arXiv:2006.06127v1 [math.GT] 11 Jun 2020

ALGEBRAIC CRITERIA FOR STABLE DIFFEOMORPHISM OF SPIN 4-MANIFOLDS

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Abstract. We study closed, connected, spin 4-manifolds up to stabilisation by connected sums with copies of $S^2 \times S^2$. For a fixed fundamental group, there are primary, secondary and tertiary obstructions, which together with the signature lead to a complete stable classification. The primary obstruction exactly detects $\mathbb{CP}^2$-stable diffeomorphism and was previously related to algebraic invariants by Kreck and the authors.

In this article we formulate conjectural relationships of the secondary and tertiary obstructions with algebraic invariants: the secondary obstruction should be determined by the (stable) equivariant intersection form and the tertiary obstruction via a τ-invariant recording intersection data between 2-spheres, with trivial algebraic self-intersection, and their Whitney discs.

We prove our conjectures for the following classes of fundamental groups: groups of cohomological dimension at most 3, right-angled Artin groups, abelian groups, and finite groups with quaternion or abelian 2-Sylow subgroups.

1. Introduction

Two closed, connected, smooth 4-manifolds are *stably diffeomorphic* if they become diffeomorphic after taking connected sums with finitely many copies of $S^2 \times S^2$, where we allow different numbers of copies for the two manifolds. An analogous notion of *stable homeomorphism* applies in the topological category. Kreck [Kre99] reduced these classification problems to computations of bordism groups, which we review in Section 1.1 in the spin case.

In this article we introduce new *algebraic* invariants for spin 4-manifolds. The invariants are independent of category (smooth or topological) as one might expect in the stable setting. We shall focus on the smooth category, and explain why our classification also applies in the topological category in Section 2.3.

Theorem 1.1. *Fix a group π with finite 3-dimensional classifying space, an abelian group, a right angled Artin group, or a finite group π with abelian or quaternion 2-Sylow subgroup.*

One can decide whether two closed, connected, spin, smooth 4-manifolds N_1 and N_2 are stably diffeomorphic over their fundamental group π by computing, for all possible spin structures α_i on N_i, the following algebraic invariants on a 1-skeleton sum $M := (N_1, \alpha_1)\#_1(N_2, -\alpha_2)$:

(0) the signature $\sigma(M) = \sigma(N_1) - \sigma(N_2)$ of M, an integer;
(1) the extension class of the stable $\mathbb{Z}\pi$-module $\pi_2(M)$, a 'linear' invariant;
(2) the stable intersection form λ_M on $\pi_2(M)$, a $\mathbb{Z}\pi$-valued 'quadratic' form;
(3) the Kervaire-Milnor invariant τ_M, a $\mathbb{Z}/2$-valued 'cubical' refinement of λ_M.

There are handle decompositions of N_i with diffeomorphic 1-skeleta (the union of the 0- and 1-handles). To obtain a 1-skeleton sum $(N_1, \alpha_1)\#_1(N_2, -\alpha_2)$, remove the 1-skeleta, and identify the

2020 *Mathematics Subject Classification.* 57K40.

boundaries by a spin structure preserving diffeomorphism. The stable diffeomorphism class turns out not to depend on the choices.

The invariants in Theorem 1.1 detect the steps in an obstruction theory to decide whether M is trivial in the spin bordism group $\Omega_4^{Spin}(B\pi)$. While the signature is an invariant of each N_i, it remains an open problem to formulate the other invariants solely in terms of the N_i, in such a way that they detect stable diffeomorphism. Our solution is to use the 1-skeleton sum described above. We shall give precise definitions of the invariants in points (1), (2), and (3) in Sections 1.3 to 1.5.

The linear invariant in (1) exactly detects $\mathbb{CP}^2$-stable diffeomorphism of closed, connected 4-manifolds. This was the focus of [KPT18], so in this article we shall focus on the other invariants.

The stable classification is particularly neat in the case of fundamental groups with 2-dimensional classifying space. To state it we recall a special case of the cubical invariant τ_M in (3). Assume that $x \in \pi_2(M)$ satisfies $\lambda_M(x, y) = 0$ for all y. Then x lies in the radical $\mathrm{Rad}(\lambda_M)$ of the intersection form, that is the kernel of the adjoint λ_M^{ad}, and x can be represented by a generic immersion $f\colon S^2 \looparrowright M$ with algebraically trivial self-intersections. This means that all double points of f can be paired by Whitney discs, which by boundary twisting and pushing down can be chosen to be disjointly embedded, framed, and to intersect f transversely. Then the Kervaire-Milnor invariant $\tau_M(x)$ counts the number, modulo two, of intersection points between the Whitney discs and $f(S^2)$. This invariant appeared previously in [FQ90], [Sto94], and [ST01], following a closely related invariant defined in [FK78, Mat78]. We will show that this number does not depend on the choices and in fact only depends on the image of x under a natural map $\mathrm{Rad}(\lambda_M) \cong H^2(\pi; \mathbb{Z}\pi) \to H^2(\pi; \mathbb{Z}/2)$. Here is the promised stable classification for fundamental groups with 2-dimensional classifying space.

Corollary 1.2. *Let π be a group with a finite 2-dimensional classifying space $B\pi$. Then closed, connected, smooth, spin 4-manifolds M_1 and M_2 with fundamental group π are stably diffeomorphic if and only if the following invariants coincide:*

(0) the signatures of M_1 and M_2;
(3) the Kervaire-Milnor invariants $\tau_{M_i} \in H_2(\pi; \mathbb{Z}/2) \cong \mathrm{Hom}(H^2(\pi; \mathbb{Z}/2), \mathbb{Z}/2)$, considered up to the natural action of $\mathrm{Aut}(\pi)$.

Note that the quotient in Corollary 1.2 (3) is necessary, because the isomorphism $\pi_1(M) \cong \pi$ can be varied arbitrarily. Examples of groups with 2-dimensional classifying spaces are $\pi := \pi_1(\Sigma)$ for a closed oriented surface Σ of genus at least one. In this case, the new invariant (3) takes two distinct values, the trivial τ_M being represented by $M = \Sigma \times S^2$ where the radical $\mathrm{Rad}(\lambda_M) = \pi_2(M) \cong \mathbb{Z}$ is generated by an embedded sphere $\{\mathrm{pt}\} \times S^2$. The second stable diffeomorphism class is represented by a 4-manifold M' constructed from $\Sigma \times T^2$ by performing surgery on framed circles representing a dual pair of generators of $\pi_1(T^2) \cong \mathbb{Z}^2$, where the framing is "twisted". For the latter manifold M', the generator of $\mathrm{Rad}(\lambda_{M'}) \cong \mathbb{Z}$ cannot be represented by an embedding, even stably.

The stable classification for 4-manifolds with surface groups as fundamental group was previously completed in [HKT09], but without using the Kervaire-Milnor invariant. Instead it was shown that (a codimension 2 Arf invariant of) λ_M determines this additional obstruction. It follows that the second stable diffeomorphism class, with nontrivial Kervaire-Milnor invariant, cannot be realised by a "minimal" 4-manifold N with $\mathrm{Rad}(\lambda_N) = \pi_2(N)$. However, a stable classification via the intersection form alone has not been proven for 2-dimensional groups other than surface groups, and in any case would not imply the simple classification of Corollary 1.2 in terms of τ_M evaluated on generically immersed spheres in the radical.

The algebraic obstructions that we will discuss are generalisations of the ones studied in [KLPT17] by the present authors together with Land. There the fundamental group was that of a closed, oriented, aspherical 3-manifold, and we obtained our results by computing the obstructions on explicit models for the stable diffeomorphism classes. The current investigation proceeds more abstractly, as suggested for the secondary obstruction in the third author's thesis [Tei92].

1.1. **Review of stable classification.** Kreck [Kre99] showed that two closed, connected, spin 4-manifolds with fundamental group π are stably diffeomorphic if and only if there are choices of spin structures and identifications of the fundamental groups with π, giving rise to equal elements in the bordism group $\Omega_4^{Spin}(\pi) := \Omega_4^{Spin}(B\pi)$, as we shall explain in Section 2. To understand this group of bordism classes of pairs (M, c), where M is a closed 4-manifold with spin structure and $c\colon M \to B\pi$ classifies the universal cover, we consider the Atiyah-Hirzebruch spectral sequence (AHSS) computing $\Omega_4^{Spin}(B\pi)$ in terms of $E^2_{p,q} = H_p(B\pi; \Omega_q^{Spin})$. The AHSS gives rise to a filtration whose iterated graded quotients are

$$\mathbb{Z} \cong \Omega_4^{Spin} \underbrace{\subseteq}_{E_{2,2}} F_2 \underbrace{\subseteq}_{E_{3,1}} F_3 \underbrace{\subseteq}_{E_{4,0}} \Omega_4^{Spin}(\pi).$$

The first isomorphism is determined by the signature. More precisely, it is given by the signature divided by 16 in the smooth case and the signature divided by 8 in the topological case; the closed topological E_8-manifold cannot be smoothed. This divisibility is the only difference between the stable classification of smooth and topological spin 4-manifolds and therefore we can ignore it in the sequel. The signature extends to the entire group $\Omega_4^{Spin}(\pi)$ and so we reduce our study to $\widetilde{\Omega}_4^{Spin}(\pi)$, the kernel of the signature map, which is independent of category. The AHSS then reduces to a shorter filtration

$$E_{2,2} \underbrace{\subseteq}_{E_{3,1}} F \underbrace{\subseteq}_{E_{4,0}} \widetilde{\Omega}_4^{Spin}(\pi),$$

where the subgroup F consists of bordism classes represented by signature zero 4-manifolds M with spin structure such that $c\colon M \to (B\pi)^{(3)}$ lands in the 3-skeleton of the classifying space $B\pi$. Similarly, the smallest filtration term $E_{2,2}$ is represented by elements (M, c) with $c\colon M \to (B\pi)^{(2)}$. Since the $E^2_{p,q}$ term of the spectral sequence is $H_p(\pi; \Omega_q^{Spin})$, the $E^\infty_{p,q}$-terms are as follows:

- $E_{2,2} := E^\infty_{2,2} = H_2(\pi; \mathbb{Z}/2)/\operatorname{im}(d_2, d_3)$;
- $E_{3,1} := E^\infty_{3,1} = H_3(\pi; \mathbb{Z}/2)/\operatorname{im}(d_2)$;
- $E_{4,0} := E^\infty_{4,0} = \ker(d_2\colon H_4(\pi; \mathbb{Z}) \to H_2(\pi; \mathbb{Z}/2))$.

Following [Tei92], we obtain the primary invariant $\mathfrak{pri}(M) = c_*[M] \in E_{4,0}$, the secondary invariant $\mathfrak{sec}(M) \in E_{3,1}$ and the tertiary invariant $\mathfrak{ter}(M) \in E_{2,2}$. The challenge is to recast these obstructions in terms of algebraic topological data of the 4-manifold. As a preliminary observation, the equivariant intersection form λ_M on $\pi_2(M)$ changes by orthogonal sum with a hyperbolic form (on a free $\mathbb{Z}\pi$-module of rank 2) if we add $S^2 \times S^2$ to M. So the signature $\sigma(M) \in \mathbb{Z}$ is a stable invariant, as is the isomorphism class of $\pi_2(M)$ up to stabilisation by free $\mathbb{Z}\pi$-modules, and the isometry class of λ_M up to stabilisation with hyperbolic forms on free modules.

1.2. **Translating the bordism invariants into algebra.** Let us discuss the three obstructions in order of appearance.

(1) The *Primary Obstruction Theorem 1.4* [KPT18] reinterprets the invariant $\mathfrak{pri}(M) = c_*[M]$ as the extension class of a short exact sequence of $\mathbb{Z}\pi$-modules whose central module is stably isomorphic to $\pi_2(M)$. In loc. cit. we gave examples of various fundamental groups for which $\mathfrak{pri}(M)$ is and is not determined by the stable isomorphism class of the $\mathbb{Z}\pi$-module $\pi_2(M)$.

(2) We say that a group π has the *Secondary Property* if $\mathfrak{sec}(M)$ is detected by $\lambda_{M,s}$ for all spin 4-manifolds M with fundamental group π. Here $\lambda_{M,s}$ is the restriction of λ_M to a certain summand of $\pi_2(M)$. This summand arises as the image of a splitting s of the extension in (1), which exists if and only if $\mathfrak{pri}(M) = 0$. We shall show in Section 1.4 that if $\mathfrak{sec}(M)$ vanishes then there are choices of a splitting for which λ_M vanishes on the image.

(3) We say that a group π has the *Tertiary Property* if $\mathfrak{ter}(M)$ is detected by the Kervaire-Milnor invariant $\tau_{M,s}$ for all spin 4-manifolds M with fundamental group π. Here $\tau_{M,s}$ records intersection data between 2-spheres (with trivial algebraic self-intersection) and their Whitney discs. It is the Kervaire-Milnor invariant τ_M from [FQ90], as corrected by [Sto94] and [ST01], restricted to the image of a splitting in (1) on which λ_M vanishes. This cubical intersection invariant is defined if and only if $\mathfrak{pri}(M) = 0 = \mathfrak{sec}(M)$, and it does not depend on the choice of splitting, as we will discuss in Section 1.5.

We will state these properties of a group in detail as Secondary Property 1.8 and Tertiary Property 1.12. Then the following statement will be made precise.

Theorem 1.3. *Consider a group π with finite 3-dimensional classifying space $B\pi$, or an abelian group, a right angled Artin group, or a finite group π with abelian or quaternion 2-Sylow subgroup. Then π has the Secondary and Tertiary properties. As a consequence, Theorem 1.1 above holds.*

Next we give more detailed descriptions of the obstructions and the statements of our theorems.

1.3. **The algebraic primary obstruction.** Let M be a closed, connected, oriented 4-manifold with fundamental group π. There is a map $c\colon M \to B\pi$ inducing the identification on fundamental groups, that is well defined up to based homotopy. The primary obstruction $\mathfrak{pri}(M)$ corresponds to the edge homomorphism in the AHSS and is given by the image $c_*[M]$ of the fundamental class $[M] \in H_4(M;\mathbb{Z})$ in $H_4(B\pi;\mathbb{Z}) = H_4(\pi;\mathbb{Z})$. We will build on the following result.

Theorem 1.4 (Primary obstruction theorem [KPT18, Theorem 1.8])**.** *Let K be a finite connected 2-complex with fundamental group π. There is an isomorphism*

$$\mathrm{Ext}^1_{\mathbb{Z}\pi}(H^2(K;\mathbb{Z}\pi), \pi_2(K)) \cong H_4(\pi;\mathbb{Z})$$

mapping $\mathfrak{pri}(M) = c_*[M]$ to an extension

$$0 \longrightarrow \pi_2(K) \longrightarrow (\mathbb{Z}\pi)^r \oplus \pi_2(M) \longrightarrow H^2(K;\mathbb{Z}\pi) \longrightarrow 0.$$

Recall that a finite presentation of π gives a 2-complex K as above by using a single 0-cell, one 1-cell for each generator and one 2-cell for each relation. Using Tietze transformations, we showed that the choice of presentation disappears when considering the extension group above. To make this precise, we need the following notion and the next lemma.

Definition 1.5. We say that two R-modules P and Q are *stably isomorphic*, and write $P \cong_s Q$, if there exist non-negative integers p and q such that $P \oplus R^p \cong Q \oplus R^q$.

Lemma 1.6. [HAM93, (40)] *Let K_i be finite connected 2-complexes with fundamental group π. Then there exist $k_i \in \mathbb{N}_0$ such that $K_1 \vee \bigvee_{r=1}^{k_1} S^2 \simeq K_2 \vee \bigvee_{r=1}^{k_2} S^2$. In particular, $\pi_2(K_1) \cong_s \pi_2(K_2)$ and $H^2(K_1;\mathbb{Z}\pi) \cong_s H^2(K_2;\mathbb{Z}\pi)$.*

In [KPT18, Lemma 5.11] we checked that wedge sum with S^2 does not change the extension group in 1.4 and that a homotopy self-equivalence of K inducing the identity on $\pi_1(K)$ determines the identity map on this extension group [KPT18, Lemma 5.12]. We concluded that the image of $\mathfrak{pri}(M)$ in this extension group is a well-defined algebraic invariant of M in a group that depends only on the group π.

1.4. **The algebraic secondary obstruction.** Consider the equivariant intersection form

$$\lambda_M\colon \pi_2(M)\times\pi_2(M)\to\mathbb{Z}\pi$$

of a closed, connected, oriented 4-manifold M with fundamental group π. It is given by either counting geometric intersections (with signs and group elements) between transverse 2-spheres in M or by identifying $\pi_2(M)\cong H_2(M;\mathbb{Z}\pi)$ via the Hurewicz homomorphism and setting

$$\lambda_M(x,y):=\langle PD^{-1}(y),x\rangle\quad\text{for } x,y\in H_2(M;\mathbb{Z}\pi),$$

where $PD\colon H^2(M;\mathbb{Z}\pi)\to H_2(M;\mathbb{Z}\pi)$ is the Poincaré duality isomorphism. The intersection form λ_M is *sesquilinear*, meaning it is additive in each variable and $\lambda_M(ax,by)=a\lambda_M(x,y)\overline{b}$ holds for all $a,b\in\mathbb{Z}\pi$, where the involution $a\mapsto\overline{a}$ on the group ring $\mathbb{Z}\pi$ is determined by $\overline{g}:=g^{-1}$ for $g\in\pi$. Note that λ_M is frequently singular. In fact, its kernel and cokernel are determined by the fundamental group π because by the universal coefficient spectral sequence, its adjoint fits into the exact sequence

$$\text{(UCSS)}\qquad 0\to H^2(\pi;\mathbb{Z}\pi)\to\pi_2(M)\xrightarrow{\lambda_M^{\mathrm{ad}}}\mathrm{Hom}_{\mathbb{Z}\pi}(\pi_2(M),\mathbb{Z}\pi)\to H^3(\pi;\mathbb{Z}\pi)\to 0.$$

The intersection form λ_M is also hermitian in the following sense. The dual λ^* of a sesquilinear form λ is given by

$$\lambda^*(x,y):=\overline{\lambda(y,x)}.$$

and λ is called *hermitian* if $\lambda^*=\lambda$. We say that λ is *even* if $\lambda=q+q^*$ for some sesquilinear form q on $\pi_2(M)$.

Remark 1.7. Such a form q, evaluated on the diagonal $x\mapsto\widehat{q}(x):=q(x,x)$, defines a *quadratic refinement* $\widehat{q}\colon\pi_2(M)\to\mathbb{Z}\pi$ of λ_M in the sense that for $x,y\in\pi_2(M)$ we have

$$\text{(qr)}\qquad \lambda_M(x,x)=q(x,x)+q^*(x,x)=\widehat{q}(x)+\overline{\widehat{q}(x)}.$$

It is important to point out that for $\widetilde{M}$ spin, every class $x\in\pi_2(M)$ can be represented by a generically immersed 2-sphere $f\colon S^2\looparrowright M$ with trivial normal bundle. Since $w_2(x)=0$, the Euler number e of the normal bundle to f is even and hence can be changed to zero by a non-regular homotopy of f, performing $e/2$ cusp homotopies. As a consequence, the intersection form λ_M has a quadratic refinement

$$\mu_M\colon\pi_2(M)\to\mathbb{Z}\pi/\{a-\overline{a}\mid a\in\mathbb{Z}\pi\}$$

given by counting self-intersections (with group elements and signs) of a generic immersion $f\colon S^2\looparrowright M$ with $e(f)=0$. The expression $\mu_M(x)+\overline{\mu_M(x)}$ can be lifted canonically to $\mathbb{Z}\pi$ and then equals $\lambda_M(x,x)$. It is not hard to show algebraically that μ_M, if it exists, is completely determined from λ_M by this property. As a consequence, we shall not keep μ_M in our notation but will remember that λ_M is a *weakly even hermitian* form, meaning by definition that a quadratic refinement exists in this sense.

In general, weakly even forms are not even. If $\pi_2(M)$ happens to be a free $\mathbb{Z}\pi$-module on an ordered basis $\{e_i\}$ then weakly even forms λ are even, as can be seen by lifting $\mu(e_i)$ to $q(e_i,e_i)\in\mathbb{Z}\pi$ and setting $q(e_i,e_j):=\lambda(e_i,e_j), q(e_j,e_i)=0$ for $i<j$. But as we shall see, for many spin 4-manifolds M the intersection form λ_M is not even. To make this precise, consider a closed, connected spin 4-manifold M with fundamental group π and $\mathfrak{pri}(M)=0$. In the notation of Theorem 1.4, where we chose a 2-complex K, also choose a splitting

$$s=(s_1,s_2)\colon H_K:=H^2(K;\mathbb{Z}\pi)\to(\mathbb{Z}\pi)^r\oplus\pi_2(M)$$

of the short exact sequence $0\longrightarrow\pi_2(K)\longrightarrow(\mathbb{Z}\pi)^r\oplus\pi_2(M)\longrightarrow H^2(K;\mathbb{Z}\pi)\longrightarrow 0$. Consider the sesquilinear form $\lambda_{M,s}\colon H_K\times H_K\to\mathbb{Z}\pi$ on H_K given by $\lambda_M(s_2(-),s_2(-))$. We write $\mathrm{Sesq}(H)$ for the group of $\mathbb{Z}\pi$-sesquilinear forms on the $\mathbb{Z}\pi$-module H. Sending a form to its hermitian conjugate induces a $\mathbb{Z}/2 = \langle T \rangle$ action on $\operatorname{Sesq}(H)$, and we consider the corresponding Tate group

$$\widehat{H}^0(\operatorname{Sesq}(H)) := \widehat{H}^0(\mathbb{Z}/2; \operatorname{Sesq}(H)) = \ker(1-T)/\operatorname{im}(1+T) = \{\text{Hermitian forms}\}/\{\text{even forms}\}.$$

In Definition 4.15 we will construct, purely algebraically, a homomorphism

$$A_K \colon H_3(\pi;\mathbb{Z}/2) \longrightarrow \widehat{H}^0(\operatorname{Sesq}(H_K)).$$

For every map $\varphi\colon K \to K'$ inducing the given identification with π on fundamental groups, the diagram

$$\begin{tikzcd} H_3(\pi;\mathbb{Z}/2) \arrow[r, "A_K"] \arrow[dr, "A_{K'}"'] & \widehat{H}^0(\operatorname{Sesq}(H_K)) \arrow[d, "\varphi_*"] \\ & \widehat{H}^0(\operatorname{Sesq}(H_{K'})) \end{tikzcd}$$

commutes by Lemma 4.16. We obtain a group $\operatorname{Tate}(\pi) := H_3(\pi;\mathbb{Z}/2)/\ker(A_K) \cong \operatorname{im} A_K$, which is independent of the choice of K. We write the projection as

$$A_\pi \colon H_3(\pi;\mathbb{Z}/2) \twoheadrightarrow \operatorname{Tate}(\pi).$$

Although it will not play a role for this article, note that $\operatorname{im} A_K$ lies in the subgroup of weakly even forms on H_K by Lemma 4.20. One might hope to identify $\operatorname{Tate}(\pi)$ with a particular class of such weakly even forms on H_K.

We will show in Section 4.4 that for every spin 4-manifold M with fundamental group π and $\mathfrak{pri}(M) = 0$, a splitting $s\colon H_K \to (\mathbb{Z}\pi)^r \oplus \pi_2(M)$ determines a map $f_s\colon M \to (B\pi)^{(3)}$, which in turn determines an element $\mathfrak{sec}(M, f_s) \in H_3(\pi;\mathbb{Z}/2)$, lifting the element $\mathfrak{sec}(M) \in H_3(\pi;\mathbb{Z}/2)/\operatorname{im}(d_2)$. We will show that the two elements $\lambda_{M,s}$ and $A_\pi(\mathfrak{sec}(M, f_s))$ agree in $\operatorname{Tate}(\pi) \subseteq \widehat{H}^0(\operatorname{Sesq}(H_K))$.

Secondary Property 1.8. *We say that a group π has the* Secondary Property *if for all closed, connected spin 4-manifolds M with fundamental group π and $\mathfrak{pri}(M) = 0$, $\lambda_{M,s} \in \operatorname{Tate}(\pi)$ is independent of the splitting s and $\mathfrak{sec}(M)$ is determined by $\lambda_{M,s} \in \operatorname{Tate}(\pi)$.*

The Secondary Property of a group π follows from the purely algebraic assertion that the differential d_2 in the AHSS makes the following sequence exact:

$$(\mathfrak{sec}_\pi) \qquad H_5(\pi;\mathbb{Z}) \xrightarrow{d_2} H_3(\pi;\mathbb{Z}/2) \xrightarrow{A_\pi} \operatorname{Tate}(\pi).$$

In Sections 9 and 10 we will prove this exactness (and hence the Secondary Property) for the following classes of groups, proving the secondary part of Theorem 1.3.

Theorem 1.9. *The Secondary Property 1.8 holds for the following classes of groups.*

- *Groups π with a finite 3-dimensional model for $B\pi$.*
- *Abelian groups and right angled Artin groups.*
- *Finite groups whose 2-Sylow subgroups are abelian or quaternion.*

When π has geometric dimension at most two, the Secondary Property holds since $H_3(\pi;\mathbb{Z}/2) = 0$. If π is the fundamental group of a closed, oriented, aspherical 3-manifold then $H_5(\pi;\mathbb{Z}) = 0$ and $H_K \cong_s I\pi$, the augmentation ideal in $\mathbb{Z}\pi$. We showed in [KLPT17] that A_π is an isomorphism between two groups of order 2.

The case of finite groups with quaternion 2-Sylow subgroups was proven in [Tei92, Theorem 6.4.1]; see also our Section 9.3. Emboldened by our success, we make the following conjecture cf. [Tei92, Conjecture B].

Conjecture 1.10. *The sequence* $\mathfrak{sec}_\pi$ *is exact for every finitely presented group* π.

1.5. **The algebraic tertiary obstruction.** Section 5 contains a detailed explanation of the τ invariant that appears in the Tertiary Property, however we give a brief description here. Recall that in the smooth category, a generic map of a surface to a 4-manifold looks locally like a linear subspace $\mathbb{R}^2 \subset \mathbb{R}^4$ or like a transverse double point $\mathbb{R}^2 \times \{0\} \cup \{0\} \times \mathbb{R}^2 \subset \mathbb{R}^4$. We know that, stably, every spin manifold with signature zero has a smooth structure, so we may assume that all our surfaces are smoothly embedded with respect to such a structure.

Let $f\colon S^2 \looparrowright M$ be a generically immersed 2-sphere with vanishing algebraic self-intersection, that is $\mu_M(f) = 0$. Then all the double points of f can be paired by generically immersed Whitney discs. They can be chosen to be disjointly embedded, framed, and to intersect f transversely, by boundary twisting to fix the framing and then pushing any intersections down to f. We count the number, modulo two, of intersection points between the Whitney discs and f. This count depends a priori on the choice of Whitney discs, so one has to be careful.

Recall that for a closed, connected 4-manifold M, since the $\mathbb{Z}/2$ intersection form is nonsingular the homomorphism

$$H_2(M;\mathbb{Z}/2) \longrightarrow \mathbb{Z}/2, \quad x \mapsto x \cdot x := \langle PD^{-1}(x), x\rangle$$

can be written as $x \cdot x = c \cdot x$ for a unique *characteristic* element $c \in H_2(M;\mathbb{Z}/2)$. This motivates the terminology in the next definition.

Definition 1.11. For $\alpha \in H_2(M;\mathbb{Z}/2)$, if $x \cdot x = \alpha \cdot x$ for all *spherical* x, meaning those x represented by a map $S^2 \to M$, we call α *S^2-characteristic*.

Similarly we call α *$\mathbb{RP}^2$-characteristic* if the same equation holds for all classes x represented by a map $\mathbb{RP}^2 \to M$. Note that $\mathbb{RP}^2$-characteristic implies S^2-characteristic.

We say that an element $\beta \in \pi_2(M)$ is S^2- or $\mathbb{RP}^2$-characteristic if its image under the modulo 2 Hurewicz map $\pi_2(M) \to H_2(M;\mathbb{Z}/2)$ is S^2- or $\mathbb{RP}^2$-characteristic respectively.

It was shown in [Sto94] and [ST01] that the modulo 2 count of intersection points between Whitney discs and f introduced above is well-defined exactly when f is $\mathbb{RP}^2$-characteristic. Moreover, the resulting invariant $\tau(f) \in \mathbb{Z}/2$ only depends on the homotopy class $[f] \in \pi_2(M)$. It was called the *Kervaire-Milnor invariant* by Freedman-Quinn, who gave a slightly incomplete account of it in [FQ90, Section 10]. This was later resolved by Stong [Sto94].

Let us return to our obstruction theory for closed, connected spin 4-manifolds M with $\mathfrak{pri}(M) = 0 = \mathfrak{sec}(M)$. For the algebraic secondary invariant, we chose a 2-complex K and a stable splitting $s\colon H_K \to (\mathbb{Z}\pi)^r \oplus \pi_2(M)$ of the short exact sequence from Theorem 1.4 and defined the hermitian form $\lambda_{M,s}$ on H_K as the composition of λ_M with $s_2\colon H_K \to \pi_2(M)$. Up to even forms, this did not depend K nor on the splitting s and we shall show in Lemma 6.10 that there are splittings s for which $\lambda_{M,s}$ actually vanishes.

We want to define τ on elements in the image of such splittings s. Start with an element $x \in H^2(\pi;\mathbb{Z}/2)$, restrict it using $K \subseteq B\pi$ to $H^2(K;\mathbb{Z}/2)$ and lift that to $\widetilde{x} \in H_K = H^2(K;\mathbb{Z}\pi)$, using that $H^3(K;-) = 0$, so the Bockstein potentially obstructing this lifting vanishes. Given a splitting s with $\lambda_{M,s} \equiv 0$, let $y := s_2(\widetilde{x}) \in \pi_2(M)$, and then compute $\tau_M(y) \in \mathbb{Z}/2$. This is well defined if y is $\mathbb{RP}^2$-characteristic.

We will show in Lemma 6.5 that the above construction determines a well-defined map

$$\tau_{M,s}\colon h^2(\pi) \to \mathbb{Z}/2$$

on the set

$$h^2(\pi) := \ker(Sq^2\colon H^2(\pi;\mathbb{Z}/2) \to H^4(\pi;\mathbb{Z}/2)).$$

Here $h^2(\pi)$ is dual to the $E^3_{2,2}$-term in our AHSS, since

$$Sq^2 = (d^2_{4,1})^*\colon H^2(\pi;\mathbb{Z}/2) \to H^4(\pi;\mathbb{Z}/2)$$

is the dual of the $d^2_{4,1}$ differential. The duals of the other relevant differentials in the AHSS for $\Omega_4^{Spin}(B\pi)$ are

$$(d^2_{5,0})^*\colon H^3(\pi;\mathbb{Z}/2) \to \operatorname{Hom}_{\mathbb{Z}}(H_5(\pi;\mathbb{Z}),\mathbb{Z}) \quad \text{and}$$

$$(d^3_{5,0})^*\colon h^2(\pi) = \ker(d^2_{4,1})^* \to \ker(d^2_{5,0})^*.$$

There is a canonical isomorphism

$$\begin{aligned}\omega\colon E_{2,2} = H_2(\pi;\mathbb{Z}/2)/\operatorname{im}(d^2_{4,1}, d^3_{5,0}) &\xrightarrow{\cong} \operatorname{Hom}_{\mathbb{Z}/2}(H^2(\pi;\mathbb{Z}/2),\mathbb{Z}/2)/\operatorname{im}(d^2_{4,1}, d^3_{5,0})\\ &\xrightarrow{\cong} \operatorname{Hom}_{\mathbb{Z}/2}(\ker(d^3_{5,0})^*,\mathbb{Z}/2).\end{aligned}$$

Since $\ker(d^3_{5,0})^* \subseteq h^2(\pi)$, we can restrict $\tau_{M,s}$ to a map $\tau_{M,s}|\colon \ker(d^3_{5,0})^* \to \mathbb{Z}/2$. We want to show that this coincides with the image of $\mathfrak{ter}(M)$ under ω. Again, we give the statement in the form of a property of a group.

Tertiary Property 1.12. *We say that π has the* Tertiary Property *if $\mathfrak{ter}(M) \in E_{2,2}$ is sent to $\tau_{M,s}|$ via ω for all closed, connected, spin 4-manifolds M with fundamental group π and $\mathfrak{pri}(M) = 0 = \mathfrak{sec}(M)$.*

We remark that $\tau_{M,s}(0) = 0$ because $0 \in h^2(\pi)$ leads to computing τ_M on $0 \in \pi_2(M)$, which is represented by a trivial (embedded) sphere. So in the case that $\ker(d^3_{5,0})^* = 0$, the Tertiary Property automatically holds. We are unable to show in general that $\tau_{M,s}|$ is a homomorphism. This follows from the Tertiary Property, when it holds, since $\mathfrak{ter}(M)$ maps to a homomorphism.

In Proposition 6.8 we will show that the Tertiary Property holds whenever the splitting s is induced by a map $f\colon M \to K$, in the sense that

$$s_2 = PD \circ f^*\colon H^2(K;\mathbb{Z}\pi) \to H^2(M;\mathbb{Z}\pi) \to H_2(M;\mathbb{Z}\pi) \cong \pi_2(M)$$

for K a 2-complex with $f_*\colon \pi_1(M) \to \pi_1(K)$ an isomorphism. In Theorem 6.21 we give a condition under which every splitting is realised in this way, and we use this to deduce the following theorem.

Theorem 1.13. *The Tertiary Property holds for the following classes of groups.*

- *Groups π with a finite 3-dimensional model for $B\pi$.*
- *Abelian groups and right angled Artin groups.*
- *Finite groups whose 2-Sylow subgroups are abelian or quaternion.*

Our verification of Theorem 1.13 in the case of abelian groups relies on computations of Whitehead's Γ-groups [Whi50] made by the first and second authors with Ben Ruppik [KPR20]. In the special case that π is the fundamental group of a closed, oriented, aspherical 3-manifold, we proved in [KLPT17] that π has the Tertiary Property 1.12. To do this, we computed $\mathfrak{ter}(M)$ and $\tau_{M,s}$ on sufficiently many concrete examples for which the primary and secondary obstructions vanish. We close this section with the following conjecture, analogous to Secondary Property 1.8.

Conjecture 1.14. *Every group has the Tertiary Property.*

1.6. **When stable homeomorphism implies homeomorphism.** By combining our results with results of Hambleton-Kreck, Khan and Crowley-Sixt, we obtain results on the unstable homeomorphism classification. If we consider manifolds that are already sufficiently stabilised, in a sense to be made precise presently, then it turns out that stable homeomorphism implies homeomorphism.

Recall that a finitely presented group π is *polycyclic-by-finite* if it has a subnormal series where the quotients are either cyclic or finite. The number of infinite cyclic quotients in such a subnormal series turns out to be an invariant of π, called the *Hirsch number* $\mathfrak{h}(\pi)$. Define $\mathfrak{h}'(\pi) = 1$ if π is finite, and define $\mathfrak{h}'(\pi) = \mathfrak{h}(\pi) + 3$ if π is infinite. The following theorem is due to Hambleton and Kreck [HK93, Theorem B] for π finite, and due to Crowley and Sixt [CS11, Theorem 1.1] for π infinite.

Theorem 1.15 ([HK93, CS11]). *Let M and N be closed, connected 4-manifolds with polycyclic-by-finite fundamental group π such that $\chi(N) + 2k = \chi(M)$ for some $k \geq \mathfrak{h}'(\pi)$. If M and N are stably homeomorphic then M is homeomorphic to $N \# k(S^2 \times S^2)$.*

Note that [CS11, Theorem 1.1] is only stated in the smooth category, but as remarked at the beginning of [CS11, Section 2.1], the theorem also holds in the topological category since polycyclic-by-finite groups are good in the sense of Freedman [FQ90, Section 2.9 and Theorem 5.1A].

For π virtually abelian and infinite, there is a similar statement to [CS11, Theorem 1.1] by Khan [Kha17, Corollary 2.4], with a slightly better bound on the number of stabilisations needed.

1.7. **Context for our work.** The stable classification of 4-manifolds was reduced to a bordism computation by Kreck in [Kre99]. 4-manifolds with finite fundamental group were studied by Hambleton and Kreck in [HK88], as well as in the PhD thesis of the third author [Tei92]. The case of geometrically 2-dimensional groups was solved by Hambleton, Kreck and the last author in [HKT09]. Groups of cohomological dimension at most 3, and in particular right angled Artin groups with this property, were studied by Hambleton and Hildum [HH19] in the case that the equivariant intersection form is even, which by our results in the current paper is equivalent to the secondary obstruction vanishing. Previous work on the stable classification question also includes work of Cavicchioli, Hegenbarth and Repovš [CHR95], Spaggiari [Spa03] and Davis [Dav05].

The papers [HK88], [HKT09] and [HH19] focussed on the normal 2-type – roughly this means they looked at the entire intersection form of the 4-manifold – also obtaining unstable classification results. On the other hand, the algebraic invariants that we study are tailored to the stable question, so we are able to handle more fundamental groups with more easily computable obstructions.

Acknowledgements. We are delighted to have the opportunity to thank Diarmuid Crowley, Jim Davis, Fabian Hebestreit, Matthias Kreck, Markus Land, Ian Hambleton, Henrik Rüping, Mark Ullmann and Christoph Winges for many useful and interesting discussions on this work.

The authors thank the Max Planck Institute for Mathematics and the Hausdorff Institute for Mathematics in Bonn for financial support and their excellent research environments. The second author gratefully acknowledges an NSERC Discovery Grant.

Contents

2. Stable diffeomorphism, bordism groups, and spectral sequences

Throughout the paper M denotes a smooth, closed, compact, connected, oriented 4-manifold with fundamental group $\pi_1(M) \cong \pi$. The universal cover of M will be denoted $\widetilde{M}$. If we fix an identification of $\pi_1(M)$ with π, the identification determines a homotopy class of maps $c\colon M \to B\pi$ classifying $\widetilde{M}$.

Two smooth 4-manifolds M and N are called *stably diffeomorphic* if there exist integers $m, n \in \mathbb{N}_0$ such that stabilising M and N with copies of $S^2 \times S^2$ yields diffeomorphic manifolds

$$M \# m(S^2 \times S^2) \cong N \# n(S^2 \times S^2).$$

We require that the diffeomorphism respects orientations. Note that, unlike elsewhere in the literature, we do not require that $m = n$.

2.1. **Bordism groups.** The starting point for our investigation, coming from Kreck's modified surgery, is that stable diffeomorphism can be understood in terms of bordism over the normal 1-type.

Definition 2.1. Let M be a closed oriented manifold of dimension n. A *normal* 1*-type* is a fibration over BSO, denoted by $\xi\colon B \to BSO$, through which the stable normal bundle $\nu_M\colon M \to BSO$ factors as:

$$\begin{tikzcd} & B \arrow[d, "\xi"] \\ M \arrow[ur, bend left, "\widetilde{\nu}_M"] \arrow[r, "\nu_M"'] & BSO \end{tikzcd}$$

such that $\widetilde{\nu}_M$ is 2-connected and ξ is 2-coconnected. A choice of a map $\widetilde{\nu}_M$ is called a *normal* 1*-smoothing* of M.

The different normal 1-types of a fixed M are fibre homotopy equivalent to one another.

Theorem 2.2 ([Kre99, Theorem C])**.** *Two closed* 4*-dimensional manifolds with the same Euler characteristic and the same normal* 1*-type* $\xi\colon B \to BSO$*, admitting bordant normal* 1*-smoothings, are diffeomorphic after connected sum of both with* r *copies of* $S^2 \times S^2$ *for some* r*.*

Here is a summary of the proof. Given a bordism over the normal 1-type, one can perform surgery below the middle dimension until the bordism becomes 1-connected. Excise 2-spheres by tubing a neighbourhood to the boundary. This makes the bordism into an s-cobordism, which is stably a product, at the expense of connect summing the boundary 4-manifolds with copies of $S^2 \times S^2$. A key point to check is that the normal smoothing data allows us to excise 2-spheres with a framed normal bundle, resulting in connect sums with $S^2 \times S^2$ and not the twisted bundle $S^2 \widetilde{\times} S^2$. The converse, that stably diffeomorphism manifolds admit bordant 1-smoothings, can be found in Crowley-Sixt [CS11, Lemma 2.3(ii)].

By taking account of the different choices of normal 1-smoothing, and allowing different numbers of $S^2 \times S^2$ to be added to either side to allow the Euler characteristics of the initial manifolds to differ, we obtain the following.

Theorem 2.3. *The stable diffeomorphism classes of* 4*-manifolds with normal* 1*-type* ξ *are in one-to-one correspondence with* $\Omega_4(\xi)/\operatorname{Aut}(\xi)$*.*

Next we need to understand the normal 1-type. The following lemma, determining the normal 1-type of spin 4-manifolds, is well-known to the experts. We refer to [KLPT17, Lemma 3.5] for a proof.

Lemma 2.4. *Let* π *be a finitely presented group. A normal* 1*-type of a spin* 4*-manifold with fundamental group* π *is given by*

$$B\pi \times BSpin \xrightarrow{\gamma\circ \mathrm{pr}_2} BSO,$$

where pr_2 *is the projection onto* $BSpin$ *and* γ *is the canonical map* $BSpin \to BSO$*.*

This article gives algebraic invariants that obstruct null-bordism in $\Omega_4(\xi)$ in the above two cases.

2.2. **Spectral sequences.** The main tool for understanding the bordism groups is the James spectral sequence. However, as explained in [KLPT17, Section 2], this is isomorphic to the Atiyah-Hirzebruch spectral sequence for $\Omega_4^{Spin}(B\pi)$.

Let h_* be a generalised homology theory (for us spin bordism Ω_*^{Spin}), and let X be a CW complex. The Atiyah-Hirzebruch spectral sequence arising from the homology Leray-Serre spectral sequence for the fibration $\mathrm{pt} \to X \to X$ has E_2 page given by

$$E^2_{p,q} := H_p(X; h_q(\mathrm{pt})).$$

The d^r differential has (p,q)-bidegree $(-r, r-1)$, and the sequence converges to $h_*(X)$. We denote the differentials by

$$d^r_{p,q} \colon H_p(X; h_q(\mathrm{pt})) \to H_{p-r}(X; h_{q+r-1}(\mathrm{pt})).$$

Denote the filtration on the abutment of an Atiyah-Hirzebruch spectral sequence by

$$0 \subset F_{0,n} \subset F_{1,n-1} \subset \cdots \subset F_{n-q,q} \subset \cdots \subset F_{n,0} = h_n(X) = \Omega_n(\xi).$$

Recall that $F_{n-q,q}/F_{n-q-1,q+1} \cong E^\infty_{n-q,q}$.

Let $B := B\pi \times BSpin$. Denote the restriction of $\mathrm{pr}_1 \colon B \to B\pi$ to the inverse image of the p-skeleton of $B\pi$ by $B|_p$, and let $\xi|_p \colon B|_p \to BSO$ be the restriction of ξ to $B|_p$. An element of $\Omega_n(\xi)$ lies in $F_{p,n-p}$ if and only if it is in the image of the map $\Omega_n(\xi|_p) \to \Omega_n(\xi)$. That is, if the element lies in the image of $\Omega_n^{Spin}(B\pi^{(p)})$. This follows from the naturality of the spectral sequence applied to the map of fibrations induced by the inclusion of $B\pi^{(p)} \to B\pi$.

In our situation, being in $F_{p,4-p}$ means that a 4-manifold M together with its classifying map $c \colon M \to B\pi$ is spin bordant to a 4-manifold M' over $B\pi$ where the map to $B\pi$ factors through the p-skeleton $B\pi^{(p)}$ of $B\pi$. For $2 \leq p \leq 4$ we can perform surgeries on M' to convert the map $M' \to B\pi^{(p)}$ to a map $M'' \to B\pi^{(p)}$ that induces an isomorphism on fundamental groups. By Kreck's modified surgery (Theorem 2.2), M and M'' are stably diffeomorphic, and thus after connected sums with copies of $S^2 \times S^2$, c is homotopic to a map factoring through $B\pi^{(p)}$.

The converse also holds, that is if one can find a map $f \colon M \to B\pi^{(p)}$ for $2 \leq p \leq 4$ inducing the chosen isomorphism on fundamental groups, then (M, c) lies in $F_{p,4-p}$.

Now, to state the next lemma, let X be any CW complex and let $X^{(p)}$ be its p-skeleton. Denote the barycentres of the p-cells $\{e_i^p\}$ of X by $\{b_i^p\}_{i\in I}$. Given an element $[M \to X^{(p)}] \in \Omega_n^{Spin}(X^{(p)})$, denote the regular preimage of the barycentre $\{b_i^p\} \in X^{(p)}$ by $N_i \subset M$. Note that $[N_i] \in \Omega_{n-p}^{Spin}$, since the normal bundle of N_i in M is trivial, and hence N_i inherits a spin structure from M.

The following lemma is a consequence of [KLPT17, Lemma 2.5].

Lemma 2.5. *The canonical map $\Omega_n^{Spin}(X^{(p)}) \to H_p(X^{(p)}; \Omega_{n-p}^{Spin})$ that comes from the spectral sequence (see below) coincides with the map*

$$\begin{array}{rcl} \Omega_n^{Spin}(X^{(p)}) & \to & H_p(X^{(p)}; \Omega^{Spin}_{n-p}) \\ {[M \to X^{(p)}]} & \mapsto & \Big[\sum_{i\in I} [N_i] \cdot e_i^p\Big]. \end{array}$$

The same holds with oriented bordism replacing spin bordism.

The map in the statement of the lemma $\Omega_n^{Spin}(X^{(p)}) \to H_p(X^{(p)}; \Omega_{n-p}^{Spin})$ is sometimes called an edge homomorphism. It arises as follows. The abutment of the Atiyah-Hirzebruch spectral sequence $\Omega_n(\xi|_{X^{(p)}}) = F_{n,0}$ maps to its quotient by the first filtration step $F_{p,n-p}$ that differs from $F_{n,0}$. This term is indeed $F_{p,n-p}$, since the homology of $X^{(p)}$ vanishes in degrees greater than p, thus $E^2_{s,t} = E^\infty_{s,t} = 0$ for all $s > p$. We have $F_{n,0}/F_{p,n-p} \cong E^\infty_{p,n-p}$. The target, $H_p(X^{(p)}; \Omega_{n-p}^{Spin})$, is the $E^2_{p,n-p}$ term of the spectral sequence. Since no differentials have image in $E^2_{p,n-p}$, we have that $E^\infty_{p,n-p} \subseteq E^2_{p,n-p} = H_p(X^{(p)}; \Omega_{n-p}^{Spin})$, and so the composition

$$\Omega_n^{Spin}(X^{(p)}) = F_{n,0} \to F_{n,0}/F_{p,n-p} \xrightarrow{\cong} E^\infty_{p,n-p} \to E^2_{p,n-p} = H_p(X^{(p)}; \Omega_{n-p}^{Spin})$$

gives the desired map.

We will need the following standard fact, so we give the argument here separately. Recall that π can be any finitely presented group.

Lemma 2.6. *The subgroup $F_{0,4} \subset \Omega_4^{Spin}(B\pi)$ is isomorphic to Ω_4^{Spin}, and is a direct summand of $\Omega_4^{Spin}(B\pi)$.*

Proof. The term $E^2_{0,4} \cong H_0(B\pi;\Omega_4^{Spin}) \cong \Omega_4^{Spin}$. The maps $\mathrm{pt} \to B\pi$ and $B\pi \to \mathrm{pt}$ induce maps between the Atiyah-Hirzebruch spectral sequences. But the spectral sequence for a point is supported in a single column, so there are maps $\Omega_4^{Spin} \to \Omega_4^{Spin}(B\pi) \to \Omega_4^{Spin}$. The composition $\mathrm{pt} \to B\pi \to \mathrm{pt}$ is the identity, and maps between the spectral sequences are natural, so the map $\Omega_4^{Spin} \to \Omega_4^{Spin}(B\pi)$ splits. Thus Ω_4^{Spin} is a direct summand as claimed. □

We consider the Atiyah-Hirzebruch spectral sequence for $\Omega_4^{Spin}(B\pi)$.

Proposition 2.7. *There is a filtration*

$$0 \subseteq F_3 \subseteq F_2 \subseteq F_1 \subseteq \Omega_4^{Spin}(B\pi)$$

by subgroups such that the following holds.

(1) The quotient $\Omega_4^{Spin}(B\pi)/F_1$ is isomorphic to

$$\ker\left(d^3_{4,0}\colon \ker\left(\mathrm{Sq}_2 \circ \mathrm{red}_2 \colon H_4(\pi;\mathbb{Z}) \to H_2(\pi;\mathbb{Z}/2)\right) \to H_1(\pi;\mathbb{Z}/2)\right).$$

The image of an element (M,c) in this subgroup of $H_4(B\pi;\mathbb{Z})$ is $c_*([M])$, and is denoted $\mathfrak{pri}(M)$. Here $\mathrm{Sq}_2\colon H_4(\pi;\mathbb{Z}/2) \to H_2(\pi;\mathbb{Z}/2)$ denotes the map on homology that is dual to the usual cohomology Steenrod operation $\mathrm{Sq}^2\colon H^2(\pi;\mathbb{Z}/2) \to H^4(\pi;\mathbb{Z}/2)$ and red_2 is the reduction modulo two of the coefficients.

(2) The quotient F_1/F_2 is isomorphic to $H_3(\pi;\mathbb{Z}/2)/\operatorname{im}(d^2_{5,0})$, where

$$d^2_{5,0} = \mathrm{Sq}_2 \circ \mathrm{red}_2 \colon H_5(\pi;\mathbb{Z}) \to H_3(\pi;\mathbb{Z}/2).$$

The image of an element (M,c) is denoted $\mathfrak{sec}(M)$. If $\mathfrak{pri}(M) = 0$, then $\mathfrak{sec}(M)$ can be computed by Lemma 2.5.

(3) The quotient F_2/F_3 is isomorphic to $H_2(\pi;\mathbb{Z}/2)/\operatorname{im}(d^2_{4,1}, d^3_{5,0})$, where

$$d^2_{4,1} = \mathrm{Sq}_2 \circ \mathrm{red}_2 \colon H_4(\pi;\mathbb{Z}/2) \to H_2(\pi;\mathbb{Z}/2).$$

If $\mathfrak{pri}(M) = \mathfrak{sec}(M) = 0$, then $\mathfrak{ter}(M)$ can be computed by Lemma 2.5.

(4) The subgroup F_3 is a direct summand isomorphic to $\mathbb{Z}$, and the map to $\mathbb{Z}$ is given by $\sigma/16$, that is take the signature and divide by 16.

Proof. The proposition follows from computing using the Atiyah-Hirzebruch spectral sequence, with $E^2_{p,q} = H_p(B\pi;\Omega_q^{Spin})$. We know from Lemma 2.6 that the coefficients split, so recalling Rochlin's theorem that $\Omega_4^{Spin} \cong 16\mathbb{Z}$, this proves the last item.

Spin bordism in the lower dimensions is given by $\Omega_3^{Spin} = 0$, $\Omega_2^{Spin} = \mathbb{Z}/2$, $\Omega_1^{Spin} = \mathbb{Z}/2$ and $\Omega_0^{Spin} \cong \mathbb{Z}$ [Mil63], [ABP67].

For the spin bordism Atiyah-Hirzebruch spectral sequence, the E^2-page differential d_2 is dual to the Steenrod square Sq^2, together with reduction of the coefficients mod 2 if necessary, according to [Tei92, Theorem 3.1.3]. This implies that the differential vanishes when the codomain is $H_p(B\pi;\Omega_q^{Spin})$ and $p < 2$, since $\mathrm{Sq}^r\colon H^n \to H^{n+r}$ is zero whenever $n < r$.

There are a couple of potentially nontrivial d^2 and d^3 differentials, that are recorded in the proposition. □

Now, computing the secondary and tertiary invariants as inverse images is not a very feasible task. A particular difficulty is that one typically has to find a bordant manifold such that the map to $B\pi$ can be homotoped onto the p-skeleton, where $p = 3$ for $\mathfrak{sec}$ and $p = 2$ for $\mathfrak{ter}$. Then one takes inverse images with framings. The goal of this paper, as explained in the introduction, is to give algebraic criteria that decide whether these invariants vanish. A particularly nice feature is that we show that the algebraic secondary obstruction does not depend on the way in which the primary invariant vanishes. That is, while it needs the primary obstruction to vanish in order to be well-defined, it does not depend on the choice of vanishing i.e. of splitting of the short exact sequence. Similarly, the algebraic tertiary obstruction needs the first two obstructions to vanish in order to be well-defined, but it does not depend on the way in which they vanish. These features increase the computability of our obstructions.

2.3. **Topological 4-manifolds and stable homeomorphism.** Our results apply to the stable homeomorphism classification of topological spin 4-manifolds, as we explain in this section. First we point out that the invariants from Theorem 1.1 apply in the topological category, as asserted in the introduction. Note that the signature $\sigma(M)$, $\pi_2(M)$, and λ_M are all homotopy invariant, so give rise to stable homeomorphism invariants. The Kervaire-Milnor invariant τ_M is defined using intersections: by topological transversality [FQ90, Theorem 9.5A], such intersections can be counted in topological manifolds. Theorem 1.1 requires that we compute these invariants on all possible 1-skeleton connected sums $(N_1, \alpha_1)\#_1(N_2, -\alpha_2)$. To avoid handle structures, one can instead take the connected sum $N_1 \# -N_2$ and then perform surgery on circles, framed using the α_i, to make the fundamental group isomorphic to π, and in this way construct the test manifolds on which we claim one must compute our algebraic invariants in order to decide stable homeomorphism.

Now we explain why the classifications coincide. To start, we have forgetful map

$$\mathbb{Z} \cong \Omega_4^{Spin} \xrightarrow{\cdot 2} \Omega_4^{TOPSpin} \cong \mathbb{Z}.$$

Recall that the isomorphism with $\mathbb{Z}$ is given by the signature, taking account of its divisibility. That is, we have isomorphisms $\sigma\colon \Omega_4^{SO} \xrightarrow{\cong} \mathbb{Z}$, $\sigma/16\colon \Omega_4^{Spin} \xrightarrow{\cong} \mathbb{Z}$ and $\sigma/8\colon \Omega_4^{TOPSpin} \xrightarrow{\cong} \mathbb{Z}$. The cokernel

$$\operatorname{coker}(\Omega_4^{Spin} \to \Omega_4^{TOPSpin}) \cong \mathbb{Z}/2$$

is detected by the Kirby-Siebenmann invariant. Thus in particular the Kirby-Siebenmann invariants must coincide for stably homeomorphic 4-manifolds, but for spin manifolds the Kirby-Siebenmann invariant is determined by the signature.

We want to apply our stable classification, but for this we need smooth manifolds. Consider two 4-manifolds M and M' that are spin and have nonvanishing Kirby-Siebenmann invariant. Suppose that $\pi_1(M) \cong \pi_1(M')$ has the Secondary and Tertiary Properties. We want to know if the two manifolds are stably homeomorphic. Let W be the E_8-manifold. The manifolds $M\#W$ and $M'\#W$ have vanishing Kirby-Siebenmann invariants, and so are stably smoothable by [FQ90, Section 8.6]. We then have smooth manifolds, and so we can apply our classification programme to decide whether they are stably diffeomorphic. This involves algebraic invariants that are independent of the smooth structure. Suppose that we discover $M\#W$ and $M'\#W$ to be stably diffeomorphic. Then $M\#W\#-W$ is stably homeomorphic to $M'\#W\#-W$. But then $W\#-W$ is B-null bordant, so is stably homeomorphic to S^4. It follows that M and M' are stably homeomorphic. Thus the same programme as in the smooth case determines whether two closed, spin topological 4-manifolds, whose fundamental groups satisfy Secondary Property 1.8 and Tertiary Property 1.12, are stably homeomorphic.

3. Whitehead's certain exact sequence and the Γ functor

In this section, we give some background on the "certain exact sequence" of Whitehead [Whi50], material that we will make extensive use of in our investigation of the Secondary Property, and its proof for some families of groups.

Let Y be a simply connected CW complex. Let $C_n := \pi_n(Y^{(n)}, Y^{(n-1)})$ be the nth cellular chain group. Let $d_n\colon C_n \to C_{n-1}$ denote the canonical boundary map arising from the long exact sequence of the pair. Let $i_n\colon \pi_n(Y^{(n)}) \to C_n$ be the map induced by the inclusion of pairs. Let $Z_n := \ker(d_n)$ and let $B_n := \operatorname{im}(d_{n+1})$. Then of course $H_n(Y;\mathbb{Z}) \cong Z_n/B_n$. Let $h_n\colon \pi_n(Y) \to Z_n/B_n$ be the Hurewicz map, and let $b_n\colon Z_n/B_n \to \ker(i_{n-1})$ be induced from the boundary map $\pi_n(Y^{(n)}, Y^{(n-1)}) \to \pi_{n-1}(Y^{(n-1)})$. Furthermore, let $\iota_n\colon \ker(i_n) \to \pi_n(Y)$ be induced by the inclusion $Y^{(n)} \to Y$. These maps enable us to formulate the "certain exact sequence" of J.H.C. Whitehead, which describes the kernel and cokernel of the Hurewicz map.

Theorem 3.1 ([Whi50, Section 10])**.** *There is a long exact sequence*

$$\cdots \to \pi_4(Y) \xrightarrow{h_4} H_4(C_n, d_n) \xrightarrow{b_4} \ker(i_3) \xrightarrow{\iota_3} \pi_3(Y) \xrightarrow{h_3} H_3(C_n, d_n) \to 0.$$

The extra ingredient that makes this sequence extremely useful is a description of $\ker(i_3)$ in terms of $\pi_2(Y)$. This uses the Γ group, which we will now define.

Definition 3.2. Let A be an abelian group. Then $\Gamma(A)$ is an abelian group with generators the elements of A. We write a as $v(a)$ when we consider it as an element of $\Gamma(a)$. The group $\Gamma(A)$ has the following relations:

$$\{v(-a) - v(a) \mid a \in A\} \quad \text{and}$$
$$\{v(a+b+c) - v(b+c) - v(c+a) - v(a+b) + v(a) + v(b) + v(c) \mid a, b, c \in A\}.$$

We remark that the symbol v has no meaning on its own, rather it is used to differentiate the generating set for $\Gamma(A)$ from the generating set for A.

The functor Γ is the universal quadratic functor, in the following sense. A map $f\colon A \to B$ of abelian groups is called quadratic if $f(a) = f(-a)$ for all $a \in A$ and for all $a, a' \in A$ the map

$$A \times A \to B,\ (a, a') \mapsto f(a + a') - f(a) - f(a')$$

is bilinear. The map $j\colon A \to \Gamma(A)$ sending a to $v(a)$ is quadratic. The functor Γ satisfies the universal property that for every quadratic map $f\colon A \to B$, there is a unique homomorphism $\Gamma(f)\colon \Gamma(A) \to B$ with $f = \Gamma(f) \circ j$.

Lemma 3.3 ([Whi50, p. 62])**.** *If A is free abelian with basis $\mathcal{B}$, then $\Gamma(A)$ is free abelian with basis*

$$\{v(b), v(b + b') - v(b) - v(b') \mid b, b' \in \mathcal{B}\}.$$

In particular, if A is free abelian, then sending $v(a)$ to $a \otimes a$ defines an isomorphism between $\Gamma(A)$ and the subgroup of symmetric tensors of $A \otimes_{\mathbb{Z}} A$, that is, the subgroup generated by $\{a \otimes a \mid a \in A\}$.

Theorem 3.4 ([Whi50, Sections 10 and 13])**.** *Let $\eta\colon S^3 \to S^2$ be the Hopf map. The map $\widehat{\eta}\colon \Gamma(\pi_2(Y)) \to \ker(i_3\colon \pi_3(Y^{(3)}) \to \pi_3(Y^{(3)}, Y^{(2)}))$ given by $v(\alpha) \mapsto \alpha \circ \eta$ is an isomorphism. In particular, for every simply-connected CW complex Y we have the exact sequence:*

$$\ldots \pi_4(Y) \to H_4(Y;\mathbb{Z}) \xrightarrow{\widehat{\eta}^{-1} \circ b_4} \Gamma(\pi_2(Y)) \xrightarrow{\iota_3 \circ \widehat{\eta}} \pi_3(Y) \to H_3(Y;\mathbb{Z}) \to 0.$$

Now let L be any CW complex and as above let $\eta\colon S^3 \to S^2$ be the Hopf map.

Lemma 3.5. [Whi50, Section 13] *The map*

$$\Gamma(\eta)\colon \Gamma(\pi_2(L)) \to \pi_3(L); v(\alpha) \mapsto \alpha \circ \eta$$

yields a well defined homomorphism.

For later use we restate the Whitehead exact sequence in the form that we will need it, namely for general CW complexes that need not be simply connected.

Theorem 3.6. *For a CW complex L, the following sequence is exact:*

$$H_4(\widetilde{L};\mathbb{Z}) \to \Gamma(\pi_2(L)) \xrightarrow{\Gamma(\eta)} \pi_3(L) \to H_3(\widetilde{L};\mathbb{Z}) \to 0.$$

Each of the terms in functorial in L and the maps in the sequence above are natural.

To finish the section, we record a couple of preliminary facts on the Γ groups that we will use throughout the rest of the paper.

Corollary 3.7. *Let K be a 2-complex. Then the map $\Gamma(\eta)\colon \Gamma(\pi_2(K)) \to \pi_3(K)$ is an isomorphism.*

Proof. For K a 2-complex, both the third and fourth homology groups of $\widetilde{K}$ vanish, so $\Gamma(\eta)$ is an isomorphism by Theorem 3.6. $\square$

Corollary 3.8. *Let K be a 2-complex. Every element in $\pi_3(K)$ is a sum of elements of the form $\beta \circ \eta$ with $\beta \in \pi_2(K)$.*

Proof. Every element of $\Gamma(\pi_2(K))$ is a sum of elements $v(a)$ with $a \in \pi_2(K)$ by definition. Since $\Gamma(\eta)$ is a surjection and $\Gamma(\eta)(v(a)) = a \circ \eta$, the corollary follows. $\square$

Let

$$\begin{array}{rcl} T\colon N \otimes_{\mathbb{Z}} N' & \to & N' \otimes_{\mathbb{Z}} N \\ n \otimes n' & \mapsto & n' \otimes n \end{array}$$

be the transposition map.

Lemma 3.9. *Let N, N' be free $\mathbb{Z}$-modules. Then*

$$\Gamma(N \oplus N') \cong \Gamma(N) \oplus (N \otimes N') \oplus \Gamma(N').$$

Moreover the inclusion into

$$(N \oplus N') \otimes_{\mathbb{Z}} (N \oplus N') \cong (N \otimes_{\mathbb{Z}} N) \oplus (N \otimes_{\mathbb{Z}} N') \oplus (N' \otimes_{\mathbb{Z}} N) \oplus (N' \otimes_{\mathbb{Z}} N')$$

is given by the direct sum of the inclusion $\Gamma(N) \to N \otimes_{\mathbb{Z}} N$, the diagonal map $(1, T)\colon N \otimes_{\mathbb{Z}} N' \to (N \otimes_{\mathbb{Z}} N') \oplus (N' \otimes_{\mathbb{Z}} N)$ and the inclusion $\Gamma(N') \to N' \otimes_{\mathbb{Z}} N'$.

Proof. Since $N \oplus N'$ is free, $\Gamma(N \oplus N')$ is the symmetric tensors. Embed $\Gamma(N \oplus N')$ into $(N \oplus N') \otimes (N \oplus N')$ and observe that the subgroup of symmetric tensors is isomorphic to $\Gamma(N) \oplus N \otimes N' \oplus \Gamma(N')$ with

$$(n_1 \otimes n_2, m \otimes m', \ell_1' \otimes \ell_2') \mapsto n_1 \otimes n_2 + m \otimes m' + m' \otimes m + \ell_1' \otimes \ell_2'$$

the inverse of the embedding. $\square$

4. The secondary obstruction

Here is an outline of this section. First we discuss the equivariant intersection form and some of its different guises. We associate an element of Tate cohomology (see Section 4.2) to the intersection form of a 4-manifold with vanishing primary obstruction. The Tate group measures whether or not the intersection form λ_M is even, when restricted to the $H^2(K;\mathbb{Z}\pi)$ summand of the stabilised second homotopy group $\pi_2(M)\oplus\mathbb{Z}\pi^n$. The Secondary Property asserts that this element of the Tate group detects $\mathfrak{sec}(M)$.

In Condition 4.17, we give a condition that we then prove in the remainder of the section is sufficient for the Secondary Property 1.8 to hold for a group π. In Sections 9 and 10, we will use this condition to show that many families of groups have the Secondary Property 1.8.

4.1. **The equivariant intersection form.** Let M be a smooth, closed, spin, based 4-manifold together with an identification $\pi_1(M)\xrightarrow{\cong}\pi$. The equivariant intersection form

$$\lambda_M\colon \pi_2(M)\times\pi_2(M)\to\mathbb{Z}\pi$$

is defined as follows. Identify $\pi_2(M)\cong H_2(M;\mathbb{Z}\pi)$ via the Hurewicz theorem. Then for classes $x,y\in H_2(M;\mathbb{Z}\pi)$, we have by definition

$$\lambda_M(x,y)=\langle PD^{-1}(y),x\rangle,$$

where $PD\colon H^2(M;\mathbb{Z}\pi)\to H_2(M;\mathbb{Z}\pi)$ is the Poincaré duality isomorphism given by cap product with the fundamental class $[M]\in H_4(M;\mathbb{Z})$. Here, and indeed throughout the article, we use the involution in the definition of cohomology $H^n(M;\mathbb{Z};\pi):=H^n(\operatorname{Hom}_{\mathbb{Z}\pi}(C_*(M;\mathbb{Z}\pi)^t,\mathbb{Z}\pi))$ to consider chains as a right $\mathbb{Z}\pi$-module $C_*(M;\mathbb{Z}\pi)^t$, so that cohomology still carries a left $\mathbb{Z}\pi$-module structure.

We identify the equivariant intersection form on M with $\mathbb{Z}\pi$ coefficients with the intersection form on $\widetilde{M}$. Pick a lift of each cell of M to obtain an identification $\theta\colon C_*(M;\mathbb{Z}\pi)\xrightarrow{\cong}C_*(\widetilde{M};\mathbb{Z})$. We also have an isomorphism

$$\begin{array}{rcl}\Psi\colon C^*_{cs}(\widetilde{M};\mathbb{Z}) & \xrightarrow{\cong} & \operatorname{Hom}_{\mathbb{Z}\pi}(C_*(\widetilde{M};\mathbb{Z}),\mathbb{Z}\pi)\\ f & \mapsto & \big(a\mapsto\sum_{g\in\pi}f(g^{-1}a)\cdot g\big)\end{array}$$

for $a\in C_*(\widetilde{M};\mathbb{Z})$. The inverse is given by sending $\varphi\in\operatorname{Hom}_{\mathbb{Z}\pi}(C_*(\widetilde{M};\mathbb{Z}),\mathbb{Z}\pi)$ to the homomorphism that maps $x\in C_*(\widetilde{M};\mathbb{Z})$ to the coefficient of the neutral group element of $\varphi(x)\in\mathbb{Z}\pi$. These two maps induce an isomorphism $\theta^*\circ\Psi_*\colon H^*_{cs}(\widetilde{M};\mathbb{Z})\xrightarrow{\cong}H^*(M;\mathbb{Z}\pi)$, which fits into the following commuting diagram.

$$\begin{tikzcd}
H_2(M;\mathbb{Z}\pi) \arrow[r, "PD^{-1}", "\cong"'] \arrow[d, "\theta"', "\cong"] & H^2(M;\mathbb{Z}\pi) \arrow[r, "\mathrm{ev}"] & \operatorname{Hom}_{\mathbb{Z}\pi}(H_2(M;\mathbb{Z}\pi),\mathbb{Z}\pi) \\
H_2(\widetilde{M};\mathbb{Z}) \arrow[r, "PD^{-1}", "\cong"'] & H^2_{cs}(\widetilde{M};\mathbb{Z}) \arrow[r, "\mathrm{ev}\circ\Psi"] \arrow[u, "\theta^*\circ\Psi"', "\cong"] & \operatorname{Hom}_{\mathbb{Z}\pi}(H_2(\widetilde{M};\mathbb{Z}),\mathbb{Z}\pi) \arrow[u, "\theta^*"', "\cong"]
\end{tikzcd}$$

In the diagram we also write $PD\colon H^2_{cs}(\widetilde{M};\mathbb{Z})\to H_2(\widetilde{M};\mathbb{Z})$ for Poincaré duality in $\widetilde{M}$.

Thus, by taking the two routes from $H_2(M;\mathbb{Z}\pi)$ to $\operatorname{Hom}_{\mathbb{Z}\pi}(H_2(M;\mathbb{Z}\pi),\mathbb{Z}\pi)$, we see that for $x,y\in H_2(M;\mathbb{Z}\pi)$, we have

$$\lambda_M(x,y)=\langle PD^{-1}(y),x\rangle=\sum_{g\in\pi}\langle PD^{-1}(\theta(y)),g^{-1}\theta(x)\rangle\cdot g.$$

Now we will consider $w = \theta(x)$ and $u = \theta(y)$ in $H_2(\widetilde{M};\mathbb{Z})$. Note that $H_0(\widetilde{M};\mathbb{Z}) \cong H^4_{cs}(\widetilde{M};\mathbb{Z}) \cong \mathbb{Z}$. In the final expression in the equation above, we write $PD\colon H^2_{cs}(\widetilde{M};\mathbb{Z}) \to H_2(\widetilde{M};\mathbb{Z})$ for Poincaré duality. Then since the cup product is signed commutative, we have, for $u, w \in H_2(\widetilde{M};\mathbb{Z})$:

$$\begin{aligned}\langle PD^{-1}(u), g^{-1}w\rangle &= PD^{-1}(u) \cap g^{-1}w \\ &= PD^{-1}(u) \cup (g^{-1}PD^{-1}(w) \cap [M]) \\ &= (gPD^{-1}(u) \cup PD^{-1}(w)) \cap [M] \\ &= (PD^{-1}(w) \cup gPD^{-1}(u)) \cap [M].\end{aligned}$$

We record the outcome in the following proposition.

Proposition 4.1. *The map $\theta_*\colon H_2(M;\mathbb{Z}\pi) \to H_2(\widetilde{M};\mathbb{Z})$ induces an isometry of the equivariant intersection form $\lambda_M(x,y) = \langle PD^{-1}(y), x\rangle$ with the form*

$$\lambda_{\widetilde{M}}(\theta_*(x), \theta_*(y)) = \lambda_{\widetilde{M}}(w,u) = \sum_{g\in\pi} \big((PD^{-1}(w) \cup gPD^{-1}(u)) \cap [M]\big)g \in \mathbb{Z}\pi.$$

It is well-known that $(PD^{-1}(x) \cup PD^{-1}(y)) \cap [M] \in H_0(\widetilde{M};\mathbb{Z}) \cong \mathbb{Z}$ agrees with the geometric intersection of S_x and S_y, where S_x and S_y are transverse, generically immersed surfaces in $\widetilde{M}$ representing x and y respectively.

Here each intersection point p is counted with a sign depending on whether the orientation of $T_pS_x \oplus T_pS_y$ agrees with T_pM or not.

Let x_0 be the chosen base point of M and let $\widetilde{x}_0$ be a chosen lift of x_0 in $\widetilde{M}$. Given two transverse, generically immersed spheres $\alpha, \beta \in \pi_2(M, x_0)$, any intersection point p determines an element of π by choosing a path in α from x_0 to p and concatenating it with a path from p to x_0 in β. This element is g precisely if the lifts $\widetilde{\alpha} \in \pi_2(\widetilde{M}, \widetilde{x}_0)$ and $g\widetilde{\beta} \in \pi_2(\widetilde{M}, gx_0)$ intersect at a lift $\widetilde{p}$ of p. Hence we have the following statement, which we will use later in our discussion of the tertiary invariant.

Proposition 4.2. *The above geometric count of intersections in $\mathbb{Z}\pi$ computes the equivariant intersection form.*

Using the description

$$\lambda_M(x,y) = \sum_{g\in\pi}((PD^{-1}(x) \cup gPD^{-1}(y)) \cap [M])g,$$

it is easy to see that the form λ_M is sesquilinear and hermitian. But it is often not nonsingular or even nondegenerate, as can be computed by the universal coefficient spectral sequence [Lev77, Theorem 2.3].

Proposition 4.3. *There is an exact sequence*

$$0 \to H^2(\pi;\mathbb{Z}\pi) \to H_2(M;\mathbb{Z}\pi) \to \operatorname{Hom}_{\mathbb{Z}\pi}(H_2(M;\mathbb{Z}\pi), \mathbb{Z}\pi) \to H^3(\pi;\mathbb{Z}\pi) \to 0$$

where the middle map is the adjoint of the intersection form.

Proof. Recall that the universal coefficient spectral sequence has E^2 page $E_2^{p,q} \cong \operatorname{Ext}^q_{\mathbb{Z}\pi}(H_p(M;\mathbb{Z}\pi), \mathbb{Z}\pi)$, differential d^r of degree $(1-r, r)$, and the sequence converges to $H^{p+q}(M;\mathbb{Z}\pi)$. Since $H_1(M;\mathbb{Z}\pi) = 0$, the spectral sequence yields a filtration $0 \subseteq F_{0,2} \subseteq H^2(M;\mathbb{Z}\pi)$ with $F_{0,2} \cong \operatorname{Ext}^2_{\mathbb{Z}\pi}(H_0(M;\mathbb{Z}\pi), \mathbb{Z}\pi)$ and

$$H^2(M;\mathbb{Z}\pi)/F_{0,2} \cong \ker\big(\operatorname{Hom}_{\mathbb{Z}\pi}(H_2(M;\mathbb{Z}\pi), \mathbb{Z}\pi) \to \operatorname{Ext}^3_{\mathbb{Z}\pi}(H_0(M;\mathbb{Z}\pi), \mathbb{Z}\pi)\big).$$

Note that $\operatorname{Ext}^i_{\mathbb{Z}\pi}(H_0(M;\mathbb{Z}\pi), \mathbb{Z}\pi) = H^i(\pi;\mathbb{Z}\pi)$ to obtain the exact sequence claimed. ∎

4.2. **Sesquilinear forms and Tate cohomology.** In this section, and throughout the rest of the article, when not specified every $\mathbb{Z}\pi$-module is assumed by default to be a left $\mathbb{Z}\pi$-module.

Let N be a left $\mathbb{Z}\pi$-module. We write N^* for $\mathrm{Hom}_{\mathbb{Z}\pi}(N, \mathbb{Z}\pi)$. This would a priori be a right $\mathbb{Z}\pi$-module, where $\mathbb{Z}\pi$ acts by right multiplication on the target, but we turn it into a left module using the involution on $\mathbb{Z}\pi$.

Below, and indeed also throughout the article, when we consider the tensor product $N \otimes_{\mathbb{Z}\pi} N'$ of two left $\mathbb{Z}\pi$-modules N, N', we use the involution to turn N into a right $\mathbb{Z}\pi$-module, so that the tensor product makes sense.

We denote the group of sesquilinear forms on N by $\mathrm{Sesq}(N)$, that is the group of maps

$$\lambda\colon N \otimes N \to \mathbb{Z}\pi$$

with $\lambda(am, bn) = a\lambda(m,n)\bar{b}$ for $a, b \in \mathbb{Z}\pi$ and $m, n \in N$. The group operation is the obvious addition, defined by $(\lambda + \lambda')(m,n) = \lambda(m,n) + \lambda'(m,n)$. Equivalently $\mathrm{Sesq}(N) \cong \mathrm{Hom}_{\mathbb{Z}\pi}(N, N^*)$ by $\lambda(n)(m) := \lambda(m,n)$. This is a contravariant functor, where a map $f\colon N \to N'$ is sent to the map $f^*\colon \mathrm{Sesq}(N') \to \mathrm{Sesq}(N)$ with $(f^*\lambda)(m,m') = \lambda(f(m), f(m'))$. The group $\mathbb{Z}/2$ acts on $\mathrm{Sesq}(N)$ via $(T\lambda)(m,n) = \overline{\lambda(n,m)}$. Thus we can form the *Tate cohomology group*

$$\widehat{H}^0(\mathrm{Sesq}(N)) := \widehat{H}^0(\mathbb{Z}/2; \mathrm{Sesq}(N)) = \ker(1-T)/\operatorname{im}(1+T).$$

Note that $\ker(1-T)$ is precisely the hermitian forms, that is those for which $\lambda(m,n) = \overline{\lambda(n,m)}$ for all $m, n \in N$. In addition, every hermitian form has order two in the Tate cohomology, since $\lambda = T\lambda$ implies that $2\lambda = \lambda + T\lambda = (1+T)\lambda$.

Definition 4.4. A hermitian form is called *even* if it is in the image of $1 + T\colon \mathrm{Sesq}(N) \to \mathrm{Sesq}(N)$, that is if it vanishes in $\widehat{H}^0(\mathrm{Sesq}(N))$.

We therefore obtain a contravariant functor $\widehat{H}^0(\mathrm{Sesq}(-))$ from $\mathbb{Z}\pi$-modules to $\mathbb{Z}/2$-modules. Let $(-)^* \otimes_{\mathbb{Z}\pi} (-)^*$ be the contravariant functor sending N to $N^* \otimes_{\mathbb{Z}\pi} N^*$, where we consider the first N^* as a right $\mathbb{Z}\pi$-module as described above; equivalently, we do not use the involution on this N^*.

Definition 4.5. For left $\mathbb{Z}\pi$-modules N and N', define

$$\begin{array}{rcl} \Phi_{N,N'}\colon N^* \otimes_{\mathbb{Z}\pi} N' & \to & \mathrm{Hom}_{\mathbb{Z}\pi}(N, N') \\ f \otimes n' & \mapsto & \big(n \mapsto f(n)n'\big). \end{array}$$

We also define

$$\begin{array}{rcl} \Phi_N := \Phi_{N,N^*}\colon N^* \otimes_{\mathbb{Z}\pi} N^* & \to & \mathrm{Hom}_{\mathbb{Z}\pi}(N, N^*) \cong \mathrm{Sesq}(N). \\ f_1 \otimes f_2 & \mapsto & \big((n_1, n_2) \mapsto f_1(n_1)\overline{f_2(n_2)}\big). \end{array}$$

The next lemma follows directly from the definition.

Lemma 4.6. *For two left $\mathbb{Z}\pi$-modules N and N', the map $\Phi_{N \oplus N'}$ is equivalent to the direct sum $\Phi_N \oplus \Phi_{N'} \oplus \Phi_{N,(N')^*} \oplus \Phi_{N',N^*}$, under the obvious isomorphisms of the domains and the codomains.*

Lemma 4.7. *The map*

$$\Phi_N\colon N^* \otimes_{\mathbb{Z}\pi} N^* \to \mathrm{Sesq}(N)$$

defines a natural transformation

$$\Phi\colon (-)^* \otimes_{\mathbb{Z}\pi} (-)^* \Rightarrow \mathrm{Sesq}(-).$$

The proof of Lemma 4.7 is straightforward and we omit the details.

Lemma 4.8. *The map*

$$\begin{array}{rcl}\Theta_N\colon \mathbb{Z}/2\otimes_{\mathbb{Z}\pi} N^* & \to & \widehat{H}^0(N^*\otimes_{\mathbb{Z}\pi} N^*)\\ 1\otimes f & \mapsto & [f\otimes f]\end{array}$$

determines a natural isomorphism

$$\Theta\colon \mathbb{Z}/2\otimes_{\mathbb{Z}\pi}(-)^*\Rightarrow \widehat{H}^0((-)^*\otimes_{\mathbb{Z}\pi}(-)^*).$$

Proof. First we show that Θ_N is well-defined. For all $f,f'\in N^*$ and for all $g\in\pi$, we have

$$\Theta_N(1\otimes gf)=[gf\otimes gf]=[fg^{-1}\otimes gf]=[f\otimes f]=\Theta_N(1\otimes f)$$

and

$$\begin{aligned}\Theta_N(1\otimes(f+f'))&=[(f+f')\otimes(f+f')]\\&=[f\otimes f+f'\otimes f'+(1+T)(f\otimes f')]\\&=[f\otimes f]+[f'\otimes f']=\Theta_N(1\otimes f)+\Theta_N(1\otimes f')\end{aligned}$$

This shows that Θ_N is well-defined as desired.

It is easy to see that Θ is natural, so it remains to prove that it is an isomorphism. For this we will define an inverse. Let $N^*/2:=\mathbb{Z}/2\otimes_{\mathbb{Z}\pi}N^*$. The projection $N^*\to N^*/2$ defines a map $N^*\otimes_{\mathbb{Z}\pi}N^*\to N^*/2\otimes_{\mathbb{Z}/2}N^*/2$, and then applying $\widehat{H}^0$ we obtain a map

$$\widehat{H}^0(N^*\otimes_{\mathbb{Z}\pi}N^*)\to\widehat{H}^0(N^*/2\otimes_{\mathbb{Z}/2}N^*/2).$$

Since $N^*/2$ is a $\mathbb{Z}/2$-vector space, we have an isomorphism $\widehat{H}^0(N^*/2\otimes_{\mathbb{Z}/2}N^*/2)\cong N^*/2$ given by sending $[[f]\otimes[f]]\to[f]$. To see that this map makes sense and is an isomorphism, observe that symmetric modulo even forms are determined by the diagonal entries. So every element in $\widehat{H}^0(N^*/2\otimes_{\mathbb{Z}/2}N^*/2)$ is a sum of elements of the form $[[f]\otimes[f]]$. The composition

$$\widehat{H}^0(N^*\otimes_{\mathbb{Z}\pi}N^*)\to\widehat{H}^0(N^*/2\otimes_{\mathbb{Z}/2}N^*/2)\xrightarrow{\cong}N^*/2$$

sends $[f\otimes f]\in\widehat{H}^0(N^*\otimes_{\mathbb{Z}\pi}N^*)$ to $[f]=1\otimes f\in N^*/2=\mathbb{Z}/2\otimes N^*$, and is therefore the desired inverse to Θ_N. □

Definition 4.9. Define the natural transformation

$$\Psi:=\widehat{H}^0(\Phi)\circ\Theta\colon\mathbb{Z}/2\otimes_{\mathbb{Z}\pi}(-)^*\Rightarrow\widehat{H}^0(\mathrm{Sesq}(-)).$$

Lemma 4.10. *Let P,N be left $\mathbb{Z}\pi$-modules, and suppose in addition that P is finitely generated projective. Then:*

(i) $\Phi_{P,N}$ *and* $\Phi_{N,P}$ *are isomorphisms.*
(ii) $\Psi_P\colon\mathbb{Z}/2\otimes_{\mathbb{Z}\pi}P^*\to\widehat{H}^0(\mathrm{Sesq}(P))$ *is an isomorphism.*
(iii) Φ_P *is an isomorphism.*

Proof.

(i) First suppose that $P\cong\mathbb{Z}\pi$. In this case $\Phi_{\mathbb{Z}\pi,N}$ and $\Phi_{N,\mathbb{Z}\pi}$ are obviously isomorphisms. Since $\Phi_{N,\mathbb{Z}\pi^n}$ is the direct sum of n times $\Phi_{N,\mathbb{Z}\pi}$, it is an isomorphism. The same holds for $\Phi_{\mathbb{Z}\pi^n,N}$.

Now let P be finitely generated projective and let Q be such that $P\oplus Q$ is finitely generated free. Then the isomorphism $\Phi_{N,P\oplus Q}$ is the direct sum of $\Phi_{N,P}$ and $\Phi_{N,Q}$. Thus both these maps have to be isomorphisms as well. By the same argument $\Phi_{P,N}$ is an isomorphism.

(ii) Since Φ_P is an isomorphism by part (i), so is $\widehat{H}^0(\Phi_P)$. Furthermore, Θ is a natural isomorphism by Lemma 4.8, and thus also the composition $\Psi_P=\widehat{H}^0(\Phi_P)\circ\Theta_P$ is an isomorphism.

(iii) Immediate from (i). □

Corollary 4.11. *Let N be a left $\mathbb{Z}\pi$-module. Then the inclusion $N \to N \oplus \mathbb{Z}\pi$ induces isomorphisms*
$$\ker \Phi_N^{\mathbb{Z}/2} \xrightarrow{\cong} \ker \Phi_{N\oplus\mathbb{Z}\pi}^{\mathbb{Z}/2} \text{ and } \ker \widehat{H}^0(\Phi_N) \xrightarrow{\cong} \ker \widehat{H}^0(\Phi_{N\oplus\mathbb{Z}\pi}),$$
where $\Phi_N^{\mathbb{Z}/2}$ is the map induced on $\mathbb{Z}/2$-fixed points by Φ_N.

This corollary will be used in the proof of Lemma 4.16.

Proof. We have a commutative diagram
$$\begin{tikzcd}
\big((N^* \oplus \mathbb{Z}\pi) \otimes_{\mathbb{Z}\pi} (N^* \oplus \mathbb{Z}\pi)\big)^{\mathbb{Z}/2} \arrow[r, "\cong"] \arrow[d, "\Phi_{N\oplus\mathbb{Z}\pi}"'] & (N^* \otimes_{\mathbb{Z}\pi} N^*)^{\mathbb{Z}/2} \oplus (\mathbb{Z}\pi \otimes_{\mathbb{Z}\pi} N^*) \oplus (\mathbb{Z}\pi \otimes_{\mathbb{Z}\pi} \mathbb{Z}\pi)^{\mathbb{Z}/2} \arrow[d] \\
\mathrm{Sesq}(N \oplus \mathbb{Z}\pi)^{\mathbb{Z}/2} \arrow[r, "\cong"] & \mathrm{Sesq}(N)^{\mathbb{Z}/2} \oplus \mathrm{Hom}_{\mathbb{Z}\pi}(\mathbb{Z}\pi, N^*) \oplus \mathrm{Sesq}(\mathbb{Z}\pi)^{\mathbb{Z}/2}.
\end{tikzcd}$$
In the two terms in the right hand column, the middle summands are isomorphic, as are the third summands by Lemma 4.10 (i). The map between the first summands is $\Phi_N^{\mathbb{Z}/2}$. Since the horizontal maps are isomorphisms, the first part of the lemma follows from the commutativity of the diagram.

The diagram descends to a similar diagram on $\widehat{H}^0$ groups. By Lemma 4.10 (ii) and Lemma 4.8, the analogous statements hold, namely the maps induced by the right hand vertical map in the diagram (with $\mathbb{Z}/2$ fixed points replaced by $\widehat{H}^0$) splits along the direct summands, and the maps on the second and third summands are isomorphisms. The second part of the lemma then also follows from commutativity. □

4.3. **Relating $H_3(\pi;\mathbb{Z}/2)$ with symmetric modulo even forms.** Let K be a connected finite 2-complex with $\pi_1(K) \cong \pi$. Let (D_*, d_*) be the cellular $\mathbb{Z}\pi$-module chain complex of K. In this subsection we establish a map $H_3(\pi;\mathbb{Z}/2) \to \widehat{H}^0(\mathrm{Sesq}(H^2(K;\mathbb{Z}\pi)))$ and we use this map to formulate Condition 4.17. This condition will then later be shown to imply the Secondary Property 1.8.

Given $x \in \ker d_2 \cong \pi_2(K)$, we define $f_x \colon \mathbb{Z}\pi \to \ker d_2$ by $f_x(a) = ax$. By dualizing we obtain a map $f_x^* \colon H^2(K, \mathbb{Z}\pi) \to \mathbb{Z}\pi^* \cong \mathbb{Z}\pi$, where the map $\mathbb{Z}\pi^* \to \mathbb{Z}\pi$ is given by $f \mapsto \overline{f(1)}$, which is a left $\mathbb{Z}\pi$-module homomorphism. Hence f_x^* is a left $\mathbb{Z}\pi$-module homomorphism. Note that for $\varphi \in H^2(K;\mathbb{Z}\pi)$ we have $f_x^*(\varphi) = \overline{\varphi(x)}$.

Lemma 4.12. *Let K be a connected finite 2-complex with $\pi_1(K) \cong \pi$. The canonical map*
$$\begin{array}{rcl} I\colon \pi_2(K) & \to & \mathrm{Hom}_{\mathbb{Z}\pi}(H^2(K;\mathbb{Z}\pi), \mathbb{Z}\pi) \\ x & \mapsto & f_x^* \end{array}$$
is an isomorphism.

Proof. To see that I is a left $\mathbb{Z}\pi$-module homomorphism, let $g \in \pi$. We compute that
$$I(gx)(\varphi) = f_{gx}^*(\varphi) = \overline{\varphi(gx)} = \overline{g \cdot \varphi(x)} = \overline{\varphi(x)} \cdot \overline{g} = f_x^*(\varphi) \cdot \overline{g} = (g \cdot f_x^*)(\varphi) = gI(x)(\varphi).$$
Now let (D_*, d_*) denote the cellular $\mathbb{Z}\pi$-module chain complex of K. Consider the exact sequence
$$D^1 \xrightarrow{d^2} D^2 \to H^2(K;\mathbb{Z}\pi) \to 0.$$
Since the functor $\mathrm{Hom}_{\mathbb{Z}\pi}(-, \mathbb{Z}\pi)$ is left exact, we obtain an exact sequence
$$0 \to \mathrm{Hom}_{\mathbb{Z}\pi}(H^2(K;\mathbb{Z}\pi), \mathbb{Z}\pi) \to \mathrm{Hom}_{\mathbb{Z}\pi}(D^2, \mathbb{Z}\pi) \xrightarrow{(d^2)^*} \mathrm{Hom}_{\mathbb{Z}\pi}(D^1, \mathbb{Z}\pi)$$
and since D_1 and D_2 are finitely generated, free $\mathbb{Z}\pi$-modules, this is isomorphic to the sequence
$$0 \to \mathrm{Hom}_{\mathbb{Z}\pi}(H^2(K;\mathbb{Z}\pi), \mathbb{Z}\pi) \to D_2 \xrightarrow{d_2} D_1.$$

Hence $\operatorname{Hom}_{\mathbb{Z}\pi}(H^2(K;\mathbb{Z}\pi),\mathbb{Z}\pi)$ is isomorphic to $\ker d_2 = H_2(K;\mathbb{Z}\pi) \cong \pi_2(K)$. The isomorphism $\operatorname{Hom}(D^2,\mathbb{Z}\pi) \to D_2$ sends f_x^*, considered as a function $D^2 \to \mathbb{Z}\pi$, to $f(1) = x$, considered as an element in D_2. □

As above, consider a CW-complex model for $B\pi$ with finite 2-skeleton K.

Definition 4.13. Consider the Leray-Serre spectral sequence computing $H_*(K;\mathbb{Z}/2)$ from the fibration $\widetilde{K} \to K \to B\pi$ with E^2-term $E^2_{p,q} := H_p(\pi; H_q(\widetilde{K};\mathbb{Z}/2))$. Identify $H_2(\widetilde{K};\mathbb{Z}/2)$ with $\mathbb{Z}/2 \otimes_{\mathbb{Z}} \pi_2(K)$. Then $H_0(\pi; H_2(\widetilde{K};\mathbb{Z}/2)) \cong \mathbb{Z}/2 \otimes_{\mathbb{Z}\pi} \pi_2(K)$. Since $H_1(\widetilde{K};\mathbb{Z}/2) = 0$, the entire $q = 1$ row of the E^2 page vanishes, so the d_3 differential gives a map:

$$\iota := d_3\colon H_3(\pi;\mathbb{Z}/2) \to \mathbb{Z}/2 \otimes_{\mathbb{Z}\pi} \pi_2(K).$$

Lemma 4.14. *The map $\iota\colon H_3(\pi;\mathbb{Z}/2) \to \mathbb{Z}/2 \otimes_{\mathbb{Z}\pi} \pi_2(K)$ is injective.*

Proof. In the spectral sequence, $\ker \iota \cong E^\infty_{3,0}$. Since $E^\infty_{3,0}$ is isomorphic to the quotient of $H_3(K;\mathbb{Z}/2)$ by largest subgroup in the filtration of it determined by the Leray-Serre spectral sequence, we obtain a surjection $H_3(K;\mathbb{Z}/2) \to \ker \iota$. Since K is 2-dimensional, we have $H_3(K;\mathbb{Z}/2) = 0$ and thus ι is injective. □

Definition 4.15. Let $H := H^2(K;\mathbb{Z}\pi)$. Define

$$A\colon H_3(\pi;\mathbb{Z}/2) \to \widehat{H}^0(\operatorname{Sesq}(H))$$

to be the composition $\Psi_H \circ (\operatorname{Id} \times I) \circ \iota$.

Lemma 4.16. *For any other choice K' as the 2-skeleton of $B\pi$ and every map $\varphi\colon K \to K'$ that induces the identity on fundamental groups, the diagram*

$$\begin{tikzcd} H_3(\pi;\mathbb{Z}/2) \arrow[r, "A"] \arrow[dr, "A'"'] & \widehat{H}^0(\operatorname{Sesq}(H)) \arrow[d, "\varphi_*"] \\ & \widehat{H}^0(\operatorname{Sesq}(H')) \end{tikzcd}$$

commutes, where A' and H' are the analogous definitions of A and H for K' instead of K. In particular, the kernel of A does not depend on the choice of finite 2-complex K with $\pi_1(K) = \pi$, but only on π.

Proof. Consider the following diagram where ι' denotes the map ι for K' instead of K.

$$\begin{tikzcd} H_3(\pi;\mathbb{Z}/2) \arrow[r, hook, "\iota"] \arrow[dr, hook, "\iota'"] & \mathbb{Z}/2 \otimes_{\mathbb{Z}\pi} \pi_2(K) \arrow[r, "\Psi_H"] \arrow[d] & \widehat{H}^0(\operatorname{Sesq}(H^2(K;\mathbb{Z}\pi))) \arrow[d] \\ & \mathbb{Z}/2 \otimes_{\mathbb{Z}\pi} \pi_2(K') \arrow[r, "\Psi_{H'}"] & \widehat{H}^0(\operatorname{Sesq}(H^2(K';\mathbb{Z}\pi))) \end{tikzcd}$$

Here we use the isomorphism $I\colon \pi_2(K) \cong H^*$ from Lemma 4.12 in order to consider Ψ_H as a map from $\mathbb{Z}/2 \otimes \pi_2(K)$ to $\widehat{H}^0(\operatorname{Sesq}(H))$, and similarly for $H' := H^2(K';\mathbb{Z}\pi)$. The triangle commutes by naturality of the spectral sequence defining ι, and the square commutes by naturality of Ψ.

It follows that the kernel of A is contained in the kernel of A'. By switching the rôles of K and K', the kernel of A is independent of the choice of K. □

In light of the previous lemma, $\operatorname{Tate}(\pi) := H_3(\pi;\mathbb{Z}/2)/\ker(A)$ is independent of the choice of K and we can formulate the following condition.

Condition 4.17. The sequence

$$H_5(\pi;\mathbb{Z}) \xrightarrow{\mathrm{Sq}_2\circ\mathrm{red}_2} H_3(\pi;\mathbb{Z}/2) \xrightarrow{A_\pi} \mathrm{Tate}(\pi)$$

is exact at $H_3(\pi;\mathbb{Z}/2)$, where A_π is the projection.

Remark 4.18. By definition, Condition 4.17 is equivalent to the exactness of the sequence

$$H_5(\pi;\mathbb{Z}) \xrightarrow{\mathrm{Sq}_2\circ\mathrm{red}_2} H_3(\pi;\mathbb{Z}/2) \xrightarrow{A} \widehat{H}^0(\mathrm{Sesq}(H))$$

for any choice of 2-complex K and $H := H^2(K;\mathbb{Z}\pi)$ as in Definition 4.15.

By the end of this section we will have proven the following theorem.

Theorem 4.19. *If Condition 4.17 holds for π, then π has the Secondary Property 1.8.*

Thus we will have reduced the Secondary Property, for a given group π, to verifying Condition 4.17 for that group. The classes of group for which we have been able to show that Condition 4.17 holds are described in Sections 9 and 10. In [KLPT17], we proved the Secondary Property by constructing model manifolds and computing their intersection forms and their secondary invariant in $H_3(\pi;\mathbb{Z}/2)$. Here we cannot construct model manifolds, but as long as we can compute $H_3(\pi;\mathbb{Z}/2)$ and the map $\mathrm{Sq}_2\circ\mathrm{red}_2$, then by 4.19 we can test the Secondary Property on "model" intersection forms produced by the map A.

The following lemma will not be used in this section, but it may become useful in the future because, unlike the Tate group $\widehat{H}^0(\mathrm{Sesq}(H))$, the subgroup of *weakly even* hermitian forms modulo even forms on a $\mathbb{Z}\pi$-module H is unchanged under adding projective modules $H \mapsto H\oplus P$.

Recall the exact sequence from the Leray-Serre spectral sequence for $\widetilde{K}\to K\to B\pi$

$$0\longrightarrow H_3(\pi;\mathbb{Z}/2)\xrightarrow{\iota}\mathbb{Z}/2\otimes_{\mathbb{Z}\pi}H_2(K;\mathbb{Z}\pi)\xrightarrow{H_2(\varepsilon_2)}H_2(K;\mathbb{Z}/2),$$

where $\varepsilon_2\colon\mathbb{Z}\pi\to\mathbb{Z}/2$ is the nontrivial ring homomorphism, inducing the map on coefficients in the final map. Also recall that $H := H^2(K;\mathbb{Z}\pi)$ has dual module $H^*\cong H_2(K;\mathbb{Z}\pi)\cong\pi_2(K)$ and that we defined a map $\Psi_H\colon\mathbb{Z}/2\otimes_{\mathbb{Z}\pi}H^*\to\widehat{H}^0(\mathrm{Sesq}(H))$ via

$$\Psi_H(1\otimes f)(h_1,h_2)=f(h_1)\cdot\overline{f(h_2)}\text{ for }f\in H^*,\ h_i\in H.$$

Lemma 4.20. *For every $f\in H^*$, the hermitian form $\Psi_H(1\otimes f)$ is weakly even if and only if $1\otimes f$ is in the image of ι, or equivalently, if and only if $H_2(\varepsilon_2)(1\otimes f)=0$.*

Proof. For $h\in H$, write $f(h)=\sum_{i=1}^n m_ig_i$ and compute in $\mathbb{Z}\pi$

$$f(h)\cdot\overline{f(h)}=\sum_{i,j=1}^n m_im_j(g_i\bar g_j)=\sum_{i=1}^n(m_i)^2+\sum_{i<j}m_im_j(g_i\bar g_j+g_j\bar g_i).$$

The second sum is of the form $a+\bar a$ and the first sum is just a multiple of the unit. That multiple $\sum_{i=1}^n(m_i)^2$ is modulo 2 the same as $\sum_{i=1}^n m_i\equiv\varepsilon_2(f(h))$, so we conclude that the above element is of the form $a+\overline{a}$ if and only if $\varepsilon_2(f(h))=0$.

The property of $\Psi_H(1\otimes f)$ being weakly even means that $\Psi_H(1\otimes f)(h,h)$ lies in the subgroup $\{a+\overline{a}\mid a\in\mathbb{Z}\pi\}$ for all $h\in H$. By the above, this is equivalent to the vanishing of the composition $\varepsilon_2\circ f\colon H\to\mathbb{Z}\pi\to\mathbb{Z}/2$. Finally, we use that $H^2(\varepsilon_2)\colon H^2(K;\mathbb{Z}\pi)\to H^2(K;\mathbb{Z}/2)$ is surjective because the next term in the Bockstein sequence is $H^3(K;\ker(\varepsilon_2))=0$. Hence the vanishing of $\varepsilon_2\circ f$ is equivalent to the vanishing of $H_2(\varepsilon_2)(f)=H_2(\varepsilon_2)(1\otimes f)$, which is our original claim. □

4.4. **Proof of Theorem 4.19.** Before we start to prove Theorem 4.19, we switch to a dual picture, that we find convenient for the statements and the proofs of some imminent lemmas. By [KPT18, Proposition 1.7], the short exact sequence

$$0 \to \ker d_2 \to C_2 \oplus H_2(M;\mathbb{Z}\pi) \to \operatorname{coker} d_3 \to 0.$$

splits if and only if the primary obstruction vanishes. Recall that we write $C_* := C_*(M;\mathbb{Z}\pi)$ for the handle chain complex coming from a choice of handle decomposition of M, with boundary maps d_i.

In the discussion of the secondary obstruction we work with a slightly different "dual" formulation. For this, let (C'_*, d'_*) denote the $\mathbb{Z}\pi$-module chain complex coming from a dual handle decomposition of M. We denote M endowed with the dual handle decomposition by M^d. The above sequence for M^d has the form

$$0 \to \ker d'_2 \to C'_2 \oplus H_2(M^d;\mathbb{Z}\pi) \to \operatorname{coker} d'_3 \to 0.$$

We can identify the chain complex of the dual handle decomposition with the dual of the chain complex of the first handle decomposition, i.e. $C'_* \cong C^{4-*}$. Apply this to the above sequence to yield

$$0 \to \ker d^3 \to C^2 \oplus H^2(M;\mathbb{Z}\pi) \to \operatorname{coker} d^2 \to 0.$$

Since the primary invariant is independent of the chosen handle decomposition of M, the last sequence splits if and only if $\mathfrak{pri}(M) = 0$. Note that $\operatorname{coker} d^2 \cong H^2(M^{(2)};\mathbb{Z}\pi)$.

Next, we show that every splitting of this dual short exact sequence can be realised by a geometric map $M^{(3)} \to M^{(2)}$.

Proposition 4.21. *For every splitting*

$$s = (s_1, s_2)\colon H^2(M^{(2)};\mathbb{Z}\pi) \to C^2 \oplus H^2(M;\mathbb{Z}\pi),$$

there is a map $f\colon M^{(3)} \to M^{(2)}$ *such that*

$$f^* = s_2\colon H^2(M^{(2)};\mathbb{Z}\pi) \to H^2(M^{(3)};\mathbb{Z}\pi) \cong H^2(M;\mathbb{Z}\pi).$$

Proof. Let $j = (j_1, j_2)\colon C^2 \oplus H^2(M;\mathbb{Z}\pi) \to H^2(M^{(2)};\mathbb{Z}\pi)$ be the map from the short exact sequence above, so j is split by $s = (s_1, s_2)$. That is, j_1 is the projection $C^2 \to C^2/\operatorname{im} d^2$, and $j_2\colon \ker d^3/\operatorname{im} d^2 \hookrightarrow C^2/\operatorname{im} d^2$ is the inclusion.

Since s is a splitting, we have

$$j_1(c) = j_1 s_1 j_1(c) + j_2 s_2 j_1(c), \text{ so that } j_1(c - s_1 j_1(c)) = j_2 s_2 j_1(c).$$

Thus $j_1(c - s_1 j_1(c)) \in \operatorname{im} j_2$, and so in particular $c - s_1 j_1(c) \in \ker d^3$.

Define a map $\rho\colon C^2 \to \ker d^3 \subset C^2$ by $c \mapsto c - s_1 j_1(c)$. Furthermore, for all $e \in C^1$,

$$(\rho \circ d^2)(e) = d^2(e) - s_1 j_1 d^2(e) = d^2(e) - s_1(0) = d^2(e).$$

Therefore, we can consider the following commutative diagram, in which we use the dual ρ^* of $\rho\colon C^2 \to C^2$, composed with the identification of C_i with its double dual.

$$\begin{tikzcd}
 & C_2 \arrow[r, "d_2"] & C_1 \arrow[r, "d_1"] & C_0 \\
C_3 \arrow[ur, "0"] \arrow[r, "d_3"'] & C_2 \arrow[u, "\rho^*"'] \arrow[r, "d_2"'] & C_1 \arrow[u, "\mathrm{Id}"'] \arrow[r, "d_1"'] & C_0 \arrow[u, "\mathrm{Id}"']
\end{tikzcd}$$

We will show that this chain map can be realised as a map $f\colon M^{(3)} \to M^{(2)}$. To see this, start with the identity map $M^{(2)} \to M^{(2)}$. Since $d_2 \circ \rho^* - d_2 = (\rho \circ d^2)^* - d_2 = (d^2)^* - d_2 = 0$, we can change this map on the 2-cells by elements of $\pi_2(M^{(2)})$ according to $\rho^* - \mathrm{Id}$, i.e. pinch off a 2-sphere from each

2-cell of $M^{(2)}$, and map this 2-sphere to the image of the 2-cell under $\rho^* - \mathrm{Id}\colon C_2 \to \ker d_2 = \pi_2(M^{(2)})$. Since $\rho^* \circ d_3 = (d^3 \circ \rho)^* = 0$, the attaching maps of the 3-cells are null homotopic under the new map $M^{(2)} \to M^{(2)}$. Therefore it can be extended to a map $f\colon M^{(3)} \to M^{(2)}$, as desired.

It remains to show that the map f^* induced by f on second cohomology is s_2. Consider the following diagram.

$$\begin{tikzcd}
C^2/\operatorname{im} d^2 \arrow[r, "s_2"] & \ker d^3/\operatorname{im} d^2 \arrow[dr, hook, "j_2"] & \\
C^2 \arrow[u, two heads, "j_1"] \arrow[d, two heads, "j_1"'] \arrow[r, "\rho=\mathrm{Id}-s_1j_1"] & \ker d^3 \subset C^2 \arrow[r, "j_1"] \arrow[d] & C^2/\operatorname{im} d^2 \\
C^2/\operatorname{im} d^2 \arrow[r, "f^*"] & \ker d^3/\operatorname{im} d^2. \arrow[ur, hook, "j_2"'] &
\end{tikzcd}$$

The bottom left square commutes since f was constructed to realise the chain level map ρ^*. The top trapezium commutes by the formula $j_1(c - s_1 j_1(c)) = j_2 s_2 j_1(c)$ from above. It is straightforward to see that the bottom right triangle commutes. Since j_1 is surjective and j_2 injective, it follows from the equality of the top and bottom routes that $f^* = s_2\colon H^2(M^{(2)};\mathbb{Z}\pi) \to H^2(M^{(3)};\mathbb{Z}\pi) \cong H^2(M;\mathbb{Z}\pi)$. □

Let X be a CW complex model for $B\pi$ with finite 2-skeleton K. Let $i\colon K \to X^{(3)}$ denote the inclusion of the 2-skeleton into the 3-skeleton. Let $\alpha \in \pi_3(M^{(3)})$ denote the class of the attaching map of the (unique) 4-cell of M.

Lemma 4.22. *The map $i\colon K \to X^{(3)}$ induces the trivial map $\pi_3(K) \xrightarrow{0} \pi_3(X^{(3)})$. In particular, for every map $f\colon M^{(3)} \to K$ we have $i_* f_*(\alpha) = 0 \in \pi_3(K)$, and therefore the map $i \circ f$ can be extended to a map $\widehat{f}\colon M \to X^{(3)}$.*

Proof. Since the maps in Whitehead's sequence (Theorem 3.6) are natural, we have a commutative diagram:

$$\begin{tikzcd}
0 = H_4(\widetilde{K};\mathbb{Z}) \arrow[r] \arrow[d] & \Gamma(\pi_2(K)) \arrow[r, "\cong"] \arrow[d, "i_*"] & \pi_3(K) \arrow[r] \arrow[d, "i^*"] & H_3(\widetilde{K};\mathbb{Z}) = 0 \arrow[d] \\
0 = H_4(\widetilde{X^{(3)}};\mathbb{Z}) \arrow[r] & \Gamma(\pi_2(X^{(3)})) \arrow[r] & \pi_3(X^{(3)}) \arrow[r] & H_3(\widetilde{X^{(3)}};\mathbb{Z}).
\end{tikzcd}$$

But $\pi_2(X^{(3)}) = 0$, and hence $i_*\colon \pi_3(K) \to \pi_3(X^{(3)})$ is trivial. □

We obtain the following corollary to Proposition 4.21.

Corollary 4.23. *There exists a map $f\colon M^{(3)} \to K$ that induces an isomorphism $\pi_1(M^{(3)}) \to \pi_1(K)$ if and only if $\mathfrak{pri}(M) = 0$.*

Proof. Recall that all 2-complexes with the same fundamental group become homotopy equivalent after wedging with sufficiently many 2-spheres, by Lemma 1.6. Let K and K' be two such 2-complexes. We have a sequence of maps

$$K' \to K' \vee \bigvee^{m} S^2 \xrightarrow{\cong} K \vee \bigvee^{m'} S^2 \to K,$$

where the first map is inclusion, the second map is a homotopy equivalence, and the final map is a collapse map. The composition induces an isomorphism on fundamental groups, so it is enough to show that there is a map $M^{(3)} \to K'$ for one 2-complex K' with $\pi_1(K') \cong \pi$. We will use $K' = M^{(2)}$. If $\mathfrak{pri}(M) = 0$, then the sequence

$$0 \to \ker d^3 \to C^2 \oplus H^2(M;\mathbb{Z}\pi) \to \operatorname{coker} d^2 = H^2(M^{(2)};\mathbb{Z}\pi) \to 0$$

splits and there is a map $f\colon M^{(3)} \to M^{(2)}$ by Proposition 4.21, as required.

On the other hand, if there is a map $f\colon M^{(3)} \to K$, then we can extend f to a map $\widehat{f}\colon M \to X^{(3)}$ by Lemma 4.22. If f is an isomorphism on fundamental groups, then so is $\widehat{f}$. By Proposition 2.7 and the discussion preceding it, this implies $\mathfrak{pri}(M) = 0$. □

From now on in this section we will now only consider manifolds M with $\mathfrak{pri}(M) = 0$. Let $f\colon M^{(3)} \to K$ be a fixed map that is an isomorphism on fundamental groups, which exists by Corollary 4.23. Let $\alpha \in \pi_3(M^{(3)})$ denote the class of the attaching map of the 4-cell of M (we may and will assume that M has a unique 4-cell).

By Lemma 4.22, f can be extended to a map $\widehat{f}\colon M \to X^{(3)}$. The invariant

$$\mathfrak{sec}(M) \in H_3(\pi;\mathbb{Z}/2)/\operatorname{im}(d^2_{5,0})$$

from Proposition 2.7 is independent of the choice of homotopy of the classifying map $M \to B\pi$ to a map $M \to B\pi^{(3)} = X^{(3)}$. If we fix a map $\widehat{f}\colon M \to X^{(3)}$, then we can consider an element $\mathfrak{sec}(M,\widehat{f}) \in H_3(\pi;\mathbb{Z}/2)$, defined as in Lemma 2.5. Note that $\mathfrak{sec}(M,\widehat{f}) \in H_3(\pi;\mathbb{Z}/2)$ is independent of the homotopy class of $\widehat{f}$ viewed as a map $M \to X^{(4)}$.

We briefly recall the construction here, since we will need the details in the near future. Take a regular preimage F_i of a barycentre for every 3-cell e_i of X. A framing of the normal bundle of this point pulls back to a framing of the normal bundle of the regular preimage $F_i \subseteq D^4 \subseteq M$ and thus gives the preimage a framing of the normal bundle. Thus we can consider $[F_i] \in \Omega_1^{fr} \cong \Omega_1^{Spin} \cong \mathbb{Z}/2$. Then $\sum_i [F_i] e_i \in \mathbb{Z}/2 \otimes C_3(X)$ lies in $\ker(\mathbb{Z}/2 \otimes C_3(X) \to \mathbb{Z}/2 \otimes C_2(X))$, and so defines a homology class $\mathfrak{sec}(M,\widehat{f}) \in H_3(\pi;\mathbb{Z}/2)$.

Lemma 4.24. *The element $\mathfrak{sec}(M,\widehat{f}) \in H_3(\pi;\mathbb{Z}/2)$, where $\widehat{f}\colon M \to X^{(3)}$ is an extension of $i \circ f\colon M^{(3)} \to K \to X^{(3)}$, only depends on the map f.*

Proof. Note that any two extensions of f to $\widehat{f}$ only differ by an element of $\pi_4(X^{(3)})$, that is obtained from the two choices of extension on the unique 4-cell of M. Consider the diagram

$$\begin{tikzcd}
\pi_4(X^{(3)}) \arrow[r] \arrow[d, "h"] & \pi_4(X^{(4)}) \arrow[d, "h", "\cong"'] \\
H_4(X^{(3)};\mathbb{Z}\pi) = 0 \arrow[r] & H_4(X^{(4)};\mathbb{Z}\pi),
\end{tikzcd}$$

where h denotes Hurewicz maps, which is commutative by naturality of the Hurewicz homomorphism. It follows that $\pi_4(X^{(3)}) \to \pi_4(X^{(4)})$ is the trivial map, so any two choices of extension $\widehat{f}$ are homotopic over $X^{(4)}$. Hence $\mathfrak{sec}(M,\widehat{f})$ only depends on f, and not on the choice of $\widehat{f}$. □

Definition 4.25. We define

$$\mathfrak{sec}(M,f) := \mathfrak{sec}(M,\widehat{f}) \in H_3(\pi;\mathbb{Z}/2),$$

for an extension $\widehat{f}\colon M \to X^{(3)}$ of $i \circ f\colon M^{(3)} \to X^{(3)}$. Such an extension exists by Lemma 4.22 and $\mathfrak{sec}(M,f)$ is independent of the choice of extension by Lemma 4.24.

Consider the following diagram, which commutes by functoriality of Tate cohomology $\widehat{H}^0(-)$, and by the definition of $A\colon H_3(\pi;\mathbb{Z}/2) \to \widehat{H}^0(\mathrm{Sesq}(H^2(K;\mathbb{Z}\pi)))$. We will explain more about the maps in the diagram below. The proof of Theorem 4.19, that Condition 4.17 implies the Secondary Property, will be based on this diagram.

(4.26)
$$\begin{tikzcd}
\mathbb{Z}\otimes_{\mathbb{Z}\pi}\pi_3(M^{(3)}) \arrow[d, "f_*"] & H_3(\pi;\mathbb{Z}/2) \arrow[r, hook, "\iota"] \arrow[ddr, dashed, "A"] & \mathbb{Z}/2\otimes_{\mathbb{Z}\pi}\pi_2(K) \arrow[d, "\Theta_{H^2(K;\mathbb{Z}\pi)}"] \\
\mathbb{Z}\otimes_{\mathbb{Z}\pi}\pi_3(K) \arrow[r, "\cong", "S"'] & \mathbb{Z}\otimes_{\mathbb{Z}\pi}\Gamma(\pi_2(K)) \arrow[r] \arrow[d, "\Phi_{H^2(K;\mathbb{Z}\pi)}"] & \widehat{H}^0(\pi_2(K)\otimes_{\mathbb{Z}\pi}\pi_2(K)) \arrow[d, "\widehat{H}^0(\Phi_{H^2(K;\mathbb{Z}\pi)})"] \\
 & (\mathrm{Sesq}(H^2(K;\mathbb{Z}\pi)))^{\mathbb{Z}/2} \arrow[r] & \widehat{H}^0(\mathrm{Sesq}(H^2(K;\mathbb{Z}\pi))) \\
 & (\mathrm{Sesq}(H^2(M;\mathbb{Z}\pi)))^{\mathbb{Z}/2} \arrow[r] \arrow[u, "\mathrm{Sesq}(f^*)"] & \widehat{H}^0(\mathrm{Sesq}(H^2(M;\mathbb{Z}\pi))) \arrow[u, "\widehat{H}^0(\mathrm{Sesq}(f^*))"']
\end{tikzcd}$$

As promised, some remarks on the diagram are in order.

(1) In the bottom row, we could have written $M^{(3)}$ instead of M, since $M^{(3)}$ is the domain of the map f. But attaching a 4-cell makes no difference to second cohomology, so we allow ourselves this slight abuse.

(2) Recall that for a set X and a group G that acts on X, X^G denotes the fixed points. Thus $(\mathrm{Sesq}(H^2(K;\mathbb{Z}\pi)))^{\mathbb{Z}/2}$ is the $\mathbb{Z}/2$-fixed points, that is the kernel of the $\mathbb{Z}[\mathbb{Z}/2]$-module homomorphism $1-T\colon \mathrm{Sesq}(H^2(K;\mathbb{Z}\pi)) \to \mathrm{Sesq}(H^2(K;\mathbb{Z}\pi))$. In other words, the hermitian forms.

(3) Also recall that, by Lemma 4.12, there is a canonical isomorphism $I\colon \pi_2(K) \xrightarrow{\cong} (H^2(K;\mathbb{Z}\pi))^*$, so that we can consider the map
$$\Phi_{H^2(K;\mathbb{Z}\pi)}\colon \pi_2(K)\otimes_{\mathbb{Z}\pi}\pi_2(K) \to \mathrm{Sesq}(H^2(K;\mathbb{Z}\pi)).$$
In the diagram, we see the map induced on Tate cohomology by $\Phi_{H^2(K;\mathbb{Z}\pi)}$ in the right hand column, and we see the map restricted to the Γ group in the middle column.

(4) The map $S := \mathrm{Id}_{\mathbb{Z}}\otimes\Gamma(\eta)$ is an isomorphism by Corollary 3.7, where we use that $H_4(\widetilde{K}) = H_3(\widetilde{K}) = 0$.

(5) The map $\iota\colon H_3(\pi;\mathbb{Z}/2) \to \mathbb{Z}/2\otimes_{\mathbb{Z}\pi}\pi_2(K)$ is an injection by Lemma 4.14.

We will show that the image of $\mathfrak{sec}(M,f) \in H_3(\pi;\mathbb{Z}/2)$ in $\widehat{H}^0(\mathrm{Sesq}(H^2(K;\mathbb{Z}\pi)))$ agrees with the image of $\lambda_M \in (\mathrm{Sesq}(H^2(M;\mathbb{Z}\pi)))^{\mathbb{Z}/2}$ in $\widehat{H}^0(\mathrm{Sesq}(H^2(K;\mathbb{Z}\pi)))$. Here the image of $\mathfrak{sec}(M,f)$ is $A(\mathfrak{sec}(M,f))$ (the map A was introduced in Definition 4.15). The image of λ_M is the intersection form restricted to $H^2(K;\mathbb{Z}\pi)$ along f^*, modulo even forms. To show that the images coincide we will show that both of them agree with a third element of $\widehat{H}^0(\mathrm{Sesq}(H^2(K;\mathbb{Z}\pi)))$, namely the image of $[\alpha] \in \mathbb{Z}\otimes_{\mathbb{Z}\pi}\pi_3(M^{(3)})$ in $\widehat{H}^0(\mathrm{Sesq}(H^2(K;\mathbb{Z}\pi)))$; recall that $\alpha \in \pi_3(M^{(3)})$ is the attaching map of the 4-cell of M. We abuse notation and write α for $1\otimes\alpha \in \mathbb{Z}\otimes\pi_3(M^{(3)})$.

Lemma 4.27. *We have an equality*
$$\Theta_{H^2(K;\mathbb{Z}\pi)} \circ \iota(\mathfrak{sec}(M,f)) = [S\circ f_*(\alpha)] \in \widehat{H}^0(\pi_2(K)\otimes_{\mathbb{Z}\pi}\pi_2(K)).$$

Proof. We have that $(M\#(S^2\times S^2))^{(3)} \simeq M^{(3)}\vee S^2\vee S^2$, and under this stabilisation the attaching map of the 4-cell changes by the Whitehead product of two identity maps $S^2 \to S^2$. Let $\alpha' \in$

$\pi_3(M^{(3)} \vee S^2 \vee S^2)$ denote the new attaching map of the 4-cell. For any two elements $a, b \in \pi_2(K)$, we can consider the map $f \vee a \vee b\colon M^{(3)} \vee S^2 \vee S^2 \to K$ and obtain

$$S \circ (f \vee a \vee b)_* \alpha' = S \circ f_* \alpha + a \otimes b + b \otimes a \in \mathbb{Z} \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2(K)).$$

Hence

$$[S \circ (f \vee a \vee b)_* \alpha'] = [S \circ f_* \alpha] \in \widehat{H}^0(\pi_2(K) \otimes_{\mathbb{Z}\pi} \pi_2(K)).$$

On the other hand, if we extend f and $f \vee a \vee b$ to $X^{(3)}$ via the inclusion $K \to X^{(3)}$, then they become bordant maps over $X^{(3)}$, since $X^{(3)}$ is built from K by adding 3-cells to kill $\pi_2(K)$. It follows that $\mathfrak{sec}(M, f) = \mathfrak{sec}(M \# (S^2 \times S^2), f \vee a \vee b)$. Thus the lemma holds for (M, f) if and only if it holds true for $(M \# (S^2 \times S^2), f \vee a \vee b)$.

Up to elements of the form $a \otimes b + b \otimes a$, every element in $\Gamma(\pi_2(K))$ can be written as $\beta \otimes \beta$ for some $\beta \in \pi_2(K)$. Thus by the previous paragraph we can and will assume that the attaching map of the 4-cell $\alpha \in \pi_3(M^{(3)})$ is mapped to $\beta \otimes \beta$ under the composition

$$\Gamma(\eta)^{-1} \circ f_*\colon \pi_3(M^{(3)}) \to \pi_3(K) \to \Gamma(\pi_2(K)).$$

It follows from Lemma 3.5 that $f_*(\alpha) = \beta \circ \eta$, where $\eta\colon S^3 \to S^2$ is the Hopf map.

Then f extends to a map $\widehat{f}\colon M \to K \cup_\beta D^3$, where the map from the top cell of M to D^3 is the cone on $\eta\colon S^3 \to S^2$.

Attach further 3-cells to turn $K \cup_\beta D^3$ into the 3-skeleton $B\pi^{(3)}$ of a model for $B\pi$. Therefore, we obtain $\mathfrak{sec}(M, f) = \mathfrak{sec}(M, \widehat{f}) \in H_3(\pi; \mathbb{Z}/2)$ by the construction given in the proof of Lemma 4.24, where technically speaking we extend $\widehat{f}$ further by the inclusion $\widehat{f}\colon M \to K \cup_\beta D^3 \to B\pi^{(3)}$. This is a particular choice of extension $\widehat{f}$, as in the proof of Lemma 4.24. Recall that for the definition of $\mathfrak{sec}(M, f)$, we take a regular preimage F_i of a barycentre for every 3-cell e_i of $X^{(3)}$, where $X \simeq B\pi$, and then $\sum_i [F_i] e_i$ determines $\mathfrak{sec}(M, \widehat{f}) \in H_3(\pi; \mathbb{Z}/2)$, where $[F_i]$ is the framed or spin bordism class of F_i.

We assert that in the above construction $[F_i] = 1$ if and only if $e_i = e_\beta$, the first cell attached along β. If $e_i \neq e_\beta$, then F_i is empty since $\widehat{f}$ factors through $K \cup_\beta D^3 \to B\pi^{(3)}$. To see that $[F_\beta] = 1$, note that it was a regular preimage of $C\eta\colon D^4 \to D^3$, and thus of $\Sigma\eta$, since $\Sigma\eta\colon S^4 \to S^3$ arises from gluing together two copies of $C\eta\colon D^4 \to D^3$, and we can assume the point whose preimage we take to be in one of the $C\eta$ halves. Hence $[F_\beta] = 1$ follows from

$$\Sigma\eta = 1 \in \pi_1^{st} \cong \Omega_1^{fr} \cong \Omega_1^{Spin} \cong \mathbb{Z}/2.$$

Hence $\mathfrak{sec}(M, f) = [e_\beta] \in H_3(\pi; \mathbb{Z}/2)$. Consider the diagram

$$\begin{tikzcd}
H_3(K;\mathbb{Z}/2) \arrow[r] \arrow[d] & H_3(\pi;\mathbb{Z}/2) \arrow[r, "\iota"] \arrow[d] & \mathbb{Z}/2 \otimes_{\mathbb{Z}\pi} \pi_2(K) \arrow[d] \\
H_3(K \cup_\beta D^3;\mathbb{Z}/2) \arrow[r] & H_3(\pi;\mathbb{Z}/2) \arrow[r, "\iota"] & \mathbb{Z}/2 \otimes_{\mathbb{Z}\pi} \pi_2(K \cup_\beta D^3)
\end{tikzcd}$$

Since $[e_\beta]$ comes from $H_3(K \cup_\beta D^3; \mathbb{Z}/2)$ it maps to zero in $\mathbb{Z}/2 \otimes_{\mathbb{Z}\pi} \pi_2(K \cup_\beta D^3)$ and thus $\iota([e_\beta]) \in \mathbb{Z}/2 \otimes_{\mathbb{Z}\pi} \pi_2(K)$ lies in the kernel of the map to $\mathbb{Z}/2 \otimes_{\mathbb{Z}\pi} \pi_2(K \cup_\beta D^3)$. This kernel precisely consists of $1 \otimes \beta$. Hence $\iota(\mathfrak{sec}(M, f)) = 1 \otimes \beta$ and thus

$$\Theta_{H^2(K;\mathbb{Z}\pi)} \circ \iota(\mathfrak{sec}(M, f)) = [\beta \otimes \beta] = [S \circ f_*(\alpha)]. \qquad \square$$

Lemma 4.28. *Let $\alpha \in \pi_3(M^{(3)})$ be the attaching map of the 4-cell of M, and let $f\colon M^{(3)} \to K$ be a map that induces an isomorphism on fundamental groups. We have an equality:*

$$[\Phi_{H^2(K;\mathbb{Z}\pi)}(S \circ f_*(\alpha))] = [\mathrm{Sesq}(f^*)(\lambda_M)] \in \widehat{H}^0(\mathrm{Sesq}(H^2(K;\mathbb{Z}\pi))).$$

Proof. Arguing as in the first two paragraphs of the proof of Lemma 4.27, it suffices to consider the case $f_*(\alpha) = \beta \circ \eta$ for some $\beta \in \pi_2(K)$. Denote the extension of $f\colon M^{(3)} \to K$ to $M \to K \cup_{\beta\circ\eta} D^4$ again by f. Then for any $x, y \in H^2(K;\mathbb{Z}\pi)$ we have

$$\begin{aligned}&\lambda_M(PD(f^*(x)), PD(f^*(y))) = \langle f^*(y), PD(f^*(x))\rangle\\ =&\langle y, f_*(f^*(x) \cap [M])\rangle = \langle y, x \cap f_*[M]\rangle.\end{aligned}$$

Denote the extension of $\beta\colon S^2 \to K$ by $\widehat{\beta}\colon \mathbb{CP}^2 \to K \cup_{\beta\circ\eta} D^4$. Then $\widehat{\beta}_*[\mathbb{CP}^2] = f_*[M]$ and we have

$$\langle y, x \cap \widehat{\beta}_*[\mathbb{CP}^2]\rangle = \langle \widehat{\beta}^*(y), \widehat{\beta}^*(x) \cap [\mathbb{CP}^2]\rangle = \widehat{\beta}^*(x)\langle 1, 1 \cap [\mathbb{CP}^2]\rangle\overline{\widehat{\beta}^*(y)},$$

where $1 \in H^2(\mathbb{CP}^2;\mathbb{Z}\pi) \cong \mathbb{Z}\pi$ denotes a chosen generator. For the second equation we used the sesquilinearity of the evaluation. By the cup product form of $\mathbb{CP}^2$ we have $\langle 1, 1 \cap [\mathbb{CP}^2]\rangle = 1$ and since $\widehat{\beta}$ is given by β when restricted to the 2-skeleton, we obtain

$$\widehat{\beta}^*(x)\langle 1, 1 \cap [\mathbb{CP}^2]\rangle\overline{\widehat{\beta}^*(y)} = \beta^*(x)\overline{\beta^*(y)} = \Phi_{H^2(K;\mathbb{Z}\pi)}(\beta^* \otimes \beta^*)(x \otimes y).$$

Combining the above equations we obtain

$$\begin{aligned}\lambda_M(PD(f^*(x)), PD(f^*(y))))\rangle &= \Phi_{H^2(K;\mathbb{Z}\pi)}(\beta^* \otimes \beta^*)(x \otimes y)\\ &= \Phi_{H^2(K;\mathbb{Z}\pi)}(S \circ (\beta \circ \eta))(x \otimes y).\end{aligned}$$

as claimed. □

Theorem 4.29. *We have an equality*

$$[\mathrm{Sesq}(f^*)(\lambda_M)] = A(\mathfrak{sec}(M, f)) \in \widehat{H}^0(\mathrm{Sesq}(H^2(K;\mathbb{Z}\pi))).$$

Proof. The theorem follows from combining Lemma 4.27 and Lemma 4.28, together with a straightforward diagram chase in diagram (4.26). □

Corollary 4.30.

(1) *If $\ker A \subseteq \mathrm{im}(Sq^2 \circ \mathrm{red}_2\colon H_5(\pi;\mathbb{Z}) \to H_3(\pi;\mathbb{Z}/2))$, then $\mathfrak{sec}(M) = 0$ if and only if there exists $f\colon M^{(3)} \to K$ as above with*

$$0 = [\mathrm{Sesq}(f^*)(\lambda_M)] \in \widehat{H}^0(\mathrm{Sesq}(H^2(K;\mathbb{Z}\pi))).$$

(2) *If $\mathrm{im}(Sq^2 \circ \mathrm{red}_2\colon H_5(\pi;\mathbb{Z}) \to H_3(\pi;\mathbb{Z}/2)) \subseteq \ker A$, then the class*

$$[\mathrm{Sesq}(f^*)(\lambda_M)] \in \widehat{H}^0(\mathrm{Sesq}(H^2(K;\mathbb{Z}\pi)))$$

is independent of the choice of f.

Proof.

(1) If $\mathfrak{sec}(M) = 0$, then there exists an f with $\mathrm{Sesq}(f^*)(\lambda_M) = 0$. This is because $\mathfrak{sec}(M) = 0$ implies that there is a map $f\colon M \to B\pi^{(2)}$, and we can take K for $B\pi^{(2)}$. The intersection form vanishes on $f^*(H^2(K;\mathbb{Z}\pi))$ by naturality of the evaluation, because $H_*(K;\mathbb{Z}) = 0$.

Conversely, if there exists a map f with

$$0 = [\mathrm{Sesq}(f^*)(\lambda_M)] \in \widehat{H}^0(\mathrm{Sesq}(H^2(K;\mathbb{Z}\pi))),$$

then by Theorem 4.29, $\mathfrak{sec}(M,f) \in \ker A \subseteq \operatorname{im}(Sq^2 \circ \mathrm{red}_2\colon H_5(\pi;\mathbb{Z}) \to H_3(\pi;\mathbb{Z}/2))$. Since $Sq^2 \circ \mathrm{red}_2$ is the second differential in the Atiyah-Hirzebruch spectral sequence computing $\Omega_4^{Spin}(B\pi)$ (see Section 2), we have $\mathfrak{sec}(M) = 0 \in H_3(\pi;\mathbb{Z}/2)/\operatorname{im}(Sq^2 \circ \mathrm{red}_2)$.

(2) Given two choices $f, f'\colon M^{(3)} \to K$, then there exists an $x \in \operatorname{im}(Sq^2 \circ \mathrm{red}_2)$ with $\mathfrak{sec}(M,f) = \mathfrak{sec}(M,f') + x$ since $Sq^2 \circ \mathrm{red}_2$ is the second differential of the Atiyah-Hirzebruch spectral sequence and $\mathfrak{sec}(M)$ is well-defined in $H_3(\pi;\mathbb{Z}/2)/\operatorname{im}(S^2 \circ \mathrm{red}_2)$. Thus by Theorem 4.29 we have

$$\begin{aligned}[\mathrm{Sesq}(f^*)(\lambda_M)] &= A(\mathfrak{sec}(M,f)) = A(\mathfrak{sec}(M,f)) + A(x) = A(\mathfrak{sec}(M,f')) \\ &= [\mathrm{Sesq}((f')^*)(\lambda_M)].\end{aligned}$$

Here we used that A is a homomorphism, and $A(x) = 0$ by assumption. □

We can now apply Corollary 4.30 to prove Theorem 4.19, which says that under Condition 4.17 the Secondary Property 1.8 holds, which we recall here in an expanded form for the convenience of the reader:

If M is a spin manifold with $\pi_1(M) \xrightarrow{\cong} \pi$ and $\mathfrak{pri}(M) = 0$, then $\mathfrak{sec}(M) = 0$ if and only if there exists a splitting

$$s = (s_1, s_2)\colon \operatorname{coker} d_3 \to C_2 \oplus H_2(M;\mathbb{Z}\pi)$$

of the extension from Theorem 1.4 such that λ_M restricted to the image of s_2 is even. The restriction of λ_M for one splitting s is even if and only if it is even for every splitting.

Moreover, for two such manifolds M, M' with the same fundamental group we have $\mathfrak{sec}(M) = \mathfrak{sec}(M')$ if and only if the restrictions of λ_M and $\lambda_{M'}$ are isomorphic modulo even forms for some splittings s and s'.

Proof of Theorem 4.19. Condition 4.17 says precisely that the hypotheses of Corollary 4.30 (1) and (2) hold. If $\mathfrak{pri}(M) = 0$, then there is a splitting $s\colon \operatorname{coker} d_3 \to C_2 \oplus H_2(M;\mathbb{Z}\pi)$.

We want to apply Proposition 4.21 to obtain a map $f\colon M^{(3)} \to K$ for a 2-complex K with $\pi_1(K) \cong \pi$. However note that Proposition 4.21 was formulated for the dual exact sequence

$$0 \to \ker d^3 \to C^2 \oplus H^2(M;\mathbb{Z}\pi) \to \operatorname{coker} d^2 \to 0.$$

So to remedy this, for a splitting $s = (s_1, s_2)\colon \operatorname{coker} d_3 \to C_2 \oplus H_2(M;\mathbb{Z}\pi)$ of the sequence

$$0 \to \ker d_2 \to C_2 \oplus H_2(M;\mathbb{Z}\pi) \to \operatorname{coker} d_3 \to 0$$

from Theorem 1.4, we apply Proposition 4.21 with the dual handle decomposition of M. Recall that M^d denotes M with the dual handle decomposition. We obtain a map $f\colon (M^d)^{(3)} \to (M^d)^{(2)}$ with $s_2 = f^*\colon H^2((M^d)^{(2)};\mathbb{Z}\pi) \to H^2(M^d;\mathbb{Z}\pi)$ by Proposition 4.21, and use the identifications $H^2((M^d)^{(2)};\mathbb{Z}\pi) \cong \operatorname{coker} d_3$ and $H^2(M^d;\mathbb{Z}\pi) \cong H_2(M;\mathbb{Z}\pi)$.

Now by Corollary 4.30 (1), $\mathrm{Sesq}(f^*)(\lambda_M) = \lambda_M|_{s_2(\operatorname{coker} d_3)}$ is even (i.e. zero in the Tate cohomology $\widehat{H}^0(\mathrm{Sesq}(H^2(K;\mathbb{Z}\pi))))$ if and only if $\mathfrak{sec}(M) = \mathfrak{sec}(M^d) = 0$. By Corollary 4.30 (2), $\mathrm{Sesq}(f^*)(\lambda_M)$ is independent of the choice of f, and is therefore independent of the choice of splitting s.

The last sentence of the Secondary Property, that $\mathfrak{sec}(M) = \mathfrak{sec}(M')$ if and only if the restrictions of λ_M and $\lambda_{M'}$ are equal for some choices of splitting s and s', also follows from Theorem 4.29. If $\mathfrak{sec}(M) = \mathfrak{sec}(M')$, then the difference lies in $\operatorname{im}(Sq^2 \circ \mathrm{red}_2) = \ker A$, so for some choice of f, f' we have $A(\mathfrak{sec}(M,f)) = A(\mathfrak{sec}(M',f'))$. Thus $[\mathrm{Sesq}(f^*)(\lambda_M)] = [\mathrm{Sesq}((f')^*)(\lambda_{M'})]$ by Theorem 4.29. On the other hand, $[\mathrm{Sesq}(f^*)(\lambda_M)] = [\mathrm{Sesq}((f')^*)(\lambda_{M'})]$ implies that $A(\mathfrak{sec}(M,f)) = A(\mathfrak{sec}(M',f'))$, so $\mathfrak{sec}(M,f) - \mathfrak{sec}(M',f') \in \ker A = \operatorname{im}(Sq^2 \circ \mathrm{red}_2)$. Thus $\mathfrak{sec}(M) = \mathfrak{sec}(M') \in H_3(\pi;\mathbb{Z}/2)/\operatorname{im} d^2_{5,0}$. □

5. The τ invariant

In this section we introduce two versions of the τ invariant that will appear in our reformulation of the tertiary obstruction. An invariant $\tau(\Sigma)$ first appeared in works of Freedman-Kirby [FK78, p. 93] and Matsumoto [Mat78], while a similar invariant was later used by Freedman and Quinn [FQ90, Definition 10.8]. In [ST01], Schneiderman and the third author defined a generalisation $\tau_1(\Sigma)$ with values in a quotient of $\mathbb{Z}[\pi \times \pi]$, that restricts to the invariant we consider here via the augmentation and reduction modulo two map $\mathbb{Z}[\pi \times \pi] \to \mathbb{Z}/2$.

In this section, all surfaces are assumed to be the images of generic maps, meaning the maps are immersions, all intersections and self-intersections are transverse double points, and there are no triple points.

5.1. The τ invariant for generically immersed spheres.

Let M denote a smooth, closed, oriented, spin 4-manifold together with an identification $\pi_1(M) \cong \pi$.

Definition 5.1 (Self-intersection number)**.** [Wal99, Chapter 5], [FQ90, Section 1.7] Let $x \in \pi_2(M)$. Since M is spin, we can represent x by a generically immersed sphere whose normal bundle has even Euler number. Add cusp homotopies in a small open set to make the Euler number of the normal bundle zero. Call the resulting sphere Σ. Now count the self intersections of Σ with sign and group elements. The attribution of signs uses the orientation of M. The group element is the image in $\pi_1(M)$ of a double point loop associated to the self-intersection point, with some choice of orientation of the double point loop. This count gives rise to an element

$$\mu(x) \in \mathbb{Z}\pi/\{g \sim g^{-1}\}.$$

The self-intersection number is valued in a quotient group of the $\mathbb{Z}\pi$-module $\mathbb{Z}\pi$ since there is no canonical way to decide whether to associate g or g^{-1} to a given double point of Σ. The number $\mu(x)$ is a well-defined invariant of the homotopy class of x.

Denote the map given by augmentation composed with reduction modulo 2 by $\mathrm{red}_2\colon \mathbb{Z}\pi \xrightarrow{\varepsilon} \mathbb{Z} \to \mathbb{Z}/2$. We abuse notation and also use $\mathrm{red}_2\colon \pi_2(M) = H_2(M;\mathbb{Z}\pi) \to H_2(M;\mathbb{Z}/2)$ to denote the induced map on homology. Let $\lambda_2\colon H_2(M;\mathbb{Z}/2) \times H_2(M;\mathbb{Z}/2) \to \mathbb{Z}/2$ be the $\mathbb{Z}/2$-valued intersection pairing.

Definition 5.2.

(1) An element $\alpha \in \pi_2(M)$ is called S^2-*characteristic* if $\mathrm{red}_2(\lambda(\alpha,\beta)) = 0 \in \mathbb{Z}/2$ for all $\beta \in \pi_2(M)$. Let $\mathcal{SC} \subseteq \pi_2(M)$ denote the subset of S^2-characteristic elements α with $\mu(\alpha) = 0$.

(2) An element $\alpha \in \pi_2(M)$ is called $\mathbb{RP}^2$-*characteristic* if $\lambda_2(\mathrm{red}_2(\alpha), [R])) = 0 \in \mathbb{Z}/2$ for every map $R\colon \mathbb{RP}^2 \to M$. Let $\mathcal{RC} \subseteq \pi_2(M)$ denote the subset of $\mathbb{RP}^2$-characteristic elements α with $\mu(\alpha) = 0$.

Lemma 5.3. *An $\mathbb{RP}^2$-characteristic sphere $\alpha \in \pi_2(M)$ is S^2-characteristic. Moreover if $\pi_1(M)$ has no elements of order two, then α is S^2-characteristic if and only if it is $\mathbb{RP}^2$-characteristic.*

Proof. A generic immersion $f\colon S^2 \looparrowright M$ determines a map $\mathbb{RP}^2 \to \mathbb{RP}^2/\mathbb{RP}^1 = S^2 \xrightarrow{f} M$, which can be perturbed to a generic map of $\mathbb{RP}^2$ with the same intersection behaviour with α as the original S^2. Thus $\mathbb{RP}^2$-characteristic implies S^2-characteristic.

On the other hand, if no element of $\pi_1(M)$ has order 2, then for every generic immersion R of $\mathbb{RP}^2$, the induced map $\pi_1(\mathbb{RP}^2) \to \pi_1(M)$ is the zero map. Therefore R is homotopic to a map that factors as $\mathbb{RP}^2 \to \mathbb{RP}^2/\mathbb{RP}^1 = S^2 \xrightarrow{f} M$, and intersections with $f(S^2)$ agree with intersections with R. It follows that S^2-characteristic implies $\mathbb{RP}^2$-characteristic. ∎

Let $S\colon S^2 \looparrowright M$ be a generically immersed 2-sphere with vanishing self-intersection number $\mu(S) = 0$. Then the self-intersection points of S can be paired up so that each pair consists of two points having oppositely signed but equal group elements associated to their double point loops. Therefore, one can choose a Whitney disc W_i for each pair of self-intersections, and arrange that all the boundary arcs are disjoint. The normal bundle to the disc W_i has a unique framing, and the Whitney framing of the normal bundle of W_i restricted to ∂W_i differs from the restriction of the disc framing by an integer $n_i \in \mathbb{Z}$. (The Whitney framing is determined by a section of the normal bundle $\nu_{W_i}|_{\partial W_i}$ that lies in $TS^2 \cap \nu_{W_i}$ along one boundary arc of ∂W_i and lies in $\nu_{S^2} \cap \nu_{W_i}$ along the other boundary arc.)

If S is $\mathbb{RP}^2$-characteristic, then the following expression is independent of the choice of Whitney discs, pairing of double points, and Whitney arcs:

$$\tau(S) := \sum_i |W_i \cap S| + n_i \mod 2.$$

Moreover, $\tau(S)$ only depends on the regular homotopy class of the generic immersion. See [ST01, Theorem 1] for a proof that this number is well-defined. We make a couple of remarks on how to translate the version in that article to the current version. First note that in the formulation of [ST01], as mentioned above the intersections were decorated with a pair of fundamental group elements, to give an invariant in a quotient of $\mathbb{Z}[\pi \times \pi]$ by certain relations. Since we consider the augmentation followed by the reduction modulo two, all but the last relation given in [ST01, Theorem 1] are vacuous. In addition their last relation is irrelevant because we consider $\mathbb{RP}^2$-characteristic elements. Secondly, the formulation of Schneidermann-Teichner requires that Whitney discs be framed, whereas we do not, and include the framing coefficient as part of the definition. However by boundary twisting [FQ90, Section 1.3], one can alter n_i to be zero at the cost of introducing $|n_i|$ intersection points in $W_i \cap S$.

We fix a regular homotopy class within the homotopy class by the requirement that the Euler number be zero. Thus τ is well-defined on $\mathcal{RC} \subset \pi_2(M)$, and so we have defined a map $\tau\colon \mathcal{RC} \to \mathbb{Z}/2$.

Remark 5.4. If S is not S^2-characteristic then $\tau(S)$ is not well-defined, since adding a sphere that intersects S in an odd number of points to one of the Whitney discs would change the sum in the definition of τ by one.

If S is S^2-characteristic but not $\mathbb{RP}^2$-characteristic, then $\tau(S)$ is also not well-defined, as observed by Stong [Sto94]. In this case, a change in choice of Whitney arcs, in the presence of 2-torsion in $\pi_1(M)$, can also change $\tau(S)$.

The following lemma is rather useful, in that it tells us that it is enough to consider the intersection pairing in order to find spheres with vanishing self-intersection number. The vanishing of the secondary obstruction gives information on the intersection pairing, and we will use this to define a tertiary obstruction in terms of τ.

Lemma 5.5. *If $\lambda(x,x) = 0$ for $x \in \pi_2(M)$, then $\mu(x) = 0$.*

Proof. Recall that $\lambda(x,x) = \mu(x) + \overline{\mu(x)}$. for spheres whose normal bundles have trivial Euler number. But $\lambda(x,x) = 0$ implies that the Euler number is even, and then cusps can be added to make the Euler number zero. This gives the regular homotopy class used to compute $\mu(x)$. Suppose that $\sum_g n_g g \in \mathbb{Z}\pi$ is a lift of $\mu(x)$. By hypothesis $\mu(x) + \overline{\mu(x)} = 0$, so $n_g + n_{g^{-1}} = 0$. But in the value group of μ we have $g \sim g^{-1}$, so $n_g g + n_{g^{-1}} g^{-1} = (n_g + n_{g^{-1}})g = 0 \cdot g = 0$. □

5.2. **The τ invariant for π_1-trivial generically immersed surfaces.** In this subsection we introduce the following extension of the τ invariant, which is defined on $\mathbb{RP}^2$-characteristic, generically immersed surfaces instead of on $\mathbb{RP}^2$-characteristic generically immersed spheres. We will not need the full version of this invariant, only the embedded version. But we anticipate that the full version might be useful in the future, so we include it here, as it requires little extra work.

We call a generically immersed surface $F\colon \Sigma \looparrowright M$ a π_1-*trivial surface* if $F_*\colon \pi_1(\Sigma) \to \pi_1(M)$ is the trivial map.

The restriction to the case that F is an embedding is similar to the version of τ from [FK78]. A π_1-trivial generically immersed surface $F : \Sigma \looparrowright M$ is said to be $\mathbb{RP}^2$-*characteristic* if it intersects every generically immersed $\mathbb{RP}^2$ in general position in an even number of points i.e. if the element of $\pi_2(M)$ determined by F via the Hurewicz isomorphism $H_2(\widetilde{M}) \cong \pi_2(M)$ is $\mathbb{RP}^2$-characteristic.

A π_1-trivial $\mathbb{RP}^2$-characteristic generically immersed surface F has a self-intersection number $\mu(F) \in \mathbb{Z}\pi/\{g \sim g^{-1} \mid g \in \pi\}$ defined as follows. Add local kinks to F until its normal bundle is trivial; this is possible since F is S^2-characteristic. Now count self-intersection points of the generically immersed surface with group elements and sign. We use π_1-triviality to see that the associated group elements do not depend on the choice of double point loop on F used to compute it.

Let $F\colon \Sigma \looparrowright M$ be a generically immersed π_1-trivial surface with $\mu(F) = 0$, and let α be an embedded circle in F. The circle α bounds a disc C in M, since F is π_1-trivial. The normal direction of α in F gives a section of the normal bundle of C at the boundary α. Therefore, the relative Euler number $e(C)$ of the normal bundle of C is a well-defined integer. We define

$$\varpi(\alpha) := \#(C \pitchfork F) + e(C) \mod 2,$$

where $\#(C \pitchfork F)$ is the number of transverse intersections between the interiors of C and F. We will show as part of the proof of the next lemma that $\varpi(\alpha)$ does not depend on the choice of C if F is $\mathbb{RP}^2$-characteristic.

Consider a hyperbolic basis of $H_1(F;\mathbb{Z})$ represented by embedded circles $a_1,\dots,a_n,b_1,\dots,b_n$ that are disjoint from each other except that a_i intersects b_i transversally in a single point.

Since $\mu(F) = 0$, all double points of F can be paired up by Whitney discs $W_1,\dots,W_m \hookrightarrow M$ whose boundary arcs on F are disjoint from each other, the a_i, and the b_i. Let n_j again denote the framing coefficient of the Whitney discs discussed in Section 5.1. Then define:

$$\tau(F) := \sum_{i=1}^{n} \varpi(a_i)\varpi(b_i) + \sum_{j=1}^{m} |W_j \cap F| + n_j \mod 2.$$

Note that in the case that F has genus zero, this reduces to the τ invariant of the previous subsection since the first sum vanishes. Also note that in the case of an embedded surface, only the first summand appears, and again the definition simplifies.

Next we will show that $\tau(F)$ is independent of the choice of basis $\{a_i, b_i\}$, as well as the choice of the discs C and W_j.

Lemma 5.6. *The element $\tau(F) \in \mathbb{Z}/2$ is independent of the choices of a_i, b_i, C and W_j made in its definition.*

Proof. Choose a path from each component of F to the base point of M. Since these paths are 1-dimensional we can choose them so that the interiors of the paths do not intersect F. Since F is π_1-trivial, it lifts to a generically immersed surface in $\widetilde{M}$, and hence defines an element of $H_2(\widetilde{M};\mathbb{Z}) \cong \pi_2(M)$. The strategy is to relate $\tau(F)$ to $\tau(S)$ for $S \in \pi_2(M)$, and use that $\tau(S)$ is well-defined.

Choose generic null-homotopies $C_i\colon D^2 \to M$ for a_i and $C_i'\colon D^2 \to M$ for b_i. We can perform boundary twists [FQ90, Section 1.3] in order to arrange that the C_i and C_i' are framed with respect to their boundaries, i.e. $e(C_i) = e(C_i') = 0$. Boundary twists do not change $\varpi(a_i)$ and $\varpi(b_i)$, since a boundary twist changes the relative Euler number of the disc by one and produces a single new intersection between the disc being twisted and F. We can turn F into a generically immersed 2-sphere S by performing surgeries along all the a_i, and gluing in two copies of each of the C_i. Each intersection $C_i \pitchfork F$ yields a pair of cancelling self-intersections of S paired by a Whitney disc constructed from (a parallel copy of) C_i' union a band. A schematic is shown in Figure 1.

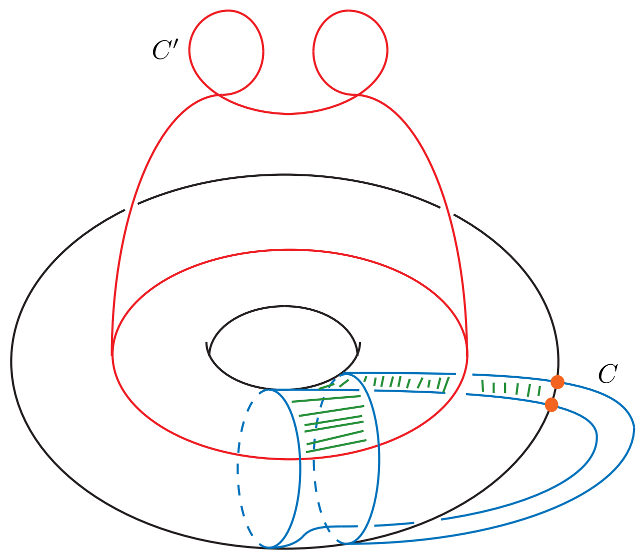

FIGURE 1. A schematic of a genus one surface F with a cap C' attached to the longitude, two parallel copies of a cap C attached to the meridian, each of which intersect F in a single point. A band is shown that, together with the cap C', forms a Whitney disc pairing the two self-intersection points of the sphere obtained from surgery on F using C.

Each self-intersection of C_i yields two pairs of cancelling self-intersections of S, each with generically immersed Whitney disc a parallel copy of C_i', union a band. A schematic is shown in Figure 2.

The boundary arcs of the new Whitney discs are disjoint from the boundary arcs of the old Whitney discs. Thus modulo two we see that

$$\begin{aligned}\tau(S) &= \sum_{i=1}^{n} 2\#(C_i \pitchfork C_i)\#(C_i' \pitchfork F) + \#(C_i \pitchfork F)\#(C_i' \pitchfork F) + \sum_{j=1}^{m} |W_j \cap F| + n_j \\ &= \sum_{i=1}^{n} \varpi(a_i)\varpi(b_i) + \sum_{j=1}^{m} |W_j \cap F| + n_j = \tau(F).\end{aligned}$$

Since S and F determine the same element of $\pi_2(M)$, with the right choice of basing paths (a choice of a collection of generic discs is needed to surger F to a sphere, but the choice of discs does not affect the homotopy class of the resulting sphere), we see that S is $\mathbb{RP}^2$-characteristic. We know that the number $\tau(S)$ only depends on the homotopy class of S [ST01], which is determined by the generic

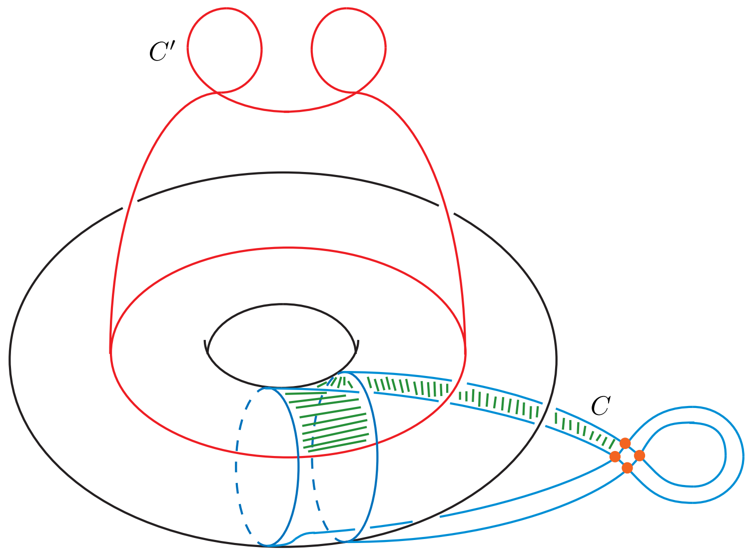

FIGURE 2. A schematic of a genus one surface F with a cap C' attached to the longitude and two parallel copies of a cap C attached to the meridian. The cap C has a single self-intersection points, which gives rise to four self-intersection points of the sphere resulting from surgery on F using C. For one pair of these four points, a band is shown, that together with the cap C', forms a Whitney disc pairing these two self-intersection points.

immersion F and does not depend on the choices of a_i, b_i, C_i, C_i', W_j. Hence the fact that $\tau(S)$ is well-defined implies that $\tau(F)$ is too. □

6. The tertiary obstruction

Let M be a spin 4-manifold with an identification $\pi_1(M) \xrightarrow{\cong} \pi$, such that $\mathfrak{pri}(M) = 0$ and $\mathfrak{sec}(M) = 0$. In this section, more precisely in Theorem 6.21, we will give an algebraic criterion on π, under which we can identify the tertiary obstruction $\mathfrak{ter}(M)$ with a τ invariant defined on a subset of $\pi_2(M)$.

Let K be a finite connected 2-complex homotopy equivalent to $M^{(2)}$. Let X be CW complex with 2-skeleton K such that X is a model for $B\pi$. Fix an identification of $\pi_1(X) \xrightarrow{\cong} \pi$. We then have a canonical map $c\colon M \to X$. Let $i\colon K \to X$ be the inclusion of the 2-skeleton.

Lemma 6.1. *Let $f\colon M^{(3)} \to K$ be a map that induces the given isomorphism on fundamental groups. Let $j\colon M^{(3)} \to M$ be the inclusion of the 3-skeleton. For every $\varphi \in \ker \mathrm{Sq}^2 \subseteq H^2(B\pi;\mathbb{Z}/2) = H^2(X;\mathbb{Z}/2)$ and every lift $\varphi' \in H^2(K;\mathbb{Z}\pi)$ of $i^*\varphi \in H^2(K;\mathbb{Z}/2)$, the element $PD \circ (j^*)^{-1} \circ f^*(\varphi') \in H_2(M;\mathbb{Z}\pi) \cong \pi_2(M)$ is $\mathbb{RP}^2$-characteristic.*

In the lemma above we used the following sequence of maps:

$$\varphi' \in H^2(K;\mathbb{Z}\pi) \xrightarrow{f^*} H^2(M^{(3)};\mathbb{Z}\pi) \xrightarrow{(j^*)^{-1}} H^2(M;\mathbb{Z}\pi) \xrightarrow{PD} H_2(M;\mathbb{Z}\pi) \cong \pi_2(M).$$

Proof. Fix a map $\beta\colon \mathbb{RP}^2 \to M$. In addition, let $\mathrm{red}_2\colon \mathbb{Z}\pi \to \mathbb{Z} \to \mathbb{Z}/2$ be the ring homomorphism given by the composition of the augmentation and reduction modulo two. The following equations prove that $PD((j^*)^{-1}f^*\varphi') \in \pi_2(M)$ is $\mathbb{RP}^2$-characteristic. We will give justification for each of the equalities afterwards.

$$\begin{array}{rcccl} & \mathrm{red}_2\big(\lambda_M(PD((j^*)^{-1}f^*\varphi'),\beta_*[\mathbb{RP}^2])\big) & = & \mathrm{red}_2\langle (j^*)^{-1}f^*\varphi', \beta_*[\mathbb{RP}^2]\rangle \\ = & \mathrm{red}_2\langle \varphi', f_*j_*^{-1}\beta_*[\mathbb{RP}^2]\rangle & = & \langle \mathrm{red}_2(\varphi'), f_*j_*^{-1}\beta_*[\mathbb{RP}^2]\rangle \\ = & \langle i^*\varphi, f_*j_*^{-1}\beta_*[\mathbb{RP}^2]\rangle & = & \langle \varphi, i_*f_*j_*^{-1}\beta_*[\mathbb{RP}^2]\rangle \\ = & \langle \varphi, c_*j_*j_*^{-1}\beta_*[\mathbb{RP}^2]\rangle & = & \langle \varphi, c_*\beta_*[\mathbb{RP}^2]\rangle = \langle \beta^*c^*\varphi, [\mathbb{RP}^2]\rangle \end{array}$$

The first equation is the algebraic definition of the intersection form (Section 4.1). The second equation uses the naturality of the evaluation. The third equation uses that reduction mod 2 of the evaluation is the same as the evaluation of the cochain reduced mod 2. The fourth equation uses that, by definition of φ', $\mathrm{red}_2(\varphi') = i^*\varphi$. The fifth equation again uses naturality. The sixth equation uses that $f^* \circ i^* = j^* \circ c^*$, which holds since the compositions $c \circ j$ and $i \circ f$ are two maps $M^{(3)} \to B\pi$ inducing the same map $\pi_1(M^{(3)}) \to \pi_1(B\pi) \cong \pi$, and so are homotopic maps.

The map $c \circ \beta\colon \mathbb{RP}^2 \to B\pi$ extends to a map $\beta'\colon \mathbb{RP}^\infty \to B\pi$. Now assume for a contradiction that $\langle \beta^*c^*\varphi, [\mathbb{RP}^2]\rangle$ is nontrivial. Then $\beta^*c^*\varphi \in H^2(\mathbb{RP}^2;\mathbb{Z}/2)$ is nontrivial and hence also $(\beta')^*\varphi \in H^2(\mathbb{RP}^\infty;\mathbb{Z}/2)$ is nontrivial. In this case, also $\mathrm{Sq}^2((\beta')^*\varphi) = (\beta')^*\mathrm{Sq}^2\varphi \in H^4(\mathbb{RP}^\infty;\mathbb{Z}/2)$ has to be nontrivial. But we assumed that φ lies in the kernel of Sq^2 and hence $\langle \beta^*c^*\varphi, [\mathbb{RP}^2]\rangle$ has to be trivial. □

The following lemma was [KLPT17, Lemma 7.3 and Lemma 7.4]. These were proven in that paper without restrictions on the group π. For the convenience of the reader we recall the statements here as Lemma 6.3.

Remark 6.2. In [KLPT17] we asserted that the invariant τ_M is well-defined, and the results of the next lemma, for S^2-characteristic spheres. But we ought to have required the stronger $\mathbb{RP}^2$-characteristic. The results of that paper are for torsion-free groups, where the two notions coincide. The proof of the next lemma goes through unchanged.

Lemma 6.3.

(1) Let $x, y \in \pi_2(M)$ be such that $\lambda(x,y) = 0$, $\mu(x)$ and $\mu(y)$ are trivial, and x is $\mathbb{RP}^2$-characteristic. Then for every element $\kappa \in \ker(\mathbb{Z}\pi \to \mathbb{Z}/2)$, we have $\tau(x) = \tau(x + \kappa y) \in \mathbb{Z}/2$.

(2) Let Y be a finite 2-dimensional CW complex with fundamental group π. Every element in the kernel of $H^2(Y;\mathbb{Z}\pi) \to H^2(Y;\mathbb{Z}/2)$ can be written as $\sum_{i=1}^n \kappa_i x_i$ with $\kappa_i \in \ker(\mathbb{Z}\pi \to \mathbb{Z}/2)$ and $x_i \in H^2(Y;\mathbb{Z}\pi)$.

As in Section 4, in this section we will consider the dual sequence

$$0 \to \ker d^3 \to C^2 \oplus H^2(M;\mathbb{Z}\pi) \to \operatorname{coker} d^2 \to 0$$

to the sequence in Theorem 1.4. As noted previously, this can be obtained from the exact sequence in Theorem 1.4 by working with the dual handle decomposition M^d. Recall that we say a map $f\colon M^{(3)} \to M^{(2)} \simeq K$ realises a splitting $s\colon \operatorname{coker} d^2 \to C^2 \oplus H^2(M;\mathbb{Z}\pi)$ if $s_2\colon \operatorname{coker} d^2 \to H^2(M;\mathbb{Z}\pi)$ coincides with the map

$$\operatorname{coker} d^2 \cong H^2(K;\mathbb{Z}\pi) \xrightarrow{f^*} H^2(M^{(3)};\mathbb{Z}\pi) \cong H^2(M;\mathbb{Z}\pi).$$

Recall that $\mathcal{RC} \subseteq \pi_2(M) \cong H_2(M;\mathbb{Z}\pi)$ denotes the subset of $\mathbb{RP}^2$-characteristic elements α with $\mu(\alpha) = 0$. The following diagram should help to read the upcoming definition.

$$\begin{tikzcd}
H^2(K;\mathbb{Z}\pi) \arrow[r, "\cong"] \arrow[dd, "p"'] & \operatorname{coker} d^2 \arrow[r, "s_2"] & H^2(M;\mathbb{Z}\pi) \arrow[r, "PD"] & H_2(M;\mathbb{Z}\pi) \\
& (PD\circ s_2)^{-1}(\mathcal{RC}) \arrow[u, hook] \arrow[rr, "(PD\circ s_2)|"] \arrow[d, "p|"'] & & \mathcal{RC} \arrow[u, hook] \arrow[d, "\tau"] \\
H^2(K;\mathbb{Z}/2) & p\big((PD\circ s_2)^{-1}(\mathcal{RC})\big) \arrow[l, hook'] \arrow[rr, "\tau'_{M,s}"] & & \mathbb{Z}/2 \\
H^2(\pi;\mathbb{Z}/2) \arrow[u, "i^*"] & \ker \mathrm{Sq}^2 \arrow[l, hook'] \arrow[u, "i^*|"']
\end{tikzcd}$$

Definition 6.4. Let $s = (s_1, s_2)\colon \operatorname{coker} d^2 \to C^2 \oplus H^2(M;\mathbb{Z}\pi)$ be a section of the sequence from Theorem 1.4 such that λ_M vanishes on $\operatorname{im} s_2$ (we will show in Lemma 6.10 that such a splitting exists stably whenever $\mathfrak{sec}(M) = 0$; see also Remark 6.6). It follows from Lemma 6.3 that the map

$$\tau \circ PD \circ s_2\colon (PD \circ s_2)^{-1}\mathcal{RC} \to \mathbb{Z}/2$$

factors through the restriction of $p\colon H^2(K;\mathbb{Z}\pi) \to H^2(K;\mathbb{Z}/2)$ to $(PD \circ s_2)^{-1}(\mathcal{RC})$. Denote the map $p((PD \circ s_2)^{-1}\mathcal{RC}) \to \mathbb{Z}/2$ arising in this factorisation by $\tau'_{M,s}$. Let $i^*\colon H^2(\pi;\mathbb{Z}/2) \to H^2(K;\mathbb{Z}/2)$ be the map induced by inclusion of the 2-skeleton. Note that

$$i^*(\ker \mathrm{Sq}^2) \subseteq p((PD \circ s_2)^{-1}(\mathcal{RC}))$$

by Lemma 6.1, since by Proposition 4.21 every splitting s can be realised by a map $f\colon M^{(3)} \to M^{(2)}$. Denote

$$\tau_{M,s} := \tau'_{M,s} \circ i^*|\colon \ker \mathrm{Sq}^2 \to \mathbb{Z}/2.$$

We quickly make the following observation.

Lemma 6.5. *We have that $\tau_{M,s}(0) = 0$ for any section s.*

Proof. Computing $\tau_{M,s}(0)$ involves computing $\tau(S)$, where S is an embedded sphere contained in a single chart. But there are no intersections, therefore no Whitney discs, and so τ evidently vanishes. □

For the convenience of the reader, we recall that the *Tertiary Property 1.12* requires that for every spin 4-manifold with $\pi_1(M) \xrightarrow{\cong} \pi$, and $\mathfrak{pri}(M) = \mathfrak{sec}(M) = 0$, we have the following, where we have switched to the dual version of the short exact sequence from Theorem 1.4 (see the preamble to Section 4.4, just before Proposition 4.21).

For every section

$$s = (s_1, s_2)\colon \operatorname{coker} d^2 \to C^2 \oplus H^2(M;\mathbb{Z}\pi)$$

of the (dual version of the) short exact sequence from Theorem 1.4 with $\lambda_M|_{\operatorname{im} PD\circ s_2} \equiv 0$, the map $\tau_{M,s}|_{\ker(d^3_{5,0})^*}$ is a homomorphism, and $\omega(\mathfrak{ter}(M)) = \tau_{M,s}|_{\ker(d^3_{5,0})^*}$, where ω is the composition:

$$\begin{aligned}\omega\colon H_2(\pi;\mathbb{Z}/2)/\operatorname{im}(d^2_{4,1}, d^3_{5,0}) &\xrightarrow{\cong} \operatorname{Hom}_{\mathbb{Z}/2}(H_2(\pi;\mathbb{Z}/2),\mathbb{Z}/2)/\operatorname{im}(d^2_{4,1}, d^3_{5,0}) \\ &\xrightarrow{\cong} \operatorname{Hom}_{\mathbb{Z}/2}(\ker(d^3_{5,0})^*,\mathbb{Z}/2).\end{aligned}$$

Remark 6.6.

(1) Since $\mathfrak{pri}(M) = 0 = \mathfrak{sec}(M)$, there exists, after stabilisation, a map $f\colon M \to K = B\pi^{(2)}$, inducing an isomorphism on fundamental groups, by Section 2. In Proposition 6.8 below, more precisely in Lemma 6.10, we show that $f^* = s\colon \operatorname{coker} d^2 = H^2(K;\mathbb{Z}\pi) \to \mathbb{Z}\pi^n \oplus H^2(M;\mathbb{Z}\pi)$ gives a map such that $\lambda_M|_{\operatorname{im} PD\circ s_2} \equiv 0$. Thus a section of the form required for the formulation of the Tertiary Property exists.
(2) Lemma 6.5 implies that the property holds when $\operatorname{Hom}_{\mathbb{Z}/2}(\ker(d^3_{5,0})^*, \mathbb{Z}/2) = 0$.

Remark 6.7. In order to prove the Tertiary Property 1.12 when the group $\operatorname{Hom}_{\mathbb{Z}/2}(\ker(d^3_{5,0})^*, \mathbb{Z}/2)$ in which $\mathfrak{ter}(M)$ lives is nontrivial, we will show the following.

(1) The map $\tau_{M,s}$ defines a homomorphism $H^2(\pi;\mathbb{Z}/2) \to \mathbb{Z}/2$.
(2) The image of $\tau_{M,s}$ under the map
$$\operatorname{Hom}(H^2(\pi;\mathbb{Z}/2);\mathbb{Z}/2) \cong H_2(\pi;\mathbb{Z}/2) \to H_2(\pi;\mathbb{Z}/2)/\operatorname{im}(d^2_{4,1}, d^3_{5,0})$$
is independent of s.
(3) The image of $\tau_{M,s}$ under the above map agrees with $\mathfrak{ter}(M)$.

Proposition 6.8 will start by showing that, in the case that the splitting s comes from a map $M \to M^{(2)}$, then the above statements hold. Theorem 6.17 gives an algebraic condition that implies s is geometrically realised in this manner. As part of the proof we will need to make precise the description of $\mathfrak{ter}(M)$ arising from taking inverse images that appeared in Lemma 2.5.

Let M be a spin 4-manifold; that is, we fix a spin structure on M. Let $f\colon M \to K$ be a map to a 2-dimensional CW complex K with $\pi_1(K) \cong \pi$, and let $\varphi \in H^2(K;\mathbb{Z}/2)$. Define $\operatorname{Arf}(f^*\varphi) \in \mathbb{Z}/2$ as follows. Represent φ by a map $K \to S^2 \subseteq K(\mathbb{Z}/2, 2)$ and let $x \in S^2$ be a regular point for $\varphi \circ f\colon M \to S^2$. Define $F := (\varphi \circ f)^{-1}(x) \subset M$. A framing of $\nu_x^{S^2}$ induces a framing of ν_F^M, and since M is spin, we obtain a spin structure on F.

Define a map $\Upsilon\colon H_1(F;\mathbb{Z}) \to \mathbb{Z}/2$ by representing $\alpha \in H_1(F;\mathbb{Z})$ by a simple closed curve α in F. Since the normal bundle ν_α^F of α in F is one dimensional, the normal bundle ν_α^F has a canonical framing, where the choice of the direction comes from the orientations. Therefore, together with the spin structure on F, this determines a spin structure on α. We define $\Upsilon(\alpha) = 0$ if and only if α is spin null-bordant. Then $\Upsilon\colon H_1(F;\mathbb{Z}/2) \to \mathbb{Z}/2$ gives rise to a quadratic refinement of the intersection form on F, and $\operatorname{Arf}(f^*\varphi)$ is defined to be the Arf invariant of this quadratic form.

By Proposition 2.7, the invariant $\mathfrak{ter}(M)$ arises as follows. Since $\mathfrak{pri}(M) = \mathfrak{sec}(M) = 0$, there exists a map $f\colon M \to K$ that is an isomorphism on fundamental groups. Now $\mathfrak{ter}(M)$ is given by the homomorphism
$$H^2(\pi;\mathbb{Z}/2) \xrightarrow{i^*} H^2(K;\mathbb{Z}/2) \to \mathbb{Z}/2,$$
with $\vartheta \mapsto \operatorname{Arf}(f^* \circ i^*(\vartheta))$. This completes our description of $\mathfrak{ter}(M)$. Now we show that the three enumerated statements above hold, in the case that the splitting s comes from a map $M \to M^{(2)} = K$.

Proposition 6.8. *If there is a map $f\colon M \to K$ that is an isomorphism on fundamental groups, then for every $\varphi \in \ker \operatorname{Sq}^2 \subseteq H^2(B\pi;\mathbb{Z}/2)$ and every lift $\varphi' \in H^2(K;\mathbb{Z}\pi)$ of $i^*\varphi \in H^2(K;\mathbb{Z}/2)$, the element $PD(f^*\varphi') \in H_2(M;\mathbb{Z}\pi) \cong \pi_2(M)$ is $\mathbb{RP}^2$-characteristic and has trivial self-intersection number. Hence $\tau(PD(f^*(\varphi'))) \in \mathbb{Z}/2$ is well-defined. Moreover, $\tau(PD(f^*\varphi'))$ agrees with $\operatorname{Arf}(f^*i^*(\varphi))$.*

Since Arf defines a homomorphism, it follows from Proposition 6.8 that so does $\tau_{M,s}$. Moreover, the Arf invariant coincides with the tertiary obstruction, and this is independent of the section s, so $\tau_{M,s}$

is independent of s too. So we know the Tertiary property whenever we know that the algebraic splitting s is geometrically realised, in the sense that it is induced from a map $f\colon M \to K$ as in Proposition 6.8. The strategy for proving the Tertiary Property that we will pursue after the proof of Proposition 6.8 mirrors the strategy for the Secondary Property. We will give an algebraic criterion, in Theorem 6.21, under which the algebraic splitting s is geometrically realised, so that Proposition 6.8 applies to prove the Tertiary property for π. In Sections 9 and 10, we will then verify the condition in Theorem 6.21 for several families of groups.

The proof of Proposition 6.8 will be broken up into a series of lemmas, and will take most of the next three pages. First we make sure that τ is well-defined on the elements $PD(f^*(\varphi'))$ that we want to compute it on.

Lemma 6.9. *The element $PD(f^*(\varphi')) \in H_2(M;\mathbb{Z}\pi) = \pi_2(M)$ is $\mathbb{RP}^2$-characteristic.*

Proof. This follows directly from Lemma 6.1 applied to the composition $f \circ j\colon M^{(3)} \to K$, where as in that lemma $j\colon M^{(3)} \to M$ is the inclusion of the 3-skeleton. □

Lemma 6.10. *For every $x, y \in H^2(K;\mathbb{Z}\pi)$, we have that $\lambda(PD(f^*(x)), PD(f^*(y))) = 0$. In particular, the self-intersection number $\mu(PD(f^*(x))) = 0$.*

Proof. Since K is 2-dimensional we have $f_*([M]) = 0$, and thus

$$\begin{aligned}\lambda(PD(f^*(x)), PD(f^*(y))) &= \langle f^*(y), PD(f^*(x))\rangle = \langle y, f_*(f^*(x) \cap [M])\rangle \\ &= \langle y, x \cap f_*([M])\rangle = \langle y, 0\rangle = 0.\end{aligned}$$ □

We have to prove that the Arf invariant $\mathrm{Arf}(f^*i^*(\varphi))$ coincides with the τ invariant $\tau(PD(f^*(\varphi')))$. For this we will use the description of τ for π_1-trivial (embedded) surfaces from Section 5.2. The definition of τ uses a quadratic refinement $\varpi\colon H_1(F;\mathbb{Z}/2) \to \mathbb{Z}/2$ of the $\mathbb{Z}/2$-intersection form of F that uses, for each curve on F, a relative Euler number and a count of intersections. We will show that the relative Euler number agrees with the spin bordism class that determines the quadratic refinement Υ that we use for computing the Arf invariant. Then we will show that the intersection component of the quadratic refinement ϖ for τ is always even, so does not contribute to the computation of τ. It will follow that the Arf and τ invariants coincide.

Since $\widetilde{K}$ is 2-dimensional, an element $\varphi' \in H^2(K;\mathbb{Z}\pi) \cong H^2_{cs}(\widetilde{K};\mathbb{Z})$ can be represented as a map $\varphi'\colon \widetilde{K} \to S^2$ with compact support (i.e. the closure of the inverse image of $S^2 \setminus \{*\}$ is compact). This follows from the fact that $\widetilde{K}$ is 2-dimensional, and the definition of cohomology with compact support as a colimit of $H^2(\widetilde{K}, \widetilde{K} \setminus L)$ over compact subsets $L \subseteq \widetilde{K}$.

Let $\widetilde{f}\colon \widetilde{M} \to \widetilde{K}$ be a lift of the map $f\colon M \to K$. Let $x \in S^2$ be a regular value of $\varphi' \circ \widetilde{f}\colon \widetilde{M} \to S^2$. Then, as we will prove in the next lemma, $F := (\varphi' \circ \widetilde{f})^{-1}(x) \subseteq \widetilde{M}$ represents $PD(\widetilde{f}^*(\varphi')) \in H_2(\widetilde{M};\mathbb{Z}) \cong H_2(M;\mathbb{Z}\pi)$. Furthermore, F is an embedded surface, and since $\pi_1(\widetilde{M}) = 0$, F is π_1-trivial.

Lemma 6.11. *The inverse image of $x \in S^2$ is a representative for $PD(\widetilde{f}^*(\varphi'))$.*

Proof. For Y a compact 4-manifold, a cohomology class y in $H^2(Y;\mathbb{Z}\pi)$ is represented by a map $f_y\colon Y \to \mathbb{CP}^2$, and the inverse image of $\mathbb{C}P^1 \subset \mathbb{C}P^2$ is the Poincaré dual to the original class y.

We may take the compact sets L in the colimit defining cohomology with compact support to be codimension zero manifolds with boundary in $\widetilde{M}$. Then for Poincaré duality, we have

$$\begin{aligned} H^2_{cs}(\widetilde{M}) = \operatorname*{colim}_L H^2(\widetilde{M}, \widetilde{M} \setminus L) &\cong \operatorname*{colim}_L H^2(L, \partial L) \\ &\cong \operatorname*{colim}_L H_2(L) \cong H_2(\widetilde{M}). \end{aligned}$$

where L belongs to the collection of compact subsets of $\widetilde{M}$ ordered by inclusions. Note that π acts on the compact subsets via the deck transformations. Tracing these isomorphisms, it follows from the corresponding fact for compact manifolds that the inverse image of $\mathbb{C}P^1 \subset \mathbb{C}P^2$ of the map $\varphi' \circ \widetilde{f} \colon \widetilde{M} \to \mathbb{C}P^2$ is the Poincaré dual to $f^*(\varphi')$. Then observe that, since the map $\varphi' \circ \widetilde{f} \colon \widetilde{M} \to \mathbb{C}P^2$ factors through S^2, the inverse image of a generic $\mathbb{C}P^1$ is the inverse image of a point in S^2. This completes the proof that the inverse image of $x \in S^2$ is a representative for $PD(\widetilde{f}^*(\varphi'))$. $\square$

We can perturb the map φ' such that $(\varphi')^{-1}(x) \subseteq \widetilde{K}$ is a finite discrete set and no two points have the same image under $\widetilde{K} \to K$. Then the image of F under $\widetilde{M} \to M$ is still an embedded π_1-trivial surface, which we again denote by F.

Lemma 6.12. *Let α be a simple closed curve on F. The spin bordism class of α, as an element of $\Omega_1^{Spin} \cong \mathbb{Z}/2$, is equal to the relative Euler number $e(C)$ for any generically immersed disc C in M with boundary α.*

Proof. Let $(v_1, v_2) \in T_xS^2 \oplus T_xS^2$ be a framing of the point x. Let $C \colon D^2 \to M$ be an generic null-homotopy of α. Use boundary twists [FQ90, Section 1.3] to arrange that the image of the normal vector of $S^1 \subseteq D^2$ in $T_{C(y)}M \cong T_{C(y)}F \oplus \nu_F^M|_{C(y)}$ agrees with $(0, (\varphi' \circ f)^*v_1)$ for every $y \in S^1$. The spin structure on α is given by the following framing on

$$\nu_\alpha^{\mathbb{R}^\infty} = \nu_\alpha^F \oplus \nu_F^M|_\alpha \oplus \nu_M^{\mathbb{R}^\infty}|_\alpha. \tag{6.13}$$

The bundle ν_α^F is 1-dimensional, and thus we obtain a canonical framing w from the orientation. On ν_F^M, we have the framing $((\varphi' \circ f)^*v_1, (\varphi' \circ f)^*v_2)$, while on $\nu_M^{\mathbb{R}^\infty}|_\alpha$ we take the framing $\underline{w}$ coming from the spin structure of M restricted to α.

We can also obtain the following framing on

$$\nu_\alpha^{\mathbb{R}^\infty} = \nu_\alpha^C \oplus \nu_C^M|_\alpha \oplus \nu_M^{\mathbb{R}^\infty}|_\alpha \tag{6.14}$$

coming from C. The bundle ν_α^C is again 1-dimensional, and by definition agrees with $(\varphi' \circ f)^*v_1$. On ν_C^M we have a canonical framing (c_1, c_2) coming from the orientation, since C is contractible. On $\nu_M^{\mathbb{R}^\infty}|_\alpha$, we again take the framing coming from the spin structure of M. When considering the spin structure on $\nu_C^{\mathbb{R}^\infty} = \nu_C^M \oplus \nu_M^{\mathbb{R}^\infty}$ given by the above framing of ν_C^M, and the spin structure of M restricted to C, this defines a spin null bordism of α, with the spin structure on α given by the framing arising from the decomposition (6.14) above. Note that to define the spin structure induced on the boundary S^1 we have to add the bundle ν_α^C, given the inwards pointing orientation.

Since $(w, (\varphi' \circ f)^*v_1, (\varphi' \circ f)^*v_2, \underline{w})$ is spin null bordant if and only if $((\varphi' \circ f)^*v_1, (\varphi' \circ f)^*v_2, w, \underline{w})$ is spin null bordant (because they only differ by an element in $SO(3)$ that is constant around S^1), the spin structure from the first framing, i.e. the framing arising from the decomposition (6.13), is null bordant if and only if the framing $((\varphi' \circ f)^*v_2, w, \underline{w}')$ can be extended over C. Since ν_C^M is 2-dimensional, the normal vector w extends over C if and only if $(\varphi' \circ f)^*v_2$ extends over C. Thus $((\varphi' \circ f)^*v_2, w)$ can stably be extended over C if and only if the relative Euler number is even, so is zero modulo 2. This completes the proof of the lemma. $\square$

We have one final lemma for the proof of Proposition 6.8.

Lemma 6.15. *The interior of the image of C intersects F transversally in an even number of points.*

Proof. The image of the boundary S^1 under $f \circ C \colon D^2 \to K$ is a point. Thus $f \circ C$ factors as $f \circ C \colon D^2 \to S^2 \xrightarrow{j} K$, where j is defined by this factorisation.

Recall that we have a map $\varphi' \colon \widetilde{K} \to S^2$ representing $\varphi' \in H^2_{cs}(\widetilde{K};\mathbb{Z}) \cong H^2(K;\mathbb{Z}\pi)$ that lifts $i^*\varphi \in H^2(K;\mathbb{Z}/2)$, where $i\colon K \to B\pi$ is the inclusion of the 2-skeleton. Let $p\colon \widetilde{K} \to K$ be the projection and define a map $\widehat{\varphi}\colon K \to S^2$ that sends the points $p((\varphi')^{-1}(x))$ to x and sends everything outside a small neighbourhood of these points to the base point of S^2. Then $\widehat{\varphi}\colon K \to S^2$ composed with the inclusion $\varrho\colon S^2 \to K(\mathbb{Z}/2, 2)$ represents $i^*\varphi \in H^2(K;\mathbb{Z}/2)$.

Since the normal bundle of $S^1 \subseteq D^2$ under $dC\colon TD^2 \to TM$ agrees with the direction of $(\varphi' \circ f)^* v_1$, and F is the preimage of the points $p((\varphi')^{-1}(x))$, the mapping degree of $\widehat{\varphi} \circ j\colon S^2 \to S^2$ agrees with the number of transverse intersections of C with F.

Compose $\widehat{\varphi} \circ j\colon S^2 \to S^2$ with the inclusion of the 2-skeleton $\varrho\colon S^2 \to K(\mathbb{Z}/2, 2)$. The map $\varrho \circ \widehat{\varphi} \circ j\colon S^2 \to K(\mathbb{Z}/2,2)$ factors through $B\pi$ by definition of $\widehat{\varphi} = i^*\varphi$, and therefore is null homotopic, since $B\pi$ is aspherical. It follows that the mapping degree of $\widehat{\varphi} \circ j$ is even, which proves the lemma. □

Proof of Proposition 6.8. We proved in Lemma 6.9 that $PD(f^*(\varphi'))$ is $\mathbb{RP}^2$-characteristic, and we showed in Lemma 6.10 that $\mu(PD(f^*(\varphi'))) = 0$. Recall that we have to show $\mathrm{Arf}(F) = \tau(F)$ for the surface F defined above Lemma 6.11. Now, the Arf invariant of F depends only on the relative Euler numbers of discs bounding curves on F, whereas the τ invariant depends on the relative Euler number *and* the intersections of the form $C \pitchfork F$. Lemma 6.15 shows that the latter do not contribute. Lemma 6.15 uses the hypothesis of the Proposition 6.8 that there is a map $M \to K$ that induces an isomorphism on fundamental groups. Therefore we have $\mathrm{Arf}(F) = \tau(F)$, as desired, which completes the proof. □

Proposition 6.8 shows that, as we are aiming for, $\tau_{M,s}$ agrees with $\mathfrak{ter}(M)$, when there is a map $M \to K$ inducing the given splitting. Now we give an algebraic criterion guaranteeing that such a map $M \to K$ exists. Note that if we did not want to show independence on the choice of splitting, we would be done. But for our obstructions to be computationally useful, it ought to be possible to choose any splitting.

Definition 6.16. Let N be a $\mathbb{Z}\pi$-module that is free as a $\mathbb{Z}$-module. Then

$$\varpi_N\colon \mathbb{Z} \otimes_{\mathbb{Z}\pi} \Gamma(N) \to N \otimes_{\mathbb{Z}\pi} N$$

denotes the map induced from the inclusion $\omega\colon \Gamma(N) \to N \otimes_{\mathbb{Z}} N$ by tensoring with $\mathbb{Z}$ over $\mathbb{Z}\pi$. As per our convention, we view the first N as a right $\mathbb{Z}\pi$-module using the involution. Here $\mathbb{Z} \otimes_{\mathbb{Z}\pi} (N \otimes_{\mathbb{Z}} N) \cong N \otimes_{\mathbb{Z}\pi} N$, since $\mathbb{Z}\pi$ acts on $N \otimes_{\mathbb{Z}} N$ on the left by the diagonal action.

Theorem 6.17. *Let M be a spin 4-manifold with $\mathfrak{pri}(M) = 0 = \mathfrak{sec}(M)$, and let $s\colon \operatorname{coker} d^2 \to C^2 \oplus H^2(M;\mathbb{Z}\pi)$ be a splitting such that $\lambda_M|_{\mathrm{im}(PD \circ s_2)} \equiv 0$ vanishes. If*

$$\Phi_{H^2(M^{(2)};\mathbb{Z}\pi)} \circ \varpi_{\pi_2(M^{(2)})}\colon \mathbb{Z} \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2(M^{(2)})) \to \mathrm{Sesq}(H^2(M^{(2)};\mathbb{Z}\pi))$$

is injective, then the map $\tau_{M,s}$ is a homomorphism, and maps to $\mathfrak{ter}(M)$ under

$$\mathrm{Hom}(\ker(\mathrm{Sq}^2), \mathbb{Z}/2) \cong \mathrm{coker}(\mathrm{Sq}_2) \cong E^3_{2,2} \to E^\infty_{2,2} = H_2(\pi;\mathbb{Z}/2)/\operatorname{im}(d^2_{4,1}, d^3_{5,0}).$$

Moreover, the image of $\tau_{M,s}$ in $E^\infty_{2,2}$ does not depend on the choice of s.

Proof. By Proposition 4.21, the splitting $s\colon \operatorname{coker} d^2 \to C^2 \oplus H^2(M;\mathbb{Z}\pi)$ can be realised as the splitting induced from a map $f\colon M^{(3)} \to M^{(2)}$, where f_* is the identity on π_1. Here a splitting is said to be realised by f if $s_2 = f^*\colon H^2(M^{(2)};\mathbb{Z}\pi) \to H^2(M;\mathbb{Z}\pi)$ after identifying $H^2(M^{(3)};\mathbb{Z}\pi) = H^2(M;\mathbb{Z}\pi)$.

The right hand side of the equation in Lemma 4.28 vanishes, since the intersection form vanishes on the $\operatorname{coker} d^2 = H^2(M^{(2)};\mathbb{Z}\pi)$ summand of $\pi_2(M)$. Thus the left hand side of the equation in Lemma 4.28 vanishes, and then by injectivity of $\Phi_{H^2(M^{(2)};\mathbb{Z}\pi)} \circ \varpi_{\pi_2(M^{(2)})}$, we have that $f_*(\alpha) = 0 \in \mathbb{Z} \otimes_{\mathbb{Z}\pi} \pi_3(M^{(2)})$; recall that α denotes the attaching map of the 4-handle of M. Here we also use the identification $S\colon \mathbb{Z} \otimes_{\mathbb{Z}\pi} \pi_3(M^{(2)}) \xrightarrow{\cong} \mathbb{Z} \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2(M^{(2)}))$ from diagram 4.26.

We claim that the images $x_n \in \mathbb{Z}\pi$ of $1 \in \mathbb{Z}\pi$ under $\mathbb{Z}\pi \cong C_4 \xrightarrow{d_4} C_3 \cong \mathbb{Z}\pi^n \xrightarrow{p_n} \mathbb{Z}\pi$, where $p_n\colon \mathbb{Z}\pi^n \to \mathbb{Z}\pi$ are the projections, generate the augmentation ideal $I\pi \subseteq \mathbb{Z}\pi$. This can be seen as follows. By Poincaré duality the cokernel of the dual map $C^3 \xrightarrow{d^4} C^4$ is $\mathbb{Z}$ and hence the image of d^4 is the augmentation ideal. The image is generated by $\{\overline{x_n}\}$ (when we view C^4 as a left module using the involution on $\mathbb{Z}\pi$). By dualising, we obtain the claim.

By changing $f\colon M^{(3)} \to M^{(2)}$ on the 3-cells, it follows from the claim that we can change $f_*(\alpha) \in \pi_3(M^{(2)})$ by elements of $I\pi \otimes_{\mathbb{Z}\pi} \pi_3(M^{(2)})$. Thus an extension $f'\colon M \to M^{(2)}$ exists if and only if $f_*(\alpha) = 0 \in \mathbb{Z} \otimes_{\mathbb{Z}\pi} \pi_3(M^{(2)})$. But that is exactly what we have, so such a map $f'\colon M \to M^{(2)}$ indeed exists.

Therefore s is realised geometrically, not just in the sense that we have $s_2 = f^*\colon H^2(M^{(2)};\mathbb{Z}\pi) \to H^2(M^{(3)};\mathbb{Z}\pi)$, but now in the stronger sense that s_2 is induced from a map $f\colon M \to M^{(2)}$. By Proposition 6.8, we therefore have $\tau_{M,s}(\varphi) = \operatorname{Arf}(PD(f^*i^*\varphi)))$ for any $\varphi \in \ker \operatorname{Sq}^2 \subseteq H^2(\pi;\mathbb{Z}/2)$. These Arf invariants induce a homomorphism, that is an element of $\operatorname{Hom}(H^2(\pi;\mathbb{Z}/2),\mathbb{Z}/2)$. In light of Proposition 2.7 and Lemma 2.5, this homomorphism maps to $\mathfrak{ter}(M)$ under the composition in the statement of the theorem. The tertiary obstruction $\mathfrak{ter}(M) \in H_2(\pi;\mathbb{Z}/2)/\operatorname{im}(d^2_{4,1}, d^3_{5,0})$ is independent of the choice of s, and therefore so is $\tau_{M,s}$. □

Next, we want to express the injectivity condition of Theorem 6.17 as a property of the group π alone.

Lemma 6.18. *Let N, N' be $\mathbb{Z}\pi$-modules that are free as $\mathbb{Z}$-modules. Then $\varpi_{N\oplus N'}$ is injective if and only if ϖ_N and $\varpi_{N'}$ are both injective.*

Proof. By Lemma 3.9, the inclusion $\Gamma(N \oplus N') \to (N \oplus N') \otimes_{\mathbb{Z}} (N \oplus N')$ is the direct sum of the inclusion $\Gamma(N) \to N \otimes_{\mathbb{Z}} N$, the diagonal map $(1+T)\colon N \otimes_{\mathbb{Z}} N' \to (N \otimes_{\mathbb{Z}} N') \oplus (N' \otimes_{\mathbb{Z}} N)$ and the inclusion $\Gamma(N') \to N' \otimes_{\mathbb{Z}} N'$. Thus, $\varpi_{N\oplus N'}$ is the direct sum of ϖ_N, $\varpi_{N'}$ and the diagonal map $(1+T)\colon N \otimes_{\mathbb{Z}\pi} N' \to (N \otimes_{\mathbb{Z}\pi} N') \oplus (N' \otimes_{\mathbb{Z}\pi} N)$. Since the diagonal map is always injective, the lemma follows. □

Lemma 6.19. *Let P be a finitely generated projective left $\mathbb{Z}\pi$-module. Then ϖ_P is injective.*

Proof. We begin with the case $P = \mathbb{Z}\pi$. The module $\mathbb{Z}\pi \otimes_{\mathbb{Z}} \mathbb{Z}\pi$ is a free $\mathbb{Z}\pi$-module with basis $\{1 \otimes g\}_{g\in\pi}$. The action of $\mathbb{Z}\pi$ is the diagonal action. The module $\Gamma(\mathbb{Z}\pi)$ is generated as a subset of $\mathbb{Z}\pi \otimes_{\mathbb{Z}} \mathbb{Z}\pi$ by the elements $1 \otimes 1$ and $1 \otimes g + g \otimes 1$. Since $1 \otimes g + g \otimes 1 = 1 \otimes g + g(1 \otimes g^{-1})$ it follows that there is a homomorphism

$$\mathbb{Z}\pi \otimes_{\mathbb{Z}} \mathbb{Z}\pi \to \bigoplus_{g\in\pi, g^2\neq 1} \mathbb{Z}\pi \oplus \bigoplus_{g^2=1, g\neq 1} \mathbb{Z}\pi/(1+g).$$

The homomorphism above sends $1 \otimes g$ to the generator of the g summand, with kernel the image of $\Gamma(\mathbb{Z}\pi)$. If $g^2 = 1$, then $1 \otimes g + g \otimes 1 = (1+g)(1 \otimes g)$, whence the quotient by $(1+g)$. Here we think of $(1+g)$ as a left ideal of the ring $\mathbb{Z}\pi$, and form the quotient *ring*. Then we consider the ring $\mathbb{Z}\pi/(1+g)$ as a left $\mathbb{Z}\pi$-module. Therefore we have a short exact sequence

$$0 \to \Gamma(\mathbb{Z}\pi) \to \mathbb{Z}\pi \otimes_{\mathbb{Z}} \mathbb{Z}\pi \to \bigoplus_{g\in\pi, g^2\neq 1} \mathbb{Z}\pi \oplus \bigoplus_{g^2=1, g\neq 1} \mathbb{Z}\pi/(1+g) \to 0.$$

Tensor this with $\mathbb{Z}$ over $\mathbb{Z}\pi$ and apply the 6-term exact sequence to obtain

$$\begin{aligned} &\mathrm{Tor}_1^{\mathbb{Z}\pi}\Big(\mathbb{Z}, \bigoplus_{g\in\pi, g^2\neq 1} \mathbb{Z}\pi \oplus \bigoplus_{g^2=1, g\neq 1} \mathbb{Z}\pi/(1+g)\Big) \to \mathbb{Z} \otimes_{\mathbb{Z}\pi} \Gamma(\mathbb{Z}\pi) \\ &\xrightarrow{\varpi_{\mathbb{Z}\pi}} \mathbb{Z}\pi \otimes_{\mathbb{Z}\pi} \mathbb{Z}\pi \to \bigoplus_{g\in\pi, g^2\neq 1} \mathbb{Z} \oplus \bigoplus_{g^2=1, g\neq 1} \mathbb{Z}/2 \to 0. \end{aligned}$$

A free $\mathbb{Z}\pi$-module resolution of $\mathbb{Z}\pi/(1+g)$, where g has order two, is given by

$$\dots \xrightarrow{1+g} \mathbb{Z}\pi \xrightarrow{1-g} \mathbb{Z}\pi \xrightarrow{1+g} \mathbb{Z}\pi \to \mathbb{Z}\pi/(1+g) \to 0.$$

Tensor this with $\mathbb{Z}$ to obtain

$$\dots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0.$$

The first homology of the corresponding deleted resolution vanishes. As $\mathrm{Tor}_1^{\mathbb{Z}\pi}(-,-)$ commutes with direct sums in the second factor, we obtain

$$\mathrm{Tor}_1^{\mathbb{Z}\pi}\Big(\mathbb{Z}, \bigoplus_{g\in\pi, g^2\neq 1} \mathbb{Z}\pi \oplus \bigoplus_{g^2=1, g\neq 1} \mathbb{Z}\pi/(1+g)\Big) = 0.$$

This concludes the proof in the case $P = \mathbb{Z}\pi$. For a general finitely generated free $\mathbb{Z}\pi$-module the lemma follows from this and Lemma 6.18 by induction on the rank.

If P is finitely generated projective, there exists P' such that $P \oplus P'$ is finitely generated free. So the injectivity of ϖ_P follows from the injectivity of $\varpi_{P\oplus P'}$, again using Lemma 6.18. □

Corollary 6.20. *Let N be a left $\mathbb{Z}\pi$-module that is free as a $\mathbb{Z}$-module. Then the map $\Phi_N \circ \varpi_{N^*}$ is injective if and only if $\Phi_{N\oplus\mathbb{Z}\pi} \circ \varpi_{N^*\oplus\mathbb{Z}\pi^*}$ is injective.*

Proof. By Lemma 4.6 and the definition of ϖ, the map $\Phi_{N\oplus\mathbb{Z}\pi} \circ \varpi_{N^*\oplus\mathbb{Z}\pi^*}$ is the direct sum of the maps $\Phi_N \circ \varpi_{N^*}$, $\Phi_{\mathbb{Z}\pi} \circ \varpi_{\mathbb{Z}\pi^*}$ and $(\Phi_{N,\mathbb{Z}\pi^*} \oplus \Phi_{\mathbb{Z}\pi,N^*}) \circ (1+T)$. It suffices to argue that the last two maps are injective. Note that $(1+T)\colon N^* \otimes_{\mathbb{Z}\pi} \mathbb{Z}\pi^* \to (N^* \otimes_{\mathbb{Z}\pi} \mathbb{Z}\pi^*) \oplus (\mathbb{Z}\pi^* \otimes_{\mathbb{Z}\pi} N^*)$ is injective, and $\varpi_{\mathbb{Z}\pi^*}$ is injective by Lemma 6.19. Moreover Lemma 4.10 says that $\Phi_{\mathbb{Z}\pi}$, $\Phi_{N,\mathbb{Z}\pi^*}$ and $\Phi_{\mathbb{Z}\pi,N^*}$ are injective. It follows that $\Phi_{\mathbb{Z}\pi} \circ \varpi_{\mathbb{Z}\pi^*}$ and $(\Phi_{N,\mathbb{Z}\pi^*} \oplus \Phi_{\mathbb{Z}\pi,N^*}) \circ (1+T)$ are injective, as required. □

We can now give the desired statement of a condition implying the Tertiary Property 1.12 that depends only on the group π.

Theorem 6.21. *Let K be a finite 2-complex with fundamental group π. Suppose that*

$$\Phi_{H^2(K;\mathbb{Z}\pi)} \circ \varpi_{\pi_2(K)} \colon \mathbb{Z} \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2(K)) \to \mathrm{Sesq}(H^2(K;\mathbb{Z}\pi))$$

is injective. Then π has the Tertiary Property 1.12.

Proof. The theorem with $K = M^{(2)}$ follows from Theorem 6.17. Since any two choices of K are homotopy equivalent after wedging with enough copies of S^2, it remains to check that the injectivity condition is preserved under wedging with S^2. But this follows from Corollary 6.20. Therefore the injectivity condition is independent of the choice of 2-complex K, as desired. □

6.1. **The Tertiary Property for finite groups.** We will prove the following corollary of Theorem 6.21, which will enable us to verify that many finite groups have the Tertiary Property 1.12.

Corollary 6.22. *Let π be a finite group and let K be a finite 2-dimensional CW complex with fundamental group π. If $\mathbb{Z}\otimes_{\mathbb{Z}\pi}\Gamma(\pi_2(K))$ is torsion-free, then π has the Tertiary Property 1.12.*

We will prove the corollary via a couple of lemmas.

Let π be a finite group and let M be a left $\mathbb{Z}\pi$-module. Consider the map of abelian groups

$$\begin{aligned} F_M\colon M^*\otimes_{\mathbb{Z}} M^* &\to \operatorname{Hom}_{\mathbb{Z}\pi}(M,\operatorname{Hom}_{\mathbb{Z}\pi}(M,\mathbb{Z}\pi\otimes_{\mathbb{Z}}\mathbb{Z}\pi)) \\ f\otimes h &\mapsto (m\mapsto (n\mapsto f(n)\otimes\overline{h(m)})). \end{aligned}$$

Here we view $\mathbb{Z}\pi\otimes\mathbb{Z}\pi$ as a left $\mathbb{Z}\pi$-module via left multiplication in the first factor. The group $\operatorname{Hom}_{\mathbb{Z}\pi}(M,\mathbb{Z}\pi\otimes_{\mathbb{Z}}\mathbb{Z}\pi)$ is a right $\mathbb{Z}\pi$-module via right multiplication in the second factor of $\mathbb{Z}\pi\otimes\mathbb{Z}\pi$ and as usual we view it as a left $\mathbb{Z}\pi$-module using the involution.

Lemma 6.23. *Let π be a finite group and let M be a finitely generated left $\mathbb{Z}\pi$-module. Then the map F_M is injective.*

Proof. Consider the abelian group homomorphism $\operatorname{ev}_1\colon M^*\to\operatorname{Hom}_{\mathbb{Z}}(M,\mathbb{Z})$ given by evaluation at the neutral group element. This map has an inverse given by sending f to the map $(x\mapsto\sum_{g\in\pi}f(g^{-1}x)g)$. Consider the commutative diagram

$$\begin{tikzcd} M^*\otimes M^* \arrow[r, "F_M"] \arrow[d, "\cong"', "\operatorname{ev}_1\otimes\operatorname{ev}_1"] & \operatorname{Hom}_{\mathbb{Z}\pi}(M,\operatorname{Hom}_{\mathbb{Z}\pi}(M,\mathbb{Z}\pi\otimes_{\mathbb{Z}}\mathbb{Z}\pi)) \arrow[d, "\operatorname{ev}_{1\otimes 1}"] \\ \operatorname{Hom}_{\mathbb{Z}}(M,\mathbb{Z})\otimes\operatorname{Hom}_{\mathbb{Z}}(M,\mathbb{Z}) \arrow[r] & \operatorname{Hom}_{\mathbb{Z}}(M,\operatorname{Hom}_{\mathbb{Z}}(M,\mathbb{Z})). \end{tikzcd}$$

Since $\operatorname{Hom}_{\mathbb{Z}}(M,\mathbb{Z})$ is a free $\mathbb{Z}$-module, the lower horizontal map is an isomorphism. Hence F_M is injective. ☐

Recall that we have an inclusion map $\omega\colon\Gamma(M^*)\to M^*\otimes_{\mathbb{Z}}M^*$, and that $\varpi\colon\mathbb{Z}\otimes_{\mathbb{Z}\pi}\Gamma(M^*)\to M^*\otimes_{\mathbb{Z}\pi}M^*$ denotes the map induced after tensoring with $\mathbb{Z}$. In the codomain, as per the conventions stipulated at the start of Section 4.2, the first M^* in the tensor product is a right $\mathbb{Z}\pi$-module and the second M^* is a left $\mathbb{Z}\pi$-module. We obtain the following commutative diagram, where the right-hand vertical map is induced by the $(\mathbb{Z}\pi,\mathbb{Z}\pi)$-bimodule homomorphism $\mathbb{Z}\pi\otimes_{\mathbb{Z}}\mathbb{Z}\pi\to\mathbb{Z}\pi$, $x\otimes y\mapsto xy$, where as above the left $\mathbb{Z}\pi$ action is via left multiplication in the first factor.

$$\begin{tikzcd} \Gamma(M^*) \arrow[r, hook, "\omega"] \arrow[d] & M^*\otimes_{\mathbb{Z}}M^* \arrow[r, hook, "F_M"] \arrow[d] & \operatorname{Hom}_{\mathbb{Z}\pi}(M,\operatorname{Hom}_{\mathbb{Z}\pi}(M,\mathbb{Z}\pi\otimes_{\mathbb{Z}}\mathbb{Z}\pi)) \arrow[d] \\ \mathbb{Z}\otimes_{\mathbb{Z}\pi}\Gamma(M^*) \arrow[r, "\varpi"] & M^*\otimes_{\mathbb{Z}\pi}M^* \arrow[r, "\Phi_M"] & \operatorname{Sesq}(M) \end{tikzcd}$$

Let $N_\pi\in\mathbb{Z}\pi$ be the norm element $N_\pi:=\sum_{g\in\pi}g$. It is easy to see that the map

$$\operatorname{tr}_M\colon\mathbb{Z}\otimes_{\mathbb{Z}\pi}M\to M,\quad 1\otimes m\mapsto N_\pi\cdot m$$

is well-defined. We obtain a map

$$\operatorname{tr}_{\mathrm{Sesq}}\colon\operatorname{Sesq}(M)\to\operatorname{Hom}_{\mathbb{Z}\pi}(M,\operatorname{Hom}_{\mathbb{Z}\pi}(M,\mathbb{Z}\pi\otimes_{\mathbb{Z}}\mathbb{Z}\pi))$$

by applying $\operatorname{tr}\colon\mathbb{Z}\pi\cong\mathbb{Z}\pi\otimes_{\mathbb{Z}\pi}\mathbb{Z}\pi\to\mathbb{Z}\pi\otimes_{\mathbb{Z}}\mathbb{Z}\pi$. Then we also have that

$$F_M\circ\omega\circ\operatorname{tr}_{\Gamma(M^*)}=\operatorname{tr}_{\mathrm{Sesq}}\circ\varpi\circ\Phi_M.$$

Lemma 6.24. *Let K be a connected, finite 2-complex with finite fundamental group π. Then the kernel of the map*

$$\Phi_{H^2(K;\mathbb{Z}\pi)} \circ \varpi \colon \mathbb{Z} \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2(K)) \to \operatorname{Sesq}(H^2(K;\mathbb{Z}\pi))$$

is the torsion in $\mathbb{Z} \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2(K))$.

Proof. We will apply the discussion above with $M = H^2(K;\mathbb{Z}\pi) \cong \pi_2(K)^*$, so that $M^* = \pi_2(K)$ by Lemma 4.12. Since $\operatorname{Sesq}(H^2(K;\mathbb{Z}\pi))$ is torsion free, all torsion elements have to lie in the kernel. Now suppose $a \in \mathbb{Z} \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2(K))$ lies in the kernel of $\Phi_{H^2(K;\mathbb{Z}\pi)} \circ \varpi$. Then

$$\begin{aligned} 0 &= \operatorname{tr}_{\operatorname{Sesq}} \circ \Phi_{H^2(K;\mathbb{Z}\pi)} \circ \varpi(a) \\ &= F_{H^2(K;\mathbb{Z}\pi)} \circ \omega \circ \operatorname{tr}_{\Gamma(\pi_2(K))}(a). \end{aligned}$$

Since $F_{H^2(K;\mathbb{Z}\pi)}$ and ω are injective, this implies that $N_\pi a = 0$ and thus $0 = |\pi| a \in \mathbb{Z} \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2 K)$. In particular, a is a torsion element. $\square$

Corollary 6.22 now follows immediately from Lemma 6.24 and Theorem 6.21.

7. Inheritance results

In this section we give inheritance results for the secondary and tertiary properties. These will be be used in the sequel to prove the corresponding properties for families of groups.

7.1. **Secondary property inheritance.** We will actually prove an inheritance result for Condition 4.17 instead of the Secondary Property 1.8.

As above, let K be a finite, connected 2-complex with $\pi_1(K) = \pi$. Let (C_*, d_*) be the cellular chain complex $C_*(K;\mathbb{Z}\pi)$, let (C^*, d^*) be the dual complex, and let $H := H^2(K;\mathbb{Z}\pi)$.

Definition 7.1. If $\varphi\colon \pi \to G$ is a group homomorphism and K_G a finite connected 2-complex with $\pi_1(K_G) = G$, we obtain a homomorphism

$$\varphi_* \colon \widehat{H}^0(\operatorname{Sesq}(H)) \to \widehat{H}^0(\operatorname{Sesq}(H^2(K_G;\mathbb{Z}G)))$$

as follows. First construct a homomorphism

$$\begin{array}{rcl} \widehat{H}^0(\operatorname{Sesq}(H)) & \to & \widehat{H}^0(\operatorname{Sesq}(\mathbb{Z}G \otimes C^2/\operatorname{im}(\operatorname{Id}_{\mathbb{Z}G} \otimes d^2))) \\ \lambda & \mapsto & [a \otimes x] \otimes [a' \otimes x'] \mapsto a\varphi(\lambda(x,x'))\overline{a'}. \end{array}$$

Then compose this with the map on $\widehat{H}^0(\operatorname{Sesq}(-))$ induced by $H^2(K_G;\mathbb{Z}G) \to H^2(K;\mathbb{Z}G) \cong \mathbb{Z}G \otimes C^2/\operatorname{im}(\operatorname{Id}_{\mathbb{Z}G} \otimes d^2)$.

Recall that for a finitely presented group π we defined a map $A\colon H_3(\pi;\mathbb{Z}/2) \to \widehat{H}^0(\operatorname{Sesq}(H^2(K_\pi;\mathbb{Z}\pi)))$ in Definition 4.15.

Lemma 7.2. *The following square commutes*

$$\begin{tikzcd} H_3(\pi;\mathbb{Z}/2) \arrow[r, "\varphi_*"] \arrow[d, "A"] & H_3(G;\mathbb{Z}/2) \arrow[d, "A"] \\ \widehat{H}^0(\operatorname{Sesq}(H)) \arrow[r, "\varphi_*"] & \widehat{H}^0(\operatorname{Sesq}(H^2(K_G;\mathbb{Z}G))) \end{tikzcd}$$

Proof. By tracing the definition of the map $A\colon H_3(\pi;\mathbb{Z}/2) \to \widehat{H}^0(\mathrm{Sesq}(H^2(K;\mathbb{Z}\pi)))$, via the maps ι, I, Θ and Φ, it is not too hard to see, for $c \in C_3(B\pi;\mathbb{Z}\pi)$, $m, m' \in C^2 = C^2(K;\mathbb{Z}\pi)$ representatives of cohomology classes in $H^2(K;\mathbb{Z}\pi)$, that

$$A([1\otimes c]) = \big(([m],[m']) \mapsto \overline{m(\partial_3(c))}m'(\partial_3(c)) \in \mathbb{Z}\pi\big).$$

Here $\partial_3\colon C_3(B\pi;\mathbb{Z}\pi) \to C_2(B\pi;\mathbb{Z}\pi)$ is the boundary map. The map

$$\varphi_*\colon C_*(B\pi;\mathbb{Z}\pi) \to \mathbb{Z}G \otimes_{\mathbb{Z}\pi} C_*(B\pi;\mathbb{Z}\pi) = C_*(B\pi;\mathbb{Z}G) \to C_*(BG;\mathbb{Z}G)$$

is given by reducing coefficients followed by the chain map induced by the map of spaces $B\pi \to BG$. In the following formulae, we also use the induced map

$$\varphi^*\colon C^2(BG;\mathbb{Z}G) \to C^2(B\pi;\mathbb{Z}G) = \mathrm{Hom}_{\mathbb{Z}\pi}(C_2(B\pi;\mathbb{Z}\pi),\mathbb{Z}G) \cong \mathbb{Z}G \otimes_{\mathbb{Z}\pi} C^2(B\pi;\mathbb{Z}\pi).$$

Then we compute, with $n, n' \in C^2(K_G;\mathbb{Z}G) = C^2(BG;\mathbb{Z}G)$, as follows.

$$\begin{aligned}
A(\varphi_*([1\otimes c])) &= A([1\otimes \varphi_*(c)]) \\
&= \big(([n],[n']) \mapsto \overline{n(\partial_3(\varphi_*(c)))}n'(\partial_3(\varphi_*(c)))\big) \\
&= \big(([n],[n']) \mapsto \overline{n(\varphi_*(\partial_3(c)))}n'(\varphi_*(\partial_3(c)))\big) \\
&= \big(([n],[n']) \mapsto \big(\overline{\varphi^*(n)(\partial_3(c))}\varphi^*(n')(\partial_3(c))\big)\big) \\
&= \varphi_*\big(([m],[m']) \mapsto \overline{m(\partial_3(c))}m'(\partial_3(c))\big) \\
&= \varphi_*(A([1\otimes c])).
\end{aligned}$$
□

Theorem 7.3. *Let G, G' be finitely presented groups satisfying Condition 4.17.*

(1) Let $p\colon \pi \to G'$ be a homomorphism, and let $x \in H_3(\pi;\mathbb{Z}/2)$ be such that $p_*(x)$ does not lie in the image of $Sq_2 \circ \mathrm{red}_2$. Then $A(x) \neq 0$.

(2) Let $i\colon G \to \pi$ be a homomorphism and let $x \in H_3(G;\mathbb{Z}/2)$ be in the image of $Sq_2 \circ \mathrm{red}_2$. Then $A(i_*x) = 0$.

Moreover, if $\pi \cong N \rtimes G$ for some normal subgroup N of π and $x \in H_3(G;\mathbb{Z}/2)$, then $i_*(x) \in H_3(\pi;\mathbb{Z}/2)$ is in the kernel of A if and only if it is in the image of $\mathrm{Sq}_2 \circ \mathrm{red}_2$. Here $i\colon G \to \pi$ denotes the inclusion of G into π that sends $g \mapsto (1_N, g)$.

Proof. Let $K, K_{G'}$ and K_G be finite 2-complexes with fundamental groups π, G' and G respectively. Consider the following diagram

$$\begin{tikzcd}
H_5(G;\mathbb{Z}) \arrow[r, "i_*"] \arrow[d, "\mathrm{Sq}_2\circ\mathrm{red}_2"] & H_5(\pi;\mathbb{Z}) \arrow[r, "p_*"] \arrow[d, "\mathrm{Sq}_2\circ\mathrm{red}_2"] & H_5(G';\mathbb{Z}) \arrow[d, "\mathrm{Sq}_2\circ\mathrm{red}_2"] \\
H_3(G;\mathbb{Z}/2) \arrow[r, "i_*"] \arrow[d, "A"] & H_3(\pi;\mathbb{Z}/2) \arrow[r, "p_*"] \arrow[d, "A"] & H_3(G';\mathbb{Z}/2) \arrow[d, "A"] \\
\widehat{H}^0(\mathrm{Sesq}(H^2(K_G;\mathbb{Z}G))) \arrow[r, "i_*"] & \widehat{H}^0(\mathrm{Sesq}(H^2(K;\mathbb{Z}\pi))) \arrow[r, "p_*"] & \widehat{H}^0(\mathrm{Sesq}(H^2(K_{G'};\mathbb{Z}G')))
\end{tikzcd}$$

The bottom two squares commute by Lemma 7.2, and the top two squares commute by naturality of Sq_2 and red_2. The left and right columns are exact by assumption, but the middle column need not be. The enumerated items in the theorem now follow from Theorem 4.19 and a diagram chase.

To see the last part, note that the homomorphism $i\colon G \to \pi = N \rtimes G$ is a splitting for the quotient map $Q\colon \pi \to \pi/N \cong G$, that is $Q \circ i = \mathrm{Id}_G$. The map Q induces a diagram similar to the diagram above.

$$\begin{tikzcd}
H_5(G;\mathbb{Z}) \arrow[r, bend left, "i_*"] \arrow[d, "\mathrm{Sq}_2 \circ \mathrm{red}_2"] & H_5(\pi;\mathbb{Z}) \arrow[l, bend left, "Q_*"] \arrow[d, "\mathrm{Sq}_2 \circ \mathrm{red}_2"] \\
H_3(G;\mathbb{Z}/2) \arrow[r, bend left, "i_*"] \arrow[d, "A"] & H_3(\pi;\mathbb{Z}/2) \arrow[l, bend left, "Q_*"] \arrow[d, "A"] \\
\widehat{H}^0(\mathrm{Sesq}(H^2(K_G;\mathbb{Z}G))) \arrow[r, shift left, "i_*"] & \widehat{H}^0(\mathrm{Sesq}(H^2(K;\mathbb{Z}\pi))) \arrow[l, shift left, "Q_*"]
\end{tikzcd}$$

The squares commute with the horizontal arrows going in both directions. The left hand column is exact. Also note that $Q_* \circ i_* = \mathrm{Id}$, but be warned that we say nothing about $i_* \circ Q_*$. Another diagram chase now shows that $i_*(x) \in H_3(\pi;\mathbb{Z}/2)$ is in the kernel of A if and only if it is in the image of $\mathrm{Sq}_2 \circ \mathrm{red}_2$, as required. $\square$

Corollary 7.4. *Condition 4.17 holds for all abelian groups if and only if it holds for all abelian groups with at most three generators.*

Proof. Let E be an abelian group. For every $y \in H_3(E;\mathbb{Z}/2)$, there exists a decomposition $E \cong E' \oplus B$, such that E' has at most 3 generators and y is in the image of i_*, where $i\colon E' \to E$ is the inclusion. Hence the sequence

$$H_5(E;\mathbb{Z}) \xrightarrow{\mathrm{Sq}_2 \circ \mathrm{red}_2} H_3(E;\mathbb{Z}/2) \xrightarrow{A} \widehat{H}^0(\mathrm{Sesq}(H^2(K_E;\mathbb{Z}E)))$$

is "exact at y," that is y lies in the kernel of A if and only if it lies in the image of $\mathrm{Sq}_2 \circ \mathrm{red}_2$, by Theorem 7.3. The corollary now follows from Theorem 4.19. $\square$

7.2. **Tertiary property inheritance.** Before we state our tertiary inheritance theorem, we have a couple of lemmas. The first is a well known fact from group theory.

Lemma 7.5. *Let $p\colon \pi \to G$ be a surjective homomorphism between finitely generated groups π and G. If G is finitely presented, then $\ker(p)$ is finitely normally generated.*

Proof. Let $\pi \cong \langle X = \{x_i\}_{i=1}^n \mid R\rangle$, where R is some set of relations. Let $G \cong \langle X' = \{x_i'\}_{i=1}^{n'} \mid R' = \{w_j'(X')\}_{j=1}^{m'}\rangle$, where w_i' is a word in n' variables and $w_i'(X')$ denotes the word given by inserting x_i' for the ith variable. Let $y_i \in \pi$ be a preimage of x_i' for all $i = 1, \ldots, n'$, $Y = \{y_i\}_{i=1}^{n'}$ and let $y_i = v_i(X)$ for some words v_i. Then

$$\pi \cong \langle X, Y \mid R, \{y_i^{-1}v_i(X)\}_{i=1}^{n'}\rangle.$$

Let $p(x_i) = v_i'(X')$ for some words v_i'. Then

$$G \cong \langle X', p(X) \mid R', \{p(x_i)^{-1}v_i'(X')\}_{i=1}^{n}\rangle.$$

Since $p(Y) = X'$ by definition and the relations of π hold in G we also have

$$G \cong \langle X, Y \mid R, \{y_i^{-1}v_i(X)\}_{i=1}^{n'}, \{w_j'(Y)\}_{j=1}^{m'}, \{x_i^{-1}v_i'(Y)\}_{i=1}^{n}\rangle.$$

To obtain this presentation for G, substitute $p(Y) = X'$ and then remove all the instances of p. In particular, the kernel of p is normally generated by the elements $\{w_j'(X')\}_{j=1}^{m'}$ and $\{x_i^{-1}v_i'(Y)\}_{i=1}^{n}$. $\square$

Lemma 7.6. *Let N be a codimension zero submanifold (possibly with boundary) of a spin 4-manifold M and let $i\colon N \to M$ be the inclusion. Suppose $\alpha \in \pi_2(N)$ has $\mu(\alpha) = 0$ and $i_*(\alpha) \in \pi_2(M)$ is $\mathbb{RP}^2$-characteristic. Then $\tau(\alpha)$ and $\tau(i_*\alpha)$ are well-defined and equal.*

Proof. From $\mu(i_*\alpha) = i_*(\mu(\alpha)) = 0$, it follows that $\tau(i_*\alpha) \in \mathbb{Z}/2$ is well-defined. The number of transverse intersection points between α and a generically immersed $\mathbb{RP}^2$ β is the same as those between $i_*\alpha$ and $i_*\beta$. Therefore α is $\mathbb{RP}^2$-characteristic and $\tau(\alpha) \in \mathbb{Z}/2$ is well-defined.

Pair up the self-intersections of α with Whitney discs in such a way that the intersection of these discs with α computes $\tau(\alpha)$. Then the same Whitney discs can be used to compute $\tau(i_*\alpha)$ inside the bigger manifold M. Hence $\tau(\alpha) = \tau(i_*\alpha)$. $\square$

Theorem 7.7. *Let π be a finitely presented group and assume that there are surjections $p_i\colon \pi \to G_i$ for some finitely presented groups $G_1, \dots, G_n$, such that*

$$\prod_{i=1}^{n} (p_i)_*\colon H_2(\pi; \mathbb{Z}/2)/\operatorname{im}(d^2_{4,1}, d^3_{5,0}) \to \prod_{i=1}^{n} H_2(G_i; \mathbb{Z}/2)/\operatorname{im}(d^2_{4,1}, d^3_{5,0})$$

is an injection. Furthermore, assume that all the G_i have the Tertiary Property 1.12. Then π has the Tertiary Property 1.12.

Proof. Let M be a spin manifold with $\pi_1(M) = \pi$ and $c_*([M]) = 0 \in H_4(B\pi; \mathbb{Z})$. Choose a model for $B\pi$ with finite 2-skeleton. Let $f\colon M^{(3)} \to B\pi^{(2)}$ be a map that induces an isomorphism on fundamental groups, and such that the equivariant intersection form vanishes on the image of

$$f^*\colon H^2(B\pi^{(2)}; \mathbb{Z}\pi) \to H^2(M^{(3)}; \mathbb{Z}\pi) \cong H^2(M; \mathbb{Z}\pi) \cong H_2(M; \mathbb{Z}\pi).$$

In particular, we assume that the primary and secondary obstructions vanish for M.

Recall that $\mathcal{RC} \subset \pi_2(M) \cong H_2(M; \mathbb{Z}\pi)$ denotes the subset of $\mathbb{RP}^2$-characteristic elements on which μ vanishes. Consider the following diagram, in which $H^2(M; \mathbb{Z}\pi)_{\mathcal{RC}}$ denotes the subset $\mathcal{RC} \cap \ker \mu$ of $H^2(M; \mathbb{Z}\pi)$, $H^2(B\pi^{(2)}; \mathbb{Z}\pi)_{\mathcal{RC}}$ denotes the preimage of $H^2(M; \mathbb{Z}\pi)_{\mathcal{RC}}$, and $H^2(B\pi^{(2)}; \mathbb{Z}/2)_{\mathcal{RC}}$ denotes the image of $H^2(M; \mathbb{Z}\pi)_{\mathcal{RC}}$. By Lemma 6.1, the maps land in the subsets claimed.

$$\begin{tikzcd}
 & H^2(B\pi^{(2)}; \mathbb{Z}\pi)_{\mathcal{RC}} \arrow[r, "f^*"] \arrow[d] & H^2(M; \mathbb{Z}\pi)_{\mathcal{RC}} \arrow[r, "\tau"] & \mathbb{Z}/2 \\
H^2(B\pi; \mathbb{Z}/2) \arrow[r] \arrow[ur, dashed] & H^2(B\pi^{(2)}; \mathbb{Z}/2)_{\mathcal{RC}} & &
\end{tikzcd}$$

Make a choice of lift, as shown by the dashed arrow; that is, for each element of $H^2(B\pi^{(2)}; \mathbb{Z}/2)_{\mathcal{RC}}$ choose an element in $H^2(B\pi^{(2)}; \mathbb{Z}\pi)_{\mathcal{RC}}$ that maps to it.

We want to show that

$$\kappa(\tau_f|_{\ker(d^3_{5,0})^*}) = \mathfrak{ter}(M) \in H_2(B\pi; \mathbb{Z}/2)/(\operatorname{im} d^2_{4,1}, \operatorname{im} d^3_{5,0})$$

where the isomorphism $\operatorname{Hom}(H^2(B\pi; \mathbb{Z}/2), \mathbb{Z}/2) \cong H_2(B\pi; \mathbb{Z}/2)$ induces an identification

$$\kappa\colon \operatorname{Hom}(\ker(d^3_{5,0})^*, \mathbb{Z}/2) \xrightarrow{\cong} H_2(B\pi; \mathbb{Z}/2)/(\operatorname{im} d^2_{4,1}, \operatorname{im} d^3_{5,0}).$$

Note that this includes showing that $\tau_f|_{\ker(d^3_{5,0})^*}$ is a homomorphism.

We will also write κ for the corresponding identification with $B\pi$ replaced by BG. Observe that it suffices, by the injectivity hypothesis of Theorem 7.7, to show that for each surjective homomorphism $p\colon \pi \to G$, where G has the Tertiary property, that $p_*\kappa(\tau_f|_{\ker(d^3_{5,0})^*}) = p_*\mathfrak{ter}(M)$. The remainder of the proof verifies this equality.

The element $p_*\mathfrak{ter}(M)$ can be computed via the bordism class $[M \xrightarrow{c} B\pi \xrightarrow{p} BG] \in \Omega_4^{Spin}(BG)$. We want to identify a class in $H_2(BG;\mathbb{Z}/2)$, from the E_2-page of the Atiyah-Hirzebruch spectral sequence for $\Omega_4^{Spin}(BG)$, with a τ invariant, using that G has the Tertiary property. For this, we need a manifold with fundamental group G.

Perform surgery on M along a normal generating set of curves $\coprod S^1 \subset M$ for $\ker(p\colon \pi \to G)$; this is a finite set by Lemma 7.5. This removes, for each surgery, a copy of $D^2 \times S^2$. Note that we will allow ourselves in the future to modify the representative circles normally generating $\ker p$ within their homotopy classes. Let

$$M' := M \setminus \Big(\coprod S^1 \times D^3\Big) \cup_{\coprod S^1\times S^2} \coprod D^2 \times S^2.$$

The gluing map in the surgery arises from the framing of the normal bundle of each S^1 i.e. the identification of a regular neighbourhood of S^1 with $S^1 \times D^3$. We use the unique (up to homotopy) identification for which the spin structure of M extends over M'.

Note that we can arrange for the surgery data $\coprod S^1 \times D^3$ to be contained in $M^{(3)}$. Choose a model for BG with finite 2-skeleton. We obtain a commutative diagram

$$\begin{tikzcd}
M^{(3)} \arrow[dr, "p\circ f"'] & M^{(3)} \setminus \Big(\coprod S^1 \times D^3\Big) \arrow[l, "i_M"'] \arrow[r, "i_{M'}"] \arrow[d, "p\circ f = f'"] & (M')^{(3)} \arrow[dl, "f'"] \\
& BG^{(2)}. &
\end{tikzcd}$$

We will view $M^{(3)} \setminus \Big(\coprod S^1 \times D^3\Big)$ as a manifold with boundary, by only removing the interiors of the $S^1 \times D^3$. In the following diagram, the boundary will be denoted by ∂.

The proof will be based on the following diagram, which we will explain in detail below.

$$\begin{tikzcd}
H^2(BG^{(2)};\mathbb{Z}/2) & H^2(BG;\mathbb{Z}/2) \arrow[l, "i^*"'] & & \\
H^2(BG^{(2)};\mathbb{Z}G) \arrow[u, two heads, "\mathrm{red}_2"] \arrow[rr, "p^*"] \arrow[dd, "(f')^*"] \arrow[ddr, "(f')^*"] & & H^2(B\pi^{(2)};\mathbb{Z}G) \arrow[d, "f^*"] & H^2(B\pi^{(2)};\mathbb{Z}\pi) \arrow[l, "\mathrm{red}"'] \arrow[dd, "f^*"] \\
& & H^2(M^{(3)};\mathbb{Z}G) & \\
H^2((M')^{(3)};\mathbb{Z}G) & H^2(M^{(3)} \setminus \coprod S^1 \times D^3, \partial;\mathbb{Z}G) \arrow[l, "\mathrm{ex}"'] \arrow[ur, "\mathrm{ex}"] & H^2(M^{(3)} \setminus \coprod S^1 \times D^3, \partial;\mathbb{Z}\pi) \arrow[l, "\mathrm{red}"'] \arrow[r, "\mathrm{ex}"] & H^2(M^{(3)};\mathbb{Z}\pi) \arrow[ul, "\mathrm{red}"'] \\
H^2(M';\mathbb{Z}G) \arrow[u, "\cong"', "i^*"] \arrow[d, "PD"] & H^2(M \setminus \coprod S^1 \times D^3, \partial;\mathbb{Z}G) \arrow[l, "\mathrm{ex}"'] \arrow[u, "\cong"', "i^*"] \arrow[d, "PD"] & H^2(M \setminus \coprod S^1 \times D^3, \partial;\mathbb{Z}\pi) \arrow[l, "\mathrm{red}"'] \arrow[r, "\mathrm{ex}"] \arrow[u, "\cong"', "i^*"] \arrow[d, "PD"] & H^2(M;\mathbb{Z}\pi) \arrow[u, "\cong"', "i^*"] \arrow[d, "PD"] \\
H_2(M';\mathbb{Z}G) \arrow[d, "h^{-1}"] & H_2(M \setminus \coprod S^1 \times D^3;\mathbb{Z}G) \arrow[l, "i_*"'] & H_2(M \setminus \coprod S^1 \times D^3;\mathbb{Z}\pi) \arrow[l, "\mathrm{red}"'] \arrow[r, "i_*"] \arrow[d, "h^{-1}"] & H_2(M;\mathbb{Z}\pi) \arrow[d, "h^{-1}"] \\
\pi_2(M') \arrow[drr, dashed, "\tau"'] & & \pi_2(M \setminus \coprod S^1 \times D^3) \arrow[ll, "i_*"'] \arrow[r, "i_*"] \arrow[d, dashed, "\tau"] & \pi_2(M) \arrow[dl, dashed, "\tau"] \\
& & \mathbb{Z}/2 &
\end{tikzcd} \tag{7.8}$$

The strategy of the proof is as follows. The passage from $H^2(BG;\mathbb{Z}/2)$ (including a choice of lift for the top left red_2 map), that goes down the left hand side of the diagram, computes $\tau_{f'}$. Note that

$$\tau_{f'}|_{\ker(d^3_{5,0}(G))^*} = \kappa^{-1}(\mathfrak{ter}(M')) = \kappa^{-1}(p_*(\mathfrak{ter}(M))).$$

The passage along the right hand side of the diagram computes $\tau_f \circ p^*$. If we show that $\tau_{f'}$ and $\tau_f \circ p^*$ are equal, this will imply that τ_f is a group homomorphism on the image of p^*. The injectivity statement in Theorem 7.7 dualises to a surjectivity statement $\prod_{i=1}^n p_i^* \colon \prod_{i=1}^n \ker(d^3_{5,0}(G_i))^* \twoheadrightarrow \ker(d^3_{5,0}(\pi))^*$ and hence we will obtain that $\tau_f|_{\ker(d^3_{5,0})^*(\pi)}$ is a homomorphism. Here $d^3_{5,0}(\pi)$ and $d^3_{5,0}(G_i)$ denote the differentials of the Atiyah-Hirzebruch spectral sequences for $B\pi$ and BG_i respectively. By naturality of the spectral sequences, $p^* \colon H^2(G;\mathbb{Z}/2) \to H^2(\pi;\mathbb{Z}/2)$ restricts to a homomorphism $p^* \colon \ker(d^3_{5,0}(G))^* \to \ker(d^3_{5,0}(\pi))^*$.

Now we know that $\tau_f \circ p^*|_{\ker(d^3_{5,0}(G))^*}$ is a homomorphism, and we can apply κ to the equality $\kappa^{-1}(p_*(\mathfrak{ter}(M))) = \tau_{f'}|_{\ker(d^3_{5,0}(G))^*} = \tau_f \circ p^*|_{\ker(d^3_{5,0}(G))^*}$ to obtain

$$\begin{aligned} p_*(\mathfrak{ter}(M)) &= \kappa(\tau_{f'}|_{\ker(d^3_{5,0}(G))^*}) = \kappa((\tau_f \circ p^*)|_{\ker(d^3_{5,0}(G))^*}) \\ &= \kappa\big((p^*)^*(\tau_f|_{\ker(d^3_{5,0}(\pi))^*})\big) = p_*(\kappa(\tau_f|_{\ker(d^3_{5,0}(\pi))^*})). \end{aligned}$$

This equality for all the surjective group homomorphisms $p_i \colon \pi \to G_i$, together with the injectivity assumption in Theorem 7.7, will then imply that $\mathfrak{ter}(M) = \kappa(\tau_f|_{\ker(d^3_{5,0})^*})$ as desired. Once we have explained the diagram and shown that it commutes, it will follow easily that $\tau_{f'}$ and $\tau_f \circ p^*$ are equal.

Next we explain the maps in the diagram.

- Arrows labelled with $f^*, p^*, (f')^*$ are the maps induced by $f \colon M^{(3)} \to B\pi^{(2)}$, $p \colon \pi \to G$ and $f' \colon (M')^{(2)} \to BG^{(2)}$ respectively.
- Arrows labelled with red are reduction of the coefficients.
- Arrows labelled with ex are given by the inverse of excision

  $$H^2(M, \coprod S^1 \times D^3) \xrightarrow{\cong} H^2(M \setminus \coprod S^1 \times D^3, \partial)$$

  composed with the map of from the long exact sequence of a pairs

  $$H^2(M, \coprod S^1 \times D^3) \to H^2(M)$$

  with the stated coefficients or by the analogous maps for $M^{(3)}$ or M' instead of M. The fact that $M \setminus \coprod S^1 \times D^3 \cong M' \setminus \coprod S^1 \times D^3$ is also used in the definition of the left-hand ex maps.
- Arrows labelled with i^* or i_* are maps induced by inclusions of the 3-skeleta.
- Arrows labelled with PD are the isomorphisms from Poincaré duality.
- Arrows labelled with h^{-1} are the inverses of the Hurewicz isomorphism.
- Arrows labelled with τ are the τ invariant. These arrows are dashed since the τ invariant is only defined on a subset.

More precisely, the map

$$(f')^* \colon H^2(BG^{(2)};\mathbb{Z}G) \to H^2(M^{(3)} \setminus \coprod S^1 \times D^3, \partial; \mathbb{Z}G)$$

is given as follows.

Consider the cohomology long exact sequence of the pair $(M'^{(3)}, \coprod D^2 \times S^2)$:

$$0 \to H^2(M'^{(3)}, \coprod D^2 \times S^2; \mathbb{Z}G) \to H^2(M'^{(3)}; \mathbb{Z}G) \to H^2(\coprod D^2 \times S^2; \mathbb{Z}G)$$

Since f' restricted to the attached copies of $D^2 \times S^2$ is null homotopic, the map $f'\colon H^2(BG^{(2)};\mathbb{Z}G) \to H^2(M'^{(3)};\mathbb{Z}G)$ lands in the image of $H^2(M'^{(3)}, \coprod D^2 \times S^2; \mathbb{Z}G)$. We can compose this $(f')^*$ further with the map induced by the map of pairs

$$H^2(M'^{(3)}, \coprod D^2 \times S^2; \mathbb{Z}G) \to H^2(M^{(3)} \setminus \coprod S^1 \times D^3, \partial; \mathbb{Z}G)$$

(by excision this is an isomorphism) to obtain the diagonal $(f')^*$ map in the big diagram

$$(f')^*\colon H^2(BG^{(2)};\mathbb{Z}G) \to H^2(M^{(3)} \setminus \coprod S^1 \times D^3, \partial; \mathbb{Z}G).$$

All quadrilaterals in the diagram commute by naturality of the involved maps. Note that one of the quadrilaterals looks like a triangle at first glance. The commutativity of the two triangles at the bottom (when they are defined) follows from Lemma 7.6.

Let $\alpha \in H^2(BG^{(2)};\mathbb{Z}G)$ be given and let $\widetilde{\alpha} \in H^2(B\pi^{(2)};\mathbb{Z}\pi)$ be a lift of $p^*\alpha$.

Claim. There exists $y \in H^2(M^{(3)} \setminus \coprod S^1 \times D^3, \partial; \mathbb{Z}\pi)$ with $\operatorname{red}(y) = (f')^*\alpha$ and $\operatorname{ex}(y) = f^*\widetilde{\alpha}$.

The claim can be seen as follows. Consider the following commutative diagram with exact rows coming from the long exact sequences of pairs.

$$\begin{tikzcd}
H^1(\coprod S^1 \times D^3;\mathbb{Z}\pi) \arrow[r] \arrow[d, "\mathrm{red}"] & H^2(M^{(3)}, \coprod S^1 \times D^3;\mathbb{Z}\pi) \arrow[r, "j"] \arrow[d, "\mathrm{red}"] & H^2(M^{(3)};\mathbb{Z}\pi) \arrow[r] \arrow[d, "\mathrm{red}"] & 0 \\
H^1(\coprod S^1 \times D^3;\mathbb{Z}G) \arrow[r] & H^2(M^{(3)}, \coprod S^1 \times D^3;\mathbb{Z}G) \arrow[r] & H^2(M^{(3)};\mathbb{Z}G) \arrow[r] & 0
\end{tikzcd} \tag{7.9}$$

The left hand vertical map is surjective, as we now argue. The circles in the left hand column represent elements of π that normally generate $\ker(\pi \to G)$. Let us denote these elements of π by $g_1, \dots, g_n$. We have that $H^1(\coprod S^1 \times D^3;\mathbb{Z}G) \cong \bigoplus^n \mathbb{Z}G$, with one summand per copy of $S^1 \times D^3$. On the other hand, $H^1(\coprod S^1 \times D^3;\mathbb{Z}\pi) \cong \bigoplus_{i=1}^n \mathbb{Z}\pi/(g_i - 1)$. The reduction map simply adds the relations $g_j - 1$ to each summand, for $j \neq i$. This shows that the left hand vertical map is surjective.

Using excision and commutativity of the big diagram (7.2), it not too hard to see, by taking $a_1 = f^*(\widetilde{\alpha})$ and $a_2 = (f')^*(\alpha)$, that the claim follows if we can show the following: for every $a_1 \in H^2(M^{(3)};\mathbb{Z}\pi)$ and for every $a_2 \in H^2(M^{(3)}, \coprod S^1 \times D^3;\mathbb{Z}G)$ with the same image in $H^2(M^{(3)};\mathbb{Z}G)$, there exists $y \in H^2(M^{(3)}, \coprod S^1 \times D^3;\mathbb{Z}\pi)$ with $j(y) = a_1$ and $\operatorname{red}(y) = a_2$. The map

$$j\colon H^2(M^{(3)}, \coprod S^1 \times D^3;\mathbb{Z}\pi) \to H^2(M^{(3)};\mathbb{Z}\pi)$$

is identified with the map

$$H^2(M^{(3)} \setminus \coprod S^1 \times D^3, \partial;\mathbb{Z}\pi) \to H^2(M^{(3)};\mathbb{Z}\pi)$$

using excision. The existence of such an element y follows from a diagram chase in the above diagram (7.9). Lift a_1 to $z \in H^2(M^{(3)}, \coprod S^1 \times D^3;\mathbb{Z}\pi)$. The element z might not map to a_2 under red. Take $\operatorname{red}(z) - a_2$, lift it to the top left corner and map it to $w \in H^2(M^{(3)}, \coprod S^1 \times D^3;\mathbb{Z}\pi)$. Define $y = z - w$. It is now straightforward to see that y has the desired properties. This completes the proof of the claim.

It follows from the claim that for any two elements $\alpha, \alpha' \in H^2(BG^{(2)};\mathbb{Z}G)$, there are elements $PD(y), PD(y') \in \pi_2(M \setminus \coprod S^1 \times D^3)$ that map to $PD((f')^*\alpha), PD((f')^*\alpha)$ and $PD(f^*\widetilde{\alpha}), PD(f^*\widetilde{\alpha}')$ under the respective inclusions. Since the equivariant intersection form vanishes on $PD \circ f^*$, we see that

$$0 = \lambda(PD(f^*\widetilde{\alpha}), PD(f^*\widetilde{\alpha}')) = \lambda(PD(y), PD(y')) = p\big(\lambda(PD((f')^*\alpha), PD((f')^*\alpha))\big).$$

Hence the equivariant intersection form of M' vanishes on $PD \circ (f')^*$. For every $x \in H^2(BG;\mathbb{Z}/2)$ and every lift $\widehat{x} \in H^2(BG^{(2)};\mathbb{Z}G)$ of i^*x we have

$$\tau_{f'}(x) = \tau(PD((f')^*\widehat{x})).$$

For every lift $\widetilde{x} \in H^2(B\pi^{(2)};\mathbb{Z}\pi)$ of $p^*\widehat{x}$ and every $y \in H^2(M^{(3)} \setminus \coprod S^1 \times D^3, \partial; \mathbb{Z}\pi)$ with $\mathrm{red}(y) = (f')^*\widehat{x}$ and $\mathrm{ex}(y) = f^*\widetilde{x}$ we have

$$\tau_f(p^*x) = \tau(PD(f^*\widetilde{x})) = \tau(PD(y)) = \tau(PD((f')^*\widehat{x})) = \tau_{f'}(x).$$

This completes the proof of the assertion that $\tau_f \circ p^* = \tau_{f'}$ and therefore completes the proof of the inheritance theorem for the Tertiary property. $\square$

Corollary 7.10. *The Tertiary Property 1.12 holds for all abelian groups if and only if it holds for all abelian groups with at most two generators.*

Proof. Assume that the Tertiary Property 1.12 holds for all abelian groups with at most two generators.

Let $\pi = \bigoplus_{i=1}^n C_i$ with C_i cyclic. Let $p_i\colon \pi \to C_i$ be the projection homomorphism and let $s_i\colon C_i \to \pi$ be the inclusion homomorphism. For $i \neq j$ let $G_{ij} := C_i \oplus C_j$, let $p_{ij}\colon G_{ij} \to \pi$ be the projection homomorphism and let $s_{ij}\colon G_{ij} \to \pi$ be the inclusion homomorphism. By the Künneth theorem

$$H_2(\pi;\mathbb{Z}/2) \cong \bigoplus_{1\le i\le n} H_2(C_i;\mathbb{Z}/2) \oplus \bigoplus_{1\le i<j\le n} H_1(C_i;\mathbb{Z}/2) \otimes H_1(C_j;\mathbb{Z}/2).$$

It follows, that

$$p := \bigoplus_{1\le i<j\le n} (p_{ij})_*\colon H_2(\pi;\mathbb{Z}/2) \to \bigoplus_{1\le i<j\le n} H_2(G_{ij};\mathbb{Z}/2)$$

is injective. Let

$$s := \bigoplus_{1\le i<j\le n} (s_{ij})_*\colon \bigoplus_{1\le i<j\le n} H_2(G_{ij};\mathbb{Z}/2) \to H_2(\pi;\mathbb{Z}/2).$$

Depending on n, the composition $s \circ p$ might not be the identity since any nontrivial elements from $H_2(C_i;\mathbb{Z}/2)$ (i.e. whenever C_i has even order) appear $(n-1)$ times in the terms $H_2(G_{ij};\mathbb{Z}/2)$ and hence get multiplied by $(n-1)$. For n even this requires no change, but for n odd this needs a remedy. We give a unified treatment. Define the $\mathbb{Z}/2$-module

$$\Lambda_n := \begin{cases} \bigoplus_{1\le i<j\le n} H_2(G_{ij};\mathbb{Z}/2) \oplus \bigoplus_{1\le i\le n} H_2(C_i;\mathbb{Z}/2) & n \text{ odd} \\ \bigoplus_{1\le i<j\le n} H_2(G_{ij};\mathbb{Z}/2) \oplus \{0\} & n \text{ even.} \end{cases}$$

For n odd, define

$$p_n' := \bigoplus_{1\le i\le n} (p_i)_*\colon H_2(\pi;\mathbb{Z}/2) \to \bigoplus_{1\le i\le n} H_2(C_i;\mathbb{Z}/2)$$

and

$$s_n' := \bigoplus_{1\le i\le n} (s_i)_*\colon \bigoplus_{1\le i\le n} H_2(C_i;\mathbb{Z}/2) \to H_2(\pi;\mathbb{Z}/2).$$

For n even, take $p_n'\colon H_2(\pi;\mathbb{Z}/2) \to \{0\}$ and $s_n'\colon \{0\} \to H_2(\pi;\mathbb{Z}/2)$ to be the zero maps. Then

$$(s \oplus s_n') \circ (p \oplus p_n')\colon H_2(\pi;\mathbb{Z}/2) \to \Lambda_n \to H_2(\pi;\mathbb{Z}/2)$$

is the identity map. If follows by naturality of the Atiyah Hirzebruch spectral sequence that this composition still induces the identity map after modding out the $d^2_{4,1}$ and $d^3_{5,0}$ differentials.

In particular, the induced map

$$p \oplus p'_n \colon H_2(\pi;\mathbb{Z}/2)/\operatorname{im}(d^2_{4,1}, d^3_{5,0}) \to \Lambda_n/\operatorname{im}(d^2_{4,1}, d^3_{5,0})$$

is injective. We can therefore apply the inheritance result Theorem 7.7 above, with G_i, in the notation of that theorem, as all the abelian groups G_{ij} or C_i with either one or two generators involved in the sum Λ_n. We conclude that π has the Tertiary Property 1.12 as asserted. ☐

8. Reduction to odd index subgroups

Let $P \leq \pi$ be a subgroup of a finitely presented group π. Let X be a CW complex with fundamental group π. There is a covering space $p\colon \widehat{X} \to X$ with fundamental group P. Both spaces have the same universal cover $\widetilde{X}$, and we can express the cohomology of the universal cover as $H^*(X;\mathbb{Z}\pi) \cong H^*(\widehat{X};\mathbb{Z}P)$, where the isomorphism is of $\mathbb{Z}P$-modules. Similarly, there are isomorphisms of the homology groups $H_*(X;\mathbb{Z}\pi) \cong H_*(\widehat{X};\mathbb{Z}P)$, where again the $\mathbb{Z}\pi$ homology is thought of as a $\mathbb{Z}P$ module by restriction. We will make use of this simple observation several times in this section.

Now suppose that $P \leq \pi$ is a finite index subgroup. If M is a right $\mathbb{Z}\pi$-module we can restrict the action to P and consider the projection

$$M \otimes_{\mathbb{Z}P} C_* \xrightarrow{p} M \otimes_{\mathbb{Z}\pi} C_*$$

inducing

$$p_*\colon H_*(\widehat{X};M) \to H_*(X;M).$$

On the chain level we obtain a *transfer map* in the other direction by

$$\operatorname{tr}\colon M \otimes_{\mathbb{Z}\pi} C_* \to M \otimes_{\mathbb{Z}P} C_*, \quad m \otimes c \mapsto \sum_{Pg \in P\backslash\pi} mg^{-1} \otimes gc.$$

This map induces a map

$$\operatorname{tr}_*\colon H_*(X;M) \to H_*(\widehat{X};M).$$

Similarly one constructs a map

$$\operatorname{tr}^*\colon H^*(\widehat{X};M) \to H^*(X;M).$$

The transfer maps have the key property that $p_* \circ \operatorname{tr}_*$ and $\operatorname{tr}^* \circ p^*$ are equal to multiplication by the index $[\pi : P]$ of P in π.

Let K be a finite 2-complex with $\pi_1(K) = \pi$. Then there is a finite cover K_P of K with fundamental group P.

Start with a map $f\colon H^2(K;\mathbb{Z}\pi) \to \operatorname{Hom}_{\mathbb{Z}\pi}(H^2(K;\mathbb{Z}\pi),\mathbb{Z}\pi)$, and define

$$\operatorname{tr}(f) \in \operatorname{Sesq}(H^2(K_P;\mathbb{Z}\pi))$$

to be the composition

$$\begin{aligned} H^2(K_P;\mathbb{Z}\pi) \xrightarrow{\operatorname{tr}^*} H^2(K;\mathbb{Z}\pi) &\xrightarrow{f} \operatorname{Hom}_{\mathbb{Z}\pi}(H^2(K;\mathbb{Z}\pi),\mathbb{Z}\pi) \\ &\xrightarrow{(\operatorname{tr}^*)^*} \operatorname{Hom}_{\mathbb{Z}\pi}(H^2(K_P;\mathbb{Z}\pi),\mathbb{Z}\pi). \end{aligned}$$

Post-compose with the isomorphism

$$\operatorname{Hom}_{\mathbb{Z}\pi}(H^2(K_P;\mathbb{Z}\pi),\mathbb{Z}\pi) \cong \operatorname{Hom}_{\mathbb{Z}P}(H^2(K_P;\mathbb{Z}\pi),\mathbb{Z}P),$$

given by post-composition with the map

$$\begin{array}{rcl} \operatorname{ev}_{\mathbb{Z}P}\colon \mathbb{Z}\pi & \to & \mathbb{Z}P \\ \sum_{g\in\pi} n_g g & \mapsto & \sum_{g\in P} n_g g, \end{array}$$

and pre-compose with the map $H^2(K_P;\mathbb{Z}P) \to H^2(K_P;\mathbb{Z}\pi)$ given by the inclusion $\mathbb{Z}P \to \mathbb{Z}\pi$, to restrict $\operatorname{tr}(f)$ to an element of $\operatorname{Sesq}(H^2(K_P;\mathbb{Z}P))$ i.e. a map $H^2(K_P;\mathbb{Z}P) \to \operatorname{Hom}_{\mathbb{Z}P}(H^2(K_P;\mathbb{Z}P),\mathbb{Z}P)$. This construction commutes with the transposition T and so defines a transfer map

$$\operatorname{tr}_*\colon \widehat{H}^0\big(\operatorname{Sesq}(H^2(K;\mathbb{Z}\pi))\big) \to \widehat{H}^0\big(\operatorname{Sesq}(H^2(K_P;\mathbb{Z}P))\big).$$

Lemma 8.1. *The following diagram commutes:*

$$\begin{tikzcd} H_3(\pi;\mathbb{Z}/2) \arrow[r, "A"] \arrow[d, "\operatorname{tr}_*"] & \widehat{H}^0(\operatorname{Sesq}(H^2(K;\mathbb{Z}\pi))) \arrow[d, "\operatorname{tr}_*"] \\ H_3(P;\mathbb{Z}/2) \arrow[r, "A"] & \widehat{H}^0(\operatorname{Sesq}(H^2(K_P;\mathbb{Z}P))) \end{tikzcd}$$

Proof. By tracing the definition of the map $A\colon H_3(\pi;\mathbb{Z}/2) \to \widehat{H}^0(\operatorname{Sesq}(H^2(K;\mathbb{Z}\pi)))$, via the maps ι, I, Θ and Φ, it is not too hard to see, for $c \in C_3(B\pi;\mathbb{Z}\pi)$, $m,m' \in C^2 = C^2(K;\mathbb{Z}\pi)$ representatives of cohomology classes in $H^2(K;\mathbb{Z}\pi)$, that

$$A([1\otimes c]) = \big(([m],[m']) \mapsto \overline{m(\partial_3(c))}m'(\partial_3(c)) \in \mathbb{Z}\pi\big).$$

Here $\partial_3\colon C_3(B\pi;\mathbb{Z}\pi) \to C_2(B\pi;\mathbb{Z}\pi)$ is the boundary map.

Similarly to the proof that the above definition of $A([1\otimes c])$ does not depend on the preimage $c \in C_3$ of $[1\otimes c]$, one checks that for $[c'] \in H_3(P;\mathbb{Z}/2)$ and for any choice of a preimage $\widetilde{c'} \in \mathbb{Z}\pi\otimes_{\mathbb{Z}P} C_3(BP;\mathbb{Z}P)$, the $\widehat{H}^0(\operatorname{Sesq} H^2(K_P;\mathbb{Z}P))$ class of the map

$$([m],[m']) \mapsto \operatorname{ev}_{\mathbb{Z}P}\big(\overline{(1\otimes m)(\partial_3(\widetilde{c'}))}(1\otimes m')(\partial_3(\widetilde{c'})) \in \mathbb{Z}\pi\big)$$

agrees with $A([c'])$.

For convenience we provide the details. For all $g,g' \in \pi$ and all $c \in C_3(BP;\mathbb{Z}P)$ we have the following equality in $\widehat{H}^0(\operatorname{Sesq}(H^2(K_P;\mathbb{Z}P)))$.

$$\begin{aligned} & \Big((m,m') \mapsto \operatorname{ev}_{\mathbb{Z}P}(\overline{(1\otimes m)((g\pm g')\otimes\partial_3(c))}(1\otimes m')((g\pm g')\otimes\partial_3(c)))\Big) \\ = \quad & \Big((m,m') \mapsto \operatorname{ev}_{\mathbb{Z}P}(\overline{m(\partial_3(c))}((\overline{g\pm g'})(g\pm g'))m'(\partial_3(c)))\Big) \\ = \quad & \Big((m,m') \mapsto \operatorname{ev}_{\mathbb{Z}P}(\overline{m(\partial_3(c))}(2\pm(g^{-1}g'+\overline{g^{-1}g'}))m'(\partial_3(c)))\Big) \\ = \quad & \Big((m,m') \mapsto \operatorname{ev}_{\mathbb{Z}P}(\overline{m(\partial_3(c))}(g^{-1}g'+\overline{g^{-1}g'})m'(\partial_3(c)))\Big) \\ = \quad & 0 \end{aligned}$$

Since any two choices of preimage of $[c'] \in H_3(P;\mathbb{Z}/2)$ in $\mathbb{Z}\pi\otimes_{\mathbb{Z}P} C_3(BP;\mathbb{Z}P)$ differ by a sum of elements of the form $(g\pm g')\otimes c$, this shows that the element in $\widehat{H}^0(\operatorname{Sesq}(H^2(K_P;\mathbb{Z}P)))$ is independent of the choice of preimage. Hence,

$$\begin{aligned} A(\operatorname{tr}_*[1\otimes c]) &= \operatorname{ev}_{\mathbb{Z}P}\big(([m],[m']) \mapsto \overline{(1\otimes m)(\partial_3(\operatorname{tr}_* c))}(1\otimes m')(\partial_3(\operatorname{tr}_* c))\big) \\ &= \operatorname{ev}_{\mathbb{Z}P}\big(([m],[m']) \mapsto \overline{\operatorname{tr}^*(1\otimes m)(\partial_3(c))}\operatorname{tr}^*(1\otimes m')(\partial_3(c))\big) \\ &= \operatorname{tr}_* A([1\otimes c]). \end{aligned}$$

$\square$

8.1. **Secondary Property inheritance from odd index subgroups.** Note that, like the Secondary inheritance from Section 7, we do not actually show that the Secondary Property 1.8 is inherited from finite odd index subgroups, but instead show inheritance for Condition 4.17. A group can have the Secondary Property without necessarily satisfying Condition 4.17.

Theorem 8.2. *Let π be a finitely presented group and let P be a finite odd index subgroup of π. Then Condition 4.17 holds for π if it holds for P.*

Proof. Let K be a finite 2-complex with $\pi_1(K) = \pi$. Then there is a finite cover K_P of K with fundamental group P. In particular, $H^2(K_P; \mathbb{Z}P) \cong H^2(K; \mathbb{Z}\pi)$, by the observation made at the beginning of this section.

Consider the following diagram. Since $i_* \circ \mathrm{tr}_*$ is multiplication by the index $[\pi : P]$, which is odd, the middle vertical composition is the identity.

$$\begin{tikzcd}
H_5(\pi;\mathbb{Z}) \arrow[r, "Sq_2\circ \mathrm{red}_2"] \arrow[d, "\mathrm{tr}_*"] & H_3(\pi;\mathbb{Z}/2) \arrow[r, "A"] \arrow[d, "\mathrm{tr}_*"] & \widehat{H}^0(\mathrm{Sesq}(H^2(K;\mathbb{Z}\pi))) \arrow[d, "\mathrm{tr}_*"] \\
H_5(P;\mathbb{Z}) \arrow[r, "Sq_2\circ \mathrm{red}_2"] \arrow[d, "i_*"] & H_3(P;\mathbb{Z}/2) \arrow[r, "A"] \arrow[d, "i_*"] & \widehat{H}^0(\mathrm{Sesq}(H^2(K_P;\mathbb{Z}P))) \arrow[d, "i_*"] \\
H_5(\pi;\mathbb{Z}) \arrow[r, "Sq_2\circ \mathrm{red}_2"] & H_3(\pi;\mathbb{Z}/2) \arrow[r, "A"] & \widehat{H}^0(\mathrm{Sesq}(H^2(K;\mathbb{Z}\pi)))
\end{tikzcd}$$

Exactness of the top and bottom rows (which are equal) now follows from the exactness of the middle row by a diagram chase using the fact that $i_* \circ \mathrm{tr}_* = \mathrm{Id}_{H_3(\pi;\mathbb{Z}/2)}$. □

8.2. **Tertiary property inheritance from odd index subgroups.**

Theorem 8.3. *Let π be a finitely presented group, and let P be a finite index subgroup of odd index. Then the Tertiary Property 1.12 holds for π if it holds for P.*

For the proof we first need a couple of lemmas.

Lemma 8.4. *Let X be a finite CW complex with fundamental group π. The composition*

$$H^2(\widehat{X}; \mathbb{Z}P) \xrightarrow{\mathrm{red}} H^2(\widehat{X}; \mathbb{Z}/2) \xrightarrow{\mathrm{tr}^*} H^2(X; \mathbb{Z}/2)$$

agrees with the reduction of coefficients

$$H^2(X; \mathbb{Z}\pi) \xrightarrow{\mathrm{red}} H^2(X; \mathbb{Z}/2)$$

under the identification of $H^2(\widehat{X}; \mathbb{Z}P)$ and $H^2(X; \mathbb{Z}\pi)$ given at the beginning of the subsection.

Proof. Let C_* denote the cellular $\mathbb{Z}\pi$-chain complex of X. On the chain level, the identification of $H^2(X; \mathbb{Z}\pi)$ with $H^2(\widehat{X}; \mathbb{Z}P)$ is given by

$$\mathrm{ev}_{\mathbb{Z}P} \colon \mathrm{Hom}_{\mathbb{Z}\pi}(C_2, \mathbb{Z}\pi) \to \mathrm{Hom}_{\mathbb{Z}P}(C_2, \mathbb{Z}P).$$

Here $\mathrm{ev}_{\mathbb{Z}P} \colon \mathbb{Z}\pi \to \mathbb{Z}P$, sending $\sum_{g\in\pi} n_g g$ to $\sum_{g\in P} n_g g$, induces the map on Hom modules above by post-composition; we abuse notation and also denote the map on Hom modules by $\mathrm{ev}_{\mathbb{Z}P}$.

The reduction of coefficients is given by post-composing with the augmentations $\varepsilon \colon \mathbb{Z}\pi \to \mathbb{Z}/2$ and $\varepsilon \colon \mathbb{Z}P \to \mathbb{Z}/2$ respectively. The lemma now follows from the straightforward computation that the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{Z}\pi}(C_2,\mathbb{Z}\pi) \arrow[r, "\mathrm{ev}_{\mathbb{Z}P}"] \arrow[d, "\varepsilon"] & \mathrm{Hom}_{\mathbb{Z}P}(C_2;\mathbb{Z}P) \arrow[d, "\varepsilon"] \\
\mathrm{Hom}_{\mathbb{Z}\pi}(C_2,\mathbb{Z}/2) & \mathrm{Hom}_{\mathbb{Z}P}(C_2,\mathbb{Z}/2) \arrow[l, "\mathrm{tr}^*"']
\end{tikzcd}$$

□

Lemma 8.5. *Let $i\colon P\to\pi$ be the inclusion. Let M be a spin manifold with fundamental group π and such that $\mathfrak{pri}(M)=\mathfrak{sec}(M)=0$. Then*

$$i_*\colon H_2(P;\mathbb{Z}/2)/\operatorname{im}(d_{5,0},d_{4,1})\to H_2(\pi;\mathbb{Z}/2)/\operatorname{im}(d_{5,0},d_{4,1})$$

maps $\mathfrak{ter}(\widehat{M})$ to $\mathfrak{ter}(M)$, where $p\colon\widehat{M}\to M$ is the cover associated to $P\le\pi$.

Proof. Stabilisation of M by a single $S^2\times S^2$ corresponds to stabilisation of $\widehat{M}$ with $[\pi:P]$ copies of $S^2\times S^2$. Since stabilisation does not change the $\mathfrak{ter}$ invariant, and since stably the map $c\colon M\to B\pi$ factors through $B\pi^{(2)}$ up to homotopy (because $\mathfrak{pri}(M)=\mathfrak{sec}(M)=0$), we will assume that we are in the situation that $c\colon M\to B\pi$ maps to the 2-skeleton $B\pi^{(2)}\subset B\pi$, and therefore also that $\widehat{c}=c\circ p\colon\widehat{M}\to M\to B\pi^{(2)}$ factors as $M\to BP^{(2)}\to B\pi^{(2)}$, through the 2-skeleton of BP. By Lemma 2.5, $\mathfrak{ter}(M)$ is given as follows. Choose a point e_i in each 2-cell c_i of $B\pi^{(2)}$, and let F_i be the regular preimage of e_i under c. The spin structure of M induces a spin structure on F_i and we have $\mathfrak{ter}(M,c)=\big[\sum_i\mathrm{Arf}(F_i)c_i\big]$. A regular preimage of e_i under $c\circ p\colon\widehat{M}\to B\pi^{(2)}$ consists of $[\pi:P]$ copies of F_i, each with the same spin structure. Therefore, when we view $\widehat{M}$ as an element of $\Omega_4^{Spin}(B\pi)$, we get

$$\mathfrak{ter}(\widehat{M},c\circ p)=[\pi:P]\Big[\sum_i\mathrm{Arf}(F_i)c_i\Big]=[\pi:P]\mathfrak{ter}(M,c)=\mathfrak{ter}(M,c),$$

where the last equality uses that the index $[\pi:P]$ is odd. Since $\mathfrak{ter}(\widehat{M},c\circ p)=i_*\mathfrak{ter}(\widehat{M},\widehat{c})$, the lemma follows. □

Proof of Theorem 8.3. Let K be a finite 2-complex with fundamental group π, and let $\widehat{K}$ be the covering space corresponding to P.

Every splitting s of $H^2(M;\mathbb{Z}\pi)\to H^2(K;\mathbb{Z}\pi)$ also is a splitting s_P of $H^2(\widehat{M};\mathbb{Z}P)\to H^2(\widehat{K};\mathbb{Z}P)$, because $H^2(\widehat{M};\mathbb{Z}P)=H^2(M;\mathbb{Z}\pi)$ and $H^2(\widehat{K};\mathbb{Z}P)=H^2(K;\mathbb{Z}\pi)$, with the latter considered as $\mathbb{Z}P$-modules.

By Lemma 8.5 it suffices to show that $i_*\tau_{s_P}=\tau_s$, since then $\tau_s=i_*\tau_{s_P}=i_*\mathfrak{ter}(\widehat{M})=\mathfrak{ter}(M)$. Here for the middle equality we used the assumption that P has the Tertiary Property 1.12.

Recall that the map $\tau_{s_P}\colon H^2(B\pi;\mathbb{Z}/2)\to\mathbb{Z}/2$ is defined as follows: restrict $x\in H^2(BP;\mathbb{Z}/2)$ to an element of $H^2(\widehat{K};\mathbb{Z}/2)$, choose a preimage $x'\in H^2(\widehat{K};\mathbb{Z}P)$ and then apply $\tau\circ PD\circ s_P$. By Lemma 8.4 with $X=K$, $x'\in H^2(\widehat{K};\mathbb{Z}P)\cong H^2(K;\mathbb{Z}\pi)$ is a preimage of the restriction of $\mathrm{tr}^*(x)\in H^2(B\pi;\mathbb{Z}/2)$ to $H^2(K;\mathbb{Z}/2)$. Hence $\tau_s(\mathrm{tr}^*(x))=\tau(PD(s(x')))=\tau_{s_P}(x)$ and we obtain that $\tau_s\circ\mathrm{tr}^*=\tau_{s_P}$. Equivalently, we have $\mathrm{tr}_*\tau_s=\tau_{s_P}$, when we view τ_s and τ_{s_P} as elements of $H_2(\pi;\mathbb{Z}/2)\cong\mathrm{Hom}(H^2(\pi;\mathbb{Z}/2),\mathbb{Z}/2)$ and $H_2(P;\mathbb{Z}/2)\cong\mathrm{Hom}(H^2(P;\mathbb{Z}/2),\mathbb{Z}/2)$ respectively. Apply i_* to obtain

$$i_*\tau_{s_P}=i_*\,\mathrm{tr}_*\,\tau_s=[\pi:P]\tau_s=\tau_s,$$

where the last equality follows from the assumption that the index of P in π is odd. □

9. The secondary and tertiary properties for three families of groups

In this section we prove that several families of groups possess the Secondary and Tertiary properties, thus classifying, at least modulo the limitations discussed in the introduction, spin 4-manifolds with these fundamental groups up to stable diffeomorphism. We consider cohomologically 3-dimensional groups, right-angled Artin groups, and generalised quarternion groups.

9.1. **Cohomologically 3-dimensional groups.**

A group G is said to have cohomological dimension at most n if $\mathbb{Z}$, thought of as a $\mathbb{Z}G$-module via the augmentation map $\mathbb{Z}G \to \mathbb{Z}$, admits a $\mathbb{Z}G$-module projective resolution of length n. A group G is said to be of type FP_n if $\mathbb{Z}$ has a projective resolution such that the first n terms are finitely generated. A group G has type FP_3 and cohomological dimension at most three if and only if there exists a finite 2-complex K with $\pi_1(K) \cong G$ such that $\pi_2(K)$ is finitely generated projective as a $\mathbb{Z}G$-module.

Theorem 9.1. *The Secondary Property 1.8 and the Tertiary Property 1.12 hold for groups of type FP_3 that have cohomological dimension at most three.*

Under the assumption that the equivariant intersection form is even, Hambleton-Hildum [HH19] give an alternative classification to ours, using the stable quadratic 2-type instead of the τ invariant. By Theorem 9.1, the intersection form being even is equivalent to $\mathfrak{sec}(M) = 0$. As discussed in the introduction, Hambleton and Hildum use the whole intersection form. This means that their results also apply to the unstable homeomorphism problem, unlike ours. On the other hand, the τ-invariant on a summand of $\pi_2(M)$ can often be an easier invariant to compute.

The proof of Theorem 9.1 is split into two parts, one for each property. Each property is dealt with in its own subsection below.

9.1.1. *Three dimensional groups have the secondary property.* For the convenience of the reader, we recall that Condition 4.17 requires the sequence

$$H_5(\pi;\mathbb{Z}) \xrightarrow{\mathrm{Sq}_2\circ \mathrm{red}_2} H_3(\pi;\mathbb{Z}/2) \xrightarrow{A} \widehat{H}^0(\mathrm{Sesq}(H))$$

be exact at $H_3(\pi;\mathbb{Z}/2)$.

For a cohomologically 3-dimensional group π of type FP_3, choose a 2-dimensional complex K with $\pi_1(K) = \pi$ and such that $\pi_2(K)$ is a finitely generated projective $\mathbb{Z}\pi$-module. Let $H := H^2(K;\mathbb{Z}\pi)$. Recall from Lemma 4.12 that $\pi_2(K) \cong H^*$. Then the double dual H^{**} is the dual of $\pi_2(K)$ and hence is a finitely generated projective $\mathbb{Z}\pi$-module. We can consider the natural evaluation map $e_H \colon H \to H^{**}$. Since $H^* \cong \pi_2(K)$ is finitely generated projective, the map $e_{H^*} \colon H^* \to H^{***}$ is an isomorphism. It is straightforward to verify that $e_H^* \circ e_{H^*} \colon H^* \to H^*$ is the identity on H^*. Thus the map $e_H^* \colon H^{***} \to H^*$ is an isomorphism.

For a $\mathbb{Z}\pi$-module M and a projective $\mathbb{Z}\pi$-module P, the canonical map

$$\mathrm{Hom}_{\mathbb{Z}\pi}(M,\mathbb{Z}\pi) \otimes_{\mathbb{Z}\pi} P \to \mathrm{Hom}_{\mathbb{Z}\pi}(M,P)$$

is an isomorphism. Hence the map $\mathrm{Hom}_{\mathbb{Z}\pi}(H^{**},H^*) \to \mathrm{Hom}_{\mathbb{Z}\pi}(H,H^*)$ given by precomposing with e_H is an isomorphism, because e_H^* is an isomorphism, and the given map can be identified with $e_H^* \otimes \mathrm{Id}_{H^*}$. This implies that

$$\begin{aligned}\mathrm{Sesq}(e_H)\colon\ \mathrm{Sesq}(H^{**}) \cong \mathrm{Hom}_{\mathbb{Z}\pi}(H^{**},H^{***}) &\xrightarrow{\mathrm{Hom}(-,e_H^*)} \\ &\mathrm{Hom}_{\mathbb{Z}\pi}(H^{**},H^*) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}\pi}(H,H^*) \cong \mathrm{Sesq}(H)\end{aligned}$$

is also an isomorphism.

Consider the diagram

$$\begin{tikzcd} H^{***} \otimes_{\mathbb{Z}\pi} H^{***} \arrow[r, "e_H^* \otimes e_H^*", "\cong"'] \arrow[d, "\Phi_{H^{**}}"] & H^* \otimes_{\mathbb{Z}\pi} H^* \arrow[d, "\Phi_H"] \\ \mathrm{Sesq}(H^{**}) \arrow[r, "\mathrm{Sesq}(e_H)", "\cong"'] & \mathrm{Sesq}(H) \end{tikzcd}$$

Now the map $\Phi_{H^{**}}$ is an isomorphism by Lemma 4.10, and hence Φ_H is also an isomorphism. It follows that $\Psi_H = \widehat{H}^0(\Phi_H) \circ \Theta_H$ is an isomorphism as well, since Θ_H is an isomorphism by Lemma 4.8. The map $A\colon H_3(\pi;\mathbb{Z}/2) \to \widehat{H}^0(\mathrm{Sesq}(H))$, which by definition is a composite $\Psi_H \circ (\mathrm{Id} \times I) \circ \iota$, where I is an isomorphism by Lemma 4.12 and ι is injective by Lemma 6.19, is therefore injective. Since $H_5(\pi;\mathbb{Z}) = 0$, this implies that Condition 4.17 holds, and therefore that π has the Secondary Property 1.8.

9.1.2. *Three dimensional groups have the tertiary property.* Since Φ_H is injective by the argument in Section 9.1.1, and ι_{H^*} is injective by Lemma 6.19, it follows from Theorem 6.17 that π has the Tertiary Property 1.12.

9.2. **Right angled Artin Groups.** Let (V,E) be a finite graph, with $V = \{v_i\}_{i\in I}$ the set of vertices and $E = \{e_j = \{v_j^1, v_j^2\}\}_{j\in J}$ the set of edges, where e_j is an edge between $v_j^1, v_j^2 \in V$. The corresponding right-angled Artin group (or RAAG for short) is $\langle V \mid \{[v_j^1, v_j^2]\}_{j\in J}\rangle$. An *$n$-clique* is a subset of n vertices $v_1,\dots,v_n \in V$, such that for every pair of vertices v_i, v_j, with $i \neq j \in \{1,\dots,n\}$, we have $\{v_i, v_j\} \in E$.

Theorem 9.2. *The Secondary Property 1.8 and the Tertiary Property 1.12 hold for right-angled Artin groups.*

We will use the following well-known theorem.

Theorem 9.3 ([CD95, Corollary 3.2.2])**.** *Let π be the RAAG associated to a graph (V,E). The dimension of $H_n(\pi;\mathbb{Z}/2)$ is the number of n-cliques in (V,E).*

9.2.1. *RAAGs have the secondary property.* For each n-clique in (V,E) with vertices v_k', $k = 1,\dots,n$, there is an inclusion $\mathbb{Z}^n = \langle x_k \mid [x_k, x_{k'}]\rangle \to \pi$ given by $x_k \mapsto v_k'$, and a projection $\pi \to \mathbb{Z}^n$ given by $v_k' \mapsto x_k$ and $v_i \mapsto 0$ for all $v_i \notin \{v_k'\}_{k=1,\dots,n}$. In particular, π is a semi-direct product $N \rtimes \mathbb{Z}^n$ of $\mathbb{Z}^n$ and a group N.

We apply this with $n = 3$, for each 3-clique. By Theorem 9.3, the rank of $H_3(\pi;\mathbb{Z}/2)$ is the same as the number of 3-cliques. We can find a basis for $H_3(\pi;\mathbb{Z})$ such that each basis element is the image $i_*(x)$ for some $x \in H_3(\mathbb{Z}^3;\mathbb{Z}/2) = \mathbb{Z}/2$. We know that $\mathbb{Z}^3$ has the Secondary property by Theorem 9.1. Then the "moreover" part of Theorem 7.3 says that $i_*(x)$ lies in the kernel of A if and only if it lies in the image of $\mathrm{Sq}_2 \circ \mathrm{red}_2$. Apply this argument at each 3-clique, and therefore for each basis element of $H_3(\pi;\mathbb{Z}/2)$, to see that Condition 4.17 holds for π. Therefore π has the Secondary Property 1.8.

9.2.2. *RAAGs have the tertiary property.* By Theorem 9.3 and the discussion at the beginning of the proof of the Secondary Property in Section 9.2.1, it follows that there are surjections $p_j\colon \pi \to \mathbb{Z}^2$ such that $H_2(\pi;\mathbb{Z}/2) \to H_2(\oplus_{j\in J}\mathbb{Z}^2;\mathbb{Z}/2)$ is injective, where as above J indexes the edges of the graph defining the RAAG π. Since all differentials in the spectral sequence for $\Omega_4^{Spin}(B\mathbb{Z}^2)$ are trivial, this map is still injective after dividing out the image of the differentials. Hence the Tertiary Property 1.12 for π follows from Theorem 7.7 and the fact that we know the property holds for $\mathbb{Z}^2$ (since $\mathbb{Z}^2$ has cohomological dimension two).

9.3. **Generalised quaternion groups.** By [Bro82, Theorem VI 9.3], every 2-group with periodic cohomology is either a cyclic group or a generalised quaternion group. In the next section, we will show that all abelian groups have the Secondary Property 1.8 and the Tertiary Property 1.12. Thus by Theorem 8.2 and Theorem 8.3 below, every finite group whose 2-Sylow subgroup has periodic cohomology has both properties once we have shown them for all generalised quaternion groups. This is the purpose of this subsection. In combination with our other results, Theorem 9.4 is therefore part of the stable diffeomorphism classification of spin 4-manifolds with finite fundamental group, whose 2-Sylow subgroup has periodic cohomology.

Note that the Secondary Property 1.8 for finite groups with quaternion 2-Sylow subgroup was already proved in [Tei92, Theorem 6.4.1]. Since the statement there is slightly different, we reprove it here for completeness, as an instructive illustration of the use of Condition 4.17.

Let n be a power of two. A presentation of the generalised quaternion group with $8n$ elements is given by

$$Q_{8n} = \langle x, y \mid x^{2n}y^{-2}, xyxy^{-1}\rangle.$$

(The quaternion group with 4 elements is omitted because it is cyclic.) Note that $xyx = y$ implies $x^{2n}yx^{2n} = y$, so that $y^4 = 1$ and therefore $x^{4n} = 1$. In the case $n = 1$, sending $x \mapsto i$, $y \mapsto j$ gives an isomorphism with the usual presentation of Q_8 given by $\langle i, j, k \mid i^2 = j^2 = k^2 = ijk\rangle$.

Theorem 9.4. *The generalised quaternion group Q_{8n} has the Secondary Property 1.8 and the Tertiary Property 1.12.*

Proof. By [Tei92, Proposition 4.2.1], the differential

$$d^3_{5,0} \colon H_5(Q_{8n};\mathbb{Z}) \to H_2(Q_{8n};\mathbb{Z}/2)$$

is an isomorphism. In particular, the $\mathfrak{ter}$ invariant lies in the trivial group. Moreover, this implies that $d^2_{5,0} \colon H_5(Q_{8n};\mathbb{Z}) \to H_3(Q_{8n};\mathbb{Z}/2)$ is the zero map. It follows immediately from the fact that $H_2(Q_{8n};\mathbb{Z}/2)/\operatorname{im}(d^3_{5,0}) = 0$ that Q_{8n} has the Tertiary Property 1.12.

Now we work on showing that the Secondary Property holds for Q_{8n}. Let $N \in \mathbb{Z}Q_{8n}$ be the norm element $N = \sum_{g\in Q_{8n}} g$ and let $\varepsilon\colon \mathbb{Z}Q_{8n} \to \mathbb{Z}$ be the augmentation. By [CE56, page 253] the beginning of a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}Q_{8n}$-module is given as follows:

$$\begin{tikzcd}
C_4 \arrow[dd, "d_4"'] & = & & \mathbb{Z}Q_{8n} \arrow[dd, "N"] & \\
& & & & \\
C_3 \arrow[dd, "d_3"'] & = & & \mathbb{Z}Q_{8n} \arrow[ddl, "x-1"'] \arrow[ddr, "1-xy"] & \\
& & & & \\
C_2 \arrow[dd, "d_2"'] & = & \mathbb{Z}Q_{8n} \arrow[dd, "\sum_{i=0}^{2n-1} x^i"'] \arrow[ddrr, "-y-1" near start] & \oplus & \mathbb{Z}Q_{8n} \arrow[dd, "x-1"] \arrow[ddll, "xy+1"' near start] \\
& & & & \\
C_1 \arrow[dd, "d_1"'] & = & \mathbb{Z}Q_{8n} \arrow[ddr, "x-1"'] & \oplus & \mathbb{Z}Q_{8n} \arrow[ddl, "y-1"] \\
& & & & \\
C_0 & = & & \mathbb{Z}Q_{8n} \arrow[d, "\varepsilon"] & \\
& & & \mathbb{Z} &
\end{tikzcd}$$

Let K denote the 2-dimensional CW complex determined by the start of this resolution (which is the same as the presentation complex for the above presentation of Q_{8n}).

One easily computes that $H_3(Q_{8n};\mathbb{Z}/2) \cong \mathbb{Z}/2 \otimes_{\mathbb{Z}Q_{8n}} \pi_2(K) \cong \mathbb{Z}/2$. Recall from the proof of the Tertiary Property for these groups, that $d^2_{5,0}$ is the zero map. To verify Condition 4.17 and therefore show that the Secondary Property 1.8 holds, we therefore have to show that the map $A\colon H_3(Q_{8n};\mathbb{Z}/2) \to \widehat{H}^0(\mathrm{Sesq}(H^2(K;\mathbb{Z}\pi)))$ is injective.

The nontrivial element in $\mathbb{Z}/2 \otimes_{\mathbb{Z}Q_8} \pi_2(K) \cong \mathbb{Z}/2 \otimes_{\mathbb{Z}Q_8} \ker d_2$ is represented by $1 \otimes d_3(1)$. By Lemma 4.12 and the definition of A (Definition 4.15), the form $\lambda := A(1 \otimes d_3(1))$ on

$$H^2(K;\mathbb{Z}Q_{8n}) \cong \operatorname{coker} d^2 = (\mathbb{Z}Q_{8n})^2 \Big/ \left\langle \Big(\sum_{i=0}^{2n-1} x^{-i}, (xy)^{-1}+1 \Big), \Big(-y^{-1}-1, x^{-1}-1 \Big) \right\rangle$$

is given by the matrix

$$L := \begin{pmatrix} (x^{-1}-1)(x-1) & (x^{-1}-1)(1-xy) \\ (1-(xy)^{-1})(x-1) & (1-(xy)^{-1})(1-xy) \end{pmatrix}.$$

Note that we used the involution on $\mathbb{Z}Q_{8n}$ to view $\operatorname{coker} d^2$ as a left module. The entries of the matrix are obtained from the third boundary map d_3 in the above resolution. This matrix a priori defines a pairing on the free module $\mathbb{Z}Q_{8n}^2$. However the matrix L determines a well-defined pairing on $H^2(K;\mathbb{Z}Q_{8n}) \cong \operatorname{coker} d^2$, due to the fact that $d_2 \circ d_3 = 0$ in the resolution above. Throughout our verification of the Secondary Property, we will often write forms on quotient modules as matrices defining forms on free modules, and it will always be necessary that these descend to well defined maps on the quotient modules.

Claim. The form $\lambda := A(1 \otimes d_3(1))$ is odd.

To see the claim, we investigate possible forms q that might possibly exhibit λ as even, and show that no such q can exist. A possible q with $q + q^* = \lambda$ can be written as

$$\begin{pmatrix} 1-x+z_1 & (x^{-1}-1)(1-xy)+z_2 \\ -\overline{z_2} & 1-xy+z_3 \end{pmatrix}$$

for some $z_1, z_2, z_3 \in \mathbb{Z}Q_{8n}$ satisfying $z_1 = -\overline{z_1}, z_3 = -\overline{z_3}$. Since

$$\big[-y^{-1}-1, x^{-1}-1 \big], \Big[\sum x^{-i}, (xy)^{-1}+1 \Big] = 0 \in \operatorname{coker} d^2,$$

the form q has to satisfy the relations

$$\begin{aligned} 0 &= q((1,0),(-y^{-1}-1, x^{-1}-1)) \\ &= (1-x+z_1)(-y-1) + (x^{-1}-1)(1-xy)(x-1) + z_2(x-1) \end{aligned}$$

and

$$0 = q\big((0,1), \big(\sum x^{-i}, (xy)^{-1}+1 \big)\big) = -\overline{z_2}\big(\sum x^i \big) + (1-xy+z_3)(xy+1).$$

To derive a contradiction, we will show that these equations cannot be solved after passing to the abelianisation $(Q_{8n})_{ab} \cong (\mathbb{Z}/2)^2$. We get that $z_1 = -\overline{z_1}$ implies $2z_1 = 0$ in $\mathbb{Z}[(\mathbb{Z}/2)^2]$, so $z_1 = 0$. Similarly $z_3 = 0$. Also $x^2 = 1$ implies that $\sum_{i=1}^{2n-1} x^i = n(1+x)$. The first relation, with $z_1 = 0$ substituted, gives

$$0 = (1-x)(-y-1) + 2(1-x)(1-xy) + z_2(x-1) = (1-x)(1+y) + z_2(x-1) \in \mathbb{Z}[(\mathbb{Z}/2)^2].$$

It follows that $z_2 = (1+y) - a(1+x)$ for some $a \in \mathbb{Z}(Q_{8n})_{ab}$, since $(1+x)$ divides any element of $\mathbb{Z}[\mathbb{Z}/2]$ that annihilates $(1-x)$. Insert $z_2 = (1+y) - a(1+x)$ into the second relation to obtain

$$\begin{aligned} 0 &= -n(1+x)(1+y-a-ax) + (1-xy)(1+xy) \\ &= -n(1+x)(1+y-a-ax) = 2na(1+x) - nN, \end{aligned}$$

where N now denotes the norm element in $\mathbb{Z}[(\mathbb{Z}/2)^2]$ of $Q_{8n}^{ab} = (\mathbb{Z}/2)^2$. But the first summand has all coefficients divisible by $2n$, while the second summand has all coefficients only divisible by n. Hence this element does not vanish in $\mathbb{Z}[(\mathbb{Z}/2)^2]$. Hence there are no solutions for z_1, z_2, z_3 that define a q as desired and so λ cannot be even as claimed. This completes the proof of the claim, so that Condition 4.17 holds for Q_{8n}, completing the proof of Theorem 9.4. □

10. Abelian groups

In this section we will focus on abelian groups. The goal is to prove the next theorem, classifying spin 4-manifolds with abelian fundamental groups up to stable diffeomorphism.

Theorem 10.1. *The Secondary Property 1.8 and the Tertiary Property 1.12 hold for all finitely generated abelian groups.*

The proof breaks up into stages, each of which is considered in one of the next few subsections. First we consider cyclic groups, then we consider abelian groups with at most two generators, then at most three generators, and finally we use the inheritance properties from Section 7 to deduce that the properties hold for any finitely generated abelian group.

We will need the following structure of certain cohomology rings of finite cyclic groups.

Theorem 10.2 ([Hat02, Example 3.41])**.** *Let C be a cyclic group of order $n = 2k$. Then the cohomology ring with $\mathbb{Z}/n$ coefficients is a quotient of the polynomial ring $(\mathbb{Z}/n)[\alpha,\beta]$, as follows:*

$$H^*(C;\mathbb{Z}/n) \cong (\mathbb{Z}/n)[\alpha,\beta]/(\alpha^2 - k\beta),$$

where $|\alpha| = 1$ and $|\beta| = 2$.

Corollary 10.3. *Let C be a cyclic group of order $n = 2k$. Then the cohomology ring with $\mathbb{Z}/2$ coefficients is a quotient of the polynomial ring $(\mathbb{Z}/2)[\alpha,\beta]$, as follows:*

$$H^*(C;\mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2[\alpha,\beta]/\alpha^2 & \text{if } k = 2m \\ \mathbb{Z}/2[\alpha] & \text{if } k = 2m+1 \end{cases}$$

where $|\alpha| = 1$ and $|\beta| = 2$.

Note that the corollary is not obtained simply by setting $n = 2$. Rather, $\alpha^2 = k\beta$ becomes either $\alpha^2 = 0$ or $\alpha^2 = \beta$, when $k = 2m$ or $k = 2m+1$ respectively.

10.1. **Cyclic groups.**

Lemma 10.4. *The Secondary Property 1.8 and the Tertiary Property 1.12 hold for all cyclic groups.*

10.1.1. *Cyclic groups have the secondary property.* Let G be a cyclic group. If the cyclic group G is of odd order or infinite order, then $H_3(G;\mathbb{Z}/2) = 0$, and there is nothing to show.

If $G = \langle T \rangle$ is cyclic of even order $2k$, then we need to check that Condition 4.17. Note that

$$\mathrm{red}_2\colon \mathbb{Z}/|G| \cong H_5(G;\mathbb{Z}) \to H_5(G;\mathbb{Z}/2) \cong \mathbb{Z}/2$$

is the projection. The generator is dual to $\alpha\beta^2$ in the notation of Corollary 10.3, if $k = 2m$ for some m, and the generator is dual to α^5 if $k = 2m + 1$. Similarly the generator of $H_3(G; \mathbb{Z}/2)$ is dual to $\alpha\beta$ when $k = 2m$ and α^3 when $k = 2m + 1$ In both cases

$$Sq^2 \colon H^3(G; \mathbb{Z}/2) \to H^5(G; \mathbb{Z}/2)$$

is an isomorphism by Corollary 10.3, together with a straightforward computation using the axioms of the Steenrod operations. In particular recall the Cartan formula $\mathrm{Sq}^n(xy) = \sum_{p+q=n} \mathrm{Sq}^p(x)\mathrm{Sq}^q(y)$. Thus

$$Sq_2 \circ \mathrm{red}_2 \colon H_5(G; \mathbb{Z}) \to H_3(G; \mathbb{Z}/2)$$

is surjective.

Therefore, to verify Condition 4.17, we have to show that

$$A \colon H_3(G; \mathbb{Z}/2) \to \widehat{H}^0(\mathrm{Sesq}(H^2(K; \mathbb{Z}G)))$$

is trivial. A free resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$ module is given by

$$\ldots \to \mathbb{Z}G \xrightarrow{N_T} \mathbb{Z}G \xrightarrow{1-T} \mathbb{Z}G \xrightarrow{N_T} \mathbb{Z}G \xrightarrow{1-T} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z},$$

where $N_T = \sum_{i=0}^{n-1} T^i$. Let K be the corresponding 2-complex with $\pi_1(K) \cong G$. Then $\mathbb{Z}/2 \otimes_{\mathbb{Z}G} \pi_2(K) \cong \mathbb{Z}/2 \otimes_{\mathbb{Z}G} \ker(N_T)$ is generated by the image of 1 under

$$d_3 = (1 - T) \colon \mathbb{Z}G \to \ker(N_T) \to \mathbb{Z}/2 \otimes_{\mathbb{Z}G} \ker(N_T).$$

Hence by Lemma 4.12, the form λ on $H^2(K; \mathbb{Z}G)$ that we have to consider is given by

$$\lambda(x, y) = x(1 - T^{-1})(1 - T)\overline{y},$$

for $x, y \in \mathbb{Z}G/(\overline{N_T}) = \mathbb{Z}G/(N_T)$. For q given by $q(x, y) = x(1 - T)\overline{y}$, we have $\lambda = q + q^*$. Thus λ is even and so the map A is trivial as required. It follows that the sequence of Condition 4.17 is exact as required, so that the cyclic group G has the Secondary Property 1.8.

10.1.2. *Cyclic groups have the tertiary property.* By Corollary 10.3, we have that $d^2_{4,1} = Sq_2 \colon H_4(G; \mathbb{Z}/2) \to H_2(G; \mathbb{Z}/2)$ is an isomorphism. Hence the group in which the $\mathfrak{ter}$ invariant resides is trivial, and the Tertiary Property 1.12 trivially holds for cyclic groups.

10.2. **Abelian groups with at most two generators.** This section proves the following lemma.

Lemma 10.5. *The Secondary Property 1.8 and the Tertiary Property 1.12 hold for all abelian groups with at most two generators.*

The proof of this lemma will require about eight pages. We consider the secondary property first.

10.2.1. *Two generator abelian groups have the secondary property.*

Claim. To prove the secondary property for two generator abelian groups, it suffices to consider the groups $\pi \cong \mathbb{Z}/2^{k_1} \times \mathbb{Z}/2^{k_2} = \langle a, b \mid a^{2^{k_1}}, b^{2^{k_2}}, [a, b]\rangle$, for some $k_1, k_2 \geq 1$, and the groups $\pi \cong \mathbb{Z} \times \mathbb{Z}/2^k$ for some $k \geq 1$.

The claim follows from the fact that we can pass to a finite odd index subgroup by Theorem 8.2, and the fact that $\mathbb{Z} \times \mathbb{Z}$ has cohomological dimension two, which we already know to have the Secondary property by Theorem 9.1. Every abelian group with two generators (as the minimal number of generators) has a finite odd index subgroup that belongs to the list in the claim. We already proved that cyclic groups have the Secondary property in Section 10.1.1. This completes the proof of the claim. □

For each of the groups in the claim, we need to check Condition 4.17. This will occupy the next few pages.

The groups $\mathbb{Z}/2^{k_1} \times \mathbb{Z}/2^{k_2}$.
We begin by considering the groups $\mathbb{Z}/2^{k_1} \times \mathbb{Z}/2^{k_2}$, with $k_1, k_2 \geq 1$. In this case $H_3(\pi;\mathbb{Z}/2) \cong (\mathbb{Z}/2)^4$. If $\alpha \in H_3(\pi;\mathbb{Z}/2)$ is in the image of the inclusion of one of the two factors, then we know exactness at α by the last subsection. Thus by symmetry we only have to consider the nontrivial element $\gamma \in H_3(\pi;\mathbb{Z}/2)$ coming from the generator of $H_1(\mathbb{Z}/2^{k_1};\mathbb{Z}/2) \otimes H_2(\mathbb{Z}/2^{k_2};\mathbb{Z}/2) \cong \mathbb{Z}/2$.

Claim. The element γ lies in $\operatorname{im}(Sq_2 \circ \mathrm{red}_2)$ if and only if $k_1 \leq k_2$.

The proof of this claim will take the next page and a half. We will show, in the case that $k_1 < k_2$, that γ lies in the image of $Sq_2 \circ \mathrm{red}_2$, and that the nontrivial element $\overline{\gamma} \in H_2(\mathbb{Z}/2^{k_1};\mathbb{Z}/2) \otimes H_1(\mathbb{Z}/2^{k_2};\mathbb{Z}/2) \subseteq H_3(\pi;\mathbb{Z}/2)$ does not lie in the image. In the case that $k_1 = k_2$, we will show that one, and hence both, of γ and $\overline{\gamma}$ lies in $\operatorname{im}(Sq_2 \circ \mathrm{red}_2)$.

First we compute the image of the $\mathrm{Sq}^2\colon H^3(\pi;\mathbb{Z}/2) \to H^5(\pi;\mathbb{Z}/2)$. A preliminary computation that we will soon need is that $\mathrm{Sq}^1(\beta_i) = 0$, with $\beta_i \in H^2(\mathbb{Z}/2^{k_i};\mathbb{Z}/2)$ as in Corollary 10.3, and $k_i > 1$. To see this, we use the fact that Sq^1 coincides with the Bockstein homomorphism $BS = \mathrm{Sq}^1\colon H^2(\mathbb{Z}/2^{k_i};\mathbb{Z}/2) \to H^3(\mathbb{Z}/2^{k_i};\mathbb{Z}/2)$ associated to the coefficient sequence $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$ (see [MT68, Chapter 3, Theorem 1]). We have that $H^2(\mathbb{Z}/2^{k_i};\mathbb{Z}/4) = \mathbb{Z}/4 = H^3(\mathbb{Z}/2^{k_i};\mathbb{Z}/4)$. The differentials in the tensored down resolutions all vanish, so maps in the Bockstein long exact sequence coincide with the maps in the short exact sequence $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$ that induces it. Thus the connecting homomorphism vanishes. This completes the proof that $\mathrm{Sq}^1(\beta_i) = 0$.

To compute the image of Sq^2, we start with the case that $1 < k_1 \leq k_2$. Then $H^3(\mathbb{Z}/2^{k_1} \times \mathbb{Z}/2^{k_2};\mathbb{Z}/2)$ is generated by elements of the form $\alpha_i\beta_j$, in the notation of Corollary 10.3 with $i, j \in \{1,2\}$. We compute:

$$\mathrm{Sq}^2(\alpha_i\beta_j) = \mathrm{Sq}^2(\alpha_i)\beta_j + \mathrm{Sq}^1(\alpha_i)\mathrm{Sq}^1(\beta_j) + \alpha_i\mathrm{Sq}^2(\beta_j) = \alpha_i\beta_j^2.$$

Next, we consider the case that $k_1 = 1 < k_2$. Now $H^3(\mathbb{Z}/2^{k_1} \times \mathbb{Z}/2^{k_2};\mathbb{Z}/2)$ is generated by elements $\alpha_1\beta_2$, $\alpha_2\beta_2$, α_1^3, and $\alpha_1^2\alpha_2$. The image of the first two cases are $\alpha_1\beta_2^2$ and $\alpha_2\beta_2^2$ by the computation above. We also have $\mathrm{Sq}^2(\alpha_1^3) = \alpha_1^5$ and $\mathrm{Sq}^2(\alpha_1^2\alpha_2) = \alpha_1^4\alpha_2$.

Finally we consider the case $k_1 = k_2 = 1$. In this case $H^3(\mathbb{Z}/2 \times \mathbb{Z}/2;\mathbb{Z}/2)$ is generated by elements $\alpha_1^i\alpha_2^j$ with $i + j = 3$. An inductive argument using the Cartan formula shows that $\mathrm{Sq}^2(\alpha^i)$ is equal to α^{i+2} if $i \equiv 2, 3 \mod 4$ and $\mathrm{Sq}^2(\alpha^i) = 0$ if $i \equiv 0, 1 \mod 4$. Thus we compute

$$\mathrm{Sq}^2(\alpha_1^2\alpha_2) = \alpha_1^4\alpha_2;\ \mathrm{Sq}^2(\alpha_i^3) = \alpha_i^5;\text{ and } \mathrm{Sq}^2(\alpha_1\alpha_2^2) = \alpha_1\alpha_2^4.$$

It follows that the map $\mathrm{Sq}_2\colon H_5(\pi;\mathbb{Z}/2) \to H_3(\pi;\mathbb{Z}/2)$ is onto in all cases. A resolution of $\mathbb{Z}$ by free $\mathbb{Z}\pi$-modules is given below. To complete the proof of the claim, one has to check whether the elements of $H_5(\pi;\mathbb{Z}/2) = H_5(\mathbb{Z}/2^{k_1} \times \mathbb{Z}/2^{k_2};\mathbb{Z}/2)$, that hit the generators of $H_3(\mathbb{Z}/2^{k_1} \times \mathbb{Z}/2^{k_2};\mathbb{Z}/2)$ under the dual of the Steenrod square, are in the image of the reduction modulo two map red_2.

First we consider the case that $k_1 < k_2$. $(\alpha_1\beta_2^2)^* \in H_5(\mathbb{Z}/2^{k_1} \times \mathbb{Z}/2^{k_2};\mathbb{Z}/2)$ maps to γ and $(\alpha_2\beta_1^2)^*$ maps to $\overline{\gamma}$. The terms in degree 4, 5 and 6 of the $\mathbb{Z}\pi$-module resolution of $\mathbb{Z}$, tensored down over $\mathbb{Z}$, are shown in the diagram below. Let $k := k_1$ and let $\ell := k_2$, so that $k \leq \ell$ and $\pi = \mathbb{Z}/2^k \times \mathbb{Z}/2^\ell$. Also denote $C_* := C_*(\pi;\mathbb{Z}\pi)$. The next diagram shows

$$\mathbb{Z} \otimes C_6 \cong \mathbb{Z}^7 \to \mathbb{Z} \otimes C_5 \cong \mathbb{Z}^6 \to \mathbb{Z} \otimes C_4 \cong \mathbb{Z}^5.$$

Ignore the underlining of two $\mathbb{Z}$ summands for now.

$$\begin{tikzcd}
\mathbb{Z} \arrow[dr, "2^k"'] & \oplus & \mathbb{Z} \arrow[dl, "0"] \arrow[dr, "0"'] & \oplus & \mathbb{Z} \arrow[dl, "2^\ell"] \arrow[dr, "2^k"'] & \oplus & \mathbb{Z} \arrow[dl, "0"] \arrow[dr, "0"'] & \oplus & \mathbb{Z} \arrow[dl, "2^\ell"] \arrow[dr, "2^k"'] & \oplus & \mathbb{Z} \arrow[dl, "0"] \arrow[dr, "0"'] & \oplus & \mathbb{Z} \arrow[dl, "2^\ell"] \\
& \mathbb{Z} \arrow[dr, "0"'] & \oplus & \underline{\mathbb{Z}} \arrow[dl, "0"] \arrow[dr, "-2^k"'] & \oplus & \mathbb{Z} \arrow[dl, "2^\ell"] \arrow[dr, "0"'] & \oplus & \mathbb{Z} \arrow[dl, "0"] \arrow[dr, "-2^k"'] & \oplus & \underline{\mathbb{Z}} \arrow[dl, "2^\ell"] \arrow[dr, "0"'] & \oplus & \mathbb{Z} \arrow[dl, "0"] \\
& & \mathbb{Z} & \oplus & \mathbb{Z} & \oplus & \mathbb{Z} & \oplus & \mathbb{Z} & \oplus & \mathbb{Z}
\end{tikzcd}$$

In $\mathbb{Z}/2 \otimes C_5$, we have $e_2 = (\alpha_2\beta_1^2)^*$, which maps to $\overline{\gamma} = (\alpha_2\beta_1)^* \in H_3(\pi;\mathbb{Z}/2)$. Similarly $e_5 = (\alpha_1\beta_2^2)^*$, which maps to $\gamma = (\alpha_1\beta_2)^* \in H_3(\pi;\mathbb{Z}/2)$. The summands generated by e_2 and e_5 are underlined in the diagram above. In the case that one or both of k, ℓ are equal to 1, replace β_i by α_i^2 in the above statements. Otherwise the computation is the same.

The relevant generator of $H_5(\pi;\mathbb{Z})$ for $\overline{\gamma}$ is $2^{k-\ell}e_2 + e_3$. If $k < \ell$ then this maps to e_3 in $\mathbb{Z}/2 \otimes C_5$, and so $\overline{\gamma}$ is not hit by $\mathrm{Sq}_2 \circ \mathrm{red}_2$. On the other hand, the relevant generator for $H_5(\pi;\mathbb{Z})$ for γ is $e_5 + 2^{k-\ell}e_4$, which maps to e_5 in $\mathbb{Z}/2 \otimes C_5$. Then e_5 maps to γ under $\mathrm{Sq}_2 \colon H_5(\pi;\mathbb{Z}/2) \to H_3(\pi;\mathbb{Z}/2)$. Thus when $k < \ell$, we see that γ lies in the image of $\mathrm{Sq}_2 \circ \mathrm{red}_2$. This proves the claim in the case that $k \neq \ell$, that is $k_1 \neq k_2$.

Now consider the case that $k_1 = k_2$, that is $k = \ell$. We just need to show that γ does lie in the image of $\mathrm{Sq}_2 \circ \mathrm{red}_2 \colon H_5(\pi;\mathbb{Z}) \to H_3(\pi;\mathbb{Z}/2)$. The relevant generator of the $\mathbb{Z}$-homology is $e_4 + e_5$. This maps to $e_4 + e_5$ in $\mathbb{Z}/2 \otimes C_5$. Now, e_4 is dual to $\beta_1\beta_2\alpha_2$ if $k = \ell \geq 2$, or $\alpha_1^2\alpha_2^3$ if $k_1 = k_2 = 1$. From the computation of the Steenrod square $\mathrm{Sq}^2 \colon H^3(\pi;\mathbb{Z}/2) \to H^5(\pi;\mathbb{Z}/2)$ that we made not long ago, it follows that $e_4 + e_5 \mapsto (\alpha_1\beta_2)^* = \gamma$ under the dual map $\mathrm{Sq}_2 \colon H_5(\pi;\mathbb{Z}/2) \to H_3(\pi;\mathbb{Z}/2)$, as required. Thus the element γ lies in $\mathrm{im}(Sq_2 \circ \mathrm{red}_2)$ if and only if $k_1 \leq k_2$. This completes the proof of the claim. □

Now we need to show that the cases in which $\gamma \in \mathrm{im}(\mathrm{Sq}_2 \circ \mathrm{red}_2)$ correspond to the cases in which $A(\gamma)$ is even. Let $n := 2^{k_1} = 2^k$ and $m := 2^{k_2} = 2^\ell$. Let $N_a := \sum_{i=0}^{n-1} a^i$ and $N_b := \sum_{i=0}^{m-1} b^i$. From the $\mathbb{Z}\pi$-chain complex

$$\begin{tikzcd}
\mathbb{Z}\pi \arrow[dr, "1-a"'] & \oplus & \mathbb{Z}\pi \arrow[dl, "1-b"] \arrow[dr, "-N_a"'] & \oplus & \mathbb{Z}\pi \arrow[dl, "N_b"] \arrow[dr, "1-a"'] & \oplus & \mathbb{Z}\pi \arrow[dl, "1-b"] \\
& \mathbb{Z}\pi \arrow[dr, "N_a"'] & \oplus & \mathbb{Z}\pi \arrow[dl, "1-b"] \arrow[dr, "a-1"'] & \oplus & \mathbb{Z}\pi \arrow[dl, "N_b"] & \\
& & \mathbb{Z}\pi \arrow[dr, "1-a"'] & \oplus & \mathbb{Z}\pi \arrow[dl, "1-b"] & & \\
& & & \mathbb{Z}\pi & & &
\end{tikzcd}$$

one computes $H^2(K;\mathbb{Z}\pi) \cong (\mathbb{Z}\pi)^3/\langle(N_a, 1-b^{-1}, 0), (0, a^{-1}-1, N_b)\rangle$, and that $A(\gamma)$ is the form represented by:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & mN_b & (1-a)N_b \\ 0 & (1-a^{-1})N_b & 2-a-a^{-1} \end{pmatrix}.$$

Claim. For $k_1 > k_2$, that is $n > m$, the form $A(\gamma)$ is not even.

A possible q with $A(\gamma) = q + q^*$ has to have the following shape:

$$\begin{pmatrix} u & v & w \\ -\overline{v} & \frac{m}{2}N_b + y & -aN_b + x \\ -\overline{w} & N_b - \overline{x} & 1 - a + z \end{pmatrix}$$

with $u, v, w, x, y, z \in \mathbb{Z}\pi$, $u + \overline{u} = y + \overline{y} = z + \overline{z} = 0$. In order to determine a form on $H^2(K; \mathbb{Z}\pi)$ it has to satisfy:

$$0 = (N_b - \overline{x})(a - 1) + (1 - a + z)N_b = (1 - a)\overline{x} + N_b z$$

and

$$0 = \tfrac{m}{2}(a - 1)N_b + (a - 1)y - amN_b + xN_b = -\tfrac{m}{2}(a + 1)N_b + (a - 1)y + xN_b.$$

Apply the involution to the second equation and then multiply by $(1 - a)$ to yield

$$\begin{aligned} 0 &= -\tfrac{m}{2}(1 - a)(a^{-1} + 1)N_b - (1 - a)(a^{-1} - 1)y + (1 - a)\overline{x}N_b \\ &= \tfrac{m}{2}(a - a^{-1})N_b + (2 - a - a^{-1})y + (1 - a)\overline{x}N_b \\ &= \tfrac{m}{2}(a - a^{-1})N_b + (2 - a - a^{-1})y - mzN_b. \end{aligned}$$

Here we used $\overline{y} = -y$, while for the last equality we used $(1 - a)\overline{x} = -N_b z$ as well as $N_b^2 = mN_b$. Reduce coefficients modulo m and take the coefficient of a to obtain:

$$0 = \frac{m}{2} + 1y_a - y_0 - y_{a^2} \mod m.$$

Since $y = -\overline{y}$ we have $y_0 = 0$ and thus $y_{a^2} = 2y_a + \frac{m}{2}$. Take the coefficient of a^k for $1 < k \leq \frac{n}{2}$ to obtain:

$$0 = 2y_{a^k} - y_{a^{k-1}} - y_{a^{k+1}} \mod m.$$

Thus by induction we have:

$$y_{a^k} = ky_a + (-1)^{\delta_{4|k}}\delta_{2|k}\tfrac{m}{2} \mod m, \tag{10.6}$$

where $\delta_{2|k}$ and $\delta_{4|k}$ are the characteristic functions of the Boolean statements $2|k$ and $4|k$ respectively. That is, for example, $\delta_{2|k} = 1$ if 2 divides k, and $\delta_{2|k} = 0$ otherwise. If $n = 2^k m$ for $k \geq 1$, then $m|\frac{n}{2}$ and

$$y_{a^{n/2}} = \pm\tfrac{m}{2} \mod m$$

But since $y = -\overline{y}$, the coefficient $y_{a^{n/2}}$ has to be zero. This shows the claim that the form $A(\gamma)$ cannot be even if $n = 2^k m$ for some $k \geq 1$, i.e. if $k_1 > k_2$. □

Claim. The form $A(\gamma)$ is even if $k_1 \leq k_2$.

First we suppose that $k_1 = k_2$, that is $n = m$. Let y be such that $y_{a^k b^l} = k - \frac{n}{2}$ for $k \neq 0$. First note that $y_{a^{-k}b^{-l}} = (n - k) - \frac{n}{2} = -(k - \frac{n}{2}) = -y_{a^k b^l}$. Therefore, $\overline{y} = -y$. Consider the form q given by

$$\begin{pmatrix} 0 & 0 & -(1 - b^{-1}) \\ 0 & \frac{m}{2}N_b + y & -aN_b + N_a \\ (1 - b) & N_b - N_a & 1 - a \end{pmatrix}$$

It follows from $N_a = \overline{N_a}$ and $\overline{y} = -y$ that $q + q^* = A(\gamma)$. It remains to show that q is indeed a form on $H^2(K;\mathbb{Z}\pi)$. For this it has to satisfy the equations:

$$0 = 0 \cdot N_a + 0 \cdot (1-b)$$
$$0 = 0 \cdot N_a + (\tfrac{m}{2} N_b + y)(1-b) \tag{10.7}$$
$$0 = (1-b) \cdot N_a + (N_b - N_a)(1-b)$$
$$0 = 0 \cdot (a-1) - (1-b^{-1})N_b$$
$$0 = (\tfrac{m}{2} N_b + y)(a-1) - (aN_b + N_a)N_b \tag{10.8}$$
$$0 = N_b(a-1) + (1-a)N_b$$

Equation (10.7) is true since y is a multiple of N_b by definition. All but Equation (10.8) are satisfied, using $(1-b)N_b = (1-b^{-1})N_b = 0$. To check (10.8), we have to show that

$$0 = (\tfrac{m}{2} N_b + y)(a-1) - (aN_b + N_a)N_b = -\tfrac{m}{2}(a+1)N_b + (a-1)y - N_a N_b.$$

Since each term is a multiple of N_b, it suffices to show that this holds for the coefficients of a^k, for every $k \geq 0$. For a^0 we have

$$0 = -\tfrac{m}{2} + y_{a^{-1}} - y_0 + 1 = -\tfrac{m}{2} + m - 1 - \tfrac{m}{2} - 0 + 1 = 0.$$

For a^1 we get

$$0 = -\tfrac{m}{2} + y_0 - y_a + 1 = -\tfrac{m}{2} + 0 - 1 + \tfrac{m}{2} + 1 = 0.$$

Finally for a^k with $k > 1$, we have

$$0 = 0 + y_{a^{k-1}} - y_{a^k} + 1 = k - 1 - \tfrac{m}{2} - k + \tfrac{m}{2} + 1 = 0.$$

This completes the verification of Equation (10.8). This completes the proof of the claim that $A(\gamma)$ is even when $k_1 = k_2$.

If $k_1 < k_2$, consider the projection $\mathbb{Z}/2^{k_2} \times \mathbb{Z}/2^{k_2} \to \mathbb{Z}/2^{k_1} \times \mathbb{Z}/2^{k_2}$ applied to the form q, to see that the form $A(\gamma)$ also has to be even in this case. This completes the proof of the claim that $A(\gamma)$ is even when $k_1 \leq k_2$, completing the proof that the groups $\mathbb{Z}/s^{k_1} \times \mathbb{Z}/2^{k_2}$ have the Secondary Property 1.8. □

The groups $\mathbb{Z} \times \mathbb{Z}/2^k$.

Next, we consider the group $\pi = \mathbb{Z} \times \mathbb{Z}/2^k = \langle a, b \mid [a,b], b^{2^k} \rangle$. Once again we consider the element $\gamma \in H_3(\pi;\mathbb{Z}/2)$ coming from $H_1(\mathbb{Z};\mathbb{Z}/2) \otimes H_2(\mathbb{Z}/2^k;\mathbb{Z}/2)$.

Claim. The form $A(\gamma)$ is not even.

A resolution of $\mathbb{Z}$ by $\mathbb{Z}\pi$ modules can be obtained from the tensor product

$$\big(\mathbb{Z}\pi \xrightarrow{1-a} \mathbb{Z}\pi\big) \otimes \big(\mathbb{Z}\pi \otimes_{\mathbb{Z}[\mathbb{Z}/2]} C_*(B\mathbb{Z}/2;\mathbb{Z}[\mathbb{Z}/2])\big).$$

This is a subcomplex of the resolution depicted above in the case $\mathbb{Z}/2^{k_1} \times \mathbb{Z}/2^{k_2}$, obtained by deleting everything apart from the two right-most summands of each chain group. We compute that $H^2(K;\mathbb{Z}\pi) \cong (\mathbb{Z}\pi)^2/\langle (1-b^{-1},0), (a^{-1}-1, N_b)\rangle$ and that $A(\gamma)$ is the form

$$\begin{pmatrix} 2^k N_b & (1-a)N_b \\ (1-a^{-1})N_b & 2-a-a^{-1} \end{pmatrix}.$$

Now let $m = 2^k$. A possible q with $A(\gamma) = q + q^*$ has to have the following shape:

$$\begin{pmatrix} \tfrac{m}{2} N_b + y & -aN_b + x \\ N_b - \overline{x} & 1-a+z \end{pmatrix}$$

with $x, y, z \in \mathbb{Z}\pi$, $y + \overline{y} = z + \overline{z} = 0$.

A verbatim repetition of the argument for $\mathbb{Z}/2^{k_1} \times \mathbb{Z}/2^{k_2}$ in the case $k_1 > k_2$, starting at "In order to determine a form on $H^2(K; \mathbb{Z}\pi)$ it has to satisfy," shows that (10.6) also has to hold in the case $\mathbb{Z} \times \mathbb{Z}/m = \mathbb{Z} \times \mathbb{Z}/2^k$. Since in this case a has infinite order, (10.6) implies that infinitely many coefficients of y are non-zero. This is a contradiction to y being an element of the group ring. Hence the form $A(\gamma)$ cannot be even for $\mathbb{Z} \times \mathbb{Z}/2^k$. This completes the proof of the claim that $A(\gamma)$ is not even. ◻

Claim. The element γ is not in the image of $\mathrm{Sq}_2 \circ \mathrm{red}_2$.

It is straightforward to compute that the map

$$\mathrm{red}_2\colon H_5(\mathbb{Z} \times \mathbb{Z}/2^k; \mathbb{Z}) \cong \mathbb{Z}/2^k \to H_5(\mathbb{Z} \times \mathbb{Z}/2^k; \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

is given by $(0, \mathrm{pr})$, where $\mathrm{pr}\colon \mathbb{Z}/2^k \to \mathbb{Z}/2$ is the reduction modulo two. The generator of the image of this map is dual to $\alpha_2\beta_2^2$. For $k > 1$, the cohomology group $H^3(\mathbb{Z}/2^k; \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ is generated by $\alpha_2\beta_2$ and $\alpha_1\beta_2$; recall that $H^*(\mathbb{Z}/2^k; \mathbb{Z}/2) = (\mathbb{Z}/2)[\alpha_2, \beta_2]/\alpha_2^2$, and note that $H^*(\mathbb{Z}; \mathbb{Z}/2) = \mathbb{Z}[\alpha_1]/\alpha_1^2$, where α_1 has degree 1. But $\mathrm{Sq}^2(\alpha_2\beta_2) = \alpha_2\beta_2^2$ and $\mathrm{Sq}^2(\alpha_1\beta_2) = \alpha_1\beta_2^2$. It follows that

$$\mathrm{Sq}_2((\alpha_2\beta_2^2)^*) = \alpha_2\beta_2 \neq \alpha_1\beta_2 = \gamma.$$

The same argument goes through in the case that $k = 1$ if we replace β_2 by α_2^2 in the above computations. This completes proof of the claim that γ is not in the image of $\mathrm{Sq}_2 \circ \mathrm{red}_2$. ◻

We have therefore completed the proof that $\mathbb{Z} \times \mathbb{Z}/2^k$ satisfies Condition 4.17 and therefore has the Secondary Property. Since this is the last case we had to check, this completes the proof that abelian groups with at most two generators have the Secondary Property 1.8. ◻

10.2.2. *Two generator abelian groups have the tertiary property.* Let us now consider the Tertiary Property 1.12. From the discussion of the Tertiary property for cyclic groups (Section 10.1.2), it follows that the $\mathfrak{ter}$ invariant of an abelian group with two generators either lives in the trivial group or in $\mathbb{Z}/2$. In more detail, consider a group $\pi := C_1 \times C_2$, where C_1 and C_2 are cyclic groups. We have that

$$H_2(\pi; \mathbb{Z}/2) \cong \bigoplus_{i=0}^{2} H_i(C_1; \mathbb{Z}/2) \otimes H_{2-i}(C_2; \mathbb{Z}/2).$$

However the parts with $i = 0$ and $i = 2$ lie in the image of $\mathrm{Sq}_2 = d^2_{4,0}\colon H_4(\pi; \mathbb{Z}/2) \to H_2(\pi; \mathbb{Z}/2)$ by the structure of the cohomology ring of cyclic groups (Corollary 10.3). Therefore if the $\mathfrak{ter}$ invariant is nontrivial, it lives in $H_1(C_1; \mathbb{Z}/2) \otimes H_1(C_2; \mathbb{Z}/2) \cong \mathbb{Z}/2$. In the case that the $\mathfrak{ter}$ invariant lives in $\mathbb{Z}/2$, the nontrivial element corresponds to the commutator relation (i.e. to the 2-cell in a model for the classifying space inducing that relation).

By [KPR20, Proposition 6.1], $\mathbb{Z} \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2(K))$ is torsion-free if π is a finite abelian group with two generators. Hence the Tertiary Property 1.12 holds for these groups by Corollary 6.22.

Any infinite abelian group with two generators admits a surjection onto $\mathbb{Z}/8 \times \mathbb{Z}/2$, mapping the commutator relation to the commutator relation. By the discussion at the beginning of this subsection, this surjection induces an injection on the $\mathfrak{ter}$ invariant if the $\mathfrak{ter}$ invariant for $\mathbb{Z}/8 \times \mathbb{Z}/2$ is nontrivial. Hence by the Inheritance Theorem 7.7, it suffices to show that the $\mathfrak{ter}$ invariant for $\mathbb{Z}/8 \times \mathbb{Z}/2$ is nontrivial.

The commutator relation is dual to the product $\alpha_1\alpha_2 \in H^2(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z}/2)$ where α_1 and α_2 are the generators of H^1 of the two cyclic subgroups. Since $\mathbb{Z}/8$ has order higher than two, α_1^2 is trivial. In particular, $Sq^2(\alpha_1\alpha_2) = \alpha_1^2\alpha_2^2 = 0$, and the commutator relation is not in the image of the dual of Sq^2.

Now consider the commutative square

$$\begin{tikzcd}
\mathbb{Z}/2 \cong H_2(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z}/2)/\operatorname{im} d^2_{4,1} \arrow[r, "\cong", "p_*"'] & \mathbb{Z}/2 \cong H_2(\mathbb{Z}/4 \times \mathbb{Z}/2; \mathbb{Z}/2)/\operatorname{im} d^2_{4,1} \\
\ker d^2_{5,0} \subseteq H_5(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z}) \arrow[r, "p_*"'] \arrow[u, "d^3_{5,0}"] & \ker d^2_{5,0} \subseteq H_5(\mathbb{Z}/4 \times \mathbb{Z}/2; \mathbb{Z}), \arrow[u, "d^3_{5,0}"]
\end{tikzcd} \tag{10.9}$$

where $p\colon \mathbb{Z}/8 \times \mathbb{Z}/2 \to \mathbb{Z}/4 \times \mathbb{Z}/2$ is the projection.

Claim. The map p_* in the bottom row of the diagram (10.9) is trivial on $\ker d^2_{5,0} \subseteq H_5(\mathbb{Z}/8\times\mathbb{Z}/2;\mathbb{Z})$.

Using the claim, it follows by commutativity that $d^3_{5,0}\colon \ker d^2_{5,0} \subseteq H_5(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z}) \to \mathbb{Z}/2 \cong H_2(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z}/2)/\operatorname{im} d^2_{4,1}$ is trivial. Thus the $\mathfrak{ter}$ invariant for $\mathbb{Z}/8 \times \mathbb{Z}/2$ is nontrivial. This completes the proof that abelian groups with at most two generators have the Tertiary Property 1.12, modulo the proof of the claim.

Before we prove the claim we give a computation of the maps between group homology that is needed. Consider the projection map $p\colon \mathbb{Z}/2^n \to \mathbb{Z}/2^{n-1}$, and view $\mathbb{Z}[\mathbb{Z}/2^{n-1}]$ as a module over $\mathbb{Z}[\mathbb{Z}/2^n]$ via this projection. Write T for the generator of $\mathbb{Z}/2^n$ and U for the generator of $\mathbb{Z}/2^{n-1}$, and note that $p_*(T) = U$. The induced map on resolutions, shown in degrees up to four, is as follows:

$$\begin{tikzcd}
\mathbb{Z}[\mathbb{Z}/2^n] \arrow[r, "N_{2^n}"] \arrow[d, "4\cdot p_*"] & \mathbb{Z}[\mathbb{Z}/2^n] \arrow[r, "1-T"] \arrow[d, "2\cdot p_*"] & \mathbb{Z}[\mathbb{Z}/2^n] \arrow[r, "N_{2^n}"] \arrow[d, "2\cdot p_*"] & \mathbb{Z}[\mathbb{Z}/2^n] \arrow[r, "1-T"] \arrow[d, "p_*"] & \mathbb{Z}[\mathbb{Z}/2^n] \arrow[d, "p_*"] \\
\mathbb{Z}[\mathbb{Z}/2^{n-1}] \arrow[r, "N_{2^{n-1}}"] & \mathbb{Z}[\mathbb{Z}/2^{n-1}] \arrow[r, "1-U"] & \mathbb{Z}[\mathbb{Z}/2^{n-1}] \arrow[r, "N_{2^{n-1}}"] & \mathbb{Z}[\mathbb{Z}/2^{n-1}] \arrow[r, "1-U"] & \mathbb{Z}[\mathbb{Z}/2^{n-1}],
\end{tikzcd}$$

where $N_{2^j} = 1 + S + \cdots + S^{2^j-1}$ is the norm element of $\mathbb{Z}/2^j$ for $(S, j) \in \{(T, n), (U, n-1)\}$. This can be seen by starting with the identity map on C_0, and then extending in order to make the diagram commute. After tensoring with $\mathbb{Z}$, the induced vertical maps are unchanged: multiplication by 2^i in degrees $2i$ and $2i + 1$.

Proof of the claim.
Now we embark upon the proof of the claim, which is elementary but giving a detailed computation will require most of the next two pages. We need to show that $p_*\colon H_5(\mathbb{Z}/8\times\mathbb{Z}/2;\mathbb{Z}) \to H_5(\mathbb{Z}/4\times\mathbb{Z}/2;\mathbb{Z})$ is trivial on $\ker d^2_{5,0} \subseteq H_5(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z})$. Here we recall the part of the chain complex

$$C_6(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z}) \cong \mathbb{Z}^7 \to C_5(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z}) \cong \mathbb{Z}^6 \to C_4(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z}) \cong \mathbb{Z}^5$$

relevant to computing the fifth homology:

$$\begin{tikzcd}[column sep=small]
\mathbb{Z} \arrow[dr, "8"] & \oplus & \mathbb{Z} \arrow[dl, "0"'] \arrow[dr, "0"] & \oplus & \mathbb{Z} \arrow[dl, "2"'] \arrow[dr, "8"] & \oplus & \mathbb{Z} \arrow[dl, "0"'] \arrow[dr, "0"] & \oplus & \mathbb{Z} \arrow[dl, "2"'] \arrow[dr, "8"] & \oplus & \mathbb{Z} \arrow[dl, "0"'] \arrow[dr, "0"] & \oplus & \mathbb{Z} \arrow[dl, "2"'] \\
& \mathbb{Z} \arrow[dr, "0"] & \oplus & \mathbb{Z} \arrow[dl, "0"'] \arrow[dr, "-8"] & \oplus & \mathbb{Z} \arrow[dl, "2"'] \arrow[dr, "0"] & \oplus & \mathbb{Z} \arrow[dl, "0"'] \arrow[dr, "-8"] & \oplus & \mathbb{Z} \arrow[dl, "2"'] \arrow[dr, "0"] & \oplus & \mathbb{Z} \arrow[dl, "0"'] & \\
& & \mathbb{Z} & \oplus & \mathbb{Z} & \oplus & \mathbb{Z} & \oplus & \mathbb{Z} & \oplus & \mathbb{Z} & &
\end{tikzcd}$$

Let $e_0, \ldots, e_5$ be the basis for $C_5(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z})$ generating the summands shown in order. So $e_i \in C_{5-i}(\mathbb{Z}/8; \mathbb{Z}) \otimes C_i(\mathbb{Z}/2; \mathbb{Z})$. We see that

$$\ker(C_5(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z}) \to C_4(\mathbb{Z}/8 \times \mathbb{Z}/2; \mathbb{Z})) = \langle e_0, e_1 + 4e_2, e_3 + 4e_4, e_5\rangle.$$

Meanwhile

$$\mathrm{im}(C_6(\mathbb{Z}/8\times\mathbb{Z}/2;\mathbb{Z})\to C_5(\mathbb{Z}/8\times\mathbb{Z}/2;\mathbb{Z}))=\langle 8e_0, 2e_1+8e_2, 2e_3+8e_4, 2e_5\rangle.$$

Therefore

$$H_5(\mathbb{Z}/8\times\mathbb{Z}/2;\mathbb{Z})\cong\mathbb{Z}/8\oplus(\mathbb{Z}/2)^3=\langle e_0, e_1+4e_2, e_3+4e_4, e_5\rangle.$$

On the other hand part of the chain complex

$$C_6(\mathbb{Z}/4\times\mathbb{Z}/2;\mathbb{Z})\cong\mathbb{Z}^7\to C_5(\mathbb{Z}/4\times\mathbb{Z}/2;\mathbb{Z})\cong\mathbb{Z}^6\to C_4(\mathbb{Z}/4\times\mathbb{Z}/2;\mathbb{Z})\cong\mathbb{Z}^5$$

relevant to computing the fifth homology is:

$$\begin{tikzcd}
\mathbb{Z} \arrow[dr,"4"'] & \oplus & \mathbb{Z} \arrow[dl,"0"] \arrow[dr,"0"'] & \oplus & \mathbb{Z} \arrow[dl,"2"] \arrow[dr,"4"'] & \oplus & \mathbb{Z} \arrow[dl,"0"] \arrow[dr,"0"'] & \oplus & \mathbb{Z} \arrow[dl,"2"] \arrow[dr,"4"'] & \oplus & \mathbb{Z} \arrow[dl,"0"] \arrow[dr,"0"'] & \oplus & \mathbb{Z} \arrow[dl,"2"] \\
 & \mathbb{Z} \arrow[dr,"0"'] & \oplus & \mathbb{Z} \arrow[dl,"0"] \arrow[dr,"-4"'] & \oplus & \mathbb{Z} \arrow[dl,"2"] \arrow[dr,"0"'] & \oplus & \mathbb{Z} \arrow[dl,"0"] \arrow[dr,"-4"'] & \oplus & \mathbb{Z} \arrow[dl,"2"] \arrow[dr,"0"'] & \oplus & \mathbb{Z} \arrow[dl,"0"] & \\
 & & \mathbb{Z} & \oplus & \mathbb{Z} & \oplus & \mathbb{Z} & \oplus & \mathbb{Z} & \oplus & \mathbb{Z} & &
\end{tikzcd}$$

Let $f_0,\dots,f_5$ be the basis for $C_5(\mathbb{Z}/4\times\mathbb{Z}/2;\mathbb{Z})$ generating the summands shown in order. We see that

$$\ker(C_5(\mathbb{Z}/4\times\mathbb{Z}/2;\mathbb{Z})\to C_4(\mathbb{Z}/4\times\mathbb{Z}/2;\mathbb{Z}))=\langle f_0, f_1+2f_2, f_3+2f_4, f_5\rangle.$$

We also have

$$\mathrm{im}(C_6(\mathbb{Z}/4\times\mathbb{Z}/2;\mathbb{Z})\to C_5(\mathbb{Z}/4\times\mathbb{Z}/2;\mathbb{Z}))=\langle 4f_0, 2f_1+4f_2, 2f_3+4f_4, 2f_5\rangle.$$

Therefore

$$H_5(\mathbb{Z}/4\times\mathbb{Z}/2;\mathbb{Z})\cong\mathbb{Z}/4\oplus(\mathbb{Z}/2)^3=\langle f_0, f_1+2f_2, f_3+2f_4, f_5\rangle.$$

The induced map on group homology sends $e_i\mapsto 4f_i$ for $i=0,1$, $e_i\mapsto 2f_i$ for $i=2,3$, and $e_i\mapsto f_i$ for $i=4,5$. We therefore compute that the induced map

$$p_*\colon H_5(\mathbb{Z}/8\times\mathbb{Z}/2;\mathbb{Z})\cong\mathbb{Z}/8\oplus(\mathbb{Z}/2)^3\to H_5(\mathbb{Z}/4\times\mathbb{Z}/2;\mathbb{Z})\cong\mathbb{Z}/4\oplus(\mathbb{Z}/2)^3$$

has kernel $\langle e_0, e_1+4e_2, e_3+4e_4\rangle\cong\mathbb{Z}/8\oplus(\mathbb{Z}/2)^2$, and is an isomorphism between the last $\mathbb{Z}/2$ summands of each, so $e_5\mapsto f_5$.

Next we need to compute the kernel of $d^2_{5,0}=\mathrm{Sq}_2\circ\mathrm{red}_2$. We start by looking at $\mathrm{red}_2\colon H_5(\mathbb{Z}/8\times\mathbb{Z}/2;\mathbb{Z})\to H_5(\mathbb{Z}/8\times\mathbb{Z}/2;\mathbb{Z}/2)$. After tensoring with $\mathbb{Z}/2$, all the maps in the chain complex above become zero maps. Therefore $H_5(\mathbb{Z}/8\times\mathbb{Z}/2;\mathbb{Z}/2)\cong(\mathbb{Z}/2)^6$. Denote the basis elements by $\widetilde{e}_i$, for $i=0,\dots,6$. The map $\mathbb{Z}/8\oplus(\mathbb{Z}/2)^3\to(\mathbb{Z}/2)^6$ is given by projection $\mathbb{Z}/8\to\mathbb{Z}/2$ on the first summand, and maps the second, third and fourth summands isomorphically to the second, fourth and sixth $\mathbb{Z}/2$ summands of $H_5(\mathbb{Z}/8\times\mathbb{Z}/2;\mathbb{Z}/2)\cong(\mathbb{Z}/2)^6$ respectively. The kernel is thus $2e_0$. It remains to compute the dual of the Steedrod square. First we look at the map

$$\mathrm{Sq}^2\colon H^3(\mathbb{Z}/8\times\mathbb{Z}/2;\mathbb{Z}/2)\cong(\mathbb{Z}/2)^4\to H^5(\mathbb{Z}/8\times\mathbb{Z}/2;\mathbb{Z}/2),$$

then we take its dual. Recall that $H^*(\mathbb{Z}/8;\mathbb{Z}/2)\cong(\mathbb{Z}/2)[\alpha_1,\beta_1]/(\alpha_1^2)$ and $H^*(\mathbb{Z}/2;\mathbb{Z}/2)\cong(\mathbb{Z}/2)[\alpha_2]$, where α_1 and α_2 have degree one and β_1 has degree two. We computed earlier that:

$$\begin{aligned}\mathrm{Sq}^2(\alpha_2^3)&=\alpha_2^5=\widetilde{e}_5^*\\ \mathrm{Sq}^2(\alpha_2^2\alpha_1)&=\alpha_2^4\alpha_1=\widetilde{e}_4^*\\ \mathrm{Sq}^2(\alpha_2\beta_1)&=\alpha_2\beta_1^2=\widetilde{e}_1^*\\ \mathrm{Sq}^2(\alpha_1\beta_1)&=\alpha_1\beta_1^2=\widetilde{e}_0^*.\end{aligned}$$

Therefore

$$\ker(\mathrm{Sq}_2\colon(\mathbb{Z}/2)^6\to(\mathbb{Z}/2)^4)=\langle\widetilde{e}_2,\widetilde{e}_3\rangle.$$

It follows from combining the maps Sq_2 and red_2 that

$$\ker d^2_{4,0}=\langle 2e_0,e_3+4e_4\rangle\cong\mathbb{Z}/4\oplus\mathbb{Z}/2\subset\mathbb{Z}/8\oplus(\mathbb{Z}/2)^3,$$

where the $\mathbb{Z}/2$ in the domain includes into the second $\mathbb{Z}/2$ summand in the codomain. This is contain in $\ker p_*$ as claimed. This completes the proof of the claim: the map p_* in the bottom row of diagram (10.9) is trivial on $\ker d^2_{5,0}\subseteq H_5(\mathbb{Z}/8\times\mathbb{Z}/2;\mathbb{Z})$. □

We have therefore completed the proof that abelian groups with at most two generators have the Tertiary Property 1.12, which completes the proof of Lemma 10.5. □

10.3. **Abelian groups with at most 3 generators.**

Lemma 10.10. *The Secondary Property 1.8 and the Tertiary Property 1.12 hold for all abelian groups with at most three generators.*

For the Tertiary property, Corollary 7.10 implies that knowing the property for abelian groups with at most two generators is sufficient. Therefore we do not need to consider the Tertiary property in this subsection. The proof for the Secondary property will take about five pages.

10.3.1. *Three generator abelian groups have the secondary property.* Once again, we can pass to finite odd index subgroups by Theorem 8.2. Thus it suffices to consider groups G of the form $\mathbb{Z}^3$, $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}/2^k$, $\mathbb{Z}\times\mathbb{Z}/2^{k_1}\times\mathbb{Z}/2^{k_2}$ or $\mathbb{Z}/2^{k_1}\times\mathbb{Z}/2^{k_2}\times\mathbb{Z}/2^{k_3}$. We already know the Secondary property for $\mathbb{Z}^3$ by Theorem 9.1. We will compute that the Secondary Property 1.8 holds for the groups $\pi=\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/2$ and $\pi=\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/4$, in a fashion similar to the computation in Section 10.2.1, and then we will deduce that this implies the Secondary property for all the groups G just listed.

The group $\pi=\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/2$.
If $\pi\cong\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/2$, in order to check Condition 4.17, we only have to consider the element $\gamma\in H_3(\pi;\mathbb{Z}/2)$ coming from the generator of $H_1(\mathbb{Z}/2;\mathbb{Z}/2)\otimes H_1(\mathbb{Z}/2;\mathbb{Z}/2)\otimes H_1(\mathbb{Z}/2;\mathbb{Z}/2)$, since all other elements come from a direct summand with at most 2 generators.

Claim. The element γ is in the image of $\mathrm{Sq}_2\circ\mathrm{red}_2$.

The $\mathbb{Z}/2$-coefficient cohomology of $\pi=\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/2$ is

$$\begin{aligned}H^*(\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/2;\mathbb{Z}/2)&\cong H^*(\mathbb{Z}/2;\mathbb{Z}/2)\otimes H^*(\mathbb{Z}/2;\mathbb{Z}/2)\otimes H^*(\mathbb{Z}/2;\mathbb{Z}/2)\\ &\cong\mathbb{Z}/2[\alpha_1]\otimes\mathbb{Z}/2[\alpha_2]\otimes\mathbb{Z}/2[\alpha_3],\end{aligned}$$

where α_i has degree one for $i = 1, 2, 3$. Computing with Steenrod squares, we obtain:

$$\begin{aligned}\mathrm{Sq}^2(\alpha_1\alpha_2\alpha_3) &= \mathrm{Sq}^2(\alpha_1\alpha_2)\alpha_3 + \mathrm{Sq}^1(\alpha_1\alpha_2)\mathrm{Sq}^1(\alpha_3)\\ &= \alpha_1^2\alpha_2^2\alpha_3 + \mathrm{Sq}^1(\alpha_1)\alpha_2\alpha_3^2 + \alpha_1\mathrm{Sq}^1(\alpha_2)\alpha_3^2\\ &= \alpha_1^2\alpha_2^2\alpha_3 + \alpha_1^2\alpha_2\alpha_3^2 + \alpha_1\alpha_2^2\alpha_3^2 \in H^5(\pi;\mathbb{Z}/2).\end{aligned}$$

Let $a_i := \alpha_i^* \in H_1(\mathbb{Z}/2;\mathbb{Z}/2)$ be the dual to α_i and let $a_i^2 = (\alpha_i^2)^* \in H_2(\mathbb{Z}/2;\mathbb{Z}/2)$ be the dual to α_i^2. From the computation of Sq^2 above, we see that the element $\gamma = a_1 \otimes a_2 \otimes a_3$ lies in the image of $\mathrm{Sq}_2 \colon H_5(\pi;\mathbb{Z}/2) \to H_3(\pi;\mathbb{Z}/2)$. We therefore if suffices to show that

$$(\alpha_1^2\alpha_2^2\alpha_3 + \alpha_1^2\alpha_2\alpha_3^2 + \alpha_1\alpha_2^2\alpha_3^2)^* = a_1^2 \otimes a_2^2 \otimes a_3 + a_1^2 \otimes a_2 \otimes a_3^2 + a_1 \otimes a_2^2 \otimes a_3^2$$

lies in the image of $\mathrm{red}_2 \colon H_5(\pi;\mathbb{Z}) \to H_5(\pi;\mathbb{Z}/2)$. Recall that a $\mathbb{Z}[\mathbb{Z}/2]$-module resolution of $\mathbb{Z}$, tensored down over $\mathbb{Z}$, is given by

$$\mathbb{Z}(6) \xrightarrow{2} \mathbb{Z}(5) \xrightarrow{0} \mathbb{Z}(4) \xrightarrow{2} \mathbb{Z}(3) \xrightarrow{0} \mathbb{Z}(2) \xrightarrow{2} \mathbb{Z}(1) \xrightarrow{0} \mathbb{Z}(0),$$

where the parenthetical numbers indicate the grading. It is then straightforward to compute that this element is a cycle with integer coefficients. Here we will also use the notation a_i and a_i^2 as above, but now this notation represents the corresponding integral chains in $C_*(\mathbb{Z}/2;\mathbb{Z})$ i.e. generators of $\mathbb{Z}(1)$ and $\mathbb{Z}(2)$ respectively. We have the following computation in $C_*(\pi;\mathbb{Z}) = \bigotimes^3 C_*(\mathbb{Z}/2;\mathbb{Z})$.

$$\begin{aligned}\partial(a_1^2 \otimes a_2^2 \otimes a_3 + a_1^2 \otimes a_2 \otimes a_3^2 + a_1 \otimes a_2^2 \otimes a_3^2) &= 2a_1 \otimes a_2^2 \otimes a_3 + 2a_1^2 \otimes a_2 \otimes a_3\\ &\quad + 2a_1 \otimes a_2 \otimes a_3^2 - 2a_1^2 \otimes a_2 \otimes a_3\\ &\quad - 2a_1 \otimes a_2 \otimes a_3^2 - 2a_1 \otimes a_2^2 \otimes a_3\\ &= 0.\end{aligned}$$

If $a_1^2 \otimes a_2^2 \otimes a_3 + a_1^2 \otimes a_2 \otimes a_3^2 + a_1 \otimes a_2^2 \otimes a_3^2$ were a boundary over $\mathbb{Z}$, then it would be a boundary over $\mathbb{Z}/2$ as well. Therefore it represents an element of $H_5(\pi;\mathbb{Z})$, that maps to $(\alpha_1^2\alpha_2^2\alpha_3 + \alpha_1^2\alpha_2\alpha_3^2 + \alpha_1\alpha_2^2\alpha_3^2)^*$ under the reduction modulo two. As we have shown above that this element maps to γ under Sq_2, this completes the proof of the claim that γ is in the image of $\mathrm{Sq}_2 \circ \mathrm{red}_2$. □

Claim. The form $A(\gamma)$ is even.

We have

$$\begin{aligned}H^2(K;\mathbb{Z}\pi) \cong (\mathbb{Z}\pi)^6/\langle &(1-b, c-1, 0, 1+a, 0, 0), (a-1, 0, 1-c, 0, 1+b, 0),\\ &(0, 1-b, a-1, 0, 0, 1+c)\rangle\end{aligned}$$

and the form $A(\gamma)$ is given by

$$\begin{pmatrix} 2(1-c) & (1-c)(1-b) & (1-c)(1-a) & 0 & 0 & 0\\ (1-b)(1-c) & 2(1-b) & (1-a)(1-c) & 0 & 0 & 0\\ (1-a)(1-c) & (1-a)(1-b) & 2(1-a) & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For q given by

$$\begin{pmatrix} 1-c & -(c+b) & -(c+a) & 0 & 0 & 0 \\ 1+cb & 1-b & -(b+a) & 0 & 0 & 0 \\ 1+ca & 1+ab & 1-a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

we have $\lambda = q + q^*$. This completes the proof of the claim that $A(\gamma)$ is even. □

This therefore completes the proof of the Secondary Property 1.8 for $\pi = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$.

The group $\pi = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/4$.
Now we consider $\pi := \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/4$. Verifying Condition 4.17 for this group will occupy the next two and a half pages.

Claim. The element $\gamma \in H_3(\pi; \mathbb{Z}/2)$ coming from the generator of

$$H_1(\mathbb{Z}/2; \mathbb{Z}/2) \otimes H_1(\mathbb{Z}/2; \mathbb{Z}/2) \otimes H_1(\mathbb{Z}/4; \mathbb{Z}/2) \cong \mathbb{Z}/2$$

is not in the image of $\mathrm{Sq}_2 \circ \mathrm{red}_2$.

The $\mathbb{Z}/2$-coefficient cohomology of $\pi = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/4$ is

$$\begin{aligned} H^*(\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/4; \mathbb{Z}/2) &\cong H^*(\mathbb{Z}/2; \mathbb{Z}/2) \otimes H^*(\mathbb{Z}/2; \mathbb{Z}/2) \otimes H^*(\mathbb{Z}/4; \mathbb{Z}/2) \\ &\cong \mathbb{Z}/2[\alpha_1] \otimes \mathbb{Z}/2[\alpha_2] \otimes \mathbb{Z}/2[\alpha_3, \beta_3]/\alpha_3^2, \end{aligned}$$

where α_i has degree one for $i = 1, 2, 3$ and β_3 has degree two. Note that $\gamma = (\alpha_1\alpha_2\alpha_3)^*$. We first compute the image of the Steenrod square map $\mathrm{Sq}^2 \colon H^3(\pi; \mathbb{Z}/2) \to H^5(\pi; \mathbb{Z}/2)$. First we have

$$\begin{aligned} \mathrm{Sq}^2(\alpha_1\alpha_2\alpha_3) &= \mathrm{Sq}^2(\alpha_1)\alpha_2\alpha_3 + \alpha_1^2\mathrm{Sq}^1(\alpha_2\alpha_3) + \alpha_1\mathrm{Sq}^2(\alpha_2\alpha_3) \\ &= \alpha_1^2\alpha_2^2\alpha_3 + \alpha_1\alpha_2\alpha_3^2 + \alpha_1\alpha_2^2\alpha_3^2 \\ &= \alpha_1^2\alpha_2^2\alpha_3 \end{aligned}$$

since $\alpha_3^2 = 0$. We also compute:

$$\mathrm{Sq}^2(\alpha_1^2\alpha_2) = \alpha_1^4\alpha_2;\ \mathrm{Sq}^2(\alpha_1\alpha_2^2) = \alpha_1\alpha_2^4;\ \mathrm{Sq}^2(\alpha_i^2\alpha_3) = \alpha_i^4\alpha_3$$

and

$$\mathrm{Sq}^2(\alpha_i\beta_3) = \alpha_i\beta_3^2;\ \mathrm{Sq}^2(\alpha_i^3) = \alpha_i^5.$$

We can thus compute the dual map $\mathrm{Sq}_2 \colon H_5(\pi; \mathbb{Z}/2) \to H_3(\pi; \mathbb{Z}/2)$. We need to show that the element $(\alpha_1^2\alpha_2^2\alpha_3)^* \in H_5(\pi; \mathbb{Z}/2)$ does not lie in the image of the reduction modulo two.

For the chain complex $C_*(\mathbb{Z}/2; \mathbb{Z})$ we will again use a_i^j to denote a generator of $C_j(\mathbb{Z}/2; \mathbb{Z}))$ in the ith copy, for $i = 1, 2$. Recall that a $\mathbb{Z}[\mathbb{Z}/4]$-module resolution of $\mathbb{Z}$, tensored down over $\mathbb{Z}$, is given by

$$\mathbb{Z}(6) \xrightarrow{4} \mathbb{Z}(5) \xrightarrow{0} \mathbb{Z}(4) \xrightarrow{4} \mathbb{Z}(3) \xrightarrow{0} \mathbb{Z}(2) \xrightarrow{4} \mathbb{Z}(1) \xrightarrow{0} \mathbb{Z}(0),$$

where the parenthetical numbers indicate the grading. We will use the notation $a_3^j b_3^k$ for the generator of $\mathbb{Z}(j + 2k)$, an integral lift of the dual element to $\alpha_3^j\beta_3^k$. In $C_*(\pi; \mathbb{Z}) = C_*(\mathbb{Z}/2; \mathbb{Z}) \otimes C_*(\mathbb{Z}/2; \mathbb{Z}) \otimes C_*(\mathbb{Z}/4; \mathbb{Z})$, we compute:

$$\partial(a_1^2 \otimes a_2^2 \otimes a_3) = 2a_1 \otimes a_2^2 \otimes a_3 + 2a_1^2 \otimes a_2 \otimes a_3.$$

So the obvious dual element to $\alpha_1^2\alpha_2^2\alpha_3$ is not a cycle with $\mathbb{Z}$ coefficients. To show that γ does not lie in the image of $\mathrm{Sq}_2 \circ \mathrm{red}_2$ however, we need to argue that there is no way to add other elements of

$C_5(\pi;\mathbb{Z})$ to make $a_1^2\otimes a_2^2\otimes a_3$ into a cycle, in such a way that we preserve the correct image γ of the reduction modulo two in $H_3(\pi;\mathbb{Z}/2)$. We can try adding linear combinations of the chains

$$a_1^i\otimes a_2^j\otimes a_3;\ a_k\otimes b_3^2;\ a_s^3\otimes b_3;\ a_t^5 \text{ or } a_\ell\otimes a_m^2\otimes b_3,$$

for some i,j with $i+j=4$, for some $k,s,t=1,2$, and for some nonempty $\{\ell,m\}\subseteq\{1,2\}$. Then compute

$$\partial(a_1^3\otimes a_2\otimes a_3)=\partial(a_1\otimes a_2^3\otimes a_3)=\partial(a_1^5)=\partial(a_2^5)=0;$$
$$\partial(a_i^4\otimes a_3)=2a_i^3\otimes a_3;\ \partial(a_j^3\otimes b_3)=-4a_j^3\otimes a_3$$

for $i,j=1,2$;

$$\partial(a_1\otimes a_2^2\otimes b_3)=-2a_1\otimes a_2\otimes b_3-4a_1\otimes a_2^2\otimes a_3;$$
$$\partial(a_1^2\otimes a_2\otimes b_3)=2a_1\otimes a_2\otimes b_3-4a_1^2\otimes a_2\otimes a_3;$$

and

$$\partial(a_k\otimes b_3^2)=-4a_k\otimes a_3b_3.$$

Since $\alpha_i\beta_3^2$ and $\alpha_j^4\alpha_3$ are in the image of Sq^2, for any $i,j=1,2$, any occurrence of their dual terms $a_i\otimes b_3^2$ or $a_j^4\otimes a_3$ in a putative linear combination must have even coefficient, or this occurrence would alter the image under Sq_2, causing it to deviate from being $\gamma=(\alpha_1\alpha_2\alpha_3)^*=a_1\otimes a_2\otimes a_3$. Therefore the boundary of every term in our linear combination must have a term with coefficient divisible by 4. On the other hand the boundary we are trying to cancel is

$$\partial(a_1^2\otimes a_2^2\otimes a_3)=2a_1\otimes a_2^2\otimes a_3+2a_1^2\otimes a_2\otimes a_3,$$

in which both terms are only divisible by 2. We see that there is no way to cancel the terms $-4a_1\otimes a_2^2\otimes b_3$ and $-4a_1^2\otimes a_2\otimes b_3$ while still having the necessary odd coefficient of $a_1^2\otimes a_2^2\otimes a_3$. It follows that $\gamma=a_1\otimes a_2\otimes a_3$ does not lie in the image of $\mathrm{Sq}_2\circ\mathrm{red}_2\colon H_5(\pi;\mathbb{Z})\to H_3(\pi;\mathbb{Z}/2)$ as claimed. For interest, we remark that $2a_1^2\otimes a_2^2\otimes a_3+a_1\otimes a_2^2\otimes b_3+a_1^2\otimes a_2\otimes b_3$ represents a homology class in $H_5(\pi;\mathbb{Z})$, but of course this does not map to γ. This completes the proof of the claim. □

We write $\pi:=\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/4=\langle a,b,c\mid [a,b],[b,c],[c,a],a^2,b^2,c^4\rangle$.

Claim. The form $A(\gamma)$ is not even.

$$H^2(K;\mathbb{Z}\pi)\cong(\mathbb{Z}\pi)^6/\langle(1-b,c^{-1}-1,0,1+a,0,0),(a-1,0,1-c^{-1},0,1+b,0),$$
$$(0,1-b,a-1,0,0,1+c+c^2+c^3)\rangle$$

and the form $A(\gamma)$ is given by

$$\begin{pmatrix}(1-c)(1-c^{-1}) & (1-c^{-1})(1-b) & (1-c^{-1})(1-a) & 0 & 0 & 0\\ (1-b)(1-c) & 2(1-b) & (1-a)(1-b) & 0 & 0 & 0\\ (1-a)(1-c) & (1-a)(1-b) & 2(1-a) & 0 & 0 & 0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\end{pmatrix}$$

Assume there exists q with $q+q^*=A(\gamma)$. Then q has to be of the form

$$\begin{pmatrix}(1-c)+z & (1-b)+x & *\\ -c(1-b)-\overline{x} & (1-b)+y & *\\ * & * & *\end{pmatrix}$$

with $z+\overline{z}=y+\overline{y}=0$. The stars mean that we only specify the four entries in the upper left. The proof will show that this matrix cannot represent any form on $H^2(K;\mathbb{Z}\pi)$, no matter how we fill in the rest of the matrix.

Now work modulo 2 and under the projection $a=1$. To represent a form on $H^2(K;\mathbb{Z}\pi)$, the following equations have to be satisfied:

$$0=(1-c+z)(1-b)+(1-b+x)(c-1)=(1-b)z+(c-1)x$$

$$0=(-c(1-b)-\overline{x})(1-b)+(1-b+y)(c-1).$$

Since $z+\overline{z}=0$, there are $z_1,z_2\in\mathbb{Z}$ with $z=(z_1+z_2b)(c-c^3)$. Thus, for $k:=z_1-z_2$, we have $(1-b)z=(k-kb)(c-c^3)$. It follows from the first equation that $x=(k-kb)(c+c^2)+x'N_c$ for some element $x'\in\mathbb{Z}\pi$. Since $(1-b)(1-b)=2(1-b)$ and we are working modulo 2, if we replace x in the second equation above by $(k-kb)(c+c^2)+x'N_c$, we see that

$$0=-\overline{x'}N_c(1-b)+(1-b+y)(c-1).$$

Multiply by $(1+c)$, to obtain

$$0=(1-b+y)(1+c^2).$$

Evaluate at the neutral group element to yield

$$0=1+y_0+y_{c^2}=1,$$

since $y_0=y_{c^2}=0$. This is a contradiction, and it follows that $A(\gamma)$ cannot be even as claimed. □

Therefore, the Secondary Property 1.8 holds for $\pi=\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/4$.

Any 3*-generator abelian group.*
Now we deduce the Secondary Property for G one of the groups on the list from the start of this subsection. For $G\cong G_1\times G_2\times G_3$, with G_i cyclic and $|G_3|\geq 4$, the element $\gamma'\in H_3(G;\mathbb{Z}/2)$ coming from the generator of

$$H_1(G_1;\mathbb{Z}/2)\otimes H_1(G_2;\mathbb{Z}/2)\otimes H_1(G_3;\mathbb{Z}/2)\cong\mathbb{Z}/2$$

is also not in the image of $\mathrm{Sq}_2\circ\mathrm{red}_2$. This can be seen by considering the projection $\varphi\colon G\to\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/4=\pi$, under which γ' is mapped to γ. By Lemma 7.2, we have a commutative diagram

$$\begin{tikzcd}
H_3(G;\mathbb{Z}/2) \arrow[r, "\varphi_*"] \arrow[d, "A"] & H_3(\pi;\mathbb{Z}/2) \arrow[d, "A"] \\
\widehat{H}^0(\mathrm{Sesq}(H^2(K_G;\mathbb{Z}G))) \arrow[r, "\varphi_*"] & \widehat{H}^0(\mathrm{Sesq}(H^2(K_\pi;\mathbb{Z}\pi))
\end{tikzcd}$$

Recall that $\gamma'\in H_3(G;\mathbb{Z}/2)$ is such that $\varphi_*(\gamma')=\gamma$, and we just showed that $A(\gamma)\neq 0$. It follows that $A(\gamma')\neq 0$. Therefore the Secondary Property 1.8 also holds for G. This completes the proof of Lemma 10.10. □

10.4. **Abelian groups with any number of generators.** Theorem 10.1 now follows by combining the work we have done above.

Proof of Theorem 10.1. The Secondary Property 1.8 holds for all finitely generated abelian groups, since by the previous subsections it holds for all abelian groups with at most three generators, and Corollary 7.4 tells us that this suffices to prove the conjecture for all finitely generated abelian groups.

The Tertiary Property 1.12 holds for all finitely generated abelian groups, since by the previous subsections it holds for all abelian groups with at most two generators, and Corollary 7.10 applies to show that the conjecture holds for all abelian groups. □

We also have the following result on the Tertiary Property.

Corollary 10.11. *If a finitely presented group G has the Tertiary Property 1.12, then so does $G \times \mathbb{Z}$.*

Proof. Let G^{ab} be the abelianisation of G and let $G^{ab} \cong \bigoplus_{i=1}^m C_i$ be a decomposition into cyclic groups C_i. Consider the projection $p_0\colon G \times \mathbb{Z} \to G =: G_0$ and the surjections

$$p_i\colon G \times \mathbb{Z} \to G^{ab} \times \mathbb{Z} \to C_i \times \mathbb{Z} =: G_i,$$

for $i = 1, \dots, m$. To apply Theorem 7.7, we need to see that the induced map

$$\prod_{i=0}^m (p_i)_*\colon H_2(G \times \mathbb{Z}; \mathbb{Z}/2)/\operatorname{im}(d^2_{4,1}, d^3_{5,0}) \to \prod_{i=0}^m H_2(G_i; \mathbb{Z}/2)/\operatorname{im}(d^2_{4,1}, d^3_{5,0})$$

is an injection. This is obvious for elements of $H_2(G\times\mathbb{Z}; \mathbb{Z}/2)$ coming from $H_2(G; \mathbb{Z}/2)$, by considering the image under $(p_0)_*$. For elements of $H_2(G \times \mathbb{Z}; \mathbb{Z}/2)$ coming from $H_1(G; \mathbb{Z}/2) \otimes_{\mathbb{Z}} H_1(\mathbb{Z}; \mathbb{Z}/2)$ in the Künneth theorem, we consider the appropriate map $(p_i)_*$, $i = 1, \dots, m$, and use that for every cyclic group C (of even or infinite order, so that $H_1(C; \mathbb{Z}/2)$ is nontrivial), the nontrivial element in $H_2(C\times\mathbb{Z}; \mathbb{Z}/2)$ coming from $H_1(C; \mathbb{Z}/2)\otimes_{\mathbb{Z}} H_1(\mathbb{Z}; \mathbb{Z}/2)$ does not lie in the image of the boundary maps $d^2_{4,1}, d^3_{5,0}$, as we showed in Section 10.2.2. Thus the corollary indeed follows from Theorem 7.7. □

Corollary 10.12. *Let π be a finite group whose 2-Sylow subgroup is abelian or has periodic cohomology. Then the Secondary Property 1.8 and the Tertiary Property 1.12 hold for π. In particular, both properties hold for every finite group with periodic cohomology.*

Proof. First, we can reduce the verification of the properties to 2-Sylow subgroups, by Theorems 8.2 and 8.3. By [Bro82, Theorem VI 9.3], every finite 2-group with periodic cohomology is either abelian or generalised quaternion. The corollary therefore follows from the statement for abelian groups, Theorem 10.1, together with Theorem 9.4 on the generalised quaternion groups.

By [Bro82, Theorem VI 9.5], for every finite group with periodic cohomology its 2-Sylow subgroup also has periodic cohomology. This implies the last statement. □

References

[ABP67] Donald W. Anderson, Edgar H. Brown, and Franklin P. Peterson. The structure of the Spin cobordism ring. *Ann. of Math. (2)*, 86:271–298, 1967.

[Bro82] Kenneth S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.

[CD95] Ruth Charney and Michael W. Davis. Finite $K(\pi, 1)$s for Artin groups. In *Prospects in topology. Proceedings of a conference in honor of William Browder, Princeton, NJ, USA, March 1994*, pages 110–124. Princeton, NJ: Princeton University Press, 1995.

[CE56] Henri Cartan and Samuel Eilenberg. *Homological algebra*. Princeton University Press, Princeton, N. J., 1956.

[CHR95] Alberto Cavicchioli, Friedrich Hegenbarth, and Dušan Repovš. On the stable classification of certain 4-manifolds. *Bull. Austral. Math. Soc.*, 52(3):385–398, 1995.

[CS11] Diarmuid Crowley and Jörg Sixt. Stably diffeomorphic manifolds and $l_{2q+1}(\mathbb{Z}[\pi])$. *Forum Math.*, 23(3):483–538, 2011.

[Dav05] James F. Davis. The Borel/Novikov conjectures and stable diffeomorphisms of 4-manifolds. In *Geometry and topology of manifolds*, volume 47 of *Fields Inst. Commun.*, pages 63–76. Amer. Math. Soc., Providence, RI, 2005.

[FK78] Michael Freedman and Robion Kirby. A geometric proof of Rochlin's theorem. In *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2*, Proc. Sympos. Pure Math., XXXII, pages 85–97. Amer. Math. Soc., Providence, R.I., 1978.

[FQ90] Michael H. Freedman and Frank Quinn. *Topology of 4-manifolds*, volume 39 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1990.

[HAM93] Cynthia Hog-Angeloni and Wolfgang Metzler. Geometric aspects of two-dimensional complexes. In *Two-dimensional homotopy and combinatorial group theory*, volume 197 of *London Math. Soc. Lecture Note Ser.*, pages 1–50. Cambridge Univ. Press, Cambridge, 1993.

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[HH19] Ian Hambleton and Alyson Hildum. Topological 4-manifolds with right-angled Artin fundamental groups. *J. Topol. Anal.*, 11(4):777–821, 2019.

[HK88] Ian Hambleton and Matthias Kreck. On the classification of topological 4-manifolds with finite fundamental group. *Math. Ann.*, 280(1):85–104, 1988.

[HK93] Ian Hambleton and Matthias Kreck. Cancellation of hyperbolic forms and topological four-manifolds. *J. Reine Angew. Math.*, 443:21–47, 1993.

[HKT09] Ian Hambleton, Matthias Kreck, and Peter Teichner. Topological 4-manifolds with geometrically two-dimensional fundamental groups. *J. Topol. Anal.*, 1(2):123–151, 2009.

[Kha17] Qayum Khan. Cancellation for 4-manifolds with virtually abelian fundamental group. *Topology Appl.*, 220:14–30, 2017.

[KLPT17] Daniel Kasprowski, Markus Land, Mark Powell, and Peter Teichner. Stable classification of 4-manifolds with 3-manifold fundamental groups. *J. Topol.*, 10(3):827–881, 2017.

[KPR20] Daniel Kasprowski, Mark Powell, and Benjamin Ruppik. Homotopy classification of 4-manifolds with finite abelian 2-generator fundamental groups. Preprint, available at arXiv:2005.00274, 2020.

[KPT18] Daniel Kasprowski, Mark Powell, and Peter Teichner. Four-manifolds up to connected sum with the complex projective plane. Preprint, available at arXiv:1802.09811, 2018.

[Kre99] Matthias Kreck. Surgery and duality. *Ann. of Math. (2)*, 149(3):707–754, 1999.

[Lev77] Jerome Levine. Knot modules. I. *Trans. Amer. Math. Soc.*, 229:1–50, 1977.

[Mat78] Yukio Matsumoto. Secondary intersectional properties of 4-manifolds and Whitney's trick. In *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2*, Proc. Sympos. Pure Math., XXXII, pages 99–107. Amer. Math. Soc., Providence, R.I., 1978.

[Mil63] John W. Milnor. Spin structures on manifolds. *Enseignement Math. (2)*, 9:198–203, 1963.

[MT68] Robert E. Mosher and Martin C. Tangora. *Cohomology operations and applications in homotopy theory*. Harper's Series in Modern Mathematics. New York-Evanston-London: Harper and Row. X, 214 p., 1968.

[Spa03] Fulvia Spaggiari. On the stable classification of Spin four-manifolds. *Osaka J. Math.*, 40(4):835–843, 2003.

[ST01] Rob Schneiderman and Peter Teichner. Higher order intersection numbers of 2-spheres in 4-manifolds. *Algebr. Geom. Topol.*, 1:1–29 (electronic), 2001.

[Sto94] Richard Stong. Existence of π_1-negligible embeddings in 4-manifolds: A correction to theorem 10.5 of Freedman and Quinn. *Proc. Am. Math. Soc.*, 120(4):1309–1314, 1994.

[Tei92] Peter Teichner. *Topological 4-manifolds with finite fundamental group*. PhD thesis, University of Mainz, Germany, 1992. Shaker Verlag, ISBN 3-86111-182-9.

[Wal99] C. Terence C. Wall. *Surgery on compact manifolds*, volume 69 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 1999. Edited and with a foreword by A. A. Ranicki.

[Whi50] J. H. C. Whitehead. A certain exact sequence. *Ann. of Math. (2)*, 52:51–110, 1950.

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