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Differential forms and 0-dimensional supersymmetric field theories

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Abstract. We show that closed differential forms on a smooth manifold X can be interpreted as topological (respectively Eudlidean) supersymmetric field theories of dimension $0|1$ over X. As a consequence, concordance classes of such field theories are shown to represent de Rham cohomology. The main contribution of this paper is to make all new mathematical notions regarding supersymmetric field theories precise.

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Contents

1. Introduction

Two of us [ST] spent the last years to find a precise notion of supersymmetric Euclidean field theories of (super) dimension $d|\delta$ and relate it to certain multiplicative cohomology theories. We showed that in dimension $1|1$ the relevant cohomology theory is K-theory, see also [HST] for a more precise account. We also conjectured

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that for dimension $2|1$ one gets elliptic cohomology, or more precisely, the cohomology theory TMF of topological modular forms. In this paper we fill the gap in dimension $0|1$ by showing that de Rham cohomology arises in this easiest case. It is a very interesting question whether cohomology theories arise from other values of $d|\delta$.

The geometric cocycles we actually get from EFTs (which is short for *Euclidean field theories*) of dimension $0|1$ are closed differential forms, just like vector bundles with connection can be used to get Euclidean field theories of dimension $1|1$, see [D]. Our goal remains to show that EFTs of dimension $2|1$ are cocycles for TMF.

Our results are consistent with the formal group point of view towards (complex oriented) cohomology theories, where the additive formal group gives ordinary rational cohomology, the multiplicative group gives K-theory and the formal groups associated to elliptic curves lead to elliptic cohomology.

Unfortunately, the precise definition of EFTs is very involved, so we won't repeat it here but refer instead to our survey [ST2]. We will summarize in Section 4 the necessary information for dimension $0|1$ where the fancy definitions reduce to well-known notions. As a consequence, most of the results in the current paper could have been written in a simpler language. However, the main point of the paper is to show how the more difficult notions, also valid in higher dimensions, reduce to these well-known simple things in dimension zero.

In our definition, an EFT has a *degree* $n \in \mathbb{Z}$ which is related to the central charge as well as to the degree of a cohomology class. If X is a smooth manifold, we also define EFTs *over* X, which can be thought of as *families* of EFTs parametrized by X. In this case, the degree n can be generalized to a *twist* over X which relates very well to twisted cohomology but will not be discussed in this paper. An EFT over X should be thought of as a *geometric object* over X. This is best explained by our main result below, Theorem 1, which says that a closed differential form over X can be interpreted as a $0|1$-dimensional EFT over X and vice versa.

Like differential forms or vector bundles with connection, EFTs over X of the same dimension $d|\delta$ can be added and multiplied. Addition preserves the degree n, whereas multiplication adds degrees as expected. Moreover, $d|\delta$-dimensional EFTs over a manifold X of degree n form a category $d|\delta\text{-EFT}^n(X)$ (in fact, a d-category, an issue we will ignore in this paper) and can be *pulled back* via smooth maps: a smooth map $f: Y \to X$ determines a functor

$$f^*\colon d|\delta\text{-EFT}^n(X) \to d|\delta\text{-EFT}^n(Y)$$

and these functors compose strictly. We call two EFTs $E_0, E_1 \in d|\delta\text{-EFT}^n(X)$ *concordant* if there exists a field theory $E' \in d|\delta\text{-EFT}^n(X \times \mathbb{R})$ and $\varepsilon > 0$ such that $E' \cong p_1^*(E_0)$ on $X \times (-\infty, \varepsilon)$ and $E' \cong p_1^*(E_1)$ on $X \times (1-\varepsilon, \infty)$.

We observe that concordance gives an equivalence relation which can be defined for geometric objects over manifolds for which *pullbacks* and *isomorphisms* make sense. By Stokes' Theorem two closed n-forms on X are concordant if and only if they represent the same de Rham cohomology class; two vector bundles with connections are concordant if and only if they are isomorphic as vector bundles (disregarding the connections). Passing from an EFT over X to its concordance class forgets the geometric information while retaining the topological information. We will write $d|\delta\text{-EFT}^n[X]$ for the set of *concordance classes* of $d|\delta$-dimensional supersymmetric EFTs of degree n over X.

Theorem 1. *For smooth manifolds X, there are natural group isomorphisms*

$$0|1\text{-EFT}^n(X) \cong \begin{cases} \Omega^{\mathrm{ev}}_{\mathrm{cl}}(X), & n \textit{ even}, \\ \Omega^{\mathrm{odd}}_{\mathrm{cl}}(X), & n \textit{ odd}, \end{cases}$$

where $\Omega^{\mathrm{ev}}_{\mathrm{cl}}(X)$, respectively $\Omega^{\mathrm{odd}}_{\mathrm{cl}}(X)$, stands for the even, respectively odd, closed differential forms on X. These isomorphisms take the tensor product of EFTs to the wedge product of differential forms.

It follows that on concordance classes we get isomorphisms

$$0|1\text{-EFT}^n[X] \cong \begin{cases} H^{\mathrm{ev}}_{\mathrm{dR}}(X), & n \text{ even}, \\ H^{\mathrm{odd}}_{\mathrm{dR}}(X), & n \text{ odd}, \end{cases}$$

where $H^{\mathrm{ev}}_{\mathrm{dR}}(X)$, respectively $H^{\mathrm{odd}}_{\mathrm{dR}}(X)$, stands for the direct sum of the even, respectively odd, de Rham cohomology groups of X.

There is a beautiful interpretation of the Chern character form of a vector bundle with connection in terms of the map from $1|1$-dimensional to $0|1$-dimensional EFTs over X, given by crossing with the standard circle, see [Ha]. It is hence essential that the result above yields differential forms of varying degrees. However, differential forms of a specific degree n arise by forgetting the Euclidean geometry (on super-points) and working with TFTs (*topological* field theories) instead. Again, there are categories $d|\delta\text{-TFT}^n(X)$ of $d|\delta$-dimensional TFTs over a manifold X of degree n as well as their concordance classes $d|\delta\text{-TFT}^n[X]$. In fact, the following result is true for any *supermanifold* X, whereas one would have to use *pseudodifferential forms* on X to make Theorem 1 hold, see Sections 3.3 and 5.2.

Theorem 2. *For supermanifolds X, there are natural group isomorphisms*

$$0|1\text{-TFT}^n(X) \cong \Omega^n_{\mathrm{cl}}(X)$$

compatible with multiplication (tensor product of TFTs versus wedge product of differential forms). Moreover, concordance classes lead to isomorphisms

$$0|1\text{-TFT}^n[X] \cong H^n_{\mathrm{dR}}(X).$$

It is well known that the canonical map induces an isomorphism

$$H^n_{\mathrm{dR}}(X) \cong H^n_{\mathrm{dR}}(X_{\mathrm{red}}),$$

so we recover information only about the underlying reduced manifold. We will show in a forthcoming paper with Chris Schommer-Pries that Theorem 2 carries over to the case of twisted topological field theories which relate to differential forms, twisted by a flat vector bundle, and the resulting twisted de Rham cohomology.

From the above theorem, it is easy to recover the entire structure of de Rham cohomology from TFTs. What is missing is the boundary map in Mayer–Vietories exact sequences for a covering of X by open sets. Equivalently, we need to express the suspension isomorphisms

$$H^n(X) \cong H^{n+1}_{\mathrm{cvs}}(X \times \mathbb{R})$$

in terms of TFTs. Here the subscript 'cvs' means *compact vertical support* (in the $\mathbb{R}$-direction). This isomorphism is given by taking the product with a particular class $u \in H^1_{\mathrm{cvs}}(\mathbb{R})$, the *Thom class* for the trivial line bundle over $X = \mathrm{pt}$. Therefore, it suffices to express the condition of compact vertical support in terms of TFTs. However, this is easy since the first part of Theorem 2 describes the *cocyles* for de Rham cohomology in terms of TFTs and compactly supported cohomology is given by concordance classes of compactly supported cocycles. The second, cohomological, part of Theorem 2 alone would not be sufficient for this argument!

Similarly, it is the description of de Rham cocylces that enables us to use TFTs for building Eilenberg–MacLane spaces $K(\mathbb{R}, n)$: Consider *extended* standard k-simplices

$$\Delta^k_e := \{(t_0, \dots, t_k) \in \mathbb{R}^{k+1} \mid \textstyle\sum_{i=0}^k t_i = 1\},$$

which are smooth manifolds (without boundary or corners). The usual face and degeneracy maps are defined on these extended simplices and hence there are simplicial sets $K_\bullet(n)$ with k-simplices $0|1\text{-TFT}^n(\Delta^k_e)$.

Corollary 3. *The geometric realization $|K_\bullet(n)|$ is an Eilenberg–MacLane space of type $K(\mathbb{R}, n)$, where $\mathbb{R}$ has the discrete topology.*

Proof. This result is well known for any ordinary cohomology theory (with arbitrary coefficients), where one replaces TFTs by the relevant cocycles for the theory. The easiest way for us to prove the result is to state the following result from [MW], Appendix: Given any sheaf (on the big site of smooth manifolds) $F : \mathsf{Man}^{\mathrm{op}} \to \mathsf{Set}$, there are natural bijections for manifolds X as follows:

$$F[X] \cong [X, |F|].$$

The left-hand side denotes concordance classes as defined above and on the right-hand side $|F|$ is the geometric realization of the simplicial set $k \mapsto F(\Delta^k_e)$. Thus it suffices to show that $F := 0|1\text{-TFT}^n \cong \Omega^n_{\text{cl}}$ is a sheaf: This means that for any open covering $\{U_i\}$ of X, a closed differential form on X is the same thing as a collection of closed differential forms on U_i that agree on intersections $U_i \cap U_j$. This is clearly true if we work with all differential forms since these are sections of a vector bundle on X. It stays true for *closed* differential forms because the de Rham operator d is defined locally. □

The very last part of the above proof is our motivation for requiring the field theories in our definition to be *local*. Currently, we express this by saying that a $d|\delta$-dimensional field theory is a (symmetric monoidal) *d-functor* from a bordism d-category to a target d-category. The precise details of this definition for $d = 2$ are far from obvious.

This paper is organized as follows. After briefly introducing the category SM of supermanifolds, we give a detailed proof that for any supermanifold X, the odd tangent bundle ΠTX represents the inner Hom $\underline{\mathsf{SM}}(\text{spt}, X)$ of maps from the odd line spt to X. We actually think of the odd line as the easiest way of thickening a point to a supermanifold and, as a consequence, view this inner Hom as the *supermanifold of superpoints* in X. We abbreviate this generalized supermanifold by

$$\text{SP}X := \underline{\mathsf{SM}}(\text{spt}, X)$$

It is characterized by its T-points $\text{SP}X(T) = \underline{\mathsf{SM}}(T \times \text{spt}, X)$ for any supermanifold T. The notation $\text{SP}X$ for the superpoints in X is similar to writing $\text{L}X = \mathsf{Man}(S^1, X)$ for the generalized manifold of loops in X. Unlike this infinite dimensional case, the superpoints in X are representable by the supermanifold ΠTX which means roughly that a map spt $\to X$ is a point in X together with an odd tangent vector, see Proposition 3.1 for a precise statement. As a consequence, differential forms on an ordinary manifold X are the functions on this supermanifold of superpoints:

$$\Omega^* X \cong C^\infty(\Pi TX) \cong C^\infty(\text{SP}X).$$

This isomorphism explains both structures, the $\mathbb{Z}$-grading and de Rham d, on differential forms via symmetries of the superpoint as follows. The (inner) diffeomorphism supergroup of the superpoint spt is

$$\underline{\text{Diff}}(\text{spt}) \cong \mathbb{R}^{0|1} \rtimes \mathbb{R}^\times.$$

Here the translational part $\mathbb{R}^{0|1}$ induces an action on functions which turns out to be infinitesimally generated by d, see Lemma 3.4. The relation $d^2 = 0$ follows because translations commute with each other. Moreover, the $\mathbb{Z}$-grading of $\Omega^*(X)$ comes from the dilation action of $\mathbb{R}^\times$ on spt, see Corollary 3.7. Finally, the usual relations between dilations and translations show that d must have degree one. We claim no originality for these results since they seem to be well known to several authors. However, we exhibit detailed arguments (including the case where X is a supermanifold) in this paper because we could not find a reference that contained a *proof* of this result, rather than just stating it.

In Sections 4 and 5 we will review the notions of field theories as functors, pioneered by Atiyah, Kontsevich in the topological case and Segal in the conformal case. We will give precise meaning for such functors to be *smooth* by introducing family versions of the relevant bordism categories. Once this is done, it will be easy to generalize this notion to *supersymmetric* field theories which we then continue to study in the simplest case, that of dimension $0|1$. We will show in Proposition 5.5 that such field theories are functions on the quotient $\mathrm{SP}X/G$ on the site SM of supermanifolds. Here G is a subgroup of the (inner) diffeomorphism group $\underline{\mathrm{Diff}}(\mathrm{spt})$ *defining* the geometry on the superpoint spt, in the spirit of Felix Klein. For a *topological* field theory, $G = \underline{\mathrm{Diff}}(\mathrm{spt})$ consists of all diffeomorphisms and hence

$$0|1\text{-TFT}^0(X) \cong C^\infty(\mathrm{SP}X)^{\underline{\mathrm{Diff}}(\mathrm{spt})} \cong \Omega^* X^{\underline{\mathrm{Diff}}(\mathrm{spt})} \cong \Omega^0_{\mathrm{cl}}(X),$$

which is the case of Theorem 2 for degree 0. A *Euclidean* field theory is defined by setting $G = \underline{\mathrm{Iso}}(\mathrm{spt}) := \mathbb{R}^{0|1} \rtimes \{\pm 1\}$, allowing only translations and reflections as isometries of spt, but not all dilations. It follows that

$$0|1\text{-EFT}^0(X) \cong C^\infty(\mathrm{SP}X)^{\underline{\mathrm{Iso}}(\mathrm{spt})} \cong \Omega^* X^{\underline{\mathrm{Iso}}(\mathrm{spt})} \cong \Omega^{\mathrm{ev}}_{\mathrm{cl}}(X),$$

which is the case of Theorem 1 for degree 0. In both Theorems, the degree n case is obtained by defining *twisted* field theories in a way that functions on the quotient $\Pi TX/G$ are replaced by sections of a line bundle given by the twist. This is explained in Section 6 for the easiest possible twists, giving the degree n field theories.

These functions on the moduli spaces $\mathrm{SP}X/G$ should be thought of as the "partition functions" of our field theories. In dimension $0|1$ they obviously contain the entire information. It is also very natural that in the twisted case these are not functions but sections of certain line bundles on the moduli spaces, just like the (integral) modular forms arising as the partition functions of $2|1$-dimensional EFTs in [ST2].

Our appendix is again expository, we will survey the notion of *Grothendieck fibrations* $\mathsf{V} \to \mathsf{S}$. In the case where S is the category of manifolds respectively supermanifolds, this notion will be later used to define *smooth* respectively *supersymmetric* field theories. Here V will be family versions of various bordism categories, respectively versions of the target categories like Pic for the field theory. The only original result of the appendix is Proposition 7.13.

2. Quick survey of supermanifolds

Following a suggestion of the referee, we have removed our original survey from this paper, it now appears as [HST1] online in the *Manifold Atlas Project* (coordinated by Kreck and Crowley at the Hausdorff Institute for Mathematics in Bonn). We encourage knowledgable readers to improve and extend this survey.

We also recommend the beautiful survey article on supermanifolds by Deligne and Morgan [DM] or one of the standard references by Leites [L], Berezin [B], Manin [M], or Voronov [V]. For the simplest supermanifold $\mathbb{R}^{0|n}$ and its diffeomorphisms we recommend [KS].

We shall just summarize the absolutely basic notions, working with the ground field $\mathbb{R}$. A *superalgebra* is a monoidal object in the category of supervector spaces and is hence the same thing as a $\mathbb{Z}/2$-graded algebra. However, the interesting symmetry operators on this monoidal category implies that a superalgebra is *commutative* if for all homogenenous $a, b \in A$ we have

$$ab = (-1)^{|a||b|} ba,$$

a very different notion than a commutative $\mathbb{Z}/2$-graded algebra. The *derivations* of such a commutative superalgebra A are endomorphisms $D \in \mathrm{End}(A)$ satisfying the Leibniz rule:[1]

$$D(a \cdot b) = Da \cdot b + (-1)^{|D||a|} a \cdot Db.$$

Der A is a super Lie algebra with respect to the bracket operation

$$[D, E] := DE - (-1)^{|D||E|} ED.$$

A *supermanifold* M of dimension $p|q$ is a pair $(|M|, \mathcal{O}_M)$ consisting of a (Hausdorff and second countable) topological space $|M|$ together with a sheaf of commutative superalgebras $\mathcal{O}_M$ that is locally isomorphic to $\mathbb{R}^{p|q}$. The latter is the space $\mathbb{R}^p$ equipped with the sheaf $\mathcal{O}_{\mathbb{R}^{p|q}}$ of commutative superalgebras $U \mapsto C^\infty(U) \otimes \Lambda^*(\mathbb{R}^q)$.

The category SM of supermanifolds is defined by using morphisms of sheaves. There is a functor $\mathsf{SM} \to \mathsf{Man}$ that associates to a supermanifold M its *reduced manifold*

$$M_{\mathrm{red}} := (|M|, \mathcal{O}_M/\mathrm{Nil})$$

obtained by dividing out the ideal of nilpotent functions. By construction, this quotient sheaf gives a smooth manifold structure on the underlying topological space $|M|$ and there is an inclusion of supermanifolds $M_{\mathrm{red}} \hookrightarrow M$. Note that the sheaf of ideals $\mathrm{Nil} \subset \mathcal{O}_M$ is generated by the odd functions.

Example 2.1. Let E be a real vector bundle of fiber dimension q over the ordinary manifold X^p and $\Lambda^*(E^*)$ the associated algebra bundle of alternating multilinear forms on E. Then its sheaf of sections gives a supermanifold of dimension $p|q$, denoted by ΠE. For example, if E is the tangent bundle of an ordinary manifold X then the functions on ΠTX are just differential forms on X:

$$C^\infty(\Pi TX) \cong \Omega^*(X).$$

[1] Whenever we write formulas involving the degree $|\cdot|$ of certain elements, we implicitly assume that these elements are homogenous.

The following proposition gives two extremely useful ways of looking at morphisms between supermanifolds. We shall use the notation $C^\infty(M) := \mathcal{O}_M(M)$ for the algebra of (global) functions on a supermanifold M.

Proposition 2.2. *For $S, M \in \mathsf{SM}$, the functor C^∞ induces natural bijections*

$$\mathsf{SM}(S, M) \cong \mathsf{Alg}(C^\infty(M), C^\infty(S)).$$

If $M \subseteq \mathbb{R}^{p|q}$ is an open supersubmanifold (a domain), $\mathsf{SM}(S, M)$ is in bijective correspondence with those $(f_1, \dots, f_p, \eta_1, \dots, \eta_q)$ in $(C^\infty(S)^{\mathrm{ev}})^p \times (C^\infty(S)^{\mathrm{odd}})^q$ that satisfy

$$(|f_1|(s), \dots, |f_p|(s)) \in |M| \subseteq \mathbb{R}^p \quad \text{for all } s \in |S|.$$

The f_i, η_j are called the coordinates of $\phi \in \mathsf{SM}(S, M)$ and are defined by

$$f_i = \phi^*(x_i) \quad \text{and} \quad \eta_j = \phi^*(\theta_j),$$

where $x_1, \dots, x_p, \theta_q, \dots, \theta_p$ are coordinates on $M \subseteq \mathbb{R}^{p|q}$. Moreover, by the first part we see that $f_i \in C^\infty(S)^{\mathrm{ev}} = \mathsf{SM}(S, \mathbb{R})$ and hence $|f_i| \in \mathsf{Man}(|S|, \mathbb{R})$.

The proof of the first part is based on the existence of partitions of unity for supermanifolds, so it is false in analytic settings. The second part always holds and is proved in [L].

Since sheaves are generally difficult to work with, one often thinks of supermanifolds in terms of their S-points, i.e., instead of M itself one considers the morphism sets $\mathsf{SM}(S, M)$, where S varies over all supermanifolds. More formally, one embeds the category SM of supermanifolds in the category of contravariant functors from SM to Set by

$$Y : \mathsf{SM} \to \mathsf{Fun}(\mathsf{SM}^{\mathrm{op}}, \mathsf{Set}), \quad Y(M) = (S \mapsto \mathsf{SM}(S, M)).$$

This Yoneda embedding is fully faithful and identifies SM with the category of *representable* functors, defined to be those in the image of Y. Following A. S. Schwarz, we will sometimes refer to an arbitrary functor $F : \mathsf{SM}^{\mathrm{op}} \to \mathsf{Set}$ as a *generalized supermanifold*.

Note that Proposition 2.2 makes it easy to describe the morphism sets $\mathsf{SM}(S, M)$. We would also like to point out that this functor of points approach is closely related to computations involving additional odd quantities (the odd coordinates of S as opposed to those of M) in many physics papers.

3. The superpoints in a supermanifold

For a supermanifold X, we would like to talk about the supermanifold $\mathrm{SP}X$ of *superpoints* in X. By definition, this is the inner Hom from the superpoint spt to X in the category SM of supermanifolds, usually denoted by $\mathrm{SP}X$, compare Remark 7.8.

More generally, for any supermanifold $M \in \mathsf{SM}$, we can consider the inner Hom $\underline{\mathsf{SM}}(M, X)$ as a generalized supermanifold given by

$$\underline{\mathsf{SM}}(M, X)(S) := \mathsf{SM}(S \times M, X) \quad \text{for } S \in \mathsf{SM}.$$

It is clear that if the dimensions of M_{red} and X are nonzero, this functor is not representable, at least not by *finite dimensional* supermanifold that we are studying here. However, it turns out that for $M = \mathbb{R}^{0|n}$ it actually is in the image of the Yoneda embedding

$$Y : \mathsf{SM} \to \mathsf{Fun}(\mathsf{SM}^{\text{op}}, \mathsf{Set}), \quad Y(M) = (S \mapsto \mathsf{SM}(S, M)).$$

The following proposition will prove the case $n = 1$, the other cases follow by induction.

Proposition 3.1. *For any supermanifold X, the odd tangent bundle ΠTX represents the inner Hom $\underline{\mathsf{SM}}(\text{spt}, X) =: \text{SP}X$. More precisely, there is an isomorphism of generalized supermanifolds*

$$(T \mapsto \text{SP}X(T)) \cong (T \mapsto \Pi TX(T)).$$

Remark 3.2. This result is mentioned as an obvious fact in many places, for example in Vaintrob [Va], p. 66, where the supermanifold ΠTX is abbreviated as $\widehat{X}$. We decided to write out the proof because we will use it later in identifying the action of the diffeomorphism group of $\mathbb{R}^{0|1}$ which is obvious on $\text{SP}X$ but a priori not on ΠTX.

Our point of view differs from Vaintrob's because we start with $\text{SP}X$ as a generalized supermanifold and then show that it is represented by ΠTX. As a consequence, in the remainder of the paper we will think of the superpoints in X as a supermanifold, i.e., $\text{SP}X \in \mathsf{SM}$ and will ignore the fact that it is actually only a generalized supermanifold.

Proof of Proposition 3.1. We split the proof of the desired bijection into the following natural correspondences, where in (3) Der_f denotes derivations $C^\infty(X) \to C^\infty(S)$ with respect to f, in the sense that $C^\infty(S)$ is a $C^\infty(X) - C^\infty(X)$-bimodule using the algebra homomorphism f.

$$\begin{aligned}
\mathsf{SM}(S \times \text{spt}, X) &\underset{(1)}{\longleftrightarrow} \mathsf{Alg}(C^\infty(X), C^\infty(S \times \text{spt})) \\
&\underset{(2)}{\longleftrightarrow} \mathsf{Alg}(C^\infty(X), C^\infty(S) \otimes \Lambda^*(\mathbb{R})) \\
&\underset{(3)}{\longleftrightarrow} \{(f, g) \mid f \in \mathsf{Alg}(C^\infty(X), C^\infty(S)),\ g \in \text{Der}_f^{\text{odd}}\} \\
&\underset{(4)}{\longleftrightarrow} \mathsf{SM}(S, \Pi TX).
\end{aligned}$$

In (3) Der_f denotes derivations $g\colon C^\infty(X) \to C^\infty(S)$ with respect to f in the sense that

$$g(ab) = g(a)f(b) + (-1)^{|g||a|} f(a)g(b).$$

In other words, these are odd sections of the pulled back tangent bundle along f. (1) follows directly from Proposition 2.2 and (2) just uses the definition of products of supermanifolds together with $C^\infty(\mathrm{spt}) = \Lambda^*(\mathbb{R})$. To see (3), decompose $\varphi\colon C^\infty(X) \to C^\infty(S) \otimes \Lambda^*(\mathbb{R}) = C^\infty(S)[\theta]$ as a sum

$$\varphi = f + \theta g \quad \text{with } f, g\colon C^\infty(X) \to C^\infty(S).$$

Here θ is the usual odd coordinate on spt. Note that f preserves the grading, whereas g reverses it. For $a, b \in C^\infty(X)$ we have $\varphi(ab) = f(ab) + \theta g(ab)$, and since φ is an algebra homomorphism this is also equal to

$$\begin{aligned}\varphi(a)\varphi(b) &= (f(a) + \theta g(a))(f(b) + \theta g(b))\\ &= f(a)f(b) + \theta(g(a)f(b) + (-1)^{|a|} f(a)g(b)).\end{aligned}$$

Comparing the coefficients we conclude that f is an algebra homomorphism and that g is an odd derivation with respect to f. Conversely, any such pair (f, g) defines an algebra map φ. It is clear that the bijection is natural with respect to superalgebra maps $C^\infty(S) \to C^\infty(S')$.

Proposition 2.2 translates f into a morphism $\hat{f}\colon S \to X$ in (4). Then g is taken to a global section $\hat{g}$ of the sheaf $\hat{f}^*(\mathcal{T}X)^{\mathrm{odd}}$. This pair $(\hat{f}, \hat{g})$ is an S-point of the supermanifold ΠTX. The last statement holds more generally: Any vector bundle over M (aka a locally free and finitely generated sheaf $\mathcal{E}$ of $\mathcal{O}_M$-modules) has a *total space* $E \in \mathsf{SM}$ that comes with a projection map $\pi\colon E \to M$. It can be most easily described in terms of its S-points

$$E(S) = \{(f, g) \mid f \in \mathsf{SM}(S, M), g \in f^*(\mathcal{E}^{\mathrm{ev}})\}.$$

So g is an even global section of the pullback bundle on S and the projection π comes from forgetting this datum. If we reverse the parity of $\mathcal{E}$ by tensoring fibrewise with spt, we obtain the sheaf $\Pi\mathcal{E}$ with total space ΠE determined by its S-points

$$\Pi E(S) = \{(f, g) \mid f \in \mathsf{SM}(S, M), g \in f^*(\mathcal{E}^{\mathrm{odd}})\}.$$

This finishes the proof of the proposition. We would like to point out that Proposition 2.2 is not crucial for the proof. One can write down the equivalences in terms of maps of sheaves (instead of their restriction to global sections), the only thing that changes is that the notation becomes more complicated. □

Let us write down the above natural bijection more explicitly for superdomains $X = U \subseteq \mathbb{R}^{p|q}$. Let $y_1, \dots, y_{p+q}$ be coordinate functions on X, where $y_1, \dots, y_p$

are even and $y_{p+1}, \dots, y_{p+q}$ are odd. Then a morphism $\varphi\colon S \times \mathrm{spt} \to U$ is given by coordinates

$$\begin{aligned} \varphi^*(y_i) &= (x_1 + \theta\hat{x}_1, \dots, x_p + \theta\hat{x}_p, \eta_1 + \theta\hat{\eta}_1, \dots, \eta_q + \theta\hat{\eta}_q) \\ &\in (\mathcal{O}^{\mathrm{ev}}_{S\times\mathbb{R}^{0|1}})^p \times (\mathcal{O}^{\mathrm{odd}}_{S\times\mathbb{R}^{0|1}})^q. \end{aligned} \tag{φ}$$

In this case we can make the identification $\Pi TU \cong U \times \mathbb{R}^{q|p}$ with coordinates $(y_i, \hat{y}_i)$ where $\hat{y}_1, \dots, \hat{y}_p$ are odd and $\hat{y}_{p+1}, \dots, \hat{y}_{p+q}$ are even. Going through the above bijections one sees that the image $\tilde{\varphi}\colon S \to U \times \mathbb{R}^{q|p}$ of the morphism φ has coordinates

$$\begin{aligned} \varphi^*(y_i, \hat{y}_i) &= (x_1, \dots, x_p, \eta_1, \dots, \eta_q, \hat{\eta}_1, \dots, \hat{\eta}_q, \hat{x}_1, \dots, \hat{x}_p) \\ &\in (\mathcal{O}^{\mathrm{ev}}_S)^p \times (\mathcal{O}^{\mathrm{odd}}_S)^q \times (\mathcal{O}^{\mathrm{ev}}_S)^q \times (\mathcal{O}^{\mathrm{odd}}_S)^p. \end{aligned} \tag{$\tilde{\varphi}$}$$

3.1. The translation action of $\mathbb{R}^{0|1}$. Addition gives $\mathbb{R}^{0|1}$ a super Lie group structure which we denote by $m\colon \mathbb{R}^{0|1} \times \mathbb{R}^{0|1} \to \mathbb{R}^{0|1}$. We get a right action of spt on itself and a left group action $\mu_0\colon \mathbb{R}^{0|1} \times \mathrm{SP}X \to \mathrm{SP}X$ by pre-composing with right translations: To an S-point

$$(\eta, \varphi) \in \mathsf{SM}(S, \mathbb{R}^{0|1}) \times \mathsf{SM}(S \times \mathbb{R}^{0|1}, X) \cong \mathsf{SM}(S, \mathbb{R}^{0|1} \times \mathrm{SP}X)$$

$\mu_0(S)$ associates the composition $\varphi_\eta \in \mathsf{SM}(S \times \mathbb{R}^{0|1}, X) \cong \mathsf{SM}(S, \mathrm{SP}X)$ given by

$$\varphi_\eta\colon S \times \mathrm{spt} \xrightarrow{\mathrm{id}, \eta\circ p_1} (S \times \mathrm{spt}) \times \mathrm{spt} = S \times (\mathrm{spt} \times \mathrm{spt}) \xrightarrow{\mathrm{id}_S \times m} S \times \mathrm{spt} \xrightarrow{\varphi} X.$$

The infinitesimal generator of this spt-action is a globally defined odd vector field D on $\mathrm{SP}X \cong \Pi TX$. Since spt is commutative, we have $D^2 = \frac{1}{2}[D, D] = 0$ (which is not always true for *odd* vector fields). We next describe D in local coordinates for superdomains $X = U \subseteq \mathbb{R}^{p|q}$.

It is easy to see that φ_η is given by replacing θ by $\theta + \eta$ in the coordinate representation (φ) above. Translating from $\mathrm{SP}X$ to ΠTX, this action becomes in coordinates for $\tilde{\varphi}$:

$$(\eta, x_1, \dots, \eta_q, \hat{\eta}_1, \dots, \hat{x}_p) \mapsto (x_1 + \eta\hat{x}_1, \dots, \eta_q + \eta\hat{\eta}_q, \hat{\eta}_1, \dots, \hat{x}_p). \tag{$*$}$$

The fact that

$$x_i = \varphi^*(y_i), \qquad \hat{x}_i = \varphi^*(\hat{y}_i), \qquad i = 1, \dots, p,$$

and

$$\eta_i = \varphi^*(y_{p+i}), \quad \hat{\eta}_i = \varphi^*(\hat{y}_{p+i}), \quad i = 1, \dots, q,$$

together with formula $(*)$ above implies that the action map $\mu_0\colon \mathrm{spt} \times \Pi TX \to \Pi TX$ pulls back these coordinate functions as follows:

$$\mu_0^*(y_i) = y_i + \eta\hat{y}_i \quad \text{and} \quad \mu_0^*(\hat{y}_i) = \hat{y}_i \quad \text{for } i = 1, \dots, p+q.$$

Abusing notation, η here denotes the standard odd coordinate function on spt. To get the infinitesimal generator D for the action, we have to differentiate this formula with respect to η and evaluate at $\eta = 0$. This gives $D(y_i) = \hat{y}_i$ and $D(\hat{y}_i) = 0$. Using $\partial_i := \frac{\partial}{\partial y_i}$, the local representation of our odd vector field D is therefore given as the derivation

$$D = \sum_{i=1}^{p+q} \hat{y}_i \partial_i .$$

3.2. The de Rham complex for supermanifolds. For a supermanifold X, the algebra of differential forms on X has two gradings, the $\mathbb{Z}/2$-parity and the (cohomological) $\mathbb{Z}$-degree. There are two conventions how to deal with this situation, we will work with one that makes $\Omega^* X$ into a $\mathbb{Z}$-graded commutative superalgebra and leads to an *odd* de Rham differential d. This seems to be a natural choice, since we want to relate d to the action of the odd vector field D on the commutative superalgebra $C^\infty(\Pi TX)$.

Let V be a module over the commutative superalgebra A. Following the convention of Bernstein–Leites we define the exterior algebra on V to be

$$\Lambda_A^*(V) := \mathsf{Sym}_A(\Pi V).$$

Here $\mathsf{Sym}_A(W)$ is the quotient of the tensor algebra on W by the ideal generated by all supercommutators $w_1 \otimes w_2 - (-1)^{|w_1||w_2|} w_2 \otimes w_1$. The commutative superalgebra $\Lambda_A^*(V)$ has the universal property that giving a superalgebra map from $\Lambda_A^*(V)$ to any commutative A-superalgebra B is the same as giving an A-module map $\Pi V \to B$.

Let $\mathbf{\Omega}^1 X := \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{T} X, \mathcal{O}_X)$ be the cotangent sheaf of X and define

$$\mathbf{\Omega}^* X := \Lambda_{\mathcal{O}_X}^*(\mathbf{\Omega}^1 X).$$

Clearly, $\mathcal{O}_X$ and $\Pi \mathbf{\Omega}^1 X$ are subsheaves of $\Lambda_{\mathcal{O}_X}^*(\mathbf{\Omega}^1 X)$ in a natural way. The universal *even* differential $d_{\mathrm{ev}} \colon \mathcal{O}_X \to \mathbf{\Omega}^1 X$ is characterized by

$$d_{\mathrm{ev}} f(\xi) = (-1)^{|f||\xi|} \xi(f) \quad \text{for all } \xi \in \mathcal{T} X.$$

Alternatively, we can think of this as an *odd* differential $d \colon \mathcal{O}_X \to \Pi \mathbf{\Omega}^1 X$. The de Rham differential on $\mathbf{\Omega}^* X$ is the extension of d whose square is zero and which satisfies the Leibniz rule.

We will denote the global sections of the sheaf $\mathbf{\Omega}^* X$ as usual by $\Omega^* X$ (non-bold).

3.3. Differential forms as functions. The next step is to interpret differential forms on X as certain functions on ΠTX. We thank the referee for pointing out that functions on ΠTX are also called *pseudodifferential forms* on X. This notation is motivated by the following well-known result.

Lemma 3.3. *There is an embedding of sheaves of $\mathcal{O}_X$-superalgebras*

$$\iota \colon \mathbf{\Omega}^* X \hookrightarrow \mathcal{O}_{\Pi TX}$$

that maps onto the functions that are polynomial on every fiber.

Proof. Let $x_1, \dots, x_p, \eta_1, \dots, \eta_q$ be local coordinates on X. We have canonically associated coordinates

$$(x_1, \dots, x_p, \eta_1, \dots, \eta_q, \hat{\eta}_1, \dots, \hat{\eta}_q, \hat{x}_1, \dots, \hat{x}_p)$$

on ΠTX. Recall that the $\hat{x}_i$'s are odd, whereas the $\hat{\eta}_j$'s are even. On the other hand, a local basis for the $\mathcal{O}_X$-module $\mathbf{\Omega}^1 X$ is given by $dx_1, \dots, d\eta_q$. According to the convention we picked for the definition of the de Rham complex, the $dx_i, d\eta_j \in \Pi\mathbf{\Omega}^1 X$ have odd and even parity, respectively. Hence we can define a map of super $\mathcal{O}_X$-modules $\iota_0\colon \Pi\mathbf{\Omega}^1 X \to \mathcal{O}_{\Pi TX}$ by prescribing $dx_i \mapsto \hat{x}_i$ and $d\eta_j \mapsto \hat{\eta}_j$. It is not hard to check that this is independent of the coordinate system chosen.[2] According to the defining property of $\mathsf{Sym}_{\mathcal{O}_X}(\Pi\mathbf{\Omega}^1 X)$ the map ι_0 extends to a unique homomorphism of $\mathcal{O}_X$-algebras $\iota\colon \mathbf{\Omega}^* X \to \mathcal{O}_{\Pi TX}$. It is clear that ι is injective with image as stated above. □

The map ι is surjective if and only if X is an ordinary manifold. For example, if $X = \mathbb{R}^{0|q}$ then $\Omega^* X = \Lambda(\mathbb{R}^q)[x_1, \dots, x_q]$, the polynomial ring on q even generators x_i over the ground ring $\Lambda(\mathbb{R}^q)$. It has to be completed in the x_i-directions to obtain

$$C^\infty(\Pi TX) = C^\infty(\mathbb{R}^{q|q}) = \Lambda(\mathbb{R}^q) \otimes_{\mathbb{R}} C^\infty(\mathbb{R}^q).$$

Since D is an odd vector field on ΠTX we have the Leibniz rule

$$D(fg) = (Df)g + (-1)^{|f|} f(Dg) \quad \text{for all functions } f, g \text{ on } \Pi TX.$$

Furthermore, we already know that $D^2 = 0$. Hence the restriction of D to $\Omega^* X$ is the de Rham differential once we have shown

Lemma 3.4. *The restriction of D to $\mathcal{O}_X \subset \mathcal{O}_{\Pi TX}$ is the odd differential d. More precisely, we have*

$$D = \iota d\colon \mathcal{O}_X \to \mathcal{O}_{\Pi TX}.$$

Proof. It is clear from the local representation of D that the image of D is contained in $\iota(\Pi\mathbf{\Omega}^1 X) \subset \mathcal{O}_{\Pi TX}$. The claim is equivalent to showing that the composition

$$\widetilde{D} := \iota^{-1} D\colon \mathcal{O}_X \to \mathbf{\Omega}^1 X$$

is equal to d, i.e., for all $f \in \mathcal{O}_X$ we have to check that

$$\widetilde{D} f(\xi) = (-1)^{|f||\xi|} \xi(f) \quad \text{for all vector fields } \xi \in \mathcal{T} X.$$

[2] In fact, one can see this using the (global!) vector field D considered in Section 3.1: The map ι_0 is equal to the composition

$$\Pi\mathbf{\Omega}^1 X \xrightarrow{\Pi} \mathbf{\Omega}^1 X \hookrightarrow \mathbf{\Omega}^1(\Pi TX) \xrightarrow{\bar{D}} \mathcal{O}_{\Pi TX},$$

where $\bar{D}(\omega) = (-1)^{|\omega|}\omega(D)$.

It suffices to prove this for (local) basis vector fields $\partial_j = \frac{\partial}{\partial y_j}$, $j = 1, \ldots, p+q$, where the y_i are local coordinates on X. We first compute

$$\widetilde{D} f = \iota^{-1}\Big(\sum_{i=1}^{p+q} \hat{y}_i \partial_i f\Big) = \iota^{-1}\Big(\sum_{i=1}^{p+q} (-1)^{(|y_i|+1)(|f|+|y_i|)} (\partial_i f)\hat{y}_i\Big).$$

Since ι is even, we get

$$\widetilde{D} f = \Big(\sum_{i=1}^{p+q} (-1)^{(|y_i|+1)(|f|+|y_i|)} (\partial_i f) dy_i\Big) = \sum_{i=1}^{p+q} (-1)^{|y_i|(|f|+1)} (\partial_i f) dy_i.$$

Applying this 1-form to ∂_j and using $(dy_i)\partial_j = (-1)^{|y_i||y_j|}\delta_{ij}$ yields

$$\widetilde{D} f(\partial_j) = (-1)^{|y_j|(|f|+1)+|y_j|} = (-1)^{|y_j||f|}\partial_i(f),$$

as desired. □

If X is purely even the cohomological degree of $\alpha \in \Omega^* X$ is equal to the parity of its image in $C^\infty \Pi T X$ modulo 2. Hence the Leibniz rule above is exactly the (graded) Leibniz rule for differential forms, and so D is equal to the usual de Rham differential on $\Omega^* X$.

3.4. The diffeomorphism group of the superpoint. We used the translation action of $\mathbb{R}^{0|1}$ on itself to define an action of $\mathbb{R}^{0|1}$ on the superpoints $\mathrm{SP}X \cong \Pi T X$. In fact, the whole super Lie group $\underline{\mathrm{Diff}}(\mathrm{spt})$ of diffeomorphisms of the supermanifold spt acts on $\mathrm{SP}X$. We briefly describe this action. By definition, $\underline{\mathrm{Diff}}(\mathrm{spt})$ is the super Lie group representing the group-valued functor

$$S \mapsto \mathrm{Diff}_S(\mathrm{spt} \times S, \mathrm{spt} \times S).$$

Here $\mathrm{Diff}_S(\mathrm{spt} \times S, \mathrm{spt} \times S)$ is the group of diffeomorphisms of $\mathrm{spt} \times S$ that are compatible with the projection to S. The following result follows from a short computation together with Proposition 3.1 for $X = \mathrm{spt}$.

Lemma 3.5. *There is an isomorphism of (generalized) super Lie groups*

$$\underline{\mathrm{Diff}}(\mathrm{spt}) \cong \mathbb{R}^\times \ltimes \mathbb{R}^{0|1}$$

where the semi-direct product is defined by the right action of $\mathbb{R}^\times$ on $\mathbb{R}^{0|1}$, given by scalar multiplication.

The right action $\mathrm{spt} \times (\mathbb{R}^\times \ltimes \mathrm{spt}) \to \mathbb{R}^{0|1}$ is on S-points given by

$$(\theta, (a, \eta)) \mapsto \theta a + \eta \quad \text{where } a \in C^\infty(S)^{\mathrm{ev}} \text{ and } \theta, \eta \in C^\infty(S)^{\mathrm{odd}}.$$

It follows that in our local coordinates $(y_i, \hat{y}_i)$ for $\Pi T X$ from Section 3.1, the action of $\lambda \in \mathbb{R}^\times$ is given by fixing the y_i and multiplying each $\hat{y}_i$ by λ.

Remark 3.6. In Lemma 3.5, we use the following convention for the (S-points of the) semi-direct product. If a group G acts on another group A on the right, written as $(a, g) \mapsto a^g$, elements of $G \ltimes A$ are just pairs (g, a) with (associative) multiplication

$$(g_1, a_1) \cdot (g_2, a_2) := (g_1 g_2, (a_1)^{g_2} a_2).$$

In Remark 3.10 it will be useful to consider the case of a left action of G on A, defined by $g(a) := a^{g^{-1}}$. In particular, this will introduce an inverse for the scalar multiplication of $\mathbb{R}^\times$.

Every function f on ΠTX which is polynomial on fibers is locally a finite sum of functions of the form

$$f = g \hat{y}_1^{i_1} \dots \hat{y}_{p+q}^{i_{p+q}} \quad \text{where } g \in \mathcal{O}_X.$$

It follows that the action of $\lambda \in \mathbb{R}^\times$ on such an f is given by the formula

$$(\lambda, f) \mapsto \lambda^{\sum_{k=1}^{p+q} i_k} f.$$

Conversely, if a function $f \in \mathcal{O}_{\Pi TX}$ has degree n in the sense that

$$(\lambda, f) \mapsto \lambda^n f \quad \text{for all } \lambda \in \mathbb{R}^\times,$$

then $f \in \mathbf{\Omega}^n X \subset \mathcal{O}_{\Pi TX}$ must be homogenous of degree n along the fibres.

Corollary 3.7. *The $\mathbb{R}^\times$-action on $\mathcal{O}_{\Pi TX}$ coming from dilations of the superpoint determines the $\mathbb{Z}$-degree operator and vice versa. More precisely, all $\lambda \in \mathbb{R}^\times$ map $f \in \mathcal{O}_{\Pi TX}$ to $\lambda^n f$ if and only if $f \in \mathbf{\Omega}^n X$.*

In the proof of Proposition 6.3 below we will need the following reformulations of the above computations.

Lemma 3.8. *Let $\mu_0 \colon \mathbb{R}^{0|1} \times \mathrm{SP}X \to \mathrm{SP}X$ be the left action given by pre-composition with right translation as in Section 3.1. Then the induced action on functions is given by*

$$\mu_0^*(f) = 1 \otimes f + \eta \otimes D(f) \in C^\infty(\mathbb{R}^{0|1} \times \mathrm{SP}X) = \Lambda^*[\eta] \otimes C^\infty(\mathrm{SP}X)$$

for all $f \in C^\infty(\mathrm{SP}X)$.

Proof. Let η be the standard coordinate on $\mathbb{R}^{0|1}$ and D the vector field on $\Pi TX \cong \mathrm{SP}X$ infinitesimally giving the action of the super Lie algebra of $\mathbb{R}^{0|1}$ on $C^\infty(\Pi TX)$. Using our explicit (local) coordinate representations of μ_0 and D in Section 3.1, the asserted equality is trivial to verify when f is equal to the local coordinates y_i and $\hat{y}_i$ of ΠTX.

Furthermore, using $\eta^2 = 0$ and that D is an odd derivation, it is easy to check that the right-hand side of the asserted equality defines an algebra homomorphism from $C^\infty(\Pi TX)$ to $C^\infty(\mathbb{R}^{0|1} \times \Pi TX)$. Since the coordinates y_i and $\hat{y}_i$ locally generate $\mathcal{O}_{\Pi TX}$, it follows that μ_0^* and this algebra homomorphism are equal. Hence the asserted equality holds for all $f \in C^\infty(\Pi TX)$. $\square$

Proposition 3.9. *Let* $\mu\colon \underline{\mathrm{Diff}}(\mathrm{spt}) \times \mathrm{SP}X \to \mathrm{SP}X$ *be the left action of* $\underline{\mathrm{Diff}}(\mathrm{spt})$ *on* $\mathrm{SP}X$ *induced by the right action of* $\underline{\mathrm{Diff}}(\mathrm{spt}) = \mathbb{R}^\times \ltimes \mathbb{R}^{0|1}$ *on* spt. *Then*

$$\mu^*(f) = t^n \otimes 1 \otimes f + t^n \otimes \eta \otimes D(f) \in C^\infty(\mathbb{R}^\times) \otimes \Lambda^*[\eta] \otimes C^\infty(\mathrm{SP}X)$$

for all $f \in \Omega^n X \subset C^\infty(\Pi TX) \cong C^\infty(\mathrm{SP}X)$. *Here* $t \in C^\infty(\mathbb{R}^\times)$ *is the standard coordinate coming from the inclusion* $\mathbb{R}^\times \subset \mathbb{R}$.

Proof. By Corollary 3.7, the dilation action

$$\nu\colon \mathbb{R}^\times \times \mathrm{SP}X \to \mathrm{SP}X$$

induces on functions $f \in \Omega^n(X)$ the action $\nu^*(f) = t^n \otimes f$, where $t \in C^\infty(\mathbb{R}^\times)$ is the standard coordinate. By Remark 3.10, our action map μ can be written as a composition

$$\mathbb{R}^\times \times (\mathbb{R}^{0|1} \times \mathrm{SP}X) \xrightarrow{\mathrm{id}\times\mu_0} \mathbb{R}^\times \times \mathrm{SP}X \xrightarrow{\nu} \mathrm{SP}X.$$

It follows that for $f \in \Omega^n(X)$ one has

$$\mu^*(f) = (\nu \circ (\mathrm{id} \times \mu_0))^*(f) = (\mathrm{id} \times \mu_0)^*(t^n \otimes f) = t^n \otimes (1 \otimes f + \eta \otimes D(f)),$$

which proves our claim. $\square$

Remark 3.10. In the above proof, we have used the following elementary fact about left actions of a semi-direct product $G \ltimes A$ on a set Y: A G-action and an A-action on Y fit together to an action of $G \ltimes A$ if and only if the A-action map $A \times Y \to Y$ is G-equivariant. Here we assume that G acts on the left on A and on Y and hence it acts on $A \times Y$ diagonally. This observation uses the conventions from Remark 3.6 for semi-direct products. In particular, the left action map

$$\delta\colon \mathbb{R}^\times \times \mathbb{R}^{0|1} \to \mathbb{R}^{0|1}$$

is determined by $\delta^*(\eta) = t^{-1} \otimes \eta$ since it comes from the right action given by scalar multiplication, see Lemma 3.5. Then the above compatibility condition for the $\mathbb{R}^{0|1}$- and $\mathbb{R}^\times$-actions comes from the commutative diagram

$$\begin{array}{ccccc}
\mathbb{R}^\times \times (\mathbb{R}^{0|1} \times \mathrm{SP}X) & \xrightarrow{\Delta_{13}} & (\mathbb{R}^\times \times \mathbb{R}^{0|1}) \times (\mathbb{R}^\times \times \mathrm{SP}X) & \xrightarrow{\delta\times\nu} & \mathbb{R}^{0|1} \times \mathrm{SP}X \\
\Big\downarrow{\scriptstyle \mathrm{id}\times\mu_0} & & & & \Big\downarrow{\scriptstyle \mu_0} \\
\mathbb{R}^\times \times \mathrm{SP}X & & \xrightarrow{\qquad\qquad\nu\qquad\qquad} & & \mathrm{SP}X
\end{array}$$

which we leave for the reader to check.

4. Topological field theories

The usual definition of a d-dimensional TFT, going back to Atiyah and Segal, is in terms of a symmetric monoidal functor $E\colon d\text{-}\mathsf{B} \to \mathsf{Vect}$ or shorter

$$E \in \mathsf{Fun}^{\otimes}(d\text{-}\mathsf{B}, \mathsf{Vect}).$$

The domain category d-B is the bordism category whose objects are closed $(d-1)$-manifolds and whose morphisms are diffeomorphism classes of compact d-dimensional bordisms. The target of the functor is the category Vect of finite dimensional vector spaces. The symmetric monoidal structures are given by disjoint union respectively tensor product. In [ST2] we explain a version of this definition using internal categories in which one can easily add several bells and whistles, for example geometry, supersymmetry and a notion of degree. We also describe what these definitions mean in dimension 0 and $0|1$, which we shall summarize in the coming subsections.

One obvious simplification in these smallest possible dimensions is that the empty set is the only manifold of dimension $(-1|\delta)$. This implies that our language of internal categories in [ST2] can be reduced to ordinary categories which we shall stick to in this paper. Instead of working exclusively in dimension 0 and $0|1$, however, we shall explain the part of our work that can be formulated in terms of categories alone. In any dimension, this is the part given by restricting attention to the empty set as the only relevant $(d-1|\delta)$-manifold.

Definition 4.1. Consider the categories d-B_{c} (respectively d-B_{cc}) with objects closed (respectively closed, connected, non-empty) d-manifolds and morphisms being diffeomorphisms. The disjoint union operation makes d-B_{c} into a symmetric monoidal category.

The subscripts ‘c’ respectively ‘cc’ stand for *closed* respectively *closed connected* manifolds but also for *category*: Notice that unlike for d-B, we are not considering diffeomorphism classes of manifolds but keep track of the diffeomorphisms as morphims. This will be essential for supersymmetric and twisted field theories discussed below. We next point out a lemma that shows how one can simplify the discussions related to the symmetric monoidal structure. It follows from the fact that any compact manifold is the disjoint union of connected manifolds and that any diffeomorphism is uniquely determined by its restriction to connected components.

Lemma 4.2. *For any symmetric monoidal category* C*, there is an equivalence of functor categories*

$$\mathsf{Fun}^{\otimes}(d\text{-}\mathsf{B}_{\mathrm{c}}, \mathsf{C}) \simeq \mathsf{Fun}(d\text{-}\mathsf{B}_{\mathrm{cc}}, \mathsf{C}).$$

In other words, d-B_{c} *is the free symmetric monoidal groupoid generated by its subcategory* d-B_{cc}*. The manifolds in* d-B_{cc} *are assumed to be non-empty so that the monoidal unit does not lie in this subcategory.*

For example, such a functor arises from a d-dimensional TFT by restricting it to the empty $(d-1)$-manifold (and hence to closed d-manifolds) and taking $\mathsf{C} := \mathbb{R}$, our chosen ground field, considered as a monoid via multiplication and as a *discrete category*, i.e., a category with identity morphisms only.

This is the well-known observation that in the top dimension d, TFTs give multiplicative diffeomorphism invariants of closed manifolds. Given $E \in \mathsf{Fun}(d\text{-}\mathsf{B}_{cc}, \mathsf{C})$, it is the locality properties of E that tell whether it can be extended to a full fledged TFT. The Atiyah–Segal axioms address the codimension 1 gluing laws and higher codimensions can be handled by using d-categories, an aspect that is very important but not relevant for the current paper.

4.1. 0-dimensional TFTs. Starting with the Atiyah–Segal definition and observing that there is only one (-1)-dimensional manifold, $\emptyset$, we have

$$0\text{-TFT} := \mathsf{Fun}^{\otimes}(0\text{-}\mathsf{B}, \mathsf{Vect}_{\mathbb{R}}) \cong \mathrm{Maps}^{\otimes}(0\text{-}\mathsf{B}(\emptyset,\emptyset), \mathbb{R}).$$

Here we have used that the monoidal unit $\emptyset \in d\text{-}\mathsf{B}$ has to be sent to the monoidal unit $\mathbb{R} \in \mathsf{Vect}$, up to canonical isomorphism. By definition $0\text{-}\mathsf{B}(\emptyset,\emptyset)$ are the isomorphism classes of objects in $0\text{-}\mathsf{B}_c$ and since the category $\mathbb{R}$ has only identity morphisms it follows that there are bijections

$$0\text{-TFT} \cong \mathrm{Maps}^{\otimes}(0\text{-}\mathsf{B}(\emptyset,\emptyset), \mathbb{R}) \cong \mathsf{Fun}^{\otimes}(0\text{-}\mathsf{B}_c, \mathbb{R}) \cong \mathsf{Fun}(0\text{-}\mathsf{B}_{cc}, \mathbb{R}),$$

where the right most bijection follows from Lemma 4.2. Since the point has no non-trivial diffeomorphisms, it follows that $0\text{-}\mathsf{B}_{cc} = \{\mathrm{pt}\}$ and hence we conclude

Lemma 4.3. *There is a bijection*

$$0\text{-TFT} \cong \mathsf{Fun}(0\text{-}\mathsf{B}_{cc}, \mathbb{R}) \cong \mathsf{Fun}(\mathrm{pt}, \mathbb{R}) \cong \mathbb{R}$$

sending a TFT E to the real number $E(\mathrm{pt})$.

Graeme Segal also introduced the notion of a *field theory over a manifold X* as a symmetric monoidal functor $E\colon d\text{-}\mathsf{B}(X) \to \mathsf{Vect}$ or

$$E \in \mathsf{Fun}^{\otimes}(d\text{-}\mathsf{B}(X), \mathsf{Vect}).$$

Here one replaces the domain category $d\text{-}\mathsf{B}$ by $d\text{-}\mathsf{B}(X)$ where both objects and bordisms are equipped with a smooth map to X. Arguing exactly as above one sees that

$$\mathsf{Fun}^{\otimes}(0\text{-}\mathsf{B}(X), \mathsf{Vect}) \cong \mathsf{Fun}(0\text{-}\mathsf{B}_{cc}(X), \mathbb{R}) \cong \mathrm{Maps}(X, \mathbb{R})$$

because $0\text{-}\mathsf{B}_{cc}(X) \cong X$ (as discrete categories). It follows that TFTs over X would be *all* real-valued maps on X. Note that there is no smoothness or continuity requirement on these functions! Our main contribution is to implement smoothness in such a way that it generalizes to

- higher dimension and to
- supersymmetry.

For this purpose, we introduce in [ST2] *family versions* of all relevant categories and require that the functors extend in a natural way to these family versions. We will now explain these *smooth* families of manifolds in a way that can easily be extended to various other settings, in particular to *super* families of supermanifolds.

4.2. Smooth 0-dimensional TFTs over X

Definition 4.4. The category $d\text{-}\mathsf{B}_{\mathrm{c}}^{\mathrm{fam}}$ (respectively $d\text{-}\mathsf{B}_{\mathrm{cc}}^{\mathrm{fam}}$) has objects smooth fibre bundles $Y \to S$ where the fibres are closed (respectively closed, connected, non-empty) d-manifolds. This is by definition a *smooth S-family* of such d-manifolds.

There are also categories $d\text{-}\mathsf{B}_{\mathrm{c}}(X)^{\mathrm{fam}}$ (respectively $d\text{-}\mathsf{B}_{\mathrm{cc}}(X)^{\mathrm{fam}}$) whose objects include in addition a smooth map $Y \to X$. In all cases, morphisms are smooth bundle maps

$$\begin{array}{ccc} Y' & \xrightarrow{\phi} & Y \\ \downarrow & & \downarrow \\ S' & \xrightarrow{f} & S \end{array}$$

that are fibrewise diffeomorphisms and commute with the map to X (if present). Note that there are projection functors $d\text{-}\mathsf{B}_{\mathrm{c}}(X)^{\mathrm{fam}} \to \mathsf{Man}$ that take a bundle $Y \to S$ and sends it to its base (or parameter space) S.

Fortunately, there is already a very well developed language that deals with *fibred categories* such as $d\text{-}\mathsf{B}_{\mathrm{c}}(X)^{\mathrm{fam}} \to \mathsf{Man}$, going back to at least Grothendieck. So we borrow some language from algebraic geometry introduced for dealing with families of schemes and import them to manifolds and supermanifolds, see our appendix for a quick survey.

Using Lemma 4.9 in the case $d = 0$ it follows that the following definition agrees with that given in [ST2] for arbitrary d. It is a much simplified version, for example the symmetric monoidal structure plays no role. In this paper we decided to give the simplest possible definitions and prove only later that they agree with the ones in arbitrary dimension.

Definition 4.5. A *smooth* 0-dimensional TFT over X is a fibred functor into the representable fibred category $\underline{\mathbb{R}}$, see Definition 7.7:

$$\begin{array}{ccc} 0\text{-}\mathsf{B}_{\mathrm{cc}}(X)^{\mathrm{fam}} & \xrightarrow{E} & \underline{\mathbb{R}} \\ & \searrow \quad \swarrow & \\ & \mathsf{Man} & \end{array}$$

In the notation introduced in the appendix, we define actually the following *category*

$$0\text{-TFT}(X) := \mathsf{Fun}_{\mathsf{Man}}(0\text{-}\mathsf{B}_{\mathrm{cc}}(X)^{\mathrm{fam}}, \underline{\mathbb{R}}).$$

However, by Lemma 7.6 this is indeed the *set* of 0-dimensional TFTs over X.

There is an equivalence of fibred categories $0\text{-}\mathsf{B}_{\mathrm{cc}}(X)^{\mathrm{fam}} \simeq \underline{X}$ over Man because any bundle $Y \to S$ with fibres a single point must be a diffeomorphism and only the map $S \to X$ remains as a datum. By the Yoneda lemma we end up with the desired result:

Lemma 4.6. *There is a bijection between smooth* 0*-dimensional TFTs over* X *and smooth functions:*

$$0\text{-TFT}(X) \cong \mathsf{Fun}_{\mathsf{Man}}(\underline{X}, \underline{\mathbb{R}}) \cong \mathsf{Man}(X, \mathbb{R}) \cong C^{\infty}(X; \mathbb{R}).$$

Since any smooth function on $X \times \{0, 1\}$ can be extended to a smooth function on $X \times [0, 1]$ it follows that no cohomological information can be derived from 0-dimensional TFTs over X:

Corollary 4.7. *There is a single concordance class of smooth* 0*-dimensional TFTs over* X*. In other words,* $0\text{-TFT}[X] = 0$.

Surprisingly, this changes as soon as we introduce one odd dimension which we shall do in the next subsection.

Remark 4.8. Restricting to closed, connected d-manifolds gives a functor

$$d\text{-TFT}(X) \to \mathsf{Fun}_{\mathsf{Man}}(d\text{-}\mathsf{B}_{\mathrm{cc}}(X)^{\mathrm{fam}}, \underline{\mathbb{R}}),$$

where the left-hand side is the category of *smooth* TFTs over X in the sense of [ST2]. The same remark holds in the conformal setting and in fact the image of the analogous functor evaluated on the moduli stack of tori gives the *partition function* of the CFT.

For the careful reader we would like to address the following subtlety. In Definition 4.5 we could have used disconnected 0-manifold fibres to obtain a symmetric monoidal family bordism category over Man which is in fact closer to the definition given in [ST2]. The following parametrized version of Lemma 4.2 shows that the outcome would not have been different because $\underline{\mathbb{R}}$ with multiplication is a symmetric monoidal stack. Since this lemma only serves to justify Definition 4.5 above, we will not be overly careful in explaining the stacky notions used in the proof.

Lemma 4.9. *If* $\mathsf{C} \to \mathsf{Man}$ *is a symmetric monoidal stack in the sense of Definition* 7.21 *then there is an equivalence of categories*

$$\mathsf{Fun}_{\mathsf{Man}}(d\text{-}\mathsf{B}_{\mathrm{cc}}(X)^{\mathrm{fam}}, \mathsf{C}) \simeq \mathsf{Fun}^{\otimes}_{\mathsf{Man}}(d\text{-}\mathsf{B}_{\mathrm{c}}(X)^{\mathrm{fam}}, \mathsf{C}).$$

Proof. We start with the observation that d-$\mathsf{B}_{\mathrm{cc}}(X)^{\mathrm{fam}}$ is the stackification of a much simpler fibred category, namely d-$\mathsf{B}_{\mathrm{cc}}(X)^{\mathrm{prfam}}$, whose objects consists of *product families* only. That is to say, all total spaces are of the form $Y = S \times F$ where F is a closed, connected, non-empty d-manifold. Stackification is left adjoint to the forgetful functor (from stacks to prestacks), so since C is a stack by assumption, there is an equivalence of categories

$$\mathsf{Fun}_{\mathsf{Man}}(d\text{-}\mathsf{B}_{\mathrm{cc}}(X)^{\mathrm{fam}}, \mathsf{C}) \simeq \mathsf{Fun}_{\mathsf{Man}}(d\text{-}\mathsf{B}_{\mathrm{cc}}(X)^{\mathrm{prfam}}, \mathsf{C}).$$

The symmetric monoidal structure on the fibred category $\mathsf{C} \to \mathsf{Man}$ gives a fibred functor

$$\otimes \colon \mathsf{C} \times_{\mathsf{Man}} \mathsf{C} \to \mathsf{C}.$$

In the case of d-$\mathsf{B}_{\mathrm{c}}(X)^{\mathrm{fam}}$ (as well as d-$\mathsf{B}_{\mathrm{c}}(X)^{\mathrm{prfam}}$) it comes from the disjoint union of two total spaces Y, Y' with a fixed base S. Note that this is *not* a symmetric monoidal structure on the category but rather of the *fibred* category d-$\mathsf{B}_{\mathrm{c}}(X)^{\mathrm{fam}} \to \mathsf{Man}$.

The symmetric monoidal fibred category d-$\mathsf{B}_{\mathrm{c}}(X)^{\mathrm{prfam}}$ is freely generated by the fibred category d-$\mathsf{B}_{\mathrm{cc}}(X)^{\mathrm{prfam}}$ and hence we obtain an equivalence as in Lemma 4.2

$$\mathsf{Fun}_{\mathsf{Man}}(d\text{-}\mathsf{B}_{\mathrm{cc}}(X)^{\mathrm{prfam}}, \mathsf{C}) \simeq \mathsf{Fun}^{\otimes}_{\mathsf{Man}}(d\text{-}\mathsf{B}_{\mathrm{c}}(X)^{\mathrm{prfam}}, \mathsf{C}).$$

Finally, the symmetric monoidal stack d-$\mathsf{B}_{\mathrm{c}}(X)^{\mathrm{fam}}$ is the stackification of the symmetric monoidal fibred category d-$\mathsf{B}_{\mathrm{c}}(X)^{\mathrm{prfam}}$, leading to the final equivalence of categories by our assumption on C:

$$\mathsf{Fun}^{\otimes}_{\mathsf{Man}}(d\text{-}\mathsf{B}_{\mathrm{c}}(X)^{\mathrm{prfam}}, \mathsf{C}) \simeq \mathsf{Fun}^{\otimes}_{\mathsf{Man}}(d\text{-}\mathsf{B}_{\mathrm{c}}(X)^{\mathrm{fam}}\mathsf{C}).$$

Putting the three equivalences together finishes the proof of our lemma. □

5. Supersymmetric field theories

In this section we will be rewarded for expressing smooth functions on a manifold X in Lemma 4.6 in very fancy language as certain fibred functors over the site Man. In fact, we can easily generalize all definitions to supermanifolds in the following straightforward way.

The naive extension of the Atiyah–Segal definition would say that a $d|\delta$-dimensional TFT associates a finite dimensional supervector space to any closed $(d-1|\delta)$-manifold and a linear map to a compact $d|\delta$-dimensional bordism, satisfying the usual gluing axioms. In the presence of a target X, all supermanifolds would be in addition equipped with a smooth map to X. Even for $d = 0$ we then run into the question how to implement the smoothness of the functor. We find it very natural use the same formalism as in the previous subsection, except for using supermanifolds S as the parameter (or base) spaces for the family versions of our categories. For $d > 0$, we also have to work with supermanifolds with boundary which were introduced in [VZ].

When trying to generalize, say, the fibred category $d\text{-}\mathsf{B}_{\mathrm{c}}^{\mathrm{fam}} \to \mathsf{Man}$, we have to generalize the notions of a fibre bundle of supermanifolds. There is an obvious way of doing that, namely to start with trivial bundles and define general bundles via gluing data. In the language of fibred categories this procedure is exactly the stackification, already used in Lemma 4.9. Keeping with our spirit of giving the simplest possible definitions and using stacks only as a motivation, see Lemma 5.3, we proceed as follows.

Definition 5.1. A (supersymmetric) $0|\delta$-dimensional TFT over X is a fibred functor:

$$\begin{array}{ccc} 0|\delta\text{-}\mathsf{B}_{\mathrm{cc}}(X)^{\mathrm{prfam}} & \xrightarrow{\;E\;} & \underline{\mathbb{R}} \\ & \searrow \quad \swarrow & \\ & \mathsf{SM} & \end{array}$$

In other words, we set

$$0|\delta\text{-TFT}(X) := \mathsf{Fun}_{\mathsf{SM}}(0|\delta\text{-}\mathsf{B}_{\mathrm{cc}}(X)^{\mathrm{prfam}}, \underline{\mathbb{R}}).$$

Since the dimension already signifies supermanifolds, the additional adjective 'supersymmetric' will be usually skipped.

The missing piece in this definition is that of the fibred category of *product families* of supermanifolds

$$d|\delta\text{-}\mathsf{B}_{\mathrm{cc}}(X)^{\mathrm{prfam}} \to \mathsf{SM} \quad \text{respectively} \quad d|\delta\text{-}\mathsf{B}_{\mathrm{c}}(X)^{\mathrm{prfam}} \to \mathsf{SM}$$

that we shall spell out for arbitrary d. Recall that, by definition, a supermanifold is closed (respectively connected) if and only if its underlying manifold is.

Definition 5.2. The category $d|\delta\text{-}\mathsf{B}_{\mathrm{c}}^{\mathrm{prfam}}$ has objects supermanifolds of the form $S \times F$ where F is a closed $d|\delta$-manifold. For a fixed supermanifold X, $d|\delta\text{-}\mathsf{B}_{\mathrm{c}}(X)^{\mathrm{prfam}}$ is the category whose objects include in addition a smooth map $S \times F \to X$. Morphisms are commutative diagrams in SM

$$\begin{array}{ccc} S' \times F' & \xrightarrow{\;\phi\;} & S \times F \\ \downarrow & & \downarrow \\ S' & \xrightarrow{\;f\;} & S \end{array} \tag{1}$$

that are fibrewise diffeomorphisms and commute with the map to X (if present). To explain the former, note that $\phi = \psi \circ (f \times \mathrm{id}_F)$ where ψ is a map over S':

$$\begin{array}{ccc} S' \times F' & \xrightarrow{\ \psi\ } & S' \times F \\ & \searrow \quad \swarrow & \\ & S' & \end{array}$$

We say that ϕ is a *fibrewise diffeomorphism* if ψ is a diffeomorphism. Alternatively, one can start with all commutative diagrams as in (1) and then restrict to the cartesian morphisms. It follows that there are fibrations

$$d|\delta\text{-}\mathsf{B}_{\mathrm{c}}(X)^{\mathrm{prfam}} \to \mathsf{SM}$$

that take $S \times F$ and send it to its base (or parameter space) S. Finally, in the case of $0|\delta\text{-}\mathsf{B}_{\mathrm{cc}}(X)^{\mathrm{prfam}}$ we only use fibres $F := \mathbb{R}^{0|\delta}$.

Definition 5.1 is justified by the following result that is proven exactly as Lemma 4.9. The last description is the one used in [ST2]. Recall that the representable stack $\underline{\mathbb{R}} \to \mathsf{SM}$ used for TFTs is symmetric monoidal with respect to multiplication.

Lemma 5.3. *If* $\mathsf{C} \to \mathsf{SM}$ *is a symmetric monoidal stack in the sense of Definition 7.21 then there are equivalences of categories*

$$\mathsf{Fun}_{\mathsf{SM}}(d|\delta\text{-}\mathsf{B}_{\mathrm{cc}}(X)^{\mathrm{prfam}}, \mathsf{C}) \simeq \mathsf{Fun}_{\mathsf{SM}}(d|\delta\text{-}\mathsf{B}_{\mathrm{cc}}(X)^{\mathrm{fam}}, \mathsf{C})$$

and

$$\mathsf{Fun}_{\mathsf{SM}}(d|\delta\text{-}\mathsf{B}_{\mathrm{cc}}(X)^{\mathrm{fam}}, \mathsf{C}) \simeq \mathsf{Fun}^{\otimes}_{\mathsf{SM}}(d|\delta\text{-}\mathsf{B}_{\mathrm{c}}(X)^{\mathrm{fam}}, \mathsf{C}),$$

where $d|\delta\text{-}\mathsf{B}_{\mathrm{c}}(X)^{\mathrm{fam}} \to \mathsf{SM}$ *is the stack of fibre bundles with closed* $d|\delta$*-dimensional fibres (obtained from stackifying* $d|\delta\text{-}\mathsf{B}_{\mathrm{c}}(X)^{\mathrm{prfam}} \to \mathsf{SM}$*).*

5.1. 0|1-dimensional TFTs over X. In the 0-dimensional case we used the equivalence of fibred categories

$$0\text{-}\mathsf{B}_{\mathrm{cc}}(X)^{\mathrm{fam}} \simeq \underline{X}$$

to complete our computation of $0\text{-TFT}(X)$. In the $0|1$-dimensional case, the corresponding result is more interesting due to the nontrivial diffeomorphisms of superpoints.

Lemma 5.4. *For every supermanifold X and every $\delta \in \mathbb{N}$ there is an equivalence*

$$0|\delta\text{-}\mathsf{B}_{\mathrm{cc}}(X)^{\mathrm{prfam}} \simeq \underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)/\underline{\mathrm{Diff}}(\mathbb{R}^{0|\delta}).$$

of fibred categories over SM.

The right-hand side is the quotient construction explained in Definition 7.12 of the appendix. Note that a priori, the two inner Homs are just presheaves on SM but that is all one needs to form the fibred quotient category. It actually turns out that both presheaves are representable but we will only discuss this in the case $\delta = 1$.

Proof. To simplify the discussion, fix a supermanifold S and only look at the fibre categories over S on both sides of the equation. The left-hand side has objects

$$f \in \mathsf{SM}(S \times \mathbb{R}^{0|\delta}, X) \cong \mathsf{SM}(S, \underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)),$$

where the right-hand side consists exactly of the objects in the quotient category. As for morphisms, Definition 5.2 explains why the left-hand side has pairs (ψ, f), where $f : S' \to S$ and $\psi : S' \times \mathbb{R}^{0|\delta} \to S' \times \mathbb{R}^{0|\delta}$ is a diffeomorphism over S'. Comparing this to the morphisms on the right-hand side, we see that we just need to translate ψ into a map

$$g : S' \to \underline{\mathrm{Diff}}(\mathbb{R}^{0|\delta}).$$

However, this translation is just the definition of the diffeomorphism group in terms of its S-points. It is not hard to see that these translations preserve the composition in the respective categories. Finally, we can use Lemma 7.4 or work things out by hand. □

We now turn to the case $\delta = 1$ and abbreviate the superpoint spt as before. We conclude from the above lemma that

$$0|1\text{-}\mathsf{B}_{\mathrm{cc}}(X)^{\mathrm{prfam}} \simeq \mathrm{SP}X / \underline{\mathrm{Diff}}(\mathrm{spt}), \tag{2}$$

where the supermanifold $\mathrm{SP}X$ of superpoints in X is represented by ΠTX by Proposition 3.1. From Lemma 3.5 we know that the diffeomorphism supergroup of spt is given by

$$\underline{\mathrm{Diff}}(\mathrm{spt}) \cong \mathbb{R}^{\times} \ltimes \mathbb{R}^{0|1},$$

where $\mathbb{R}^{\times}$ is the even dilational part and $\mathbb{R}^{0|1}$ are the odd translations of spt.

Proposition 5.5. *For any supermanifold X, there is a bijection between $0|1$-dimensional TFTs over X and closed 0-forms on X:*

$$0|1\text{-TFT}(X) \cong \mathsf{Fun}_{\mathsf{SM}}(\mathrm{SP}X / \underline{\mathrm{Diff}}(\mathrm{spt}), \underline{\mathbb{R}}) \cong \Omega^0_{\mathrm{cl}}(X).$$

The right-hand side equals $\{f \in C^\infty(X) \mid df = 0\} = \{f \in C^\infty(X_{\mathrm{red}}) \mid df = 0\}$.

Proof. By Corollary 7.17 we just need to determine those functions on the supermanifold $\mathrm{SP}X$ that are fixed by the supergroup $\underline{\mathrm{Diff}}(\mathrm{spt})$. For the even action of $\mathbb{R}^{\times}$ by dilations this is literally the fixed point set of the action on $C^\infty(\mathrm{SP}X)$. As explained in Section 3.4, the only functions on $\mathrm{SP}X$ that are fixed by all dilations are the functions on X. For the odd part of the action, namely by translations of $\mathbb{R}^{0|1}$, it is not hard to see that being 'invariant' in the sense of Corollary 7.17 is equivalent to being annihilated by the infinitesimal generators of translation. By Lemma 3.4, this infinitesimal generator of translation is just the de Rham d on $C^\infty(X)$. □

Corollary 5.6. *Concordance classes of* $0|1$*-dimensional TFTs over X agree with degree* 0 *de Rham cohomology:*

$$0|1\text{-TFT}[X] \cong H^0_{\mathrm{dR}}(X) \cong H^0_{\mathrm{dR}}(X_{\mathrm{red}}).$$

This finishes the proof of the degree 0 case of Theorem 2. Before going to non-trivial degrees, we shall prove the degree 0 case of Theorem 1.

5.2. 0|1-dimensional EFTs over X. In the spirit of Felix Klein's Erlangen program, we introduce a *Euclidean geometry* on the superpoint by specifying its isometry group to be

$$\underline{\mathrm{Iso}}(\mathrm{spt}) := \{\pm 1\} \ltimes \mathbb{R}^{0|1} \leq \mathbb{R}^\times \ltimes \mathbb{R}^{0|1} = \underline{\mathrm{Diff}}(\mathrm{spt}),$$

given by translations and reflections of the superpoint. This is analogous to the Euclidean group of $\mathbb{R}$ inside all diffeomorphisms. It leads to a Euclidean bordism category $0|\delta\text{-}\mathsf{EB}_{\mathrm{cc}}$ and its family version and also to the notion of a *Euclidean* field theory by following the same steps as for the case of TFTs. So we define

$$0|1\text{-EFT}(X) := \mathsf{Fun}_{\mathsf{SM}}(0|\delta\text{-}\mathsf{EB}_{\mathrm{cc}}(X)^{\mathrm{prfam}}, \underline{\mathbb{R}})$$

and compute exactly as in Proposition 5.5 that it is isomorphic to

$$\mathsf{Fun}_{\mathsf{SM}}(\mathrm{SP}X/\underline{\mathrm{Iso}}(\mathrm{spt}), \underline{\mathbb{R}}) \cong \{f \in C^\infty(\mathrm{SP}X)^{\mathrm{ev}} | D(f) = 0\}.$$

Here D is the infinitesimal generator of translations, acting on $C^\infty(\mathrm{SP}X)$ as explained in Section 3.1. One can thus think of such field theories as closed pseudodifferential forms on a supermanifold X. For example, if $X = \mathbb{R}^{0|1}$ then $\mathrm{SP}X = \mathbb{R}^{1|1}$ and

$$0|1\text{-EFT}(\mathbb{R}^{0|1}) \cong C^\infty(\mathbb{R}).$$

If X is an ordinary manifold then Proposition 3.1 and Lemma 3.4 imply

$$C^\infty(\mathrm{SP}X) \cong \Omega^* X \quad \text{and} \quad D = d$$

and hence the degree 0 case of Theorem 1 follows.

6. Twisted field theories

Recall from Definition 5.1 and Lemma 7.20 that we can express untwisted field theories as fibred natural transformations over SM as follows:

$$0|\delta\text{-TFT}(X) \cong \left\{ 0|\delta\text{-}\mathsf{B}_{\mathrm{cc}}(X)^{\mathrm{prfam}} \underset{\mathbb{1}}{\overset{\mathbb{1}}{\underset{\Downarrow E}{\rightrightarrows}}} \mathsf{Pic} \right\} := \mathrm{Nat}_{\mathsf{SM}}(\mathbb{1}, \mathbb{1}),$$

where $\mathsf{Pic} \to \mathsf{SM}$ is the symmetric monoidal stack of $\mathbb{Z}/2$-graded real line bundles and $\mathbb{1}\colon \mathsf{SM} \to \mathsf{Pic}$ is the monoidal unit, giving the trivial bundle for each $S \in \mathsf{SM}$. Hence the following definition is not surprising.

Definition 6.1. A *twist* for $0|\delta$-dimensional TFTs over a supermanifold X is a fibred functor

$$\mathcal{T} \in \mathsf{Fun}_{\mathsf{SM}}(0|\delta\text{-}\mathsf{B}_{\mathrm{cc}}(X)^{\mathrm{prfam}}, \mathsf{Pic}).$$

Moreover, a $\mathcal{T}$-twisted TFT over X is a fibred natural transformation

$$E \in \mathrm{Nat}_{\mathsf{SM}}(\mathbb{1}, \mathcal{T}) = \left\{ 0|\delta\text{-}\mathsf{B}_{\mathrm{cc}}(X)^{\mathrm{prfam}} \underset{\mathcal{T}}{\overset{\mathbb{1}}{\underset{\Downarrow E}{\rightrightarrows}}} \mathsf{Pic} \right\}.$$

We write $0|\delta\text{-TFT}^{\mathcal{T}}(X)$ for the set of $\mathcal{T}$-twisted TFTs over X.

In the case $\delta = 1$ we computed the bordism category in equation (2) to be equivalent to the quotient fibration

$$QX := \mathrm{SP}X/\underline{\mathrm{Diff}}(\mathrm{spt}) \simeq 0|1\text{-}\mathsf{B}_{\mathrm{cc}}(X)^{\mathrm{prfam}}$$

and hence twists $\mathcal{T} \in \mathsf{Fun}_{\mathsf{SM}}(QX, \mathsf{Pic})$ are by definition *line bundles* over the quotient QX, see Section 7.4 of our appendix. Moreover, $\mathcal{T}$-twisted field theories are *sections* of this line bundle:

$$0|1\text{-TFT}^{\mathcal{T}}(X) \cong \Gamma(QX; \mathcal{T}) := \mathrm{Nat}_{\mathsf{SM}}(\mathbb{1}, \mathcal{T}). \tag{3}$$

We will compute all twists and their sections in the forthcoming paper with Chris Schommer-Pries, here we will finish by studying the simplest twists, namely those that do not depend on X and give the notion of "degree".

Definition 6.2. Consider the composition of supergroup homomorphisms

$$\rho : \underline{\mathrm{Diff}}(\mathbb{R}^{0|\delta}) \twoheadrightarrow GL_\delta(\mathbb{R}) \overset{\det}{\twoheadrightarrow} \mathbb{R}^\times$$

and recall from Section 7.4 that there is a corresponding even line bundle L_ρ on the quotient fibration

$$\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)/\underline{\mathrm{Diff}}(\mathbb{R}^{0|\delta}).$$

The *degree* 1 *twist* is defined to be the odd partner of that line bundle, $\mathcal{T}_1 := \Pi L_\rho$. For each $n \in \mathbb{Z}$ we define $\mathcal{T}_n$ to be the n-th power of $\mathcal{T}_1$ with respect to the symmetric monoidal structure on Pic. We denote by

$$0|\delta\text{-TFT}^n(X) := 0|\delta\text{-TFT}^{\mathcal{T}_n}(X)$$

the set of $0|\delta$-dimensional TFTs of *degree* n over X. In particular, since $\mathcal{T}_0 = \mathbb{1}$ we see that degree 0 TFTs are by definition untwisted.

Even though this definition applies for all δ, we shall only continue the discussion for $\delta = 1$. By Section 7.4, the projection $\rho\colon \underline{\mathrm{Diff}}(\mathrm{spt}) \to \mathbb{R}^\times$ gives a canonical line bundle L on QX:

$$L\colon QX = \mathrm{SP}X/\underline{\mathrm{Diff}}(\mathrm{spt}) \to \mathrm{pt}/\underline{\mathrm{Diff}}(\mathrm{spt}) \xrightarrow{\rho} \mathrm{pt}/\mathbb{R} \to \mathrm{pt}/\!\!/\mathbb{R} = \mathsf{Pic}.$$

Then $\mathcal{T}_1 := \Pi L$ is the odd partner of this line bundle and $\mathcal{T}_n = \mathcal{T}_1^{\otimes n}$. Specializing equation (3) to $\mathcal{T}_n$, we see that the set $0|1\text{-TFT}^n(X)$ of $0|1$-dimensional *degree* n TFTs over X can be identified with the set $\Gamma(QX;\mathcal{T}_n)$ of sections of the line bundle $\mathcal{T}_n$ over the quotient fibration QX. Hence the following proposition implies Theorem 2.

Proposition 6.3. *For any supermanifold X, there is a canonical bijection between sections of the line bundle $\mathcal{T}_n$ on $QX = \mathrm{SP}X/\underline{\mathrm{Diff}}(\mathrm{spt})$ and closed differential forms of degree n on X:*

$$\Gamma(QX;\mathcal{T}_n) \cong \Omega^n_{\mathrm{cl}}(X).$$

Proof. We apply Corollary 7.19 in the case where $G := \underline{\mathrm{Diff}}(\mathrm{spt})$ acts on the superpoints $\mathrm{SP}X$ and ρ is the n-th power map on $\mathbb{R}^\times$. We conclude

$$\begin{aligned}\Gamma(\mathrm{SP}X/G;\mathcal{T}_n) \cong \{f \in C^\infty(\mathrm{SP}X) \mid \mu^*(f) &= p_1^*(f)\cdot p_2^*(\rho)\\ &= f\otimes\rho \in C^\infty(\mathrm{SP}X\times G)\}.\end{aligned}$$

Expanding $C^\infty(\mathrm{SP}X \times G) = C^\infty(\mathrm{SP}X)\otimes\Lambda^*[\eta]\otimes C^\infty(\mathbb{R}^\times)$, the right-hand side is given as $p_1^*(f)\cdot p_2^*(\rho) = f\otimes\rho = f\otimes 1\otimes t^n$, where $t \in C^\infty(\mathbb{R}^\times)$ is the standard coordinate. By Corollary 3.7 this can only equal the left-hand side if $f \in \Omega^n X \subset C^\infty(\mathrm{SP}X)$. Moreover, Proposition 3.9 says that

$$\mu^*(f) = f\otimes 1\otimes t^n + D(f)\otimes\eta\otimes t^n \in C^\infty(\mathrm{SP}X)\otimes\Lambda^*[\eta]\otimes C^\infty(\mathbb{R}^\times)$$

for any $f \in \Omega^n X$. Comparing coefficients shows that $D(f) = 0$ and by Lemma 3.4 this is equivalent to $d(f) = 0$. □

6.1. Twisted Euclidean field theories. Recall that the Eudlidean structure on the superpoint $\mathrm{spt} = \mathbb{R}^{0|1}$ is defined by its isometry group

$$\underline{\mathrm{Iso}}(\mathrm{spt}) := \{\pm 1\}\ltimes\mathbb{R}^{0|1} \le \mathbb{R}^\times\ltimes\mathbb{R}^{0|1} = \underline{\mathrm{Diff}}(\mathrm{spt})$$

given by translations and reflections. This leads to

$$0|1\text{-EFT}^n(X) := \mathrm{Nat}_{\mathsf{SM}}(\mathbb{1},\mathcal{T}_n) = \left\{\ 0|1_{\mathrm{cc}}\text{-}\mathsf{EB}(X)^{\mathrm{prfam}} \underset{\mathcal{T}_n}{\overset{\mathbb{1}}{\Downarrow E}} \mathsf{Pic}\ \right\},$$

which can be computed exactly as in Proposition 6.3 to be isomorphic to

$$\Gamma(\mathrm{SP}X/\underline{\mathrm{Iso}}(\mathrm{spt});\mathcal{T}_n) \cong \{f \in C^\infty(\mathrm{SP}X)^{\mathrm{ev}/\mathrm{odd}} \mid D(f) = 0\},$$

where the functions are even respectively odd depending on the parity of n. Specializing to an ordinary manifold X, we get $\Omega_{cl}^{ev/odd}(X)$ as before. This finishes the proof of Theorem 1.

7. Appendix: Grothendieck fibrations

In this appendix we will give a survey of the language used in this paper to discuss smooth and supersymmetric *families* of categories. Excellent references on these (Grothendieck) fibrations are [C] and [Vi], here we only recall the main aspects for the convenience of the reader who does not want to look at other sources. We only claim originality for Proposition 7.13 which we generalize from groupoids to categories.

We will typeset categories in sans-serif C, S, $\mathsf{V}, \dots$ and abbreviate $C \in \mathsf{C}$ to mean that C is an object in C. Similarly, $\phi \in \mathsf{C}(C', C)$ will denote a morphism in C, $\phi\colon C' \to C$.

Consider a functor $p\colon \mathsf{V} \to \mathsf{S}$ where S will later be the category Man of manifolds or SM of supermanifolds. We also use the letter S to remind the reader that this is the (Grothendieck) *site* over which everything is happening. One motivating example to keep in mind is when V is the category of (super) vector bundles over (super) manifolds and p is the map that takes a vector bundle to its base.

7.1. Pullbacks and categories of fibred functors. In the following diagrams, an arrow going from an object V of V to an object S of S, written as $V \mapsto S$, will mean that $p(V) = S$ or in words, that "V lies over S". Furthermore, the commutativity of the diagram

$$\begin{array}{ccc} V & \xrightarrow{\ \phi\ } & W \\ \downarrow & & \downarrow \\ S & \xrightarrow{\ f\ } & T \end{array} \tag{4}$$

means that $p(\phi) = f$, or that "ϕ lies over f".

Definition 7.1. A morphism $\phi \in \mathsf{V}(V, W)$ is *cartesian* if for any $\psi \in \mathsf{V}(U, W)$ and any $g \in \mathsf{S}(p(U), p(V))$ with $p(\phi) \circ g = p(\psi)$, there exists a unique $\theta \in \mathsf{V}(U, V)$ with $p(\theta) = g$ and $\phi \circ \theta = \psi$, as in the commutative diagram

$$\begin{tikzcd} U \arrow[d, mapsto] \arrow[dr, dashed, "\theta"'] \arrow[drr, bend left, "\psi"] & & \\ R \arrow[dr, "g"'] \arrow[drr, bend left, "h"] & V \arrow[r, "\phi"] \arrow[d, mapsto] & W \arrow[d, mapsto] \\ & S \arrow[r, "f"'] & T . \end{tikzcd}$$

If $\phi\colon V \to W$ is cartesian, we say that the diagram (4) is a *cartesian square*. It is easy to see that cartesian morphisms are closed under composition. They should be thought of as the "fibrewise isomorphisms". We refer to [C], 3.1.2, for a careful comparison of this notion and the one used by Grothendieck where $g = \mathrm{id}_S$ in the diagram above.

For example, any isomorphism ϕ is clearly cartesian and in fact, this notion is meant to formalize that of 'fibrewise isomorphisms', compare also Definition 7.9.

Definition 7.2. A functor $p\colon \mathsf{V} \to \mathsf{S}$ is a (Grothendieck) *fibration* if pullbacks exist: for every object $W \in \mathsf{V}$ and every morphism $f \in \mathsf{S}(S, p(W))$, there is a cartesian square

$$\begin{array}{ccc} V & \overset{\phi}{\dashrightarrow} & W \\ \vdots & & \downarrow \\ S & \xrightarrow{f} & p(W)\,. \end{array}$$

One can think of this property as existence of "categorical path-lifting". We will define the fibres of such a fibration below and we shall see in Definition 7.5 that in the *discrete* case, path-lifting is unique, just like for covering maps (also known as fibre bundles with discrete fibres).

A *fibred category over* S is a category V together with a functor $p\colon \mathsf{V} \to \mathsf{S}$ which is a fibration. If $p_{\mathsf{V}}\colon \mathsf{V} \to \mathsf{S}$ and $p_{\mathsf{W}}\colon \mathsf{W} \to \mathsf{S}$ are fibered categories over S, then a *fibred functor* $F\colon \mathsf{V} \to \mathsf{W}$ is a functor with $p_{\mathsf{V}} \circ F = p_{\mathsf{W}}$ that sends cartesian morphisms to cartesian morphisms, i.e., that preserves pullbacks.

For an object $S \in \mathsf{S}$ the *fibre* of the fibration $p\colon \mathsf{V} \to \mathsf{S}$ over S is by definition the subcategory $\mathsf{V}_S \subseteq \mathsf{V}$ whose objects are those $V \in \mathsf{V}$ lying over S and whose morphisms lie over id_S. For example, $\mathsf{Vect} \to \mathsf{Man}$ is a fibration with a fibre Vect_S being the category of vector bundles over the manifold S.

After the choice of a *cleavage*, i.e., a certain collection of pullbacks, these fibres assemble into a pseudo-functor (or lax 2-functor)

$$\mathsf{S}^{\mathrm{op}} \to \mathsf{Cat}, \qquad S \mapsto \mathsf{V}_S,$$

which is a different way of looking at the fibration condition. This point of view leads naturally to the following

Definition 7.3. A *fibred natural transformation* $\alpha \in \mathrm{Nat}_{\mathsf{S}}(F, G)$ between two fibred functors $F, G\colon \mathsf{V} \to \mathsf{W}$ is a natural transformation $\alpha\colon F\mathbb{R} \to G$ such that for any object $V \in \mathsf{V}$, the morphism

$$\alpha_V \in \mathsf{W}(F(V), G(V))$$

lies in W_S, or equivalently α_V lies over id_S, where

$$S := p_{\mathsf{V}}(V) = p_{\mathsf{W}}(F(V)) = p_{\mathsf{W}}(G(V)) \in \mathsf{S}.$$

We shall write $\mathsf{Fun}_{\mathsf{S}}(\mathsf{V}, \mathsf{W})$ for the category of fibred functors and fibred natural transformations. The notion of an *equivalence of fibred categories* arises in the usual way from observing that fibred categories over S form a (strict) 2-category.

The following lemma is very useful, see [Vi], Prop. 3.36, for a proof.

Lemma 7.4. *A fibred functor $F: \mathsf{V} \to \mathsf{W}$ is an equivalence of fibred categories if and only if the restrictions $F_S: \mathsf{V}_S \to \mathsf{W}_S$ are equivalences for all objects $S \in \mathsf{S}$.*

The next three definitions will give different ways of constructing fibred categories. An easy class of fibrations are those with *discrete fibres*, i.e., those where all fibres V_S have only identity morphisms. These are sometimes also referred to as *categories fibred in sets*. By [Vi], Prop. 3.26, up to equivalence these always arise from a presheaf (also known as functor) $\mathcal{F}: \mathsf{S}^{\mathrm{op}} \to \mathsf{Set}$ as follows.

Definition 7.5. Define the objects of $\underline{\mathcal{F}}$ to be pairs (S, g) where $S \in \mathsf{S}$ and $g \in \mathcal{F}(S)$ and morphisms by

$$\underline{\mathcal{F}}((S', g'), (S, g)) := \{f \in \mathsf{S}(S', S) \mid \mathcal{F}(f)(g) = g'\}.$$

The forgetful map $p: \underline{\mathcal{F}} \to \mathsf{S}$ is easily seen to be a fibration, in fact, there are *unique* pullbacks in this case.

Lemma 7.6. *The only fibred natural transformation between functors F, $G \in \mathsf{Fun}_{\mathsf{S}}(\mathsf{V}, \underline{\mathcal{F}})$ is the identity. In particular, this category is discrete* (*in the sense that it has only identity morphisms*).

The easiest examples of fibrations with discrete fibres come from *representable* presheaves, i.e., where the presheaf arises from a fixed object $M \in \mathsf{S}$ via $\mathcal{F}(S) := \mathsf{S}(S, M)$.

Definition 7.7. For $M \in \mathsf{S}$, we write $\underline{M} \to \mathsf{S}$ for the resulting *representable fibration*.

The 2-Yoneda's lemma [Vi], 3.6.2, gives natural equivalences of categories

$$\mathsf{Fun}_{\mathsf{S}}(\underline{M}, \mathsf{W}) \simeq \mathsf{W}_M. \tag{5}$$

In the case $\mathsf{W} = \underline{N}$ these reduce to natural bijections of sets

$$\mathsf{Fun}_{\mathsf{S}}(\underline{M}, \underline{N}) \cong \underline{N}(M) = \mathsf{S}(M, N).$$

Remark 7.8. We have used the notation of 'underlining' in two different contexts: above we used it to get fibred categories over S from presheaves respectively objects of S. In Section 3 we used it for distinguishing Hom from *inner Hom* in the category SM. Inner Hom can also be discussed in our current general context, assuming that S has a monoidal structure $\otimes$. For two fixed objects $M, N \in \mathsf{S}$ one can then consider the presheaf

$$\mathsf{S}^{\mathrm{op}} \to \mathsf{Set}, \quad S \mapsto \mathsf{S}(S \otimes M, N).$$

If this presheaf is representable then the representing object is determined uniquely up to canonical isomorphism by the usual Yoneda lemma. It is denoted by $\underline{\mathsf{S}}(M, N) \in \mathsf{S}$ and has the S-points

$$\mathsf{S}(S, \underline{\mathsf{S}}(M, N)) \cong \mathsf{S}(S \otimes M, N).$$

We hope that these two distinct underlining conventions will not confuse the reader, one makes fibred categories from objects, the other objects from morphism sets.

Definition 7.9. Given a functor $\mathcal{G} \colon \mathsf{S}^{\mathrm{op}} \to \mathsf{Cat}$, there is a corresponding fibred category $\widehat{\mathcal{G}} \to \mathsf{S}$. The objects of $\widehat{\mathcal{G}}$ are pairs (S, g) where $S \in \mathsf{S}$ and $g \in \mathcal{G}(S)$. Moreover, define

$$\widehat{\mathcal{G}}((S', g'), (S, g)) = \{(f, \phi) \in \mathsf{S}(S', S) \times \mathrm{mor}\, \mathcal{G}(S') \mid s(\phi) = g', t(\phi) = \mathcal{G}(f)(g)\},$$

where s, t are the source respectively target maps of the category $\mathcal{G}(S')$. The composition law is not hard to guess but surprisingly can also be written down in the case $\mathcal{G}$ is just a pseudo-functor, see [Vi], 3.1.3. In both cases, the reader is invited to check that a morphism (f, ϕ) is cartesian if and only if ϕ is an isomorphism.

Remark 7.10. Every fibred category $p \colon \mathsf{V} \to \mathsf{S}$ is canonically equivalent to one of the form $\widehat{\mathcal{G}}$, see [Vi], Thm. 3.45. Namely, one takes $\mathcal{G}$ to be the functor that sends S to $\mathsf{Fun}_{\mathsf{S}}(\underline{S}, \mathsf{V})$.

7.2. Stacks. If S is a (Grothendieck) *site* in the sense of [Vi], Def. 2.24, i.e., it carries the notion of *coverings* of objects, one can ask for a generalization of the sheaf property of a presheaf $\mathcal{F}$ as above. It turns out that $\mathcal{F}$ is a sheaf if and only if the fibration $\underline{\mathcal{F}}$ is a stack in the following sense.

Definition 7.11. A fibred category $\mathsf{V} \to \mathsf{S}$ over a site S is called a *stack* if for every covering $S_i \to S$ of an object $S \in \mathsf{S}$ the natural functor

$$\mathsf{V}_S \to \mathsf{V}_{S_i \to S} \tag{6}$$

is an equivalence of categories. The right-hand side is the *descent category* of the covering $S_i \to S$ whose objects are gluing data, see [Vi], 4.1.2.

Let us explain the descent category in the case of a fibred category $\widehat{\mathcal{G}}$ coming from a functor $\mathcal{G}\colon \mathsf{S}^{\mathrm{op}} \to \mathsf{Cat}$ and where the covering is given by a single morphism $U \to S$. By the properties of a site, the pullbacks

$$U^{[n+1]} = \underbrace{U \times_S \cdots \times_S U}_{n+1}$$

exist for all $n \geq 0$ and they form the n-simplices of a (Čech-like) simplicial object $U^\bullet$ in S, resolving the object $S \in \mathsf{S}$, thought of as a constant simplicial object $S^\bullet$:

$$U^\bullet = (\cdots\ U \times_S U \times_S U \rightrightarrows U \times_S U \rightrightarrows U\ ) \to S^\bullet.$$

If we apply the functor $\mathcal{G}$ to this simplicial map, we get a cosimplicial functor $\mathcal{G}(S^\bullet) \to \mathcal{G}(U^\bullet)$. Using a version of Definition 7.14 below, we can form its homotopy limit

$$\widehat{\mathcal{G}}_S = \mathcal{G}(S) = \mathsf{hlim}^\times(\mathcal{G}(S^\bullet)) \to \mathsf{hlim}^\times(\mathcal{G}(U^\bullet)) =: \widehat{\mathcal{G}}_{U\to S},$$

which is the functor in equation (6) that is required to be an equivalence for $\widehat{\mathcal{G}}$ to be a stack. Here the version of the homotopy limit $\mathsf{hlim}^\times$ is defined exactly like the homotopy limit hlim in Definition 7.14 below, except that $\phi \in \mathsf{C}_1(s(C), t(C))$ is assumed to be an *isomorphism*. This is important for being able to glue objects together consistently. In particular, the right-hand side $\widehat{\mathcal{G}}_{U\to S}$ is by definition the descent category of the covering $U \to S$.

We shall only use these *descent conditions* for motivating our definitions in the next sections, see Lemmas 4.9 and 5.3. The main tool we will need is the *stackification* functor from fibred categories to stacks over S, see [C], 4.2.2. It is left adjoint to the forgetful functor in analogy to sheafification.

The example of vector bundles fibred over $\mathsf{S} = \mathsf{Man}$ is a stack. The descent conditions just formalize the fact that bundles can be constructed from their restriction to open subsets via gluing data. Here we use the usual notion of a *covering* of a manifold $S \in \mathsf{Man}$, namely where $S_i \subseteq S$ are open subsets with union S. If we form $U := \amalg_i S_i$ then we get a single covering $U \to S$ in Man and the reader is invited to check that the above definition of a descent category indeed equals the category formed by gluing data for vector bundles over S with respect to the covering S_i.

7.3. Internal categories as fibrations. An important special case in the construction of $\widehat{\mathcal{G}}$ in Definition 7.9 is the case where the functor $\mathcal{G} : \mathsf{S}^{\mathrm{op}} \to \mathsf{Cat}$ takes values in *small* categories. Then $\mathcal{G}$ is given by an *internal category* in presheaves on S. So

$$\operatorname{obj}\mathcal{G}(S) = \mathcal{F}_0(S), \quad \operatorname{mor}\mathcal{G}(S) = \mathcal{F}_1(S)$$

for presheaves $\mathcal{F}_i : \mathsf{S}^{\mathrm{op}} \to \mathsf{Set}$ and there are various structure maps defining the structure of the categories $\mathcal{G}(S)$. These can be most easily summarized by the following diagram where $\mathcal{F}_2 := \mathcal{F}_1 \times_{\mathcal{F}_0} \mathcal{F}_1$ is the pullback in $\mathsf{Presheaf}(\mathsf{S})$ formed using the maps s, t below:

$$\mathcal{F}_2 \overset{p_1}{\underset{c}{\overset{p_2}{\rightrightarrows\!\!\!\to}}} \mathcal{F}_1 \overset{s}{\underset{u}{\overset{t}{\rightrightarrows}}} \mathcal{F}_0 \tag{7}$$

Here p_i are the two projections, c is the composition map, s, t are source and target maps and u is the unit (or identity) map. Some of the relations between these structure maps are elegantly expressed by the *simplicial identities* present in the above diagram, namely

$$t \circ u = \mathrm{id} = s \circ u, \quad t \circ p_1 = t \circ c, \quad s \circ p_2 = s \circ c, \quad t \circ p_2 = s \circ p_1.$$

However, to formulate the associativity of c, the above diagram is not sufficient and one needs to extend it to the *nerve* $\mathcal{F}_\bullet$ of the category by setting for $n \geq 1$

$$\mathcal{F}_n := \underbrace{\mathcal{F}_1 \times_{\mathcal{F}_0} \cdots \times_{\mathcal{F}_0} \mathcal{F}_1}_{n}, \tag{8}$$

with the well-known structure maps that make this into a *simplicial* presheaf. Diagram 7 above is its initial segment which we shall refer to as its 2-*skeleton* (even though we are missing two degeneracy maps $\mathcal{F}_1 \to \mathcal{F}_2$). It is an important elementary fact that all identities between the structure maps of our category are expressed in terms of simplicial identities (all appearing in the 3-skeleton). Vice versa, every simplicial presheaf satisfying equation 8 comes from an internal category. We will therefore use the notation $\mathcal{F}_\bullet$ for the internal category in presheaves on S.

As in Definition 7.9, the functor $\mathcal{G}$ gives a fibration $\widehat{\mathcal{G}} \to \mathsf{S}$ which we shall abbreviate as $\mathcal{F}_0/\mathcal{F}_1$ in the case at hand. The common notation for the associated stack is $\mathcal{F}_0/\!\!/\mathcal{F}_1$.

Definition 7.12. In our applications to field theories, the most important example of an internal category will arise from a monoid object $G \in \mathsf{S}$ that acts (from the right) on another object $M \in \mathsf{S}$. This action has an associated internal *transport category* in S (and hence in presheaves on S)

$$M \times G \overset{p}{\underset{\mu}{\rightrightarrows}} M$$

and therefore the *quotient fibration* $M/G := M/(M \times G)$ is defined as above.

For example, for $\mathsf{S} = \mathsf{Man}$ and G a Lie group acting on a manifold M, the quotient *space* is not a manifold in general and usually not even Hausdorff. It is therefore often wise to study instead the quotient *fibration* M/G. This is in general not a stack but its stackification, the *quotient stack* $M/\!\!/G$, has objects (P, m) where $P \to S$ is a G-principal bundle and $m \in \mathsf{S}(P, M)$ is G-equivariant, see [C], 4.4.9.

The objects of M/G correspond to trivial bundles and general bundles are obtained by the gluing construction.

More generally, if pt is a terminal object in a site S and $G \in \mathsf{S}$ is a group object, one can *define* the category of G-principal bundles (sometimes also referred to as G-torsors) in S to be the stackification of the fibration pt/G. Then the discussion in the previous paragraph applies to describe the quotient stack $M/\!\!/G$ in this setting.

The following computation of the functor category between two fibred categories will be essential in our applications to field theories. We state a very general result also for the purpose of referring to it in future papers.

Proposition 7.13. *For any fibration* $\mathsf{W} \to \mathsf{S}$ *and any internal category* $\mathcal{F}_\bullet$ *in presheaves on* S*, there is an isomorphism of categories*

$$\mathsf{Fun}_{\mathsf{S}}(\mathcal{F}_0/\mathcal{F}_1, \mathsf{W}) \simeq \mathsf{hlim}(\mathsf{Fun}_{\mathsf{S}}(\underline{\mathcal{F}_\bullet}, \mathsf{W})),$$

where the right-hand side is the homotopy limit (*defined below*) *of the cosimplicial category with* n*-simplices consisting of fibred functors* $\underline{\mathcal{F}_n} \to \mathsf{W}$.

Unfortunately, the left-hand side above is computed wrongly in [C], Prop. 3.7.5, where the homotopy pullback needs to be replaced by the *homotopy limit* defined as follows.

Definition 7.14. Let $\mathsf{C}_\bullet$ be a cosimplicial category with 2-skeleton

$$\mathsf{C}_2 \;\overset{c}{\underset{p_1}{\underset{p_2}{\Longleftarrow\!\!\!-}}}\; \mathsf{C}_1 \;\overset{u}{\underset{s}{\underset{t}{\rightleftarrows}}}\; \mathsf{C}_0 \;.$$

Then its *homotopy limit* $\mathsf{hlim}(\mathsf{C}_\bullet)$ is the category with objects (C, ϕ) where $C \in \mathsf{C}_0$ and $\phi \in \mathsf{C}_1(s(C), t(C))$ is a morphism such that

$$u(\phi) = \mathrm{id}_C \quad \text{and} \quad p_1(\phi) \circ p_2(\phi) = c(\phi).$$

Note that the cosimplicial identity $p_1 \circ s = p_2 \circ t$ implies that the above composition exists in C_2 and the other two simplicial identities say that it has the same source and target as $c(\phi)$.

Morphisms from (C', ϕ') to (C, ϕ) are $f \in \mathsf{C}_0(C', C)$ such that

$$t(f) \circ \phi' = \phi \circ s(f).$$

Note that the homotopy limit only depends on the 2-skeleton of $\mathsf{C}_\bullet$. Moreover, if C_1 is discrete, then so is $\mathsf{hlim}(C_\bullet)$. It is then just the equalizer of s and t on objects of C_0.

Remark 7.15. Given a cosimplicial *set* $C_\bullet$, we can form its limit in the category of sets. This is a certain subset of the direct product of all sets C_n. Using the cosimplicial identities, it turns out that this limit is canonically isomorphic to the limit (or equalizer) of the 1-skeleton

$$\mathsf{C}_1 \underset{t}{\overset{s}{\leftleftarrows}} \mathsf{C}_0 \ .$$

This is the analogue of the fact that the homotopy limit of a cosimplicial category in Definition 7.14 only depends on its 2-skeleton.

To make this analogy precise we would have to introduce an appropriate Quillen model structure on cosimplicial categories and discuss homotopy limits with respect to it. This was done for cosimplicial groupoids in [Ho] and in this setting Proposition 7.13 can be proven as follows:

- Define a Quillen model structure on categories fibred in groupoids.
- Show that in this model structure, $\mathcal{F}_0/\mathcal{F}_1$ is the homotopy limit of the diagram from equation 7.
- Show that the functor $\mathsf{Fun}_\mathsf{S}(-, \mathsf{W})$ takes homotopy limits to homotopy colimits (of cosimplicial groupoids as defined above).

This conceptional background might help some readers, even though we prefer the following direct argument. It also constructs an isomorphism of categories, rather than just an equivalence.

Proof of Proposition 7.13. Let $Z \in \mathsf{Fun}_\mathsf{S}(\mathcal{F}_0/\mathcal{F}_1, \mathsf{W})$. Define $C \in \mathsf{Fun}_\mathsf{S}(\underline{\mathcal{F}_0}, \mathsf{W})$ as the composition

$$C : \underline{\mathcal{F}_0} \to \mathcal{F}_0/\mathcal{F}_1 \to \mathsf{W},$$

where the first arrow is the fibered functor including the subcategory $\underline{\mathcal{F}_0}$ in $\mathcal{F}_0/\mathcal{F}_1$. Next recall that a morphism in $\mathcal{F}_0/\mathcal{F}_1$ is given by a triple (f, g, X), where $f : S' \to S$ is a morphism in S, $X \in \mathcal{F}_0(S)$, and $g \in \mathcal{F}_1(S')$ such that $t(g) = \mathcal{F}_1(f)(X)$. Now define an injection

$$\iota\colon \ \mathrm{obj}(\underline{\mathcal{F}_1}) \hookrightarrow \mathrm{mor}(\mathcal{F}_0/\mathcal{F}_1), \quad (S, g) \mapsto (\mathrm{id}_S, g, t(g)).$$

Note that for elements $(\mathrm{id}_S, g, t(g))$ in the image of ι the third entry in the triple is redundant. In order to avoid simplify notation, we will use the abbreviation $(\mathrm{id}_S, g, \_)$ for such morphisms. The key observation we will use is that the morphisms in $\mathrm{im}(\iota)$ and $\mathrm{mor}(\underline{\mathcal{F}_0})$ generate all of $\mathrm{mor}(\mathcal{F}_0/\mathcal{F}_1)$. Namely, any $(f, g, X) \in \mathrm{mor}(\mathcal{F}_0/\mathcal{F}_1)$ can uniquely be written as

$$(f, g, X) = (f, \mathrm{id}_{\mathcal{F}_1(f)(X)}, X) \circ (\mathrm{id}_{S'}, g, \mathcal{F}_1(f)(X)) = (f, \mathrm{id}, X) \circ (\mathrm{id}, g, \_).$$

The asserted isomorphism of categories is given (on objects) by

$$\mathsf{Fun}_\mathsf{S}(\mathcal{F}_0/\mathcal{F}_1, \mathsf{W}) \ni Z \mapsto (C, Z\iota) \in \mathrm{hlim}(\mathsf{Fun}_\mathsf{S}(\underline{\mathcal{F}_\bullet}, \mathsf{W})).$$

Of course, we have to verify that the assignment $Z\iota\colon \mathrm{obj}(\underline{\mathcal{F}_1}) \to \mathrm{mor}(\mathsf{W})$ indeed defines a natural transformation between the functors $C \circ s$ and $C \circ t\colon \underline{\mathcal{F}_1} \to \mathsf{W}$

and hence an element in $\mathsf{Fun}_{\mathsf{S}}(\underline{\mathcal{F}_1}, \mathsf{W})(s(C), t(C))$. To see this, let $f: S' \to S$, $X \in \mathcal{F}_0(S)$, and $g: X \to Y$ in $\overline{\mathcal{F}_1}(S)$. Then $F = (f, \mathrm{id}, X)$ and $G = (\mathrm{id}, g, Y)$ are composable and

$$G \circ F = (f, \mathcal{F}_1(f)(g), Y) = (f, \mathrm{id}, Y) \circ (\mathrm{id}, \mathcal{F}_1(f)(g), \_) =: \widetilde{F} \circ \widetilde{G}.$$

Since Z is a functor, we have $Z(G) \circ Z(F) = Z(G \circ F) = Z(\widetilde{F}) \circ Z(\widetilde{G})$. Equivalently, the relation

$$(Z\iota)(S, g) \circ C(f, \mathrm{id}, X) = C(f, \mathrm{id}, Y) \circ (Z\iota)(S', \mathcal{F}_1(f)(g))$$

holds. This equality, in turn, exactly expresses the naturality of the transformation $Z\iota$. To see this, note that the fibration $\underline{\mathcal{F}_1} \to S$ has discrete fibres. Hence any morphism in $\underline{\mathcal{F}_1}$ is of the form $(f, \mathrm{id}, g): \overline{(S', \mathcal{F}_1(f)(g))} \to (S, g)$ with $f: S' \to S$ and $g \in \mathcal{F}_1(S)$. It is now easy to check that the naturality of $Z\iota$ for (f, id, g) precisely amounts to the previous equation with $X = s(g)$, $Y = t(g)$.

Finally, since Z is a functor, $Z\iota$ respects compositions and identities as required in the definition of the hlim. We give the argument for the composition and leave the easier identity $u(Z\iota) = \mathrm{id}_C$ to the reader. The equality $p_1(Z\iota) \circ p_2(Z\iota) = c(Z\iota)$ amounts to showing that for all $(S, g_1), (S, g_2) \in \mathrm{obj}(\underline{\mathcal{F}_1})$ that are composable in the sense that $t(S, g_1) = s(S, g_2)$ (such elements necessarily live over the same base S) we have

$$Z\iota p_1((S, g_2), (S, g_1)) \circ Z\iota p_2((S, g_2), (S, g_1)) = Z\iota c((S, g_2), (S, g_1))$$

or equivalently

$$Z(\mathrm{id}, g_2, \_) \circ Z(\mathrm{id}, g_1, \_) = Z(\mathrm{id}, g_2 g_1, \_) = Z(\mathrm{id}, g_1, \_) \circ Z(\mathrm{id}, g_2, \_),$$

which holds since Z is a functor.

It is now easy to see how the inverse of the functor $Z \mapsto (C, Z\iota)$ is defined (on objects): its inverse takes $(C, \phi) \in \mathsf{hlim}(\mathsf{Fun}_{\mathsf{S}}(\underline{\mathcal{F}_\bullet}, \mathsf{W}))$ and builds a functor Z as above. Using the unique factorization of a morphism $(f, g, X) \in \mathrm{mor}(\mathcal{F}_0/\mathcal{F}_1)$, we can extend the definition of Z to all of $\mathrm{mor}(\mathcal{F}_0/\mathcal{F}_1)$ by letting

$$Z(f, g, X) := C(f, \mathrm{id}, X) \circ \phi(S', g),$$

where $f: S' \to S$ and $g \in \mathcal{F}_1(S')$. It remains to check that this yields a fibered functor Z. Functoriality holds automatically on $\mathrm{im}(\iota)$ and $\mathrm{mor}(\underline{\mathcal{F}_0})$ and for a general morphism it comes down to checking

$$\phi(S, g) \circ C(f, \mathrm{id}, X) = C(f, \mathrm{id}, X) \circ \phi(S', \mathcal{F}_1(f)(g))$$

for $f: S' \to S$ and $g \in \mathcal{F}_1(S)$. However, as above, this is precisely the condition that ϕ is a natural transformation.

It remains to prove that Z preserves cartesian morphisms. It follows readily from the definitions that a morphism (f, g, X) is cartesian if and only if g is an isomorphism. Since the composition of two cartesian morphisms is cartesian, we only have to check that for such an (f, g, X) the morphisms $C(f, \mathrm{id}, X)$ and $\phi(S', g)$ are cartesian. The former holds since C is a fibered functor and (f, id, X) is cartesian. The latter follows since we already know that Z is a functorial on $\mathrm{im}(\iota)$: this implies that $\phi(S', g)$ is also invertible and thus cartesian.

The definition of the functor $\mathsf{Fun}_{\mathsf{S}}(\mathcal{F}_0/\mathcal{F}_1, \mathsf{W}) \to \mathsf{hlim}(\mathsf{Fun}_{\mathsf{S}}(\underline{\mathcal{F}_\bullet}, \mathsf{W}))$ on morphisms is easy. In both categories a morphism is a fibered natural transformation, given by an assignment

$$N\colon \mathrm{obj}(\mathcal{F}_0/\mathcal{F}_1) = \mathrm{obj}(\underline{\mathcal{F}_0}) \to \mathrm{mor}(\mathsf{W}).$$

Finally, we check that the naturality condition agrees in both cases. Using the same factorization as above, we see that N is natural when considered a transformation between functors Z_1 and Z_2 on $\mathcal{F}_0/\mathcal{F}_1$ if and only if the diagram expressing naturality holds for all morphisms in $\mathrm{mor}(\underline{\mathcal{F}_0})$ and $\mathrm{im}\,\iota$. The condition for $\mathrm{mor}(\underline{\mathcal{F}_0})$ precisely expresses that N is a natural transformation between the functors $C_1 = Z_1|_{\underline{\mathcal{F}_0}}$ and $C_2 = Z_2|_{\underline{\mathcal{F}_0}}$. We claim that naturality for $(\mathrm{id}, g, \_) \in \mathrm{im}(\iota)$ is equivalent to the condition $t(N) \circ \phi_2 = \phi_1 \circ s(N)$ in the definition of the hlim, where $\phi_i = Z_i\iota$. We leave this simple verification to the reader. □

Example 7.16. There are various important special cases of this result:

(1) If W is discrete then so is $\mathsf{Fun}_{\mathsf{S}}(\mathsf{V}, \mathsf{W})$ by Lemma 7.6. Therefore, the homotopy limit is just an equalizer and we get a bijection of objects

$$\mathsf{Fun}_{\mathsf{S}}(\mathcal{F}_0/\mathcal{F}_1, \mathsf{W}) \cong \lim\ \mathsf{Fun}_{\mathsf{S}}(\underline{\mathcal{F}_0}, \mathsf{W}) \overset{s^*}{\underset{t^*}{\rightrightarrows}} \mathsf{Fun}_{\mathsf{S}}(\underline{\mathcal{F}_1}, \mathsf{W})\ .$$

(2) If the simplicial presheaf $\mathcal{F}_\bullet$ is represented by a simplicial object $G_\bullet$ in S then the 2-Yoneda lemma (5) implies $\mathsf{Fun}_{\mathsf{S}}(\underline{\mathcal{F}_i}, \mathsf{W}) \simeq \mathsf{W}_{G_i}$ and hence

$$\mathsf{Fun}_{\mathsf{S}}(G_0/G_1, \mathsf{W}) \simeq \mathsf{hlim}(\mathsf{W}_{G_\bullet}).$$

(3) If $G_\bullet$ is as in (2) and $\mathcal{W}$ is a presheaf on S with corresponding discrete fibration $\underline{\mathcal{W}}$, then (1) and (2) above lead to a bijection

$$\mathsf{Fun}_{\mathsf{S}}(G_0/G_1, \underline{\mathcal{W}}) \cong \{w \in \mathcal{W}(G_0) \mid \mathcal{W}(s)(w) = \mathcal{W}(t)(w) \in \mathcal{W}(G_1)\}.$$

(4) If S has a terminal object pt and $H \in \mathsf{S}$ is a monoid in S with unit $\mathrm{pt} \to H$ then we can form the quotient fibration $\mathsf{W} = \mathrm{pt}/H$ and get an equivalence of categories

$$\mathsf{Fun}_{\mathsf{S}}(G_0/G_1, \mathrm{pt}/H) \simeq \mathsf{hlim}((\mathrm{pt}/H)_{G_\bullet}).$$

Since the categories $(\mathrm{pt}/H)_S$ have only one object, the objects of the homotopy limit are morphisms ρ in the category $(\mathrm{pt}/H)_{G_1}$ where the composition in

$$\mathrm{mor}((\mathrm{pt}/H)_S) = \mathsf{S}(S, H)$$

is given by the monoid structure on H. Moreover, the requirements on $\rho\colon G_1 \to H$ from Definition 7.14 reduce to saying that ρ is an (internal) homomorphism. If $\rho', \rho \in \mathsf{S}(G_1, H)$ are two such internal homomorphisms then the result above says that the morphisms in the category $\mathsf{Fun}_{\mathsf{S}}(G_0/G_1, \mathrm{pt}/H)$ from ρ' to ρ are in bijective correspondence with certain $\alpha\colon G_0 \to H$,

$$\mathrm{Nat}_{\mathsf{S}}(\rho', \rho) \cong \{\alpha \in \mathsf{S}(G_0, H) \mid \rho \cdot s^*(\alpha) = t^*(\alpha) \cdot \rho' \in \mathsf{S}(G_1, H)\},$$

where $s^*(\alpha) = \alpha \circ s$ is multiplied ‘pointwise’ by ρ.

7.4. Quotients of supermanifolds and line bundles. Let us apply the above results in some cases needed later. Consider a group action of G on M in the site $\mathsf{S} = \mathsf{SM}$ of supermanifolds, for example $G = \underline{\mathrm{Diff}}(\mathrm{spt})$ acting on the superpoints $\mathrm{SP}X$. Then 7.16, (3), above has the following corollary, using for $\mathcal{W}$ the presheaf represented by $\mathbb{R}$ respectively $\mathbb{R}^{0|1}$ and recalling that $\mathsf{SM}(M, \mathbb{R}) = C^\infty(M)^{\mathrm{ev}}$ and $\mathsf{SM}(M, \mathbb{R}^{0|1}) = C^\infty(M)^{\mathrm{odd}}$.

Corollary 7.17. *Functions on the quotient are invariants by the group:*

$$\mathsf{Fun}_{\mathsf{S}}(M/G, \underline{\mathbb{R}}) \cong \{f \in C^\infty(M)^{\mathrm{ev}} \mid \mu^*(f) = p^*(f) \in C^\infty(M \times G)^{\mathrm{ev}}\}$$

To obtain odd functions, we need to replace $\underline{\mathbb{R}}$ *by* $\underline{\mathbb{R}^{0|1}}$ *in the above.*

Remark 7.18. If $\mathfrak{g}$ is the super Lie algebra of a super Lie group G with connected underlying manifold G_{red} then the infinitesimal action of $\mathfrak{g}$ on the algebra $C^\infty(M)$ of functions has the same fixed point set as described in Corollary 7.17 above (where we have to add the even and odd parts).

Let $\mathsf{Pic} \to \mathsf{SM}$ be the symmetric monoidal stack of real line bundles, compare Definition 7.21. An object in Pic is a line bundle over a supermanifold and on each connected component, it can have superdimension $1|0$ respectively $0|1$. We refer to these cases as *even* respectively *odd* line bundles. Let $\mathbb{1}\colon \mathsf{SM} \to \mathsf{Pic}$ be the monoidal unit, giving the trivial even line bundle for each $S \in \mathsf{SM}$. If M/G is a quotient fibration over SM then we define the *category of line bundles* over M/G to be

$$\mathsf{Fun}_{\mathsf{SM}}(M/G, \mathsf{Pic}),$$

extending the case $\mathsf{Fun}_{\mathsf{SM}}(\underline{M}, \mathsf{Pic}) \simeq \mathsf{Pic}_M$. The stack Pic contains the fibred subcategory $\mathrm{pt}/\mathbb{R}$ of trivial line bundles, where $\mathbb{R}$ is a monoid via multiplication. Since every line bundle is locally trivial, it follows that Pic is the stackification of $\mathrm{pt}/\mathbb{R}$. A functor $M/G \to \mathrm{pt}/\mathbb{R}$ can be given by a homomorphism $\rho\colon G \to \mathbb{R}^\times$ because this induces a morphism of internal categories in SM. We denote the corresponding even line bundle on M/G by L_ρ. For example, the trivial bundle $\mathbb{1}$ comes from the trivial homomorphism. Again extending the non-equivariant case, we define the *sections* of L_ρ to be

$$\Gamma(M/G; L_\rho) := \mathrm{Nat}_{\mathsf{SM}}(\mathbb{1}, \rho) = \left\{ M/G \underset{L_\rho}{\overset{\mathbb{1}}{\underset{\Downarrow E}{\rightrightarrows}}} \mathsf{Pic} \right\}.$$

We note that both functors $\mathbb{1}$ and ρ take values in $\mathrm{pt}/\mathbb{R} \subset \mathsf{Pic}$ which, for each $S \in \mathsf{SM}$, is the *full* subcategory of trivial bundles over S. Therefore, it does not matter whether the above natural transformations have target $\mathrm{pt}/\mathbb{R}$ or Pic and Example 7.16, (4), above leads to the following computation.

Corollary 7.19. *Consider a quotient fibration M/G over SM with an even line bundle L_ρ given by a homomorphism $\rho : G \to \mathbb{R}^\times$. Then*

$$\Gamma(M/G; L_\rho) \cong \{f \in C^\infty(M)^{\mathrm{ev}} \mid \mu^*(f) = p_1^*(f) \cdot p_2^*(\rho) \in C^\infty(M \times G)^{\mathrm{ev}}\}.$$

To obtain odd functions, we need to replace L_ρ by its odd partner ΠL_ρ.

7.5. Natural transformations as functors. In order to motivate twisted field theories in Section 6, we will need the following yoga. Let $w_0, w_1 \in \mathsf{Fun}_{\mathsf{S}}(\mathsf{S}, \mathsf{W})$ be two sections of a fibred category $\mathsf{W} \to \mathsf{S}$. Then one can define the presheaf

$$\mathcal{F}^{w_0,w_1} : \mathsf{S}^{\mathrm{op}} \to \mathsf{Set}, \qquad S \mapsto \mathsf{W}(w_0(S), w_1(S)),$$

using the existence and uniqueness properties of cartesian morphisms. The following result can be derived directly from the definitions.

Lemma 7.20. *For any fibred category $p : \mathsf{V} \to \mathsf{S}$, there is a natural bijection*

$$\mathrm{Nat}_{\mathsf{S}}(p \circ w_0, p \circ w_1) \cong \mathsf{Fun}_{\mathsf{S}}(\mathsf{V}, \underline{\mathcal{F}}^{w_0,w_1}).$$

We shall use this lemma in the case where $w_0 = w_1 = \mathbb{1}$ is the monoidal unit in a *fibred monoidal category* W. This is a category fibred over S together with fibred functors

$$\otimes : \mathsf{W} \times_{\mathsf{S}} \mathsf{W} \to \mathsf{W} \quad \text{and} \quad \mathbb{1}\colon \mathsf{S} \to \mathsf{W}$$

and fibred natural transformations (associator etc.) that satisfy the usual properties (pentagon etc.). In this case, the following picture describes Lemma 7.20 well if one remembers that everything is over S:

$$\left\{ \mathsf{V} \underset{\mathbb{1}}{\overset{\mathbb{1}}{\underset{\Downarrow E}{\rightrightarrows}}} \mathsf{W} \right\} \cong \{\mathsf{V} \xrightarrow{E} \underline{\Omega_{\mathbb{1}}\mathsf{W}}\}.$$

Here the 'based loops' $\Omega_{\mathbb{1}}\mathsf{W}$ signify the presheaf $S \mapsto \mathsf{W}(\mathbb{1}(S), \mathbb{1}(S))$. For example, if $\mathsf{W} = \mathsf{Pic}$ is the category of real line bundles over SM then this is the presheaf represented by $\mathbb{R}$ and we get the motivation for the definition of twisted field theories in Section 6.

The following definition will be used in Lemmas 4.9 and 5.3.

Definition 7.21. A *monoidal stack* is a fibred monoidal category $\mathsf{W} \to \mathsf{S}$ that satisfies the descent conditions in the monoidal sense: Both sides of (6) are by assumption monoidal categories and we require that the natural functor is a monoidal equivalence. It is also clear how to define a *symmetric monoidal stack* over S.

We note that none of this requires S to be monoidal, the definition only captures monoidal structures *along the fibres* of a fibration.

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