Intro to Whitney towers
Part 1B

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Intro to Whitney towers continued:

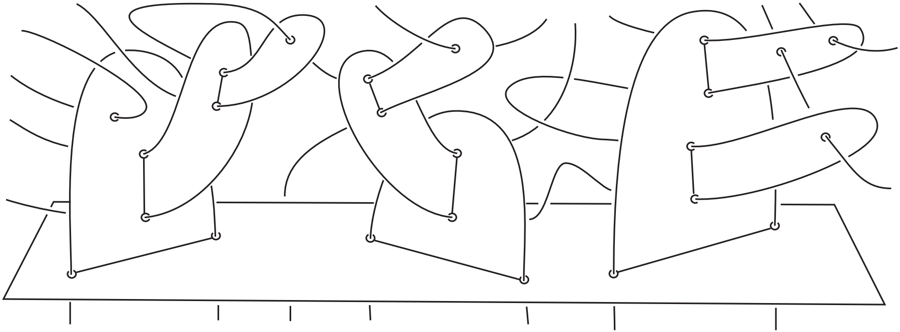
1. Recall definition of Whitney towers
2. Trees and intersection forests
3. Gradings of Whitney towers
4. 4-dimensional Jacobi identity

Recall:

Definition:

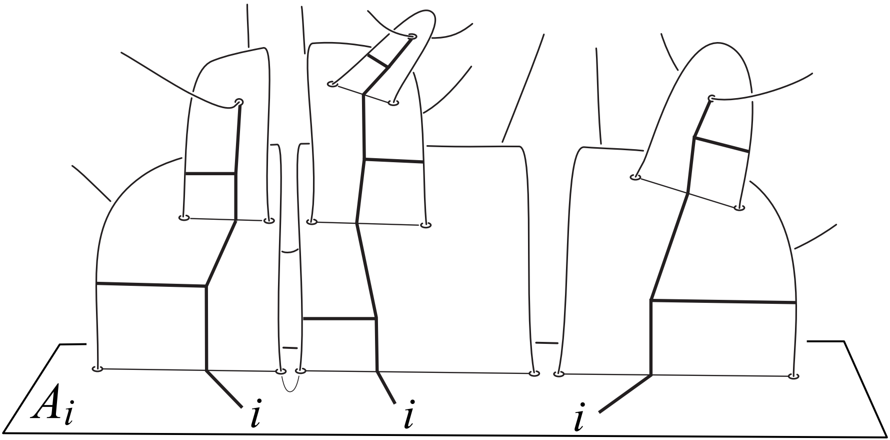
A *Whitney tower* on an immersed surface $A^2 \looparrowright X^4$ is defined by:

1. A itself is a Whitney tower.
2. If $\mathcal{W}$ is a Whitney tower and W is a Whitney disk pairing intersections in $\mathcal{W}$, then the union $\mathcal{W} \cup W$ is a Whitney tower.



Part of a Whitney tower!

All singularities in split Whitney towers are near **trivalent trees**:



Trees 'bifurcate down' from unpaired intersections.
Univalent vertices inherit labels from components of the underlying properly immersed surface $A = A_1 \cup A_2 \cup \cdots \cup A_m$.

Rooted trees

Identify non-associative bracketings of elements of $\{1, 2, \ldots, m\}$ with rooted unitrivalent trees (labeled and vertex-oriented):

$$(i, j) \quad \longleftrightarrow \quad -\!\!< {}^{j}_{i}$$

and recursively

$$(I, J) \quad \longleftrightarrow \quad -\!\!< {}^{J}_{I}$$

Here a singleton is identified with a rooted edge:

$$(i) = i \quad \longleftrightarrow \quad \text{---}\; i$$

Un-rooted trees = *inner products* of rooted trees

Gluing two rooted trees I and J together at their roots yields an un-rooted tree $\langle I, J\rangle := I \;—\; J$.

Example:

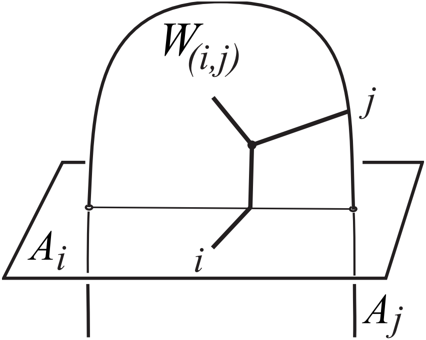
$$\langle (i,k),(j,l)\rangle \quad = \quad {}^{i}_{k}\!>\!\!—\!\!<\!{}^{l}_{j}$$

Example:

$$\langle (I,J),K\rangle \quad = \quad {}^{I}_{J}\!>\!— \; K$$

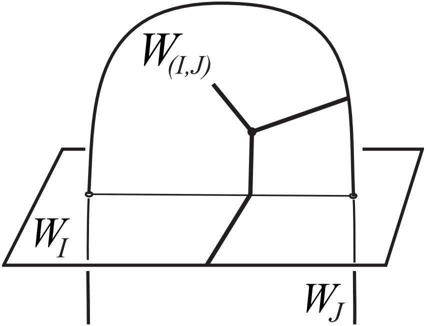
Paired intersections $\longrightarrow$ rooted trees

Whitney disk $W_{(i,j)}$ pairing $A_i \pitchfork A_j \quad \longmapsto \quad$ rooted tree $-\!\!<^{j}_{i}$



Paired intersections → rooted trees

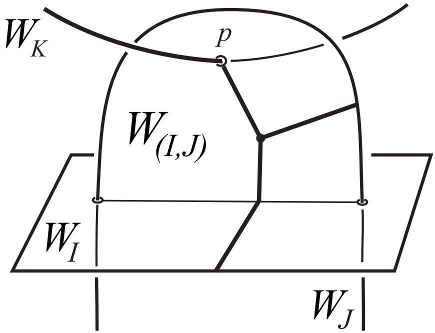
Recursively: $W_{(I,J)}$ pairing $W_I \pitchfork W_J \quad \longmapsto \quad -\!\!<^{J}_{I}$



root edge of (I, J) contained in interior of $W_{(I,J)}$

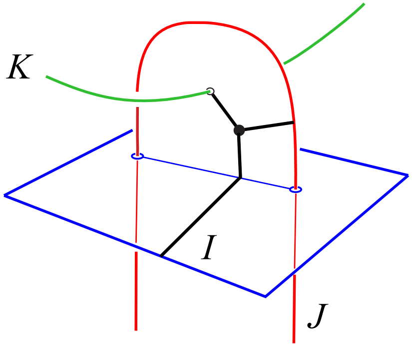
Un-paired intersections → un-rooted trees

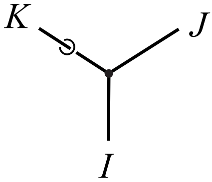
$$p \in W_{(I,J)} \pitchfork W_K \quad \longmapsto \quad t_p = \langle (I,J), K \rangle = {}^{I}_{J}\!>\!\!-\, K$$



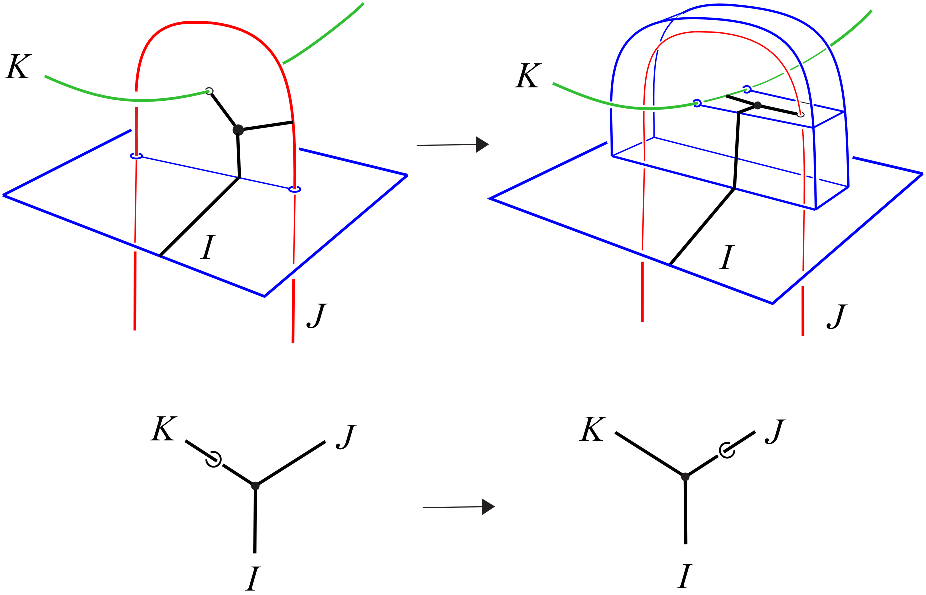
Glue together root vertices of (I,J) and K at $p \in W_{(I,J)} \pitchfork W_K$

Why not keep track of edge in t_p corresponding to p?

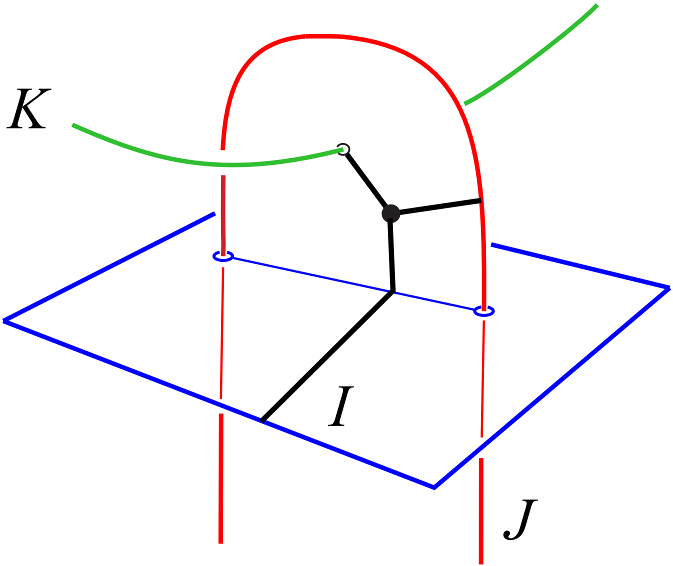




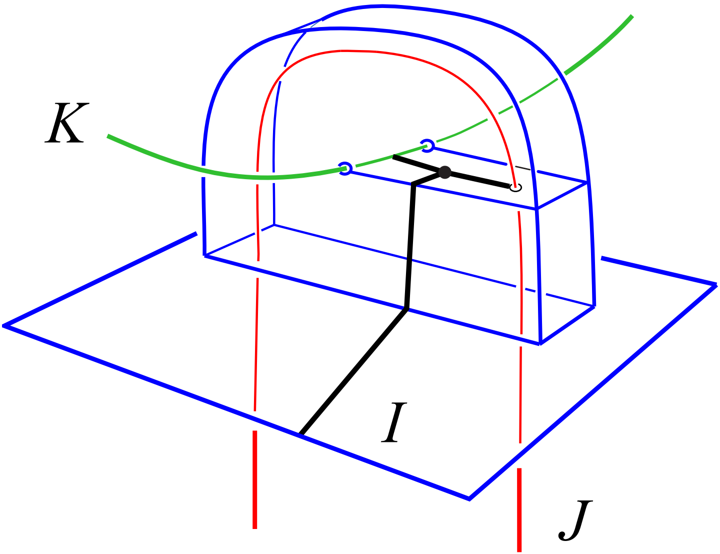
Because can 'move' un-paired intersection to any edge of its tree!



Close-up view before Whitney move

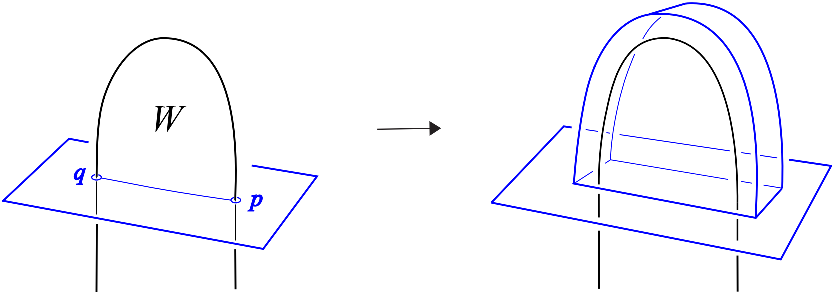


Close-up view after Whitney move

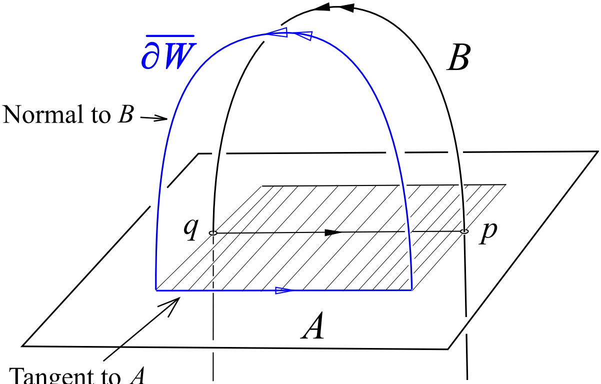


Towards 'twisted' trees for twisted Whitney disks...

Recall: Whitney move guided by W uses two parallel copies of W:



The *twisting* $\omega(W) \in \mathbb{Z}$ of W is the relative Euler number of a normal section $\overline{\partial W}$ over ∂W determined by the sheets:



If $\omega(W) = 0$, then W is *framed*.
If $\omega(W) \neq 0$, then W is *twisted* and a W-Whitney move will create intersections between the parallel copies of W...

Twisted Whitney disks $\to$ twisted trees

Define the ∞-*tree*

$$J^{\infty} := J \text{---} \infty$$

by labeling the root of J with the 'twist' symbol ∞.

These ∞-trees are called 'twisted trees' since they are associated to twisted Whitney disks:

$$W_J \quad \mapsto \quad J^{\infty} \quad \text{if } \omega(W_J) \neq 0.$$

So we sometimes refer to the un-rooted t_p as 'framed trees'...

Definition:

The *intersection forest* $t(\mathcal{W})$ of a Whitney tower $\mathcal{W}$ is the multiset:

$$t(\mathcal{W}) := \sum \epsilon_p \cdot t_p + \sum \omega(W_J) \cdot J^\infty$$

where 'formal sum' is over all unpaired p and all twisted W_J in $\mathcal{W}$.

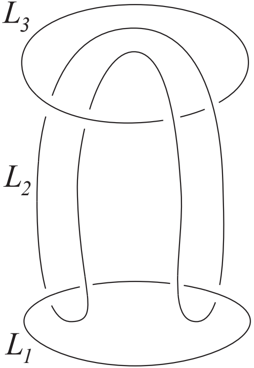
$\epsilon_p = \pm$ is usual sign of the unpaired transverse intersection point p (orientation conventions suppressed).

$\omega(W_J) \in \mathbb{Z}$ is twisting of W_J.

Think of $t(\mathcal{W}) \subset \mathcal{W}$.

Example: L **bounds** $\mathcal{W} = D_1 \cup D_2 \cup D_3 \cup W_{(1,2)}$ **with** $t(\mathcal{W}) = \frac{1}{2} \succ\!\!-\, 3$

Moving into B^4 from left to right, starting with $L \subset S^3 = \partial B^4$:



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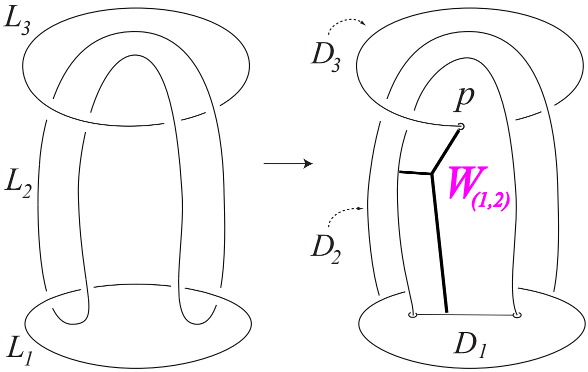

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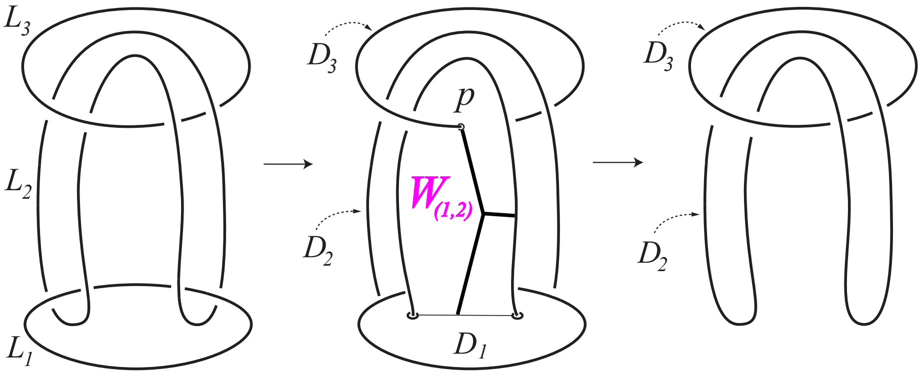
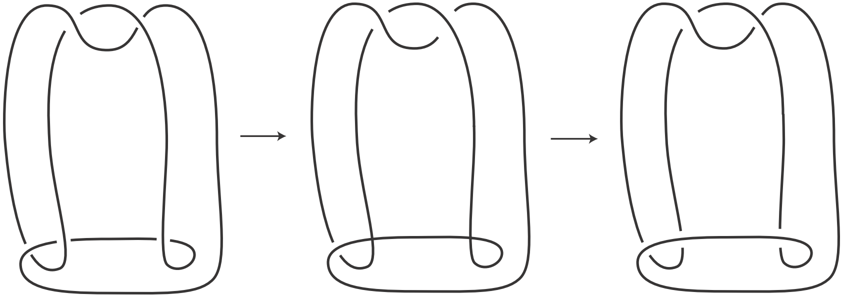


$p = W_{(1,2)} \pitchfork D_3 \quad \mapsto \quad t_p = \langle (1,2), 3 \rangle = \frac{1}{2} \succ\!\!- 3 = t(\mathcal{W})$

Example: Fig-8 knot bounds $\mathcal{W} = D_1 \cup W_{(1,1)}$ with $t(\mathcal{W}) = +(1,1)^\infty$

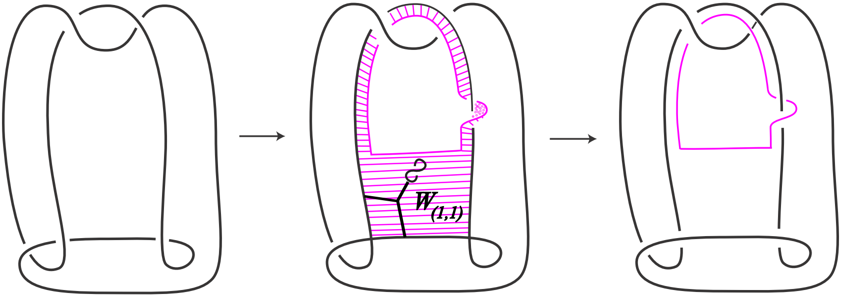
Moving into B^4, D_1 is the track of a null-homotopy of K:



$K = \partial D_1 \subset S^3$

Example: Fig-8 knot bounds $\mathcal{W} = D_1 \cup W_{(1,1)}$ with $t(\mathcal{W}) = +(1,1)^{\infty}$

Moving into B^4, D_1 is the track of a null-homotopy of K:



$K = \partial D_1 \subset S^3$ part of $W_{(1,1)}$ cap off unlink...

Realization

- By iterated Bing-doubling can realize any collection of signed trees as $t(\mathcal{W})$ for $\mathcal{W}$ on 2-disks $\looparrowright B^4$ bounded by $L \subset S^3$.

- Exist restrictions on possible $t(\mathcal{W})$ for $\mathcal{W}$ on 2-spheres $\looparrowright B^4$. (See later...)

No trees = No problems = Embedding!

If $\mathcal{W}$ is a Whitney tower on A such that $t(\mathcal{W}) = \emptyset$,

then A is regularly homotopic to an embedding:

$t(\mathcal{W}) = \emptyset \implies$ no unpaired intersections and no twisted Whitney disks.
Do the clean framed Whitney moves on all the Whitney disks in $\mathcal{W}$ starting at the 'top level'...

Next, will introduce gradings to filter the condition of being homotopic to an embedding.

Higher-order Whitney disks and intersections

Definition:

- The *order* of a tree is the number of trivalent vertices.
- The *order* of a Whitney disk or an intersection point is the order of the corresponding tree.

Order n framed Whitney towers

Definition:
$\mathcal{W}$ is an *order n framed Whitney tower* if

- every framed tree t_p in $t(\mathcal{W})$ is of order $\geq n$, and
- there are no ∞-trees in $t(\mathcal{W})$.

So in an order n framed $\mathcal{W}$ all unpaired intersections have order $\geq n$, and all Whitney disks are framed.

Order n twisted Whitney towers

Definition:
$\mathcal{W}$ is an *order n twisted Whitney tower* if

- every framed tree t_p in $t(\mathcal{W})$ is of order $\geq n$,
- every twisted ∞-tree in $t(\mathcal{W})$ is of order $\geq \frac{n}{2}$.

Intersection invariants from $t(\mathcal{W})$ and order-raising obstruction theory

Let $\mathcal{W}$ be an order n twisted Whitney tower on $A \looparrowright X$.

Will (eventually) define abelian groups $\mathcal{T}_n^\infty$ such that if the order n twisted intersection invariant $\tau_n^\infty(\mathcal{W}) := [t(\mathcal{W})] \in \mathcal{T}_n^\infty$ vanishes, then A is homotopic to A' supporting an order $n+1$ twisted Whitney tower.

Classification of order n twisted $\mathcal{W}$ on $\cup_i D_i^2 \looparrowright B^4$

Theorem

A link $L \subset S^3$ bounds immersed disks supporting an order $n+1$ twisted Whitney tower $\mathcal{W} \subset B^4$ if and only if L has vanishing Milnor invariants and higher-order Arf invariants through order n.

Idea of proof: Identify the order-raising intersection invariants τ_n^∞ with Milnor and higher-order Arf invariants. (Will do this later.)

General classification of order n Whitney towers?

Open Problem:

Find invariants of order n $\mathcal{W}$ on immersed surfaces in 4-manifolds.

Partial results so far. Can formulate similar tree-valued invariants as for links. Need to understand relations in target groups...

Note: An embedded surface is a Whitney tower of order n for all n. So related to the (difficult!) embedding problem.

Other complexity gradings: *Non-repeating* order n Whitney towers

$\mathcal{W}$ is an order n *non-repeating* Whitney tower if all $t_p \in t(\mathcal{W})$ having distinctly-labeled vertices are of order $\geq n$.

Non-repeating Whitney towers characterize being able to 'pull apart' components:

Theorem:

$$A = \cup_{i=1}^{m} A_i \looparrowright X \text{ bounds an order } m-1 \text{ non-repeating } \mathcal{W}$$

if and only if

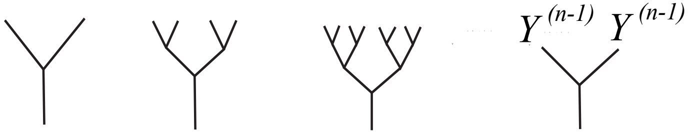
$$A \text{ is homotopic to } A' = \cup_{i=1}^{m} A'_i \text{ with } A'_i \cap A'_j = \emptyset \text{ for all } i \neq j.$$

Other complexity gradings: *Symmetric* Whitney towers

A Whitney tower $\mathcal{W}$ is *symmetric* if the interiors of all Whitney disks in $\mathcal{W}$ only intersect Whitney disks of the same order.

A symmetric Whitney tower of order $2^n - 1$ has *height* n.

The Whitney disks in a symmetric Whitney tower correspond to symmetric rooted trees:



The symmetric rooted-trees of height 1, 2, 3, and n

Other complexity gradings: *Symmetric* Whitney towers

A Whitney tower $\mathcal{W}$ is *symmetric* if the interiors of all Whitney disks in $\mathcal{W}$ only intersect Whitney disks of the same order.

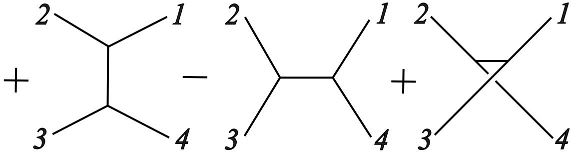
A symmetric Whitney tower of order $2^n - 1$ has *height n*.

Theorem: (Cochran–Teichner)
If $L \subset S^3$ bounds $\mathcal{W} \subset B^4$ of height $n+2$, then L is n-solvable in the sense of Cochran–Orr–Teichner.

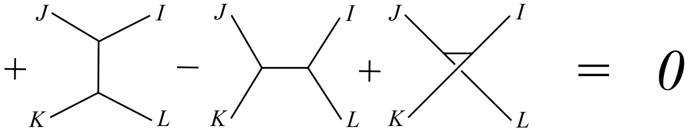
Open Problem:
Formulate invariants corresponding to a complete 'height-raising' obstruction theory for symmetric Whitney towers.

Geometric Jacobi Identity in 4-dimensions

There exist four 2-spheres in 4-space supporting $\mathcal{W}$ with intersection forest $t(\mathcal{W})$ equal to:

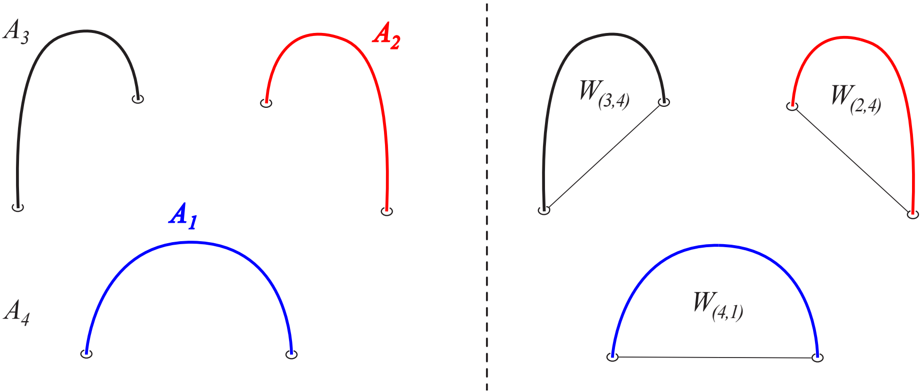


Conclude: The local 'IHX relation' of finite type theory is needed in the target of any invariant represented by $t(\mathcal{W})$:



Geometric Jacobi Identity in 4-dimensions

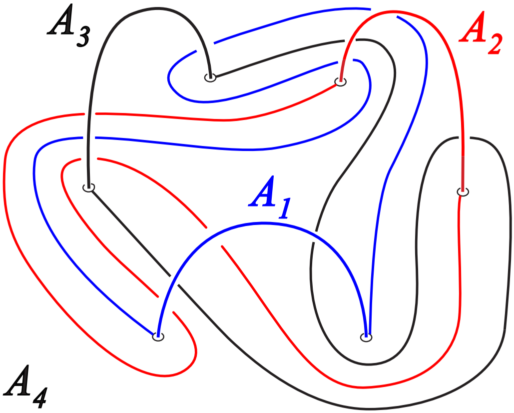
Start with disjoint embeddings $A_i : S^2 \to B^4$, $i = 1, 2, 3, 4$.
Then do finger moves of A_1, A_2, A_3 into A_4:



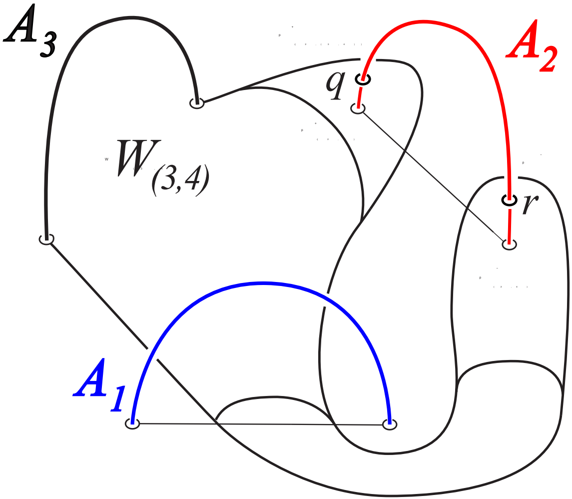
Whitney disks on the right are inverse to the finger moves.

Geometric Jacobi Identity in 4-dimensions

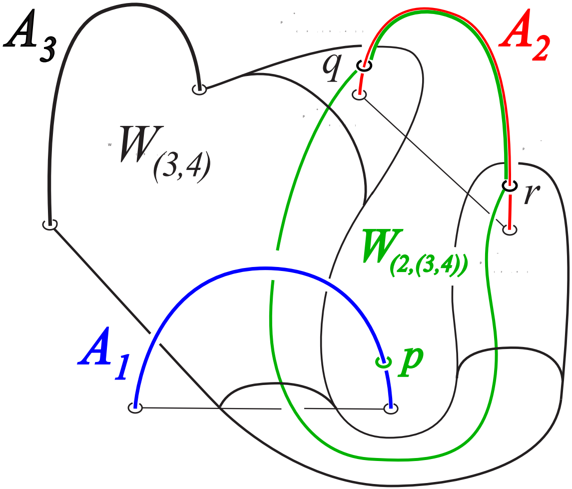
Will construct new Whitney disks with these boundaries:



First change collar of $W_{(3,4)}$; creating $\{q, r\} = A_2 \pitchfork W_{(3,4)}$:



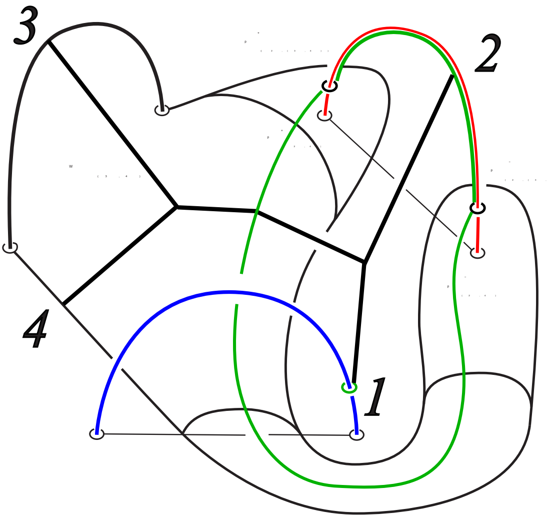
Then add $W_{(2,(3,4))}$ pairing $\{q, r\} = A_2 \pitchfork W_{(3,4)}$:



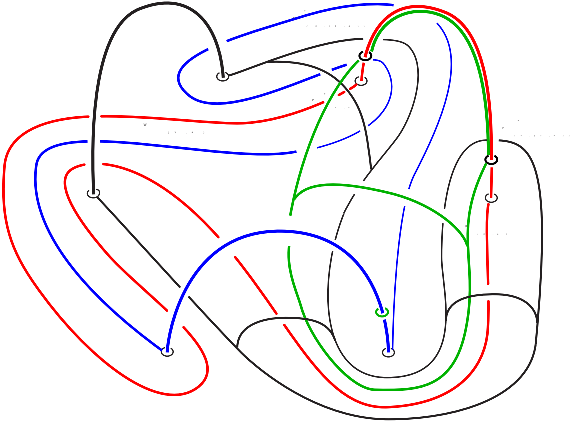
$W_{(3,4)}$ and $W_{(2,(3,4))}$ are contained in the 'present' slice of $B^4 = B^3 \times I$

Creates $p = A_1 \cap W_{(2,(3,4))}$.

$p = A_1 \cap W_{(2,(3,4))} \mapsto t_p = {}^{3}_{4}\!\succ\!\!-\!\!\prec^{2}_{1}$:



Exercise: Construct other two trees of the IHX relation analogously using past and future...



HINT: Here in 'present' red and blue Whitney disks have clean collars along horizontal A_4-sheet.

(See *Jacobi identities in Low-dimensional Topology*, Compositio Mathematica vol. 143, no. 3 May 2007.)