

# BETTI NUMBER ESTIMATES FOR NILPOTENT GROUPS

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ABSTRACT. We prove an extension of the following result of Lubotzky and Madid on the rational cohomology of a nilpotent group  $G$ : If  $b_1 < \infty$  and  $G \otimes \mathbb{Q} \neq 0, \mathbb{Q}, \mathbb{Q}^2$  then  $b_2 > b_1^2/4$ . Here the  $b_i$  are the rational Betti numbers of  $G$  and  $G \otimes \mathbb{Q}$  denotes the Malcev-completion of  $G$ . In the extension, the bound is improved when the relations of  $G$  are known to all have at least a certain commutator length. As an application we show that every closed oriented 3-manifold falls into exactly one of the following classes: It is a rational homology 3-sphere, or it is a rational homology  $S^1 \times S^2$ , or it has the rational homology of one of the oriented circle bundles over the torus (which are indexed by an Euler number  $n \in \mathbb{Z}$ , e.g.  $n = 0$  corresponds to the 3-torus) or it is of *general type* by which we mean that the rational lower central series of the fundamental group does not stabilize. In particular, any 3-manifold group which allows a maximal torsion-free nilpotent quotient admits a rational homology isomorphism to a torsion-free nilpotent group.

## 1. THE MAIN RESULTS

We analyze the rational lower central series of 3-manifold groups by an extension of the following theorem of Lubotzky and Magid [15, (3.9)].

**Theorem 1.** *If  $G$  is a nilpotent group with  $b_1(G) < \infty$  and  $G \otimes \mathbb{Q} \neq \{0\}, \mathbb{Q}, \mathbb{Q}^2$ , then*

$$b_2(G) > \frac{1}{4} b_1(G)^2$$

Here  $b_i(G)$  denotes the  $i$ th rational Betti number of  $G$  and  $G \otimes \mathbb{Q}$  the Malcev-completion of  $G$ . (We refer the reader to Appendix A, where we have collected the group theoretic definitions.)

This Theorem is an analogue of the Golod-Shafarevich Theorem (see [9] or [14, p. 186]), which states that if  $G$  is a finite  $p$ -group, then the inequality  $r > d^2/4$  holds, where  $d$  is the minimal number of generators in a presentation of  $G$ , and  $r$  is the number of relations in any presentation. It can be used to derive a result for finitely generated nilpotent groups similar to the one above, but with  $b_2(G)$  replaced by the minimal number of relations in a presentation for  $G$ , see [6, p. 121].

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However, in applications (compare Section 2) only information about the Betti numbers may be available. Also, the presence of the (necessary) assumptions in Theorem 1 shows that the proof in the nilpotent case must at some point differ from the one for  $p$ -groups.

We also need a refined version of this result in the case where the relations of  $G$  are known to have a certain commutator length. In order to make this precise, let

$$H_2(G; \mathbb{Q}) =: \Phi_2^{\mathbb{Q}}(G) \supseteq \Phi_3^{\mathbb{Q}}(G) \supseteq \Phi_4^{\mathbb{Q}}(G) \supseteq \dots$$

be the rational Dwyer filtration of  $H_2(G; \mathbb{Q})$  (defined in Appendix A).

**Theorem 2.** *If  $G$  is a nilpotent group with  $b_1(G) < \infty$ ,  $H_2(G; \mathbb{Q}) = \Phi_r^{\mathbb{Q}}(G)$  and  $G \otimes \mathbb{Q} \neq \{0\}, \mathbb{Q}, \mathbb{Q}^2$ , then*

$$b_2(G) > \frac{(r-1)^{(r-1)}}{r^r} b_1(G)^r.$$

Note that Theorem 1 is the special case of Theorem 2 with  $r = 2$ .

The intuitive idea behind these results is that  $H_1(G)$  corresponds to generators of  $G$  and  $H_2(G)$  to its relations. For example, if  $G$  is abelian, then the number of (primitive) relations ( $\approx b_2$ ) grows quadratically in the number of generators ( $\approx b_1$ ) because the commutator of a pair of generators has to be a consequence of the primitive relations. Similarly, if  $G$  is nilpotent, then the relations have to imply that for some  $r$ , all  $r$ -fold commutators in the generators vanish. If  $H_2(G) = \Phi_r(G)$ , then all relations are in fact  $r$ -fold commutators because, by Dwyer's Theorem (see Appendix A), this condition is equivalent to  $G/\Gamma_r$  being isomorphic to  $F/\Gamma_r$  where  $F$  is the free group on  $b_1(G)$  generators. Thus  $b_2$  should grow as the  $r$ -th power of  $b_1$ . But if some of the relations are shorter commutators, then they can imply a vast number of relations among the  $r$ -fold commutators. Therefore, one can only expect a lower order estimate in this case.

**Example .** Let  $x_1, \dots, x_4$  be generators of  $H^1(\mathbb{Z}^4; \mathbb{Z})$ . Then the central extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow G \longrightarrow \mathbb{Z}^4 \longrightarrow 1$$

classified by the class  $x_1x_2 + x_3x_4 \in H^2(\mathbb{Z}^4; \mathbb{Z})$  satisfies  $b_1(G) = 4, b_2(G) = 5$ . This shows that the lower bound for  $b_2$  in Theorem 1 cannot be of the form  $b_1(b_1 - 1)/2$  with equality in the abelian case.

This paper is organized as follows. In Section 2 we give applications of the above results to 3-dimensional manifolds. Section 3 explains how to derive the nilpotent classification of 3-manifolds from Theorem 2. In Section 4 we give further examples and in Section 5 we prove Theorem 2 modulo two key lemmas. These are proven in Sections 6 and 7. Appendices A and B contain background information from group theory respectively rational homotopy theory.

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2. APPLICATIONS TO 3-MANIFOLDS

We will look at a 3-manifold through nilpotent eyes, observing only the tower of nilpotent quotients of the fundamental group, but never the group itself. This point of view has a long history in the study of link complements and it arises naturally if one studies 3- and 4-dimensional manifolds together. For example, Stallings proved that for a link  $L$  in  $S^3$  the nilpotent quotients  $\pi_1(S^3 \setminus L)/\Gamma_r$  are invariants of the topological concordance class of the link  $L$ . These quotients contain the same information as Milnor's  $\mu$ -invariants which are generalized linking numbers. For precise references about this area of research and the most recent applications to 4-manifolds see [7] and [8].

Let  $M$  be a closed oriented 3-manifold and  $\{\Gamma_r^{\mathbb{Q}} \mid r \geq 1\}$  the rational lower central series (see Appendix A) of  $\pi_1(M)$ . Similarly to Stallings' result, the quotients  $(\pi_1(M)/\Gamma_r^{\mathbb{Q}}) \otimes \mathbb{Q}$  are invariants of rational homology  $H$ -cobordism between such 3-manifolds.

**Definition .** A 3-manifold  $M$  is of *general type* if

$$\pi_1(M) = \Gamma_1^{\mathbb{Q}} \supsetneq \Gamma_2^{\mathbb{Q}} \supsetneq \Gamma_3^{\mathbb{Q}} \supsetneq \dots$$

and *special* if, for some  $r > 0$ ,  $\Gamma_r^{\mathbb{Q}} = \Gamma_{r+1}^{\mathbb{Q}}$ . (Following the terminology used in group theory, the fundamental group of a special 3-manifold is called *rationally prenilpotent*, compare the Appendix A.)

Our nilpotent classification result reads as follows.

**Theorem 3.** *If a closed oriented 3-manifold  $M$  is special, then the maximal torsion-free nilpotent quotient of its fundamental group is isomorphic to exactly one of the groups*

$$\{1\}, \quad \mathbb{Z} \quad \text{or} \quad H_n.$$

*In particular, this quotient is a torsion-free nilpotent 3-manifold group of nilpotency class  $< 3$ .*

Here the groups  $H_n, n \geq 0$ , are the central extensions

$$1 \longrightarrow \mathbb{Z} \longrightarrow H_n \longrightarrow \mathbb{Z}^2 \longrightarrow 1$$

classified by the Euler class  $n \in H^2(\mathbb{Z}^2; \mathbb{Z}) \cong \mathbb{Z}$ . This explains the last sentence in our theorem because  $H_n$  occurs as the fundamental group of a circle bundle over the 2-torus with Euler class  $n$ .

Since the groups above have nilpotency class  $< 3$ , it is very easy to recognize the class to which a given 3-manifold belongs; one simply has to compute  $\pi_1(M)/\Gamma_3^{\mathbb{Q}}$ . Note, in particular, that a 3-manifold  $M$  is automatically of general type if its first rational Betti number satisfies  $b_1 M > 3$ .

If  $b_1 M = 0$ , then  $M$  is a rational homology sphere, if  $b_1 M = 1$ , then it is a rational homology  $S^1 \times S^2$ . In the case  $b_1 M = 2$ , the cup-product between the two 1-dimensional cohomology classes vanishes and one can compute a Massey triple-product to obtain the integer  $n \in \mathbb{Z}$  (note that  $H^1(M; \mathbb{Z}) \cong \mathbb{Z}^2$

and thus one can do the computation integrally). It determines whether  $M$  is of general type ( $n = 0$ ) or whether it belongs to one of the groups  $H_n$ ,  $n > 0$ . Finally, if  $b_1 M = 3$ , then  $M$  is of general type if and only if the triple cup-product between the three 1-dimensional cohomology classes vanishes, otherwise it is equivalent to the 3-torus, i.e., to  $H_0$ .

One should compare the above result with the list of nilpotent 3-manifold groups:

$$\frac{\text{finite}}{\text{infinite}} \mid \begin{array}{c|c} \mathbb{Z}/n & Q_n \\ \hline \mathbb{Z} & H_n \end{array}$$

In the case of finite groups,  $n \geq 0$  is the order of the cyclic group  $\mathbb{Z}/n$  and the quaternion group  $Q_n$ . In the infinite case, it is the Euler number that determines  $H_n$ . (See Section 3 for an argument why no other nilpotent groups occur.)

The reason we used the rational version of the lower central series is that our Betti number estimates only give rational information. The integral version of Theorem 3 was in the meantime proven by the third author [21] using completely different methods.

The following result is the rational version of Turaev's Theorem 2 in [23].

**Theorem 4.** *A finitely generated nilpotent group  $G$  is isomorphic to  $\pi_1(M)/\Gamma_r^{\mathbb{Q}}$  for a closed oriented 3-manifold  $M$  if and only if there exists a class  $m \in H_3(G; \mathbb{Q})$  such that the composition*

$$H^1(G; \mathbb{Q}) \xrightarrow{\cap m} H_2(G; \mathbb{Q}) \twoheadrightarrow H_2(G; \mathbb{Q})/\Phi_r^{\mathbb{Q}}(G)$$

*is an epimorphism.*

Here  $\Phi_r^{\mathbb{Q}}(G)$  is the  $r$ -th term in the rational Dwyer filtration of  $H_2(G; \mathbb{Q})$  (defined in the Appendix A).

### 3. LOW-DIMENSIONAL SURGERY

In this section we first show how Theorems 4 and 1 imply Theorem 3. Then we recall the proof of Theorem 4 and finally we give a short discussion of nilpotent 3-manifold groups.

First note that by definition  $\Phi_r^{\mathbb{Q}} = 0$  if  $\Gamma_{r-1}^{\mathbb{Q}} = \{0\}$ . So by Theorem 4, in order for a group  $G$  to be the maximal nilpotent quotient of a rationally prenilpotent 3-manifold group, it must possess the following property  $\cap$  (here rational coefficients are to be understood):

$$\cap : \quad \begin{cases} \text{There exists } m \in H_3(G) \text{ such that} \\ \cap m : H^1(G) \longrightarrow H_2(G) \text{ is an epimorphism.} \end{cases}$$

Theorem 3 now follows immediately from the following:

**Proposition 1.** *A finitely generated torsion-free nilpotent group satisfying property  $\cap$  is isomorphic to exactly one of the groups:*

$$\{1\}, \quad \mathbb{Z} \quad \text{or} \quad H_n.$$

Observe that  $H_0 \otimes \mathbb{Q} = \mathbb{Z}^3 \otimes \mathbb{Q} = \mathbb{Q}^3$ . When  $n > 0$  each of the groups  $H_n \otimes \mathbb{Q}$  is isomorphic to the  $\mathbb{Q}$  points of the Heisenberg group  $H_{\mathbb{Q}}$ . On the other hand, the groups  $\mathbb{Z}^k$  are the only finitely generated torsion-free nilpotent groups whose Malcev completion is  $\mathbb{Q}^k$  and, for  $n > 0$ , the groups  $H_n$  are the only finitely generated torsion-free nilpotent groups with Malcev completion  $H_{\mathbb{Q}}$ . Therefore, Proposition 1 follows from the following result and the fact that the Malcev completion of a finitely generated nilpotent group is a uniquely divisible nilpotent group with  $b_1 < \infty$ .

**Proposition 2.** *A uniquely divisible nilpotent group with  $b_1 < \infty$  satisfying property  $\cap$  is isomorphic to exactly one of the groups:*

$$\{0\}, \quad \mathbb{Q}, \quad \mathbb{Q}^3 \quad \text{or} \quad H_{\mathbb{Q}}.$$

Property  $\cap$  implies  $b_1 \geq b_2$ . This, combined with the inequality  $b_2 > b_1^2/4$  from Theorem 1, implies that Proposition 2 follows from the result below. The proof will be given in Section 4.

**Proposition 3.** *Suppose that  $G$  is a uniquely divisible nilpotent group with  $b_1(G) < \infty$  satisfying property  $\cap$ . If  $b_1 = b_2 = 2$ , then  $G$  is isomorphic to  $H_{\mathbb{Q}}$ . If  $b_1 = b_2 = 3$ , then  $G$  is isomorphic to  $\mathbb{Q}^3$ .*

This finishes the outline of the proof of Theorem 3.

*Proof of Theorem 4.* Given a closed oriented 3-manifold  $M$ , we may take a classifying map  $M \rightarrow K(\pi_1(M), 1)$  of the universal covering and compose with the projection  $\pi_1(M) \rightarrow \pi_1(M)/\Gamma_r$  to get a map  $u : M \rightarrow K(\pi_1(M)/\Gamma_r, 1)$  and a commutative diagram

$$\begin{array}{ccc} H^1(M) & \xrightarrow[\cong]{\cap [M]} & H_2(M) \\ u^* \uparrow & & u_* \downarrow \\ H^1(\pi_1(M)/\Gamma_r) & \xrightarrow{\cap u_*[M]} & H_2(\pi_1(M)/\Gamma_r) \end{array}$$

Clearly  $u$  induces an isomorphism on  $H_1$ , and therefore  $u^*$  is an isomorphism. Thus the “only if” part of our theorem follows directly from Dwyer’s theorem (see the Appendix A). Here we could have worked integrally or rationally.

To prove the “if” part, we restrict to rational coefficients. Let  $u : M \rightarrow K(G, 1)$  be a map from a closed oriented 3-manifold with  $u_*[M] = m$ , the given class in  $H_3(G)$ . Such a map exists because rationally oriented bordism maps onto homology by the classical result of Thom. Now observe that we are done by Dwyer’s theorem (and the above commutative diagram) if the map  $u$  induces an isomorphism on  $H_1$ . If this is not the case, we will change the map  $u$  (and the 3-manifold  $M$ ) by surgeries until it is an isomorphism on  $H_1$ : We describe the surgeries as attaching 4-dimensional handles to the upper boundary of  $M \times I$ . Then  $M$  is the lower boundary of this 4-manifold and the upper boundary is denoted by  $M'$ . If the map  $u$  extends to the 4-manifold, then the image of the fundamental class of the new 3-manifold  $(M', u')$  is still the given class  $m \in H_3(G)$ .

First add 1-handles  $D^1 \times D^3$  to  $M \times I$  and extend  $u$  to map the new circles to the (finitely many) generators of  $G$ . This makes  $u'$  an epimorphism (on  $H_1$ ). Then we want to add 2-handles  $D^2 \times D^2$  to  $M' \times I$  to kill the kernel  $K$  of

$$u'_* : H_1(M') \longrightarrow H_1(G).$$

We can extend the map  $u'$  to the 4-manifold

$$W := M' \times I \cup \text{2-handles}$$

if we attach the handles to curves in the kernel of  $u : \pi_1(M) \rightarrow G$ .

Now observe that the new upper boundary  $M''$  of  $W$  still maps onto  $H_1(G)$  because one can obtain  $M'$  from  $M''$  by attaching 2-handles ( $2 + 2 = 4$ ). But the problem is that  $\text{Ker}\{u''_* : H_1(M'') \rightarrow H_1(G)\}$  may contain new elements which are meridians to the circles  $c$  we are trying to kill. If  $c$  has a dual, i.e. if there is a surface in  $M'$  intersecting  $c$  in a point, then these meridians are null-homologous. But since we only work rationally we may assume all the classes in the kernel to have a dual and we are done.

The more involved integral case is explained in detail in [23] but we do not need it here. ■

We finish this section by explaining briefly the table of nilpotent 3-manifold groups given in Section 2. The finite groups  $G$  in the table are all subgroups of  $SU(2)$  and thus the corresponding 3-manifolds are homogenous spaces  $SU(2)/G$ . Embeddings of cyclic groups into  $SU(2)$  are given by matrices of the form  $\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}$ , and the image of the quaternion groups in  $SU(2)$  is generated by the above matrix together with the matrix  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ .

To explain why only cyclic and quaternion groups occur, first recall that a finite group is nilpotent if and only if it is the direct product of its  $p$ -Sylow subgroups [13]. Now it is well known [2] that the only  $p$ -groups with periodic cohomology are the cyclic groups and, for  $p = 2$ , the quaternion groups. The only fact about the group we use is that it acts freely on a homotopy 3-sphere, and thus has 4-periodic cohomology.

To understand why only the groups  $\mathbb{Z}$  and  $H_n$  occur as infinite nilpotent 3-manifold groups, first notice that a nilpotent group is never a nontrivial free product. Except in the case  $\pi_1(M) = \mathbb{Z}$ , the sphere theorem implies that the universal cover of the corresponding 3-manifold must be contractible. In particular, the nilpotent fundamental groups ( $\neq \mathbb{Z}$ ) of a 3-manifolds must have homological dimension 3. It is then easy to see that the groups  $H_n$  are the only such groups. For more details see [22].

#### 4. EXAMPLES

In this section we prove Proposition 3 and give an example concerning the difference between integral and rational prenilpotence of 3-manifold groups.

*Proof of Proposition 3.* The key to both cases is the following commutative diagram (later applied with  $r = 2, 3, 4$ ). Here all homology groups, as well

as the groups  $\Gamma_r$  and  $\Phi_r$ , have rational coefficients, i.e., we suppress the letter  $\mathbb{Q}$  from the notation.

$$\begin{array}{ccccccc} \Phi_{r+1}(G) & \longrightarrow & H_2(G) & \xrightarrow{p_*} & H_2(G/\Gamma_r) & \longrightarrow & \Gamma_r(G)/\Gamma_{r+1}(G) \\ & & \cap m \uparrow \cong & & \cap p_*(m) \uparrow & & \\ & & H^1(G) & \xleftarrow[p_*]{\cong} & H^1(G/\Gamma_r) & & \end{array}$$

The upper line is *short* exact by the 5-term exact sequence and the definition of  $\Phi_{r+1}$ . Since we consider cases with  $b_1 = b_2$ , the map  $\cap m$  is actually an isomorphism.

Consider first the case  $b_1(G) = b_2(G) = 3$  and  $r = 2$  in the above diagram. Since  $H_3(G/\Gamma_2) \cong H_3(\mathbb{Q}^3) \cong \mathbb{Q}$ , there are only two cases to consider:

- (i)  $p_*(m) \neq 0$ : Then, by Poincaré duality for  $\mathbb{Q}^3$ , the map  $\cap p_*(m)$  is an isomorphism and thus  $p_* : H_2(G) \rightarrow H_2(G/\Gamma_2)$  is also an isomorphism. By Stallings' Theorem this implies that  $G \cong G/\Gamma_2 \cong \mathbb{Q}^3$ .
- (ii)  $p_*(m) = 0$ : Then  $p_* : H_2(G) \rightarrow H_2(G/\Gamma_2)$  is the zero map and thus  $\Phi_3(G) = \mathbb{Q}^3$ . This contradicts Theorem 2 with  $r = 3$ , since  $3 \not\asymp 4$ .

Now consider the case  $b_1(G) = b_2(G) = 2$ . If  $p_* : H_2(G) \rightarrow H_2(G/\Gamma_2) \cong \mathbb{Q}$  is onto, then by Stallings' Theorem,  $G$  would be abelian and thus have  $b_2(G) = 1$ , a contradiction. Therefore,  $H_2(G) = \Phi_3(G)$  and  $G/\Gamma_3$  is the rational Heisenberg group  $H_{\mathbb{Q}}$ . This follows from Dwyer's Theorem by comparing  $G$  to the free group  $F$  on 2 generators and noting that  $H_{\mathbb{Q}} \cong F/\Gamma_3 \otimes \mathbb{Q}$ .

Now consider the above commutative diagram for  $r = 3$ . Since  $H_3(H_{\mathbb{Q}}) \cong \mathbb{Q}$  there are again only two cases to consider:

- (i)  $p_*(m) \neq 0$ : Then by Poincaré duality for  $H_{\mathbb{Q}}$ , the map  $\cap p_*(m)$  is an isomorphism and thus  $p_* : H_2(G) \rightarrow H_2(G/\Gamma_2)$  is also an isomorphism. By Stallings' Theorem this implies that  $G \cong G/\Gamma_3 \cong H_{\mathbb{Q}}$ .
- (ii)  $p_*(m) = 0$ : Then  $p_* : H_2(G) \rightarrow H_2(G/\Gamma_2)$  is the zero map and thus  $H_2(G) = \Phi_4(G)$ .

Unfortunately, this does not contradict Theorem 2 with  $r = 4$  (since  $2 > 27/16$ ) and we have to go one step further. Again by Dwyer's Theorem  $G/\Gamma_4$  is isomorphic to  $K := F/\Gamma_4 \otimes \mathbb{Q}$ . One easily computes that the cap-product map

$$\cap : H_3(K) \otimes H^1(K) \longrightarrow H_2(K)$$

is identically zero and thus the above commutative diagram for  $r = 4$  shows that  $H_2(G) = \Phi_5(G)$ . Now Theorem 2 does lead to the contradiction  $2 > 8192/3125$ .  $\blacksquare$

We believe that the above proof is unnecessarily complicated because Theorem 2 does not give the best possible estimate. In fact, we conjecture that there is no nilpotent group with  $b_1 = b_2 = 2$  and  $\Phi_4 \cong \mathbb{Q}^2$ , and that a nilpotent group with  $b_1 = b_2 = 3$  is always rationally  $\mathbb{Q}^3$  (without assuming property  $\cap_{\mathbb{Q}}$ ). However, we found the following

**Example .** Besides the Heisenberg group  $H_{\mathbb{Q}}$ , there are other nilpotent groups with  $b_1 = b_2 = 2$ . For example, take a nontrivial central extension

$$0 \longrightarrow \mathbb{Q} \longrightarrow G \longrightarrow H_{\mathbb{Q}} \longrightarrow 0.$$

By Proposition 3,  $G$  cannot satisfy property  $\cap_{\mathbb{Q}}$ , which can be checked directly.

We finish this section by discussing the 2-torus bundle over the circle with holonomy given by

$$(z_1, z_2) \mapsto (\bar{z}_1, \bar{z}_2), \quad z_i \in S^1 \subset \mathbb{C}.$$

The fundamental group  $G$  of this 3-manifold is the semidirect product of  $\mathbb{Z}$  with  $\mathbb{Z}^2$ , where a generator of the cyclic group  $\mathbb{Z}$  acts as minus the identity on the normal subgroup  $\mathbb{Z}^2$ . One computes that

$$\Gamma_r(G) = 2^{r-1} \cdot \mathbb{Z}^2$$

and thus  $G$  is not prenilpotent. However, one sees that  $\Gamma_r^{\mathbb{Q}}(G) = \mathbb{Z}^2$  for all  $r$  — i.e.,  $G$  is rationally prenilpotent with maximal torsion-free nilpotent quotient  $\mathbb{Z}$ .

This example illustrates three phenomena. Firstly, the rational lower central series stabilizes more often than the integral one, even for 3-manifold groups. Secondly, going to the maximal torsion-free nilpotent quotient kills many details of the 3-manifold group which can be still seen in the nilpotent quotients. Finally, unlike in the geometric theory, even a 2-fold covering can alter the class (Theorem 3) to which a 3-manifold belongs.

## 5. THE PROOF OF THEOREM 2

The proof of the estimate in Theorem 2 consists of three steps which are parallel to those in Roquette's proof of the Golod-Shafarevich Theorem, as presented in [14]. We begin by recalling the structure of the proof:

1. The  $\mathbb{F}_p$ -homology of a  $p$ -group  $G$  can be calculated using a free  $\mathbb{F}_p[G]$ -resolution of the trivial module  $\mathbb{F}_p$  in place of a free  $\mathbb{Z}[G]$ -resolution of  $\mathbb{Z}$ . This puts us in the realm of linear algebra.
2. There is a *minimal resolution* of  $\mathbb{F}_p$  over  $\mathbb{F}_p[G]$ . This is an exact sequence

$$\cdots \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow \mathbb{F}_p \longrightarrow 0$$

of free  $\mathbb{F}_p[G]$ -modules  $C_i$  such that, for all  $i \geq 0$ , one has

$$\text{rank}_{\mathbb{F}_p[G]} C_i = \dim_{\mathbb{F}_p} H_i(G; \mathbb{F}_p).$$

3. Let  $I$  be the kernel of the augmentation  $\mathbb{F}_p[G] \rightarrow \mathbb{F}_p$ . Define a generating function  $d(t)$  by

$$d(t) := \sum_{k \geq 0} \dim_{\mathbb{F}_p} (I^k / I^{k+1}) t^k$$



This function is a polynomial which is positive on the unit interval. The proof is completed by filtering the first two pieces of a minimal resolution by powers of  $I$ . This leads to an inequality

$$(\text{a quadratic polynomial})d(t) \geq 1$$

for  $t \in [0, 1]$  from which one deduces the result by looking at the discriminant of the quadratic.

We now consider the analogues of these for a finitely generated torsion free nilpotent group  $G$ . We reduce to the case of a uniquely divisible group with  $b_1 < \infty$  by replacing  $G$  by  $G \otimes \mathbb{Q}$ .

**Ad(1)** The  $\mathbb{Q}$ -homology of  $G$  can be clearly calculated using a free resolution of  $\mathbb{Q}$  over the group ring  $\mathbb{Q}[G]$  rather than over  $\mathbb{Z}[G]$ . More importantly, we can also resolve  $\mathbb{Q}$  as a module over the  $I$ -adic completion of  $\mathbb{Q}[G]$ , where  $I$  is again the augmentation ideal. This step involves an Artin-Rees Lemma proved in this setting in [3]. Let  $A$  denote the completion  $\mathbb{Q}[G]^\wedge$ .

**Ad(2)** There is a minimal resolution of  $\mathbb{Q}$  over  $A$  in the same sense as above. More precisely, there is minimal resolution which takes into account the Dwyer filtration of  $H_2(G)$ . The construction of such a resolution will be given in Section 6. It is crucial that one uses the completed group ring  $A$  rather than  $\mathbb{Q}[G]$ .

**Ad(3)** Define a generating function  $d(t)$  by

$$d(t) := \sum_{k \geq 0} \dim_{\mathbb{Q}}(I^k/I^{k+1}) t^k$$

The proof of the following result is at the end of this section.

**Proposition 4.** *With notation as above,*

$$d(t) = \prod_{r \geq 1} \frac{1}{(1 - t^r)^{g_r}}$$

where  $g_r$  is the dimension of  $\Gamma_r(G)/\Gamma_{r+1}(G)$ . In particular,  $d(t)$  is a rational function.

Define  $p_r$  to be the dimension of  $\Phi_r^{\mathbb{Q}}(G)/\Phi_{r+1}^{\mathbb{Q}}(G)$ . Recall that, roughly speaking,  $p_r$  is the number of relations which are  $r$ -fold commutators but not  $(r + 1)$ -fold commutators. Let  $b_i$  be the  $i$ th rational Betti numbers of  $G$ . Set

$$p(t) := \sum_{r \geq 2} p_r t^r - b_1 t + 1.$$

The following result will be proven in Section 7 using the resolution constructed in Section 6.

**Proposition 5.** *For all  $0 < t < 1$  one has the inequality  $p(t)d(t) \geq 1$ .*

Now it is easy to prove Theorem 2: First note that when  $0 < t < 1$ , the generating function  $d(t)$  is positive, and thus  $p(t)$  is also positive in this interval. Now assume that  $H_2(G; \mathbb{Q}) = \Phi_r^{\mathbb{Q}}(G)$  for some  $r \geq 2$  and that  $G \neq \{0\}, \mathbb{Q}$ . Then  $b_1 > 1, b_2 > 0$  and  $p(t) \leq q(t)$  for each  $t$  in  $0 < t < 1$ , where

$$q(t) := b_2 t^r - b_1 t + 1.$$

The polynomial  $q(t)$  has a minimum at

$$t_0 := \sqrt[r-1]{\frac{b_1}{r b_2}}.$$

The desired inequality follows from the inequality  $q(t_0) > 0$  after a little algebraic manipulation. But in order to do this we need  $t_0 < 1$ , which is equivalent to the condition  $b_1 < r b_2$ .

We know that  $0 \leq q(1) = b_2 - b_1 + 1$ , and thus  $b_2 \geq b_1 - 1$ . Therefore, the only case where  $t_0 < 1$  is not satisfied is  $(b_1, b_2) = (2, 1)$  (and  $r = 2$ ). We claim that  $(b_1, b_2) = (2, 1)$  implies  $G \cong \mathbb{Q}^2$ .

Consider the projection  $G \rightarrow G/\Gamma_2 \cong \mathbb{Q}^2$ . If the induced map on  $H_2(\ ; \mathbb{Q})$  is onto, then we are done by Stallings' Theorem. If not then we know that  $H_2(G; \mathbb{Q}) = \Phi_3(G)$ , and are in a case where  $t_0 < 1$  since  $r = 3$ . But then  $q(t_0) > 0$  leads to the contradiction  $b_2 = 1 > 32/27$ . ■

*Proof of Proposition 4.* Set

$$\mathrm{Gr}(G) = \bigoplus_{r \geq 1} \Gamma_r(G)/\Gamma_{r+1}(G).$$

This is a graded Lie algebra. Observe that  $\mathrm{Gr}(A) \cong \mathrm{Gr}(\mathbb{Q}[G])$ . Then, by a theorem of Quillen [18], we have an isomorphism  $\mathrm{Gr}(\mathbb{Q}[G]) \cong U(\mathrm{Gr}(G))$  of graded Hopf algebras. Here  $U(\mathrm{Gr}(G))$  denotes the universal enveloping algebra of  $\mathrm{Gr}(G)$ . We shall write  $\mathcal{G}$  for  $\mathrm{Gr}(G)$  and  $\mathcal{G}_r$  for  $\Gamma_r(G)/\Gamma_{r+1}(G)$ .

By the Poincaré-Birkhoff-Witt Theorem, there is a graded coalgebra isomorphism of  $U\mathcal{G}$  with the symmetric coalgebra  $S\mathcal{G}$  on  $\mathcal{G}$ . Note that we have the isomorphism

$$S\mathcal{G} \cong \bigotimes_{r \geq 1} S\mathcal{G}_r,$$

of graded vector spaces, where the tensor product on the right is finite as  $\mathcal{G}$  is nilpotent. Since  $\mathcal{G}_r$  has degree  $r$ , the generating function of the symmetric coalgebra  $S\mathcal{G}_r$  is  $1/(1 - t^r)^{g_r}$ . The result follows. ■

## 6. THE MINIMAL RESOLUTION

In this section we use techniques from rational homotopy theory to prove the existence of the minimal resolutions needed in the proof of Theorem 2. We obtain the minimal resolution using Chen's method of formal power series connections, which provides a minimal associative algebra model of the loop space of a space. Chen's theory is briefly reviewed in Appendix B.

The precise statement of the main result is:

**Theorem 5.** *If  $G$  is a nilpotent group with  $b_1(G) < \infty$ , then there is a free  $\mathbb{Q}[G]^\wedge$  resolution*

$$\cdots \xrightarrow{\delta} C_2 \xrightarrow{\delta} C_1 \xrightarrow{\delta} C_0 \rightarrow \mathbb{Q} \rightarrow 0$$

of the trivial module  $\mathbb{Q}$  with the properties:

1.  $C_k$  is isomorphic to  $H_k(G) \otimes \mathbb{Q}[G]^\wedge$ ;
2.  $\delta(\Phi_m^\mathbb{Q}) \subseteq H_1(G) \otimes I^{m-1}$ , where  $I$  denotes the augmentation ideal of  $\mathbb{Q}[G]^\wedge$ .

Denote the nilpotent Lie algebra associated to  $G$  by  $\mathfrak{g}$ . This is a finite dimensional Lie algebra over  $\mathbb{Q}$  as  $G$  is nilpotent and the  $\mathbb{Q}$  Betti numbers of  $G$  are finite. Denote the standard (i.e., the Chevelley-Eilenberg) complex of cochains of  $\mathfrak{g}$  by  $\mathcal{C}^\bullet(\mathfrak{g})$ . By Sullivan's theory of minimal models, this is the minimal model of the classifying space  $BG$  of  $G$ .

Let  $(\omega, \partial)$  be a formal power series connection associated to  $\mathcal{C}^\bullet(\mathfrak{g})$ .<sup>1</sup> Note that there is a natural isomorphism between  $\tilde{H}_\bullet(\mathcal{C}^\bullet(\mathfrak{g}))$  and the reduced homology groups  $\tilde{H}_\bullet(G, \mathbb{Q})$ . Set

$$A_\bullet(G) = (T(\tilde{H}_\bullet(G)[1])^\wedge, \partial).$$

Since  $A_\bullet(G)$  is complete, we may assume, after taking a suitable change of coordinates of the form

$$Y \mapsto Y + \text{higher order terms}, \quad Y \in \tilde{H}_\bullet(G)[1],$$

that

$$(1) \quad \partial Y \in I^k \iff k = \max\{m \geq 1 : \partial(Y + \text{higher order terms}) \in I^m\}.$$

Since  $\mathcal{C}^\bullet(\mathfrak{g})$  is a minimal algebra in the sense of Sullivan, we can apply the standard fact [1, (2.30)] (see also, [11, 2.6.2]) to deduce that  $H^k(B(\mathcal{C}(\mathfrak{g})))$  vanishes when  $k > 0$ . It follows from Theorem B.2 that  $H_k(A_\bullet(G))$  also vanishes when  $k > 0$ .

When  $k = 0$ , Chen's Theorem B.1 yields a complete Hopf algebra isomorphism

$$H_0(A_\bullet(G)) \cong U\mathfrak{g} \cong \mathbb{Q}[G]^\wedge.$$

These two facts are key ingredients in the construction of the resolution. Another important ingredient is the Adams-Hilton construction:

**Proposition 6.** *There is a continuous differential  $\delta$  of degree  $-1$  on the free  $A_\bullet(G)$ -module*

$$W := H_\bullet(G) \otimes A_\bullet(G)$$

such that:

1. the restriction of  $\delta$  to  $A_\bullet = 1 \otimes A_\bullet$  is  $\partial$ ;
2.  $\delta$  is a graded derivation with respect to the right  $A_\bullet(X)$  action;
3.  $(W, \delta)$  is acyclic.

<sup>1</sup>The definition and notation can be found in Appendix B.

*Proof.* To construct the differential, it suffices to define  $\delta$  on each  $Y \otimes 1$ , where  $Y \in H_\bullet(G)$ . To show that the tensor product is acyclic, we will construct a graded map  $s : W \rightarrow W$  of degree 1, such that  $s^2 = 0$  and

$$(2) \quad \delta s + s\delta = \text{id} - \epsilon.$$

Here  $\epsilon$  is the tensor product of the augmentations of  $H_\bullet(G)$  and  $A_\bullet(G)$ .

Define  $s(1) := 0 =: \delta(1)$ . For  $Y \in \tilde{H}_\bullet(G)$ , we shall denote the corresponding element of  $\tilde{H}_\bullet(G)[1]$  by  $\bar{Y}$ . For  $Y, Y_1, Y_2, \dots, Y_k \in \tilde{H}_\bullet(G)$ , define

$$s(1 \otimes \bar{Y}_1 \bar{Y}_2 \dots \bar{Y}_k) := Y_1 \otimes \bar{Y}_2 \dots \bar{Y}_k \text{ and } s(Y \otimes \bar{Y}_1 \bar{Y}_2 \dots \bar{Y}_k) := 0.$$

We shall simply write  $Y$  for  $Y \otimes 1$  and  $\bar{Y}$  for  $1 \otimes \bar{Y}$ . Since  $\delta$  is a derivation, it suffices to define it on the  $A_\bullet(G)$ -module generators  $1, Y, \bar{Y}$  of  $W$ . We already know how to define  $\delta$  on the  $\bar{Y}$ 's. In order that (2) holds when applied to  $Y$ , we have to define

$$\delta Y := \bar{Y} - s\delta\bar{Y}.$$

Then (2) also holds when applied to  $\bar{Y}$  and this implies  $\delta^2 Y = 0$ . One can now verify (2) by induction on degree using the fact that  $s$  is right  $A_\bullet(G)$ -linear.  $\blacksquare$

The way to think of this result is that  $A_\bullet(G)$  is a homological model of the loop space  $\Omega_x BG$  of the classifying space of  $G$ . One can take  $H_\bullet(G)$  with the trivial differential to be a homological model for  $BG$ . The complex  $(W, \delta)$  is a homological model of the path-loops fibration whose total space is contractible. This picture should help motivate the next step, the construction of the minimal resolution.

Filter the complex  $(W, \delta)$  by degree in  $H_\bullet(G)$ . This leads to a spectral sequence

$$E_{p,q}^1 = H_p(G) \otimes H_q(A_\bullet(G)) \implies H_{p+q}(W, \delta).$$

(This is the algebraic analogue of the Serre spectral sequence for the path-loops fibration.) But since  $H_q(A_\bullet(G))$  vanishes when  $q > 0$ , the spectral sequence collapses at the  $E^1$ -term to a complex

$$\dots \rightarrow H_2(G) \otimes H_0(A_\bullet(G)) \rightarrow H_1(G) \otimes H_0(A_\bullet(G)) \rightarrow H_0(A_\bullet(G)) \rightarrow \mathbb{Q} \rightarrow 0$$

which is acyclic as  $(W, \delta)$  is. The existence of the resolution now follows from the fact that

$$H_0(A_\bullet(G)) \cong \mathbb{Q}[G]^\wedge.$$

The  $\mathbb{Q}[G]^\wedge$  linearity of the differential follows directly from the second assertion of Proposition 6.

It remains to verify the condition satisfied by the differential. Observe that  $H_0(A_\bullet(G))$  is the quotient of  $T(H_1(G))^\wedge$  by the closed ideal generated by the image of

$$\partial : H_2(G) \rightarrow T(H_1(G))^\wedge.$$

Define a filtration

$$H_2(G; \mathbb{Q}) = L_2 \supseteq L_3 \supseteq \dots$$

by  $L_k = \partial^{-1}I^k$ . It is not difficult to verify that in the complex  $H_\bullet(G) \otimes H_0(A_\bullet(G))$  we have

$$\delta(L_k \otimes \mathbb{Q}) \subseteq H_1(G) \otimes I^{k-1}.$$

So, to complete the proof, we have to show that  $\Phi_k^{\mathbb{Q}} = L_k$ .

Consider the spectral sequence dual to the one in the proof of Theorem B.2. It has the property that  $E_{-1,2}^r$  is a subspace of  $H_2(G; \mathbb{Q})$  and  $E_{-p,p}^r$  is a quotient of  $H_1(G; \mathbb{Q})^{\otimes p}$ . We thus have a filtration

$$H_2(G; \mathbb{Q}) = E_{-1,2}^1 \supseteq E_{-1,2}^2 \supseteq E_{-1,2}^3 \supseteq \cdots$$

of  $H_2(G; \mathbb{Q})$ . From the standard description of the terms of the spectral sequence associated to a filtered complex, it is clear from condition (1) that when  $r \geq 2$

$$L_r = E_{-1,2}^{r-1}.$$

We know from [4, (2.6.1)] that

$$E_{-p,p}^p = E_{-p,p}^\infty \cong I^p/I^{p+1},$$

where  $I$  denotes the augmentation ideal of  $\mathbb{Q}[\pi_1(G)]$ . Consequently, a group homomorphism  $G \rightarrow H$  induces an isomorphism

$$\mathbb{Q}[G]/I^k \rightarrow \mathbb{Q}[H]/I^k$$

if and only if the induced map  $H_2(G; \mathbb{Q})/L_k \rightarrow H_2(H; \mathbb{Q})/L_k$  is an isomorphism. The proof is completed using the fact (cf. [11, (2.5.3)]) that a group homomorphism  $G \rightarrow H$  induces an isomorphism  $(G \otimes \mathbb{Q})/\Gamma_k \rightarrow (H \otimes \mathbb{Q})/\Gamma_k$  if and only if the induced map  $\mathbb{Q}[G]/I^k \rightarrow \mathbb{Q}[H]/I^k$  is an isomorphism, [17].

## 7. THE INEQUALITY

In this section, we prove Proposition 5. We suppose that  $G$  is a nilpotent group with  $b_1(G) < \infty$ . Denote  $\mathbb{Q}[G]^\wedge$  by  $A$  and its augmentation ideal by  $I$ . Recall that

$$d(t) = \sum_{l \geq 0} d_l t^l,$$

where  $d_l = \dim I^l/I^{l+1}$ , and that this is a rational function whose poles lie on the unit circle.

Consider the part

$$H_2(G) \otimes A \xrightarrow{\delta} H_1(G) \otimes A \rightarrow A \rightarrow \mathbb{Q} \rightarrow 0$$

of the resolution constructed in Section 6. Define a filtration

$$E^0 \supseteq E^1 \supseteq E^2 \supseteq \cdots$$

of  $H_2(G) \otimes A$  by

$$E^l := \delta^{-1} \left( H_1(G) \otimes I^l \right).$$

Then we have an exact sequence

$$(3) \quad 0 \rightarrow E^l/E^{l+1} \rightarrow H_1(G) \otimes I^l/I^{l+1} \rightarrow I^{l+1}/I^{l+2} \rightarrow 0$$

for all  $l \geq 0$ . Set  $e_l := \dim E^l/E^{l+1}$  and

$$e(t) := \sum_{l \geq 0} e_l t^l \in \mathbb{R}[[t]].$$

**Proposition 7.** *The series for  $e(t)$  converges to the rational function*

$$\frac{b_1(G) td(t) - d(t) + 1}{t}$$

*all of whose poles lie on the unit circle.*

*Proof.* Because the sequence (3) is exact, we have  $e_l = b_1(G) d_l - d_{l+1}$ . This implies that

$$te(t) = b_1(G) td(t) - d(t) + 1.$$

from which the result follows. ■

Our final task is to bound the  $e_l$ . Our resolution has the property that

$$\delta(\Phi_m \otimes I^{l-m+2}) \subseteq H_1(G) \otimes I^{l+1}.$$

This implies that

$$\sum_{m \geq 2} \Phi_m \otimes I^{l-m+2} \subseteq E^{l+1},$$

so that the linear map

$$H_2(G) \otimes A / \left( \sum_{m \geq 2} \Phi_m \otimes I^{l-m+2} \right) \rightarrow H_2(G) \otimes A / E^{l+1}$$

is surjective. This implies that

$$(4) \quad \dim \left( H_2(G) \otimes A / \left( \sum_{m \geq 2} \Phi_m \otimes I^{l-m+2} \right) \right) \geq e_0 + e_1 + \cdots + e_l.$$

To compute the dimension of the left hand side, we apply the following elementary fact from linear algebra.

**Proposition 8.** *Suppose that*

$$B = B^0 \supseteq B^1 \supseteq B^2 \supseteq \cdots$$

*and*

$$C = C^0 \supseteq C^1 \supseteq C^2 \supseteq \cdots$$

*are two filtered vector spaces. Define a filtration  $F$  of  $B \otimes C$  by*

$$F^k := \sum_{i+j=k} B^i \otimes C^j.$$

*Then there is a canonical isomorphism*

$$F^k / F^{k+1} \cong \bigoplus_{i+j=k} A^i / A^{i+1} \otimes B^j / B^{j+1}. \quad \blacksquare$$

Applying this with  $B = H_2(G)$  with the filtration  $\Phi_\bullet$ , and  $C = A$  with the filtration  $I^\bullet$ , we deduce that

$$\begin{aligned} \dim \left( H_2(G) \otimes A / \left( \sum_{m \geq 2} \Phi_m \otimes I^{l-m+2} \right) \right) &= \sum_{m+k \leq l+1} (\dim \Phi_m / \Phi_{m+1}) (\dim I^k / I^{k+1}) \\ &= \sum_{m+k \leq l+1} p_m d_k \end{aligned}$$

where  $p_m = \dim \Phi_m$ . Combined with (4), this implies that

$$\sum_{m+k \leq l+1} p_m d_k \geq e_0 + \cdots + e_l.$$

Using geometric series, this can be assembled into the following inequality which holds when  $0 < t < 1$ :

$$\frac{(p_2 t + p_3 t^2 + \cdots + p_{l+1} t^l) d(t)}{1-t} \geq \frac{e(t)}{1-t}.$$

Plugging in the formula for  $e(t)$  given in Proposition 7, we deduce the desired inequality

$$d(t) \left( \sum_{m \geq 2} p_m t^m - b_1(G) t + 1 \right) \geq 1$$

when  $0 < t < 1$ .

#### APPENDIX A.

Here we collect the necessary group theoretic definitions. Let  $G$  be a group.

- $\Gamma_r(G)$  denotes the  $r$ -th term of the *lower central series* of  $G$  which is the subgroup of  $G$  generated by all  $r$ -fold commutators. We abbreviate  $G/\Gamma_r(G)$  by  $G/\Gamma_r$ . We say  $G$  is *nilpotent* if  $\Gamma_r(G) = \{1\}$  for some  $r$  and it is then said to have *nilpotency class*  $< r$ . For example, abelian groups have  $\Gamma_2 = \{1\}$  and are of nilpotency class 1.
- $G$  is *pre-nilpotent* if its lower central series stabilizes at some term  $\Gamma_r$ , i.e.,  $\Gamma_r(G) = \Gamma_{r+1}(G)$ . This happens if and only if  $G$  has a maximal nilpotent quotient (which is then isomorphic to  $G/\Gamma_r$ ).

The main homological tool in dealing with nilpotent groups is the following result of Stallings:

**Theorem A.1.** [20] *If  $f : G \rightarrow H$  is a homomorphism of groups inducing an isomorphism on  $H_1$  and an epimorphism on  $H_2$ , then the induced maps  $G/\Gamma_r \rightarrow H/\Gamma_r$  are isomorphisms for all  $r \geq 1$ .*

- The *Dwyer filtration* of  $H_2(G; \mathbb{Z})$

$$H_2(G; \mathbb{Z}) =: \Phi_2(G) \supseteq \Phi_3(G) \supseteq \Phi_4(G) \supseteq \cdots$$

is defined by (see [8] for a geometric definition of  $\Phi_r$  using gropes)

$$\Phi_r(G) := \text{Ker}(H_2(G; \mathbb{Z}) \longrightarrow H_2(G/\Gamma_{r-1}; \mathbb{Z})).$$

This filtration is used in Dwyer's extension of Stallings' Theorem:

**Theorem A.2.** [5] *If  $f : G \longrightarrow H$  induces an isomorphism on  $H_1(\ ; \mathbb{Z})$ , then for  $r \geq 2$  the following three conditions are equivalent:*

1.  $f$  induces an epimorphism  $H_2(G; \mathbb{Z})/\Phi_r(G) \rightarrow H_2(H; \mathbb{Z})/\Phi_r(H)$ ;
2.  $f$  induces an isomorphism  $G/\Gamma_r \rightarrow H/\Gamma_r$ ;
3.  $f$  induces an isomorphism  $H_2(G; \mathbb{Z})/\Phi_r(G) \rightarrow H_2(H; \mathbb{Z})/\Phi_r(H)$ , and an injection  $H_2(G; \mathbb{Z})/\Phi_{r+1}(G) \rightarrow H_2(H; \mathbb{Z})/\Phi_{r+1}(H)$ .

There are rational versions of the above definitions.

- $\Gamma_r^{\mathbb{Q}}$  denotes the  $r$ -th term of the *rational lower central series* of  $G$ , which is defined by

$$\Gamma_r^{\mathbb{Q}}(G) := \text{Rad}(\Gamma_r(G)) := \{g \in G \mid g^n \in \Gamma_r \text{ for some } n \in \mathbb{Z}\}.$$

It has the (defining) property that  $G/\Gamma_r^{\mathbb{Q}} = (G/\Gamma_r)/\text{Torsion}$ . (Note that the torsion elements in a nilpotent group form a subgroup.)  $G$  is *rationally nilpotent* if  $\Gamma_r^{\mathbb{Q}}(G) = \{1\}$  for some  $r$ .

- $G$  is *rationally prenilpotent* if  $\Gamma_r^{\mathbb{Q}}(G) = \Gamma_{r+1}^{\mathbb{Q}}(G)$  for some  $r$ . This happens if and only if  $G$  has a maximal torsion-free nilpotent quotient (which is then isomorphic to  $G/\Gamma_r^{\mathbb{Q}}$ ).
- The *rational Dwyer filtration* of  $H_2(G; \mathbb{Q})$

$$H_2(G; \mathbb{Q}) =: \Phi_2^{\mathbb{Q}}(G) \supseteq \Phi_3^{\mathbb{Q}}(G) \supseteq \Phi_4^{\mathbb{Q}}(G) \supseteq \dots$$

is defined by

$$\Phi_r^{\mathbb{Q}}(G) := \Phi_r(G) \otimes \mathbb{Q} = \text{Ker}(H_2(G; \mathbb{Q}) \longrightarrow H_2(G/\Gamma_{r-1}^{\mathbb{Q}}; \mathbb{Q})).$$

- The *Malcev completion* [16]  $G \otimes \mathbb{Q}$  of a nilpotent group  $G$  may be defined inductively through the central extensions determining  $G$  as follows: If  $G$  is abelian then one takes the usual tensor product of abelian groups to define  $G \otimes \mathbb{Q}$ . It comes with a homomorphism  $\epsilon : G \rightarrow G \otimes \mathbb{Q}$  which induces an isomorphism on rational cohomology. Using the fact that the cohomology group  $H^2$  classifies central extensions, one can then define the Malcev completion for a group which is a central extension of an abelian group. It comes again with a map  $\epsilon$  as above. The Serre spectral sequence then shows that  $\epsilon$  induces an isomorphism on rational cohomology. Therefore, one can repeat the last step to define  $(G \otimes \mathbb{Q}, \epsilon)$  for an arbitrary nilpotent group  $G$ .

The map  $\epsilon : G \rightarrow G \otimes \mathbb{Q}$  is universal for maps of  $G$  into uniquely divisible nilpotent groups and it is characterized by the following properties:

1.  $G \otimes \mathbb{Q}$  is a uniquely divisible nilpotent group.
2. The kernel of  $\epsilon$  is the torsion subgroup of  $G$ .
3. For every element  $x \in G \otimes \mathbb{Q}$  there is a number  $n \in \mathbb{N}$  such that  $x^n$  is in the image of  $\epsilon$ .



A version of Stallings' Theorem holds in the rational setting:

**Theorem A.3.** [20] *If  $f : G \rightarrow H$  is a homomorphism of groups inducing an isomorphism on  $H_1(\ ;\mathbb{Q})$  and an epimorphism on  $H_2(\ ;\mathbb{Q})$ , then the induced maps  $(G/\Gamma_r^{\mathbb{Q}}) \otimes \mathbb{Q} \rightarrow (H/\Gamma_r^{\mathbb{Q}}) \otimes \mathbb{Q}$  are isomorphisms for all  $r \geq 1$ .*

A good example to keep in mind when trying to understand this theorem is the inclusion of the integral Heisenberg group  $H_1$  into the rational Heisenberg group  $H_{\mathbb{Q}}$ . This induces an isomorphism on rational homology. Both groups are nilpotent of class 2.

There is also a rational analogue of Dwyer's theorem.

**Theorem A.4.** [5] *If  $f : G \rightarrow H$  induces an isomorphism on  $H_1(\ ;\mathbb{Q})$ , then for  $r \geq 2$  the following three conditions are equivalent:*

1.  *$f$  induces an epimorphism  $H_2(G; \mathbb{Q})/\Phi_r^{\mathbb{Q}}(G) \rightarrow H_2(H; \mathbb{Q})/\Phi_r^{\mathbb{Q}}(H)$ ;*
2.  *$f$  induces an isomorphism  $(G/\Gamma_r) \otimes \mathbb{Q} \rightarrow (H/\Gamma_r) \otimes \mathbb{Q}$ ;*
3.  *$f$  induces an isomorphism  $H_2(G; \mathbb{Q})/\Phi_r^{\mathbb{Q}}(G) \rightarrow H_2(H; \mathbb{Q})/\Phi_r^{\mathbb{Q}}(H)$ , and an injection  $H_2(G; \mathbb{Q})/\Phi_{r+1}^{\mathbb{Q}}(G) \rightarrow H_2(H; \mathbb{Q})/\Phi_{r+1}^{\mathbb{Q}}(H)$ .*

APPENDIX B.

In this appendix, we give a brief review of Chen's method of formal power series connections. Two relevant references are [4] and [10]. There is also an informal discussion of the ideas behind Chen's work in [12].

Fix a field  $\mathbb{F}$  of characteristic zero. Suppose that  $\mathcal{A}^{\bullet}$  is an augmented commutative d.g. algebra over  $\mathbb{F}$ . Suppose in addition that  $\mathcal{A}^{\bullet}$  is positively graded (i.e.,  $\mathcal{A}^k = 0$  when  $k < 0$ ) and that  $H^{\bullet}(\mathcal{A}^{\bullet})$  is connected (i.e.,  $H^0(\mathcal{A}^{\bullet}) = \mathbb{F}$ ). For simplicity, we suppose that each  $H^k(\mathcal{A}^{\bullet})$  is finite dimensional. Typical examples of such  $\mathcal{A}^{\bullet}$  in this theory are the  $\mathbb{F}$  ( $=\mathbb{Q}, \mathbb{R}$  or  $\mathbb{C}$ ) de Rham complex of a connected space with finite rational betti numbers, or the Sullivan minimal model of such an algebra.

Set

$$H_k(\mathcal{A}^{\bullet}) = \text{Hom}_{\mathbb{F}}(H^k(\mathcal{A}^{\bullet}), \mathbb{F}).$$

and

$$\tilde{H}_{\bullet}(\mathcal{A}^{\bullet}) = \sum_{k>0} H_k(\mathcal{A}^{\bullet})$$

Denote by  $\tilde{H}_{\bullet}(\mathcal{A}^{\bullet})[1]$  the graded vector space obtained by taking  $\tilde{H}_{\bullet}(\mathcal{A}^{\bullet})$  and lowering the degree of each of its elements by one.<sup>2</sup>

Denote the free associative algebra generated by the graded vector space  $V$  by  $T(V)$ . Assume that  $V_n$  is non-zero only when  $n \geq 0$ . Denote the ideal generated by  $V_0$  by  $I_0$ . Let  $T(V)^{\wedge}$  denote the  $I_0$ -adic completion

$$\lim_{\leftarrow} T(V)/I_0^m.$$

By defining each element of  $V$  to be primitive, we give  $T(V)^{\wedge}$  the structure of a complete graded Hopf algebra. Recall that the set of primitive elements

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<sup>2</sup>Here we are using the algebraic geometers' notation that if  $V$  is a graded vectorspace, then  $V[n]$  is the graded vector space with  $V[n]_m = V_{m+n}$ .

$PT(V)$  of  $T(V)$  is the free Lie algebra  $\mathbb{L}(V)$ , and the set of primitive elements of  $T(V)^\wedge$  is the closure  $\widehat{\mathbb{L}(V)}$  of  $\mathbb{L}(V)$  in  $T(V)^\wedge$ .

Set  $A_\bullet = T(\widetilde{H}_\bullet(\mathcal{A}^\bullet)^\wedge)$ . Choose a basis  $\{X_i\}$  of  $\widetilde{H}_\bullet(\mathcal{A}^\bullet)$ . Then  $A_\bullet$  is the subalgebra of the non-commutative power series algebra generated by the  $X_i$  consisting of those power series with the property that each of its terms has a fixed degree. For each multi-index  $I = (i_1, \dots, i_s)$ , we set

$$X_I = X_{i_1} X_{i_2} \dots X_{i_s}.$$

If  $X_i \in \widetilde{H}_{p_i}(\mathcal{A}^\bullet)$ , we set  $|X_I| = -s + p_1 + \dots + p_s$ .

A *formal power series connection* on  $\mathcal{A}^\bullet$  is a pair  $(\omega, \partial)$ . The first part

$$\omega = \sum_I w_I X_I$$

is an element of the completed tensor product  $\mathcal{A}^\bullet \widehat{\otimes} PA_\bullet$ , where  $w_I$  is an element of  $\mathcal{A}^\bullet$  of degree  $1 + |X_I|$ . The second is a graded derivation

$$\partial : A_\bullet \rightarrow A_\bullet$$

of degree  $-1$  with square 0 and which satisfies the minimality condition  $\partial(IA_\bullet) \subseteq I^2 A_\bullet$ . Here  $I^k A_\bullet$  denotes the  $k$ th power of the augmentation ideal of  $A_\bullet$ . These are required to satisfy two conditions. The first is the ‘‘integrability condition’’

$$\partial\omega + d\omega + \frac{1}{2}[J\omega, \omega] = 0.$$

Here  $J : \mathcal{A}^\bullet \rightarrow \mathcal{A}^\bullet$  is the linear map  $a \mapsto (-1)^{\deg a} a$ . The value of the operators  $d$ ,  $\partial$  and  $J$  on  $\omega$  are obtained by applying the operators to the appropriate coefficients of  $\omega$ :

$$d\omega = \sum dw_I X_I, \quad \partial\omega = \sum w_I \partial X_I, \quad J\omega = \sum Jw_I X_I.$$

The second is that if

$$(5) \quad \sum_i w_i X_i$$

is the reduction of  $\omega \bmod I^2 A_\bullet$ , then each  $w_i$  is closed and the  $[w_i]$  form a basis of  $H^{>0}(\mathcal{A}^\bullet)$  dual to the basis  $\{X_i\}$  of  $\widetilde{H}_\bullet(\mathcal{A}^\bullet)$ .

Such formal connections always exist in the situation we are describing. To justify the definition, we recall one of Chen’s main theorems. It is the analogue of Sullivan’s main theorem about minimal models.

**Theorem B.1.** *Suppose that  $X$  is a smooth manifold with finite betti numbers,  $x$  a fixed point of  $X$ , and suppose that  $\mathcal{A}^\bullet$  is the  $\mathbb{F}$ -de Rham complex of  $X$ , with the augmentation induced by  $x$ . If  $X$  is simply connected, the connection gives a natural Lie algebra isomorphism*

$$\pi_\bullet(X, x)[1] \otimes \mathbb{F} \cong H_\bullet(PA_\bullet, \partial).$$

If  $X$  is not simply connected, then the connection gives a Lie algebra isomorphism

$$\mathfrak{g}(X, x) \cong H_0(PA_\bullet, \partial)$$

and complete Hopf algebra isomorphisms

$$\mathbb{Q}[\pi_1(X, x)]^\wedge \cong U\widehat{\mathfrak{g}}(X, x) \cong H_0(A_\bullet, \partial).$$

Here  $\mathfrak{g}(X, x)$  denotes the  $\mathbb{F}$  form of the Malcev Lie algebra associated to  $\pi_1(X, x)$  and  $U\widehat{\mathfrak{g}}(X, x)$  is the completion of its enveloping algebra with respect to the powers of its augmentation ideal.  $\blacksquare$

We can apply the bar construction to the augmented algebra  $\mathcal{A}^\bullet$  to obtain a commutative d.g. Hopf algebra  $B(\mathcal{A}^\bullet)$ . (We use the definition in [4].) One can define the *formal transport map* of such a formal connection. It is defined to be the element

$$T = 1 + \sum [w_I] X_I + \sum [w_I|w_J] X_I X_J + \sum [w_I|w_J|w_K] X_I X_J X_K + \cdots$$

of  $B(\mathcal{A}^\bullet) \widehat{\otimes} A_\bullet$ . It induces a linear map

$$\Theta : \text{Hom}_{\mathbb{F}}^{\text{cts}}(A_\bullet, \mathbb{F}) \rightarrow B(\mathcal{A}^\bullet).$$

Here  $\text{Hom}_{\mathbb{F}}^{\text{cts}}(A_\bullet, \mathbb{F})$  denotes the continuous dual

$$\lim_{\rightarrow} \text{Hom}_{\mathbb{F}}(A_\bullet / I^k A_\bullet, \mathbb{F})$$

of  $A_\bullet$ . It is a commutative Hopf algebra. The map  $\Theta$  takes the continuous functional  $\phi$  to the result of applying it to the coefficients of  $T$ :

$$\begin{aligned} \Theta(\phi) = \\ 1 + \sum [w_I] \phi(X_I) + \sum [w_I|w_J] \phi(X_I X_J) + \sum [w_I|w_J|w_K] \phi(X_I X_J X_K) + \cdots \end{aligned}$$

(Note that this is a finite sum as  $\phi$  is continuous.) The properties of the formal connection imply that  $\Theta$  is a d.g. Hopf algebra homomorphism (cf. [10, (6.17)]). The basic result we need is:

**Theorem B.2.** *The map  $\Theta$  induces an isomorphism on homology.*

We conclude by giving a brief sketch of the proof. One can filter  $A_\bullet$  by the powers of its augmentation ideal. This gives a dual filtration of  $\text{Hom}_{\mathbb{F}}^{\text{cts}}(A_\bullet, \mathbb{F})$ . The corresponding spectral sequence has  $E_1$  term

$$E_1^{-s,t} = [\widetilde{H}^\bullet(\mathcal{A}^\bullet)^{\otimes s}]^t.$$

On the other hand, one can filter  $B(\mathcal{A}^\bullet)$  by the “bar filtration” to obtain a spectral sequence, also with this  $E_1$  term. It is easy to check that  $\Phi$  preserves the filtrations and therefore induces a map of spectral sequences. The condition on the  $w_i$  in (5) implies that the map on  $E_1$  is an isomorphism. The result follows.

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