Invent. math. 122, 509–529 (1995)

Inventiones mathematicae

4-Manifold topology I: Subexponential groups

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Oblatum 20-II-1995 & 26-V-1995

Abstract. The technical lemma underlying the 5-dimensional topological s-cobordism conjecture and the 4-dimensional topological surgery conjecture is a purely smooth category statement about locating π_1-null immersions of disks. These conjectures are theorems precisely for those fundamental groups ("good groups") where the π_1-null disk lemma (NDL) holds. We expand the class of known good groups to all groups of subexponential growth and those that can be formed from these by a finite number of application of two operations: (1) extension and (2) direct limit. The finitely generated groups in this class are amenable and no amenable group is known to lie outside this class.

Introduction

The 4-dimensional surgery and 5-dimensional s-cobordism theorems are known to be true in the topological category for a certain class of groups called "good groups". The class of previously known good groups coincided with the elementary amenable groups [F2] which is the class of groups containing all groups of polynomial growth and closed under (1) extension and (2) direct limit. We expand the class of good groups to contain all groups of subexponential growth (and still closed under (1) and (2)). This is a nontrivial expansion because it is known that elementary amenable groups grow either polynomially or exponentially. Moreover, there are (uncountably many) groups of intermediate growth [G], i.e. groups that grow faster than any polynomial but slower than any exponential function. Thus they give new good groups. The known groups of intermediate growth are finitely generated but not finitely

The first author is supported by the IHES, the Guggenheim foundation and the NSF.
The second author is supported by the IHES and the Humboldt foundation.

presentable, one therefore has to go to the noncompact setting in order to get specific new applications of surgery or s-cobordisms.

In the smooth setting, the surgery and s-cobordism theorems are known to be false even in the simply-connected case [D]. This fact is extremely hard to prove. The seemingly obvious failure of the Whitney-trick (i.e. locating *embedded* 2-disks on null homotopic circles) is not at all easy to nail down since in the proofs of the surgery and s-corbordism theorems every Whitney circle (for which one needs an embedded 2-disk to change the algebraic intersection information into actual geometric intersections) comes equipped with some additional data which can be used for geometric constructions. For example, in the simply-connected case these "Whitney data" suffice to construct a Casson handle on the Whitney circle [C] which is an infinite union of thickenings of immersed disks proper homotopy equivalent to an open 2-handle $D^2 \times \mathring{D}$ By the work of the first author [F1] a Casson handle is actually homeomorphic to the open 2-handle and thus a topological Whitney trick can be performed. But even in the presence of a fundamental group π, the Whitney data might suffice to locate a (smooth) Casson-handle (and thus a topological 2-handle). We now formulate the precise conditions on π under which one does find the desired Casson-handle. As made precise in the definition below, a group π is good if the following lemma holds for it.

π_1-Null Disk Lemma (NDL). *Let (G^c, γ) be a Capped Grope of height 2 and $\varphi : \pi_1 G^c \to \pi$ a group homomorphism. Then γ bounds a disk $\Delta : D^2 \to G^c$ which is π_1-null under φ.*

Here a Capped Grope G^c of height 2 is the standard thickening of a specified 2-complex (made from two surface stages and one layer of disks, the "caps") and γ is a special circle in ∂G^c. The simplest example is given in Fig. 0.1. (Also see Fig. 1.1.)

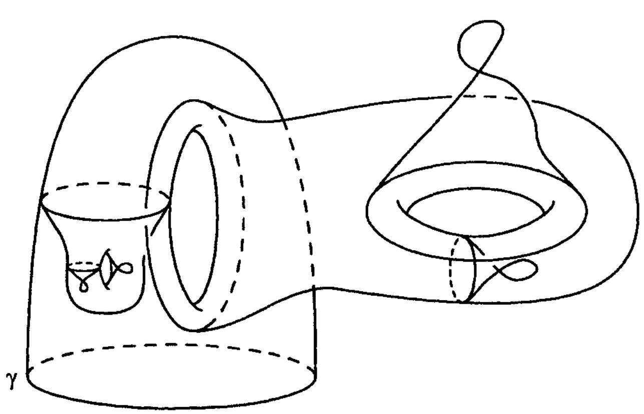

Fig. 0.1.

A disk $\Delta : D^2 \to M^4$ is π_1-null if all loops ℓ in $\Delta(D^2)$ are contractible in M, and Δ is π_1-null under $\varphi : \pi_1 M \to \pi$ if all $\varphi(\ell) = 1$.

Definition. *A group π is* **good** *if the* (NDL) *holds for all $\{(G^c, \gamma), \varphi\}$ as above.*

Theorem 0.1. *Groups of subexponential growth are good.*

Remarks. (i) The (NDL) is a purely smooth lemma because Capped Gropes are smooth 4-manifolds and any π_1-null disk can be perturbed to become a smooth π_1-null disk (small loops are contractible). Note that a topologically embedded disk is π_1-null but in general it cannot be perturbed to become smoothly embedded.

(ii) As explained above, the topological surgery and s-cobordism theorems hold for good groups, (compare with [FQ]). Since noncompact formulations are absent from the standard texts we formulate the simplest such application.

s-cobordisms. Let (W^5, M_0, M_1) be a 5-dimensional (noncompact) topological h-cobordism. Assume W^5 has a product structure near infinity; i.e. for some compactum $K \subset W$ there is a homeomorphism

$$(W \backslash K, M_0 \backslash K, M_1 \backslash K) \approx (M \times I, M \times \{0\}, M \times \{1\}) .$$

Then the torsion $\tau(W^5) \in \mathrm{Wh}(\pi_1(W^5))$ is defined. If $\tau(W^5) = 0$ and $\pi_1(W^5)$ is good then some restriction of the product structure to a smaller neighborhood of infinity extends to a product structure over all of W^5.

Surgery. Let $f : (M^4, \partial M) \to (X, \partial X)$ be a degree 1 normal map from a (noncompact) topological 4-manifold to a 4-dimensional Poincaré duality pair (using cohomology with compact supports and locally finite homology). Assume f is already a [proper] homotopy equivalence near infinity; i.e. for some compacta $K_1 \subset K_2 \subset X$ there is a $g : (X \backslash K_2, \partial X \backslash K_2) \to (M^4, \partial M)$ so that $f \circ g$ is relatively [properly] homotopic in $X \backslash K_1$ to $\mathrm{id}_{X \backslash K_2}$. Then the Wall obstruction θ to normally bording f, rel boundary, to a [proper] homotopy equivalence is defined in $L_4(\pi_1 W)$. If $\pi_1 W$ is good then the vanishing of θ implies that f has a compactly supported normal bordism, rel boundary, to a [proper] homotopy equivalence $f' : (M', \partial M') \to (X, \partial X)$.

Since the fundamental group of a Capped Grope is free, one immediately sees that the (NDL) holds for all groups if and only if it holds for free groups. At the time of writing there is no proof that the (NDL) requires any fundamental group restriction. However, there is a program for locating an obstruction in the free case. This program does not involve gauge theory, the most powerful known source of restrictions on smooth 4-manifolds and subsurfaces, but perhaps it should. We wish to advertise the (NDL) as a fascinating open problem.

The paper is organized as follows: In Sect. 1 we give the definitions of (Capped) Gropes, the growth rate of a group, and outline the proof of Theorem 0.1. This proof uses improvements of two basic constructions previously used to verify the (NDL) for groups of polynomial growth: Grope height raising and Contraction/Pushoff. In Sect. 2 we describe the new "linear" Grope height raising. Section 3 contains the new "exponential" Contraction/Pushoff which is derived from computations in the (nilpotent) theory of "colored" link homotopy. We would like to point out that both these generalizations are best possible in the sense that if they could be improved by an $\varepsilon > 0$ (which is made precise in Sect. 1) then the (NDL) would hold for the simplest "model" gropes (and the identity homomorphism φ) where we strongly believe it to be false.

We thank Richard Stong for many useful comments on earlier versions of this paper.

1. Outline of the proof of Theorem 0.1

The *derived series* of a group G is defined by $G^{(1)} := [G,G], G^{(h+1)} := [G^{(h)}, G^{(h)}]$ for $h \geqq 1$. There is an equivalent geometric formulation in terms of maps of *gropes*.

Definition. *A* **grope** *is a special pair (2-complex, base circle). A grope has a* **height** $h \in \mathbb{N}$. *For* $h = 1$ *a grope is precisely a compact oriented surface* Σ *with a single boundary component which is the base circle. A grope of height* $(h+1)$ *is defined inductively as follows: Let* $\{\alpha_i, i = 1, \ldots, 2\,genus\}$ *be a standard symplectic basis of circles for* Σ. *Then a grope of height* $(h+1)$ *is formed by attaching gropes of height* h *to each* α_i *along the base circles.*

Thus a grope of height h has h surface stages and its fundamental group is freely generated by the circles of the symplectic basis for all the surfaces in the top stage. To be precise, these circles have to be connected by arcs to a base-point (on the base circle). The π_1-elements represented by these circles do depend on these arcs. For example, if all the surfaces in the grope have genus 1 then there are $2^{(h-1)}$ top stage surfaces each giving 2 free generators. We leave the following lemma as an exercise to the reader.

Lemma 1.1. *For a space X, a loop* γ *lies in* $\pi_1(X)^{(h)}$ *if and only if* γ *bounds a map of a grope of height* h *(i.e.* γ *becomes the base circle of that grope). Moreover, the height of a grope* (g, γ) *is the maximal* $h \in \mathbb{N}$ *such that* $\gamma \in \pi_1(g)^{(h)}$.

As one can see from Fig. 1.1. every grope (g, γ) embeds properly (i.e. boundary goes to boundary) into $(\mathbb{R}^3_+, \mathbb{R}^2 \times \{0\})$ mapping γ to the unit circle in $\mathbb{R}^2$.

Definition. *A* **Grope** *is a special pair (4-manifold, base circle). It is obtained from a grope* $g \subset \mathbb{R}^3_+$ *by first thickening* g *into a 3-manifold in* $\mathbb{R}^3_+$ *(with a preferred annulus around* γ *in the boundary) and then crossing with the interval* $I = [0,1]$ *to get a 4-manifold G. It has a preferred solid torus (around the base circle* γ*) in the boundary.*

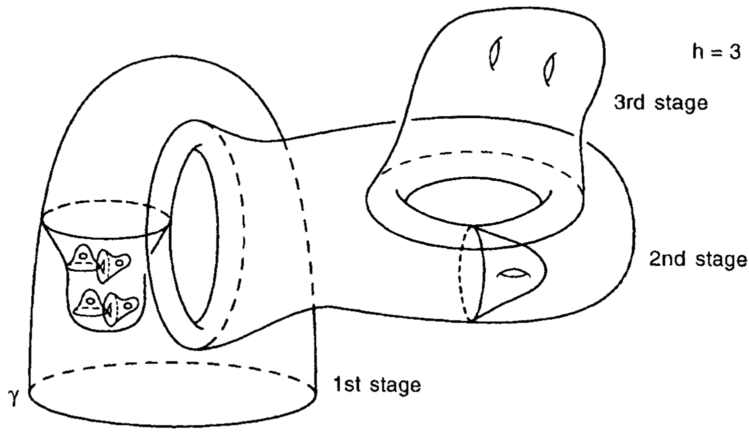

Fig. 1.1.

Note that the 3-dimensional thickening of a grope does not depend on the choice of the embedding into $\mathbb{R}^3_+$. This is obvious for all the surfaces Σ_i in the grope (they all give $\Sigma_i \times I$). But the gluings are only along the preferred annuli and are thus uniquely determined by the gluings of the cores of these annuli (= base circles) and thus by the grope. Therefore, a Grope G is the unique "untwisted" thickening of a grope g (which constitutes the *spine* of G). In particular, all the π_1-data carry over from g to G.

Definition. *A* **capped grope** *is a special pair* (*2-complex, base circle*)*. It is obtained from a grope g by first attaching 2-cells* (=: **caps**) *to the circles of the symplectic basis for all the surfaces in the top stage of g. Then one introduces finitely many double points among caps.*

We do not allow double points between a cap and the grope. The fundamental group of a capped grope is freely generated by the *double point loops*. These start at the base point (on the base circle) and run up through the grope to the attaching circle of a cap. Then they pass through exactly one double point of this cap bringing them back to the same cap or to some other cap. Finally, the loops run down the grope to the base point without passing through further double points. Note that the π_1-element represented by a double point loop does only depend on the double point it uses (and the direction it passes through it) because the inclusion map $\pi_1(\text{grope}) \to \pi_1$ (capped grope) is trivial. Changing the direction in which a cap-cap double point is passed changes the double point loop to its inverse.

Note that a capped grope without double points also embeds properly into the pair $(\mathbb{R}^3_+, \mathbb{R}^2 \times \{0\})$.

Definition. *A* **Capped Grope** *is a special pair* (*4-manifold, base circle*)*. It is obtained from a capped grope $g^c \subset \mathbb{R}^3_+$ without double points by first thickening it into a 3-manifold in $\mathbb{R}^3_+$, then crossing with I, and finally introducing finitely many plumbings among caps.*

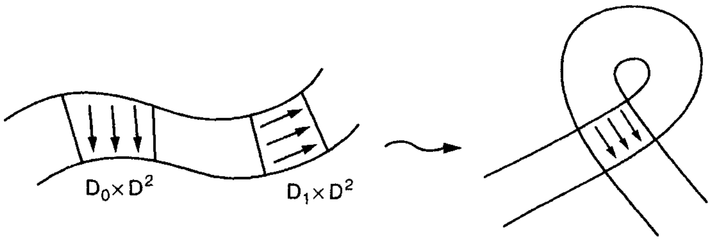

Fig. 1.2.

A *plumbing* is an identification of $D_0 \times D^2$ with $D_1 \times D^2$ where D_0, D_1 are subdisks of some caps of g^c and the second factor D^2 refers to the thickened normal directions. In complex coordinates the plumbing may be written as $(z_1, z_2) \mapsto (z_2, z_1)$ or $(z_1, z_2) \mapsto (\bar{z}_2, \bar{z}_1)$ creating either a positive or negative double point on the caps of g^c.

If one just thickened a capped grope (with double points) in $\mathbb{R}^3$, one would completely loose the feature of transversality at the double point. But one might still think of a Capped Grope G^c as the unique "untwisted" thickening of a capped grope g^c (which constitutes the spine of G^c). In particular, the π_1-information discussed above carries over from g^c to G^c.

Definition. *The* **height** *of a Grope, capped grope or Capped Grope is the height of the underlying grope.*

Remark. Our terminology slightly differs from the one in [FQ]. There a capped grope has no double points at all while a Capped Grope is the image of a "properly immersed capped grope" in [FQ]. Finally, the term "grope" was not explicitly defined in [FQ] but it was sometimes abused to mean a capped grope. The object "grope" in our sense, which we now think of the fundamental object, was referred to as the "body of the capped grope".

As an exercise in getting used to the definitions we next want to prove (compare [F]) the following

Lemma 1.2. *The class of good groups is closed under* (1) *extension and* (2) *direct limit.*

Proof. Let $1 \to N \to \pi \to Q \to 1$ be a group extension and assume that N and Q are good groups. Let (G^c, γ) be a Capped Grope of height 2 and $\varphi : \pi_1 G^c \to \pi$ a group homomorphism. Using any kind of "Grope height raising" (e.g. Theorem 2.1 or [FQ, 2.7.]) we may construct a Capped Grope $(G_1^c, \gamma) \subseteq (G^c, \gamma)$ of height 4. Use the composition $\pi_1 G_1^c \to \pi_1 G^c \to \pi \to Q$ (and the fact that Q is good) at the 2 top stages of G_1^c to find disks inside these top stages which map trivially into Q. Thus (G_1^c, γ) contains a Capped

Grope (G_2^c, γ) of height 2 such that the following diagram commutes:

$$\begin{array}{ccc} \pi_1 G_2^c & \longrightarrow & N \\ {\scriptstyle i_*}\downarrow & \varphi & \downarrow \\ \pi_1 G^c & \longrightarrow & \pi \end{array}$$

Since N is good G_2^c (and thus G^c) contains a disk on γ which maps π_1-null under φ. Thus π is a good group.

To prove (2) just observe that π_1(Capped Grope) is finitely generated and thus any homomorphism to a direct limit $\pi = \varinjlim \pi_{(i)}$ will eventually factor through a map $\pi_{(N)} \longrightarrow \pi$. This implies that π is good if all groups $\pi_{(N)}$ are good. □

We can now start the outline of the proof of Theorem A. Let (G^c, γ) be a Capped Grope of height 2. Instead of directly trying to find a π_1-null disk on γ in G^c, we first construct Capped Gropes $(G_h^c, \gamma) \subseteq (G^c, \gamma)$ of arbitrary large height h. We need some control on the π_1-elements that occur in the new Capped Gropes. This is measured as follows: Any double point loop in G_h^c can be expressed as a unique reduced word in the double point loops (and their inverses) of G^c. Let $\ell(h)$ be the maximum length of these words. Then our Linear Grope Height Raising (Theorem 2.1) produces for each height $h \geqq 2$ Capped Gropes $(G_h^c, \gamma) \subseteq (G^c, \gamma)$ with $\ell(h) < 200h$, i.e. the maximum word length $\ell(h)$ is bounded by a linear function in the height h.

Remark. At the end of this section it will be clear that if one found a "sublinear" Grope height raising, i.e. one in which $\ell(h)$ was bounded by a function that grew slower than linear in h, then one could prove that all free groups, and thus all groups, are good. The same remark applies to the "model" gropes (indicated in Fig. 1.1 with $\varphi = \mathrm{id}_{\pi_1 G^c}$ in mind) if one could replace our bound $2^h - 1$ by the bound 2^h in the following result, proved in Sect. 3.

Exponential contraction/Pushoff (Corollary 3.5). *Let $\varphi : \pi_1 G^c \longrightarrow \pi$ be a group homomorphism with (G^c, γ) a Capped Grope of height h. If φ maps the double point loops of G^c to a set of cardinality at most $2^h - 1$ in π then G^c contains a disk on γ which is π_1-null under φ.*

This result is exponentially better than the old Contraction/Pushoff which had the number h instead of $2^h - 1$ as an upper bound. The main reason for getting this improvement is the group theoretic rebracketing fact $\pi^{(h)} \subseteq \pi^{2^h}$, i.e. the h-th derived group of π lies inside the 2^h-th term of the lower central series of π. In Sect. 3 we will explain in detail why the nilpotent quotients rather than the solvable quotients of π measure the existence of π_1-null disks. Roughly speaking, finger moves on disks in 4-manifolds correspond to link homotopies in 3-manifolds and this theory can be computed from nilpotent quotients of the fundamental group.

Proof of Theorem 0.1. Let $\varphi : \pi_1 G^c \longrightarrow \pi$ be a group homomorphism with (G^c, γ) a Capped Grope of height 2. Let $S \subseteq \pi$ be the (finite) image of the double point loops of G^c under φ. Use linear Grope height raising to find Capped Gropes $(G_h^c, \gamma) \subseteq (G^c, \gamma)$ of height h whose maximal word length in π (relative to S) is smaller than $200h$. If π grows subexponentially then (for large h) the number of distinct group elements in π which are images of double point loops in G_h^c is less than b^{200h} for any $b > 1$. Setting $b < 2^{(1/200)}$ exponential contraction/pushoff implies that G_h^c eventually contains a disk on γ which is π_1-null under φ. □

We close this section by reminding the reader of the necessary definitions concerning the growth of groups, compare [M2]: If S is a finite subset of a group π, one may define a *growth function* $f_S : \mathbb{N} \to \mathbb{N}$ by sending a "radius" $r \in \mathbb{N}$ to the number of distinct group elements in π which can be written as words of length $\leqq r$ in the elements of S and their inverses.

Definition. (i) *π has* **polynomial growth** *if for all finite subsets $S \subseteq \pi$ there exists a degree $d \in \mathbb{N}$ such that $f_S(r) \leqq r^d$ for all sufficiently large r.*

(ii) *π has* **subexponential growth** *if for all finite subsets $S \subseteq \pi$ and all basis $b > 1$ one has $f_S(r) \leqq b^r$ for all sufficiently large r.*

Note that for a finitely generated group it suffices to check the conditions on any generating set S. However, from our point of view S is always given in a natural way as the image of some finite set of double point loops. Therefore, the above definition captures exactly the property of π that we need.

2. Linear grope height raising

To fix notation let g (resp. g^c) be the 2-dimensional spine of a Grope (resp. Capped Grope) of height $k \geqq 2$ and let the corresponding capital letters G and G^c denote the standard untwisted 4-dimensional thickenings. In this section, we require that the number of selfintersections of each cap is zero when counted by sign. Since selfintersections can be introduced locally this represents no loss of generality. We remind the reader that caps are permitted to have transverse intersections with (1) themselves, (2) other caps. Thus body-body intersections in g and cap-body intersections are prohibited. Sometimes, to emphasize the distinguished circle γ in g we will write $(g, \gamma), (G^c, \gamma)$ etc.

The term *grope height raising* (compare [F2]) refers to the location of new Capped Gropes $(G_h^c, \gamma) \subset (G^c, \gamma)$ of large height $h \gg k$. Given $(g_h^c, \gamma) \subset (G^c, \gamma)$ and a base point $* \in \gamma$ each double point loop (see Sect. 1) of g_h^c determines a reduced *word* w in $\pi_1(G^c)$ which is well defined up to inverse. We call w a word rather than element because $\pi_1(G^c)$ has a preferred free basis (consisting of double point loops of G^c). This basis can be described dually by marking with a letter either one of the two canonical solid tori τ at each plumbing (these are associated to the double points of G^c) and then reading that signed letter as a (base pointed) loop crosses that solid torus. Let ℓ be the maximum length among reduced words w which express the double point loops of g_h^c in terms of the double point loops of G^c. The goal is to produce embeddings

$(g^c_h, \gamma) \subset (G^c, \gamma)$ where word length ℓ grows as slowly as possible relative to the height h. The previous best estimate [Sto] produces arbitrarily large values of h with $\ell <$ const $\cdot\, h^\delta$, $\delta \approx 1.94$, that is, length grows slightly slower than height squared. Similar power laws with larger δ's are implicit in [F2] and [FQ].

Theorem 2.1 (Linear grope height raising). *Given a Capped Grope (G^c, γ) of height $k \geqq 2$. Then there exist Capped Gropes $(G^c_h, \gamma) \subset (G^c, \gamma)$ of height h and word length ℓ satisfying $\ell < 200\,h$ for all values of $h \geqq k$.*

It is convenient to describe the constructions of geometric topology as if smooth structures and even Riemannian metrics are present. It is routine to remedy this small abuse using the foundational results [FQ, Chapters 8 and 9] on smoothings, normal bundles and transversality for topological 4-manifolds.

Proof. The core of the proof is an inductive argument for raising height from k to $k+1$ to $\dots k+r$ for any positive integer r. Bracketing this core are "warm up" and "warm down" steps whose numerical effects are summarized in Table 2.1 below.

If $k = 2$, the "warm up" begins by raising height to 3. The grope height raising algorithm of [FQ, Exercise 2.7] produces a Capped Grope of height 3 and with word length of its double points $\leqq 5^2 = 25$ (not 7^2 as suggested by a slight miscounting in [FQ, 2.8]) in the fundamental group of the original G^c_2. Next non-disjoint immersed dual spheres $\{S\}$ to the second stage surfaces $\{\Sigma_2\}$ of the grope are constructed. The collections $\{S\}$ and $\{\Sigma_2\}$ are geometrically paired δ_{ij} and $\{S\}$ has no other intersections with the (modified) Capped Grope. A sphere S is formed as follows. Choose a "branch" b of g^c issuing from a symplectic basis curve p_i or q_i on the base surface Σ_1 and let b' be a parallel copy. "Contract" the caps on b' to get b'' and then "push off" all caps of g^c from $\{b''\}$ (see [FQ, 2.3]). Now the sphere S is formed by joining two copies of b'' by an annulus. According to the counting rules explained in [FQ], the length of double point words in $\{S\}$ is increased by at most a factor of 3 and the length of the double point words of $(g^c_k)_{\text{pushed off}}$ is increased by at most a factor of 2. Thus the new g^c has word length $\leqq 50$ and the dual spheres have word length $\leqq 75$.

Table 2.1.

grope height	word length of g^c	
$k \geqq 2$	1	initial state
$k \geqq 3$	$25 \cdot 2 = 50$	made duals $\{S\}$
$h = k + r$	$2(50r) + 50$	core construction
$h = k + r$	$2[2(50r) + 50] + 75 < 200\,h$	removed intersections using $\{S\}$

In the above table, the last ("warm down") step is necessary because the core construction gives "capped gropes" with cap-grope intersections which are not allowed.

As we start the core construction we have a Capped Grope $G^c := G_k^c$ of height $k \geqq 3$ and word length $\leqq 50$. The inductive set up is a Grope G_{h-1} of height $h-1 \geqq k$ and an imbedding $(G_{h-1}, \gamma) \hookrightarrow (G^c, \gamma)$. One works with the spines, proceeding from g_{h-1} to g_h in two steps: Step 1 raises height by one but creates (illegal) double points between the top stage surfaces and stages at various heights above

$$Y := \text{base stage} \cup \text{second stage surfaces of } g^c =: \Sigma_1 \cup \{\Sigma_2\}\ .$$

The subspace Y is protected in the construction so that the dual spheres $\{S\}$ will remain geometrically dual to $\{\Sigma_2\}$, the second stages, and disjoint from everything else. Step 2 only changes the new top stage to remove double points of the top stage with itself and with earlier stages (and in the process increases the genus of the surfaces in that top stage). Every application of Step 1 involves grabbing some obvious surface (often a disk) so, formally, the presence of these obvious surfaces is an inductive hypothesis which must be propagated in passing from g_{h-1} to g_h. These surfaces are of three types:

(1) the initial caps $g^c \backslash g$, used only to get from $g = g_k$ to g_{k+1},

(2) meridinal disks M_α and M_β described below, and

(3) "parallel" copies of stages of lower height which have been previously constructed.

Every application of Step 2 is accomplished by a finite number of moves called a *lollipop move* or a *double lollipop move*. These moves change some surface to avoid an intersection; but we do not rename the modified surface in our notation. At an intersection between a new stage and an old stage the new stage is moved. At a new stage-new stage intersection, we will see that at least one of the two sheets of intersection lies on a stage which is topologically a disk. The Step 2 algorithm moves one such sheet to eliminate each new stage-new stage intersection. The caps $g_h^c \backslash g_h$, necessary to define $\ell(g_h)$, are constructed last and in two steps. The preliminary caps cross all grope stages above Y (stages $\geqq 3$); these are refined to caps disjoint from the grope using the duals $\{S\}$.

The first application of Step 1 simply attaches the caps to g to get $g_{k+1} := g^c$. Thus g^c is regarded as a "singular" uncapped grope of height $k+1$ by noting that disks are surfaces. It is singular because when regarded as grope stages the crossings in the caps $g^c \backslash g$ are impermissible.

Every surface stage Σ in a grope except for top surface stages has a symplectic basis of circles $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ where g is the genus of Σ, along which higher stages have been attached. We consider tori T_{α_i} (resp. T_{β_i}), $i = 1, \ldots, g$, which are normal ε- (resp. 2ε-) circle bundles to Σ in G^c restricted to α_i (resp. β_i) where ε is a small positive number depending on Σ. Notice that all these tori are disjoint. Suppose x is a double point with local sheets $S_\gamma \subset \Sigma_\gamma$ and $S_\beta \subset \Sigma_\beta$, and that $S_\gamma \subset \Sigma_\gamma$ is the sheet selected for modification by the Step 2 algorithm, and finally that Σ_β is attached to Σ along β. Symmetrically, if the sheet not selected for modification attaches to Σ along α then lable it by Σ_α and interchange α and β (also ε and 2ε) in the next paragraph.

The *lollipop move* replaces a disk neighborhood S_γ of x with a copy of T_α, made by taking normal ε-circle bundle over a parallel displacement (depending on x) of α in Σ, boundary connected summed to S_γ along a tube which is the normal $\varepsilon/10$-bundle of Σ_β in G^c restricted to an arc $\lambda \subset \Sigma_\beta$ from $T_{\alpha(\text{displaced})} \cap \Sigma_\beta$ to x. Denote the lollipop by L_α. It is the punctured torus made by attaching the tube to $T_{\alpha(\text{displaced})}$.

In Fig.2.1 the lollipop L_α is shown in bold. Surfaces drawn as 2-dimensional lie (at least locally) in the 3-space model. Surfaces drawn as 1-manifolds are being observed in a cross section or a "time frame". Surfaces will only be represented in cross section when their global behavior is quite simple so that the figure is informationally complete. For example, to see all of $T_{\alpha(\text{displaced})}$ imagine the two circles indicated in Fig. 2.1 converging to α(displaced) in $\Sigma \times \{\text{present time}\}$ both forward and backward in time. The curve α(displaced) lies slightly below α and parallel to the plane of α.

We specify that the initial application of Step 2 replaces all intersection in the $(k+1)^{\text{st}}$ stage (formerly $g^c \backslash g$) using lollipop moves.

In subsequent applications of Step 1 we must specify which surfaces we grab and what the intersections are. Each lollipop $L_{\alpha(\beta)}$ contains a meridinal circle $m_{\alpha(\beta)}$ to which we attach the meridinal disk $M_{\alpha(\beta)}$ and a longitude $\ell_{\alpha(\beta)}$, see Fig. 2.1, to which we attach a parallel (in the 3-space present of Fig. 2.2) copy Σ'_α of the surface stage $\Sigma_{\alpha(\beta)}$. The rest of the top stage is constructed from parallel copies of previous stages. By the framing assumption on G^c these surface stages and its parallels are disjoint. Note that such surfaces "moving up" the original Grope G will collide with the $M_{\alpha(\beta)}$ and that $M_{\alpha(\beta)}$ will intersect older (lower) stages as well. The reader may expect that the next application of Step 2 will use lollipop moves on the meridinal disks to remove the grope-grope intersection points. This is part of the picture, but there is a difficulty. The lollipop moves, if repeated, produce a branch heading inexorably down G: namely resolving $M_{\alpha(\beta)} \cap \Sigma_i$ (Σ_i being a surface at stage i in G^c) with a lollipop capped by a meridinal disk meeting a Σ_{i-1} leads toward the base of G which is Σ_1. There is no way of using a lollipop to remove a point of $M_{\alpha(\beta)} \cap \Sigma_1$.

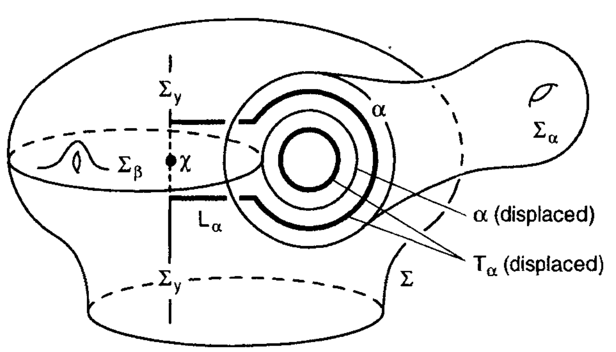

Fig. 2.1.

The solution is to use the *double lollipop move* to resolve any intersection of a current top stage meridinal disk with a third stage surface Σ_3. This move turns the branch of the growing grope back "upward" to avoid the base Σ_1 and also to avoid the second stages $\{\Sigma_2\}$ for which we have duals $\{S\}$ which need to be protected from additional intersections.

The *double lollipop move* replaces an intersection x of a top stage Σ_{t} with a third story surface Σ_3. This move replaces a small disk neighborhood $S_{\mathrm{t}} \subset \Sigma_{\mathrm{t}}$ of x with L_α/Σ_α. The notation assumes Σ_3 attaches to β (otherwise reverse the labels α and β), L_α is the lollipop made from T_α as described above. (Here and later we simplify the notation "α(displaced)" to α.) Moreover, Σ_α is the third story surface attached to α and finally L_α/Σ_α denotes the imbedded surface that results by surgering L_α along a parallel copy Σ'_α of Σ_α, i.e.

$$L_\alpha/\Sigma_\alpha := (L_\alpha \setminus \text{neib.}(l_\alpha)) \cup \text{two copies of } \Sigma'_\alpha .$$

Because we have assumed G^c is an untwisted thickening the two copies of Σ'_α are disjoint from each other and the original. We claim that if Step 2 is done with care the result is an embedding $g_h \hookrightarrow G^c$. Since the various lollipops are easily made disjoint the chief concern is disjointness of the copies of Σ_α and Σ_β from the stems of the lollipops. The solution to this positioning problem is described by Fig. 2.2 below. The same figure solves the closely related disjointness problems of the later applications of Step 2.

In later applications of Step 2 meridinal disks cross other top surface stages (growing up the original grope). For later convenience in counting the word length ℓ we make the necessary lollipop moves on these disks. In general, Step 2 removes double points by replacing sheets of top stage surfaces. It prefers to replace sheets lying on disks. The replacement is with a lollipop except in the case of intersection with a third stage surface Σ_3 in which case a double lollipop is used.

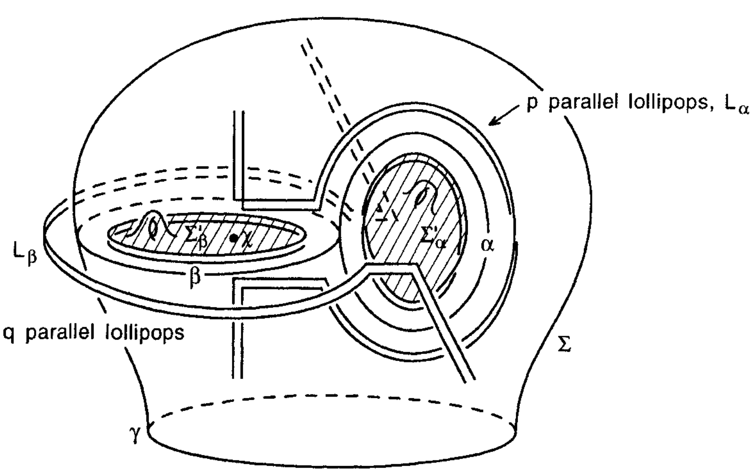

Fig. 2.2.

The operations of any given Step 1 or Step 2 are performed simultaneously. After finishing a Step 1, we have a new object, a singular grope, and all the operations in the next Step 2 (constructing $T_{\alpha(\beta)}$, normal tubes, parallel copies of surfaces etc.) take place in a small regular neighborhood of the new object. After finishing a Step 2, the latest grope g_h has been constructed. The next application of Step 1 adds meridinal disks $M_{\alpha(\beta)}$ and higher genus surfaces both of which leave the neighborhood of g_h. Fig. 2.2 can be used to verify (1) that, in a Step 2, two or more double-lollipop moves across a stage two surface can be made disjoint from each other and the first three stages of g_h and (2) that at any stage, the surfaces added to various parallel L_α's and L_β's can be made disjoint from each other and the lower stages. Property (2) is not crucial since the additional surface-surface intersection points could be replaced during the next Step 2 but (1) seems critical for Step 2 to make progress.

The fig. 2.2 indicates two parallel lollipops resolving two intersections with both Σ_α and Σ_β but these should be interpreted as parallel packets of p and q lollipops respectively. To visualize, the L_β's vary vertically and the L_α's vary into the page. The key point is that the parallel copies of $\Sigma_{\alpha(\beta)}$ which will variously surger or attach to $L_{\alpha(\beta)}$, in cases (1) and (2) above, slip through the stems of the lollipops $L_{\beta(\alpha)}$ without intersection.

After r cycles through steps 1 and 2 we have an imbedded grope (g_{k+r}, γ) of height $h := k + r$ into (G^c, γ). We now check the normal framings. A lollipop move on a $\pm$-self-intersection changes the relative Euler class by ± 2. This is best checked in the closed case, $S^2 \times S^2$, where adding the framed dual $0 \times S^2$ to $S^2 \times 0$ gives the diagonal. The restriction on cap selfintersections, that they sum to zero, shows that the totality of lollipop moves on the stages at height $k + 1$ leave the relative Euler class trivial so the neighborhood agrees with the standard model. Subsequent stages are clearly 0-framed. This allows us to thicken any g_{k+r} to a G_{k+r}. Write

$$G^c = \text{neighborhood}(\text{base } \Sigma_1) \cup G^c_{\text{left}} \cup G^c_{\text{right}}$$

where $G^c_{\text{left(right)}}$ is a disjoint union of genus(Σ_1) Capped Gropes of height $k - 1$. Although any individual surface stage of g_{k+r} may be very "spread out" and reside partially in all three pieces of G^c, the natural symplectic basis $\mathscr{B}$ of simple closed curves on the top stages of g_{k+r} is disjoint from Σ_1 so we may write $\mathscr{B} = \mathscr{B}_{\text{left}} \coprod \mathscr{B}_{\text{right}}$, with $\mathscr{B}_{\text{left(right)}} \subset G_{\text{left(right)}}$. Since $G_{\text{left(right)}}$ is π_1-null in $G^c_{\text{left(right)}}$ we can cap off each element of $\mathscr{B}_{\text{left(right)}}$ by an immersed disk δ in $G^c_{\text{left(right)}}$, which will be specified in more detail below. Note already that the left-right distinction implies that δ does not intersect the base Σ_1. Without loss of generality we may spin δ near $\partial\delta$ (see [FQ, Chapter 1]) to ensure that the relative Euler class of $\nu_{\delta \hookrightarrow G^c}$ vanishes and also stabilize δ to ensure that it has equally many + and − double points. We set

$$g^\bullet_{k+r} := g_{k+r} \cup \{\delta\} .$$

The superscript • warns the reader that $g^\bullet_{k+r}$ does not satisfy the definition of a capped grope owing to the cap-grope intersections. These will be removed at the last step.

Let us next bound the word length $\ell(g^\bullet_{k+r})$. We must, mentally, mark the 1-dimensional submanifolds σ of $g^\bullet_{k+r}$ which are inverse images of the solid tori $\{\tau\}$ coming from the double points of G^c, see the beginning of this section. Then we count how many times a preferred generating loop of $g^\bullet_{k+r}$ crosses σ. The first cycle of Steps 1 and 2 produces unnested σ-loops at the stems of the attached lollipops. These loops are unnested in the sense that any point on the surface may be joined to the boundary by an arc which crosses the union of the σ's in only one point. In subsequent Steps 1, meridinal disks M satisfy $M \cap \{\tau\} = \emptyset$ but they may at the next Step 2 have a sheet replaced with a lollipop whose stem is a normal circle bundle over an arc $\lambda \subset \Sigma$ which crosses $\{\tau\}$ once. Thus these modified meridinal disks may also have unnested σ-loops. Also, Steps 1 grab parallels of earlier non-disk surfaces and these contain unnested σ-loops but the following Step 2 will not modify these surfaces at all since the Step 2 algorithm replaces only sheets lying on disks. Finally the double lollipop move is too low in the grope to produce σ loops. What this analysis shows is that any point on g_{k+r} can be joined by an arc $\omega \subset g_{k+r}$ to the base γ of g_{k+r} which crosses σ in at most r points. The universal cover $\widetilde{G^c}$ of G^c is built from $\pi_1(G^c)$-many copies of $\overline{G} := G^c$ with its plumbings separated, by replumbing distinct copies together according to the group element of the plumbing. In this picture ω lifts to an arc in $\widetilde{G^c}$ which traverses through at most $r+1$ copies of $\overline{G}$. Now suppose that ω connects a circle of the top symplectic basis $\mathscr{B}$ to γ and that this circle bounds its cap δ in $G^c_{\text{left(right)}}$. We may choose δ such that the lift $\tilde{\delta}$ lies in a single copy of $\overline{G}$. Consider two such assemblages $\omega \cup \delta$ and $\omega' \cup \delta'$. Their lifts each lie in at most $r+1$ copies of $\overline{G}$ so if they intersect the union lies in at most $2r+2$ copies of $\overline{G}$. This represents the longest possible lift of a preferred path in $g^\bullet_{k+r}$ and gives the estimate: $\ell(g^\bullet_{k+r}) \leqq 2r+1$, in terms of the modified g^c_k. Measuring with respect to the original g^c_k (the one before the "warm up" step) we get:

$$\ell(g^\bullet_{k+r}) \leqq 2(50r) + 50\,.$$

Recall that the caps are disjoint from Y. Push all cap-grope intersections in $g^\bullet_{k+r}$ down to the second stage $\{\Sigma_2\}$ by pushing the sheet on the cap and then remove these by tubing (ambient connected sum) into copies of the dual spheres $\{S\}$. Note that our r steps of grope height raising took place "above Y" leaving $\{S\}$ disjoint from all but the second stages and geometrically dual to them. This changes $g^\bullet_{k+r}$ to g^c_{k+r} since there are now only cap-cap intersections. Since $\{S\}$ are framed the untwisted framings of the caps are preserved by the tubings and g^c_{k+r} may be thickened to G^c_{k+r}. The longest words associated to the cap-cap double points of g^c_{k+r} result from tubing two double points of word length $2(50r)+50$ into dual spheres S_1 and S_2 which intersect with a double point of length 75. Thus recalling that $h = k+r$ and $k \geqq 2$ we get

$$\ell(g^c_h) \leqq 2[2(50r)+50]+75 < 200\,h\,. \qquad \square$$

3. Exponential contraction/Pushoff

We first collect the necessary group theoretic facts: The *lower central series* of a group G is defined by $G^1 := G$, $G^{k+1} := [G, G^k]$ for $k \geqq 1$. All nilpotent quotients of a group factor through some G/G^k, however these are not the only interesting nilpotent quotients. If G is provided with a set of normal generators, $G = \langle\langle x_1, \ldots, x_n \rangle\rangle$, then there is the Milnor quotient or *Milnor group*

$$MG := G/\langle\langle [x_i, x_i^y] \rangle\rangle, \quad i = 1, \ldots, n \text{ and } y \in G\,.$$

These quotients were introduced in J. Milnor's thesis [M1] for the study of *link homotopy*. A useful generalization of the Milnor quotient is the *colored Milnor group CMG*. Here a set of normal generators is provided in batches or *colors* determined by the first of two indices, i.e. $G = \langle\langle x_{ij} \rangle\rangle$, $1 \leqq i \leqq n$, $1 \leqq j \leqq n_i$, and x_{ij} are thought of as having the color i. Define

$$CMG := G/\langle\langle [x_{ij}, x_{ij'}^y] \rangle\rangle, \quad 1 \leqq i \leqq n, \ 1 \leqq j, j' \leqq n_i \text{ and } y \in G\,.$$

Therefore, n_i is the number of generators with the same color and if each $n_i = 1$ then $CMG = MG$. Moreover, n is the *number of colors* of G. The following lemma about colored Milnor groups uses the fact that we have chosen the elements x_{ij} to be normal generators for G.

Lemma 3.1. *CMG has nilpotency class* $\leqq n$, *i.e.* $CMG^{n+1} = \{1\}$.

Proof. We will use an induction on the number n of colors in CMG. If $n = 1$ the relations imply that all x_{ij}^y commute in CMG. Since by assumption these elements generate G it follows that all commutators vanish in CMG.

Now assume the statement holds for groups with $n - 1$ colors and let G be normally generated by x_{ij}, $1 \leqq i \leqq n$, $1 \leqq j \leqq n_i$. Define $A_i \trianglelefteq CMG$ to be the normal closure of the elements $x_{i1}, \ldots, x_{in_i}$. Since all the conjugates of these elements commute, A_i is abelian. Moreover, the intersection A of all A_i lies obviously in the center of CMG. Now consider a commutator $[x, y]$ with $x \in CMG$, $y \in CMG^n$. Since all quotients CMG/A_i are Milnor groups with $n - 1$ colors it follows by induction that $y \in A$ and thus y is central i.e. $[x, y] = 1$ in CMG. This shows that $CMG^{n+1} = \{1\}$. □

Corollary 3.2. *CMG is (finitely) generated by* x_{ij} *and is also finitely presented.*

Proof. The statement for the generators follows from the standard *rewriting process* in nilpotent groups: If a nilpotent group N is normally generated by x_i then it is also generated by these elements. One uses an induction on the nilpotency class of N based on the fact that $x \equiv y \bmod N^k$ implies $a^x \equiv a^y \bmod N^{k+1}$ for all $a, x, y \in N$. Moreover, the fact that N^k is generated by k-fold commutators $[n_{i_1}, \ldots, n_{i_k}]$ if the n_i generate N shows that N^k is finitely generated if N is (compare [V, 2.1.10]). An induction on the nilpotency class together with the fact that a (central) extension of finitely presented groups is finitely presented implies that a finitely generated nilpotent group is also finitely presented. □

Definition. *A* **colored link** $L = (\ell_{ij}) \subset S^3$, $1 \leqq i \leqq n$, $1 \leqq j \leqq n_i$ *is a link whose components are batched into colors as above. A* **colored link homotopy** *is a motion of the link during which only components with identical colors may cross.*

There is a generalization of Milnor's theorem on link homotopy [M1] and its extension to the 4-ball [L].

Lemma 3.3. *If γ is a loop in $S^3 \backslash L$, a colored link complement, then the following conditions are equivalent:*

(1) *γ is trivial in $CM\pi_1(S^3 \backslash L)$ (use one meridian m_{ij} per component ℓ_{ij} for the definition of this colored Milnor group. These obviously generate $\pi_1(S^3 \backslash L)$ normally).*

(2) *γ is null homotopic in $S^3 \backslash L'$ for some link L' colored homotopic to L.*

(3) *There exists properly immersed annuli $A_{ij} : S^1 \times I \to S^3 \times I$ restricting to L at top and bottom and a proper disk $\Delta : D^2 \to S^3 \times I$ bounding γ with the following disjointness properties: $A_{ij} \cap A_{i'j'} = \phi$ for $i \neq i'$ and $\Delta \cap A_{ij} = \phi$ for all i, j.*

Proof. (1) $\Rightarrow$ (2) is a straightforward generalization of Milnor's original argument: One writes γ as a product of colored Milnor relations $[m_{ij}^g, m_{ij'}^h]$ and one reduces the number of these relations one at a time using the following figure (or Figs. 1, 2 in [M1, p. 183]). (This move is sometimes referred to as a (colored) *elementary homotopy*).

Note that we allow components ℓ_{ij} and $\ell_{ij'}$ of the same color to cross which is precisely reflected in the choice of the colored Milnor relations.

(2) $\Rightarrow$ (3) The annuli A_{ij} and the disk Δ may be constructed in three steps: First run the (colored) link homotopy from L to L' keeping γ fixed. (The time parameter is thought of as the I-direction to get annuli in $S^3 \times I$, one of them being $\gamma \times I \subseteq S^3 \times I$.) Then keep L' fixed and run the null homotopy for γ in $S^3 \backslash L'$ to cap off the annulus $\gamma \times I$ into Δ. Finally, run the first homotopy backwards, with γ out of the picture, to get back to the link L. The conditions on the colored homotopies translate precisely into the disjointness properties in (3).

(3) $\Rightarrow$ (1) Let $A \subseteq S^3 \times I$ be the union of the annuli A_{ij}. There is a homomorphism

$$i_* : CM\pi_1(S^3 \backslash L) \to CM\pi_1(S^3 \times I \backslash A)$$

of colored Milnor groups (formed w.r.t. the meridians m_{ij}) coming from the inclusion $i : S^3 \backslash L \to S^3 \times I \backslash A$. Since the meridians m_{ij} normally generate both fundamental groups, i_* is onto by Corollary 3.2. The existence of Δ (disjoint from A) shows that $i_*(\gamma) = 1$ and thus it suffices to show that i_* is injective. This will be proved using a theorem of Stallings (which can be derived by iterated applications of the 5-term exact sequence for group extensions involving the terms of the lower central series).

Stallings' Theorem [St]. *If $\sigma \to \pi$ is a homomorphism of groups inducing an isomorphism on H_1 and an epimorphism on H_2 then the induced maps $\sigma/\sigma^k \to \pi/\pi^k$ are isomorphisms for all $1 \leqq k < \omega$.*

The above inclusion i induces an isomorphism on H_1 because by Alexander duality the meridians m_{ij} freely generate $H_1(S^3 \setminus L)$ as well as $H_1(S^3 \times I \setminus A)$. By Lemma 3.1 all colored Milnor groups are nilpotent and thus it suffices to show that i_* is an epimorphism in H_2. Let us for convenience introduce finger moves (see [FQ, 1.5]) between A_{ij} and $A_{ij'}$, to achieve

$$\pi_1(S^3 \times I \setminus A) \cong CM\pi_1(S^3 \times I \setminus A)$$

(without changing the disjointness properties of the A_{ij}). Note that a finger move as above introduces precisely the relation $[m^g_{ij}, m_{ij'}] = 1$ into π_1 of the complement [C] and by Corollary 3.2 finitely many finger moves give the desired isomorphism. Now $H_2(S^3 \times I \setminus A)$ maps onto $H_2(CM\pi_1(S^3 \times I \setminus A))$ and it is Alexander dual to $H_1(A, \partial)$. This group is freely generated by (a) arcs $a_{ij} \subset A_{ij}$ connecting top and bottom of the boundary of each annulus A_{ij} (and missing the double points) and (b) the double point loops, i.e. circles on A passing through precisely one double point (as in the case of capped gropes). Obviously, the arcs a_{ij} are Alexander dual to *boundary tori* $T_{ij} \subset S^3 \setminus L$ which can be obtained as the boundary of $S^3 \setminus N(L)$ where $N(L)$ is a small open tubular neighborhood of the link L in S^3. Moreover, if a neighborhood of the k^{th} (transverse) intersection point p_k is parametrized by $\mathbb{R}^2 \times \mathbb{R}^2$ then $S^1 \times S^1$ is called the *linking torus* T_k of p_k. It lies in the complement of A and links $\delta(k, \ell)$ with the double point loops for p_ℓ (and does not link the arcs a_{ij}). Therefore, the linking tori T_k are Alexander dual to the double point loops. We conclude that $H_2(S^3 \times I \setminus A)$ is (freely) generated by the tori T_{ij} and T_k. By construction, the boundary tori T_{ij} come from $H_2(S^3 \setminus L)$ and thus it suffices to show that the linking tori T_k are in the image of i_*. If p_k is an intersection point between A_{ij} and $A_{ij'}$ then a symplectic basis on T_k is given by meridians m^g_{ij} and $m^h_{ij'}$ for some $g, h \in \pi_1(S^3 \times I \setminus A)$. But since i_* is onto, there must be a relation $[m^{\bar{g}}_{ij}, m^{\bar{h}}_{ij'}]$ in $CM\pi_1(S^3 \setminus L)$ which maps homologically onto T_k. Thus i_* is indeed onto in H_2 and by Stallings' theorem i_* is thus an isomorphism. This concludes the proof of Lemma 3.3. □

Remark. An elementary link homotopy as in Fig.3.1. is the moving picture for "half" a finger move. To get a full finger move one has to run the elementary link homotopy forward and then backward, allowing circles like γ to slip off the link components in the middle of the move. This explains the presence of the Milnor relation in π_1 of the 4-dimensional complement after the finger move.

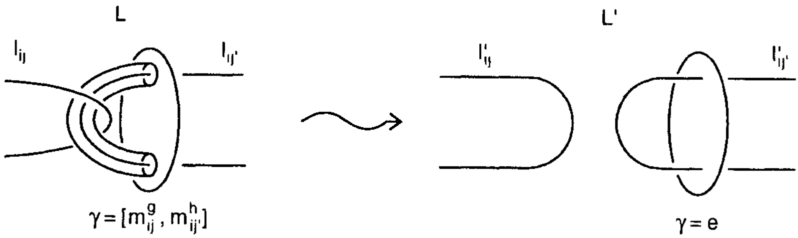

Fig. 3.1.

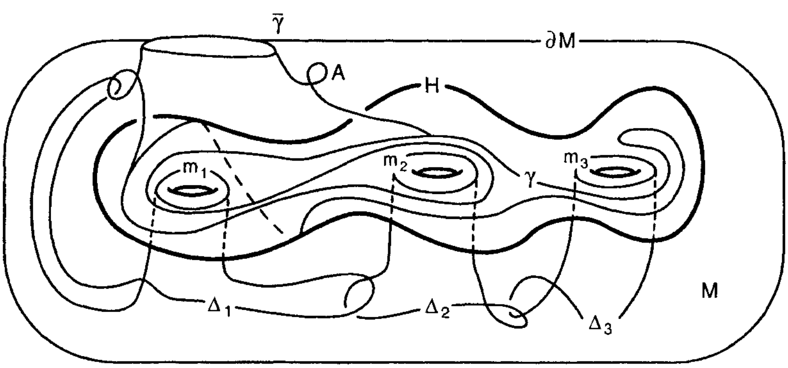

Fig. 3.2.

We are now ready to prove the main result of this section. Let $H := \natural_{i=1}^{g}(S^1 \times D^3)$ be a 1-handlebody imbedded in the interior of a 4-manifold M with boundary ∂M. Let γ be a simple closed curve in ∂H and A a π_1-null general position annulus in $\mathring{H}$ which connects γ to a simple closed curve $\overline{\gamma}$ in ∂M. Let m_i, $i = 1, \dots, g$ be standard representatives for $\pi_1(H)$ lying in $\partial H \setminus \gamma$. We use this notation to suggest the m_i are the meridians to the unlink in a standard 1-handle diagram for H. Now suppose that the m_t bound general position disks $\Delta_i \subset M \setminus H$, $i = 1, \dots, g$. Picking a base point in H and an (arbitrary) ordering of the sheets at each intersection point between Δ_i and Δ_j or Δ_i and A, each double point is associated with a group element in $\pi_1(M)$, the double point loop. The situation is pictured in Fig. 3.2.

Lemma 3.4. (Exponential contraction/Pushoff) *In the above notation suppose that γ lies in $\pi_1(H)^k$, the k-th term of the lower central series of $\pi_1(H)$, and that the cardinality of the set of double point loops in $\pi_1 M$ is at most $k-1$. Then $\overline{\gamma}$ bounds a π_1-null disk in M.*

Proof. Formally add untwisted 2-handles h_i to the $m_i \subset \partial H$ to produce a 4-ball $B = H \cup \{h_i\}$ containing standard slices {co-core (h_i)} for the unlink $\{\ell_i\}$ where $\ell_i \subset \partial H$ is parallel in S^3 to a meridian to m_i. Note that the 1-handlebody H also contains slices for $\{\ell_i\}$.

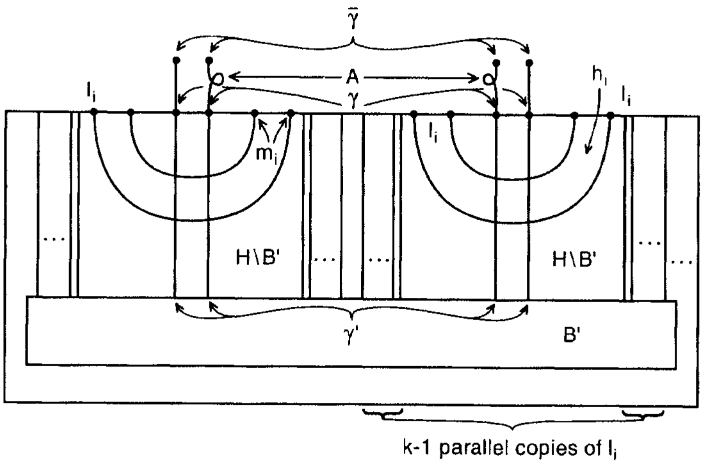

Fig. 3.3.

Take $k-1$ parallel copies of each ℓ_i to form a $(k-1)g$ component unlink J in ∂B^4. Erect cylinders on $J \cup \gamma$ using a standard product structure near ∂B^4. This can be done so that (1) the inner boundaries of the cylinders are lower than the slices $\{h_i\}$ and (2) only the cylinder C_γ on γ meets the slices $\{h_i\}$. Let $B' := B^4 \setminus$ collar and denote by J' resp. γ' the inner boundaries of the cylinders.

Now suppose $k-1$ colors are used to label the parallels of $\{\ell_i\}$ so that each subscript has a parallel of each color. From $\gamma \in \pi_1(H)^k$ it follows that $\gamma' \in \pi_1(\partial B' \setminus J')^k$ and therefore by Lemmas 3.1 and 3.3 that there is a colored null homotopy of $J' \cup \gamma'$ in B'. (This means that each component of $J' \cup \gamma'$ bounds a map of a disk into B' and that endowing γ' with a k-th distinct color, intersection between disks are permitted if and only if their boundaries have the same color.) Let Δ_γ be the null homotopy of γ in B^4 obtained by gluing the cylinder C_γ to the disk map bounding γ' in B'.

Recall that B^4 was only a formal construct, but there is an obvious quotient map $B^4 \xrightarrow{\pi} H \cup \{\Delta_i\}$. The map π is obtained by collapsing each co-core(h_i) to one point and then introducing the double point structure of $\{\Delta_i\}$ into the resulting $\{\text{core}(h_i)\}$.

The composition:

$$\Delta_\gamma \longrightarrow B^4 \xrightarrow{\pi} H \cup \{\Delta_i\} \xrightarrow{inc} M$$

may be perturbed near the inverse image of $\{\Delta_i\}$ to obtain a disk $\Delta'_\gamma \subset M$ whose double point loops in $\pi_1(M)$ are identical to those of $\{\Delta_i \cup A\}$ (since H is π_1-null in M) and thus compress to a set of cardinality $< k$. By examining the figure below, the reader may see that the $k-1$ colored null homotopies for parallels of the ℓ_i may be capped off (along the slices for $\{\ell_i\}$ in H) to obtain (inside H) $k-1$ π_1-null dual spheres [FQ, 1.9] to each Δ_i. These $k-1$ spherical duals each have a different color, and satisfy the

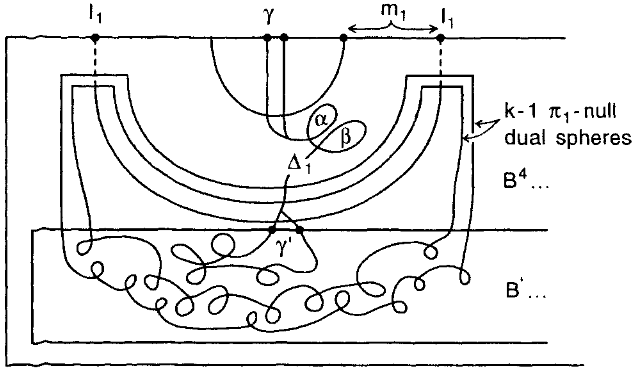

Fig. 3.4.

condition that no two spheres of different color intersect. Moreover, the entire image of the spheres of any fixed color i is π_1-null since H is π_1-null in M.

Now use the formula $(\alpha, \beta) \mapsto \alpha\beta^{-1}$ for the transition of the group elements when a double point loop α and a double point loop β are removed by sum with intersecting π_1-null duals. We see that by assigning a distinct color to each group element in the set of double point loops of $\Delta'_\gamma \cup A$, and this is possible since this set has cardinality $\leqq k-1$, summing with copies of the spherical duals will produce a π_1-null disk Δ''_γ on γ. Finally, we obtain a disk on $\bar{\gamma} \subseteq \partial M$ by taking $\Delta''_\gamma \cup_\gamma A$. This disk is π_1-null since A was π_1-null to start with and we made the double point loops of Δ''_γ-A intersections trivial in $\pi_1 M$. □

The standard application of Lemma 3.4 works as follows: Take a capped grope of height h and thicken it to a 4-manifold (not necessarily the untwisted thickening!). The grope body thickens to a 1-handlebody H with the base circle γ in ∂H. Define the manifold M to be $H \cup_{\partial H} (\partial H \times I)$, i.e. add a little collar to H. The annulus A is just $\gamma \times I$ and the disks Δ_i have their interiors disjoint from H. Translating the height h to the nilpotency class $\gamma \in \pi_1(H)^{2^h}$ gives the following

Corollary 3.5. *Let $\varphi : \pi_1 g^c \to \pi$ be a group homomorphism with (g^c, γ) a capped grope of height h. If φ maps the double point loops of g^c to a set of cardinality at most $2^h - 1$ in π then any 4-dimensional thickening of g^c contains a disk on γ which is π_1-null under φ.*

The reader should convince him/herself that Lemma 3.4 works without changes if one adds a group homomorphism $\varphi : \pi_1 M \to \pi$ to the assumption and statement. This concludes the proof of Theorem 0.1. □

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