Twisting q-holonomic sequences by complex roots of unity

Stavros Garoufalidis[*]
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332-0160, USA
stavros@math.gatech.edu

Christoph Koutschan[†]
MSR-INRIA Joint Centre
INRIA-Saclay
91893 Orsay Cedex, France
koutschan@risc.jku.at

ABSTRACT

A sequence $f_n(q)$ is q-holonomic if it satisfies a nontrivial linear recurrence with coefficients polynomials in q and q^n. Our main theorems state that q-holonomicity is preserved under twisting, i.e., replacing q by ωq where ω is a complex root of unity, and under the substitution $q \to q^\alpha$ where α is a rational number. Our proofs are constructive, work in the multivariate setting of ∂-finite sequences and are implemented in the Mathematica package `HolonomicFunctions`. Our results are illustrated by twisting natural q-holonomic sequences which appear in quantum topology, namely the colored Jones polynomial of pretzel knots and twist knots. The recurrence of the twisted colored Jones polynomial can be used to compute the asymptotics of the Kashaev invariant of a knot at an arbitrary complex root of unity.

Categories and Subject Descriptors

G.2.1 [**Discrete Mathematics**]: Combinatorics—*Recurrences and difference equations*; G.4 [**Mathematical Software**]: Algorithm design and analysis; I.1.2 [**Symbolic and Algebraic Manipulation**]: Algorithms—*Algebraic algorithms*

General Terms

Algorithms, Theory

Keywords

q-holonomic sequence, ∂-finite sequence, multivariate recurrence, twisting, colored Jones polynomial, pretzel knot, twist knot, quantum topology

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1. INTRODUCTION

A univariate sequence $\big(f_n(q)\big)_{n\in\mathbb{N}}$ is called *q-holonomic* if it satisfies a nontrivial linear recurrence with coefficients that are polynomials in q and q^n; the indeterminate q here is assumed to be transcendental over $\mathbb{K}$ which, for the moment, is an arbitrary but fixed field of characteristic zero. More precisely, $f_n(q)$ is q-holonomic if there exists a nonnegative integer d and bivariate polynomials $c_j(u,v) \in \mathbb{K}[u,v]$ for $j = 0,\dots,d$ with $c_d(u,v) \neq 0$ such that for all $n \in \mathbb{N}$ the following recurrence is satisfied:

$$\sum_{j=0}^{d} c_j(q,q^n) f_{n+j}(q) = 0. \tag{1}$$

The notion of q-holonomic sequences was introduced by Zeilberger [40] in the early 1990s and occurs frequently in enumerative combinatorics [9, 34] and more recently also in quantum topology [16]. Zeilberger and Wilf [38] proved a *Fundamental Theorem* (i.e., multisums of q-proper hypergeometric terms are q-holonomic), and their proof was algorithmic and computer-implemented; an excellent introduction into the subject is given in [31].

It is well known that the class of q-holonomic sequences is closed under certain operations that include addition and multiplication [27, 25]. These operations can be executed algorithmically on the level of recurrences, i.e., given recurrences for two q-holonomic sequences $f_n(q)$ and $g_n(q)$, a recurrence for $f_n(q) + g_n(q)$ and one for $f_n(q) \cdot g_n(q)$ can be computed; see the packages `qGeneratingFunctions` [25] and `HolonomicFunctions` [28] for implementations in Mathematica, as well as the Maple package `Mgfun` [2].

The aim of the present article is to establish two new closure properties for q-holonomic sequences. The first one is *twisting by roots of unity*: for a given complex number $\omega \in \mathbb{C}$, we call $f_n(\omega q)$ the *twist* of the sequence $f_n(q)$ by ω. Closure of q-holonomicity under twisting by ω requires that ω is a complex root of unity as the example of $f_n(q) = q^{n^2}$ shows; see Remark 1.5 and Section 3.2 of [19]. The second closure property introduced here is the substitution of q by q^α where α is a rational number.

So far the discussion was about univariate sequences. A generalization of q-holonomy to a multivariate setting was given in [32]. The theory of q-holonomic sequences parallels to the geometric theory of holonomic systems, see [33] and references therein. A different generalization of univariate q-holonomic sequences to several variables is given by the class of *∂-finite functions* introduced by Chyzak [4, 3]. This notion is a little weaker than q-holonomicity but very useful in practice, as the execution of closure properties (e.g., addition and multiplication) is rather simple and requires merely linear algebra. In our q-setting the definition can be stated as follows: a multivariate sequence $f_{\mathbf{n}}(\mathbf{q})$ is ∂-finite if for every variable $\mathbf{n} = n_1, \ldots, n_r$ it satisfies a linear recurrence of the form (1):

$$\sum_{j=0}^{d_k} c_{k,j}(\mathbf{q}, q_{a_1}^{n_1}, \ldots, q_{a_r}^{n_r}) f_{\mathbf{n}+j\mathbf{e}_k}(\mathbf{q}) = 0 \tag{2}$$

for $k = 1, \ldots, r$. We use bold letters for vectors and denote by $\mathbf{e}_k$ the k-th unit vector of length r. As above, the d_k's are nonnegative integers and the $c_{k,j}$'s are multivariate polynomials in $\mathbb{K}[\mathbf{u}, \mathbf{v}]$ with $c_{k,d_k} \neq 0$. The indeterminates $\mathbf{q} = q_1, \ldots, q_s$ with $1 \leqslant s \leqslant r$ are assumed to be transcendental over $\mathbb{K}$ and the indices $a_1, \ldots, a_r$ need to be between 1 and s. In most applications just a single indeterminate q occurs, i.e., $s = 1$. From the definitions (1) and (2) it is immediately clear that for univariate sequences (i.e., for $r = 1$) the notions q-holonomic and ∂-finite coincide. A more detailed exposition on holonomy and ∂-finiteness can be found in [28].

The twist of the sequence $f_{\mathbf{n}}(\mathbf{q})$ by complex numbers $\boldsymbol{\omega} = \omega_1, \ldots, \omega_s$ is the sequence $f_{\mathbf{n}}(\omega_1 q_1, \ldots, \omega_s q_s)$; Theorem 1 states that ∂-finiteness is preserved under twisting by complex roots of unity. On the other hand, one may be interested in the sequence $f_{\mathbf{n}}(q_1^{\alpha_1}, \ldots, q_s^{\alpha_s})$ for rational numbers $\alpha_1, \ldots, \alpha_s \in \mathbb{Q}$; Theorem 3 states that ∂-finiteness is also preserved under this substitution.

A motivation for our work was the effective computation of the expansion of the Kashaev invariant of a knot, i.e., its colored Jones polynomial around complex roots of unity, that was initiated by Zagier [8, 39]; see also [12]. Using our results, such an expansion can now be achieved and will be the focus of several separate publications [7, 20]. More details and some examples are given in Section 3.

2. TWISTING PRESERVES ∂-FINITENESS

2.1 Operator Notation and Left Ideals

To state our results, it will be helpful to write recurrences like (1) in operator form. For this purpose consider the operators L and M which act on a sequence $f_n(q)$ by

$$L f_n(q) = f_{n+1}(q),$$
$$M f_n(q) = q^n f_n(q),$$

and satisfy the q-commutation relation $LM = qML$. The noncommutative algebra that is generated by L and M modulo q-commutation is denoted by $\mathbb{W} = \mathbb{K}(q)[M]\langle L\rangle$ and is called the *first q-Weyl algebra*. If one wants to allow division by M then it is convenient to utilize a noncommutative *Ore algebra* (see [4, 3] for more details) which is denoted by $\mathbb{O} = \mathbb{K}(q, M)\langle L\rangle$. Clearly the inclusion $\mathbb{W} \subset \mathbb{O}$ holds.

Similarly, for representing the system of recurrences (2), the operators $\mathbf{L} = L_1, \ldots, L_r$ and $\mathbf{M} = M_1, \ldots, M_r$ are introduced, which act on a multivariate sequence $f_{\mathbf{n}}(\mathbf{q})$ by

$$\begin{aligned} L_k f_{\mathbf{n}}(\mathbf{q}) &= f_{\mathbf{n}+\mathbf{e}_k}(\mathbf{q}), \\ M_k f_{\mathbf{n}}(\mathbf{q}) &= q_{a_k}^{n_k} f_{\mathbf{n}}(\mathbf{q}), \end{aligned} \tag{3}$$

for $k = 1, \ldots, r$ and with the same notation as in (2). Again the above operators q-commute, i.e., they satisfy

$$L_k M_k = q_{a_k} M_k L_k,$$
$$L_j M_k = M_k L_j \quad \text{for } j \neq k.$$

More generally, we can state the q-commutation for arbitrary expressions in $\mathbf{M}$:

$$L_k F(\mathbf{M}) = F(M_1, \ldots, M_{k-1}, q_{a_k} M_k, M_{k+1}, \ldots, M_r) L_k.$$

In operator form, Equation (2) is written as $P_k f = 0$ where

$$P_k = \sum_{j=0}^{d_k} c_{k,j}(\mathbf{q}, \mathbf{M}) L_k^j \tag{4}$$

for $k = 1, \ldots, r$. The operators $P_1, \ldots, P_r$ are regarded as elements of the Ore algebra $\mathbb{O} = \mathbb{K}(\mathbf{q}, \mathbf{M})\langle \mathbf{L}\rangle$. The algebra $\mathbb{O}$ can be viewed as the multivariate polynomial ring in the indeterminates $L_1, \ldots, L_r$ with coefficient field being the rational functions in $\mathbf{q}$ and $\mathbf{M}$, subject to the above stated q-commutation relations. Given a multivariate sequence $f_{\mathbf{n}}(\mathbf{q})$, the set

$$\mathrm{Ann}_{\mathbb{O}}(f) = \{P \in \mathbb{O} \mid Pf = 0\}$$

is a left ideal of $\mathbb{O}$, the so-called annihilator of f with respect to the algebra $\mathbb{O}$. Left ideals in $\mathbb{O}$ have well-defined *dimension* and *rank* which can be computed for instance by (left) Gröbner bases. In this terminology, a multivariate sequence $f_{\mathbf{n}}(\mathbf{q})$ is *∂-finite* with respect to $\mathbb{O}$ if $\mathrm{Ann}_{\mathbb{O}}(f)$ is a zero-dimensional left ideal in $\mathbb{O}$. For example, if $f_{\mathbf{n}}(\mathbf{q})$ satisfies (2), then it is annihilated by the operators $P_1, \ldots, P_r$ of Equation (4). The latter generate a zero-dimensional ideal of rank at most $\prod_{k=1}^{r} d_k$. Note, however, that the set $\{P_1, \ldots, P_r\}$ is not a left Gröbner basis of that ideal in general (Buchberger's product criterion does not hold in noncommutative rings).

2.2 Main Theorems

We have now prepared the stage for stating our main results. To keep the presentation concise, it is assumed from now on that the field $\mathbb{K}$ contains all complex roots of unity.

THEOREM 1. *Let $f_{\mathbf{n}}(\mathbf{q}) = f_{n_1,\ldots,n_r}(q_1, \ldots, q_s)$ be a multivariate ∂-finite sequence, and let $\omega_j \in \mathbb{C}$ be an m_j-th root of unity for $j = 1, \ldots, s$. Then, the twisted sequence $g_{\mathbf{n}}(\mathbf{q}) = f_{\mathbf{n}}(\omega_1 q_1, \ldots, \omega_s q_s)$ is ∂-finite as well.*

Moreover, let I be a zero-dimensional left ideal of rank R such that $If = 0$. From a generating set of I, a Gröbner basis of a zero-dimensional left ideal J with $Jg = 0$ can be obtained and its rank is at most $R \cdot m_{a_1} \cdots m_{a_r}$.

PROOF. With the notation introduced in (2) and (3) we fix the Ore algebra $\mathbb{O} = \mathbb{K}(\mathbf{q}, \mathbf{M})\langle \mathbf{L}\rangle$ so that I is a left ideal in $\mathbb{O}$. We now shall show that sufficiently many operators in $\mathbb{O}$ can be found which annihilate the sequence $g_{\mathbf{n}}(\mathbf{q})$. A naive attempt to obtain some recurrences for g is to substitute q_j by $\omega_j q_j$ (for $1 \leqslant j \leqslant s$) in the recurrences for f. Indeed, the result are valid recurrences for g, but in general they cannot be represented in the algebra $\mathbb{O}$ since they contain terms of the form $\omega_j^{n_k}$. However, for an operator $P \in I$ this substitution is admissible (in the sense that the result is in $\mathbb{O}$) if for each k the variable M_k appears in P only with powers that are multiples of m_{a_k} (for sake of readability we will write $m(k)$ instead of m_{a_k}). The idea of the proof is to show that such operators exist and that they generate a zero-dimensional ideal of rank at most $R \cdot m(1) \cdots m(r) =: \tilde{R}$.

First we introduce a new set of variables $\mathbf{N} = N_1, \dots, N_r$ such that $N_k = M_k^{m(k)}$. In this notation the goal is to obtain a set of generators for the left ideal

$$J = I \cap \mathbb{K}(\mathbf{q}, \mathbf{N})\langle \mathbf{L} \rangle.$$

For this purpose, fix k and consider an ansatz operator of the form

$$A = \sum_{j=0}^{d} c_j(\mathbf{q}, \mathbf{N}) L_k^j$$

where the unknowns $\mathbf{c} = c_0, \dots, c_d$ are assumed to be rational functions in $\mathbf{q}$ and $\mathbf{N}$. The remainder of A modulo the left ideal I can be computed by reducing it with a left Gröbner basis of I. After clearing denominators, this remainder is a linear combination of R different power products $\mathbf{L}^{\boldsymbol{\alpha}}$; its coefficients are polynomials in $\mathbf{q}$ and $\mathbf{M}$, and in the unknowns $\mathbf{c}$ which occur linearly. The claim that A be an annihilating operator for f is achieved by equating all those coefficients to zero. This yields a system of R equations in the unknowns $\mathbf{c}$. By making use of the new variables $\mathbf{N}$ and simple rewriting, it can be achieved that the degree of M_k is smaller than $m(k)$ for $1 \leqslant k \leqslant r$. Coefficient comparison w.r.t. the variables $\mathbf{M}$ enforces that the unknowns $\mathbf{c}$ depend only on $\mathbf{q}$ and $\mathbf{N}$, and converts each equation into a set of at most $m(1) \cdots m(r)$ equations. Choosing $d = \tilde{R}$ in A therefore produces a linear system with d equations in $d+1$ unknowns. Thus the existence of a nontrivial solution is guaranteed. The substitutions $q_j \to \omega_j q_j$ can now be performed without problems and yield an annihilating operator for g. Repeating the above procedure for $k = 1, \dots, r$ shows that g is ∂-finite.

However, in practice one would not proceed along these lines. Instead of pure recurrence operators A (i.e., univariate polynomials in $\mathbb{O}$), it is advantageous to loop over the support of A and increase it according to the FGLM algorithm (this is made explicit in Algorithm 1 below). This procedure guarantees that the resulting operators form a Gröbner basis, and at the same time shows that the rank of the ideal they generate is at most $\tilde{R}$. For the contrary, let R' denote the rank of J and assume that it is strictly greater than $\tilde{R}$; this means that a Gröbner basis of J has R' irreducible monomials under its stairs, i.e., there is no operator in J whose support is a subset of these monomials. On the other hand, an ansatz A (as above) whose support consists of all irreducible monomials will lead to a linear system with $\tilde{R}$ equations and R' unknowns. By the assumption $R' > \tilde{R}$ a nontrivial solution exists, in contradiction to the fact that the support of A consists of irreducible monomials only. $\square$

Since many applications deal with sequences in a single variable, and in order to justify the title of this paper, the following corollary is stated explicitly, even though it is a trivial consequence of Theorem 1.

Corollary 2. *Let $f_n(q)$ be a q-holonomic sequence that satisfies a recurrence of the form* (1) *of order d. Then for any root of unity $\omega \in \mathbb{C}$ of order m the sequence $f_n(\omega q)$ is q-holonomic as well and satisfies a recurrence of order at most $m \cdot d$.*

In [19, Thm. 1.5] it was shown that the specialization of a q-holonomic sequence $f_n(q) \in \mathbb{Z}[q^{\pm 1}]$ to a complex root of unity ω is a holonomic sequence, in other words, that $f_n(\omega)$ satisfies a linear recurrence with coefficients polynomials in n. The present paper reduces the proof of the above result to the case of $\omega = 1$.

Theorem 3. *Let $f_{\mathbf{n}}(\mathbf{q}) = f_{n_1,\dots,n_r}(q_1, \dots, q_s)$ be a multivariate ∂-finite sequence, and let $\alpha_1, \dots, \alpha_s \in \mathbb{Q}$. Then, the sequence $g_{\mathbf{n}}(\mathbf{q}) = f_{\mathbf{n}}(q_1^{\alpha_1}, \dots, q_s^{\alpha_s})$ is ∂-finite as well.*

Moreover, let I be a zero-dimensional left ideal of rank R such that $If = 0$. From a generating set of I, a Gröbner basis of a zero-dimensional left ideal J with $Jg = 0$ can be obtained and its rank is at most $R \cdot m_1 \cdots m_s \cdot m_{a_1} \cdots m_{a_r}$, where $m_j \in \mathbb{N}$ denotes the denominator of α_j.

Proof. Employing the notation from (2) and (3) so that I is a left ideal in $\mathbb{O} = \mathbb{K}(\mathbf{q}, \mathbf{M})\langle \mathbf{L} \rangle$, it has to be shown that there are sufficiently many elements in I for which the result of the substitutions $q_j \to q_j^{\alpha_j}$, $1 \leqslant j \leqslant s$, is still in $\mathbb{O}$. This condition is equivalent to claiming that all powers of q_j are divisible by m_j and that all powers of M_k are multiples of m_{a_k}, for $1 \leqslant j \leqslant s$ and $1 \leqslant k \leqslant r$.

The rest of the proof is analogous to the proof of Theorem 1. The only difference is that in addition to $\mathbf{N}$, one has to introduce a second set of new variables $\mathbf{Q} = Q_1, \dots, Q_s$ such that $Q_j = q_j^{m_j}$, and that the coefficient comparison then has to be performed w.r.t. $\mathbf{M}$ and $\mathbf{q}$. $\square$

Corollary 4. *Let $f_n(q)$ be a q-holonomic sequence that satisfies a recurrence of the form* (1) *of order d. Then for $\alpha \in \mathbb{Q}$ the sequence $f_n(q^\alpha)$ is q-holonomic as well and satisfies a recurrence of order at most $m^2 \cdot d$, where $m \in \mathbb{N}$ is the denominator of α.*

It is now natural to ask whether Corollaries 2 and 4 can be extended to q-holonomic sequences in more than one variable. Unfortunately the study of multivariate q-holonomic sequences is much more involved (we even didn't give a precise definition in this paper), and therefore the following statement appears without proof; it is a stronger version of Theorems 1 and 3.

Conjecture 5. *Multivariate q-holonomic sequences are closed under twisting by complex roots of unity and under substitutions of the form $q \to q^\alpha$ for $\alpha \in \mathbb{Q}$.*

At this point it may be beneficial to discuss some simple examples to illustrate Theorems 1 and 3 and their implementation in our software package. Recall the definitions for the q-Pochhammer symbol

$$(a; q)_n := \prod_{k=0}^{n-1} \left(1 - aq^k\right)$$

and the q-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

Example 1. Let $f_n(q)$ be the central q-binomial coefficient $\begin{bmatrix} 2n \\ n \end{bmatrix}_q$. It satisfies the recurrence

$$(1 - q^{n+1}) f_{n+1}(q) = (1 + q^{n+1} - q^{2n+1} - q^{3n+2}) f_n(q)$$

which translates to the operator

$$(qM - 1)L - q^2M^3 - qM^2 + qM + 1. \tag{5}$$

We choose $\omega = -1$; the substitution $q \to -q$ in the above operator is not admissible because of the odd powers of M. On the other hand, Theorem 1 guarantees that $f_n(-q)$ is also q-holonomic. Indeed, the twisted sequence $f_n(-q)$ is annihilated by the operator

$$\left(q^4M^2-1\right)L^2+\left(\left(q^7-q^6\right)M^4-q+1\right)L-\\ q^7M^6-\left(q^6-q^5+q^4\right)M^4+\left(q^4-q^3+q^2\right)M^2+q.$$

Note that it contains only even powers of M, at the cost of increasing the order. Using the Mathematica package `HolonomicFunctions`, these results can be obtained by the following commands:

```
qbin = Annihilator[QBinomial[2n, n, q], QS[qn,q^n]]
DFiniteQSubstitute[qbin, {q, 2}]
```

The first line determines the input operator (5) from the given mathematical expression. The second line computes the twisted recurrence; the substitution is given as a pair (q, m) and by default $\omega = e^{2\pi i/m}$ is chosen.

Example 2. The q-Pochhammer symbol satisfies the simple recurrence

$$(q;q)_{n+1} = (1-q^{n+1})(q;q)_n.$$

We want to study the twisted sequence $(\omega q;\omega q)_n$ for ω being a third root of unity. Therefore we have to compute a recurrence for $(q;q)_n$ in which all exponents of $M = q^n$ are divisible by 3:

$$(q;q)_{n+3}-\left(q^2+q+1\right)(q;q)_{n+2}+\\ \left(q^3+q^2+q\right)(q;q)_{n+1}+\left(q^{3n+6}-q^3\right)(q;q)_n=0. \tag{6}$$

Substituting $q \to \omega q$ into (6) delivers a q-holonomic recurrence for the twist $(\omega q;\omega q)_n$. The commands to compute it are the following:

```
qp = Annihilator[QPochhammer[q, q, n], QS[qn,q^n]]
DFiniteQSubstitute[qp, {q, 3},
   Return -> Backsubstitution]
```

The option `Return -> Backsubstitution` in this instance tells the program to return the recurrence before performing the substitution $q \to e^{2\pi i/3}q$ (see the last but one line of Algorithm 1); this is exactly recurrence (6) in operator form.

Example 3. The substitution $q \to \sqrt{q}$ is performed on the q-Pochhammer symbol $(q;q)_n$ (see Example 2). Theorem 3 predicts that the resulting recurrence is of order at most 4, which is sharp in this case. As an intermediate result, the operator

$$L^4-(q^2+1)L^3-(q^8M^2+q^6M^2-q^4-q^2)L\\ -q^{10}M^4+q^8M^2+q^6M^2-q^4$$

is found in the annihilator of $(q;q)_n$. Note that both q and M appear with even powers. The final result is the recurrence

$$f_{n+4}-(q+1)f_{n+3}-(q^{n+4}+q^{n+3}-q^2-q)f_{n+1}\\ +\left(-q^{2n+5}+q^{n+4}+q^{n+3}-q^2\right)f_n=0$$

where $f_n = \left(\sqrt{q};\sqrt{q}\right)_n$. This recurrence is obtained as the output of the command

```
DFiniteQSubstitute[qp, {q, 1, 2}]
```

where the triple (q, m, k) encodes the substitution $q \to \omega q^{1/k}$ with $\omega = e^{2\pi i/m}$.

2.3 Algorithms

The proof of Theorem 1 gives an algorithm to construct the left ideal J of annihilating operators for the twisted sequence. To formulate this algorithm in pseudo-code, the notations from (2) and (3) and from Theorem 1 are employed; additionally, if T is a set, we refer to its elements by $\{T_1, T_2, \dots\}$, and we use $\mathrm{lm}_\prec(P)$ to denote the leading monomial of the operator P with respect to the monomial order $\prec$.

ALGORITHM 1.

Input: $r, s \in \mathbb{N}$,
for $1 \leqslant j \leqslant s$: $m_j \in \mathbb{N}$, $\omega_j \in \mathbb{C}$ with $\omega_j^{m_j} = 1$ and $\omega_j^\ell \neq 1$ for all $\ell < m_j$,
$\mathbb{O} = \mathbb{K}(q_1,\dots,q_s,M_1,\dots,M_r)\langle L_1,\dots,L_r\rangle$,
a monomial order $\prec$ for $\mathbb{O}$,
a finite set $F \subset \mathbb{O}$ such that F is a left Gröbner basis w.r.t. $\prec$ and the left ideal ${}_{\mathbb{O}}\langle F\rangle$ is zero-dimensional

Output: a finite set $G \subset \mathbb{O}$ such that G is a left Gröbner basis w.r.t. $\prec$ and such that for any sequence $f_{\mathbf{n}}(q_1,\dots,q_s)$ with $F(f_{\mathbf{n}}(\mathbf{q})) = 0$ we have $G(f_{\mathbf{n}}(\omega_1q_1,\dots,\omega_sq_s)) = 0$

$G = \emptyset$
$U =$ set of monomials under the stairs of F
$T = \{1\}$
$V = \emptyset$
while $T \neq \emptyset$
  $T_0 = \min_\prec T$
  $T = T \setminus \{T_0\}$
  $A = c_0T_0 + \sum_{j=1}^{|V|} c_jV_j$
  $A' = A$ reduced with F
  clear denominators of A'
  substitute $M_k^a \to M_k^{a \bmod m(k)} N_k^{\lfloor a/m(k)\rfloor}$ in A'
  write A' as $\sum_{i=1}^{|U|}\sum_{j_1=0}^{m(1)-1}\cdots\sum_{j_r=0}^{m(r)-1} d_{i,\mathbf{j}}M_1^{j_1}\cdots M_r^{j_r}U_i$
  equate all $d_{i,\mathbf{j}}$ to zero
  solve this linear system for $c_0,\dots,c_{|V|}$ over $\mathbb{K}(\mathbf{q},\mathbf{N})$
  if a solution exists **then**
    substitute the solution into A
    $G = G \cup \{A\}$
    $T = T \cup \{T_0L_k : 1 \leqslant k \leqslant r\}$
    $T = T \setminus \{T_j : 1 \leqslant j \leqslant |T| \wedge \exists_k\, \mathrm{lm}_\prec(G_k) \mid T_j\}$
  else
    $V = V \cup \{T_0\}$
substitute $N_k \to M_k^{m(k)}$ and $q_j \to \omega_jq_j$ in G
return G

Similarly, Theorem 3 yields Algorithm 2 which, however, is just a light variation of Algorithm 1 and therefore not displayed explicitly here. Both algorithms are implemented in `HolonomicFunctions` as the command `DFiniteQSubstitute`, see [29] and Examples 1–3.

With slight modifications Algorithms 1 and 2 can be applied to inhomogeneous recurrences as well. Algebraically, an inhomogeneous recurrence of the form

$$\sum_{j=0}^{d} c_j(q,q^n)f_{n+j}(q) = b(q,q^n)$$

can be represented as $\left(\sum_{j=0}^d c_jL^j, b\right)$ in the left module $\mathbb{O}^2$, modulo the relation $(0, L-1)$. To make the algorithms work a POT ordering has to be used. The option `ModuleBasis` of the command `DFiniteQSubstitute` serves this purpose.

Given a root of unity $\omega \in \mathbb{C}$ and a univariate operator $P \in \mathbb{W}$ such that $P(f_n(q)) = 0$ for some sequence $f_n(q)$, let $\tau_\omega(P) \in \mathbb{W}$ denote the annihilating operator for the twisted sequence $f_n(\omega q)$ that is produced by Algorithm 1 (in order to represent its output in $\mathbb{W}$, one has to clear denominators). Additionally we claim that $\tau_\omega(P) = \sum_{j=0}^{d} c_j(q, M)L^j$ is *content-free*, i.e., $\gcd(c_0, \ldots, c_d) = 1$. The following result about the nature of $\tau_w(P)$ is easily obtained.

Proposition 6. *Let*

$$P(M, L, q) = \sum_{j=0}^{d} c_j(q, M)L^j \in \mathbb{W}$$

such that $\gcd(c_0, \ldots, c_d) = 1$ *and let* $\omega \in \mathbb{C}$ *be a root of unity of order* m. *Define* $\ell \in \mathbb{N}$ *to be the largest integer such that* $P \in \mathbb{K}(q)[M^\ell]\langle L\rangle$. *Then*

$$Q(M, L)\big(\tau_\omega(P)\big)(M, L, \omega^{-1}) = R(M) \prod_{k=1}^{m/\gcd(\ell,m)} P(\omega^k M, L, 1)$$

for some polynomial $Q \in \mathbb{K}[M, L]$ *and some rational function* $R \in \mathbb{K}(M)$.

2.4 Behavior of the Newton Polygon Under Twisting

In this section it is studied how the Newton polygon of a univariate operator behaves under twisting. Following [13], consider the *Newton polygon* $N(P)$ of an operator $P \in \mathbb{W}$, i.e., the convex hull of the exponents (a, b) of the monomials $M^b L^a$ of P. The Newton polygon of a (possibly inhomogeneous) recurrence $P(f_n(q)) = b(q, q^n)$, $P \in \mathbb{W}$, is defined to be $N(P)$. Let $LN(P)$ denote the *lower convex hull* of $N(P)$. $LN(P)$ consists of a finite union of non-vertical line segments together with two vertical rays. Each line segment has a *slope* and we denote by $S(P)$ the *set of slopes* of $LN(P)$. An example will clarify these notions.

Example 4. Consider the inhomogeneous recurrence

$$\begin{aligned}
&q^{2n+2}\big(q^{n+2}-1\big)\big(q^{2n+1}-1\big)f(n+2) - \\
&\big(q^{4n+4}-q^{3n+3}-q^{2n+3}-q^{2n+1}-q^{n+1}+1\big) \\
&\quad\times\big(q^{n+1}-1\big)^2\big(q^{n+1}+1\big)f(n+1) + \\
&q^{2n+2}\big(q^{n}-1\big)\big(q^{2n+3}-1\big)f(n) = \\
&q^{n+1}\big(q^{n+1}+1\big)\big(q^{2n+1}-1\big)\big(q^{2n+3}-1\big)
\end{aligned} \tag{7}$$

whose left-hand side is $P(f_n(q))$ where the operator P is given by

$$\begin{aligned}
&\big(q^5M^5 - q^3M^4 - q^4M^3 + q^2M^2\big)\,L^2 + \\
&\big(-q^7M^7 + 2q^6M^6 + (q^6+q^4)M^5 - (q^5+q^4+q^3)M^4 - \\
&\quad(q^4+q^3+q^2)M^3 + (q^3+q)M^2 + 2qM - 1\big)\,L + \\
&q^5M^5 - q^5M^4 - q^2M^3 + q^2M^2.
\end{aligned}$$

Then $N(P)$ is the hexagon with vertex set

$$\big\{(0,2), (1,0), (2,2), (2,5), (1,7), (0,5)\big\},$$

which corresponds to the smallest polygon depicted in Figure 2. The lower Newton polygon $LN(P)$ consists of the two line segments which connect the points $(0, 2)$, $(1, 0)$, and $(2, 2)$, as well as the two vertical rays starting from $(0, 2)$ and $(2, 2)$. The set of slopes $S(P)$ is easily seen to be $\{-2, 2\}$.

Proposition 7. *Fix* $P \in \mathbb{W}$ *and* $\omega \in \mathbb{C}$ *a complex* m*-th root of unity. Then* $\tau_\omega(P) \in \mathbb{K}(q)[M^m]\langle L\rangle$ *and* $S(P) \subset S(\tau_\omega(P))$.

Proof. By definition, our algorithm finds a polynomial $Q \in \mathbb{W}$ such that $\tau_\omega(P) = QP \in \mathbb{K}(q)[M^m]\langle L\rangle$. In [13, Prop.2.2] it is shown that $LN(QP) = LN(Q) + LN(P)$, where the plus operation is the *Minkowski sum*. Since the slopes of the Minkowski sum is the union of the slopes, it follows that $S(P) \subset S(\tau_\omega(P))$. $\square$

Using Proposition 6 one even gets equality instead of the inclusion. However, if the Newton polygons of inhomogeneous recurrences are considered, the set of slopes can strictly grow under twisting; this will be demonstrated in Section 3.3.

Note that every edge of $N(P)$ is either an edge of $LN(P)$, or an edge of $UN(P)$ (the *upper convex hull* of the exponents of P), or a vertical edge. Proposition 7 applies to $UN(P)$ as well, by reversing q to $1/q$.

3. APPLICATIONS IN QUANTUM TOPOLOGY

3.1 The Colored Jones Polynomial of a Knot

Quantum knot theory is a natural source of q-holonomic sequences. A knot K is the smooth embedding of a circle in 3-dimensional space $\mathbb{R}^3$, up to isotopy. The *colored Jones polynomial*

$$\big(J_{K,n}(q)\big)_{n\in\mathbb{N}} \in \big(\mathbb{Z}[q^{\pm1}]\big)^{\mathbb{N}}$$

of a knot K is a sequence of Laurent polynomials with the normalization that $J_{K,1}(q) = 1$ and $J_{\mathrm{Unknot},n}(q) = 1$ for all n. $J_{K,2}(q)$ is the famous *Jones polynomial* [23]. For an introduction to the polynomial invariants of knots that originate in quantum topology see [26, 23, 36, 37] and the book [22] where all the details of the quantum group theory can be found. Up-to-date computations of several polynomial invariants of knots are available in [1]. The colored Jones polynomial $J_{K,n}(q)$ is a q-holonomic sequence [16]; as a canonical (homogeneous) recurrence relation we choose the one with minimal order; this is the so-called *noncommutative A-polynomial* $A_K(M, L, q) \in \mathbb{W}$ of a knot K [11]. An inhomogeneous recurrence is often available, typically of smaller size [18, 14]. Theorem 1 has the following corollary.

Corollary 8. *There exists a twisting map*

$$\textit{Knots} \times \{\textit{complex roots of } 1\} \longrightarrow \mathbb{W}$$

defined by $(K, \omega) \mapsto A_{K,\omega}(M, L, q)$ *with the following properties:*

(a) $A_{K,\omega}(M, L, q) = \tau_\omega(A_{K,1}(M, L, q))$ *and the base case* $A_{K,1}(M, L, q) = A_K(M, L, q)$ *is determined by the colored Jones polynomial* $J_{K,n}(q)$.

(b) *For every complex root of unity* ω, $J_{K,n}(\omega q)$ *is annihilated by* $A_{K,\omega}(M, L, q)$.

(c) *If* ω *has order* m, *then* $A_{K,\omega}(M, L, q) \in \mathbb{K}(q)[M^m]\langle L\rangle$.

The above corollary also holds for the inhomogeneous noncommutative A-polynomial, too.

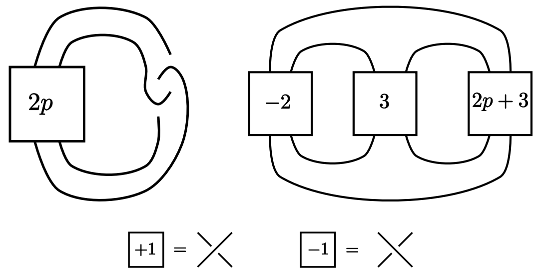

Figure 1: Twist knot K_p (left) and $(-2,3,2p+3)$ pretzel knot KP_p (right) where an integer m inside a box indicates the number of $|m|$ half-twists, right-handed (if $m>0$) or left-handed (if $m<0$).

Table 1: Data for the twisted inhomogeneous recurrences of the 4_1 knot; the integer m denotes the order of the root of unity by which the recurrence is twisted and its size is given in terms of Mathematica ByteCount.

m	1	2	3	4	5
size in KB	3	80	3867	13460	68477
q-exponent	7	58	327	698	1661
L-exponent	2	5	8	11	14
M-exponent	7	22	81	124	235

3.2 Examples of Noncommutative A-Polynomials of Knots

Although the noncommutative A-polynomial of a knot is essentially a three-variate polynomial, it is a difficult one to compute or to guess. In fact, a conjectured two-variate specialization of it, the so-called *A-polynomial* of a knot (defined in [5]) is already hard to compute and even unknown for some knots with only 9 crossings. For an updated list of A-polynomials of knots, see [6]. There are two 1-parameter families of knots with known A-polynomials, namely the twist knots K_p [21] and the $(-2,3,3+2p)$ pretzel knots KP_p [17], depicted in Figure 1. For these two families of knots, the (inhomogeneous) noncommutative A-polynomials have been computed or guessed only for a few particular values of the parameter p. For the twist knots K_p, they were computed with a certificate in [18] for $p=-14,\dots,15$. For the pretzel knots $KP_p=(-2,3,3+3p)$, they were guessed by the authors in [15] for $p=-5,\dots,5$. The results of twisting these recurrences by $\omega=-1$ can be found in

`www.math.gatech.edu/~stavros/publications/twisting.qholonomic.data/`

3.3 The 4_1 Knot

As a case study we investigate the twist knot K_{-1} which appears as knot 4_1 in the knot atlas [1]. The inhomogeneous recurrence for its colored Jones polynomial is given by (7); see [16, 11]. Table 1 shows the sizes and exponents of the twisted recurrences and demonstrates that they grow rapidly with the order m of the root of unity.

The Newton polygons of the twisted (inhomogeneous) recurrences for the orders $m=1,\dots,5$ are given in Figure 2 (recall that for the Newton polygon of an inhomogeneous recurrence, we consider just the homogeneous part of that recurrence). They are plotted in (L,M^m) coordinates, which means that a point (a,b) in the Newton polygon for a certain m represents the monomial $M^{bm}L^a$. Note that the set of slopes is $\{-2,2\}$ for the input recurrence (7), but that it is $\{-2,0,2\}$ for the Newton polygons of the twisted recurrences.

3.4 An Application of Twisting in Quantum Topology

In this section we discuss in brief an application of twisting to asymptotics questions in quantum topology. For further details and the role of recurrences, see [8, 7, 10].

The Kashaev invariant $\langle K\rangle_n$ of a knot K is given by [24, 30]

$$\langle K\rangle_n = J_{K,n}(e^{2\pi i/n}).$$

The *Volume Conjecture* relates the leading asymptotics of the Kashaev invariant to hyperbolic invariants of the knot complement. More precisely, the Volume Conjecture states that for a hyperbolic knot K we have:

$$\lim_n \frac{1}{n}\log|\langle K\rangle_n| = \frac{\mathrm{vol(K)}}{2\pi}$$

where $\mathrm{vol}(K)$ is the hyperbolic volume of K [35]. It was observed by Zagier and the first author that one can numerically compute $\langle K\rangle_n$ in $O(n)$ time given a recurrence relation for $J_{K,n}(q)$. Zagier raised questions concerning the expansion of the Kashaev invariant around other roots of unity (the original Volume Conjecture is centered around $\omega=1$). Given a recurrence relation for $J_{K,n}(\omega q)$, one can compute those asymptotics in linear time. This will be studied in detail in forthcoming work [7, 20].

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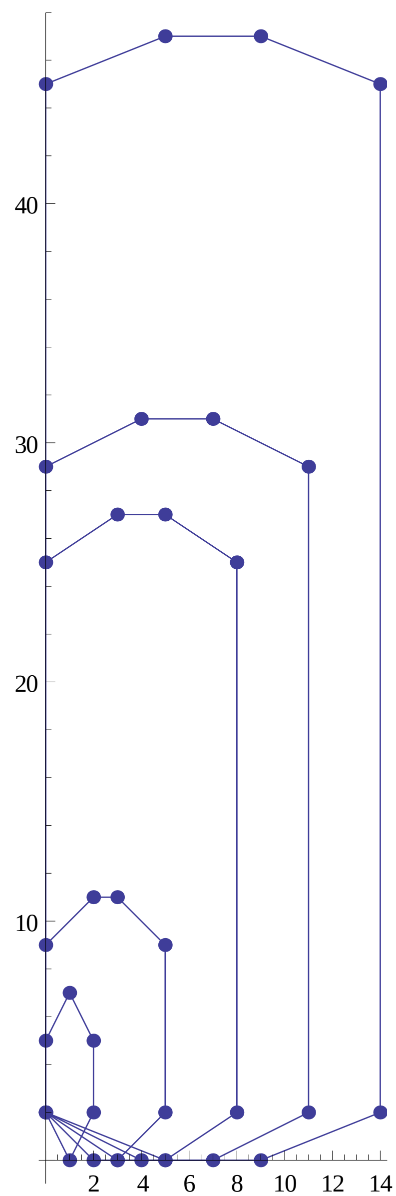

Figure 2: The Newton polygon of the twisted (inhomogeneous) recurrences for the knot 4_1 in (L, M^m)-space; note that the slopes appear jolted due to the use of (L, M^m) coordinates.

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