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4-Manifold topology II: Dwyer's filtration and surgery kernels

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Abstract. Even when the fundamental group is intractable (i.e. not "good") many interesting 4-dimensional surgery problems have topological solutions. We unify and extend the known examples and show how they compare to the (presumed) counterexamples by reference to Dwyer's filtration on second homology. The development brings together many basic results on the nilpotent theory of links. As a special case, a class of links only slightly smaller than "homotopically trivial links" is shown to have (free) slices on their Whitehead doubles.

Introduction

In dimension four, the basic machinery of manifold theory, surgery and (5-dimensional) s-cobordism theorems, exist in the topological category when the fundamental group π is "good" [FT] and is expected to fail for π free (and nonabelian) and in fact to fail for the "random" group. Nevertheless, even when π is arbitrary many special surgery problems can profitably be solved. The theorem [F2] that the Whitehead double of any boundary link is (freely) slice is an example. These applications all involve some representation of the surgery kernel by a submanifold M whose inclusion $M \subset N$ into the source of the surgery problem is π_1-null. Whereas all previous applications ([F2, F3, FQ] Chapter 6) required the second homology of M to be spherical, we find here (see Theorem 1.1 and Corollary 1.2) that the important condition is only that $H_2(M) = \phi_\omega(M)$, i.e. that the second homology lies in the ω-term of the Dwyer [D] filtration as discussed in Sect. 2. This is an important philosophical point since for any $n > 1$, the "canonical" (or "atomic" compare [CF]) surgery

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problems – to which all others restrict – can be chosen so that the kernel is carried by a π_1-null submanifold M with $H_2(M) = \phi_n(M)$. As elsewhere in this subject, taking the limit is the essential problem.

On the way to the main theorem we develop in Sect. 2 the nilpotent theory of links and their (immersed) slices in a compact contractible 4-manifold, using only group theoretic methods (and not Massey products). This unified perspective contains many previous results (e.g. from [M1, M2, D, T, K, L or C]) but uses only the largest possible indeterminancy for the invariants.

A special case of our method shows in Sect. 3 that a class of links, larger than "boundary-links" and slightly smaller than "homotopically trivial links" have (free) slices bounding their Whitehead doubles (Theorem 3.1). This generalizes the main results of both [F2] and [F3] .

1. New surgery theorems

We describe a naive (map-less) surgery theorem and then its corollary in the formal setting of normal maps.

Let N be a compact connected topological 4-manifold, possibly with boundary. Let $M \subset \overset{\circ}{N}$ be a connected codimension 0 submanifold with connected boundary. Assume that M is π_1-null in N, i.e. the inclusion induces the zero map $\pi_1(M) \to \pi_1(N)$. Assume that $H_1(\partial M) \cong H_1(M)$. Then elementary calculations (see Sect. 3) show that $H_2(M)$ is free. This says roughly that homologically M resembles a thickening of a 2-complex. Note that the triviality of $\pi_1(M) \to \pi_1(N)$ implies a natural factorisation $H_2(M) \to \pi_2(N) \to H_2(N)$ which we may use to define $N^+ := N \cup_\beta (3\text{-cells})$ where the attachment is to the image in $\pi_2(N)$ of a free basis β for $H_2(M)$. If β_1 and β_2 are two free bases they differ by a nonsingular integral linear transformation. Since any such transformation is a product of elementary matrices there exists a "deformation" of the 3-cells realizing a (simple) homotopy equivalence $N^+_{\beta_1} \simeq N^+_{\beta_2}$. Thus N^+ is well defined. It has the same 2-skeleton as N and satisfies

$$H_2(N^+; \mathbb{Z}[\pi_1]) \oplus H_2(M) \otimes_{\mathbb{Z}} \mathbb{Z}[\pi_1] \cong H_2(N; \mathbb{Z}[\pi_1]) .$$

The nonsingularity of the intersection form on M (see Sect. 3) makes N^+ a Poincaré space, but since Theorem 1.1 puts a manifold structure directly on N^+, we will not offer a separate proof for this fact. With this notation, we state a naive surgery theorem for producing a manifold with the simple homotopy type of N^+. Sect. 2 treats Dwyer's [D] filtration of $H_2, \pi_2 \subset \phi_\omega \subset \cdots \phi_k \subset \phi_{k-1} \cdots \subset \phi_2 = H_2$, appearing in the statement below.

Theorem 1.1. *If the second homology of $M \subset N$ is in the ω-term of Dwyer's filtration, $\phi_\omega(M) = H_2(M)$ then there exists a 4-manifold N', with $\partial N' = \partial N = \partial N^+$ and a (simple) homotopy equivalence* (rel ∂) $(N', \partial N') \xrightarrow{h} (N^+, \partial N^+)$, *i.e. a manifold structure* (rel ∂) *on N^+.*

Now consider the formal setting of surgery. Suppose that $N \xrightarrow{f} X$ is a degree 1 normal map from a topological manifold to a Poincaré space X. There is the classical surgery obstruction $\theta \in L_4^{(s)}(\pi_1 X)$ to constructing a normal bordism to a (simple) homotopy equivalence $N' \xrightarrow{f'} X$. (We suppose here that if $\partial N \neq \phi$ then $f|_{\partial N} : \partial N \to \partial X$ is already an equivalence and then the normal bordism mentioned above is required to be relative to the boundary.) It is always possible to normally bord f to a π_1-isomorphism with

$$K := \ker(H_2(N; \mathbb{Z}[\pi_1 X]) \to H_2(X; \mathbb{Z}[\pi_1 X]))$$

a free $\mathbb{Z}[\pi_1 X]$-module so we assume that this has been done. By definition, the surgery obstruction θ vanishes if there is a (preferred) basis for the kernel K in which the intersection form is hyperbolic. We say that $M \subset N$ represents the surgery kernel if $H_2(M) \otimes_{\mathbb{Z}} \mathbb{Z}[\pi_1 X] \to H_2(N; \mathbb{Z}[\pi_1 X])$ maps isomorphically to K. We have:

Corollary 1.2. *Given $M \subset N$ representing a standard surgery kernel and satisfying the hypotheses above: π_1-null, $H_1(M) \cong H_1(\partial M)$, and $\phi_\omega(M) = H_2(M)$ then there is a normal bordism from $f : N \to X$ to a (simple) homotopy equivalence $f' : N' \to X$.*

The essential improvement over Chapter 6 of [FQ] is that to be useful for surgery a π_1-null 2-complex K representing the surgery kernel (a neighborhood of K corresponds to M above) need not be spherical but only satisfy $\phi_\omega(K) = H_2(K)$. We now explain this condition in detail.

2. Group homology and the lower central series

The *lower central series* of a group G is defined by $G^1 := G, G^{k+1} := [G, G^k]$ for $k \geqq 1$ and may be extended to all ordinals by defining $G^\alpha := \bigcap_{\beta<\alpha} G^\beta$ for α a limit ordinal. We will be primarily interested in the cases k finite and the first limit ordinal ω. There is an equivalent geometric formulation in terms of maps of *gropes*.

Definition. *A* **grope** *is a special pair (2-complex, circle). A grope has a class $k = 1, 2, \ldots, \infty$. For $k = 1$ a grope is defined to be the pair (circle, circle). For $k = 2$ a grope is precisely a compact oriented surface Σ with a single boundary component. For k finite a k-grope is defined inductively as follow: Let $\{\alpha_i, \beta_i, i = 1, \ldots, \text{genus}\}$ be a standard symplectic basis of circles for Σ. For any positive integers p_i, q_i with $p_i + q_i \geqq k$ and $p_{i_0} + q_{i_0} = k$ for at least one index i_0, a k-grope is formed by gluing p_i-gropes to each α_i and q_i-gropes to each β_i. Finally, an ∞-grope is a nested union of (k-gropes, fixed circle) for all $k > 1$.*

The important information about the "branching" of a grope can be very well captured in a rooted tree as follows: For $k = 1$ this tree consists of a single vertex v_0 which is the root. For $k = 2$ one adds $2 \cdot \text{genus}(\Sigma)$ edges to

v_0 and may label the new vertices by α_i, β_i. Inductively, one gets the tree for a k-grope which is obtained by attaching p_i-gropes to α_i and q_i-gropes to β_i by identifying the roots of the p_i-(resp. q_i-)gropes with the vertices labeled by α_i(resp. β_i). The below figure should explain the correspondence between gropes and trees.

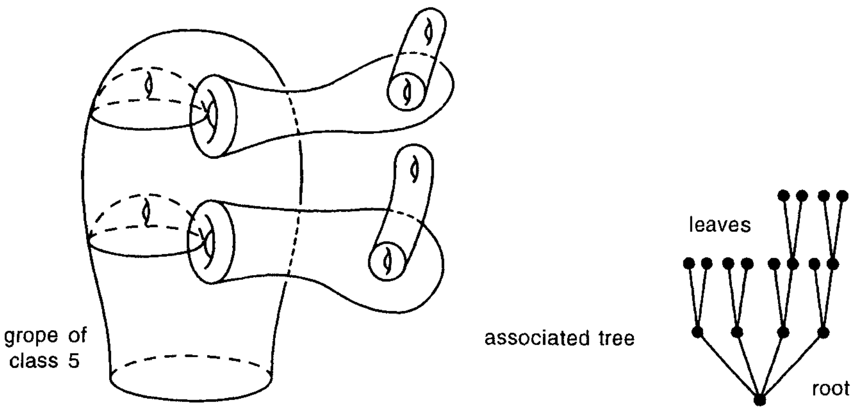

Fig. 2.1.

Note that the vertices of the tree which are above the root v_0 come in pairs corresponding to the symplectic pairs of circles in a surface stage and that such rooted paired trees correspond bijectively to gropes. Under this bijection, the *leaves* (:= non-root 1-valent vertices) of the tree correspond to circles on the grope which freely generate its fundamental group. We will sometimes refer to these circles as the *tips* of the grope. The boundary of the first stage surface Σ will be referred to as the *bottom* of the grope.

Lemma 2.1. *For a space X, a loop $\gamma \in \pi_1(X)^k$, $1 \leqq k < \omega$, if and only if γ bounds a map of some k-grope. Moreover, the class of a grope (G, γ) is the maximal k such that $\gamma \in \pi_1(G)^k$.*

Proof. The first statement is proven by an induction on k, starting with the fact that the boundary circle γ of a compact oriented surface Σ with standard symplectic basis $\{\alpha_i, \beta_i\}$ is the product of commutators $\gamma = \prod[\alpha_i, \beta_i]$. Note that the "if-direction" is harder and uses the non-obvious re-bracketing fact $[G^i, G^j] \subseteq G^{i+j}$, see e.g. [V, p. 27]

For the second statement we only have to show that the boundary circle γ of a k-grope (G, γ) does not lie in $\pi_1(G)^{k+1}$: Again this is best proven by an induction on k, starting with the fact that $\pi_1(\Sigma)$ is freely generated by all α_i and β_i. The Magnus expansion shows that $\gamma = \prod[\alpha_i, \beta_i]$ does not lie in $\pi_1(\Sigma)^3$. Similarly, for $k > 2$, $\pi_1(G)$ is freely generated by those circles in a standard symplectic basis of a surface stage in G to which nothing else is attached. Now assume that the k-grope (G, γ) is obtained by attaching p_i-gropes G_{α_i} to α_i and q_i-gropes G_{β_i} to β_i, $p_i + q_i \geqq k$. By induction, $\alpha_i \notin \pi_1(G_{\alpha_i})^{p_i+1}$

and $\beta_i \notin \pi_1(G_{\beta_i})^{q_i+1}$ since $p_i, q_i \geqq 1$. But the free generators of $\pi_1(G_{\alpha_i})$ and $\pi_1(G_{\beta_i})$ are contained in the set of free generators of $\pi_1(G)$ and therefore $\gamma = \prod[\alpha_i, \beta_i] \notin \pi_1(G)^{k+1}$. Again, this may be easiest seen by applying the Magnus expansion to $\pi_1(G)$, compare [MKS, Chapter 5]. □

Remark. Given $\gamma \in \pi_1(X)^k$, one may write it as a product of "right normed" commutators of the form $[x_1, [x_2, [\ldots, x_k], \ldots]]$. Such a commutator bounds a map of a very special k-grope, namely one whose rooted tree looks like

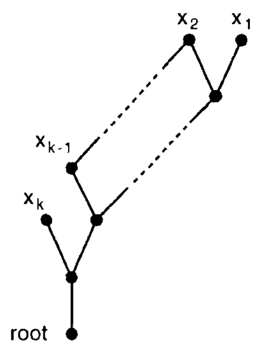

Fig. 2.2.

This implies that γ bounds a map of a 1/2-*grope* of order k (which is inductively defined to be obtained from a surface Σ by attaching 1/2-gropes of order $(k-1)$ to a 1/2-symplectic basis $\{\alpha_i\}$ of Σ). This gives a particularly symmetric class of gropes.

For each group G there is a least ordinal α such that $G^\alpha = G^\beta$ for all $\beta > \alpha$. Call this stable stage $G^{\max}$.

Lemma 2.2. $\gamma \in \pi_1(X)^{\max}$ *if and only if* γ *bounds a map of an* ∞-*grope in* X.

Proof. Follows from the definitions. □

Note. There are finitely presented groups with $G^{\max}$ strictly smaller than G^ω. Geometrically, lying in G^ω is equivalent to bounding maps of possibly unrelated k-gropes for each $k < \omega$.

The lower central series is connected to homology and thus the rest of topology by Stallings' theorem and Dwyer's extension. Both of these theorems are formal derivations of the *5-term exact sequence* for groups: Given a short exact sequence

$$1 \to N \to G \to Q \to 1$$

of groups, the bottom part of the corresponding Lerray-Serre spectral sequence is an exact sequence of homology groups (with integral coefficients)

$$H_2(G) \to H_2(Q) \to N/[N,G] \to H_1(G) \to H_1(Q) \to 0 .$$

Stallings' Theorem [St]. *If $\sigma \to \pi$ is a homomorphism of groups inducing an isomorphism on H_1 and an epimorphism on H_2 then the induced maps $\sigma/\sigma^k \to \pi/\pi^k$ are isomorphisms for all $1 \leqq k < \omega$. If $\sigma \to \pi$ is an epimorphism then $\sigma/\sigma^\omega \to \pi/\pi^\omega$ is also an isomorphism.*

For $k \geqq 2$, let $\phi_k(G)$ denote the kernel of $H_2(G) \to H_2(G/G^{k-1})$.

Dwyer's Theorem [D]. *Let $\sigma \to \pi$ induce an isomorphism on H_1. Then for $2 \leqq k < \omega$ the following three conditions are equivalent:*

(1) *f induces an epimorphism $H_2(\sigma)/\phi_k(\sigma) \to H_2(\pi)/\phi_k(\pi)$.*
(2) *f induces an isomorphism $\sigma/\sigma^k \to \pi/\pi^k$.*
(3) *f induces an isomorphism $H_2(\sigma)/\phi_k(\sigma) \to H_2(\pi)/\phi_k(\pi)$ and an injection $H_2(\sigma)/\phi_{k+1}(\sigma) \to H_2(\pi)/\phi_{k+1}(\pi)$.*

The previous two theorems apply directly to spaces (attaching cells of dimension $\geqq 3$ to make spaces into $K(\pi, 1)$'s does not affect these low dimensional statements) and we shall freely apply them in that context. For example, $\phi_k(X)$ is defined to be the kernel of the composition

$$H_2(X) \to H_2(K(\pi_1 X, 1)) = H_2(\pi_1(X)) \to H_2(\pi_1(X)/\pi_1(X)^{k-1}) .$$

Then Dwyer's functor ϕ_k has a more geometric interpretation:

Definition. *A* **closed k-grope** *is a 2-complex made by replacing a 2-cell in S^2 with a k-grope.*

Lemma 2.3. *Dwyer's subspace $\phi_k(X)$ of $H_2(X)$ is precisely the subset of homology classes represented by maps of closed k-gropes into X.*

Proof. Let $\pi := \pi_1(X)$. We first observe that a homology class which is represented by a map of a closed k-grope is also represented by a map of a closed 1/2-grope of order k. It is enough to show this for X a closed k-grope and the homology class the generator of $H_2(X) \cong \mathbb{Z}$. Cut out a 2-cell from the bottom stage of X. Then the boundary γ of this 2-cell lies in $\pi_1(X \setminus \text{2-cell})^k$. Now use the remark after Lemma 2.1 to get a map of a 1/2-grope of order k bounding γ and reglue the 2-cell to it. To see that a homology class of a space X represented by a closed 1/2-grope of order k lies in $\phi_k(X)$, take a representative bottom surface Σ of the 1/2-grope of order k. The 1/2-symplectic basis of curves of Σ to which the next grope-stages attach are mapped to π^{k-1} and so are trivial under the inclusion $K(\pi, 1) \subset K(\pi/\pi^{k-1}, 1)$. This means that the homology class of Σ becomes spherical and hence trivial in $K(\pi/\pi^{k-1}, 1)$. Now consider a surface Σ' mapped into X which is null homologous in $K(\pi/\pi^{k-1}, 1)$. Let the null homology be represented by a map F of a 3-manifold W. Think of $K(\pi/\pi^{k-1}, 1)$ as $X \cup$2-cells $\cup$ 3-cells $\cup \ldots$ or by thickening as $X \cup$2-handles

$\cup$ 3-handles $\cup \ldots$. Make F transverse to the ascending (singular) manifold A of the attached handlebody. The compact 3-manifold $W \setminus f^{-1}(F^{-1}(A))$ is a bordism in X between Σ' and another map of a surface $f : \Sigma \to X$ which, by inspection, has the property that a 1/2-symplectic-basis of Σ maps to ∂ (core 2-handle), i.e. to the relations π^{k-1}. It follows that (Σ, f) (which is homologous to Σ') extends to a map of a 1/2-grope of order k into X. $\square$

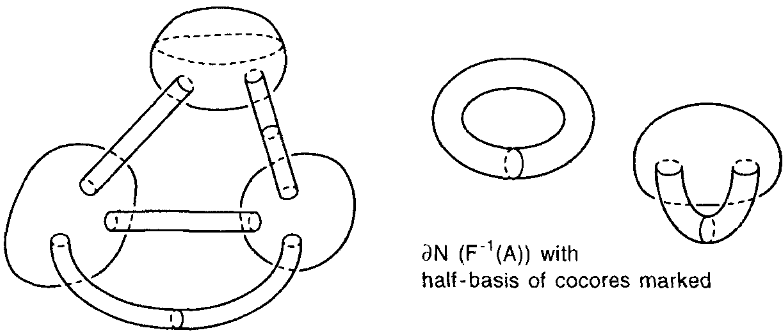

Fig. 2.3.

John Milnor [M2] defined certain numerical invariants $\overline{\mu}_L$ for a link L in S^3 which extend without difficulty to the case of a link in an integral homology 3-sphere Σ. We will define these invariants after recording some relations between the size of $\phi_k(\Sigma \setminus L)$ and the comparison of the groups $\pi_1(\Sigma \setminus L)$ and $\pi_1(\mathcal{S}^0(L))$ with the free group on n generators. The symbol $\mathcal{S}^0(L)$ denotes 0-framed surgery on an n-component link L in Σ. Let $V \subset \Sigma \setminus L$ denote a bouquet of meridians to L and set $F := \pi_1(V)$.

Lemma 2.4. *In the above setting, the following statements are equivalent for* $k \geqq 2$:

(i) *All (untwisted) longitudes of* L *bound maps of* k*-gropes in* $\Sigma \setminus L$.
(ii) $H_2(\Sigma \setminus L) = \phi_{k+1}(\Sigma \setminus L)$.
(iii) *The inclusion* $V \subset \Sigma \setminus L$ *induces an isomorphism*

$$F/F^{k+1} \xrightarrow{\cong} \pi_1(\Sigma \setminus L)/\pi_1(\Sigma \setminus L)^{k+1} .$$

(iv) $H_2(\mathcal{S}^0(L)) = \phi_k(\mathcal{S}^0(L))$ *(for* $k = 2$ *this should be read as* $H_2(\mathcal{S}^0(L)) \cong \mathbb{Z}^n$*).*
(v) *The inclusion* $V \subset \mathcal{S}^0(L)$ *induces an isomorphism*

$$F/F^k \xrightarrow{\cong} \pi_1(\mathcal{S}^0(L))/\pi_1(\mathcal{S}^0(L))^k .$$

Proof.

(i) $\Rightarrow$ (ii) By Alexander duality, $H_2(\Sigma\backslash L)$ is generated by the n tori parallel to the components of L. These are obviously in $\phi_{k+1}(\Sigma\backslash L)$ if the longitudes bound maps of k-gropes.

(ii) $\Leftrightarrow$ (iii) follows directly from Dwyer's theorem since $H_2(V) = 0$.

(iii) $\Rightarrow$ (i) Using (iii) we may write any (untwisted) longitude l as $\bar{l} \in F/F^{k+1}$. If $m \in F$ is the corresponding meridian, the relation $[m, l] = 1$ in the link complement becomes $[m, \bar{l}] = 1$ in F/F^{k+1}. Using the Magnus expansion [MKS, Chapter 5] (which is explained in detail below) this implies that $\bar{l} \in F^k$ and by (iii) $l \in \pi_1(\Sigma\backslash L)^k$ implying (i).

(i) $\Rightarrow$ (iv) Since $k \geqq 2$, the longitudes of L bound surfaces in $\Sigma\backslash L$. These can be capped off by the cores of the 2-handles in $\mathscr{S}^0(L)$ to get a basis for $H_2(\mathscr{S}^0(L)) \cong \mathbb{Z}^n$. This construction shows that $H_2(\mathscr{S}^0(L)) = \phi_k(\mathscr{S}^0(L))$ if the longitudes bound maps of k-gropes.

(iv) $\Leftrightarrow$ (v) is again Dwyer's theorem.

(v) $\Rightarrow$ (i) the commutative triangle

$$\begin{tikzcd} F \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \pi_1(\Sigma\backslash L) \arrow[d, "i"] \\ & \pi_1(\mathscr{S}^0(L)) \end{tikzcd}$$

leads to three isomorphisms if one divides by the k^{th} stage of the lower central series: This is true by assumption for β and therefore α becomes injective. Moreover, α also becomes surjective in any nilpotent quotient because the meridians become normal gener'ators since $H_1(\Sigma) = 0$. But in any nilpotent group normal generators are also generators which can be proved by an induction on the nilpotency class using the fact that $a \equiv b \bmod N$ implies $x^a \equiv x^b \bmod [G,N]$ for any elements a, b, x in a group $G, N \subset G$. This proves that (v) implies an isomorphism

$$i_* : \pi_1(\Sigma\backslash L)/\pi_1(\Sigma\backslash L)^k \xrightarrow{\cong} \pi_1(\mathscr{S}^0(L))/\pi_1(\mathscr{S}^0(L))^k$$

from which (i) follows since the longitudes become trivial in $\pi_1(\mathscr{S}^0(L))$. □

We may now define weak (with large indeterminancy) $\bar{\mu}$-invariants of an oriented link $L \subset \Sigma$ inductively as follows: The $\bar{\mu}$-invariants of length 1 are defined to be zero. Assume that statement (iii) of Lemma 2.4. holds for some $k \geqq 1$. We define integral valued $\bar{\mu}_L$-invariants of length $(k+1)$ using the isomorphism from (iii) to get well defined elements $\ell_i \in F/F^{k+1}$ from the (untwisted) longitudes of L. Then the *Magnus expansion* (given by $m_i \mapsto 1 + x_i$)

$$M : F = F(m_1, \ldots, m_n) \to \mathbb{Z}\{x_1, \ldots, x_n\}$$

into the (units in the) ring of non-commuting power series can be used to define the numbers $\overline{\mu}_L(I,j)$ via

$$\sum_I \overline{\mu}_L(I,j)x_I := M(\ell_j)\,.$$

Here I runs through all possible multi-indices but only those with exactly k entries (leading to the invariants $\overline{\mu}_L(I,j)$ of length $(k+1)$) are interesting: This follows from the fact that the Magnus expansion maps F^i to power series of the form

$$1 + \text{terms of degree} \geqq i \quad (\text{all } x_i \text{ have degree } 1)\,.$$

By assumption $\ell_j \in F^k/F^{k+1}$ and thus exactly the terms of degree k in $M(\ell_j)$ are well defined and interesting. One knows [MKS, Chapter 5] that the Magnus expansion is injective and that the associated graded map (given by $\overline{a} \mapsto \overline{M(a)-1}$)

$$\mathrm{Gr}(M) : \mathrm{Gr}(F) := \bigoplus_{k\geqq 1} F^k/F^{k+1} \to \bigoplus_{k\geqq 1} \text{degree } k/\text{degree } (k+1) =: A_n$$

into the free associative ring A_n in $x_1,\ldots,x_n$ is a Lie algebra isomorphism from $\mathrm{Gr}(F)$ (with its Lie bracket induced by the group commutator $[a,b] = aba^{-1}b^{-1}$) onto the (free) Lie algebra inside A_n (with Lie bracket $[\alpha,\beta] = \alpha\beta - \beta\alpha$) generated by $x_1,\ldots,x_n$. This implies that the $\overline{\mu}_L$-invariants of length $(k+1)$ vanish if and only if the (equivalent) conditions of Lemma 2.4. hold for $k+1$. Moreover, it implies that the $\overline{\mu}_L$-invariants satisfy certain relations (which Milnor called *shuffle symmetries*) to keep $\mathrm{Gr}(M)(\ell_j) = \overline{M(\ell_j)-1}$ in the Lie algebra on the x_i.

Lemma 2.5. *The $\overline{\mu}_L$-invariants are* **cyclically symmetric**, *i.e.*

$$\overline{\mu}_L(i_1,\ldots,i_k,j) = \overline{\mu}_L(j,i_1,\ldots,i_k)$$

if all $\overline{\mu}_L$-invariants of length $\leqq k$ vanish.

Remark. For $k=1$ this is the well known symmetry of linking numbers since one easily checks that $\overline{\mu}_L(i,j)$ is the linking number between the i-th and j-th component of L.

Proof (of cyclic symmetry). The longitudes of L give elements $\ell_j \in F^k/F^{k+1}$ which lead to elements $[m_j,\ell_j] \in F^{k+1}/F^{k+2}$. The 5-term exact sequence for the extension

$$1 \to F^{k+1} \to F \to F/F^{k+1} \to 1$$

gives an isomorphism $H_2(F/F^{k+1}) \cong F^{k+1}/F^{k+2}$ which we compose with the isomorphism of statement (iii) to get

$$H_2(\pi_1(\Sigma\backslash L)/\pi_1(\Sigma\backslash L)^{k+1}) \cong F^{k+1}/F^{k+2}\,.$$

It is easy to check that the elements $[m_j,\ell_j]$ correspond (under this isomorphism) to the image of the tori $T_j \subset \Sigma\backslash L$ parallel to the components of L

under the natural maps

$$H_2(\Sigma\backslash L) \to H_2(\pi_1(\Sigma\backslash L)) \to H_2(\pi_1(\Sigma\backslash L)/\pi_1(\Sigma\backslash L)^{k+1}).$$

The cyclic symmetry of the $\overline{\mu}_L$ can now be derived from the relation $\sum_{j=1}^n T_j = 0$ in $H_2(\Sigma\backslash L)$ as follows: mapping this relation to F^{k+1}/F^{k+2} and then applying the graded Magnus expansion gives the following relation in A_n:

$$0 = \sum_{j=1}^n x_j \mathrm{Gr}(M)(\ell_j) - \mathrm{Gr}(M)(\ell_j)x_j = \sum_{j=1}^n \sum_{|I|=k} \overline{\mu}_L(I,j)(x_j x_I - x_I x_j).$$

Focusing on the coefficients at $x_I x_j$ for some fixed index $I = (i_1, \ldots, i_k)$ one immediately recovers the relations

$$\overline{\mu}_L(i_1, \ldots, i_k, j) = \overline{\mu}_L(j, i_1, \ldots, i_k). \qquad \square$$

Let Z be the unique contractible 4-manifold with boundary Σ, see [F1]. We will say that a link $L \subset \Sigma$ is *4D-homotopically trivial* if it extends to maps $\Delta_i : D^2 \to Z$ with disjoint images.

Remark. For $(Z, \Sigma) = (D^4, S^3)$ this notion agrees with Milnor's [M1] "link homotopy" as we shall prove below. But if Σ is not simply connected then there are knots in Σ which are not null-homotopic but they bound a map $\Delta : D^2 \to Z$ since $\pi_1(Z) = \{1\}$.

Definition. *Let a group G be normally generated by elements $g_i, i \in I$. Define the* **Milnor group** *of G (rel g_i) to be the quotient*

$$MG := G \Big/ [g_i^{x_i}, g_i^{y_i}] = 1 \quad \forall x_i, y_i \in G,\ i \in I.$$

In [FT, Sect. 3] we show that for $|I| = k$ the Milnor group MG is a finitely presented nilpotent group of class $\leqq k$, see [M1] in the case of link groups. In particular, MG is a quotient of the free Milnor group

$$MF_k := F(m_1, \ldots, m_k) \Big/ [m_i^{x_i}, m_i^{y_i}] = 1 \,.$$

Usually the normal generators are clear from the context, for example $M\pi_1(S^3\backslash L)$ or $M\pi_1(Z\backslash\Delta)$ always refer to the meridians (to L_i resp. Δ_i) as normal generators.

Lemma 2.6. *Let $L \subset \Sigma$ be 4D-null homotopic and let $\Delta = (\Delta_1, \ldots, \Delta_k)$ be a null homotopy. Then the meridian map induces an isomorphism*

$$MF_k \xrightarrow{\cong} M\pi_1(Z\backslash\Delta).$$

Proof. The second homology of $Z\backslash\Delta$, if Δ is in general position, is generated by the "Clifford tori" linking the transverse double points of Δ. As in [FT, Corollary 3.2], working modulo any term N of the lower central series, a bouquet of meridians W to L induces an epimorphism θ:

$$\begin{tikzcd}
\pi_1(W)/\pi_1(W)^N \arrow[r, "\theta"] \arrow[d, "\gamma"] & \pi_1(Z\setminus\Delta)/\pi_1(Z\setminus\Delta)^N \\
\pi_1(W\cup 2\text{-cells})/\pi_1(W\cup 2\text{-cells})^N \arrow[ur, "\theta'"'] &
\end{tikzcd}$$

The "meridians" and "longitudes" of the Clifford tori now lift along θ to conjugates of the basic meridians of W, m_i^α and m_i^β. The 2-cells in the above diagram are attached to the commutators $[m_i^\alpha, m_i^\beta]$ and the map on space level extends to an epimorphism on H_2. By Stallings' theorem, θ' is an isomorphism while γ, by the nature of the relation 2-cells, induces an isomorphism on Milnor groups. Since we may assume N larger than the nilpotency class $= k$ of the Milnor group MF_k, θ induces an isomorphism $MF_k \to M\pi_1(Z\setminus\Delta)$. □

Remark. Note that $M\pi_1(Z\setminus\Delta)$ is obtained from $\pi_1(Z\setminus\Delta)$ by adding finitely many relations of the form $[m_i^\alpha, m_i] = 1$ and these may be realized by introducing additional self-fingermoves to the Δ_i. Therefore we may always assume that the null homotopy Δ satisfies

$$M\pi_1(Z\setminus\Delta) \cong \pi_1(Z\setminus\Delta)\,.$$

In [M1] Milnor showed that the Magnus expansion induces a well defined (and still injective!) homomorphism

$$MM : MF_k \to R_k$$

into the (units of the) ring R_k which is the quotient of the free associative ring A_k by the ideal generated by the monomials $x_{i_1}\dots x_{i_l}$ with one index occuring at least twice. If $l_{k+1} \subset \Sigma\setminus L$ is an additional component in the complement of a 4D-null homotopic link, $L^+ := L \cup l_{k+1}$, then we define the μ-invariants of L^+ by the equation

$$MM(l_{k+1}) = \sum_I \mu_{L^+}(I, k+1)x_I \;\in R_k$$

where I is a multi-index with nonrepeating entries from $\{1,\dots,k\}$ and $l_{k+1} \in MF_k$ is obtained via the isomorphism in Lemma 2.6. (and the inclusion $\Sigma\setminus L \to Z\setminus\Delta$). Using the injectivity of MM and the remark after lemma 2.6 we conclude that L^+ is 4D-homotopically trivial if and only if all μ_{L^+}-invariants vanish. But Milnor showed in [M1] that this is also true for his notion of homotopy for links in S^3. Therefore, 4D-homotopy and link homotopy agree in this case. The commutative diagram

$$\begin{tikzcd}
\pi_1(\Sigma\setminus L) \arrow[d] & F \arrow[l] \arrow[r, "M"] \arrow[d] & \mathbb{Z}\{x_i\} \arrow[d] \\
M\pi_1(Z\setminus\Delta) & MF \arrow[l] \arrow[r, "MM"] & R
\end{tikzcd}$$

shows that $\mu(I)=\overline{\mu}(I)$ if both invariants are defined, in particular I must have nonrepeating indices.

Lemma 2.7. *For a link $L \subset \Sigma$ and $k \geqq 2$ the following two statements are equivalent to all statements in 2.4.*

(a) *All $\overline{\mu}_L$-invariants of length $\leqq k$ vanish.*
(b) *If $\widehat{L}$ is any link of k (or fewer) components made from untwisted parallels of L, then $\widehat{L}$ is 4D-homotopically trivial.*

Proof. The equivalence of (a) with the statements in 2.4. follows from the injectivity of the Magnus expansion as discussed above.

(a) $\Rightarrow$ (b) First note that if the longitudes of L bound maps of k-gropes in $\Sigma\backslash L$ then the longitudes of $\widehat{L}$ bound maps of k-gropes in $\Sigma\backslash\widehat{L}$ (and vice versa if $\widehat{L}$ contains each component of L). By Lemma 2.4, the $\overline{\mu}_{\widehat{L}}$-invariants of length $\leqq k$ vanish. Choosing an ordering of the components of $\widehat{L}$ we can now inductively prove that also all the $\mu_{\widehat{L}}$-invariants vanish and thus $\widehat{L}$ is 4D-homotopically trivial.

(b) $\Rightarrow$ (a) By a simple induction on $|I|$ we may assume that all $\overline{\mu}_L$-invariants of length $\leqq n < k$ vanish and thus we can use our definition for $\overline{\mu}_L(I)$ for $|I| = n+1$. To prove that the invariant $\overline{\mu}_L(I)$ is trivial choose the link $\widehat{L}=L(I)$ to be formed from n_i parallels of L_i where n_i is the number of times the index i occurs in I. Also form a new multi-index $\widehat{I}$ from I by replacing the n_i occurrences of the index i with distinct indices $i_{/1},\dots,i_{/n_i}$. By definition, $\widehat{I}$ has nonrepeating indices and labels the $(n+1)$-component link $\widehat{L}$. Therefore, the invariant $\mu_{\widehat{L}}(\widehat{I})$ vanishes since we are assuming (b). It follows that the invariant $\overline{\mu}_{\widehat{L}}(\widehat{I})$ vanishes. But a straightforward checking of definitions shows that this invariant equals $\overline{\mu}_L(I)$. One only has to observe that the relabelling of L to $\widehat{L}$ changes a meridian m_i to the product $m_{i_{/1}}\dots m_{i_{/n_i}}$ and thus $x_i=\mathrm{GR}(M)(m_i)$ is replaced by the sum $x_{i_{/1}}+\dots+x_{i_{/n_i}}$. $\square$

For the final construction in Sect. 5 we need one more Lemma on links in a homology 3-sphere Σ which uses the cyclic symmetry of $\overline{\mu}$-invariants in this setting.

Lemma 2.8. *Let $L \subset \Sigma$ have vanishing $\overline{\mu}$-invariants of length $\leqq k$ and let $R' \subset \Sigma\backslash L$ be a link with each component lying in $\pi_1(\Sigma\backslash L)^k$. Then there is a "weak" homotopy (not a Milnor link homotopy avoiding certain collision but just a general 1-parameter motion in $\Sigma\backslash L$) of R' to R so that the link $L\cup R$ has vanishing $\overline{\mu}$-invariants of length $\leqq k$.*

Proof. Each component r_i' of R' bounds a singular grope of class k in $\Sigma\backslash L$. By general position these gropes have disjointly imbedded 1-spines S_i. The desired weak homotopy pushes each r_i' to a neighborhood of S_i where it bounds an imbedded grope G_i of class k. Set $r_i := \partial G_i$. Since the various G_i are disjoint and in particular $G_i \cap (r_j \cup L) = \emptyset$ for $i \neq j$, we see that the longitude of r_i lies in $\pi_1(\Sigma\backslash L\cup R)^k$ so all $\overline{\mu}$-invariants of $L\cup R$ of length $\leqq k$ ending in an

R-index vanish. By cyclic symmetry of the $\bar{\mu}$ (Lemma 2.5), the only $\bar{\mu}$-invariants of $L \cup R$ of length $\leqq k$ which could be nonzero would be those with only L indices. But the $\bar{\mu}$-invariants are natural under filling in the R-components so this would imply a nonzero $\bar{\mu}$-invariant of length $\leqq k$ for the original link L against our hypothesis.

3. Whitehead doubles of links

To introduce the construction used to prove Theorem 1.1 we present a much simpler application which generalizes the main theorems of [F2] and [F3]. Let $L \subset S^3$ be a smooth link. $\mathrm{Wh}(L)$ denotes an unramified, untwisted Whitehead double. This means each component ℓ_i has been replaced within its neighborhood by a component $\mathrm{Wh}(\ell_i)$ of either form

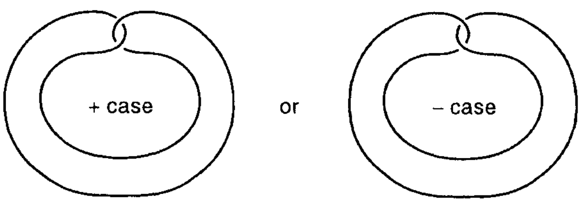

Fig. 3.1.

so that the Seifert form on the punctured torus T_i bounding $\mathrm{Wh}(\ell_i)$ within a neighborhood of ℓ_i is of the form $\left|\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right|$. This last condition makes "untwisted" precise. The $\pm$ ambiguity in the choice of the clasp in Fig. 3.1 relates as we will soon see to the sign of a double point in 4 dimensions. This sign will not be relevant in our discussion so the term "Whitehead double" refers to any of the $2^{\#(L)}$ possible links. In [F2] it is shown that if L is a boundary link then $\mathrm{Wh}(L)$ is slice (i.e. bounds disjoint flat disks in D^4) with $\pi_1(D^4\backslash\text{slices})$ freely generated by meridians ("free slice"). In [F3] it was shown that the same conclusion applied to exactly those two component links with trivial linking number.

Definition. *A smooth link $L = (\ell_1, \ldots, \ell_n)$ in S^3 is (homotopically trivial)$^+$ if the n links of $(n+1)$-components obtained by adding a parallel copy of a single component $\ell_i, i = 1, \ldots, n$ are each homotopically trivial in the sense of* [M1] *(or 4D-homotopically trivial as in Sect.* 2*).*

Theorem 3.1. *If $L \subset S^3$ is (homotopically trivial)$^+$ then* $\mathrm{Wh}(L)$ *is freely slice.*

Remark. It is unknown whether homotopically trivial is an adequate hypothesis for this theorem. The still weaker hypothesis that all linking numbers of L vanish would suffice if surgery "worked" for free groups. We note that (homotopically trivial)$^+$ is equivalent to the vanishing of all $\bar{\mu}$-invariants with at most a single repetition of one index. It is also equivalent to the meridian map inducing an isomorphism of Milnor groups $MF \to M\pi_1(\mathscr{S}^0(L))$.

Lemma 3.2. *Let $L = (\ell_1, \ldots, \ell_n)$ and $L' = (\ell'_1, \ldots, \ell'_n)$ be a parallel copy. The condition L is (homotopically trivial)$^+$ is equivalent to the existence of disks $\{\Delta_i\} \cup \{\Delta'_i\}$, $1 \leqq i \leqq n$, properly mapping into B^4 with $\partial\Delta_i = \ell_i, \partial\Delta'_i = \ell'_i$ and disjointness assumptions: $\Delta_i \cap \Delta_j = \phi$ for $i \neq j$ and $\Delta'_i \cap \Delta_j = \phi$ for all $1 \leqq i, j \leqq n$.*

Proof. The condition L homotopically trivial is known (see [L] or [FT, Lemma 3.3]) to be equivalent to the existence of disjoint $\{\Delta_i, 1 \leqq i \leqq n\}$ as above. In these terms, the $+$ condition means that for each $1 \leqq j \leqq n$ the $\{\Delta_i\}$ can be chosen so that $[\ell'_j] = e \in \pi_1(B^4 \setminus \{\Delta_i, 1 \leqq i \leqq n\})$. However, by Lemma 2.6 these groups for different choices of $\{\Delta_i\}$ all have a common quotient MF_n, the free Milnor group on n generators which is itself realized as $\pi_1(B^4 \setminus \{\Delta_i, 1 \leqq i \leqq n\})$ for sufficiently complicated choices of Δ_i. Thus the null homotopies $\{\Delta'_i, 1 \leqq i \leqq n\}$ all exist disjoint from $\{\Delta_i, 1 \leqq i \leqq n\}$. □

With the notation below the pair $(B^4; \mathcal{N}(h_1) \cup \mathcal{N}(h_2))$ is a concrete model of a (positive) plumbed pair of 2-handles, see [FT]. The notation is: h_1 and h_2 are two Hopf circles in $S^3 = \partial B^4 \subset \mathbb{C}^2$ and the $\mathcal{N}$'s are 3-dimensional solid torus neighborhoods of these. All orientations are taken standard with respect to complex multiplication. Reversing the orientation along one Hopf circle gives a negative plumbed pair. More generally, handles may be plumbed $+$ or $-$ together in many (disjoint) places and self-plumbed to produce the kinky handles of [F1].

Lemma 3.3. *The effect of introducing a $\pm$ plumbing (or self-plumbing) on the underlying (Kirby) handle diagram of a handlebody is to introduce a new 1-handle and a $\pm$ clasp of the attaching curve(s) of the 2-handle(s) being plumbed over this 1-handle as shown below*

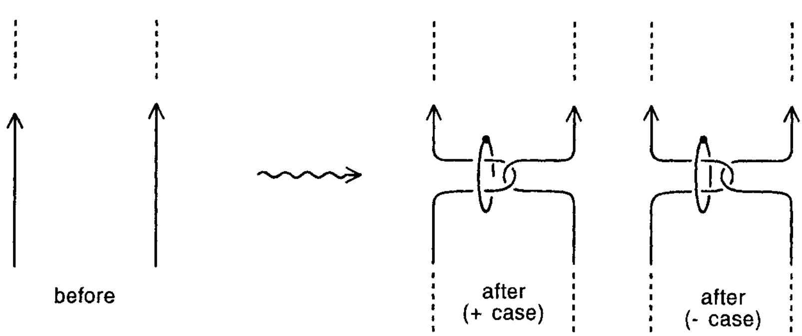

Fig. 3.2.

Proof (*sketch*). First identify a 1-handle in the plumbed handlebody by taking a neighborhood of two arcs leaving the "origin" of the plumbing on the two core sheets. The new attaching curves for two-handles are as before except that the attaching curves induced in the plumbing must now run up this 1-handle, i.e. through the dotted circle, in the diagram, clasp, and return. The sign of the clasp is worked out from the Hopf link model introduced above. □

Proof of Theorem 3.1. We introduce a specific 4-manifold N whose boundary ∂N is 0-framed surgery on $\mathrm{Wh}(L), \partial N \cong \mathcal{S}^0(\mathrm{Wh}(L))$. Set $N_0 := B^4 \cup_{L \cup L'}$ 2-handles, the result of attaching $2n$ 2-handles with framing $= 0$ to the link L and its parallel copy L'. Now set $N := N_0$/plumbings where for each $1 \leqq i \leqq n$ one plumbing is introduced between the 2-handles attached to ℓ_i and ℓ'_i. The sign of the plumbing is, for each i, chosen to agree with the sign of the Whitehead doubling of the i-th component.

Lemma 3.4. $\partial N \cong \mathcal{S}^0(\mathrm{Wh}(L))$ *with the isomorphism carrying the meridians to the* 1*-handles* (*see Lemma* 3.3) *to the meridians to* $\mathrm{Wh}(L)$.

Proof. Inside a solid torus we have the following Kirby calculus identity:

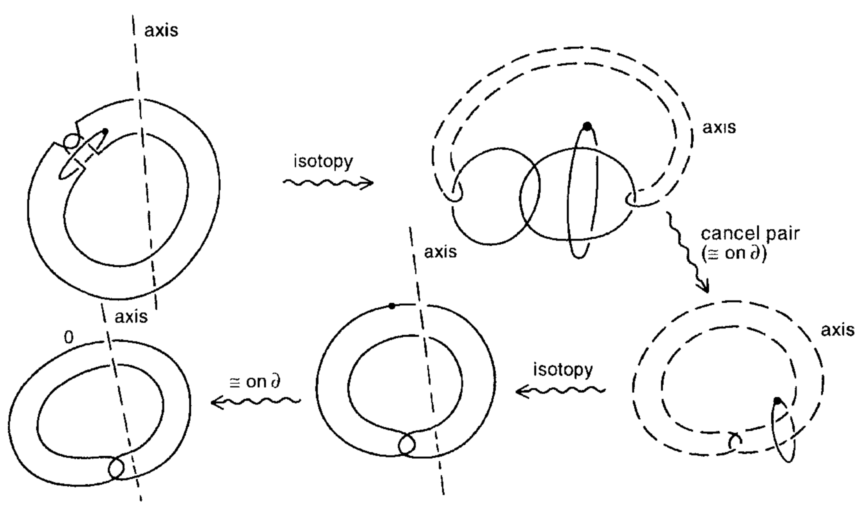

Fig. 3.3.

Note that the z-axis initially marks the complement of the solid torus. All clasps have been drawn ambiguously to imply the uniform treatment of both cases. Now apply this identity simultaneously in the n solid torus neighborhoods of $\{\ell_i\}$ to finish the proof. □

Next we cap off the cores of the $2n$ plumbed handles with the disks $\{\Delta_i, \Delta'_i, 1 \leqq i \leqq n\}$ produced by Lemma 3.2 to obtain an immersed union S of $2n$ spheres in N. (In analogy with Theorem 1.1 we would call the closed regular neighborhood of these spheres M but in this simple case the homology of M is spherical and we may finish directly). The fundamental group of N is freely generated by the n plumbings, that is, the solid torus neighborhoods $\mathcal{N}(\ell_i), 1 \leqq i \leqq n$ give a disjoint collection of 3-manifolds which by the Pontrjagin-Thom construction determine a map to $\bigvee_{i=1}^n S_i^1$ which classifies the fundamental group of N. It is easily seen that the disjointness conditions of Lemma 3.2 assure that every loop in S reads the trivial word as it crosses the various $\mathcal{N}(\ell_i)$. Thus S is π_1-null. Also the intersection form carried by S is $\bigoplus_n \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$. The plumbings contribute the nontrivial entries; the double points $d_i' \cap d_j'$ contribute nothing since $\operatorname{link}(\ell_i', \ell_j') = 0 \ \forall i,j$. While it is unknown whether S is homotopic to a disjoint collection of imbedded hyperbolic pairs, it is shown in [FQ, Chapter 6] that such π_1-null collections of spheres are s-cobordant to disjointly imbedded hyperbolic pairs. This is adequate to "complete surgery", that is to produce a 4-manifold N' with $\partial N' = \partial N$ and with a map θ to a wedge of circles $N' \xrightarrow{\theta} \bigvee_{i=1}^n S_i^1$ which is an isomorphism on π_1 and with the meridians to the 1-handle curves for the diagram of $\partial N = \partial N'$ mapping degree 1 to the n circle factors. Clearly $H_2(N') = 0$ and therefore by [FQ, 11.6C(1)] θ is a homotopy equivalence. It is now standard to observe that

$$S^3 \times I \cup_{\mathrm{Wh}(L) \times 1} \ (\text{framed 2-handles}) \cup N' \cong B^4$$

where the last union uses the identification of Lemma 3.4 and $\partial N' = \partial N$. The 0-framed 2-handles now extend through the product collar $S^3 \times I$ to become the desired free slices on $\mathrm{Wh}(L) \times 0$. □

4. A special case

This section contains the proof of Theorem 1.1 in the presence of the extra assumption that $\pi_1(N \backslash M) \to \pi_1(N)$ is an isomorphism. This is often described by saying M is π_1-*negligible* and is a very familiar condition in 4-manifold theory. Removing this assumption adds a final layer of subtlety to the argument which will be defered until Sect. 5.

But let us first collect the elementary homological consequences of the other assumption, namely that $H_1(\partial M) \to H_1(M)$ is an isomorphism. Considering the hom-dual of the isomorphism and the universal coefficient theorem, we see that $H^1(M) \to H^1(\partial M)$ is an isomorphism. By Lefschetz duality $H_3(M, \partial M) \to H_2(\partial M)$ is also an isomorphism. From the exact sequence of the pair $(M, \partial M)$ we now conclude that $H_2(M) \to H_2(M, \partial M)$ is an isomorphism and $H_1(M, \partial M) = 0$. Similarly,

$$0 = H_1(M, \partial M) \cong H^3(M) \cong \operatorname{free}(H_3(M)) \oplus \operatorname{torsion}(H_2(M)),$$

so $H_2(M)$ is free and

$$H_2(M) \cong H_2(M, \partial M) \cong H^2(M) \cong \operatorname{free}(H_2(M)) \oplus \operatorname{torsion}(H_1(M)),$$

and therefore $H_1(M)$ is also free.

Definition. *We say a group π is* **weakly-para-free** *if there is a map from a free group $F \to \pi$ inducing isomorphism on all the finite quotients of the lower central series $F/F^k \xrightarrow{\cong} \pi/\pi^k$, $k = 2,3,\ldots$.*

We know that $H_1(M) \cong H_1(\partial M)$ are free abelian so let $\varepsilon : V \to \partial M$ be any map from the wedge of n-circles to ∂M inducing an isomorphism on H_1. Now considering $\varepsilon_\# : \pi_1(V) \to \pi_1(\partial M)$ one finds

Lemma 4.1. *$\pi_1(\partial M)$ and $\pi_1(M)$ are both weakly-para-free with the inclusion of the wedge V inducing the required map. Also the map $\pi_1(\partial M) \to \pi_1(M)$ is an isomorphism modulo any finite term of the lower central series.*

Proof. Consider the composition $V \xrightarrow{\varepsilon} \partial M \xrightarrow{i} M$. Since the homology of M is "nearly spherical" in the sense that $\phi_\omega M = H_2(M)$, Dwyer's theorem tells us that both maps i and $i \circ \varepsilon$ with target M (which we already known induce isomorphisms on H_1) induce isomorphisms on all finite quotients of the lower central series. Applying the functors $\pi_1/(\pi_1)^k$ $k \geqq 2$ to the diagram proves the Lemma. □

Define $M_1 \subset N$ to be

$$M_1 := M \bigcup_{\{\gamma_i\}} \text{2-handles/plumbings and self-plumbings}\,. \tag{4.1}$$

The submanifold M_1 is simply what can be produced from M by using the π_1-null and π_1-negligible hypotheses on M. Specifically, take any collection of n disjointly imbedded circles $\{\gamma_1,\ldots,\gamma_n\}$ in ∂M homotopic to the petals of V and cap these off by n general position null homotopies $\delta_1,\ldots,\delta_n$ whose interiors are disjoint from M. The submanifold M_1 is simply a regular neighborhood of $M \cup \left(\bigcup_{i=1}^n \delta_i\right)$. By the basic theory of topological immersions [FQ, Chapter 8] M_1 may be described combinatorially as in line (4.1) where the n 2-handles determine definite normal framings f_i on γ_i (γ_i and its parallel γ_i' should bound chains with intersection number 0 in the plumbed 2-handles). Let $\Sigma := \mathscr{S}_{\partial M}((\gamma_i, f_i),\ i = 1,\ldots,n)$ be the abstract homology sphere which results as the boundary of the abstract 4-manifold $M \cup_{\{\gamma_i,f_i\}}$ (n 2-handles). It is abstract in the sense that it is not a submanifold of N, but our strategy is to construct another abstract manifold M_2 with a canonical isomorphism $\partial M_2 \cong \partial M_1$ and such that $H_2(M_2)$ is represented by a π_1-null collection of spheres.

To begin the construction of M_2, let Z be the unique contractible manifold bounded by Σ, see [F1]. We may consider (Kirby) handle diagrams drawn in $\Sigma = \partial Z$. To start notice the n-component link $(m_1,\ldots,m_n) \subset \Sigma$ consisting of meridians to $(\gamma_1,\ldots,\gamma_n) \subset \partial M$. 0-framed surgery on $(m_1,\ldots,m_n)$ is naturally identified with ∂M, reversing the initial surgery. Furthermore, the 0-framed meridians $\mu_1',\ldots,\mu_n'$ to $m_1,\ldots,m_n$ become $(\gamma_1,f_1),\ldots,(\gamma_n,f_n)$ under this identification. Now Lemma 3.3 can be exploited to give a Kirby diagram for ∂M_1

in Σ as shown below

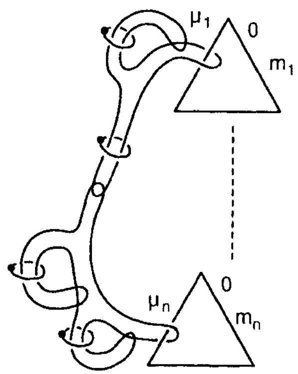

Fig. 4.1.

What we see are the meridians m_i, their meridians μ_i' modified to μ_i by clasps (induced by some number of (self-)plumbings and 1-handle curves linking these (self-)clasps. The figure shows μ_1 and μ_n with one clasp and one, respectively two, selfclasps. The geometric shape of the m_i reminds us that their detailed position in Σ is unknown. As in Sect. 3 the sign $\pm$ of the clasp is intentionally suppressed in the figure.

We define M_2 as realizing the boundary equivalent link diagram in $\partial Z = \Sigma$ where each clasp has been "blown up" to a 0-framed Hopf link and each $\mu_i, 1 \leqq i \leqq n$, has been made into a 1-handle (given a dot) as shown (locally) below.

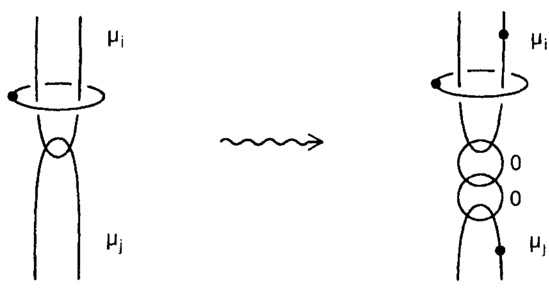

Fig. 4.2.

Thus M_2 is defined as $Z \cup$ 1-handles $\cup$ 2-handles according to Fig. 4.1, as modified locally in Fig 4.2.

We may change the handlebody structure of M_2 (but not its homeomorphism type) by Morse cancelling each μ_i with $m_i, 1 \leqq i \leqq n$. The result (using the same multiplicities as in Fig. 4.1)

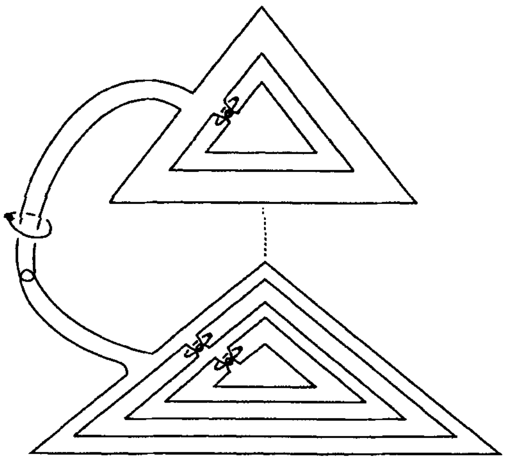

Fig. 4.3.

Lemma 4.2. *Any link consisting of untwisted parallel copies of $L := \{m_1, \ldots, m_n\}$ in Σ bounds disjoint maps of disks into Z.*

Proof. According to Lemmas 2.4 and 2.7 exactly the links $L \subset \Sigma$ with $\pi_1(\Sigma \backslash L)$ weakly-para-free admit disjoint maps of disks on any number of parallels. Since $\mathscr{S}^0_\Sigma(L) = \partial M$, Lemma 4.1 says that $\pi_1(\mathscr{S}^0_\Sigma(L))$ is weakly-para-free. By Lemma 2.4 this is equivalent to $\pi_1(\Sigma \backslash L)$ being weakly-para-free. □

Using the 2-handles in Fig. 4.3 to form the "northern hemisphere" and the singular disks in Z located by Lemma 4.2 as "southern hemisphere" we see that the entire second homology of M_2 (with group-ring coefficients) is freely generated by a π_1-null collection of spheres with hyperbolic intersection form. The verification is much the same as in Sect. 3.

As in Sect. 3 we use [FQ, Chapter 6] to s-cobord the spheres to a disjointly imbedded collection of hyperbolic pairs. Removing these pairs by surgery yields a 4-manifold M_3 with $\partial M_3 = \partial M_2 = \partial M_1$, and as at the end of Chapter 3 a homotopy equivalence $M_3 \xrightarrow{h} W$ to a wedge of circles which takes each meridian to a 1-handle of Fig. 4.3 degree 1 to a distinct circle factor. The free generators correspond to the double points of the null homotopies $\{\delta_1, \ldots, \delta_n\}$. The manifold N' asserted by Theorem 1.1 is defined as $(N \backslash M_1) \bigcup_{\mathrm{id}} M_3$.

We now construct a map $g : N^+ \to N'$. Set $g|_{N \backslash M_1} :=$ identity. Near each self-plumbing of the 2-handles in the combinatorial description (4.1) of M_1 we may locate a solid torus dual to the arc which changes sheets at the selfplumbing. These tori, by the Pontrjagin-Thom construction, determine a map $M_1 \to W$ which extends (uniquely up to homotopy) to a map $M_1 \cup$ 3-cells $\to W$. On $\partial M_1 = \partial M_3$ this map restricts to h and thus provides an extension of the identity on $\partial M_1 = \partial M_3$ to a map $M_1 \cup$ 3-cells $\xrightarrow{\eta} M_3$. Let this last map be g restricted to $M_1 \cup$ 3-cells.

To check that g induces an isomorphism on π_1 we ignore the 3-cells and consider two interconnected pushout diagrams of groups.

$$\begin{tikzcd}
 & \pi_1(M_3) = \text{free} \arrow[d, bend left, "5"] \arrow[dr] & \\
 & \pi_1(M_1) \arrow[u, "4"] \arrow[ddr, "8"] & \pi_1(N') \arrow[dd, bend left, "7"] \\
\pi_1(\partial M_1) \arrow[uur, "3"] \arrow[ur, "1"] \arrow[ddr, "2"'] & & \\
 & \pi_1(M) \arrow[uu, "9"] \arrow[r, "10"] & \pi_1(N) \arrow[uu, "6"] \\
 & \pi_1(N \setminus M_1) \arrow[uuur] \arrow[ur] &
\end{tikzcd}$$

The groups $\pi_1(N)$ and $\pi_1(N')$ are pushouts of maps $(1,2)$ and $(3,2)$ respectively. Map 4 is the restriction of η and map 5 is any splitting of map 4. By construction, map 6 is the restriction of g and we want to show that map 7 is its inverse. By the pushout property map 4 (resp. map 5) induces map 6 (resp. map 7). Since map 4 $\circ$ map 5 $=$ id, map 6 $\circ$ map 7 $=$ id on $\pi_1(N')$. To check that map 7 $\circ$ map 6 is also the identity we need to know:

$$\ker(\text{map } 4) \subset \ker(\text{map } 8) \tag{4.2}$$

for map 7 $\circ$ map 6 would then change a standard-form word for $\pi_1(N)$ only on the letters in $\pi_1(M_1)$ and these letters would only change by an element of ker(map 5 $\circ$ map 4) which is no change at all in $\pi_1(N)$. To check (4.2) observe that ker(map 4) $=$ normal closure (image map 9) and that π_1-nullity states that map 10 is zero. With $\pi := \pi_1(N^+) \cong \pi_1(N')$, define

$$K_i := \ker(H_i(N;\mathbb{Z}[\pi]) \to H_i(N';\mathbb{Z}[\pi])),$$

$$K^i := \operatorname{coker}(H^i(N',\mathbb{Z}[\pi]) \quad \to H^i(N,\mathbb{Z}[\pi]))\,.$$

Since $N \to N'$ is a degree 1 map cap product with the fundamental class induces isomorphisms $K_i \cong K^{4-i}$, $i = 0,\dots,4$. Also since $H^1(\,\cdot\,;\mathbb{Z}[\pi])$ is the first cohomology of the π-cover with compact supports it depends only on π implying $K_3 \cong K^1 = 0$. Thus K_2 is the only non-trivial homology kernel. Set $\widetilde{M}_1 \subset \widetilde{N}$ to be the inverse image of M_1. Then $\widetilde{M}_1$ consists of $\widetilde{M} = (\pi\text{-copies of } M)$ union various singular disks Δ with $H_2(\Delta,\partial) \to H_1(\widetilde{M})$ an isomorphism. Thus $H_2(\widetilde{M}) \xrightarrow{\cong} H_2(\widetilde{M}_1)$ is an isomorphism. The image of $\widetilde{M}_1$ under $\tilde{g}$ is some cover of $M_3 \simeq W$ and so has no second homology. Excision then implies

$$K_2 = \operatorname{image}(H_2(\widetilde{M}_1) \to H_2(\widetilde{N}) \cong H_2(N;\mathbb{Z}[\pi_1]))\,, \tag{4.3}$$

and also

$$H_2(\widetilde{M}_1) \cong H_2(\widetilde{M}) \cong H_2(M) \otimes_{\mathbb{Z}} \mathbb{Z}[\pi]. \tag{4.4}$$

According to the beginning of Sect.4, our hypothesis $H_1(\partial M) \cong H_1(M)$ implies that $H_2(\partial M) \to H_2(M)$ is zero and by the Mayer-Vietoris exact sequence the map $H_2(M) \otimes_{\mathbb{Z}} \mathbb{Z}[\pi] \to H_2(\widetilde{N})$ induced by the inclusion must be an injection. It follows from lines (4.3) and (4.4) that $K_2 \cong H_2(M) \otimes_{\mathbb{Z}} \mathbb{Z}\pi$. Precisely a free basis for this module is killed by 3-cells in passing to N^+. It follows that

$$g : N^+ \to N'$$

induces an isomorphism on $H_i(\ ;\mathbb{Z}[\pi])$, $i = 0,\ldots,4$ and by Whitehead's Theorem g is a homotopy equivalence. Since the only interesting part of g is a $\mathbb{Z}\pi_1(M_3)$-homology equivalence $M_1 \cup$ 3-cells$\to M_3$ any torsion would come from $\mathrm{Wh}(\pi_1(M_3)) = \mathrm{Wh}(\text{free}) = 0$. Thus g is a simple equivalence. This completes the proof of Theorem 1.1 under the assumption that M is π_1-negligible. □

5. The proof of Theorem 1.1

This section completes the proof of Theorem 1.1. Let $\delta' = \{\delta'_1,\ldots,\delta'_n\}$ be null homotopies in N for $\gamma_1,\ldots,\gamma_n$ which initially leave M in an outward normal direction but may return to M. Since $H_1(\partial M) \to H_1(M)$ is injective each component C' of $\delta'^{-1}(M)$ may be replaced with an orientable surface C, $\partial C = \partial C'$ and C mapping to ∂M, extending δ' on $\partial C'$. Set $\delta = (\delta' \backslash \cup C') \cup (\cup C)$ together with the map to N. By construction, $H_1(\cup C) \to H_1(\delta)$ is an epimorphism so a symplectic basis (α, β) for $H_1(\delta)$ may be chosen to consist of imbedded hyperbolic pairs of loops in $\cup C$; by π_1-nullity the loops bound singular disks ε' in N. Thus we have capped off $\{\gamma_i\}$ with capped surfaces $\delta \cup \varepsilon'$. Putting things in general position we have $\delta \cap M = \partial\delta$ and $\varepsilon' \cap M =$ some planar surfaces in M. If $\pi_1(M, \partial M)$ was trivial we could homotope ε' to achieve that $\mathring{\varepsilon}' \cap M$ is a disjoint union of disks, an advantageous condition. By Lemma 4.1 we do know however, that for all k

$$\pi_1(\partial M)/\pi_1(\partial M)^k \to \pi_1(M)/\pi_1(M)^k$$

is an isomorphism. Let c denote the number of components of $\mathring{\varepsilon}' \cap M$. We fix a large number K, to be specified later, and add tubes along ∂M to ε', to form ε, so that $\mathring{\varepsilon} \cap \partial M$ consists of a collection of c circles which lie in $\pi_1(\partial M)^K$ and so that $\varepsilon \backslash M$ is a collection of disks with a total of c punctures.

It is now possible to draw a handle diagram relative to some contractible manifold Z, as in Fig. 4.1 to describe the boundary of a neighborhood of $(M \cup \delta \cup \varepsilon)$. Some "small" changes made to δ and ε improve the diagram to the type drawn below. After these changes we think (roughly) of the diagram as representing a 4-manifold M_2, although we have yet to interpret the dotted components $\mu_1,\ldots,\mu_n$ which may not form a slice link. If these components are not a slice link in Z the diagram only makes sense as the description of a 3-manifold. Later we will arrange that these components are slice in a space derived from Z, namely in $Z \# S^2 \times S^2$'s, allowing a 4-dimensional interpretation of these components as "pseudo-1-handles" in a stabilized diagram for a manifold

M_2^+. Below we have the analogue of Fig. 4.1 as modified by Fig. 4.2. The background space for Fig. 5.1 is the homology sphere $\Sigma = \partial Z = \mathscr{S}_{\partial M}\{\gamma_1, \ldots, \gamma_n\}$.

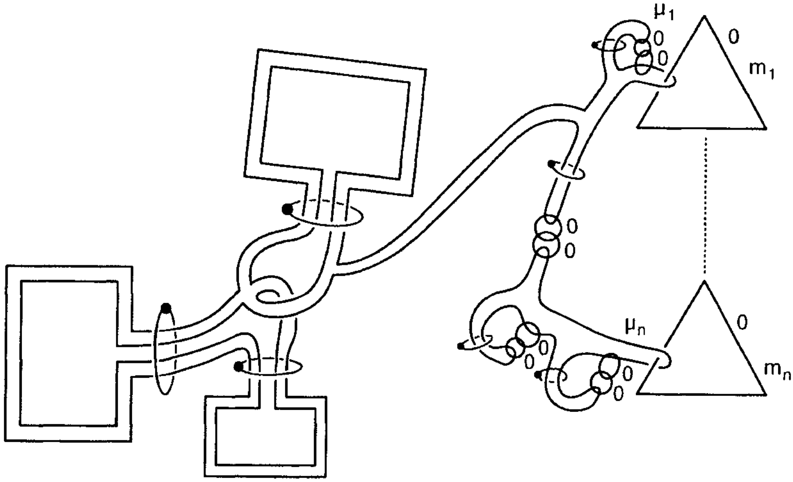

Fig. 5.1.

The new feature in Fig. 5.1 is the "tumor" on the left. It arises (along with many similar copies on all μ_i which have not been drawn) by cancelling the "Bing pairs", associated in the diagram with the surface stage δ, with the 2-handles corresponding to ε. The punctures in ε give rise to the new 1-handles and "rectangular" components as in Fig. 5.2 below.

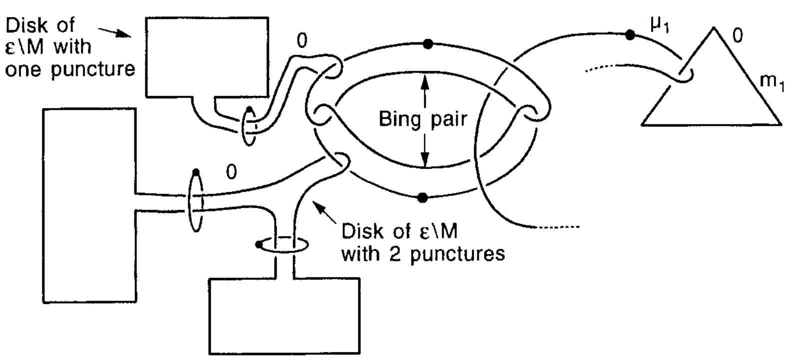

Fig. 5.2.

Absent from Fig. 5.1 are representations of:

(1) $(\mathring{\varepsilon}, \delta)$ intersections,

(2) $(\varepsilon, \varepsilon)$ (self)-intersections, and

(3) nonzero ε-framings.

By "spinning" (see [FQ, 1.3]) the framings of the 2-handles in Fig. 5.2 may be made zero. This introduces (new) intersections of type (1). All intersections of type (1) and (2) can be removed by a move in which a sheet of δ containing a bit of $\partial\varepsilon$ is pushed along an arc in ε through the troublesome intersection point (see [FQ, 2.5]). The cost of this move is additional (δ, δ)-intersection points but we have allowed for these in Fig. 5.1.

A final improvement (not actually visible in the diagram) will be to arrange that the link $L \cup R$, the union of $L :=$ the "triangular" meridians m_i and $R :=$ the "rectangular" components of Fig. 5.2 is a rather weak link. Recall that in our construction of M_2 a large number K was fixed. Using Lemma 2.8, weakly homotope the "rectangular punctures" R of ε in ∂M, extending ε continuously in the normal direction, so as to make the link $L \cup R$ have vanishing $\bar{\mu}$-invariants of length $\leqq K$. This creates new $(\varepsilon, \varepsilon)$-intersections which are removed as before. Choose the number K such that

$$K > (1 + \#_\delta)n + 2c \tag{5.1}$$

where $\#_\delta$ is the number of group elements in $\pi_1 N$ represented in the double points of δ and c is the number of components of R. In moving R' to R many new double points of ε (and thus δ) are created and we have no way to estimate the number. However since $\partial M \subset N$ is π_1-null and we count group elements only in the group $\pi_1 N$, at most $2 \cdot \binom{c}{2}$ distinct group elements arise from these double points, and precisely these same elements arise when the $(\varepsilon, \varepsilon)$-intersections are transformed into (δ, δ)-intersections. Thus it is possible to pick the number K early in the construction of M_2 (as we did) for the necessary value can be estimated from the number of components ($= c$) of $\mathring{\varepsilon}' \cap M$ and the original double points of δ.

We can now turn to the construction of an M_3 satisfying $\partial M_3 \cong \partial M_2 = \partial \mathrm{neib}(M \cup \delta \cup \varepsilon)$ and $M_3 \simeq \vee S^1$. The first point to address is that in Fig. 5.1 we formally changed 0-framed 2-handles μ_i to 1-handles μ_i (note the dot) as in the passage from Fig. 4.1 to Fig. 4.2. To justify this we must find slice disks for these components. This requires an $S^2 \times S^2$-stablization which we will remove later.

Consider the visible (band-like) Seifert surfaces $T = \{T_1, \ldots, T_n\}$ for $\mu_1, \ldots, \mu_n$ in Fig. 5.1. These meet $m_1, \ldots, m_n$ dually (δ_{ij}) and would be suitable for cancellation if they were disks. Push the interiors of T_i slightly into the contractible manifold Z. Let (α, β) denote the obvious symplectic basis; this is actually the one fixed earlier on δ. Because of the 0-framings, we may perform along $\{\alpha\}$ an abstract surgery of pairs on (Z, T^{pushed}) to obtain $(Z \# S^2 \times S^2\text{'s}, D)$ where D is a disjoint union of n imbedded slice disks for $\mu_1, \ldots, \mu_n$. Morally, M_2^+ is M_2 stablized by these surgeries. To be precise, Fig. 5.1 finally has a meaning as a 4-manifold M_2^+ since we now have a place, $Z^+ := Z \# S^2 \times S^2$'s, to locate the slices indicated by the dots on $\mu_1, \ldots, \mu_n$. We assume without loss of generality that each $\delta_1, \ldots, \delta_n$ has at least one selfintersection as shown in Fig. 5.1. This enables us to compute $\pi_1(M_2^+)$.

Lemma 5.1. *$\pi_1(M_2^+)$ is freely generated by the 1-handles of Fig. 5.1 (We exclude here the pseudo-1-handles $\mu_1, \ldots, \mu_n$ which bound slices in Z^+ and are not technically 1-handles.)*

Proof. For each $\mu_i, 1 \leqq i \leqq n$, choose a 2-handle from a hyperbolic pair generated by resolving a selfintersection of δ_i. Cancel this 2-handle with μ_i. Literally this means filling in the slice under μ_i with the handle. Fig. 5.1 loses the canceled components and the partner of the canceled 2-handle is joined to a parallel copy of μ_i by a band. In this reconstituted Fig. 5.1 the attaching regions of all 2-handles can be *homotoped* off the standard disks spanning the 1-handles. These homotopies, because of the "rectangles" in Fig. 5.1, do not exist in $\Sigma \backslash$(1-handles) but rather in $Z^+ \backslash$(1-handle slice disks). After these homotopies we see a homotopy equivalent space whose fundamental group is as claimed. □

The homology of Z^+ is conveniently represented in the complement of D by singular spheres of types A and B. An A-sphere has "northern" hemisphere the core-D^2 bounding (a parallel of) α provided by surgery and "southern" hemisphere a null homotopy of α descending further into Z. A B-sphere is made from the torus of length 2ε normal vectors of $T \subset Z$ restricted to β by removing an annular neighborhood around the "lowest" longitudinal copy of β and gluing in two "southern" null homotopies of β descending into Z. In order to get the above torus (and thus the B-sphere) inside Z^+, the support of the abstract surgery (turning Z into Z^+) should be in an ε-neighborhood of α.

Because μ_i and $m_i, 1 \leqq i \leqq n$, cancel homologically, the second homology of M_2^+ is freely generated by singular spheres of types A, B, and C where the spheres of type C are constructed as follows: Consider the n singular punctured spheres $V_1, \ldots, V_n$ in M_2^+ made by capping off "southern" null homotopies for $m_1, \ldots, m_n$ in Z with "northern" core disks of the attached 2-handles. Each V_i acquires a single puncture where it crosses the slice disk for μ_i. To construct a sphere of type C resolve the puncture on a parallel copy of some V_i by tubing along μ_i to the longitude of the attaching circle of one of the 0-framed 2-handles linking μ_i which were introduced (as Hopf links) to resolve the clasps (coming from the double points of δ), see Fig. 5.1. In this way we get two spheres of type C for each such Hopf link.

If all the "southern" null homotopies Δ in Z (more precisely $Z \backslash$collar ∂Z) used in the construction of A, B, and C are disjoint then

(5.2) $A \cup B \cup C$ is a π_1-null collection of spheres representing the basis of a hyperbolic form in $\pi_2(M_2^+)$.

Actually, less disjointness is required to obtain (5.2). Let $\Delta = \Delta_A \cup \Delta_B \cup \Delta_C$ be the null homotopies needed to form "southern" portions for classes A, B, and C respectively. We make Δ_C from many parallel copies of a collection $\Delta_{C,0}$ of disjointly immersed disks described below. Each C-hyperbolic pair derives from a double point of δ. The set $\mathscr{S}$ of elements of $\pi_1 N$ represented by these double points has cardinality $\#_\delta$, see (5.1). Assume that $\Delta_{C,0}$ are disjoint null homotopies in Z for $(1 + \#_\delta)$ parallel copies of $L = \{m_1, \ldots, m_n\}$, divided into

$\Delta_{C,0} = \Delta^1_{C,0} \cup \Delta^2_{C,0}$ where $\Delta^1_{C,0}$ consists of null homotopies on one copy of L and $\Delta^2_{C,0}$ consists of null homotopies on $\#_\delta$ copies of L. Arbitrarily divide each C-hyperbolic pair into a first and second partner. This division produces, on southern hemispheres, $\Delta_C = \Delta^1_C \cup \Delta^2_C$. Assume all the Δ^1_C are made from near parallel copies of $\Delta^1_{C,0}$ and also that Δ^2_C consists of near parallel copies of $\Delta^2_{C,0}$. It is important to make precise which copy of $\Delta^2_{C,0}$ these additional null homotopies should be parallel to. The choice is made by looking at the group element in $\pi_1 N$ of the corresponding double point of δ and using a bijection of $\mathscr{S}$ with the parallels of L. Finally assume that all null homotopies Δ_A and Δ_B are disjoint from each other and $\Delta_{C,0}$ (and thus Δ_C).

With these conditions (5.2) continues to hold, but the required collection of disjoint null homotopies $\Delta_0 := \Delta_{C,0} \cup \Delta_A \cup \Delta_B \subset \Delta$, is much smaller. In fact,

$$\operatorname{card}(\Delta_{C,0}) \leqq (1 + \#_\delta)n\,.$$

Furthermore, $\partial\Delta_{C,0}$ is made from parallel copies of the link L and $\partial(\Delta_A \cup \Delta_B)$ is made from at most 2 parallel copies of the link R by "fusing" (banding together) certain pairs of components (two copies of R are needed to produce $\partial\Delta_B$). Thus Δ_0 may be constructed if the link made by taking $(1 + \#_\delta)$ parallel copies of L and 2 parallel copies of R is 4D-homotopically trivial in Z. But by Lemma 2.8 this is assured by the condition that $L \cup R$ has vanishing $\bar{\mu}$-invariants of length $\leqq (1 + \#_\delta)n + 2c$, compare (5.1).

We finish as in Sects. 3 and 4: Use [FQ, Chapter 6] to s-cobord $A \cup B \cup C$ to disjointly imbedded hyperbolic pairs and remove these by surgery. The result M_3 of surgery is homotopy equivalent to a wedge of circles with generators corresponding to the meridians to all the one-handles in Fig. 5.1 (excluding $\mu_1, \ldots, \mu_n$). The manifold $N' = (M \backslash M_1) \cup M_3$ is shown to be simply homotopy equivalent to N^+ by the direct analogs of diagram 4.4, to compute π_1, and lines (4.3) and (4.4) to compute homology. □

6. The normal bordism

This section gives the proof of Corollary 1.2. We suppress the boundaries ∂N and ∂X from the notation since they play only a small role.

Given a degree 1 normal map $f : N \to X$ inducing π_1-isomorphism and kernel K_2 represented by M, construct N^+ and N' as in Sect. 5. Consider following diagram

$$\begin{tikzcd}
& \nu(N) \arrow[r, "F"] \arrow[d] & \xi \arrow[d] \\
& N \arrow[r, "f"] \arrow[d, "g"] & X \arrow[d, bend left, "h_2"] \\
\eta \arrow[r] & N' \arrow[r, "h_4"'] & N^+ \arrow[u, "h_1"] \arrow[l, bend right, "h_3"']
\end{tikzcd}$$

Above (h_1, h_2) and (h_3, h_4) are two pairs of (simple) homotopy inverses. The bundle ξ has structure group TOP, $\nu(N)$ is the TOP stable normal bundle to N, and $\eta := (h_4 \circ h_1)^* \xi$.

We next check that η is isomorphic to the normal bundle of N' (with the isomorphism covering $\mathrm{id}_{N'}$). To compare pull $\nu(N')$ and ξ back to N^+; $h_1^* \xi$ and $h_3^* \nu(N')$ are identical over $N \setminus M_1$ so the difference is measured in $H^i(M_1, \partial M_1; \pi_i \mathrm{BTOP}) \cong H_{4-i}(M_1; \pi_i \mathrm{BTOP})$. Since M_1 has the integral homology of a wedge of circles and $\pi_3 \mathrm{BTOP} = 0$ the only possible difference lies in $H_0(M_1; \pi_4 \mathrm{BTOP}) \cong \mathbb{Z}$. But this obstruction is associated to the first Pontrjagin class or (eight times) the signature and must vanish since $\mathrm{sign}(X) = \mathrm{sign}(N')$. It follows that $h_1^* \xi \cong h_3^* \nu(N')$ and therefore $\eta \cong \nu(N')$.

Certainly $g: \begin{smallmatrix}\nu(N)\\ \downarrow\\ N\end{smallmatrix} \to \begin{smallmatrix}\eta\\ \downarrow\\ N'\end{smallmatrix}$ is equivalent to $f: \begin{smallmatrix}\nu(N)\\ \downarrow\\ N\end{smallmatrix} \to \begin{smallmatrix}\xi\\ \downarrow\\ X\end{smallmatrix}$, so to prove Corollary 1.2 – that f is normally cobordant to a (simple) homotopy equivalence – it suffices to show that g is normally cobordant to $\mathrm{id}_{N'}$. Since $\mathrm{sign}(N) = \mathrm{sign}(N')$ the normal invariant n of g lies in $H^2(N'; \mathbb{Z}/2)$. Since g and $\mathrm{id}_{N'}$ agree over $\overline{(N' \setminus M_3)}$ the image $i^*\ (n) = 0 \in H^2(N' \setminus M_3; \mathbb{Z}/2)$. On the other hand, combining the exact sequence of the pair with Lefschetz Duality and excision we obtain

$$\begin{array}{ccccc} H^2(N', \overline{N' \setminus M_3}; \mathbb{Z}/2) & \longrightarrow & H^2(N'; \mathbb{Z}/2) & \xrightarrow{i^*} & H^2(\overline{N' \setminus M_3}; \mathbb{Z}/2) \\ \| & & & & \\ H^2(M_3, \partial M_3; \mathbb{Z}/2) & \xrightarrow{\cong} & H_2(M_3; \mathbb{Z}/2) = 0 & & \end{array}$$

This shows that i^* is an injection. Thus $n = 0 \in H^2(N'; \mathbb{Z}/2)$ proving Corollary 1.2. □

References

[CF] A. Casson, M.H. Freedman: Atomic surgery problems. Cont. Math. **35**, 181–199 (1984)

[C] T. Cochran: Derivative's of links: Milnor's Concordance invariants and Massey products, Vol. **84**, # 427, Memoirs of the A.M.S., Providence, Rhode Island, 1990

[D] W. Dwyer: Homology, Massey products and maps between groups. J. Pure Appl. Algebra **6**, 177–190 (1975)

[F1] M.H. Freedman: The topology of 4-dimensional manifolds. J. Diff. Geom. **17**, 357–453 (1982)

[F2] M.H. Freedman: A new technique for the link slice problem. Invent. Math. **80**, 453–465 (1985)

[F3] M.H. Freedman: Whitehead$_3$ is a slice link. Invent. Math. **94**, 175–182 (1988)

[FQ] M.H. Freedman, F. Quinn: The topology of 4-manifolds. Princeton Math. Series 39, Princeton, NJ, 1990

[FT] M.H. Freedman, P. Teichner: 4-Manifold Topology I: Subexponential Groups (to appear)

[K] Kojima: Milnor's $\bar{\mu}$-invariants, Massey products and Whitney's trick in 4 dimensions. Top. Appl. **16**, 43–60 (1983)

[L] X.S. Lin: On equivalence relations of links in 3-manifolds. Preprint (1985)

[MKS] W. Magnus, A. Karras, D. Solitar: Combinatorial group theory. Dover Publications, New York, 1966

[M1] J. Milnor: Link Groups. Ann. Math. **59**, 177–195 (1954)
[M2] J. Milnor: Isotopy of links, Algebraic geometry and Topology. Princeton Univ. Press, pp. 280–306, 1957
[St] J. Stallings: Homology and central series of groups. J. Algebra **2**, 1970–1981 (1965)
[T] V.G. Turaev: Milnor's invariants and Massey products, Studies in topology II. Zap. Nauc. Sem Leningrad (LOMI) **66**, 189–203, 209–210 (1976)
[V] M. Vaughan-Lee: The restricted Burnside problem. London Math. Soc. Monographs New Series 8, Clarendon Press, 1993