_2)$

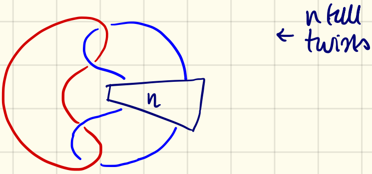
This is where we see two intersection points and the Whitney disk.

after



Remark. If W is embedded, with twisting number $t(W) = n$ we can still draw the symmetric version :

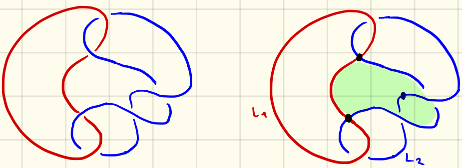
$L^{(n)} :=$



Not nice for $n \neq 0$!

Check that : $\mathcal{K}(L_n) = n \cdot t$

$\boxed{n=1}$



Can still see a Whitney disk but not framed !

Can do a boundary twist so that framed, but have intersection with L_2.

HIGH DIMENSIONAL CASE: $f: S^3 \looparrowright M^6$

$\hat{\mu}(f) = 0 \iff f$ represented by an embedding.

proof.

pair up double points: $q_1^+, q_2^+ \longmapsto p_+$

$q_1^-, q_2^- \longmapsto p_-$

get $W^2 \looparrowright M^6$ s.t.
(i) W embedded ✓
(ii) $\mathring{W} \cap f = \emptyset$ ✓
(iii) non-vanishing normal vector field: νW is 4-dim'l
$\Rightarrow$ no obstruction to finding a section ξ. □

<u>Theorem</u> [Schneiderman-Teichner, AGT 2001.]
Assume $f: (D^2 \sqcup D^2 \sqcup D^2, \partial) \looparrowright (M^4, \partial)$, where M oriented, satisfies:

$$\lambda_M(f_i, f_j) = 0 \quad \forall i \neq j$$

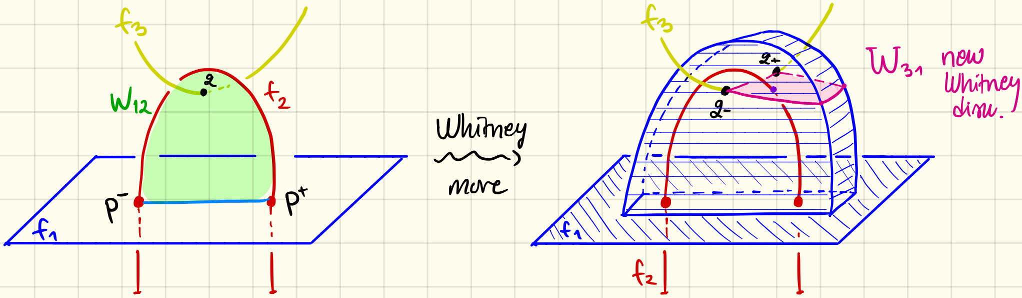
Then:
f is homotopic to f' with f_i' all disjoint $\iff$ $\lambda(f_1, f_2, f_3) = 0 \in \dfrac{\mathbb{Z}[\pi_1 M \times \pi_1 M]}{\text{relations coming from } \lambda(f_i, S) \text{ for } S \in \pi_2 M}$.

a trilinear hermitian form

Example. $M = D^4$ and $\partial f =$ Borromean rings
then:
$$\lambda(f_1, f_2, f_3) = \pm 1 \in \frac{\mathbb{Z}[1\times 1]}{\lambda(f_i, S) = 0} = \mathbb{Z}$$

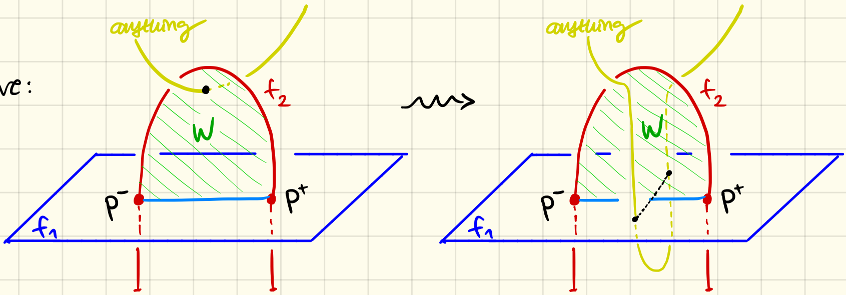
In fact this is called the Milnor invariant $\mu_{123}(L)$ of this 3-component link $L = \partial f$ with linking numbers $lk(L_i, L_j) = 0$

If $F_i \hookrightarrow S^3$ are Seifert surfaces for L_i with $F_i \cap L_j = \emptyset \quad \forall i \neq j$
then:
$$\mu_{123}(L) = \# F_1 \pitchfork F_2 \pitchfork F_3$$



Idea: Count such intersections $f_3 \cap W_{12}$ <=> $f_2 \cap W_{31}$

Recall the push-down move:



Can't do this now! Since we would get new intersection $f_1 \cap f_3$.

Warm-up: Let $\pi_1 M = \pi$

$$\mathbb{Z}[\pi \times \pi / \Delta\pi] \xrightarrow{\cong} \mathbb{Z}\pi$$

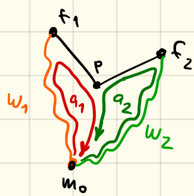
defined via $\quad (a_1, a_2) = (a_1 g, a_2 g) \longmapsto a_1 a_2^{-1}$

Involution:

$$\overline{(a_1, a_2)} = (a_2, a_1) \quad \longleftarrow - - \rightarrow \quad \overline{g} = g^{-1}$$

Now the relation:

$$\lambda(f_1, f_2) = \overline{\lambda(f_2, f_1)} \quad \text{follows from} \quad g_p = a_1 a_2^{-1}$$



Can see this as $S_2 \curvearrowright \mathbb{Z}[\pi \times \pi / \Delta\pi]$

and for $\sigma \in S_2$ have: $\lambda(f_{\sigma(1)}, f_{\sigma(2)}) = \lambda(f_1, f_2)^{\sigma}$

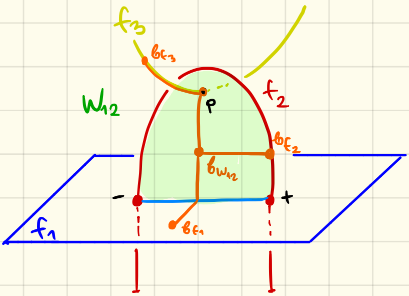
Now define:

$$\mathbb{Z}\left[\pi\times\pi\times\pi/\Delta\pi\right] \circlearrowleft S_3 \qquad (g_1,g_2,g_3)^\sigma := \operatorname{sgn}(\sigma)\cdot(g_{\sigma(1)},g_{\sigma(2)},g_{\sigma(3)})$$

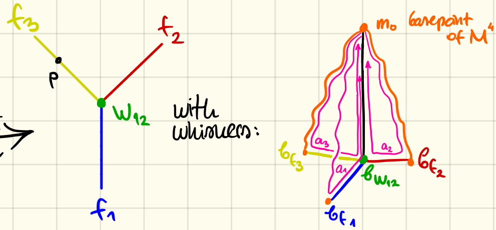
We will define $\lambda(f_1,f_2,f_3)$ living here which is "S_3-hermitian"

i.e.

$$\lambda(f_{\sigma(1)},f_{\sigma(2)},f_{\sigma(3)}) = \lambda(f_1,f_2,f_3)^\sigma$$



$\rightsquigarrow$ read off a tree



Definition.

$$\lambda(f_1,f_2,f_3) = \sum_{\sigma\in S_3}\ \sum_{p\in W_{\sigma(1)\sigma(2)}\pitchfork f_{\sigma(3)}} \overset{\{\pm1\}}{\underset{}{\overset{\cup}{\varepsilon_p}}}\cdot(a_1(p),a_2(p),a_3(p))$$

$$\in \mathbb{Z}\left[\pi\times\pi\times\pi/\underbrace{\Delta\pi}\right]$$

we had to mod out to get independence of the choice of a whisker for W_{12}

where we denote:

$a_i(p) = (\text{whisker to } b_{f_i})\cdot(\text{path from } b_{f_i} \text{ to } b_{W_{ij}})\cdot(\text{path from } b_{W_{ij}} \text{ to } m_0)$.

Class 12

Nov 20 Tue

Theorem. [ST2001] M^4 conn. oriented, $\pi = \pi_1 M$

whitney chosen

$\hookrightarrow$ $f : (\mathbb{D}^2 \sqcup \mathbb{D}^2 \sqcup \mathbb{D}^2, \partial) \looparrowright (M, \partial)$ is represented (up to homotopy rel ∂)

by a map with 3 disjoint images iff $\lambda(f_i, f_j), \forall i \neq j$ and $\lambda_3(f)$ vanish.

where:

$\lambda(f_i, f_j) = \sum_{p \in f_i \pitchfork f_j} \varepsilon_p g_p \in \mathbb{Z}\pi$ and $\lambda_3(f_1, f_2, f_3) := \sum_{p \in W_{ij} \pitchfork f_k} \varepsilon_p \cdot [g_1(p), g_2(p), g_3(p)]$

$\Leftrightarrow g_p = $ f_i — f_j, m_0

$T = $ $f_3, g_3, g_2, f_2, g_1, f_1$ $\in \dfrac{\mathbb{Z}[\pi^3/\Delta\pi]}{INT}$

These are homotopy rel ∂ invariants.

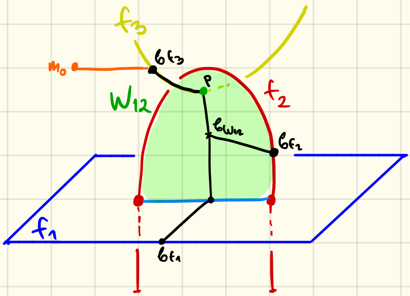
Recall. $\lambda(f_i, f_j) = 0 \iff \exists W_{ij}$ = Whitney disc pairing intersections $f_i \pitchfork f_j$

(can make it embedded & framed & disjoint from f_i & f_j)

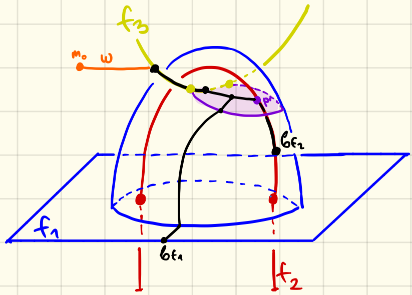
but can't control $W_{ij} \cap f_k$!

Hence: λ_3 will precisely measure this

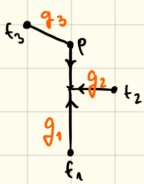
and will not change under Wh. move:

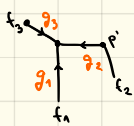


Whitney move



assoc. tree:



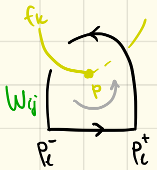


Change notation:

$$\lambda_3(f_1, f_2, f_3) := \sum_{T = \,f_3, g_3, g_2, f_2, g_1, f_1} \varepsilon_T \cdot [g_1(T), g_2(T), g_3(T)] \in \frac{\mathbb{Z}[\pi^3/\Delta\pi]}{INT}$$

Orientations:

f_i are oriented $\circlearrowright$ $\xrightarrow[\text{whitney and orient. of } M]{\text{using}}$ $f_i \pitchfork f_j$ are oriented i.e. Have pairs $p_\ell^+, p_\ell^- \in f_i \pitchfork f_j$ with $g_{p_\ell^+} = g_{p_\ell^-}$



$\xrightarrow[i<j]{\text{assume}}$ W_{ij}^ℓ is oriented by going from p^- to p^+ along f_i and p^+ to p^- along f_j.

$\Longrightarrow$ get a sign of any intersection of W_{ij}^ℓ with f_k

Lemma. $\varepsilon_p = \varepsilon_{p'}$. **Consequence.** $\varepsilon_p [g_1(p), g_2(p), g_3(p)]$ is unchanged under Wh. move.

Claim. Fixing the choice of Whitney discs: $W_{12}^i, W_{31}^i, W_{23}^i$ pairing all (non-repeating) intersections (so not self-intersections) then:

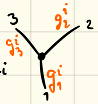
$$\lambda_3(\underbrace{f_1, f_2, f_3, W_{12}^i, W_{31}^i, W_{23}^i}_{\text{this is called a \underline{Whitney tower} } \mathcal{W} \text{ of order 1 on } f.}) \in \mathbb{Z}[\pi^3/\Delta\pi]$$

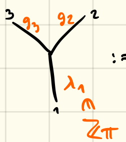
is well-defined!

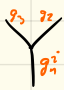
Claim. $\lambda_3(f; \mathcal{W}) = \lambda_3(f; \mathcal{W}')$ in $\dfrac{\mathbb{Z}[\pi^3/\Delta\pi]}{\mathrm{INT}}$

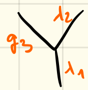
Remark. $\lambda_3(f_{\sigma(1)}, f_{\sigma(2)}, f_{\sigma(3)}) = \lambda_3(f_1, f_2, f_3)^\sigma$

Def. of $\mathrm{INT} \subseteq \mathbb{Z}[\pi^3/\Delta\pi]$ as follows:

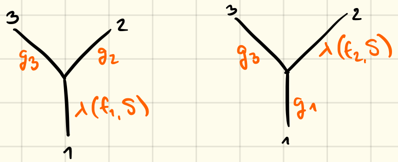
typical element: $\sum_i n_i$ 

let 

$:= \sum_i n_i$ 

and similarly 

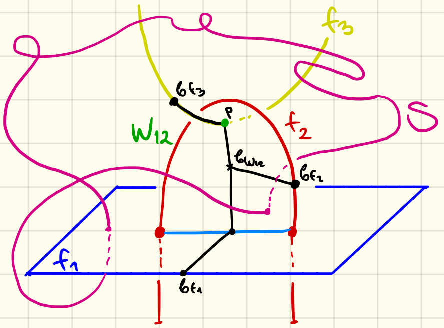
INT is the $\mathbb{Z}$-span of :

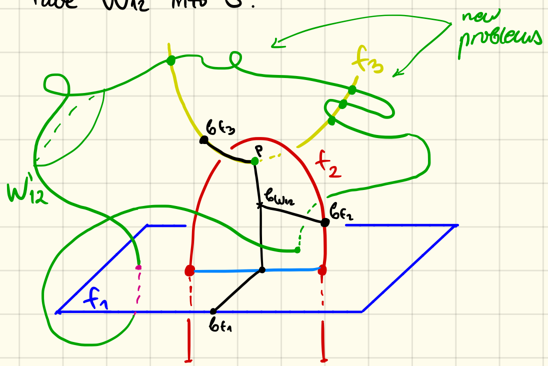


for all $S \in \pi_2 M$
$g_1, g_2, g_3 \in \pi$.

proof of THM:

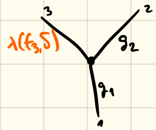
Step 1. These relations are necessary : Let S be any generic sphere in M.
Tube W_{12} into S:





Get W'_{12} so that :

$$\lambda_3(f;W') = \lambda_3(f;W) +$$

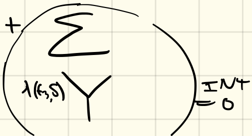


Step 2. $\lambda_3(f;W) = \lambda_3(f;W')$
if W' changed by the following choices :

a) ways to pair $f_1 \pitchfork f_2$
b) choices of Whitney arcs ∂W^e_{ij}
c) choices of the interior W^e_{ij}.

less obvious.
Next time.

make a sphere S from two choices. $\lambda_3(f;W') = \lambda_3(f;W)$



Step 3. $\lambda_3(f) = \lambda_3(f')$ if $f \underset{rel\,\partial}{\simeq} f'$

proof. f' differs from f by:

- ISOTOPY
- CUSP HOMOTOPY
- FINGER / WHITNEY MOVES.

ISOTOPY: Ambient Isotopy Theorem for stable maps.

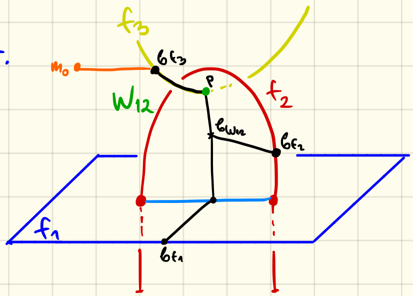
saying: isotopy embeds into an ambient isotopy
i.e. diffeotopies on domain and range realizing isotopy $f \rightsquigarrow f'$.

So take W for f and drag it along using the ambient isotopy.

CUSP HOMOTOPY: only produces self-intersections, so irrelevant.

FINGER MOVE: fixed f, W.
do finger move, get a clean Wh. disc   NEXT TIME.

Step 4.



Theorem: [S-T] M connected, oriented 4-mfd, $\pi := \pi_1 M$

$f : (D^2 \amalg D^2 \amalg D^2, \partial) \hookrightarrow (M^4, \partial)$ is

homotopic rel. ∂ to a map with 3 disjoint images iff the following invariants vanish:

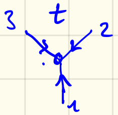
Quadratic: $\lambda(f_i, f_j) = \sum_{f_i \xrightarrow{t} f_j} \varepsilon_t \cdot g_t \in \mathbb{Z}\pi$

$\lambda(f_i, f_j) = 0 \iff \exists$ Whitney disks W_{ij}^{ℓ} pairing all points in $f_i \pitchfork f_j$.

Choose an order 1 non-repeating Whitney tower

$$\mathcal{W} = (f_1, f_2, f_3, W_{12}^{\ell}, W_{23}^{\ell}, W_{31}^{k}) \text{ for } f$$

$$\lambda_3(\mathcal{W}) := \sum_{t} \varepsilon_t \cdot [g_t^1, g_t^2, g_t^3] \in \mathbb{Z}\pi^2$$

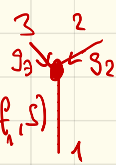


is the cubic intersection invariant for $\mathcal{W}$.

Main Lemma: $\lambda_3(f) := [\lambda_3(\mathcal{W})] \in \dfrac{\mathbb{Z}\pi^2}{\text{INT}}$

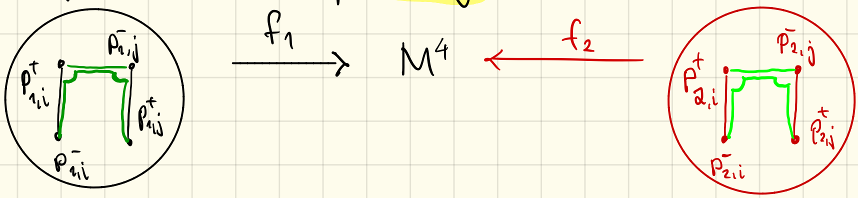
only depends on

$[f_1], [f_2], [f_3] \in \pi_2(M, \partial)$

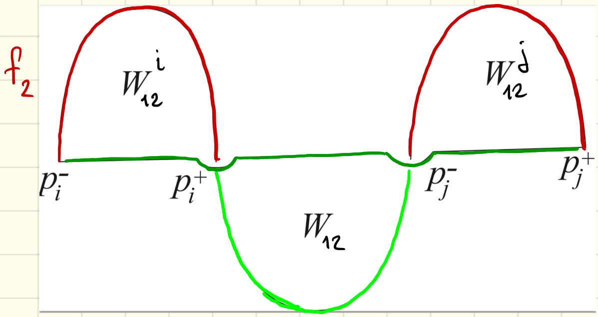
$\langle$ , ... / $S \in \pi_2 M \rangle$

Proof takes a number of steps:

1. Independence of pairing



$$g_i^+ = g_i^- \,,\; g_j^+ = g_j^- \iff W^i_{12} \text{ resp. } W^j_{12} \text{ exist}$$



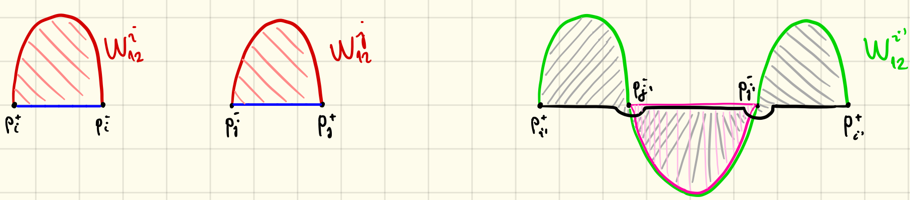
$$W'_{12} := W_{12} \sqcup W^i_{12} \sqcup W^j_{12}$$

If $g_i^+ = g_j^-$ then **W_{12} also exists**!

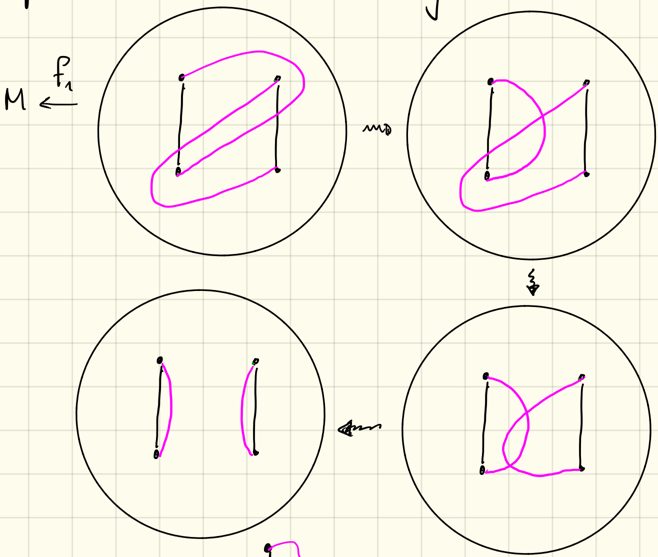
$$\lambda(f_3, W^i_{12}) + \lambda(f_3, W^j_{12}) = \lambda(f_3, W'_{12}) + \lambda(f_3, W_{12}) \qquad \blacksquare$$

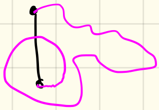
first choice of pairing:

second choice of pairing:



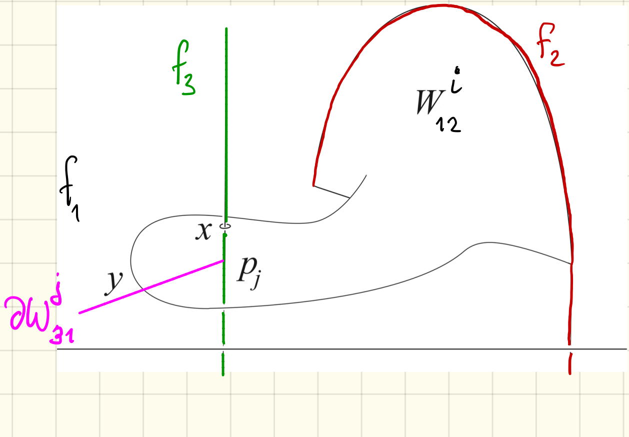
2. Independence of Whitney arcs



Any pair of arcs  in D^2 are isotopic but we have to cross other arcs.

$\Longrightarrow$ Need to allow ∂W^k to intersect!

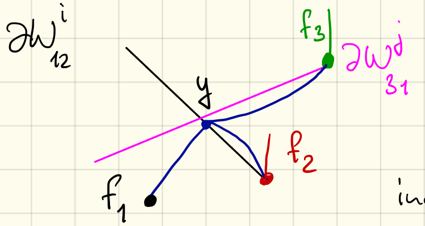
We can carry the W-disks along during the isotopy of arcs:



At a critical moment ∂W^i_{12} crosses some p_j and we see two new problems:

x and y

$\Longrightarrow$ Need to include ∂W^i_{12} & ∂W^j_{31} into our count using the foll. tree:

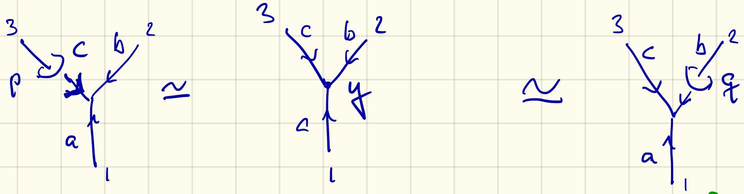


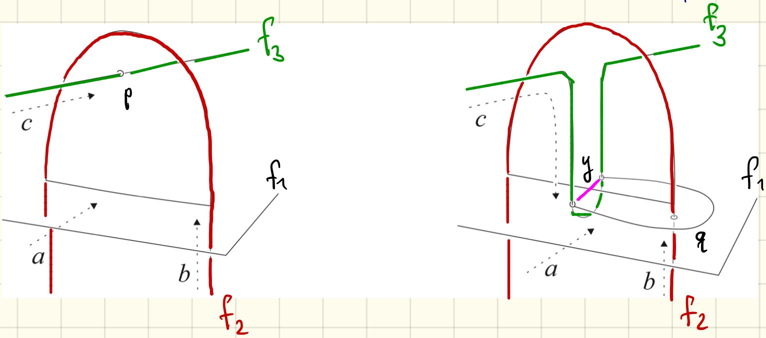
Easy check:

$$t_y = t_x$$

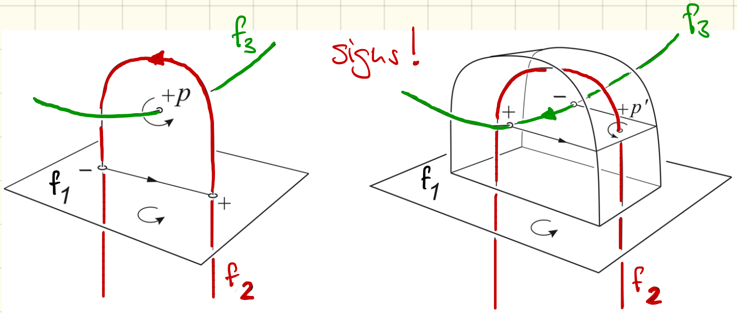
including signs.

Note that pushing down intersections is now an in between step for

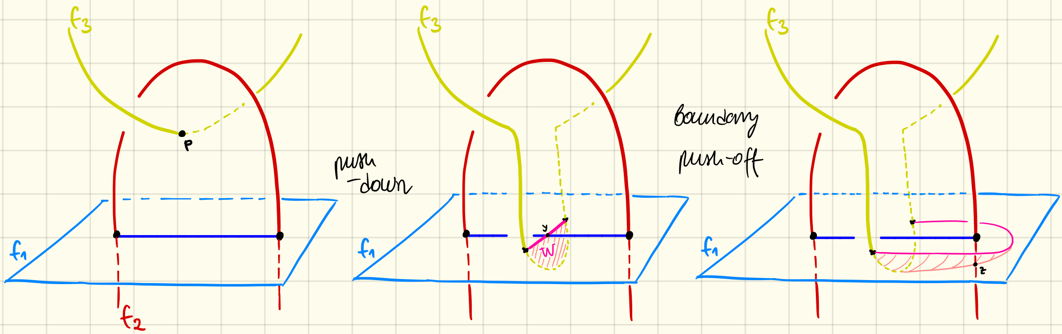




This is isotopic to our previous move:



choices of Whitney arcs (another picture for ☆)



$t_p = t_y = t_z$

3. Independence of $\mathring{W}_{ij}^{k}$ forced us to introduce INT-relation

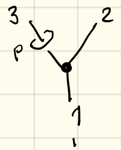
This finishes proof of main lemma ∎

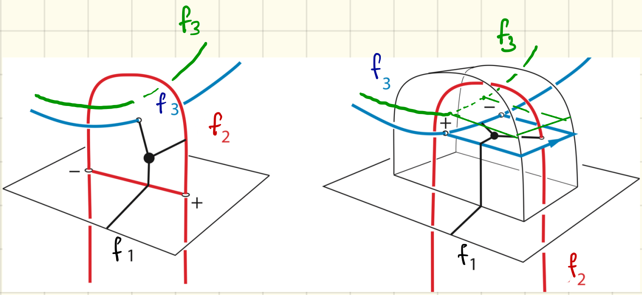
Proof of Theorem:

Assume $\lambda_3(f) = 0$. Then we can find a Whitney tower $\mathcal{W}$ (after adding spheres!) and that

$$\lambda_3(\mathcal{W}) = 0 \in \mathbb{Z}\pi^2.$$

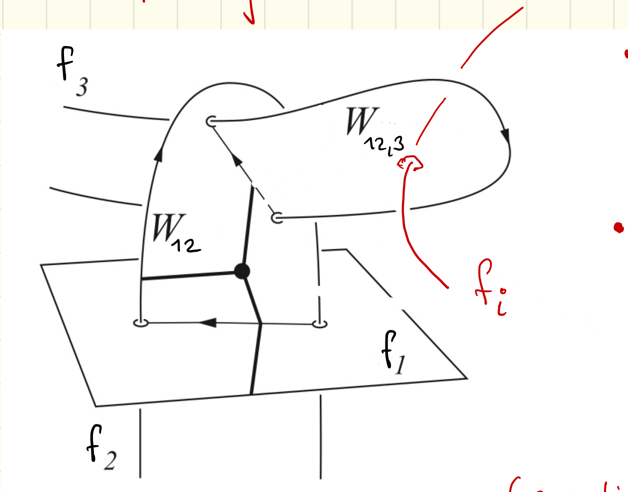
After a number of our moves,
may assume that all trees are of the form





Finally, remove all of $f_1 \pitchfork f_2$:
(assuming all trees lie on one W_{12})

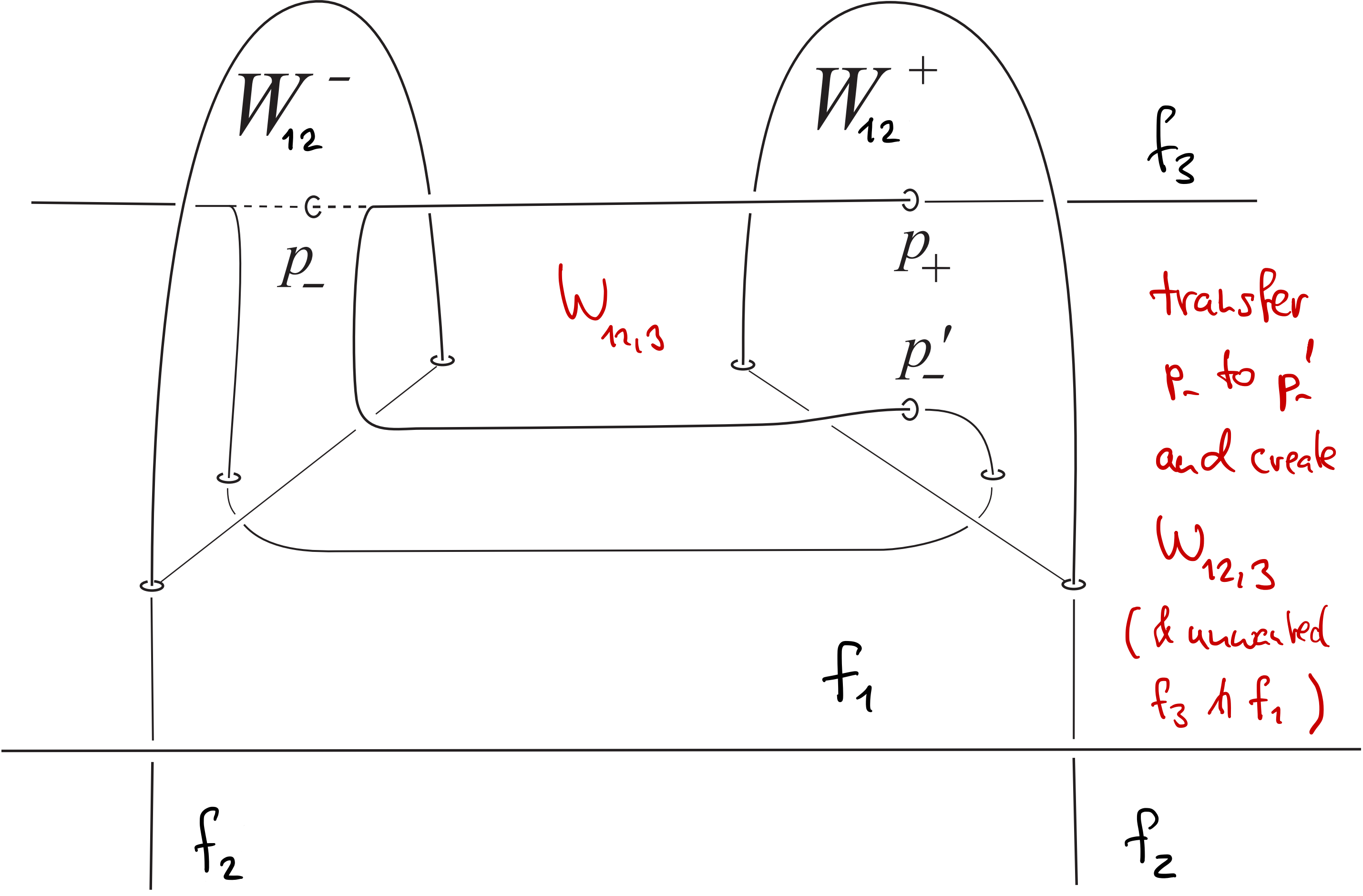
Depending on the index i we push f_i down across



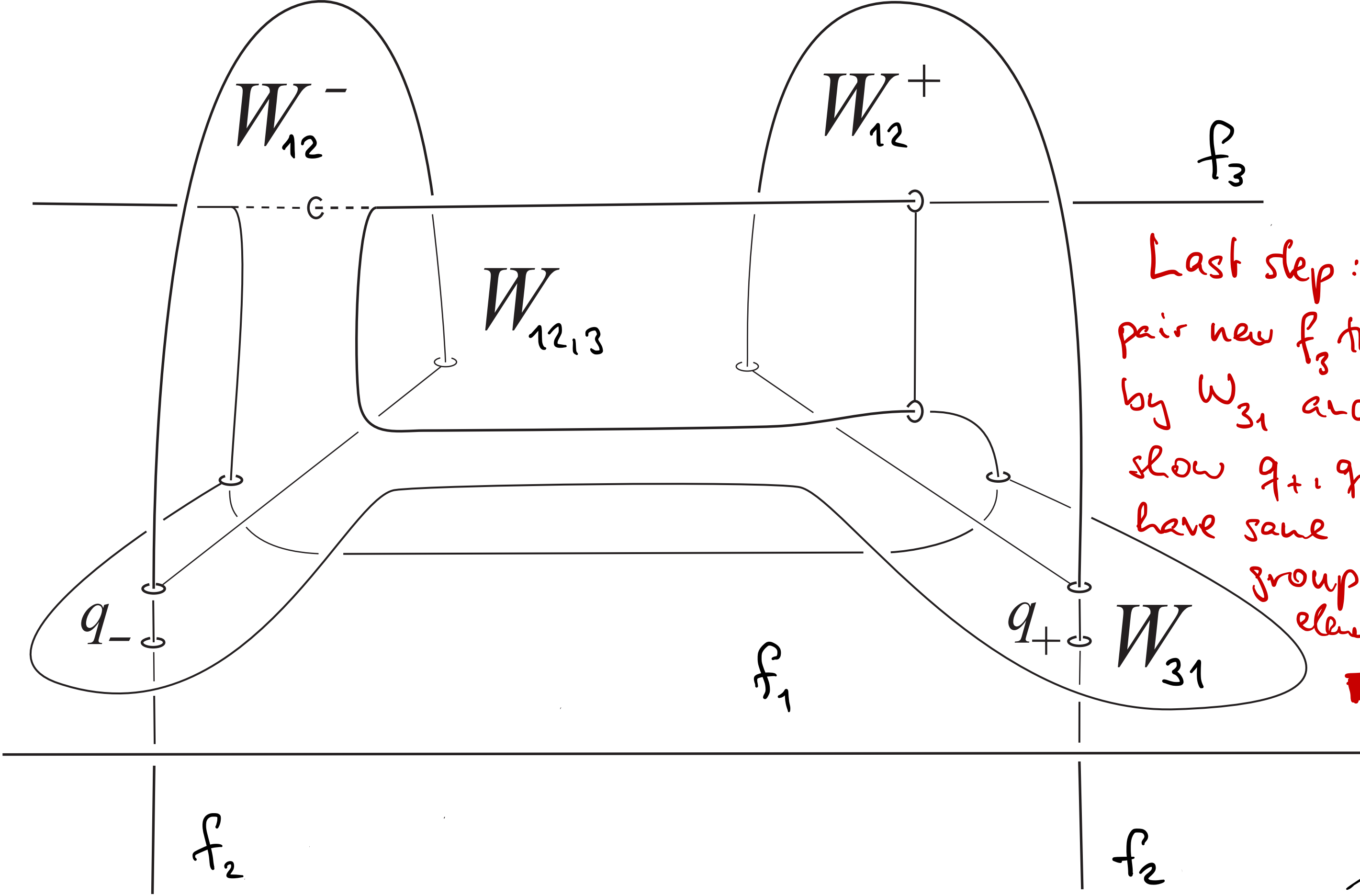
- $\partial W_{12,3} \cap f_3$ for $i = 3$
  (creating new $f_3 \pitchfork f_3$)
- $\partial W_{12,3} \cap W_{12}$ for $i = 1, 2$

In this case we push both new intersections with W_{12} down across $\partial W_{12} \cap f_i$ ($i = 1, 2$).
(creating self-intersections of f_i).

Then $W_{(1,2),3}$ is clean and a Whitney move gives clean W_{12}, further W. move makes f_i disjoint.



W_{12}^-
W_{12}^+
f_3
p_-
p_+
p_-'
$W_{12,3}$
transfer p_- to p_-' and create $W_{12,3}$ (& unmarked $f_3 \pitchfork f_1$)
f_1
f_2
f_2



Last step: pair new $f_3 \pitchfork f_1$ by W_{31} and show q_+, q_- have same group element. ∎

Theorem 3. $f:(\amalg^3 D^2, \partial) \looparrowright (M^4, \partial)$, M connected, $\pi := \pi_1 M$

The following are equivalent:

(i) $f \underset{\text{rel}\,\partial}{\simeq} f'$ and f_1', f_2', f_3' have disjoint images.

obv. $\Downarrow$ $\Uparrow$ push down

(ii) f extends to a non-repeating Whitney tower of order 2

obv. $\Downarrow$ $\Uparrow$ transfer move: transfer pair of inter. to the same Wh. disc

(iii) f extends to a non-repeating Whitney tower of order 1 $\mathcal{W}_1 = \{W_{12}^i, W_{23}^i, W_{31}^i\}$

with $(g_1, g_2, g_3) \longmapsto$ 

$$\lambda_3(f, \mathcal{W}_1) = 0 \quad \text{in} \quad \mathbb{Z}[\pi^3/\Delta\pi] =: \Lambda_3(\pi)$$

obv. $\Downarrow$ $\Uparrow$ add spheres to a choice of $\mathcal{W}_1$ to get $\lambda_3(f, \mathcal{W}_1) = 0$ in $\mathbb{Z}\pi^2$.

(iv) Both $\lambda_2(f) = 0$ and $\lambda_3(f) = 0 \in \dfrac{\mathbb{Z}[\pi^2]}{\text{INT}}$

Theorem 4. $f:(\amalg^m D^2, \partial) \looparrowright (M^4, \partial)$, M connected, $\pi := \pi_1 M$

[S-T 2014] The following are equivalent:

(i) $f \underset{\text{rel}\,\partial}{\simeq} f'$ and f_i' have disjoint images.

obv. $\Downarrow$ $\Uparrow$ push down

(ii) f extends to a non-repeating Whitney tower of order $m-1$

obv. $\Downarrow$ $\Uparrow$ transfer move: transfer pair of inter. to the same Wh. disc

(iii) f extends to a non-repeating Whitney tower of order $m-2$ $\mathcal{W}_{m-2}$

with

$$\lambda_m(f, \mathcal{W}_{m-2}) = 0 \quad \text{in} \quad \Lambda_m(\pi)$$

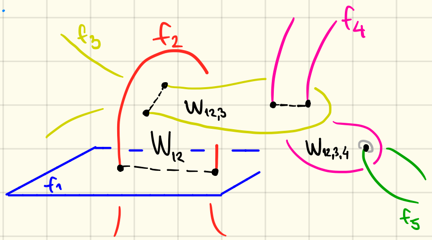
(iv) UNKNOWN where should $\lambda_m(f)$ live?

example.

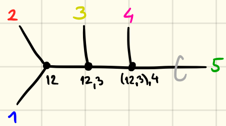
$m=5$

order 3

Wh. tower

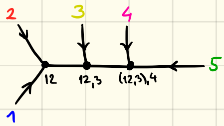


get the intersection tree:



Recall:

Lemma. The little circle 0 can be moved anywhere in the tree.



Def. $\Lambda_m(\pi)$ is the abelian group generated by

labelled oriented uni-trivalent trees

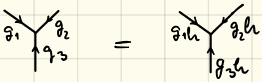
- univalent vertices labelled by elts of $\{1...m\}$
- edges labelled by $g \in \pi$

- edges oriented
- trivalent vertices have cyclic order on incident edges

two types of vertices univalent ($\leq m$ of them) & trivalent

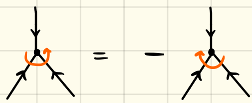
modulo:

holonomy relation:

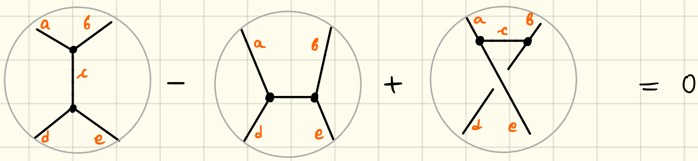


orientation relation:

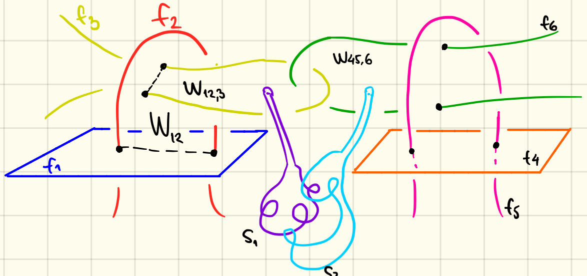
anti-symmetry relation:

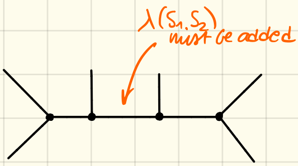


IHX (Jacobi) identity:



(iv)?





examples. (HW8 #2)

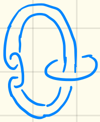
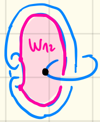
All elements in $\Lambda_m(\pi)$ are realized in any given $(M^4_\#,\partial)$!

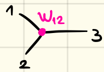
Take $M = D^4$, $f: (\coprod^m D^2, \partial) \hookrightarrow (D^4, \partial)$

Theorem. $\lambda_k(f)$ for $k \le m$ carries the same information as $\mu_{i_1 \dots i_k}(\partial f)$!

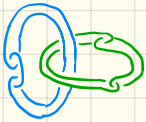
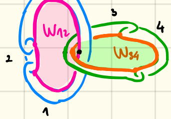
Milnor invariants. — link in S^3

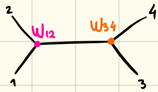
e.g. $\lambda_2 = 1$  Hopf link

$\lambda_2 = 0 \quad \mu_3 = 1$  Borr. rings 



$\downarrow$ Bing double

$\lambda_2 = 0 \quad \lambda_3 = 0 \quad \mu_4 = 1$  BD (Borr) 



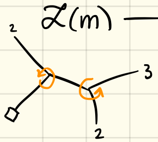
Def. $T_k(m)$ is defined similarly as $\Lambda_m(\pi = \{1\})$ but now allow repeating indices from the set $\{1, 2, \dots m\}$ for labels of univalent vertices with k trivalent vertices.

$$\mathcal{L}(m) := \bigoplus_{k=0}^{\infty} \mathcal{L}_k(m) \qquad \mathcal{L}_k(m) := \text{just like } T_k(m) \text{ except trees have a root}$$

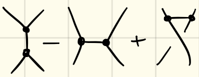
Lemma. $\mathcal{L}(m)$ is the free Lie algebra on m generators.

proof.

$$\mathcal{L}(m) \longrightarrow \mathbb{L}(x_1, \dots, x_m)$$



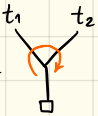
$$[[x_2, x_3]\, x_2]$$

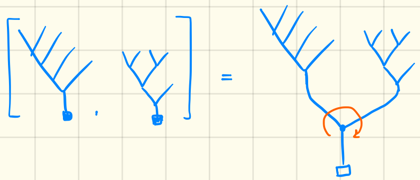
IHX 

$$[[x_3 x_2] x_1] - [x_3 [x_2 x_1]] + [x_2, [x_3, x_1]] \quad \text{Jacobi}$$

$$AS \longmapsto AS$$

Define the Lie bracket on $\mathcal{L}(m)$ by:

$t_1, t_2 \in \mathcal{L}(m) \rightsquigarrow [t_1, t_2] :=$ 

e.g. 

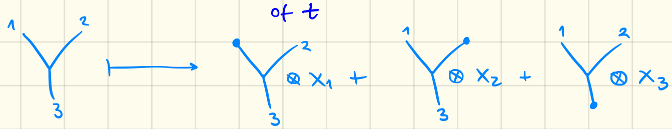
Finally, define the map  $\longleftarrow\!\mapsto X_i$ this is the inverse! $\square$

Remark. $\mathcal{L}(m)$ is $\mathbb{N}$-graded by weight: $\deg(\,\overset{i}{|}\,) = \deg(X_i) = 1$

$\deg(t) = \#$trivalent vertices $+1$

J. Levine map: $\eta : \mathcal{T}_k(m) \longrightarrow \mathcal{L}_k(m) \otimes \mathbb{Z}^m$

$$t \longmapsto \sum_{\substack{\text{choice of} \\ \text{root } r \\ \text{of } t}} t_r \otimes X_{i(r)}$$

e.g. 

This is a well-defined group homomorphism.

Milnor: $\mathcal{L}_k(m) \otimes \mathbb{Z}^m \longrightarrow$ free assoc. alg on $X_1 \dots X_m$ ...

Theorem. [ST '14] η takes $\lambda_{k+2}(f_1 \dots f_m)$ to ...

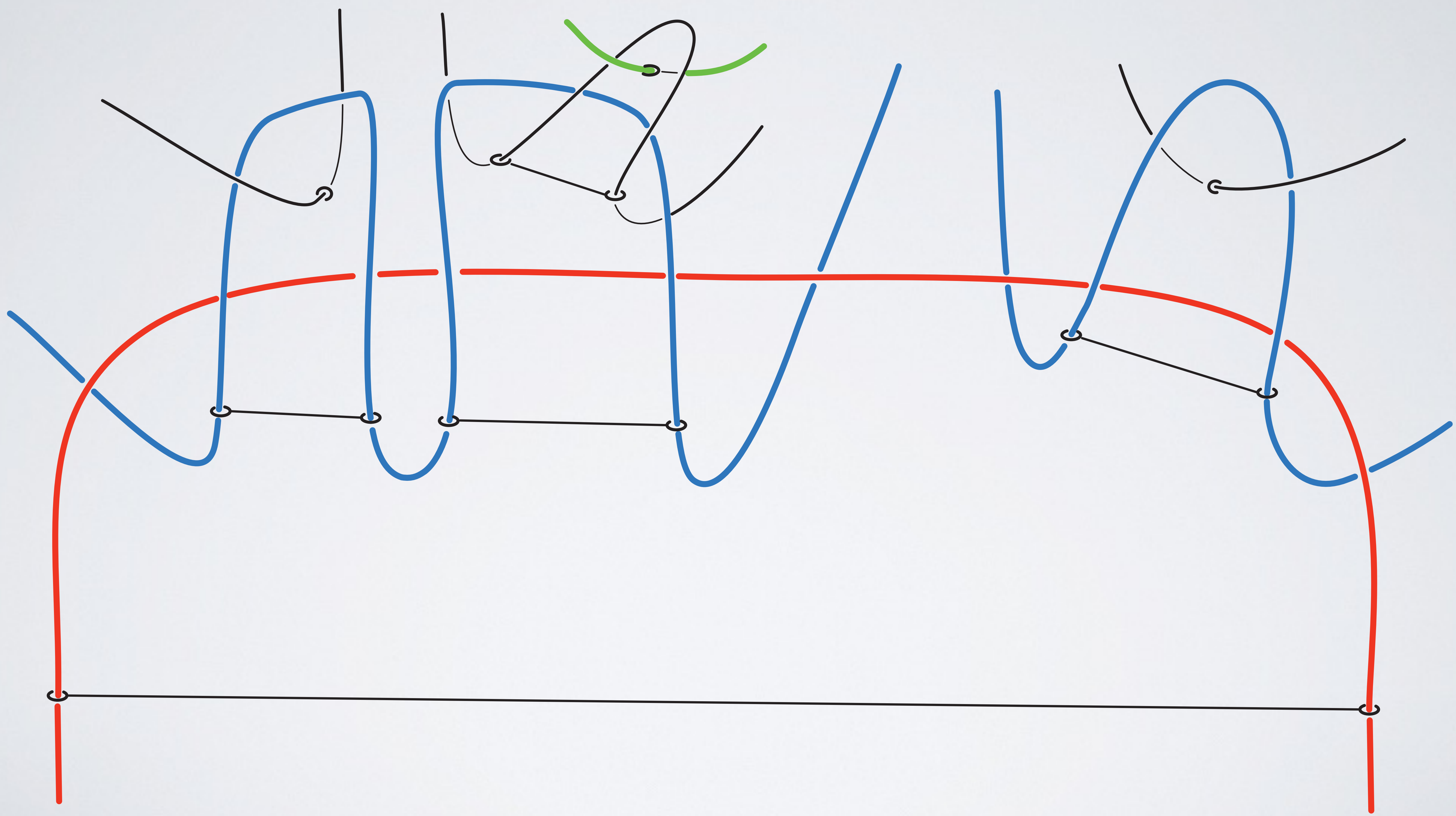
e.g. $\lambda_3(\mathrm{Borr}) =$  $\xmapsto{\text{Milnor}}$ $\mu^{\mathrm{Borr}}_{321} X_3X_2X_1 + \mu^{\mathrm{Borr}}_{231} X_2X_3X_1 + \mu^{\mathrm{Borr}}_{132} X_1X_3 \otimes X_2 - \mu^{\mathrm{Borr}}_{312} + \dots$

Rem. All the symmetries of Milnor invts come from the fact they come from trees! //

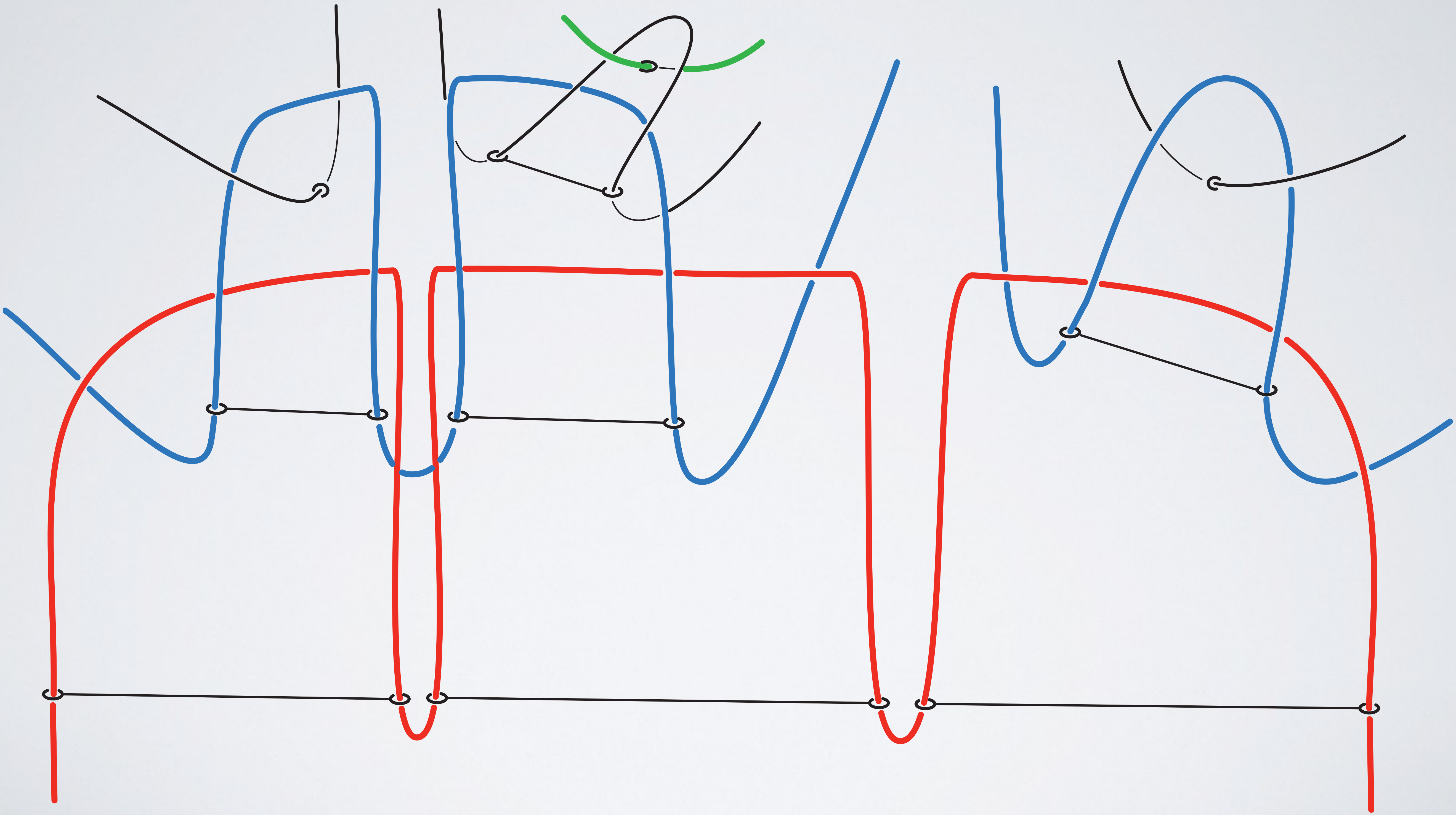
PART OF A WHITNEY TOWER W $\looparrowright$ M^4 WHICH IS CLEAN: ALL W-DISKS ARE FRAMED AND EMBEDDED

Class 15
Nov 29

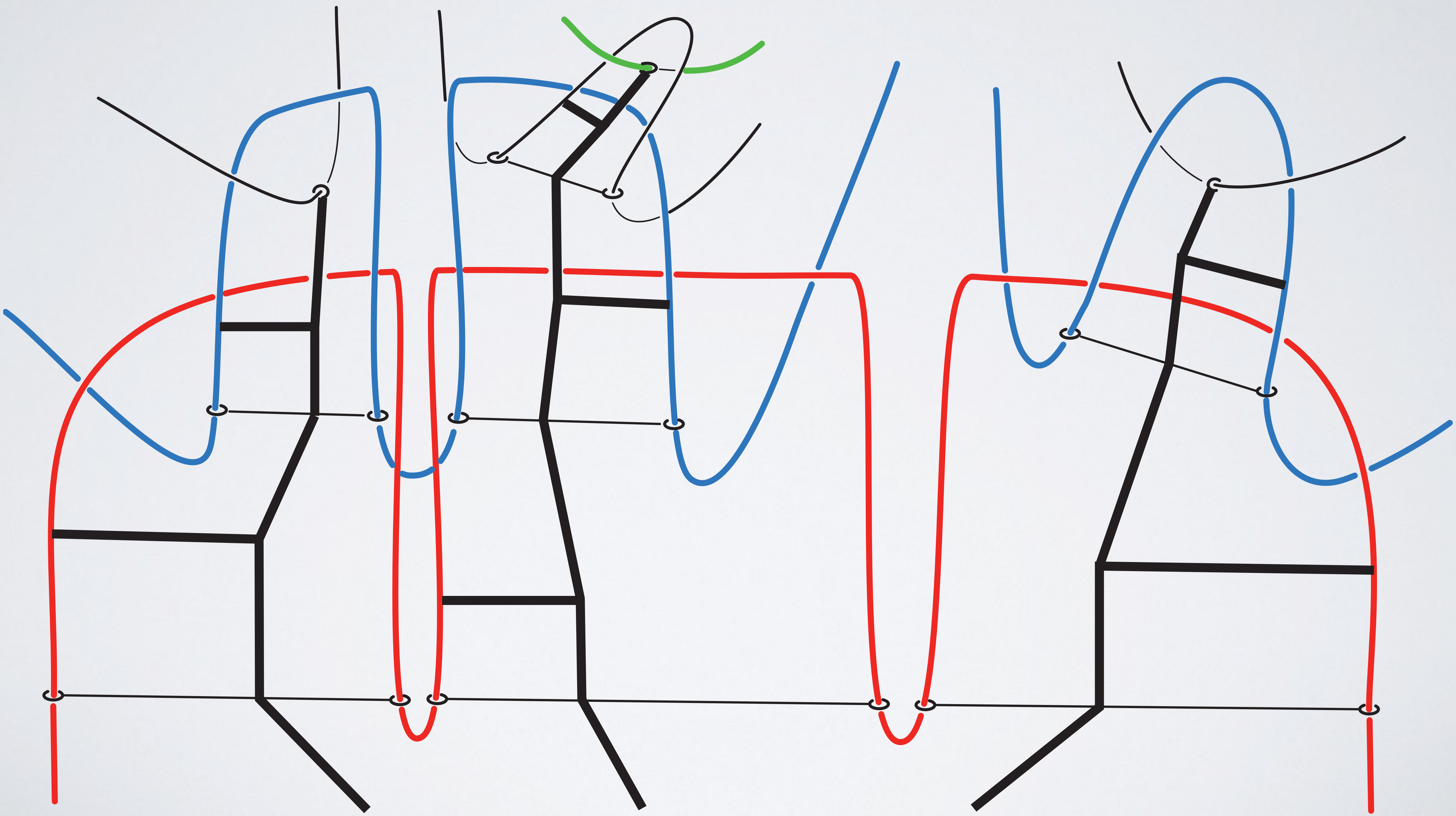
IT CAN BE SPLIT...



...BY FINGER MOVES

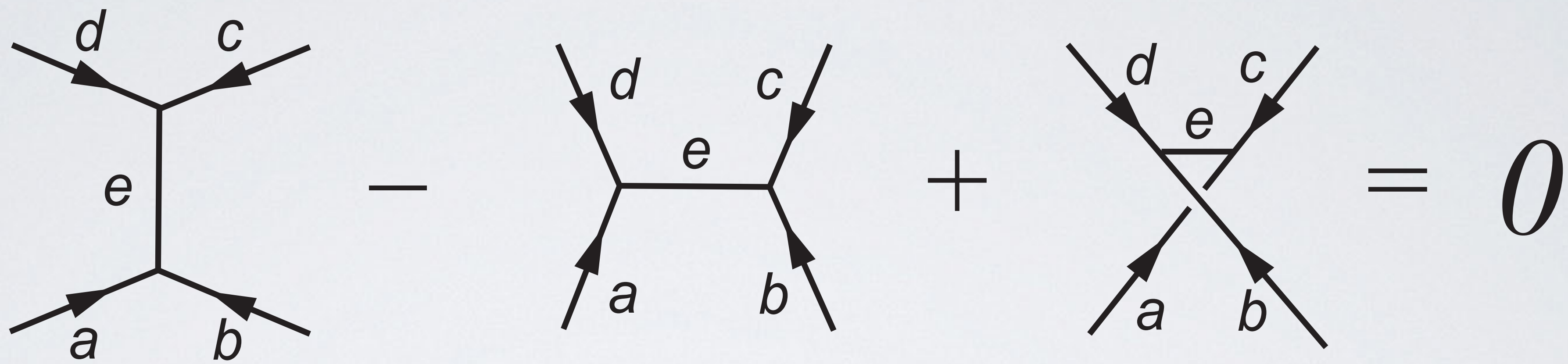


GET ONE TREE FOR EACH TOP ORDER INTERSECTION POINT

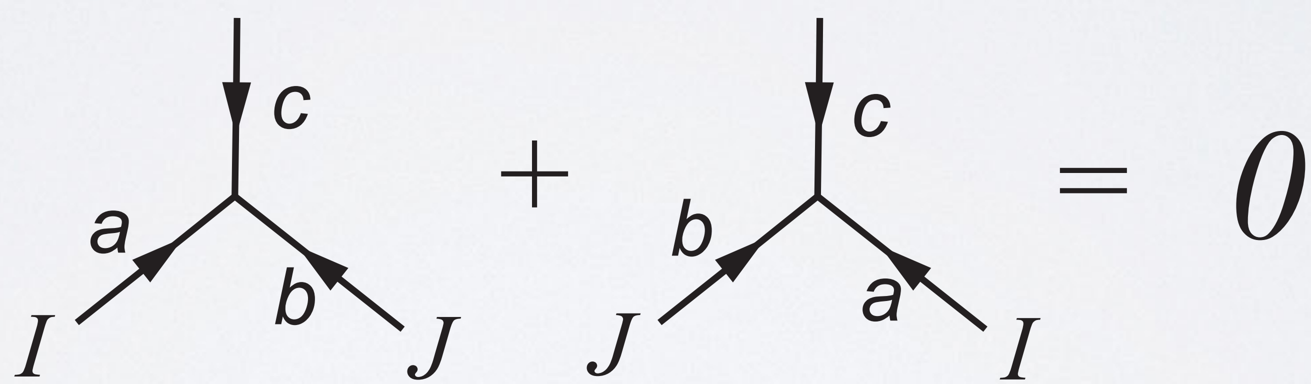


RELATIONS ON ORIENTED TREES

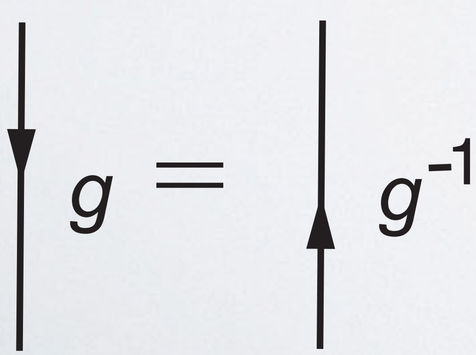
IHX:



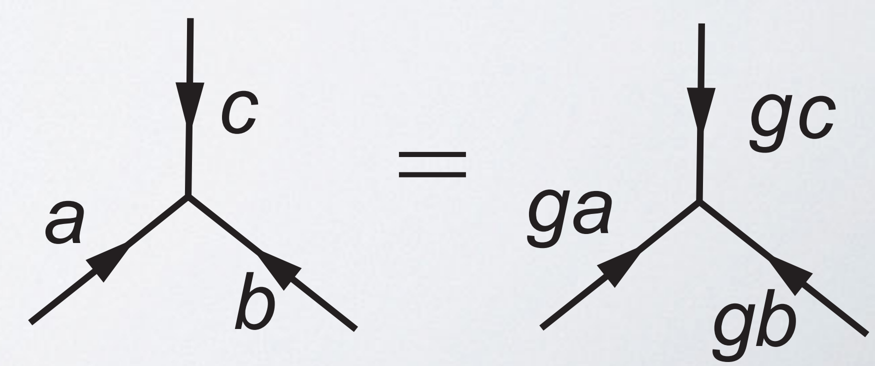
AS:



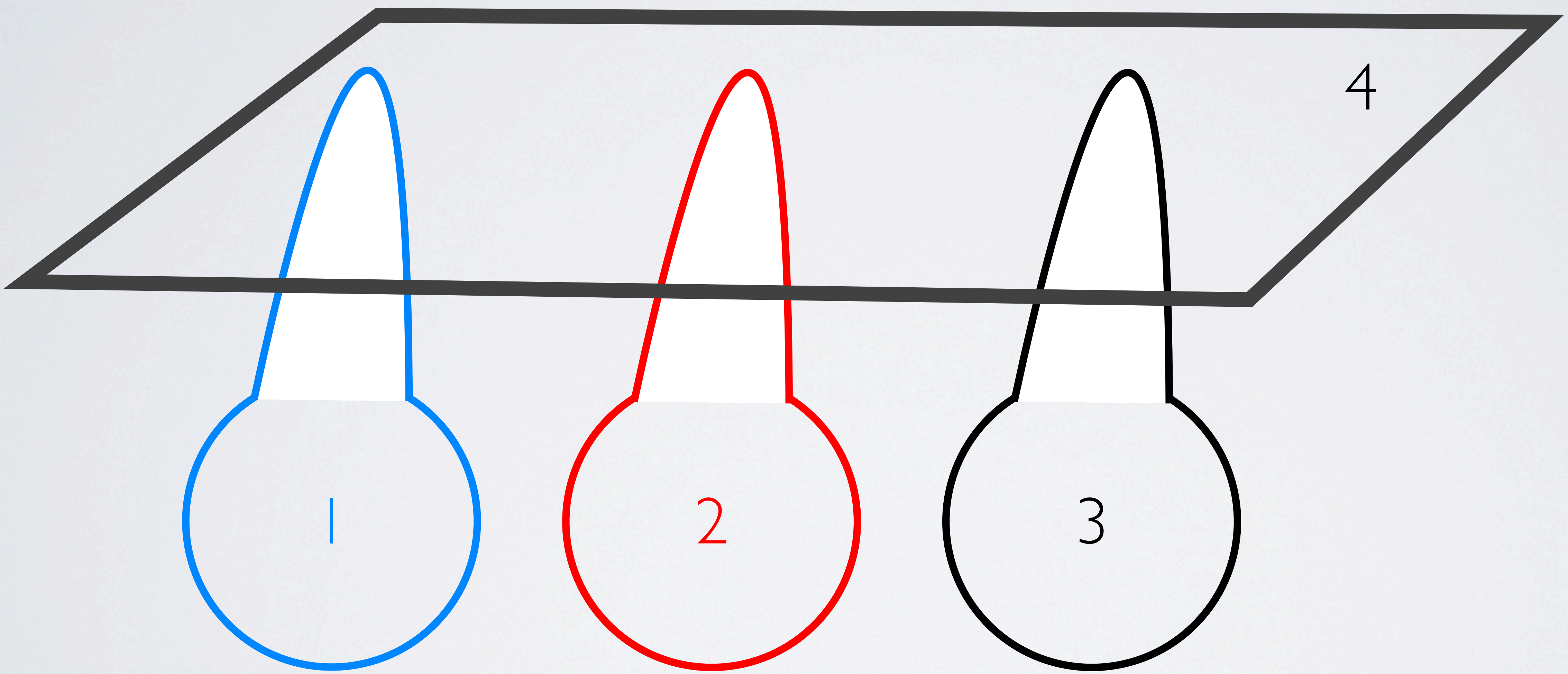
OR:



HOL:

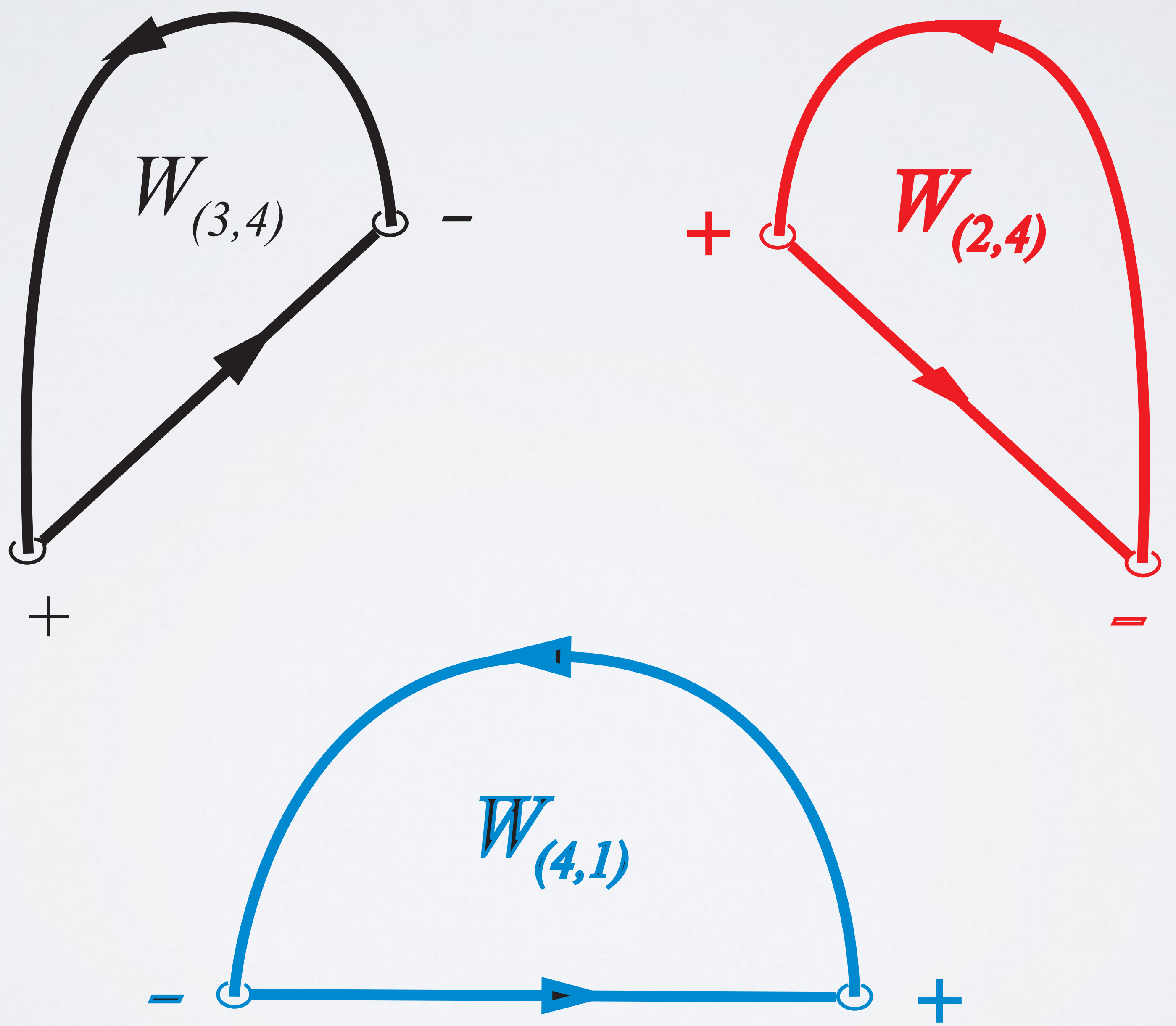


TREES ARE NOT PRESERVED WHEN MOVING WHITNEY ARCS

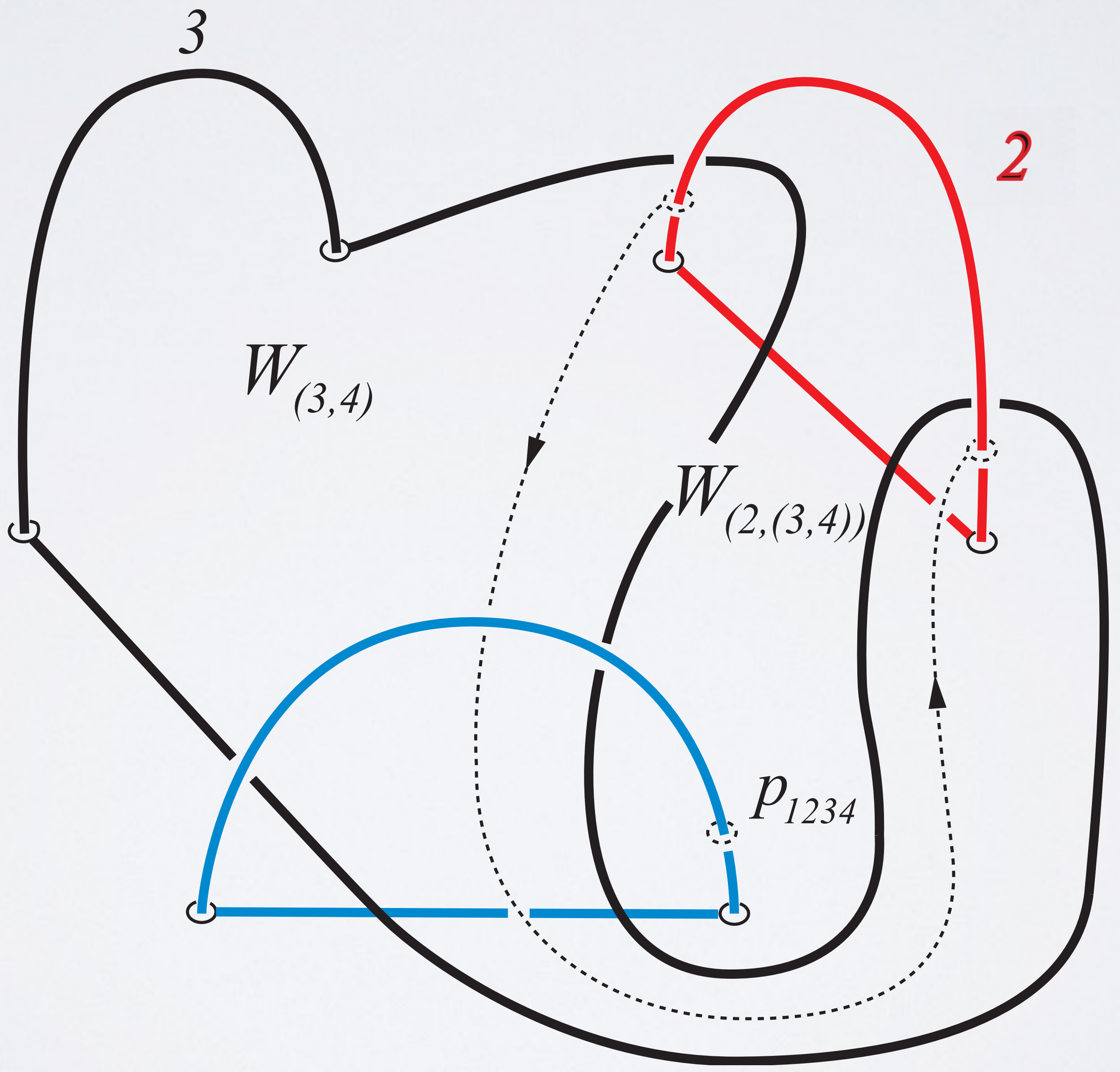


4 small spheres in the 4-ball will be made complicated:

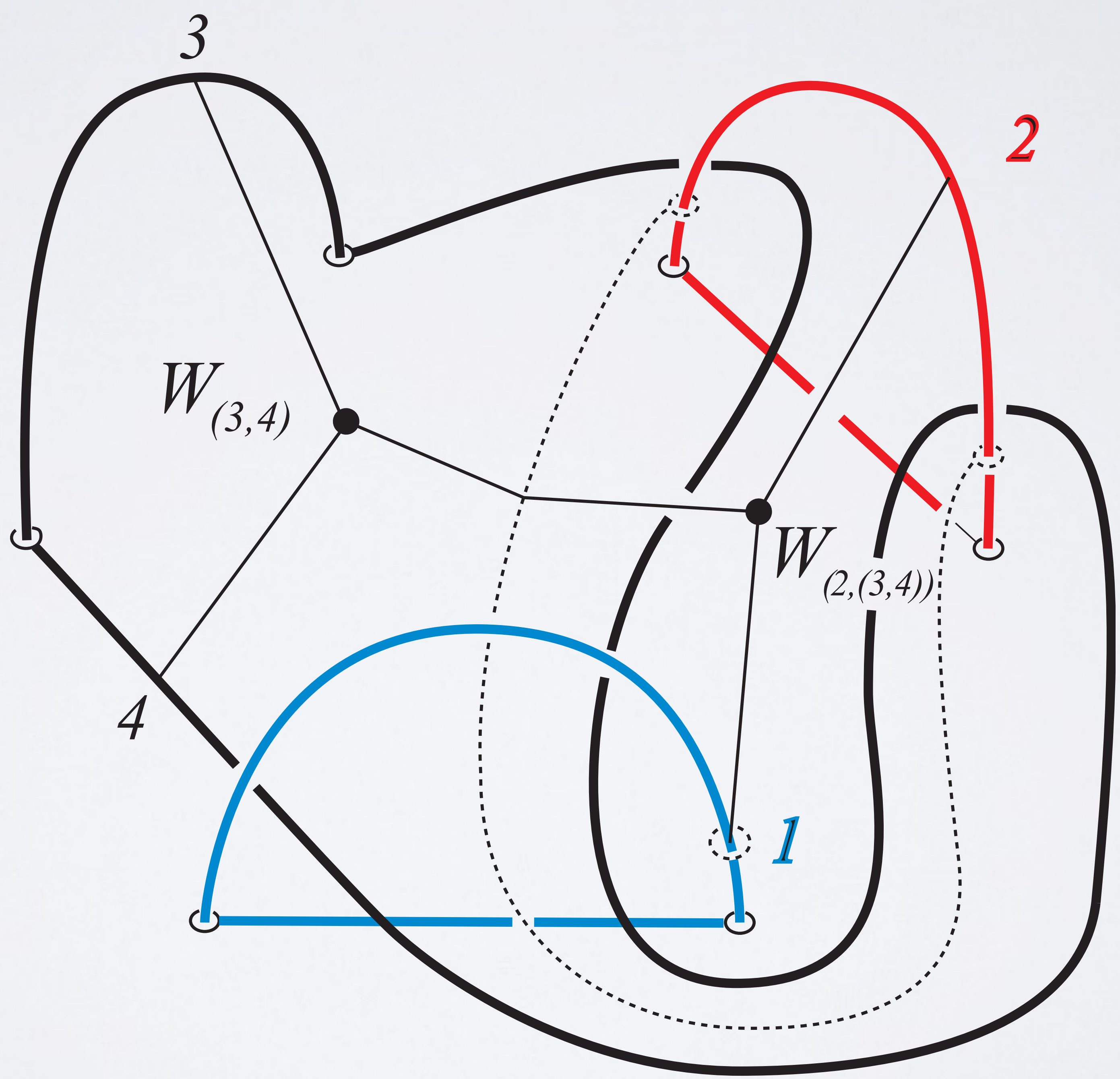
TO GIVE A 4-DIMENSIONAL JACOBI IDENTITY



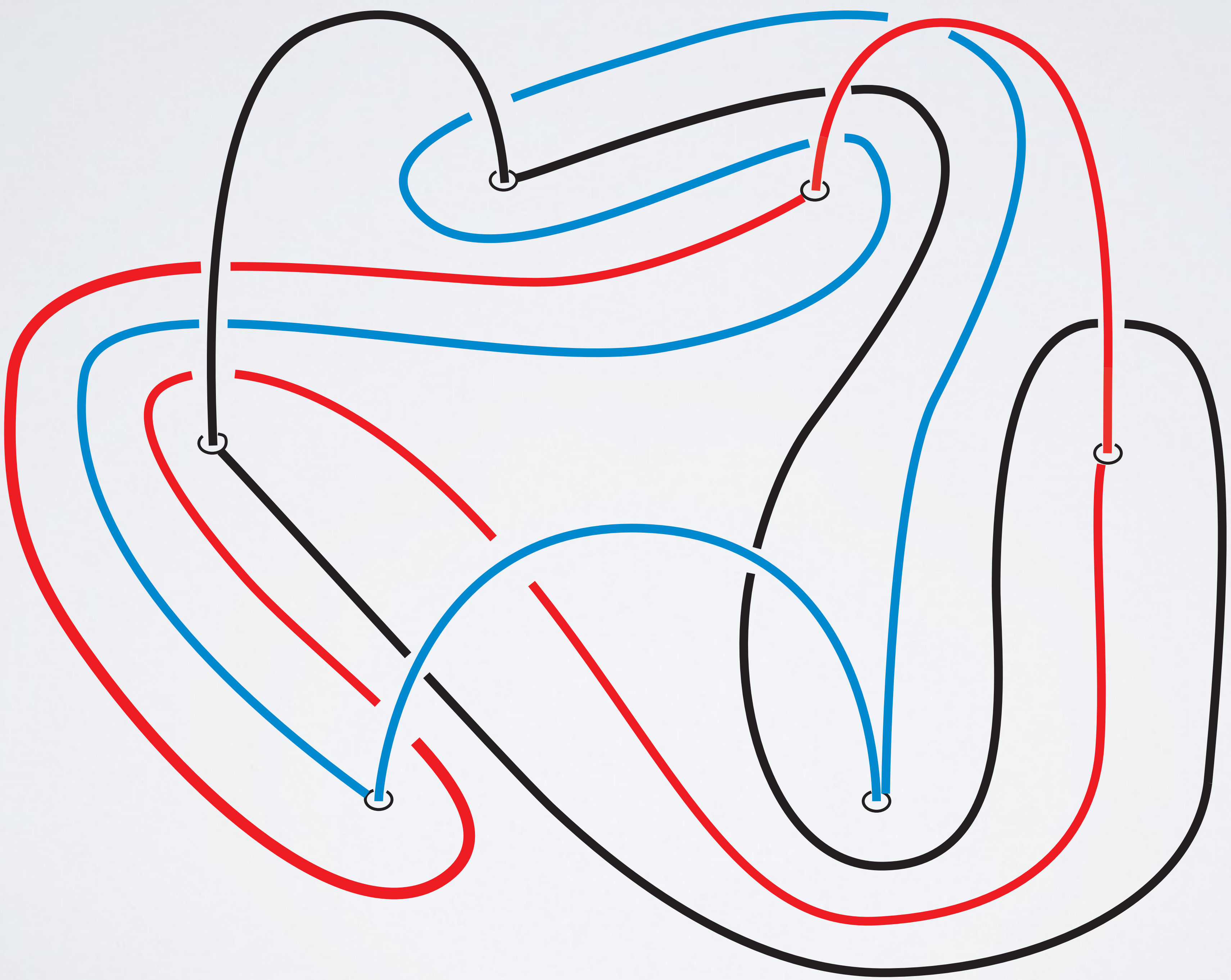
MOVE WHITNEY ARCS



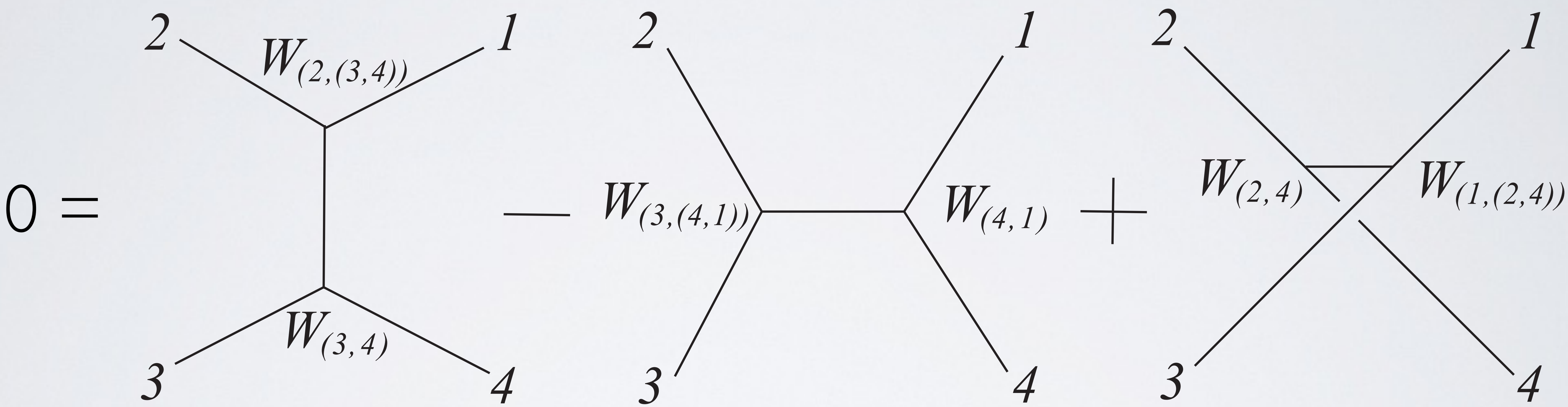
A WHITNEY TOWER OF ORDER 2 ARISES



MAKE ALL ARCS INTO DISJOINT



GET IHX-RELATION OR 4-DIM JACOBI IDENTITY



FILTERING THE SET OF LINKS IN S^3

Order n Whitney towers in the 4-ball, clean, and up to isotopy and Whitney moves $\xrightarrow{\partial_n}$ **Links in the 3-sphere** that are boundaries of order n Whitney towers

intersection tree $\big\downarrow \tau_n$

associated $\big\downarrow$ graded

T_n := abelian group generated by labelled oriented trees of order n, up to AS- and IHX-relations $\xrightarrow{R_n}$ W_n := links that bound order n Whitney towers, up to those bounding order n+1.

exists by geometric obstruction theory:

If a Whitney tower W of order n has vanishing intersection invariant, τ_n (W)=0, then it extends to order n+1, after some Whitney moves.

Base of our tower (f, W) where $f=(f_1 \dots f_m) : (\bigsqcup^m D^2, \partial) \hookrightarrow (D^4, \partial)$ is an m-component link ∂f.

$\exists W$ order 1 tower $\Leftrightarrow$ $\lambda_2(f) = 0$ $\Leftrightarrow$ $lk(\partial f) = 0$

continue this to get a filtration:

$$\underbrace{\mathrm{im}(\partial_0)}_{= \frac{\text{links}}{\text{isotopy}}} \supsetneq \underbrace{\mathrm{im}(\partial_1)}_{= \frac{lk \# \text{ zero links}}{\text{isotopy}}} \supsetneq \mathrm{im}(\partial_2) \supsetneq \cdots$$

Now look at the associated graded:

$$\mathbb{W}_n = \text{"}\frac{\mathrm{im}\,\partial_n}{\mathrm{im}\,\partial_{n+1}}\text{"} := \frac{\mathrm{im}\,\partial_n}{\sim_{n+1}}$$

Here $L_1 \underset{n}{\sim} L_2 \Leftrightarrow \exists W$ Wh. tower of order $n-1$ in $S^3 \times I$ with base $\bigsqcup^m S^1 \times I$ s.t. $\partial_0 W = L_1$, $\partial_1 W = L_2$

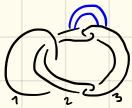
Class $0 \in \mathbb{W}_n$ is represented by links that bound order $n+1$ Wh. towers.

Realization map $R_n : T_n \longrightarrow \mathbb{W}_n$

$T \longmapsto$ Bing double according to the tree (connect sum components corresponding to repetitions)

It is surjective! Can use IHX to write T as a lin. comb. of ...ruted trees and those are realized.

Example

$R_1(Y) =$  $=$  $\in \mathbb{W}_1(2)$

Whitehead double!

Theorem [Conant, Schneiderman, T., 2010]:

For even n, R_n is an isomorphism and $W_n = W_n(m)$ is a free abelian group of known rank. For odd n, there is at most 2-torsion:

number m of components of the link

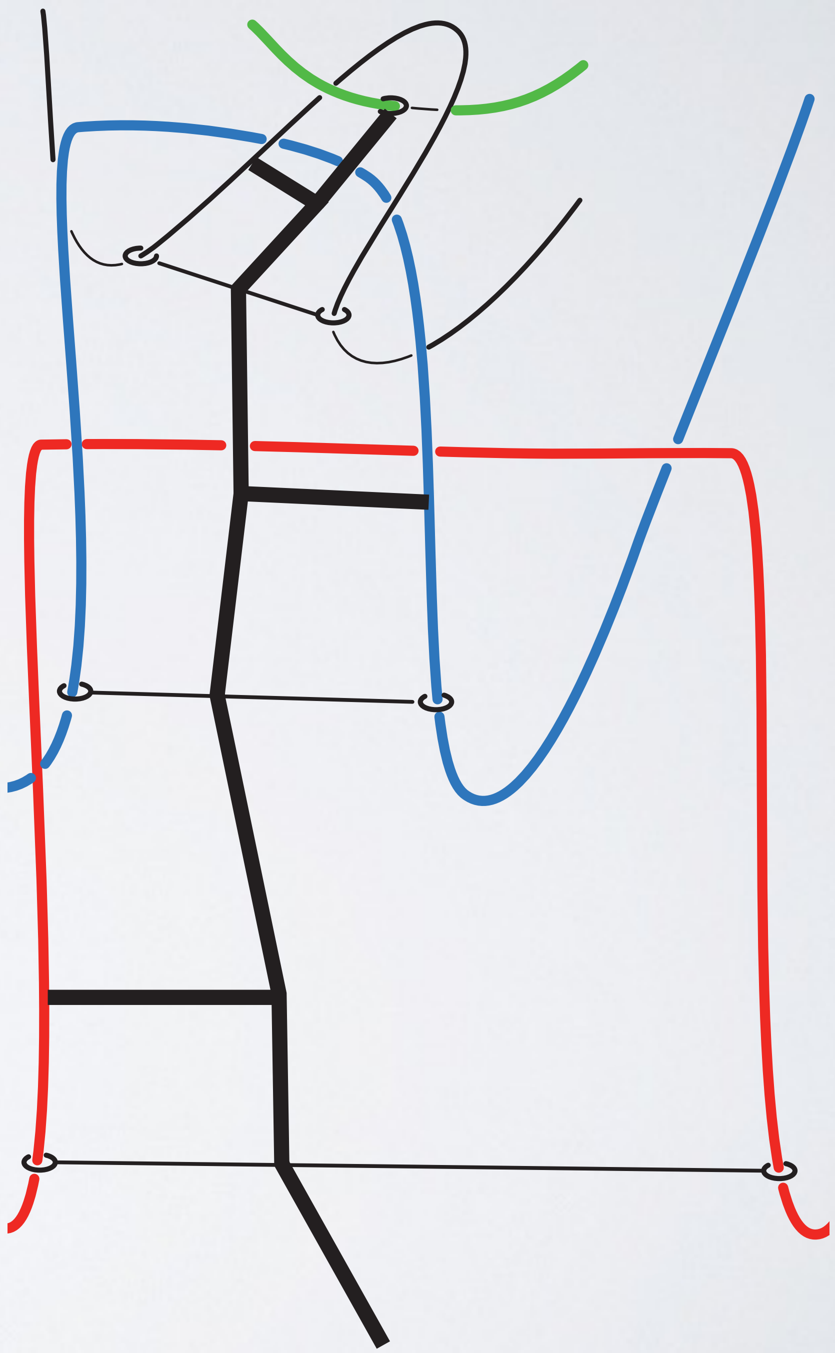
$W_n(m) =$

order n	1	2	3	4	5
0	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^6$	$\mathbb{Z}^{10}$	$\mathbb{Z}^{15}$
1	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z} \oplus \mathbb{Z}_2^6$	$\mathbb{Z}^4 \oplus \mathbb{Z}_2^{10}$	$\mathbb{Z}^{10} \oplus \mathbb{Z}_2^{15}$
2	0	$\mathbb{Z}$	$\mathbb{Z}^6$	$\mathbb{Z}^{20}$	$\mathbb{Z}^{50}$
3	0	$\mathbb{Z}_2^2$	$\mathbb{Z}^6 \oplus \mathbb{Z}_2^8$	$\mathbb{Z}^{36} \oplus \mathbb{Z}_2^{20}$	$\mathbb{Z}^{126} \oplus \mathbb{Z}_2^{40}$
4	0	$\mathbb{Z}^3$	$\mathbb{Z}^{28}$	$\mathbb{Z}^{146}$	$\mathbb{Z}^{540}$
5	0	$\mathbb{Z}_2^{e_2}$	$\mathbb{Z}^{36} \oplus \mathbb{Z}_2^{e_3}$	$\mathbb{Z}^{340} \oplus \mathbb{Z}_2^{e_4}$	$\mathbb{Z}^{1740} \oplus \mathbb{Z}_2^{e_5}$
6	0	$\mathbb{Z}^6$	$\mathbb{Z}^{126}$	$\mathbb{Z}^{1200}$	$\mathbb{Z}^{7050}$

$\mathrm{Arf}_2: \quad 3 \leq e_2 \leq 4, \quad 18 \leq e_3 \leq 21, \quad 60 \leq e_4 \leq 66, \quad 150 \leq e_5 \leq 160.$

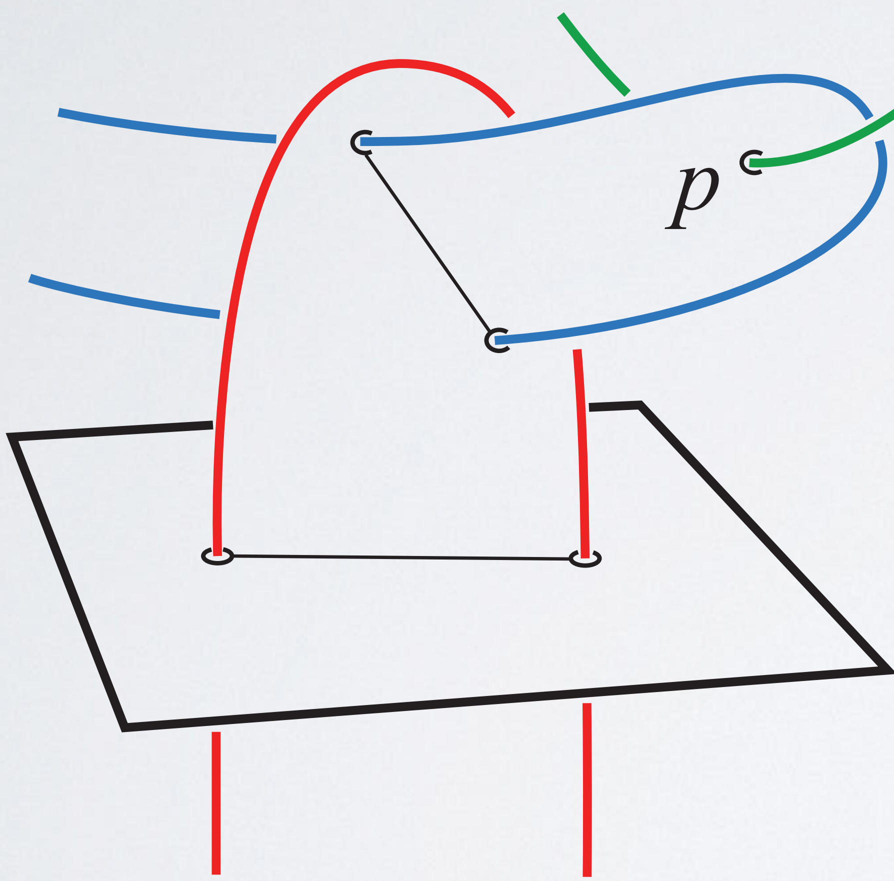
SURJECTIVITY OF MAP GIVEN BY THE INTERSECTION TREE

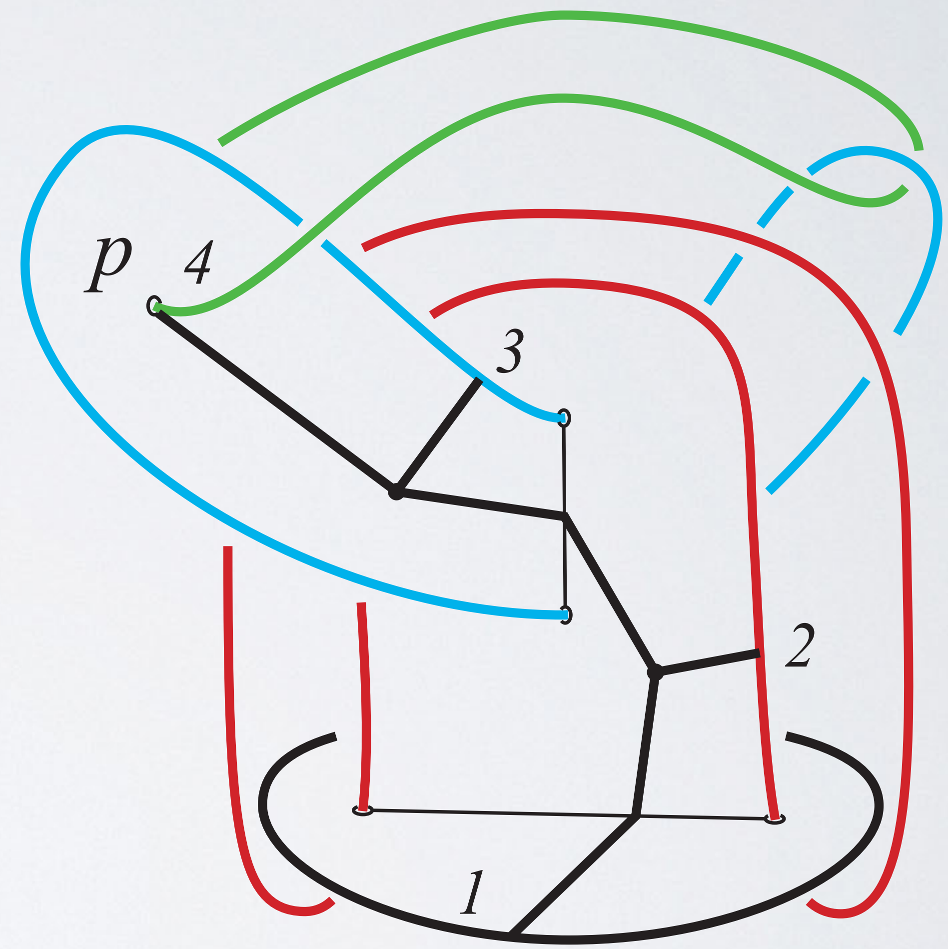
Take the Whitney towers W in our standard pictures:



Then $t = \boldsymbol{\tau}_n$ (W) runs through trees that generate T_n and the link on the boundary is $R_n(t)$.

LINK ON THE BOUNDARY IS A BING DOUBLE:





MASTER DIAGRAM

$$\begin{tikzcd} T_n \arrow[r, two heads, "R_n"] \arrow[d, "\eta_n"'] & W_n \arrow[d, "\mu_n"] \\ D'_n \arrow[r, two heads, "p_n"] & D_n \end{tikzcd}$$

T_n := abelian group generated by trees of order n, up to AS- and IHX-relations

W_n := links that bound order n Whitney towers, up to those bounding order n+1.

discrete Morse theory: $\cong$

η_n = sum over roots

μ_n

D'_n < abelian group generated by rooted labelled oriented trees of order n, up to AS- and IHX-relations

p_n

kernel is 2-torsion: [x,x]

D_n , a free abelian group of known rank, target of order n Milnor invariants μ_n

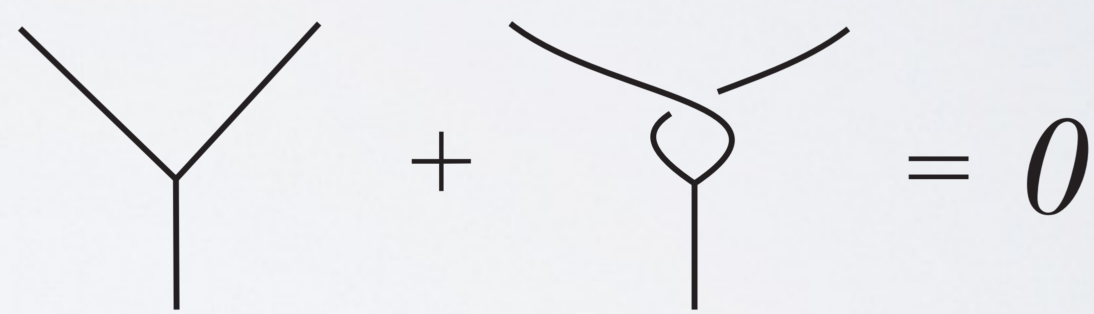
free quasi Lie algebra: [x,y]=-[y,x]

free Lie algebra: [x,x]=0

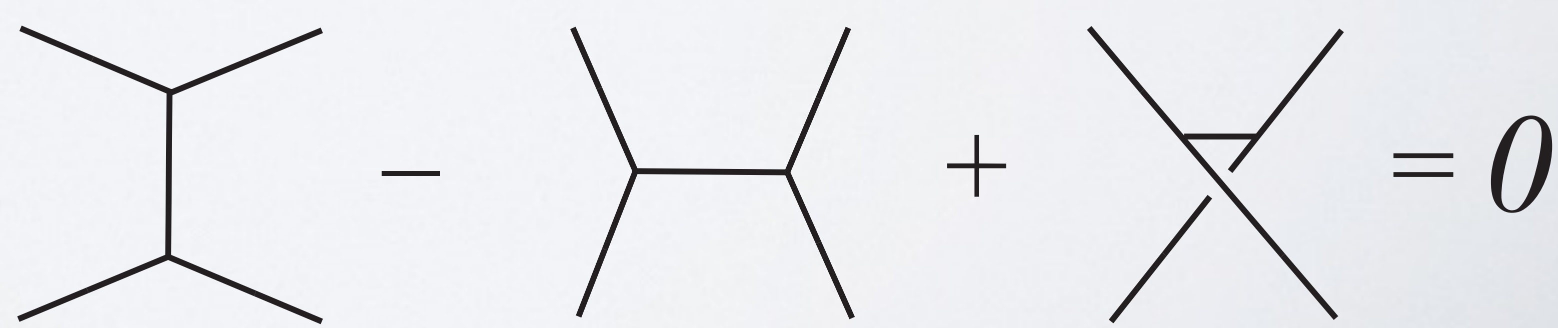
MILNOR INVARIANTS VIA TREE GROUPS

Recall that T(m) is the abelian group generated by oriented trivalent trees, with leaves labelled by {1, 2, ..., m}, modulo the two local relations:

Anti-symmetry:



Jacobi Identity:

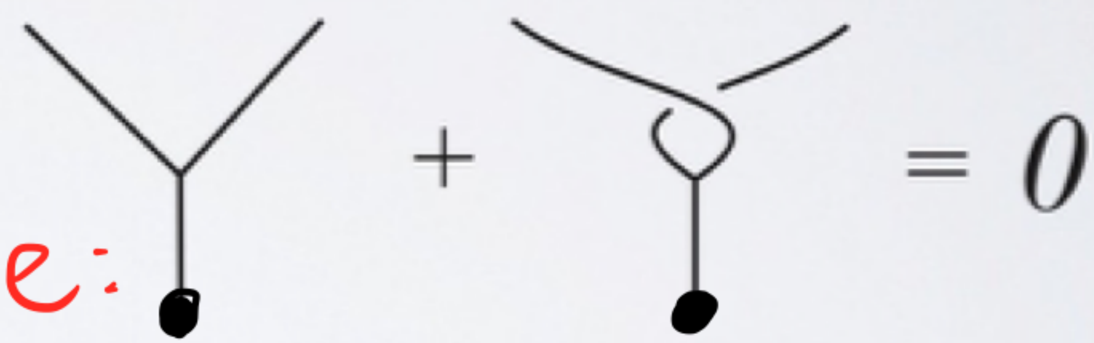


RECALL THE FREE resp. quasi- LIE ALGEBRA $/\mathbb{Z}$

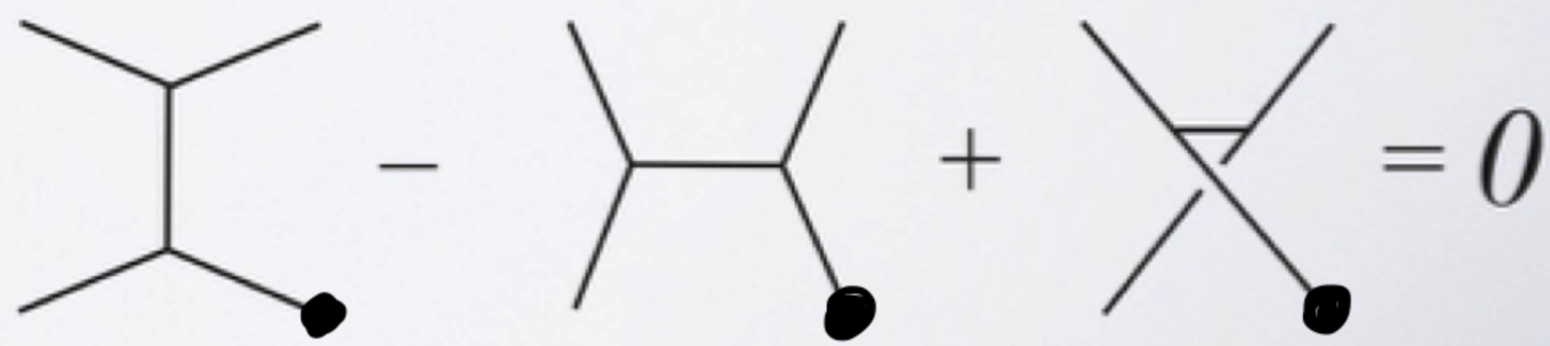
L(m) resp. L'(m) is the abelian group generated by oriented trivalent trees, with leaves labelled by {1, 2, ..., m} and one root, modulo the two local relations:

Anti-symmetry:

Lie: [x,x]=0 quasi-Lie:



Jacobi Identity:



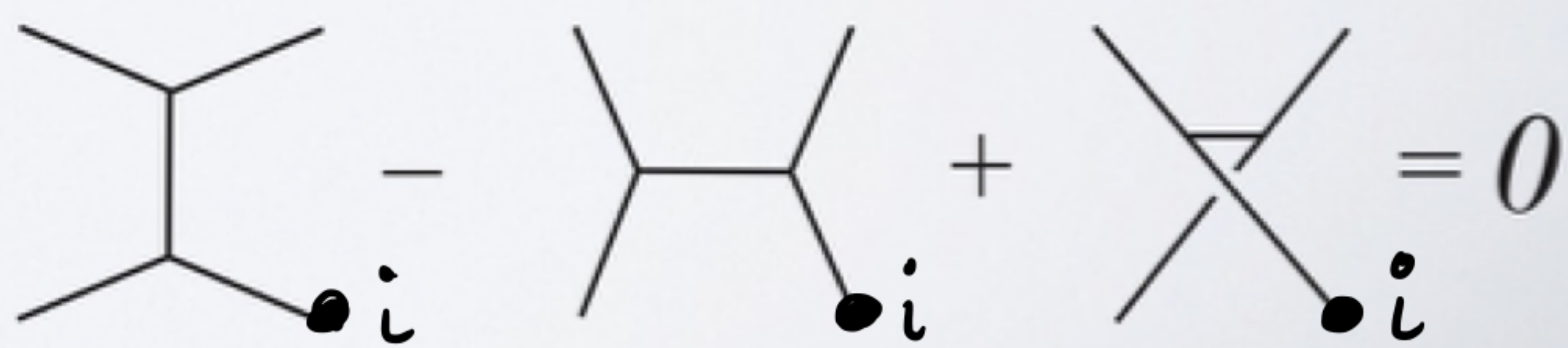
YET ANOTHER DIAGRAMMATIC GROUP

resp. $L'(m)\otimes\mathbb{Z}^m$

$L(m)\otimes\mathbb{Z}^m$ is the abelian group generated by oriented trivalent trees, leaves labelled by {1, 2, ..., m} and one labelled root, modulo the two local relations:

Anti-symmetry:
[x,x]=0, respectively:

Jacobi Identity:



$$D_n := \mathrm{Ker}\left(L_{n+1} \otimes \mathbb{Z}^m \xrightarrow{[\,,\,]} L_{n+2}\right)$$

$\cup$ cyclic symmetry

$$\mu_n(\text{link}) := \sum_{i=1}^{m} \mu_n^i \otimes m_i, \qquad \mu_n(I,i) \text{ length } n+2$$

is the order n Milnor invariant.

This is indeed in the kernel, because:

$$\partial(S^3 \setminus \mathring{\nu} L) = \bigsqcup_{i=0}^{m} T_i$$

torus nbhd of i-th comp. has longitude l_i, meridian m_i

Main Theorem:

$$\begin{tikzcd} T_n \arrow[r, "\eta_n", "\cong"'] \arrow[d, "R_n"'] \arrow[dr, phantom, "\#"] & D_n' \arrow[d, "\text{can.}"] \\ W_n \arrow[r, "\mu_n"] & D_n \end{tikzcd}$$

Define total Milnor invariant:

$$G = \pi_1(S^3 \setminus (l_1, \dots, l_m))$$

$\uparrow$ meridians m_i $\uparrow$

$$F = \text{free group on } x_1, \dots, x_m$$

If $l_i \in G_{n+1}$ then G_{n+1}/G_{n+2}

$$\sum_{I=(i_0,\dots,i_n)} \mu(I,i) \cdot X_I \longleftarrow \mu_n^i(\text{link})$$

$\cap \qquad \cap \qquad \uparrow \cong$

$$\mathbb{Z}[F] \hookleftarrow L_{n+1} \cong F_{n+1}/F_{n+2}$$

$$1 + x_i \longleftarrow x_i$$

$$x_1 x_2 - x_2 x_1 \longleftarrow [x_1, x_2] = l_3 \text{ for Bor.}$$

$$\mu_n(L) := \sum_{i=1}^{m} \mu_n^i \otimes m_i$$

image of l_i in L_{n+1} obtained from

$$\frac{(\pi_1(S^3 \setminus L))_{n+1}}{(\pi_1(S^3 \setminus L))_{n+2}} \cong \frac{F_{n+1}}{F_{n+2}} \cong L_{n+1}$$