Invariants of knots and 3-manifolds

Tentative plan of the course

title	date	lecturer
Introduction to 3-manifolds I & II	April, 10 & 12	Lück
Cobordism theory and the s-cobordism theorem	April, 17	Lück
Introduction to Surgery theory	April 19	Lück
L^2-Betti numbers	April, 24 & 26	Lück
Introduction to Knots and Links	May 3	Teichner
Knot invariants I	May, 8	Teichner
Knot invariants II	May,15	Teichner
Introduction to knot concordance I	May, 17	Teichner
Whitehead torsion and L^2-torsion I	May 29th	Lück
L^2-signatures I	June 5	Teichner
tba	June, 7	tba

May 3: Introduction to knots & links

or

How too see low-dimensional manifolds

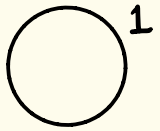
Goal for today:

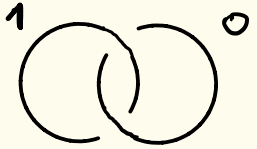
Discuss how a framed, dotted link L in $\mathbb{R}^3$

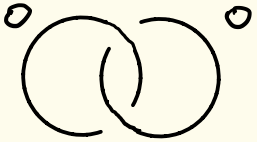
- defines a (compact, connected, oriented) 4-mfd. M_L^4
- also defines the (closed, -11-, -11-) 3-mfd ∂M_L

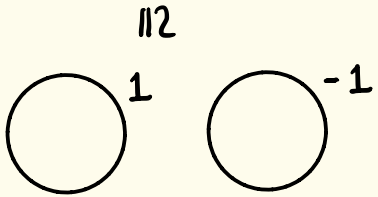
and which operations on L preserve the diffeomorphism class of M_L.

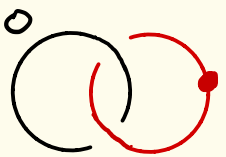
Note: ∂M_L is the result of surgery on $L \subseteq S^3$ and the above ∂-operations are called Kirby moves.

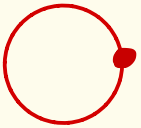
Examples: $S^4 \cong \emptyset$, $\mathbb{CP}^2 \cong$ 

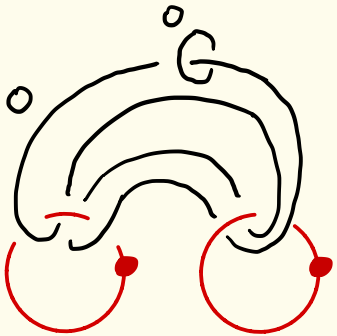
$S^2 \tilde{\times} S^2 \cong$ 

$S^2 \times S^2 \cong$ 

$\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2 \cong$ 

$S^4 \cong$ 

$S^1 \times S^3 \cong$ 

$S^1 \times S^1 \times S^2 \cong$ 

$L^3_{n,1} \cong$

Def.: • Knot, link, unlink, isotopy

- connected sum, long knots, prime decomposition
- Link complements, surgery on framed links
- Link group, homology, $\mathbb{Z}^m$-cover
- Projections, Reidemeister moves & singularities
- Wirtinger presentation, word problem

Morse theory: $M \xrightarrow{f} \mathbb{R}$ generic $\Leftrightarrow$ f is a Morse fct. ($\Rightarrow$ crit.pt.s discrete) with distinct crit. values

Key point: f + **gradient-like vector field for f** determines a handle decomposition of M:

Def.: A **handle decomposition** of $(M^d, \partial_- M, \partial_+ M)$ is

$$\partial_- M \times D^1 =: M^{(-1)} \subseteq M^{(0)} \subseteq M^{(1)} \subseteq \dots \subseteq M^{(d)} = M \quad \text{with}$$

$$M^{(k)} = M^{(k-1)} \cup I_k \text{ handles of index } k$$

$$= M^{(k-1)} \cup \coprod_{i \in I_k} S^{k-1} \times D^{d-k} \xrightarrow{\varphi_i} \partial_+ M^{(k-1)} \quad \coprod_{I_k} D^k \times D^{n-k}$$

Here we assume that $\partial M = \partial_- M \amalg \partial_+ M$ and

$\partial_+ M^{(-1)} := \partial_- M \times \{+1\}$ $\qquad$ $\partial_- M^{(-1)} := \partial_- M \times \{-1\}$

By construction, we see that $\partial_- M^{(k)} = \partial_- M^{(k)} \ \forall k$

$\partial_+ M^{(k)}$ is obtained from $\partial_+ M^{(k-1)}$ by surgery on I_k framed $(k-1)$-spheres $S^{k-1} \times D^{d-k} \xrightarrow{\varphi_i} \partial_+ M^{(k-1)}$.

Handles are the building blocks of manifolds

dim. \ index	0	1	2	3	4
1					
2					
3					
4	$(D^4, S^3, \emptyset)$	$(D^1 \times D^3, D^1 \times S^2, S^0 \times D^3)$	$(D^2 \times D^2, D^2 \times S^1, S^1 \times D^2)$	$(D^3 \times D^1, D^3 \times S^0, S^2 \times D^1)$	$(D^4, \emptyset, S^3)$

Important facts (from Morse/Cerf theory):

- handle decompositions exist in the smooth an PL-categories: The attaching maps φ_i, $i \in I_k$, are smooth resp. PL, inducing the respective structures on M (straightening of corners).
- for dim. $d \neq 4$, they also exist for topological mfd.s
- M^4 has a handle decomp. $\Leftrightarrow$ M is smooth $\Leftrightarrow$ M is PL
- handle decompositions are unique up to

(i) isotopy of φ_i in $\partial_+ M^{(k-1)} \cup (I_k \setminus i)$ handles of index k ("handle cancellation" / "includes handle slides")

(ii) birth/death

Exercise for smooth manifolds M with $\partial_- M$ compact

(1) a handle decomp. has **finitely many handles** (i.e. $\left| \coprod_{k=0}^{d} I_k \right| < \infty$) $\Leftrightarrow$ M is compact

(2) M has a handle decomposition with $\left| \pi_0 M \smallsetminus \text{image}(\pi_0 \partial_- M) \right|$ many **0-handles**.

(3) **M is orientable** $\Leftrightarrow$ all attaching maps $\varphi_i : S^0 \times D^{d-1} \hookrightarrow \partial M^{(0)}$ for 1-handles are **untwisted**, i.e. induce "same" sign on $\varphi_i |_{\{\pm 1\} \times D^{d-1}}$ if it is defined!

(4) Turning a handle decomp. of M **upside down** interchange $\partial_+ M$ & $\partial_- M$ and I_k & I_{d-k}.

Outline for May 8:

1) Recall handle decomposition for compact, connected mfds with $\partial M = \emptyset$
2) Apply to class. of d-mfds for $d = 1, 2, 3$
3) Explain dotted unlink as 1-handles $\subseteq M^4$
4) Read off 4-manifold invariants from the dotted framed link presentation of M^4:
   - $\pi_1 M$, $H_* M$, $H_* \tilde{M}$
   - intersection form on H_2 via linking numbers (first invariant!)
   - equivariant intersection form on π_2.

May 8: The **classification** of c.c.o. d-manifolds for $d = 1, 2, 3$: $\partial M =: \partial_- M$

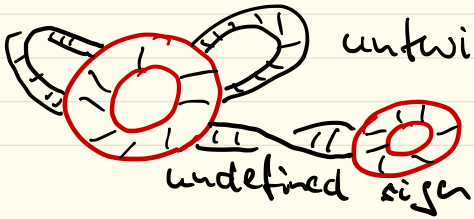
$d = 1$: M^1 compact connected $\overset{(2)}{\Longrightarrow}$

(i) $\partial M \neq \emptyset$, only need 1-handles ∩, and M connected implies $M \cong [0,1]$

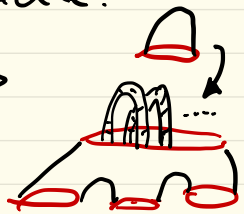
(ii) $\partial M = \emptyset$, have one 0-handle ∪ and one 1-handle ∩, i.e. $M \cong S^1$ ■

$d = 2$: $\partial M = \coprod^n S^1$ and as above we have zero ($n>0$) or one ($n=0$) 0-handle.

twisted 1-handle

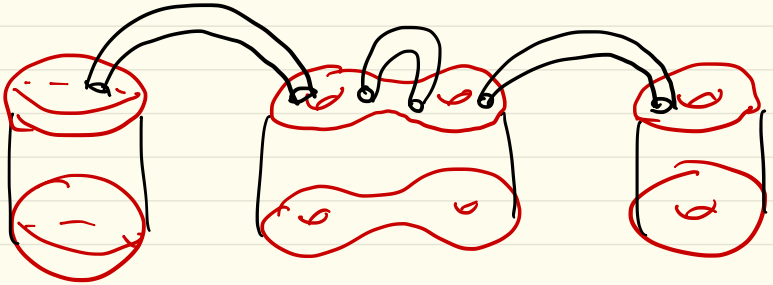


untwisted. M c.c.o. $\Rightarrow$ $M \cong \Sigma_g \setminus n$ disks $\cong$



$d=3$: $\partial M = \partial_- M = \coprod_{i=1}^{n} \Sigma_{g_i}$ and zero one 0-handle

one 3-handle



$= M^{(1)}$, $\partial_+ M^{(1)} = \Sigma_g$

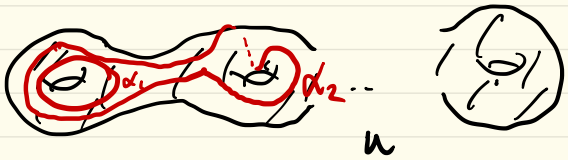
where $g \geq \sum_{i=1}^{n} g_i$

2-handle attachments $\varphi_i : S^1 \times D^1 \hookrightarrow \partial_+ M^{(1)}$ are given by $|I_2| \coprod_g$ disjointly embedded simple closed curves in Σ_g

Σ_g

Surgery on $\varphi_1, \dots, \varphi_g$ gives $S^2 = \partial_-(3\text{-handle})$ ■

Starting with $\partial_- M^3 = \emptyset$, we get the following presentation in the oriented case:

$h^0 \cup_n h^1 \cong$ 

Then 2-handles are attached along a link in $\partial_+ M^{(1)} = \Sigma_n$. New boundary $\partial_+ M^{(2)}$ = surgery on $\alpha_1, \dots, \alpha_m$ is ∂M (if $\neq \emptyset$).

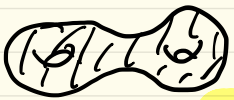
If $\partial M = \emptyset$ then $m = n$ and $\partial_+ M^{(2)} \cong S^2$ which is (uniquely) filled by a 3-handle.

Corollary: $\pi_1 M^3 \cong (x_1, \dots, x_n \mid \alpha_1, \dots, \alpha_m)$ and $m - n$ can be computed from ∂M, e.g. $m = n$ for $\partial M = \emptyset$.

Special case: **Heegaard decomposition** of a c.c.o. 3-manifold $M =$

$$(0\text{-handle} \cup g\ 1\text{-handles}) \underset{\Sigma_g}{\cup} (g\ 2\text{-handles} \cup 3\text{-handle})$$

$$\| \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$$

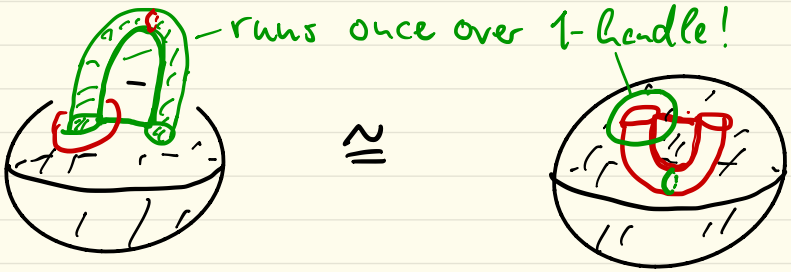
 glued along a

diffeomorphism of Σ_g

Gluing is determined by $\varphi_1^c, \dots, \varphi_g^c : S^1 \hookrightarrow \Sigma_g$ ■

Remark: We could have assumed $\partial M = \emptyset$ and hence zero or one 3-handle and a single 0-handle. Then 1-handles give $M^{(1)} = \natural^g S^1 \times D^2$ and we have $|I_2| \leq g$ with $\varphi_i : S^1 \times D^1 \hookrightarrow \partial M^{(1)} \cong \Sigma_g$ ■

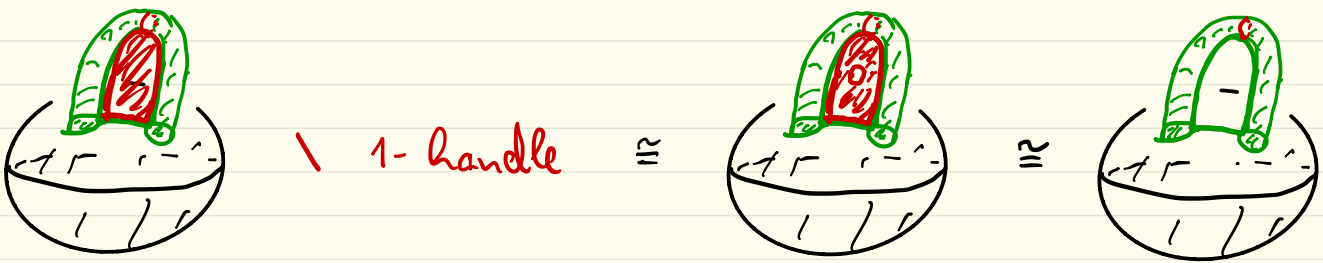
May 15 : The diffeo. $D^3 \cup$ 1-handle $\cong D^3 \setminus$ 1-handle



handle cancellation

generalizes to $D^d \cup$ 1-handle $\cup$ 2-handle $\cong D^d$

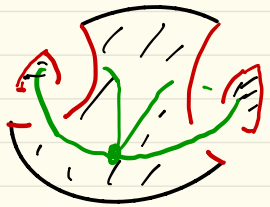
$$\Rightarrow D^d \cup_{S^0 \times D^{d-1} \hookrightarrow S^{d-1}} D^1 \times D^{d-1} \cong \left(D^d \cup_{S^0 \times D^{d-1} \hookrightarrow S^{d-1}} D^1 \times D^{d-1} \cup_{S^1 \times D^{d-2} \hookrightarrow S^1 \times S^{d-2}} D^2 \times D^{d-2} \right) \setminus \underset{D^2 \times D^{d-2}_{\varepsilon}}{(d-2)\text{-handle}}$$



In dim. $d=4$, we use

unlink on boundary!

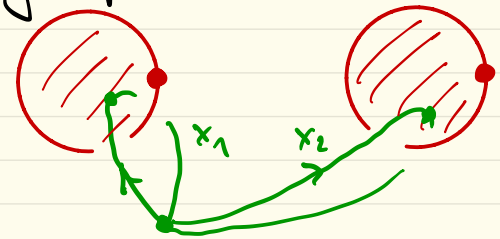
$$h^0 \cup \underset{n}{\cup} h^1 \cong D^4 \setminus \underset{n}{\cup} h^2 \cong$$



where we can see the boundary:

The group isomorphism

$S^3 \supseteq \mathbb{R}^3 \supseteq$

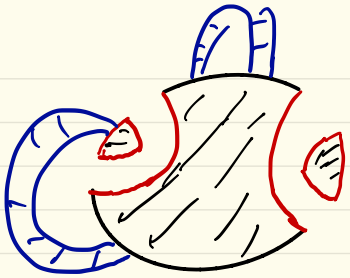


$\pi_1(h^0 \cup \underset{n}{\cup} h^1) \cong$
$\pi_1(\overset{n}{\vee} S^1) \cong F_n :=$
free group on $x_1, \dots, x_n$

is given by reading off words of intersections with spanning disks for the dotted unlink.

Corollary: We get a presentation of

$$\pi_1(D^4 \cup \underset{n}{\cup} h^1 \cup \underset{m}{\cup} h^2 \cup \underset{i}{h^3} \cup \underset{j}{h^4}) \cong (x_1, \dots, x_n \mid r_1, \dots, r_m).$$



$= M_L^4 = D^4 \setminus$ 2-handles $\cup$ 2-handles
dotted part of L — framed part of L

$\cong$ 0-handle $\cup$ untwisted 1-handles $\cup$ 2-handles

$\partial M_L =$ surgery on (S^3, L°) where $L^\circ = L$ with dots replaced by 0 framings.

Thm. [Laudenbach]: $\partial M_L \cong \#^k S^1 \times S^2$ iff $M^{(2)}$ extends to a handle dec. of $M = M_L \cup_{\#^k S^1 \times D^3} 4$ a closed mfd. Moreover, there is a **unique** way of attaching these 3- & 4-handles.

Cor.: There is a well-defined **surjective map**

$$\{\text{dotted, framed links } L \text{ with } \partial M_L \cong \#^k S^1 \times S^2\} \twoheadrightarrow \{\text{diff. classes of closed, connected, oriented 4-manifold}\}$$

Cor. [Kirby calculus]: Every cl.c.o. 3-mfd. can be obtained by surgery on a framed link. Moreover,

$$\partial M^4_L \cong \partial M^4_{L'} \iff L \text{ \& } L' \text{ differ by Kirby moves}$$

handle slides and deletion/creation of $\bigcirc^{\pm 1}$

Proof: Given N^3 we choose any co.c.o. M^4 with $\partial M = N$ (using $\Omega_3 = 0$). $\Rightarrow$

$M = M_L \cup$ 3-handles. φ_i are non-separating since ∂M is connected

$\coprod_{I^3} \varphi_i : S^2 \times D^1 \hookrightarrow \partial M_L$

$$\Rightarrow \partial M_L = \partial M \# \#_{I^3} S^1 \times S^2$$

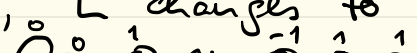
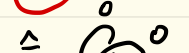
and ∂M = surgery on $|I^3|$ 0-framed S^1's in ∂M_L

= surgery on $(S^3, L^0 \cup |I^3| \text{ circles})$.

This proves existence of surgery descriptions. Idea for uniqueness: Assume that $S(L) \cong S(L')$ for framed links L, L' in S^3. Then form the 1-connected 4-mfds M_L, $M_{L'}$ with their given boundary. $\mathbb{CP}^2$-stable classification shows that

$$M_L \# \overset{n_1}{\#} \mathbb{CP}^2 \# \overset{n_2}{\#} \overline{\mathbb{CP}}^2 \cong M_{L'} \# \overset{n_1'}{\#} \mathbb{CP}^2 \# \overset{n_2'}{\#} \overline{\mathbb{CP}}^2 .$$

So adding (± 1)-framed unlinks to L & L', we assume $M_L \cong M_{L'}$. Cerf theory shows that L' is obtained from L by:
- handle slides
- birth / death

For 1-2 cancelling pair, L changes to $L \cup$ and we know that   2-3 handle pairs more tricky! ∎

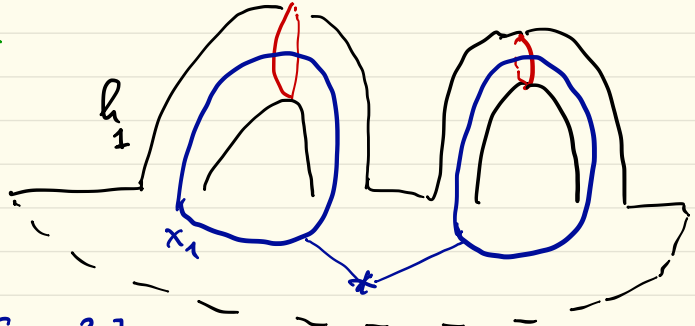
How to cancel 1-handles for simply-connected M^d, $d \geq 5$?

3D-P.C.!

Step 1: Create a 2-3 handle pair $h^2 \cup h^3$.

Step 2: Given a 1-handle h^1_1, there are (original) h^2_i s.t. in $\partial_+\left(h^0 \cup (h^1_1 \cup \dots \cup h^1_n)\right) = \#^n S^1 \times S^{d-2}$ $\quad d \geq 4 \leftrightarrow$ 1-con.

$\varphi_i : S^1 \times \mathbb{D}^{d-2}$



Free generator x_1 of $\pi_1 \partial_+$ is a word in $[\varphi_i]^{g_i}$ and we can slide the new h^2 over h^2_i along g_i to get $[\partial h^2] = x_1 \in \pi_1 \partial_+$.

If $d \geq 5$ then homotopy implies isotopy in ∂_+ so h^2 can be isotoped s.t. $\partial h^2 \cap S^{d-2}_1 = \{pt\} \Rightarrow h^1_1$ can be cancelled ∎

Corollary: M^5 1-connected, closed. $\Rightarrow$

$M^5 =$ 0-handle $\cup$ n·2-handles $\cup$ n·3-handles $\cup$ 5-handle

$$M^5 = \natural^n S^2 \tilde{\times} D^3 \;\cup\; \natural^u S^2 \tilde{\times} D^3$$

$$\# S^2 \tilde{\times} S^2$$

where $\tilde{x}$ denotes one of the two oriented D^3-bundles over S^2 ($\pi_1 SO(3) = \mathbb{Z}/2$).

M^5 is spin $\Leftrightarrow$ all $\tilde{x} = x$. Exercise:

Thm [Wall]: M^5 spin $\Rightarrow$ diffeom. type is completely determined by isom. type of $H_2 M$!

$lk/_{\text{Tors } H_2 M}$ is skew $\Rightarrow$ torsion $H_2 M \cong A \oplus A$ ■

Outline for May 17 — Top. invariants of 4-manifolds

1) $H_* M$ computed from the handle chain complex where $C_k(M, \partial_- M) :=$ free abelian group on k-handles, and $\partial \downarrow$

$C_{k-1}(M, \partial_- M)$, $\partial h^i_k : h^j_{k-1} =$ # transverse intersections $\varphi_i(S^{k-1} \times 0) \pitchfork \varphi_j(0 \times S^{d-k})$ in $\partial_t M^{(k-1)}$.

1') $H_* \tilde{M}$ computed from equiv. version over $\mathbb{Z}[\pi_1 M]$. Tricky in small degree.

2) Intersection form $\lambda : H_2 M \otimes H_2 M \longrightarrow \mathbb{Z}$ given by linking numbers (& framings) of linear combinations of 2-handles.

3) Many examples, def. of linking number.