A VERY INFORMAL INTRODUCTION TO WHITNEY TOWERS, PART 2

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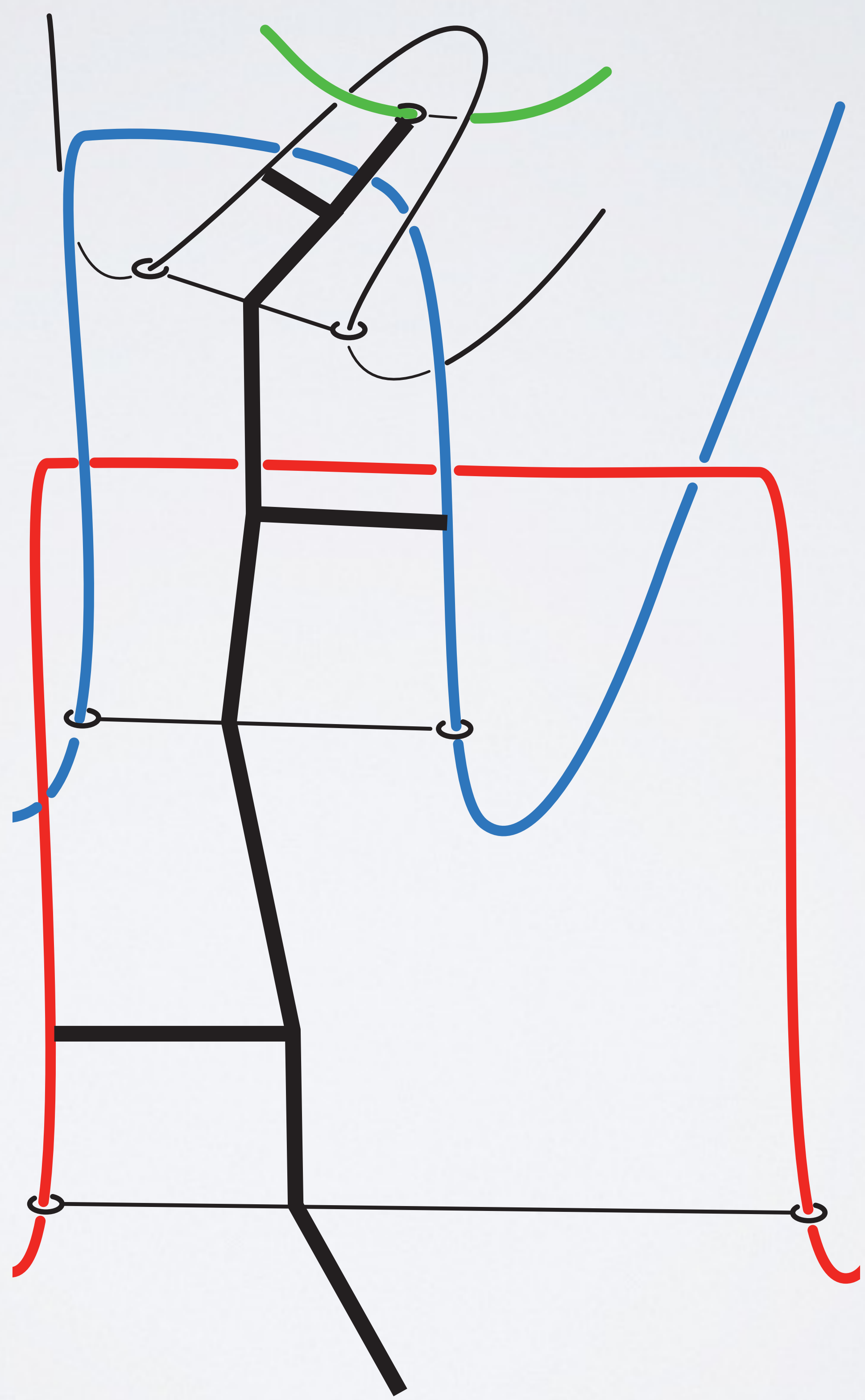
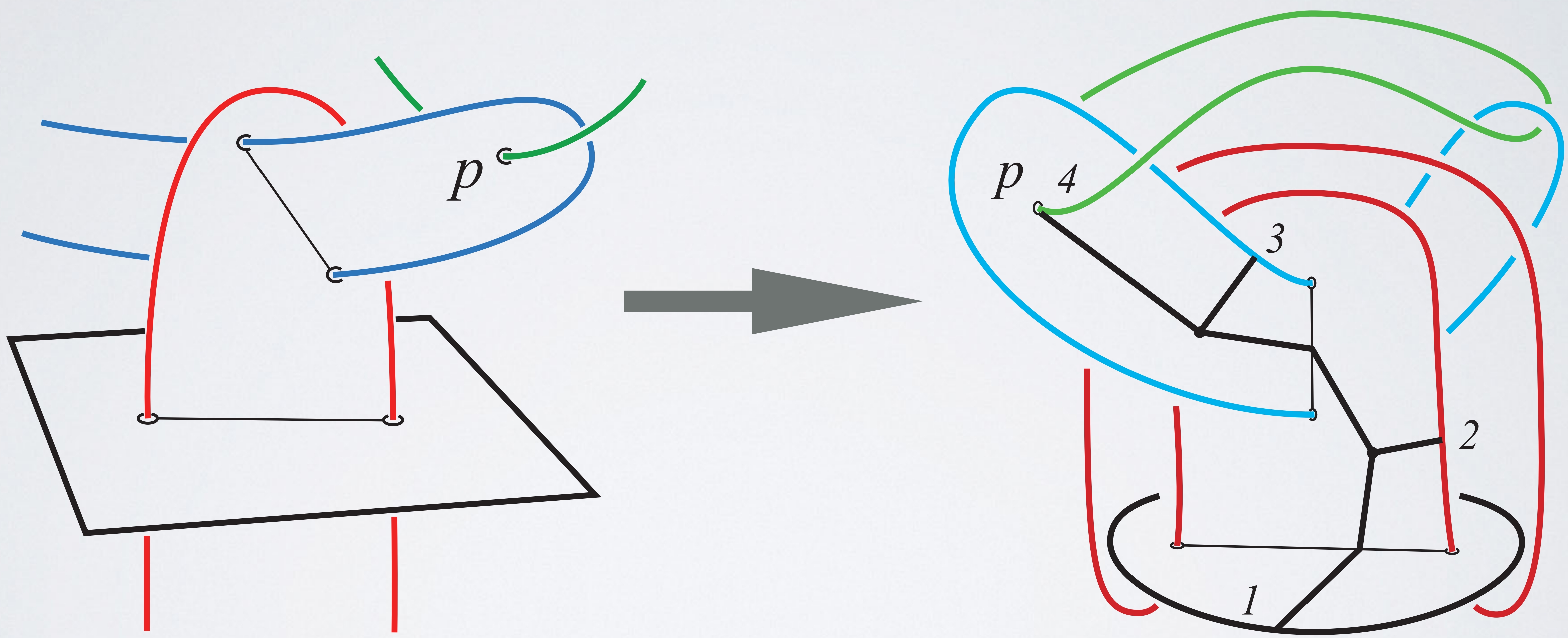
Joint work with Jim Conant and Rob Schneiderman

HIM, September 2016

RECALL SPLIT WHITNEY TOWERS IN THE 4-BALL, THEIR INTERSECTION TREES …



... AND THE LINK ON THE BOUNDARY 3-SPHERE



LEADING TO MASTER DIAGRAM

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\begin{array}{l}\text{Whitney towers of order n}\\ \text{in the 4-ball, up to regular homotopy}\end{array} \arrow[r, two heads, "\partial_n"] \arrow[d, "\text{intersection forest}"', "\tau_n"] & \begin{array}{l}\text{Links in the 3-sphere}\\ \text{that are boundaries of}\\ \text{order n Whitney towers}\end{array} \arrow[d, "\text{associated}"', "\text{graded}"] \\
\begin{array}{l}\mathcal{T}_n = \text{abelian group generated}\\ \text{by trees of order n, up to the}\\ \text{AS- and IHX-relations}\end{array} \arrow[r, dotted, two heads, "R_n"] & \begin{array}{l}\mathbb{W}_n = \text{links that bound order n}\\ \text{Whitney towers, up order n+1}\\ \text{Whitney concordance}\end{array}
\end{tikzcd}$$

geometric

obstruction theory:

If a Whitney tower W of order n has vanishing intersection forest τ_n (W)=0, then it extends to order n+1 (up to Whitney moves).

MASTER DIAGRAM CONTINUED

T_n = abelian group generated by trees of order n, up to the AS- and IHX-relations

W_n = links that bound order n Whitney towers, up to those bounding order n+1.

D'_n = subgroup of abelian group generated by rooted trees of order n, up to the AS- and IHX-relations

D_n = free abelian group of known rank, target of order n total Milnor invariant μ_n

$$\begin{tikzcd}
T_n \arrow[r, "R_n"] \arrow[d, "\eta_n"'] & W_n \arrow[d, "\mu_n"] \\
D'_n \arrow[r, "p_n"] & D_n
\end{tikzcd}$$

discrete Morse theory: $\cong$

η_n = sum over roots

for odd n, kernel is 2-torsion: [x,x]

free quasi Lie algebra: [x,y]=-[y,x]

free Lie algebra: [x,x]=0

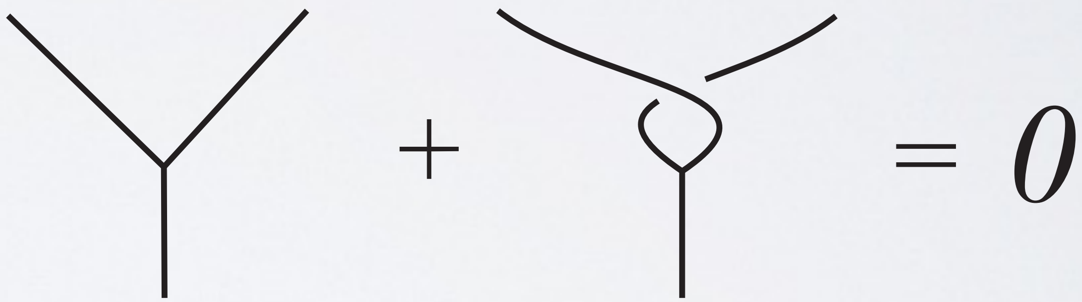
GOALS FOR TODAY

- Explain quasi Lie algebras, the groups D_n and D'_n, and how Milnor invariants arise in this language.
- Show how to read off Milnor invariants from the intersection forest of a Whitney tower.
- Compute $W_n(m)$ = associated graded groups of links.
- Discuss the open problem of higher order Arf invariants.

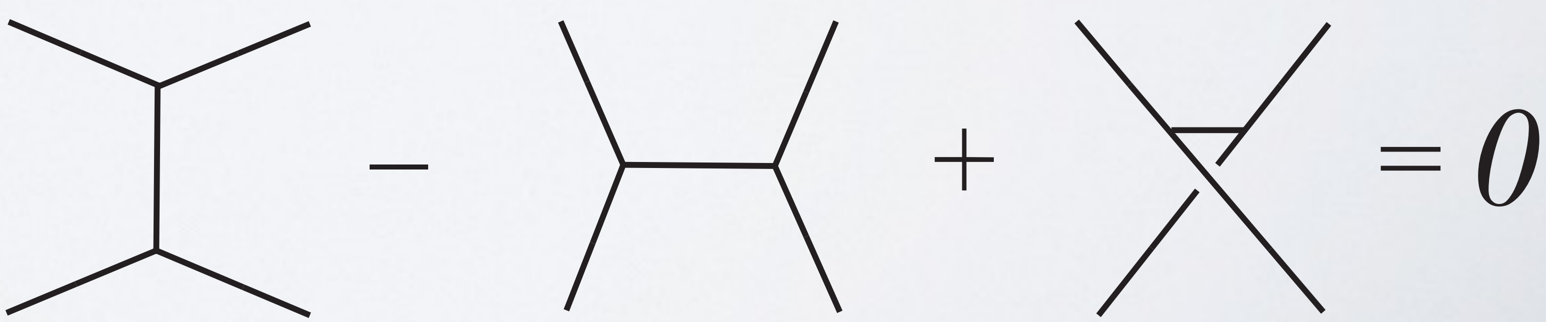
RECALL OUR TREE GROUPS

T(m) is the abelian group generated by oriented trivalent trees, with leaves labelled by {1, 2, ..., m}, modulo the two local relations:

Anti-symmetry:



Jacobi Identity:



RECALL THE FREE LIE ALGEBRA $/\mathbb{Z}$

quasi-

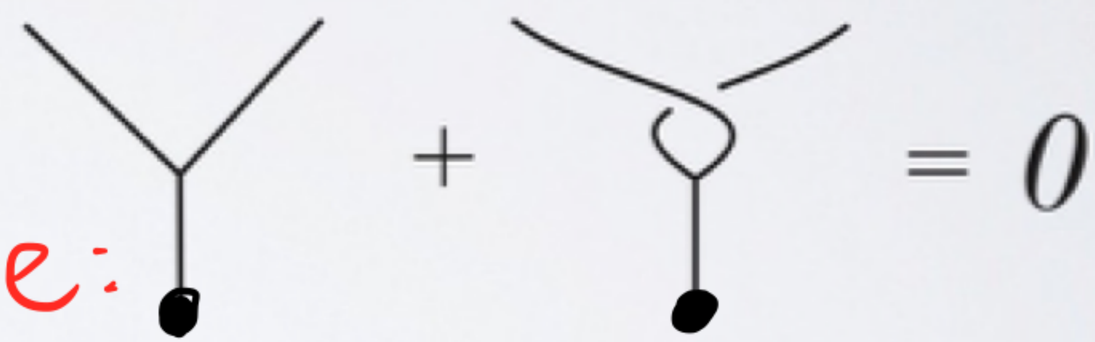
$L'(m)$

L(m) is the abelian group generated by oriented trivalent trees, with leaves labelled by {1, 2, ..., m} and one root, modulo the two local relations:

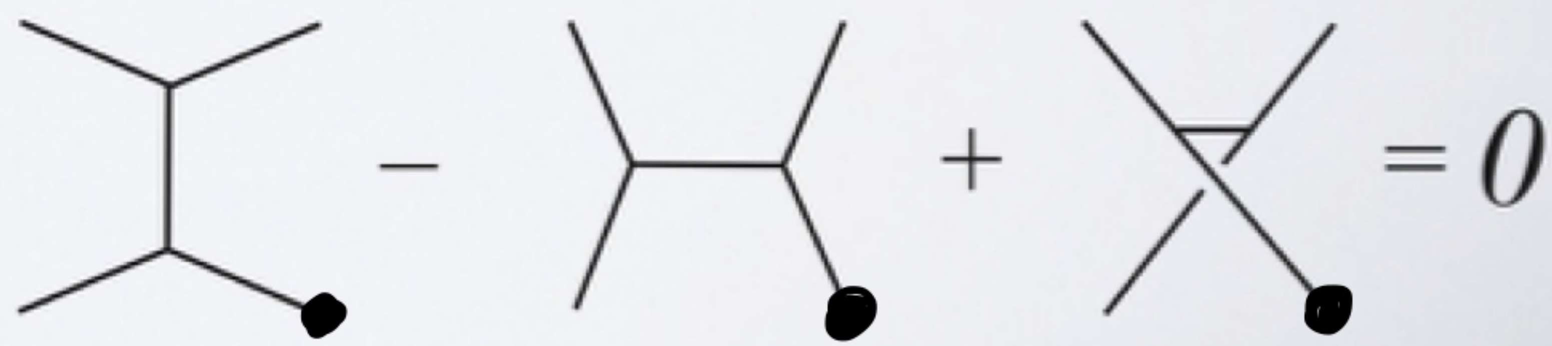
Anti-symmetry:

Lie: $[x,x]=0$

quasi-Lie:



Jacobi Identity:



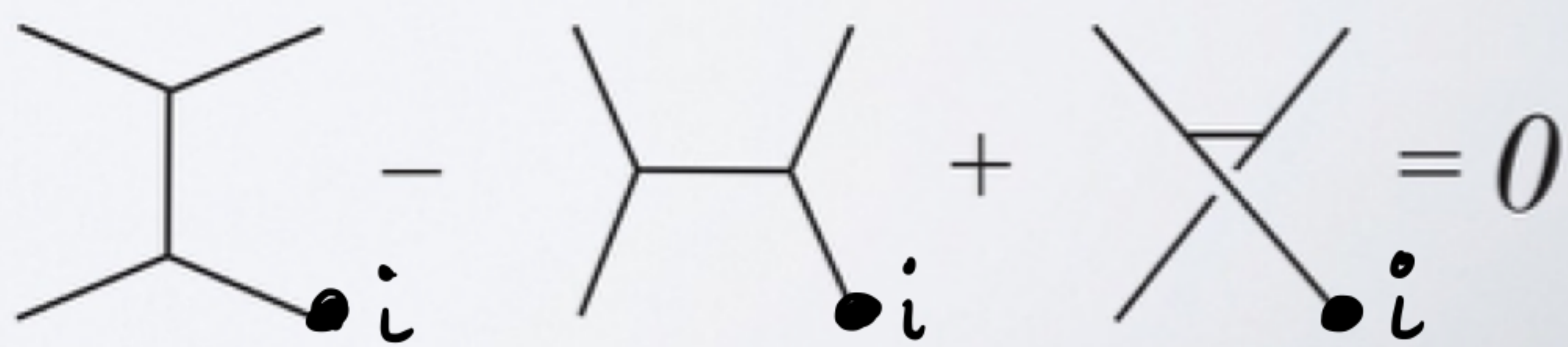
YET ANOTHER DIAGRAMMATIC GROUP

resp. $L'(m)\otimes\mathbb{Z}^m$

$L(m)\otimes\mathbb{Z}^m$ is the abelian group generated by oriented trivalent trees, leaves labelled by {1, 2, ..., m} and one labelled root, modulo the two local relations:

Anti-symmetry:
[x,x]=0, respectively:

Jacobi Identity:



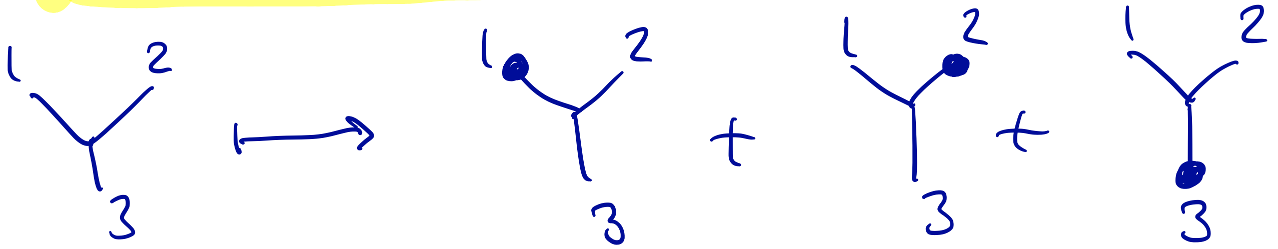
Precise relation between the various tree groups is given by the following

Levine Conjecture [C.S.T in G&T 2014]:

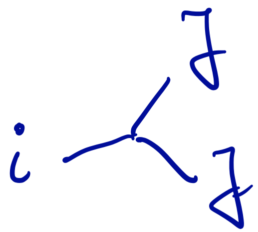
$T_n(m) \xrightarrow[\cong]{\eta} D'_n(m)$ "average over roots"

n = order = # trivalent vertices

$n+1$ = class = # non-root leaves



$$D'_n(m) := \mathrm{Ker}\left(L'_{n+1}(m)\otimes \mathbb{Z}^m \xrightarrow{[\,,\,]} L'_{n+2}(m)\right)$$

Cor.:  span torsion in $T_n^{(m)}$ (it's all 2-torsion)

How to understand $D_n'(m)$ and prove Cor.?

Start with $L_{n+1}'(m)$ or better, $L_{n+1}(m)$:

Easy Fact: $UL(m)$ is the ring freely generated by $x_1, \dots, x_m$, in part. $L(m)$ is a free abelian group and

P-B-W: $m^n = \sum_{d|n}' d \cdot \ell_d(m)$, $\ell_d(m) =$ rank of $L_d(m)$

$\ell_d(m)$	m	$\frac{m^2-m}{2}$	$\frac{m^3-m}{3}$	$\frac{m^4-m^2}{4}$	$\frac{m^5-m}{5}$	$\frac{m^6-m^3-m^2+m}{6}$	....
degree d	1	2	3	4	5	6	

m^n = # words of length n in alphabet $\{x_1, \dots, x_m\}$

A **Lyndon word** is one that's smallest among its cyclic rotations $\iff$ word is non-periodic

$\ell_d(m)$ = # Lyndon words of length d $\Rightarrow$ $m^n = \sum_{d \mid n} d \cdot \ell_d(m)$

Basis for $L_n(m)$ from Lyndon words given by algorithm:

$m = 2$, $n \leq 4$:

a, aaab, aab, aabb,
ab, abb, abbb, b

a, $[a,[a,[a,b]]]$, $[a,[a,b]]$, $[a,[[a,b],b]]$,
$[a,b]$, $[[a,b],b]$, $[[[a,b],b],b]$, b

Stallings' theorem: X a connected space

If $[g_i]$ generate $H_1 X \cong G/G_2$, $G := \pi_1 X$

and $[r_j]$ — " — $H_2 X$, $G_n \in$ l.c.s.

$$G/G_n = (g_i \mid r_j , G_n) \quad \forall n, \text{ e.g.}$$

- $L(F(m)) \cong L(m)$ is the free Lie alg.

where $L(G) := \bigoplus_{n \geq 1} G_n / G_{n+1}$ for any G

- $$\frac{\pi_1(S^3 \setminus \overset{m\text{-comp.}}{\text{link}})}{\text{— " —}\ \ n} \cong \left(\begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{matrix} \;\middle|\; [x_i, l_i], G_n \right) \forall n$$

$$G = \pi_1\left(S^3 \setminus (\ell_1, \dots, \ell_m)\right)$$

$$\uparrow \qquad\qquad \text{meridians } m_i$$

$$\qquad\qquad\qquad\qquad \uparrow$$

$$F = \text{free group on } x_1, \dots, x_m$$

If $\ell_i \in G_{n+1}$ then G_{n+1} / G_{n+2}

$$\sum_{I=(i_0,\dots,i_n)} \mu(I,i)\cdot x_I \longleftarrow \mu_n^i(\text{link}) \qquad \xrightarrow{\cong} G_{n+1}/G_{n+2}$$

$$\mathbb{Z}[F] \hookleftarrow L_{n+1} \cong F_{n+1} / F_{n+2}$$

$$1 + x_i \longleftarrow x_i$$

$$x_1 x_2 - x_2 x_1 \longleftarrow [x_1, x_2] = \ell_3 \quad \text{for Bor}:$$

order $= n$

class $= n+1$

length $= n+2$

$\mu(I,i)$

$\mu(123) = 1$

$\mu(213) = -1$

$$\mathcal{D}_n := \operatorname{Ker}\left(L_{n+1} \otimes \mathbb{Z}^m \xrightarrow{[\,,\,]} L_{n+2}\right)$$

free of rank $\ell_{n+1}(m)\cdot m - \ell_{n+2}(m)$

$\uplus$ cyclic symmetry

$\mu_n(\text{link}) := \sum_{i=1}^{m} \mu_n^i \otimes m_i$ is the total order n Milnor invariant, containing all $\mu(i_0,\dots,i_n,i)$

Master diagram:

$$\begin{tikzcd}
T_n \arrow[r, two heads, "R_n"] \arrow[d, "\eta_n"', "\cong"] \arrow[dr, phantom, "\#"] & W_n \arrow[d, "\mu_n"] \\
\mathcal{D}'_n \arrow[r, "\text{can}"', "\text{iso. mod 2-torsion}"' yshift=-12pt] & \mathcal{D}_n
\end{tikzcd}$$

Thm.: $\boxed{=}$

In part. if $(l_1,\dots,l_m) = \partial WT_n$ then $l_i \in G_{n+1}$ and μ_n is defined!

Jerry Levine: $L(m)\otimes \mathbb{Z}/2 \overset{\text{Square}}{\rightarrowtail} L'(m) \overset{\text{can.}}{\twoheadrightarrow} L(m)$ is exact.

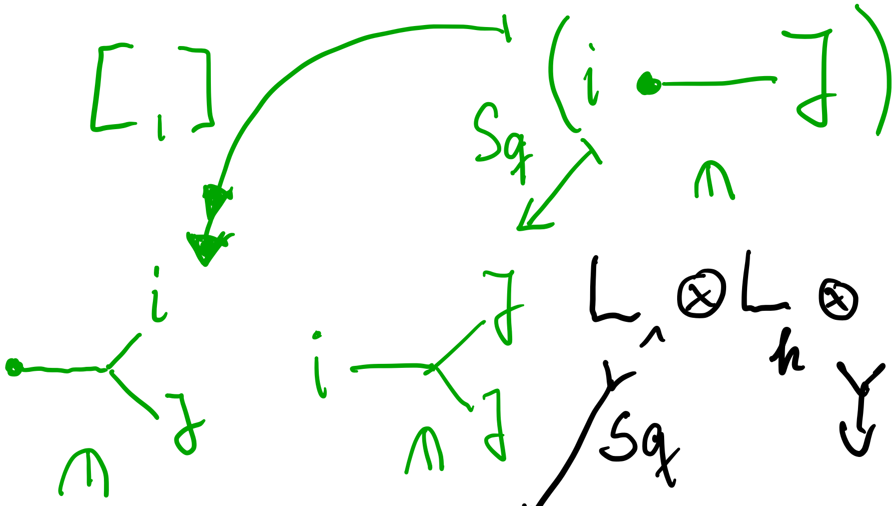
$$
\begin{tikzcd}
 & & & L_{k+1}\otimes \mathbb{Z}/2 \arrow[d, tail, "\text{Square}"] \\
T_{2k} \arrow[r, "\eta_{2k}", "\cong"'] \arrow[d, tail, "R_{2k}"', "\cong"] & D'_{2k} \arrow[r, tail] \arrow[d, "\text{can.}"] & L_1\otimes L'_{2k+1} \arrow[r, two heads, "{[\,,\,]}"] \arrow[d, "\text{can.}"', "\cong"] & L'_{2k+2} \arrow[d, "\text{can.}"] \\
W_{2k} \arrow[r, tail, "\mu_{2k}"] & D_{2k} \arrow[r, tail] \arrow[d, two heads, "sl_{2k}"] & L_1\otimes L_{2k+1} \arrow[r, two heads, "{[\,,\,]}"] & L_{2k+2} \\
 & L_{k+1}\otimes \mathbb{Z}/2 & &
\end{tikzcd}
$$

exact

torsion-free!

In even order $n=2k$, Milnor invariants rule.

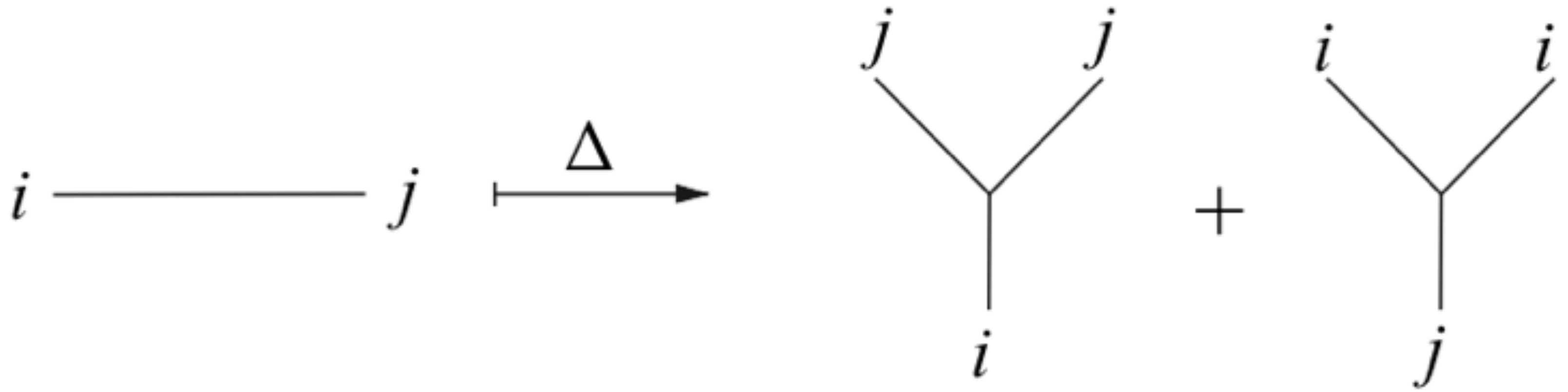
Odd orders contain 2-torsion:



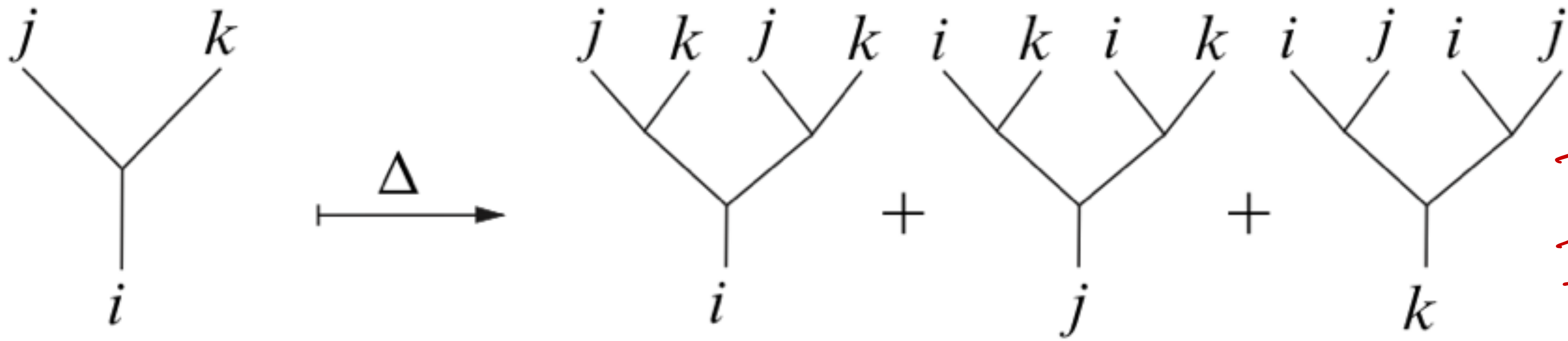
$$
\begin{tikzcd}
 & & L_1 \otimes L_k \otimes \mathbb{Z}/2 \arrow[dl, tail, "Sq"'] \arrow[d, tail] & L_1 \otimes L_k \otimes \mathbb{Z}/2 \arrow[l, phantom, "="] \arrow[d, "Square"] & \\
L_{k+1} \otimes \mathbb{Z}/2 & T_{2k-1} \arrow[r, "\cong", "\eta"'] \arrow[d, two heads, "R"] & D'_{2k-1} \arrow[r, tail] \arrow[d, two heads] & L_1 \otimes L'_{2k} \arrow[r, two heads] \arrow[d, two heads] & L'_{2k+1} \arrow[d, "\cong"] \\
K^{\mu}_{2k-1} \arrow[u, "sl \circ \mu_{2k}"] \arrow[r, tail] & W_{2k-1} \arrow[r, two heads, "\mu_{2k-1}"'] & D_{2k-1} \arrow[r, tail] & L_1 \otimes L_{2k} \arrow[r, two heads] & L_{2k+1}
\end{tikzcd}
$$

Sato-Levine invariants of order $2k-1$, defined on $\mathrm{Ker}(\mu_{2k-1})$.

Use framing relations in odd orders:



two boundary twists



twisted IHX

The map $\Delta_{2n-1} \colon \mathcal{T}_{n-1} \overset{\otimes \mathbb{Z}/2}{\longrightarrow} \mathcal{T}_{2n-1}$ in the cases $n = 1$ and $n = 2$

$n \equiv 3 \ (4)$

exact for even k

$$
\begin{tikzcd}
 & L_1 \otimes L_k \otimes \mathbb{Z}/2 \ / \ \mathrm{im}\,\Delta \arrow[d, tail, "Sq"] \arrow[dr, "\cong"] & \\
T_{k-1} \otimes \mathbb{Z}/2 \arrow[ur, "\eta \otimes \mathbb{Z}/2"] \arrow[r, "\Delta"] \arrow[dr] & T_{2k-1} \ / \ \mathrm{im}\,\Delta \arrow[d, "R", "\cong"'] \arrow[dr, phantom, "\#"] & L_{k+1} \otimes \mathbb{Z}/2 \arrow[l, dashed, tail] \\
 & W_{2k-1} & K^{\mu}_{2k} \arrow[l, tail] \arrow[u, tail, "sl_{2k} \circ \mu|_{2k}"', "\cong"]
\end{tikzcd}
$$

geometric manouvers : 0

Note : $R \circ Sq \neq 0$

PROOF THAT THE MASTER DIAGRAM COMMUTES

(a) *If L bounds an order n split ~~twisted~~ Whitney tower $\mathcal{W}$, then L bounds a dyadic class $n+1$ ~~twisted~~ capped grope G^c such that:*

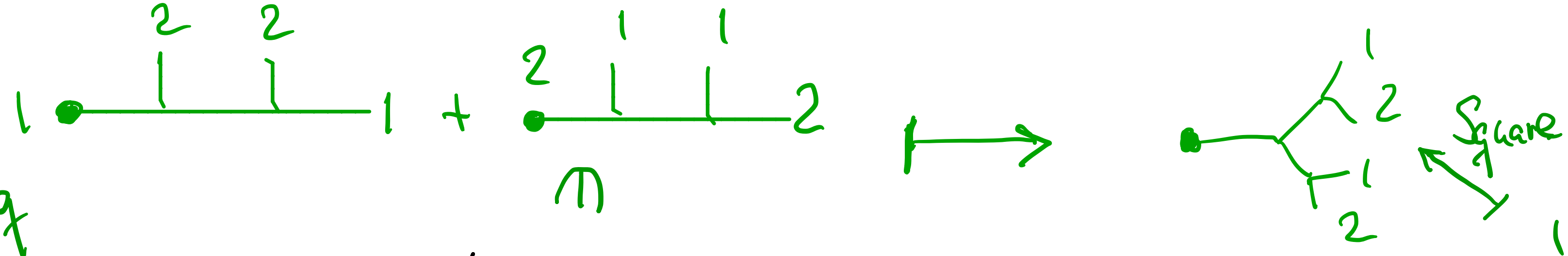
(i) *$t(\mathcal{W})$ is isomorphic to $t(G^c)$;*
(ii) *each framed cap of G^c has intersection $+1$ with a bottom stage of G, except that one framed cap in each dyadic branch of G^c with signed tree $\epsilon_p \cdot t_p$ has intersection ϵ_p with a bottom stage;*

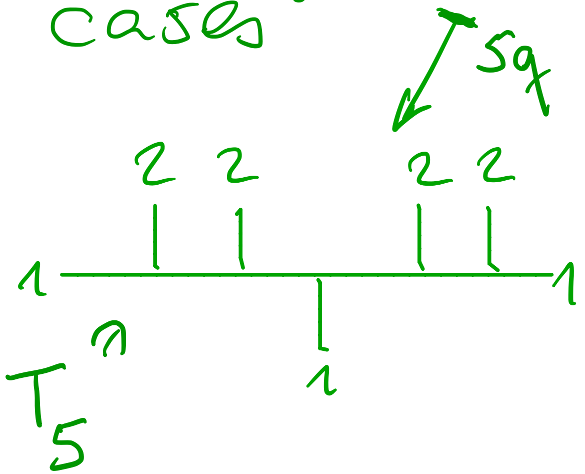
(b) *If $L \subset S^3$ bounds a class $(n+1)$ ~~twisted~~ capped grope $G^c \subset B^4$, then the inclusion $S^3 \setminus L \hookrightarrow B^4 \setminus G^c$ induces an isomorphism*

$$\frac{\pi_1(S^3 \setminus L)}{\pi_1(S^3 \setminus L)_{n+2}} \cong \frac{\pi_1(B^4 \setminus G^c)}{\pi_1(B^4 \setminus G^c)_{n+2}}.$$

We'll show below that this implies that all longitudes of L lie in the (n+1)-st term of the l.c.s. and how they can be computed in the capped grope complement.

Last cases:



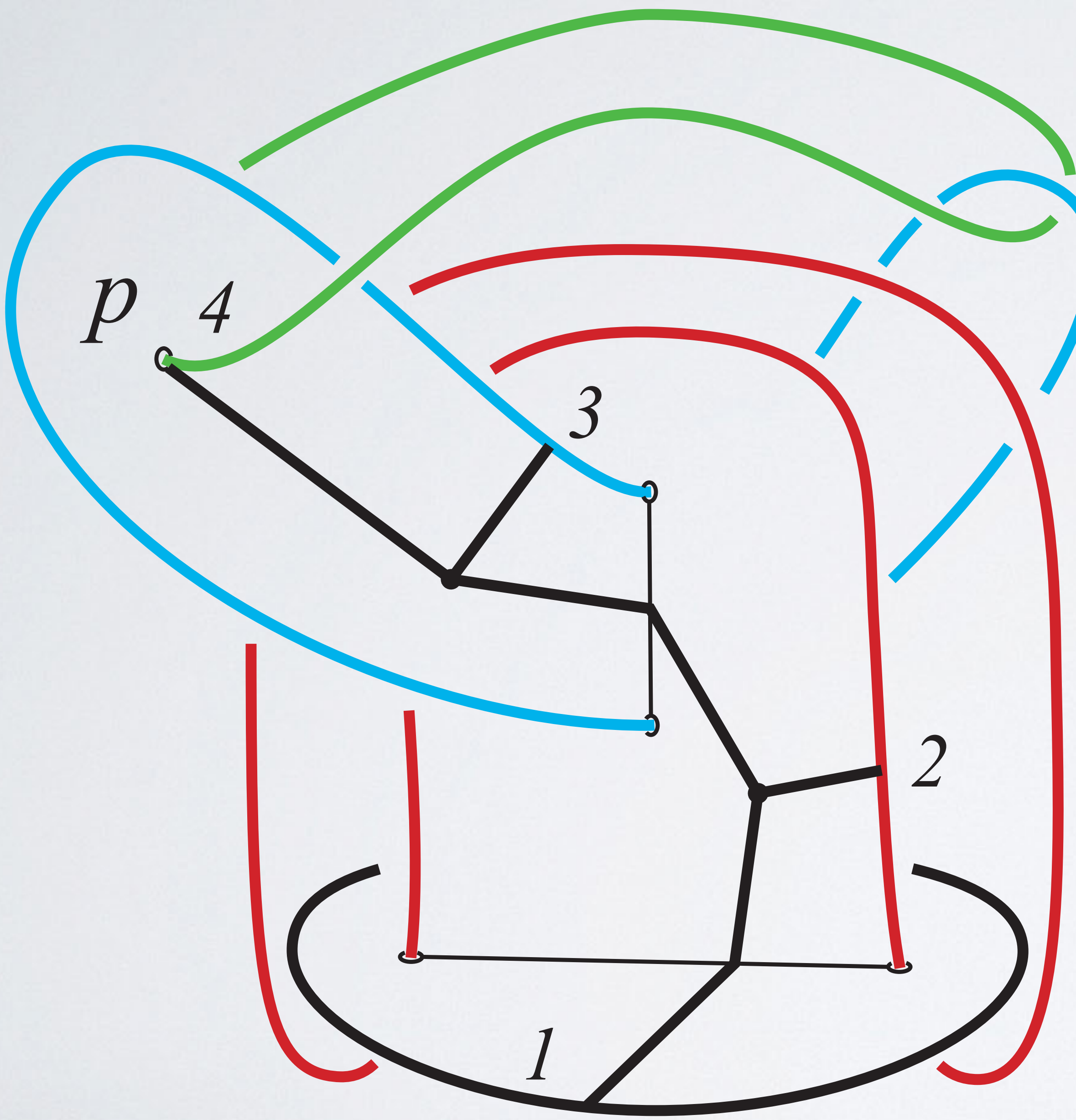


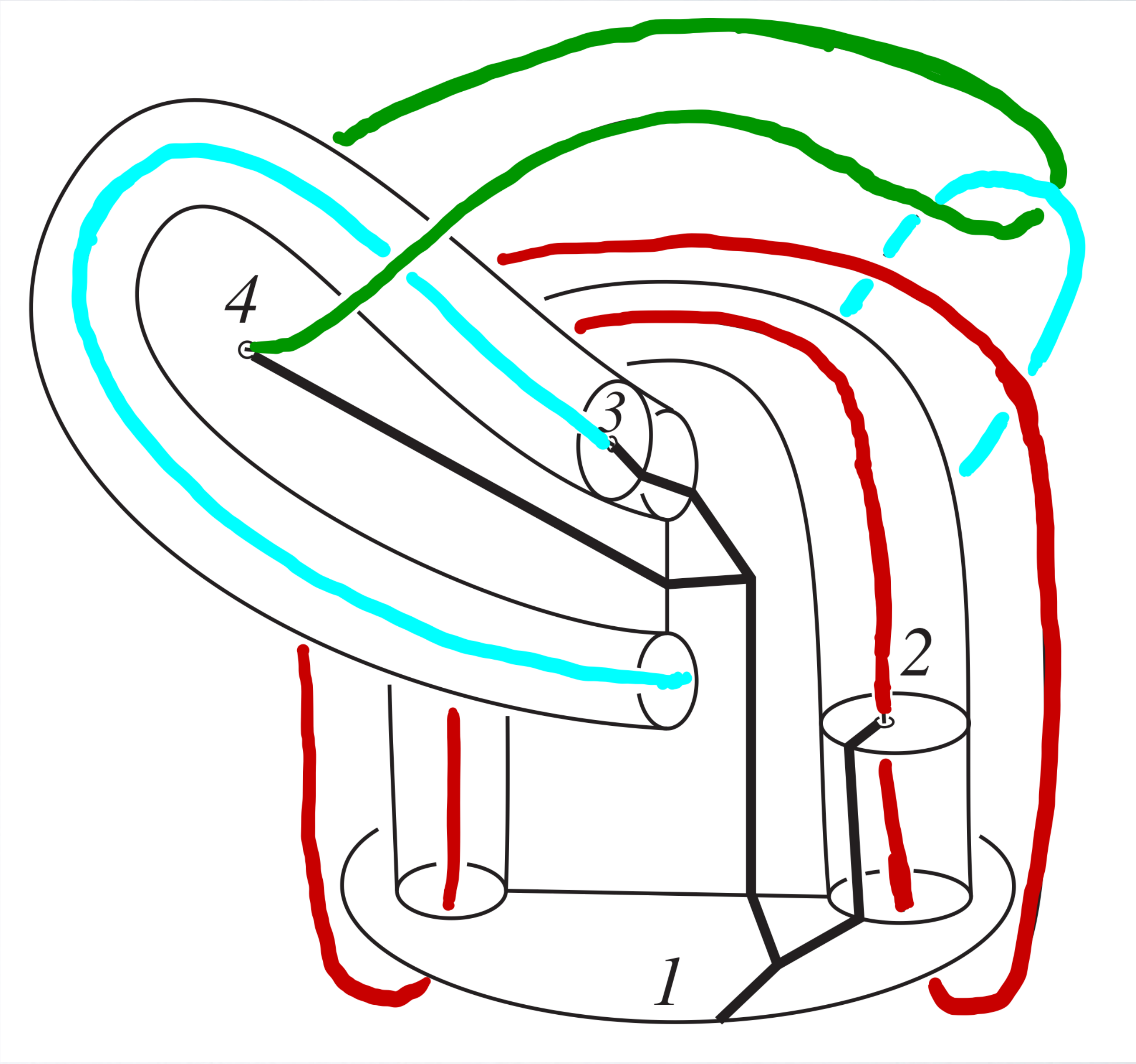
$$
\begin{tikzcd}
 & L_1 \otimes L_{2k-1} \otimes \mathbb{Z}/2 \arrow[d, tail, "Sq"'] \arrow[dr, "\cong", "/\mathrm{im}\,\Delta"'] & L_k \otimes \mathbb{Z}/2 \arrow[d, tail, "\text{Square}"] \arrow[dddd, bend left, "\alpha"] \\
T_{2k-2} \otimes \mathbb{Z}/2 \arrow[ur, "\eta \otimes \mathbb{Z}/2"] \arrow[r, "\Delta"] \arrow[ddr] & T_{4k-3} \arrow[dd, "R"] & L'_{2k} \otimes \mathbb{Z}/2 \arrow[l, dashed, "/\mathrm{im}\,\Delta"'] \arrow[d] \\
 & & L_{2k} \otimes \mathbb{Z}/2 \\
 & W_{4k-3} & K^{\mu}_{4k-3} \arrow[l] \arrow[u, "sl \circ \mu_{4k-2}"'] \\
 & & K^{sl}_{4k-3} \arrow[u]
\end{tikzcd}
$$

geometric manouvers: O

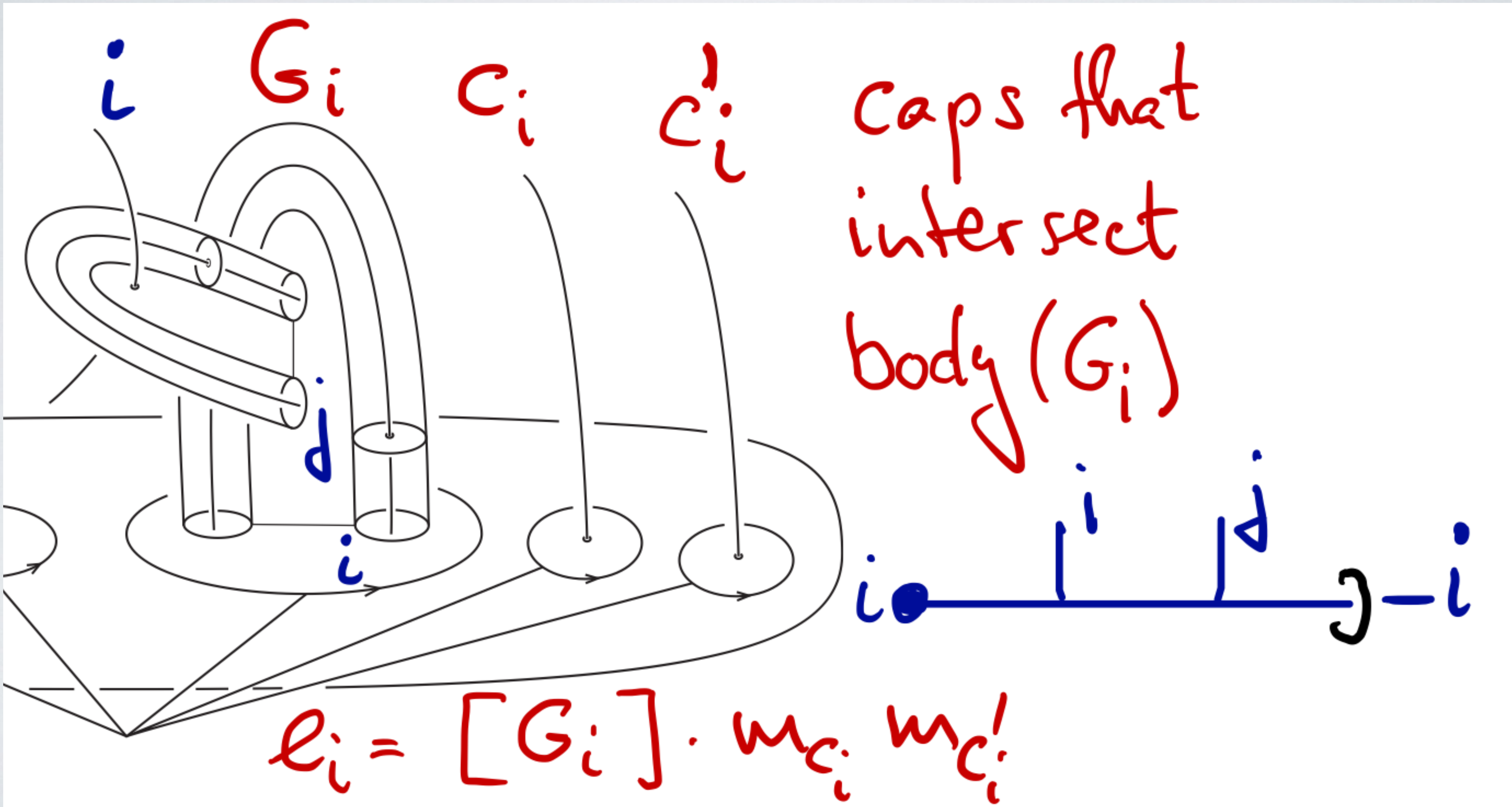
Arf_k : $K^{sl}_{4k-3} \xrightarrow[\alpha^{-1}]{\cong} L_k \otimes \mathbb{Z}/2 \,/\, \mathrm{Ker}\,\alpha \cong$ Conj. 0

WHITNEY TOWERS TO CAPPED GROPES

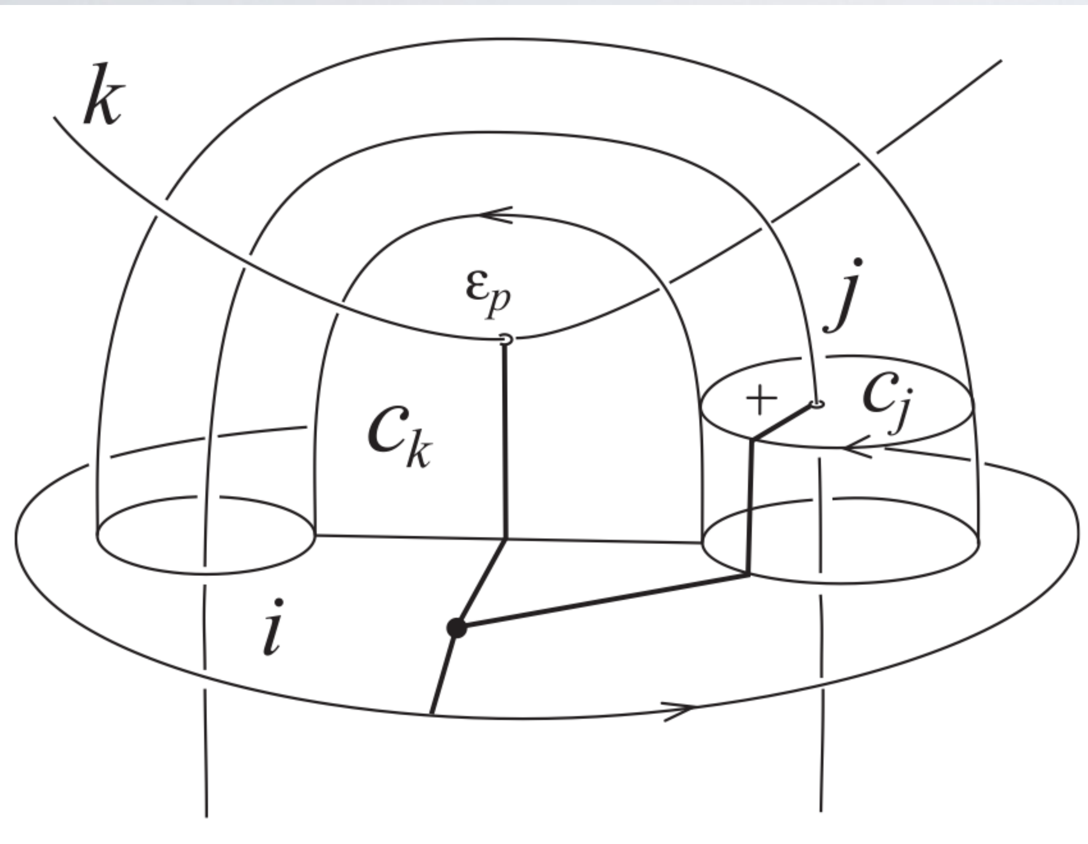


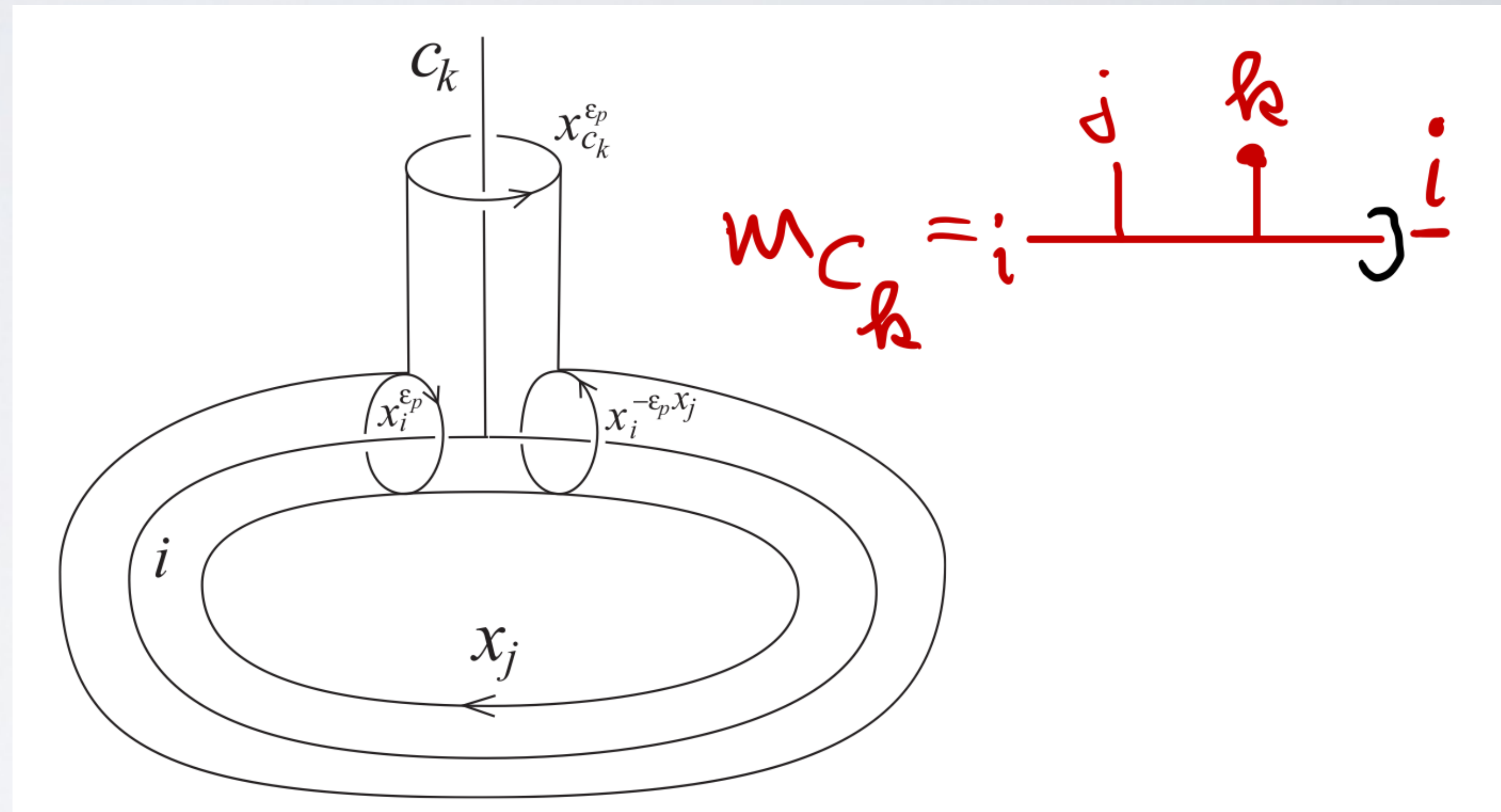


READING OFF THE LONGITUDES



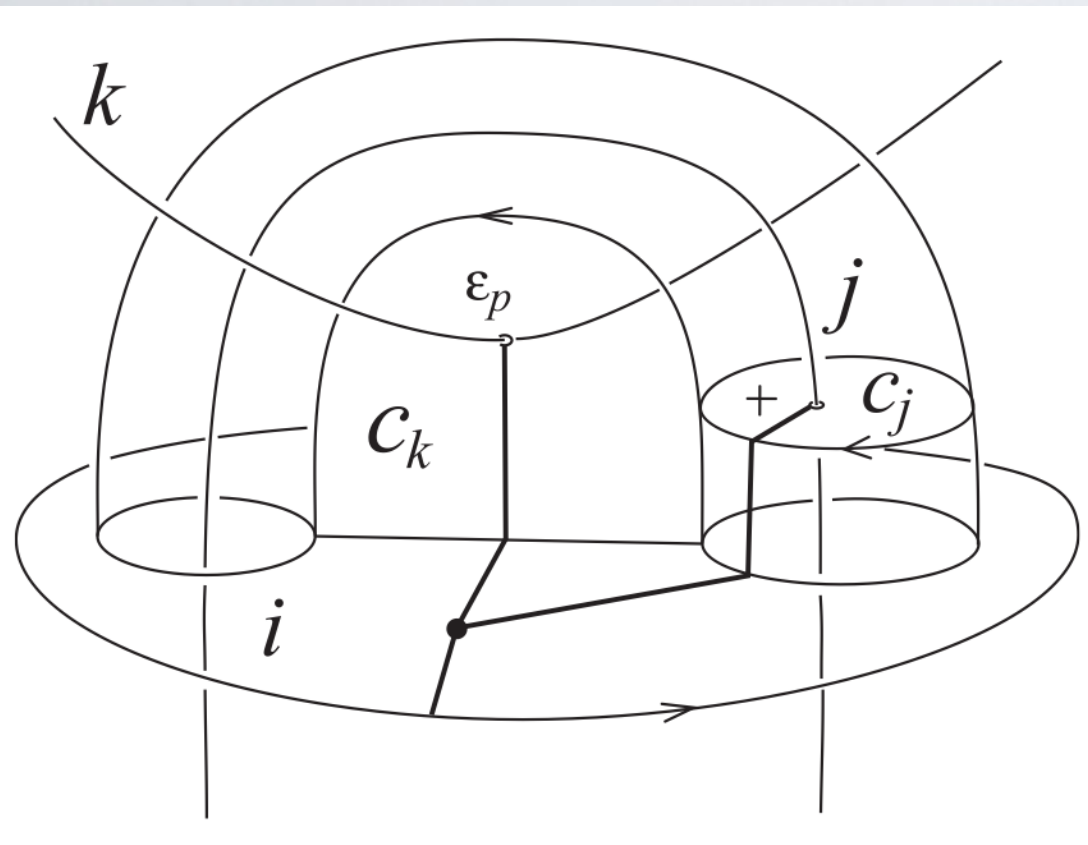
GROPE DUALITY OR SHAKING THE TREE:

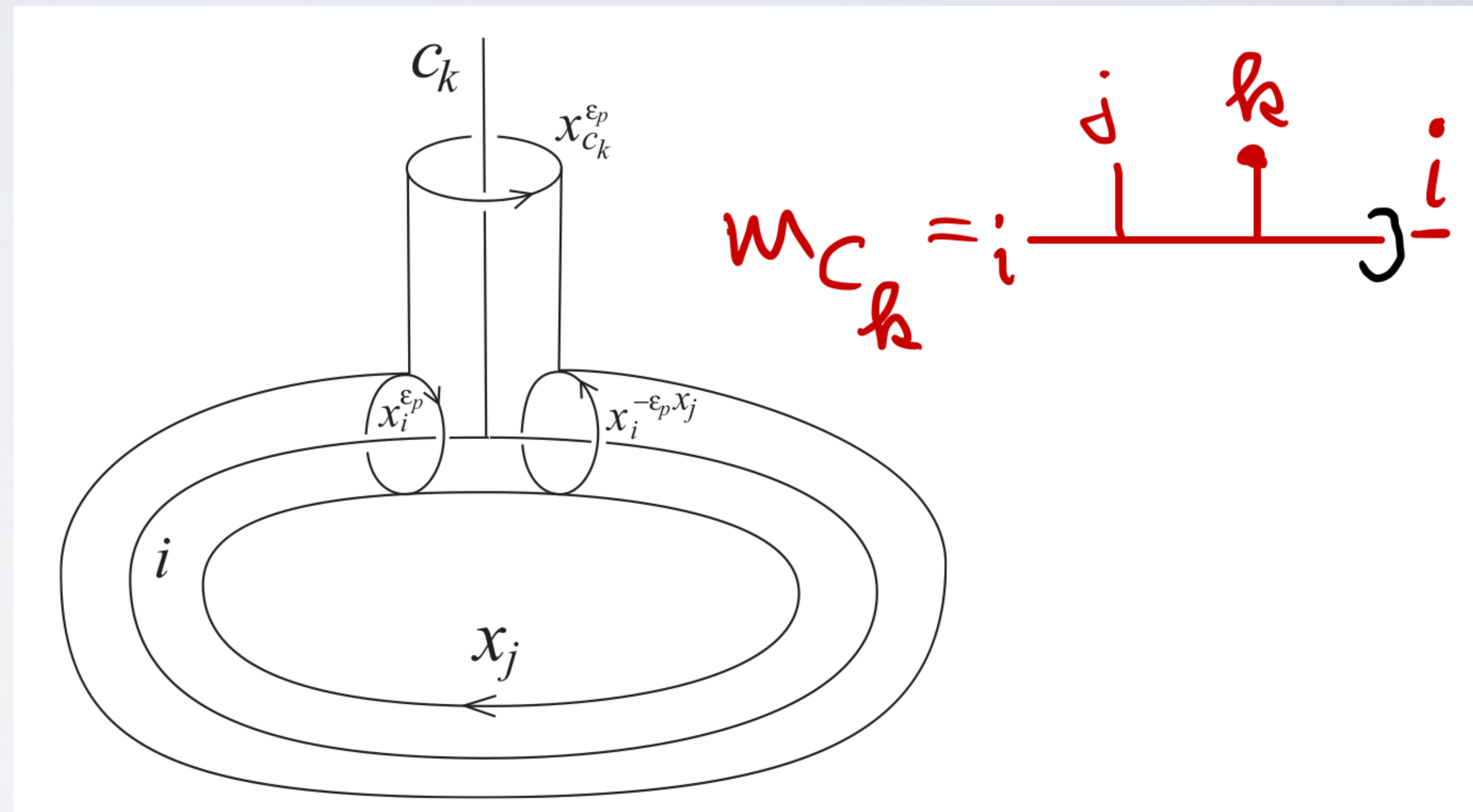




It follows that the i-th longitude is given by the summand of $\eta(L)$ that corresponds to the leaves labelled by i. QED

GROPE DUALITY OR SHAKING THE TREE:





It follows that the i-th longitude is given by the summand of $\eta(L)$ that corresponds to the leaves labelled by i. QED