# THE (TWISTED/$L^2$)-ALEXANDER POLYNOMIAL OF IDEALLY TRIANGULATED 3-MANIFOLDS

STAVROS GAROUFALIDIS AND SEOKBEOM YOON

ABSTRACT. We establish a connection between the Alexander polynomial of a knot and its twisted and $L^2$-versions with the triangulations that appear in 3-dimensional hyperbolic geometry. Specifically, we introduce twisted Neumann–Zagier matrices of ordered ideal triangulations and use them to provide formulas for the Alexander polynomial and its variants, the twisted Alexander polynomial and the $L^2$-Alexander torsion.

CONTENTS



## 1. INTRODUCTION

The Alexander polynomial is a fundamental invariant of knots that dates back to the origins of algebraic topology [Ale28]. It has been studied time and again from various points of view that include twisting by a representation [Wad94, Lin01], or considering $L^2$-versions [Lü02, DFL15]. There are numerous results and surveys to this subject that the reader may consult that include [FV11, DFJ12, Kit15].

Let $M$ be an oriented compact 3-manifold with torus boundary and $\mathcal{T}$ a concrete (in the sense of [GGZ15, Defn.2.1]) ideal triangulation of the interior of $M$. One can think of $\mathcal{T}$ as tetrahedra with their vertices removed, whose faces are identified in pairs. Under such an identification, an edge can lie in more than one tetrahedron, or said differently, going around an edge, one traverses several tetrahedra, possibly with repetition. Such combinatorial data gives rise to a pair of integer matrices, known as Neumann–Zagier matrices [NZ85]. The ideal

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triangulation lifts to the interior of any cover of $M$, and this gives rise to a twisted version of the Neumann–Zagier matrices. In particular, from the universal cover we obtain *twisted Neumann–Zagier matrices*, whose entries are in the group ring $\mathbb{Z}[\pi_1(M)]$; see Section 2 for details.

Our main theorems provide explicit relations of the twisted Neumann–Zagier matrices with the Alexander polynomial and its variants, the twisted Alexander polynomial and the $L^2$-Alexander torsion (Theorems 4.1, 4.4, 4.5). These relations follow from a connection between twisted Neumann–Zagier matrices and Fox calculus [Fox53]. More precisely, we will show in Section 3 that for an *ordered* ideal triangulation, one of the twisted Neumann–Zagier matrices can be explicitly computed from the dual complex of the triangulation by using Fox calculus. We use this fact to relate the twisted Neumann–Zagier matrix to the matrices that define the Alexander polynomial and its variants; see Section 4 for details. One important aspect of our results is the use of ordered ideal triangulations, which breaks the symmetry between the two (twisted) Neumann–Zagier matrices.

The paper is organized as follows. In Section 2, we briefly recall Neumann–Zagier matrices and introduce their twisted version. In Section 3, we show that twisted Neumann–Zagier matrices can be obtained from Fox calculus. In Section 4, we present and prove our main theorems. In Section 5, we give an explicit computation for the figure-eight knot and verify our theorems.

## 2. Twisted Neumann–Zagier matrices

In this section we briefly recall ideal triangulations of 3-manifolds, their gluing equation and Neumann–Zagier matrices following [Thu77, NZ85], and introduce their twisted versions.

Fix an oriented compact 3-manifold $M$ with torus boundary and a concrete (in the sense of [GGZ15, Defn.2.1]) ideal triangulation $\mathcal{T}$ of the interior of $M$. Note that such a triangulation is oriented, but not necessarily ordered, and that such triangulations are the ones used in `SnapPy` [CDGW]. Ordered triangulations, which we do not use, are also known as $\Delta$-complexes. We denote the edges and the tetrahedra of $\mathcal{T}$ by $e_i$ and by $\Delta_j$, respectively, for $1 \leq i, j \leq N$. Note that the number of edges is equal to that of tetrahedra. Every tetrahedron $\Delta_j$ is equipped with shape parameters, i.e. each edge of $\Delta_j$ is assigned to one shape parameter among $z_j, z_j'$ and $z_j''$ with opposite edges having same parameters as in Figure 1. If $\mathcal{T}$ is *ordered*, i.e. if every tetrahedron has (ideal) vertices labeled with $\{0, 1, 2, 3\}$ and every face-pairing respects the vertex-order, then we assign the edges (01) and (23) of each tetrahedron $\Delta_j$ with the shape parameter $z_j$.

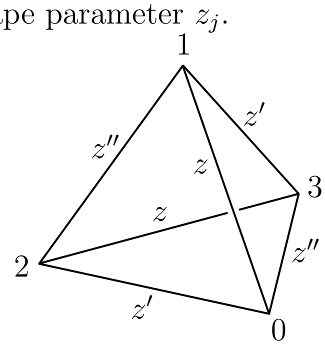


Figure 1. A tetrahedron with shape parameters.

The *gluing equation matrices* $G, G'$ and $G''$ of $\mathcal{T}$ are $N \times N$ integer matrices whose rows and columns are indexed by the edges and by the tetrahedra of $\mathcal{T}$, respectively. The $(i,j)$-entry of $G^\square$ for $\square \in \{\ ,', ''\}$ is the number of edges of $\Delta_j$ assigned to the shape parameter $z_j^\square$ and identified with the edge $e_i$ in $\mathcal{T}$. The *Neumann–Zagier matrices* of $\mathcal{T}$ are defined as the differences of the gluing equation matrices:

$$A := G - G', \qquad B := G'' - G' \in M_{N\times N}(\mathbb{Z})\,. \tag{1}$$

We now define a twisted version of the above matrices. These are essentially the Neumann–Zagier matrices of the ideal triangulation $\widetilde{\mathcal{T}}$ of the universal cover of $M$ obtained by pulling back $\mathcal{T}$. We choose a lift $\widetilde{e}_i$ of $e_i$ and $\widetilde{\Delta}_j$ of $\Delta_j$ for all $1 \leq i, j \leq N$ so that every edge and tetrahedron of $\widetilde{\mathcal{T}}$ is expressed as $\gamma \cdot \widetilde{e}_i$ or $\gamma \cdot \widetilde{\Delta}_j$ for $\gamma \in \pi := \pi_1(M)$. Analogous to the gluing equation matrices, for $\square \in \{\ ,', ''\}$ and $\gamma \in \pi$ let $G_\gamma^\square$ be $N \times N$ integer matrices whose $(i,j)$-entry is the number of edges of $\gamma \cdot \widetilde{\Delta}_j$ assigned to the shape parameter $z_j^\square$ and identified with the edge $\widetilde{e}_i$ in $\widetilde{\mathcal{T}}$. We define the *twisted gluing equation matrices* of $\mathcal{T}$ by

$$\mathbf{G}^\square := \sum_{\gamma\in\pi} G_\gamma^\square \otimes \gamma \in M_{N\times N}(\mathbb{Z}[\pi]) \tag{2}$$

and the *twisted Neumann–Zagier matrices* of $\mathcal{T}$ by

$$\mathbf{A} := \mathbf{G} - \mathbf{G}', \qquad \mathbf{B} := \mathbf{G}'' - \mathbf{G}' \in M_{N\times N}(\mathbb{Z}[\pi])\,. \tag{3}$$

The above notation differs slightly from the one used in [GY23]; hopefully this will not cause any confusion. Note that $G_\gamma^\square$ is the zero matrix for all but finitely many $\gamma$, hence the sum in (2) is finite. Since the above matrices are well-defined after fixing lifts of each edge and tetrahedron of $\mathcal{T}$, a different choice of lifts changes $\mathbf{G}^\square$, $\mathbf{A}$ and $\mathbf{B}$ by multiplication from the left or right by the same diagonal matrix with entries in $\pi$.

The Neumann–Zagier matrices of an ideal triangulation satisfy a key symplectic property [NZ85] which has been the source of many invariants in quantum topology. In particular, it follows that $AB^T$ is a symmetric matrix. This property generalizes for twisted Neumann–Zagier matrices

$$\mathbf{A}\,\mathbf{B}^* = \mathbf{B}\,\mathbf{A}^* \tag{4}$$

where the adjoint $X^*$ of a matrix $X \in M_{N\times N}(\mathbb{Z}[\pi])$ is given by the transpose followed by the involution of $\mathbb{Z}[\pi]$ defined by $\gamma \mapsto \gamma^{-1}$ for all $\gamma \in \pi$. The above equation can be proved by repeating the same argument as in the proof of [GY23, Theorem 1.2] or [Cho06].

## 3. Fox calculus and twisted NZ matrices

In this section, we discuss a connection between twisted Neumann–Zagier matrices and Fox calculus.

3.1. **Fox calculus.** Let $M$ be an oriented compact 3-manifold with torus boundary and $\mathcal{T}$ an ideal triangulation of the interior of $M$ with $N$ tetrahedra. The dual complex $\mathcal{D}$ of $\mathcal{T}$ is a 2-dimensional cell complex with $2N$ edges and $N$ faces. We fix an orientation of each edge and let $\mathcal{F}_\mathcal{D}$ be the free group generated by the edges of $\mathcal{D}$; if $\mathcal{T}$ is ordered, we fix the orientation by the one induced from the vertex-order.

Each face of $\mathcal{D}$ correspond to a word generated by the edges, hence from the faces of $\mathcal{D}$ we obtain words $r_1, \dots, r_N \in \mathcal{F}_\mathcal{D}$ well-defined up to conjugation. Two consecutive letters[1] of $r_i$ for $1 \leq i \leq N$ represent two adjacent face pairings of $\mathcal{T}$, and there is a unique shape parameter lying in between. Inserting such shape parameters between the letters of $r_i$, we obtain a word $R_i$ whose length is two times that of $r_i$. Precisely, let $\mathcal{F}_{\hat{z}}$ be the free group generated by$\hat{z}_j^\square$ for $1 \leq j \leq N$ and $\square \in \{\ ,', ''\}$ where $\hat{z}_j^\square$ is a formal variable corresponding to a shape parameter $z_j^\square$. The word $R_i \in \mathcal{F}_\mathcal{D} * \mathcal{F}_{\hat{z}}$ is defined so that

- its $2k$-th letter is the $k$-th letter of $r_i$ and
- its $(2k-1)$-st letter is a generator of $\mathcal{F}_{\hat{z}}$ corresponding to the shape parameter lying between the $(k-1)$-st and the $k$-th letters of $r_i$.

Here $k \geq 1$ and the 0-th letter of $r_i$ means the last letter of $r_i$.

We choose $N-1$ generators of $\mathcal{F}_\mathcal{D}$ forming a spanning tree in $\mathcal{D}$ and define a map

$$p : \mathcal{F}_\mathcal{D} * \mathcal{F}_{\hat{z}} \to \pi_1(M) \tag{5}$$

by eliminating those $N-1$ generators of $\mathcal{F}_\mathcal{D}$ and all generators $\hat{z}_j^\square$ of $\mathcal{F}_{\hat{z}}$. Note that the rest $N+1$ generators of $\mathcal{F}_\mathcal{D}$ with $N$ relators $p(r_1), \dots, p(r_N)$ give a presentation of $\pi = \pi_1(M)$.

**Theorem 3.1.** *The twisted gluing equation matrices $\mathbf{G}^\square$ of $\mathcal{T}$ agree with*

$$\begin{pmatrix} p\left(\dfrac{\partial R_1}{\partial \hat{z}_1^\square}\right) & \cdots & p\left(\dfrac{\partial R_1}{\partial \hat{z}_N^\square}\right) \\ \vdots & & \vdots \\ p\left(\dfrac{\partial R_N}{\partial \hat{z}_1^\square}\right) & \cdots & p\left(\dfrac{\partial R_N}{\partial \hat{z}_N^\square}\right) \end{pmatrix} \in M_{N\times N}(\mathbb{Z}[\pi]) \tag{6}$$

*up to left multiplication by a diagonal matrix with entries in $\pi$.*

*Proof.* Let $\widetilde{\mathcal{T}}$ be the ideal triangulation of the universal cover of $M$ induced from $\mathcal{T}$. For two tetahedra $\Delta$ and $\Delta'$ of $\widetilde{\mathcal{T}}$ let $d(\Delta, \Delta') \in \mathcal{F}_\mathcal{D}$ be a word representing an oriented curve that starts at $\Delta$ and ends at $\Delta'$. We choose a lift $\widetilde{\Delta}_j$ of each tetrahedron $\Delta_j$ of $\mathcal{T}$ such that

$$p\left(d(\widetilde{\Delta}_{j_0}, \widetilde{\Delta}_{j_1})\right) = 1 \tag{7}$$

for all $1 \leq j_0, j_1 \leq N$. We also choose any lift $\widetilde{e}_i$ of each edge $e_i$ of $\mathcal{T}$ so that the twisted gluing equation matrices $\mathbf{G}^\square$ are determined. Precisely, the $(i,j)$-entry of $\mathbf{G}^\square$ is given by

$$\sum_\Delta p\left(d(\widetilde{\Delta}_1, \Delta)\right) \in \mathbb{Z}[\pi] \tag{8}$$

where the sum is taken over all tetrahedra $\Delta$ of $\widetilde{\mathcal{T}}$ contributing $z_j^\square$ to $\widetilde{e}_i$. The index of $\widetilde{\Delta}_1$ can be replaced by any $1 \leq j \leq N$ due to Equation (7).

On the other hand, there is an initial tetrahedron, say $\hat{\Delta}_i$, around $\widetilde{e}_i$ such that the word $r_i \in \mathcal{F}_\mathcal{D}$ is obtained by winding around the edge $\widetilde{e}_i$ starting from $\hat{\Delta}_i$. Then it follows from

[1]Here we regard that the first and the last letter of $r_i$ are also consecutive.

the definition of $R_i$ that

$$p\left(\frac{\partial R_i}{\partial \hat{z}_j^{\square}}\right) = \sum_{\Delta} p\left(d(\hat{\Delta}_i, \Delta)\right) \in \mathbb{Z}[\pi] \tag{9}$$

where the sum is taken over all tetrahedra $\Delta$ of $\widetilde{\mathcal{T}}$ contributing $z_j^{\square}$ to $\widetilde{e}_i$. Since

$$p\left(d(\widetilde{\Delta}_1, \Delta)\right) = p\left(d(\widetilde{\Delta}_1, \hat{\Delta}_i)\right)\, p\left(d(\hat{\Delta}_i, \Delta)\right) \tag{10}$$

for any $\Delta$, we deduce from (8) and (9) that the matrix (6) agrees with $\mathbf{G}^{\square}$ up to left multiplication by a diagonal matrix with entries in $\pi$. $\square$

3.2. **Curves in triangulations.** The 1-skeleton $\mathcal{D}^{(1)}$ of the dual complex $\mathcal{D}$ intersects with a tetrahedron in four points. Hence there are three ways of smoothing it in each tetrahedron as in Figure 2. Each smoothing makes two curves in a tetrahedron winding two edges with the same shape parameter. We thus refer to it as $Z$, $Z'$, or $Z''$-smoothing accordingly. Applying $Z$-smoothing to $\mathcal{D}^{(1)}$ for all tetrahedra, we obtain finitely many loops, which we call $Z$-curves of $\mathcal{T}$. We define $Z'$ and $Z''$-curves of $\mathcal{T}$ similarly.

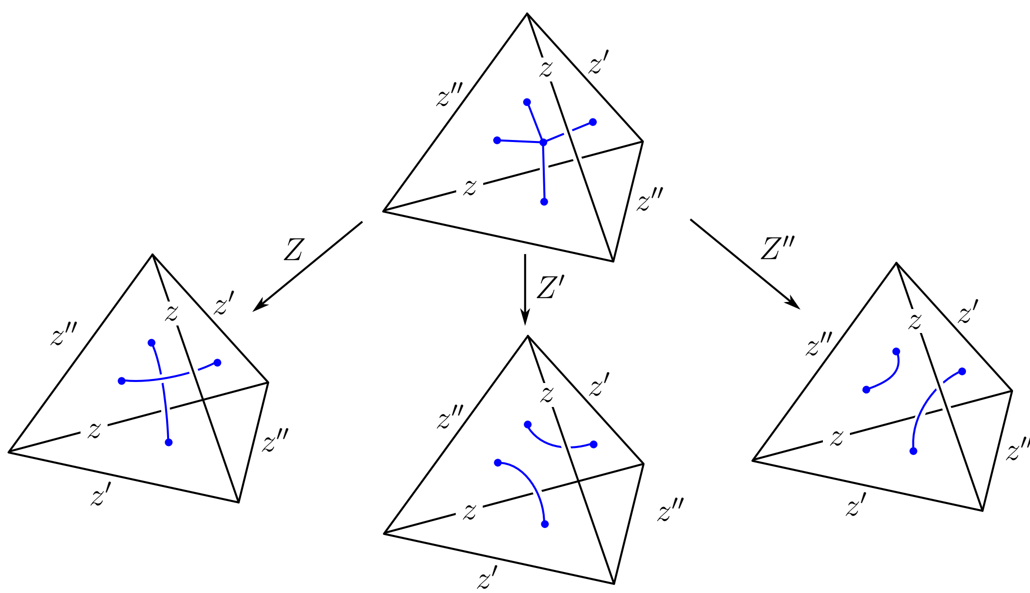


FIGURE 2. Three ways of smoothing $\mathcal{D}^{(1)}$.

**Proposition 3.2.** If $\mathcal{T}$ is ordered, the $Z$-curves homotope to disjoint peripheral curves.

*Proof.* For ordered $\mathcal{T}$, each face of $\mathcal{T}$ has a "middle" vertex, the one whose label is neither greatest nor smallest among the three vertices of the face. Recall that the $Z$-curves intersect with each face $f$ of $\mathcal{T}$ in a point. We push the intersection point toward the middle vertex of $f$. Doing so for all faces of $\mathcal{T}$, the $Z$-curves homotope to disjoint peripheral curves. Note that then the $Z$-curves make two small curves in each tetrahedron lying in a neighborhood of the vertices 1 and 2 as in Figure 3. $\square$

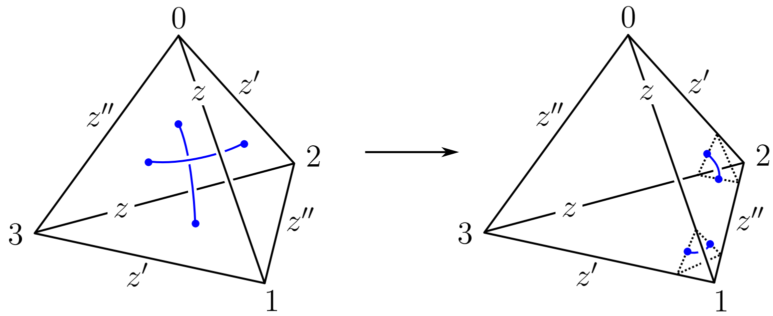


FIGURE 3. Homotope $Z$-curves to peripheral curves.

We now fix a tetrahedron $\Delta_j$ of $\mathcal{T}$. Recall that the free group $\mathcal{F}_{\mathcal{D}}$ has $2N$ generators, say $g_1, \ldots, g_{2N}$, and that a face $f$ of $\Delta_j$ corresponds to one generator $g_i$, oriented either inward or outward to $\Delta_j$. We define a column vector $v_f \in \mathbb{Z}[\pi]^{2N}$

$$v_f = \begin{cases} p(g_i)\, e_i & \text{if } g_i \text{ is inward to } \Delta_j \\ -e_i & \text{if } g_i \text{ is outward to } \Delta_j \end{cases} \tag{11}$$

where $(e_1, \ldots, e_{2N})$ is the standard basis of $\mathbb{Z}^{2N}$. We say that two faces of $\Delta_j$ are $Z$-*adjacent* if they are joined by one of two curves in $\Delta_j$ obtained from $Z$-smoothing (see Figure 2). Note that $\Delta_j$ has two pairs of $Z$-adjacent faces.

**Theorem 3.3.** *If $\mathcal{T}$ is ordered, the column vector*

$$\begin{pmatrix} p\left(\dfrac{\partial r_1}{\partial g_1}\right) & \cdots & p\left(\dfrac{\partial r_1}{\partial g_{2N}}\right) \\ \vdots & & \vdots \\ p\left(\dfrac{\partial r_N}{\partial g_1}\right) & \cdots & p\left(\dfrac{\partial r_N}{\partial g_{2N}}\right) \end{pmatrix} (v_{f_0} + v_{f_1}) \in \mathbb{Z}[\pi]^N \tag{12}$$

*is equal to the $j$-th column of*

$$\begin{pmatrix} p\left(\dfrac{\partial R_1}{\partial \hat{z}''_1}\right) & \cdots & p\left(\dfrac{\partial R_1}{\partial \hat{z}''_N}\right) \\ \vdots & & \vdots \\ p\left(\dfrac{\partial R_N}{\partial \hat{z}''_1}\right) & \cdots & p\left(\dfrac{\partial R_N}{\partial \hat{z}''_N}\right) \end{pmatrix} - \begin{pmatrix} p\left(\dfrac{\partial R_1}{\partial \hat{z}'_1}\right) & \cdots & p\left(\dfrac{\partial R_1}{\partial \hat{z}'_N}\right) \\ \vdots & & \vdots \\ p\left(\dfrac{\partial R_N}{\partial \hat{z}'_1}\right) & \cdots & p\left(\dfrac{\partial R_N}{\partial \hat{z}'_N}\right) \end{pmatrix} \tag{13}$$

*up to sign. Here $f_0$ and $f_1$ are the $Z$-adjacent faces of $\Delta_j$.*

*Proof.* Two faces of $\Delta_j$ are $Z$-adjacent if and only if they are adjacent to either the edge (01) or (23). We first consider two faces adjacent to the edge (01). One of the two faces is oriented inward to $\Delta_j$, and the other is oriented outward. Let $f_0$ and $f_1$ be the former and the latter, respectively, as in Figure 4. Note that the orientation of every edge of $f_0$ and $f_1$ is determined, regardless of the vertices 2 and 3 of $\Delta_j$.

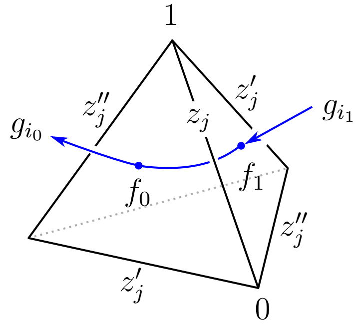


FIGURE 4. Two generators joined by B-smoothing.

From the edges of $f_0$ and $f_1$, we deduce that the generators $g_{i_0}$ and $g_{i_1}$ corresponding to $f_0$ and $f_1$ respectively appear in the words $R_1, \dots, R_N$ as follows.

$$\begin{aligned} &\cdots g_{i_1} \hat{z}_j \, g_{i_0} \cdots \\ &\cdots \hat{z}''_j \, g_{i_0} \cdots \\ &\cdots g_{i_0}^{-1} \, \hat{z}'_j \cdots \\ &\cdots g_{i_1} \, \hat{z}'_j \cdots \\ &\cdots \hat{z}''_j \, g_{i_1}^{-1} \cdots \end{aligned} \tag{14}$$

We stress that $g_{i_0}$ and $g_{i_1}$ do not appear elsewhere other than listed above, and neither do $\hat{z}'_j$ and $\hat{z}''_j$. It follows that for all $1 \le k \le N$

$$p\left(\frac{\partial r_k}{\partial g_{i_0}} - \frac{\partial r_k}{\partial g_{i_1}} \, g_{i_1}\right) = p\left(\frac{\partial R_k}{\partial \hat{z}''_j} - \frac{\partial R_k}{\partial \hat{z}'_j}\right) . \tag{15}$$

Writing the above equation in a matrix form, we obtain the proposition. We prove similarly for two faces adjacent to the edge (23), in which case the left-hand side of (15) is equal to negative of the right-hand side. □

**Remark 3.4.** One can deduce similar equations for $Z'$ and $Z''$-adjacent faces, but the equations only hold modulo 2. Namely, for $Z'$-adjacent faces $f_0$ and $f_1$ of $\Delta_j$, the column vector (12) and the $j$-th column of

$$\begin{pmatrix} p\left(\dfrac{\partial R_1}{\partial \hat{z}_1}\right) & \cdots & p\left(\dfrac{\partial R_1}{\partial \hat{z}_N}\right) \\ \vdots & & \vdots \\ p\left(\dfrac{\partial R_N}{\partial \hat{z}_1}\right) & \cdots & p\left(\dfrac{\partial R_N}{\partial \hat{z}_N}\right) \end{pmatrix} - \begin{pmatrix} p\left(\dfrac{\partial R_1}{\partial \hat{z}''_1}\right) & \cdots & p\left(\dfrac{\partial R_1}{\partial \hat{z}''_N}\right) \\ \vdots & & \vdots \\ p\left(\dfrac{\partial R_N}{\partial \hat{z}''_1}\right) & \cdots & p\left(\dfrac{\partial R_N}{\partial \hat{z}''_N}\right) \end{pmatrix}$$

are congruent modulo 2, i.e. they induce the same vector over $(\mathbb{Z}/2\mathbb{Z})[\pi]$. A similar equation modulo 2 holds for $Z''$-adjacent faces.

## 4. ALEXANDER INVARIANTS FROM TWISTED NZ MATRICES

In this section we express the Alexander polynomial and its twisted and $L^2$-versions in terms of the twisted Neumann–Zagier matrix $\mathbf{B}$. Throughout the section, we fix

(†) an oriented compact 3-manifold $M$ with torus boundary, an *ordered* ideal triangulation $\mathcal{T}$ of the interior of $M$ and a group homomorphism $\alpha : \pi \to \mathbb{Z}$.

Note that it is known that every 3-manifold with nonempty boundary has such a triangulation [BP97].

4.1. **Alexander polynomial.** The homomorphism $\alpha$ in (†) gives rise to a homomorphism $\alpha : \mathbb{Z}[\pi] \to \mathbb{Z}[\mathbb{Z}] \simeq \mathbb{Z}[t^{\pm 1}]$ of group rings, and we define

$$\mathbf{A}_\alpha(t) := \alpha(\mathbf{A}), \qquad \mathbf{B}_\alpha(t) := \alpha(\mathbf{B}) \in M_{N\times N}(\mathbb{Z}[t^{\pm 1}])\,. \tag{16}$$

Theorem 4.1 below relates the determinant of $\mathbf{B}_\alpha(t)$ with the Alexander polynomial $\Delta_\alpha(t)$ associated with $\alpha$, assuming that this is well-defined, that is, the (cellular) chain complex of $M$ with local coefficient twisted by $\alpha$ is acyclic. A typical case is $M$ being the complement of a knot in a homology sphere with $\alpha$ being the abelianization map. Note that the determinant of $\mathbf{B}_\alpha(t)$ and $\Delta_\alpha(t)$ are well-defined up to multiplication by $\pm t^k$ for $k \in \mathbb{Z}$. Below, we denote by $\doteq$ the equality of Laurent polynomials up to multiplication by $\pm t^k$.

**Theorem 4.1.** *Fix $M$,$\mathcal{T}$ and $\alpha$ as in (†). Then*

$$\det \mathbf{B}_\alpha(t) \doteq \frac{\Delta_\alpha(t)}{t-1} \prod_i (t^{\alpha(Z_i)} - 1) \tag{17}$$

*where $Z_i$ are the $Z$-curves of $\mathcal{T}$.*

*Proof.* Let $\mathcal{D}$ be the dual cell complex of $\mathcal{T}$ and consider the cellular chain complex of $\mathcal{D}$ with local coefficient $\mathbb{Z}[t^{\pm 1}]$ twisted by $\alpha : \pi \to \mathbb{Z} \simeq t^{\mathbb{Z}}$:

$$0 \longrightarrow C_2(\mathcal{D}; \mathbb{Z}[t^{\pm 1}]_\alpha) \xrightarrow{\partial_2} C_1(\mathcal{D}; \mathbb{Z}[t^{\pm 1}]_\alpha) \xrightarrow{\partial_1} C_0(\mathcal{D}; \mathbb{Z}[t^{\pm 1}]_\alpha) \longrightarrow 0\,. \tag{18}$$

Here $C_i(\mathcal{D}; \mathbb{Z}[t^{\pm 1}]_\alpha) := C_i(\widetilde{\mathcal{D}}; \mathbb{Z}) \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[t^{\pm 1}]$, where $\widetilde{\mathcal{D}}$ is the universal cover of $\mathcal{D}$, is a free $\mathbb{Z}[t^{\pm 1}]$-module of rank $N$ for $i = 0, 2$ and of rank $2N$ for $i = 1$.

We choose a spanning tree of $\mathcal{D}$, hence $N-1$ edges of $\mathcal{D}$. Lifting the tree to $\widetilde{\mathcal{D}}$, we obtain a basis of $C_i(\mathcal{D}; \mathbb{Z}[t^{\pm 1}]_\alpha)$. It is well-known that the boundary map $\partial_2$ in (18) is given by the Fox derivative

$$\partial_2 = \begin{pmatrix} \alpha(p(\frac{\partial r_1}{\partial g_1})) & \cdots & \alpha(p(\frac{\partial r_1}{\partial g_{2N}})) \\ \vdots & & \vdots \\ \alpha(p(\frac{\partial r_N}{\partial g_1})) & \cdots & \alpha(p(\frac{\partial r_N}{\partial g_{2N}})) \end{pmatrix}^T \in M_{2N\times N}(\mathbb{Z}[t^{\pm 1}]) \tag{19}$$

where $p$ is the map eliminating all generators in the tree. Also, the boundary map $\partial_1$ can be expressed in terms of the vector described in (11). Precisely, the $j$-th row of $\partial_1$ is

$$\alpha(v_{f_0}^T) + \cdots + \alpha(v_{f_3}^T) \in \mathbb{Z}[t^{\pm 1}]^{2N} \tag{20}$$

where $f_0, \ldots, f_3$ are the faces of $\Delta_j$. Recall that $v_f$ is a column vector, hence its transpose $v_f^T$ is a row vector. Since $Z$-smoothing couples the faces $f_0, \ldots, f_3$ of $\Delta_j$ into pairs, we can decompose $\partial_1$ into

$$\partial_1 = \partial_{1,B} + (\partial_1 - \partial_{1,B}) \tag{21}$$

where the $j$-th rows of both $\partial_{1,B}$ and $\partial_1 - \partial_{1,B}$ are of the form $\alpha(v_f^T) + \alpha(v_{f'}^T)$ for $Z$-adjacent faces $f$ and $f'$ of $\Delta_j$. Then Theorems 3.1 and 3.3 imply that

$$\partial_2^T\, \partial_{1,B}^T = D\mathbf{B}_\alpha(t) \tag{22}$$

where $D$ is a diagonal matrix with entries in $\{\pm t^k \mid k \in \mathbb{Z}\}$. It follows that for any $N$-tuple $b = (b_1, \ldots, b_N)$ of column vectors in $C_1(\mathcal{D}; \mathbb{Z}[t^{\pm 1}]_\alpha)$, we have

$$\left(\begin{array}{c} \partial_2^T \\ \hline b^T \end{array}\right) \left( \partial_{1,B}^T \;\middle|\; \partial_1^T \right) = \left(\begin{array}{c|c} D\mathbf{B}_\alpha(t) & 0 \\ \hline \partial_{1,B}(b)^T & \partial_1(b)^T \end{array}\right) \tag{23}$$

and thus

$$\frac{\det\left( \partial_2 \mid b \right)}{\det \partial_1(b)} \det \left(\begin{array}{c} \partial_{1,B} \\ \hline \partial_1 \end{array}\right) \doteq \det \mathbf{B}_\alpha(t) \tag{24}$$

provided that $\det \partial_1(b) \neq 0$.

The first term of the left-hand side of (24) is by definition $\Delta_\alpha(t)/(t-1)$ where $\Delta_\alpha(t)$ is the Alexander polynomial associated with $\alpha$. The second term obviously satisfies

$$\det \left(\begin{array}{c} \partial_{1,B} \\ \hline \partial_1 \end{array}\right) = \det \left(\begin{array}{c} \partial_{1,B} \\ \hline \partial_1 - \partial_{1,B} \end{array}\right). \tag{25}$$

Recall that each row of $\partial_{1,B}$ and $\partial_1 - \partial_{1,B}$ is of the form $v_f^T + v_{f'}^T$ for some faces $f$ and $f'$ and that each column of $\partial_1$ has at most two non-trivial entries. It follows that each row and column of the matrix in the right-hand side of (25) has at most two non-trivial entries. Such a matrix after changing some rows and columns can be expressed as a direct sum of matrices of the form

$$\begin{pmatrix} x_1 & -y_1 & & & \\ & x_2 & -y_2 & & \\ & & \ddots & \ddots & \\ & & & x_{n-1} & -y_{n-1} \\ -y_n & & & & x_n \end{pmatrix} \tag{26}$$

whose determinant is $x_1 \cdots x_n - y_1 \cdots y_n$. In our case, expressing the matrix in the right-hand side of (25) as in the form (26) is carried out by following the $Z$-curves. In particular, all $x_i$ are of the form $t^{\alpha(g_i)}$ and all $y_i$ are 1. It follows that the right-hand side of (25) equals to $\prod(t^{\alpha(Z_i)} - 1)$ where the product is over all components $Z_i$ of the $Z$-curves. Therefore, we obtain

$$\det \mathbf{B}_\alpha(t) \doteq \frac{\Delta_\alpha(t)}{t-1} \prod_i (t^{\alpha(Z_i)} - 1)\,.$$

This completes the proof. $\square$

**Remark 4.2.** Proposition 3.2 says that the $Z$-curves homotope to disjoint peripheral curves. If one component is homotopically trivial, we have $\det \mathbf{B}_\alpha(t) = 0$. Otherwise, the $Z$-curves are $m$-parallel copies of a peripheral curve $\gamma$ for $m \geq 1$, hence we obtain

$$\det \mathbf{B}_\alpha(t) \doteq \frac{\Delta_\alpha(t)}{t-1} (t^{\alpha(\gamma)} - 1)^m\,. \tag{27}$$

**Remark 4.3.** Applying the same argument as in the proof of Theorem 4.1, we deduce similar equations in $(\mathbb{Z}/2\mathbb{Z})[t^{\pm 1}]$ from Remark 3.4:

$$\det \mathbf{A}_\alpha(t) \equiv \frac{\Delta_\alpha(t)}{t-1} \prod_i (t^{\alpha(Z''_i)} - 1) \pmod 2\,, \tag{28}$$

$$\det(\mathbf{A}_\alpha(t) - \mathbf{B}_\alpha(t)) \equiv \frac{\Delta_\alpha(t)}{t-1} \prod_i (t^{\alpha(Z'_i)} - 1) \pmod 2\,. \tag{29}$$

Here $Z'_i$ and $Z''_i$ are the $Z'$ and $Z''$-curves of $\mathcal{T}$, respectively. These equations usually fail in $\mathbb{Z}[t^{\pm 1}]$; see Section 5 for an example.

4.2. **Twisted Alexander polynomial.** The homomorphism $\alpha$ in Theorem 4.1 can be replaced by $\alpha \otimes \rho$ for any representation $\rho : \pi \to \mathrm{SL}_n(\mathbb{C})$, provided that the twisted Alexander polynomial $\Delta_{\alpha\otimes\rho}(t)$ associated with $\alpha \otimes \rho$ is defined. This happens when the (cellular) chain complex of $M$ with local coefficient twisted by $\alpha \otimes \rho$ is acyclic. A typical case is $M$ being the complement of a hyperbolic knot in a homology sphere with $\rho : \pi \to \mathrm{SL}_2(\mathbb{C})$ being a lift of the geometric representation.

**Theorem 4.4.** *Fix $M$,$\mathcal{T}$ and $\alpha$ as in (†) and a representation $\rho : \pi \to \mathrm{SL}_n(\mathbb{C})$. Then*

$$\det \mathbf{B}_{\alpha\otimes\rho}(t) \doteq \Delta_{\alpha\otimes\rho}(t) \prod_i \det(\rho(Z_i)\, t^{\alpha(Z_i)} - I_n)$$

*where $Z_i$ are the $Z$-curves of $\mathcal{T}$, and $I_n$ is the identity matrix of rank $n$.*

Note that if $\rho$ is the trivial 1-dimensional representation, we have $\mathbf{B}_{\alpha\otimes\rho}(t) = \mathbf{B}_\alpha(t)$ and $\Delta_\alpha(t)/(t-1) = \Delta_{\alpha\otimes\rho}(t)$ [Wad94]. Hence Theorem 4.1 is a special case of Theorem 4.4.

*Proof.* We obtain Theorem 4.4 by simply replacing $\alpha$ in the proof of Theorem 4.1 by $\alpha \otimes \rho$. We omit details, as this is indeed a repetition with only obvious variants. For instance, the coefficient of the chain complex (18) is replaced by $(\mathbb{Z}[t^{\pm 1}] \otimes \mathbb{C}^n)_{\alpha\otimes\rho}$, and the matrix (26) becomes a block matrix. □

4.3. **$L^2$-Alexander torsion.** In [DFL15] Dubois–Friedl–Lück introduced the $L^2$-Alexander torsion as an $L^2$-version of the Alexander polynomial

$$\tau^{(2)}(M,\alpha) : \mathbb{R}^+ \to [0,\infty), \qquad t \mapsto \tau^{(2)}(M,\alpha)(t)\,. \tag{30}$$

As the Alexander polynomial, $\tau^{(2)}(M,\alpha)$ is well-defined up to multiplication by a function $t \mapsto t^r$ for $r \in \mathbb{R}$. We will write $f \doteq g$ for functions $f$ and $g : \mathbb{R}^+ \to [0,\infty)$ if $f(t) = t^r g(t)$ for some $r \in \mathbb{R}$. Briefly, for fixed $t > 0$, $\tau^{(2)}(M,\alpha)(t)$ is defined to be the $L^2$-torsion of the chain complex of $\mathbb{R}[\pi]$-modules

$$\mathbb{R}[\pi] \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{M};\mathbb{Z}) \tag{31}$$

where $\widetilde{M}$ is the universal cover of $M$ and $\mathbb{R}[\pi]$ is viewed as a $\mathbb{Z}[\pi]$-module using the homomorphism

$$\alpha_t : \mathbb{Z}[\pi] \to \mathbb{R}[\pi], \quad g \mapsto t^{\alpha(g)} g\,. \tag{32}$$

The $L^2$-torsion of the above complex is defined in terms of the Fulgede-Kadison determinant of matrices with entries in $\mathbb{R}[\pi]$. Roughly speaking, the Fulgede-Kadison determinant of a matrix $X$ is defined in terms of the spectral density function of $X$, viewed as a map between direct sums of the Hilbert space $\ell^2(\pi)$ of squared-summable formal sums over $\pi$. We refer to [LÖ2, DFL15] for the precise definition. However, we will not use the definition, but only some basic properties for square matrices, such as

$$\begin{aligned} \det^r_{\mathcal{N}(\pi)}(XY) &= \det^r_{\mathcal{N}(\pi)}(X)\det^r_{\mathcal{N}(\pi)}(Y)\,, \\ \det^r_{\mathcal{N}(\pi)}\begin{pmatrix} X & 0 \\ Z & Y \end{pmatrix} &= \det^r_{\mathcal{N}(\pi)}(X)\det^r_{\mathcal{N}(\pi)}(Y)\,. \end{aligned} \tag{33}$$

Here $X$ and $Y$ are square matrices with entries in $\mathbb{R}[\pi]$, and $\det^r_{\mathcal{N}(\pi)}(X)$ denotes the regular Fuglede-Kadison determinant of $X$, which equals to the Fuglede-Kadison determinant of $X$ if $X$ has full rank, and zero otherwise.

We now consider the Fuglede–Kadison determinant of the twisted Neumann–Zagier matrices and relate it with the $L^2$-Alexander torsion. Recall that the twisted Neumann–Zagier matrix $\mathbf{B}$ is a square matrix with entries in the group ring $\mathbb{Z}[\pi]$. We define a function

$$\det(\mathbf{B},\alpha):\mathbb{R}^+\to[0,\infty), \qquad t\mapsto \det^r_{\mathcal{N}(\pi)}(\alpha_t(\mathbf{B})) \tag{34}$$

where $\alpha_t:\mathbb{Z}[\pi]\to\mathbb{R}[\pi]$ is the homomorphism given in (32).

**Theorem 4.5.** *Fix $M$,$\mathcal{T}$ and $\alpha$ as in (†). Suppose that every component of the $Z$-curves of $\mathcal{T}$ has infinite order in $\pi$. Then we have*

$$\det(\mathbf{B},\alpha)\doteq\tau^{(2)}(M,\alpha)\max\{1,t^n\} \tag{35}$$

*for some $n\in\mathbb{Z}$.*

*Proof.* Let $\mathcal{D}$ be the dual cell complex of $\mathcal{T}$. The universal cover $\widetilde{\mathcal{D}}$ of $\mathcal{D}$ has the cellular chain complex of left $\mathbb{Z}[\pi]$-modules

$$0\longrightarrow C_2(\widetilde{\mathcal{D}};\mathbb{Z})\xrightarrow{\partial_2}C_1(\widetilde{\mathcal{D}};\mathbb{Z})\xrightarrow{\partial_1}C_0(\widetilde{\mathcal{D}};\mathbb{Z})\longrightarrow 0 \tag{36}$$

where $C_i:=C_i(\widetilde{\mathcal{D}};\mathbb{Z})$ has rank $N$ for $i=0,2$ and rank $2N$ for $i=1$. The boundary maps $\partial_i:C_i\to C_{i-1}$ act on the right, i.e., we have

$$\partial_2\in M_{N,2N}(\mathbb{Z}[\pi]),\quad \partial_1\in M_{N,2N}(\mathbb{Z}[\pi])\,. \tag{37}$$

As in the proof of Theorem 4.1, we decompose $\partial_1$ as $\partial_1=\partial_{1,B}+(\partial_1-\partial_{1,B})$ where the $j$-th columns of both $\partial_{1,B}$ and $\partial_1-\partial_{1,B}$ are of the form $v_f+v_{f'}$ for $Z$-adjacent faces $f$ and $f'$ of $\Delta_j$. Then Theorems 3.1 and 3.3 imply that

$$\partial_2\,\partial_{1,B}=D\mathbf{B} \tag{38}$$

where $D$ is a diagonal matrix with entries in $\pm\pi$.

We now fix $t\in\mathbb{R}^+$ and twist the coefficient of $C_i$ by using the homomorphism $\alpha_t$, i.e. consider the chain complex $C'_i:=\mathbb{R}[\pi]\otimes_{\mathbb{Z}[\pi]}C_i$ where $\mathbb{R}[\pi]$ is viewed as a $\mathbb{Z}[\pi]$-module using the homomorphism $\alpha_t$. Note that the boundary maps of $C'_i$ are given by $\partial'_i=\alpha_t(\partial_i)$. It follows from Equation (38) that for any $N$-tuple $b=(b_1,\dots,b_N)$ of (row) vectors, we have

$$\left(\frac{\partial'_2}{b}\right)\left(\,\partial'_{1,B}\;\middle|\;\partial'_1\,\right)=\left(\begin{array}{c|c}\alpha_t(D\mathbf{B}) & 0\\ \hline \partial'_{1,B}(b) & \partial'_1(b)\end{array}\right) \tag{39}$$

where $\partial'_{1,B} = \alpha_t(\partial_{1,B})$. Therefore, we obtain

$$\frac{\det^r_{\mathcal{N}(\pi)}\begin{pmatrix}\partial'_2 \\ b\end{pmatrix}}{\det^r_{\mathcal{N}(\pi)}(\partial_1(b))} \det^r_{\mathcal{N}(\pi)}\left(\partial'_{1,B} \mid \partial'_1\right) = t^k \det^r_{\mathcal{N}(\pi)}(\alpha_t(\mathbf{B})) \tag{40}$$

for fixed $k \in \mathbb{Z}$, provided that $\det^r_{\mathcal{N}(\pi)}(\partial_1(b)) \neq 0$. The first term of the left-hand side of (40) is $\tau^{(2)}(M,\alpha)(t)$ (see [DFL15, Lemma 3.1]), and the second term satisfies

$$\det^r_{\mathcal{N}(\pi)}\left(\partial'_{1,B} \mid \partial'_1\right) = \det^r_{\mathcal{N}(\pi)}\left(\partial'_{1,B} \mid \partial'_1 - \partial'_{1,B}\right). \tag{41}$$

Recall that the matrix $(\partial_{1,B} \mid \partial_1 - \partial_{1,B})$ after changing some rows and columns is the direct sum of matrices of the form (26) with $x_i \in \pi$ and $y_i = 1$. Such matrices decompose into

$$\begin{pmatrix} x_1 & & & & \\ & x_2 & & & \\ & & \ddots & & \\ & & & x_{n-1} & \\ & & & & x_n \end{pmatrix}\begin{pmatrix} 1 & -x_1^{-1} & & & \\ & 1 & -x_2^{-1} & & \\ & & \ddots & \ddots & \\ & & & 1 & -x_{n-1}^{-1} \\ & & & & 1 \end{pmatrix}\begin{pmatrix} 1 - x_1^{-1}\cdots x_n^{-1} & & & & \\ -x_2^{-1}\cdots x_n^{-1} & 1 & & & \\ \vdots & & \ddots & & \\ -x_{n-1}^{-1}x_n^{-1} & & & 1 & \\ -x_n^{-1} & & & & 1 \end{pmatrix},$$

hence we deduce that

$$\det^r_{\mathcal{N}(\pi)}\left(\partial'_{1,B} \mid \partial'_1 - \partial'_{1,B}\right) = \prod_i \det^r_{\mathcal{N}(\pi)}(1 - \alpha_t(Z_i^{-1})) = \prod_i \det^r_{\mathcal{N}(\pi)}(1 - t^{-\alpha(Z_i)} Z_i^{-1}) \tag{42}$$

where the products are over all the components $Z_i$ of the $Z$-curves of $\mathcal{T}$. Since we assumed that each component $Z_i$ has infinite order in $\pi$, $\det^r_{\mathcal{N}(\pi)}(1-t^{-\alpha(Z_i)}Z_i^{-1})$ is the Mahler measure of $Z_i - t^{-\alpha(Z_i)}$, viewed as a polynoimal in $Z_i$, which equals to $\max\{1, t^{-\alpha(Z_i)}\}$. It follows that

$$\det^r_{\mathcal{N}(\pi)}(\alpha_t(\mathbf{B})) = t^{-k}\, \tau^{(2)}(M,\alpha)(t) \prod_i \max\{1, t^{-\alpha(Z_i)}\} \tag{43}$$

for fixed $k \in \mathbb{Z}$. Since each component $Z_i$ is of infinite order and, in particular, non-trivial, Proposition 3.2 implies that all $\alpha(Z_i)$ should be the same up to sign. Thus Equation (43) implies Theorem 4.5. □

## 5. Example

As is customary in hyperbolic geometry, in this section we give an example of a cusped hyperbolic 3-manifold $M$, the complement of the knot $4_1$ in $S^3$. The default `SnapPy` triangulation $\mathcal{T}$ of $M$ consists of two ideal tetrahedra $\Delta_1$ and $\Delta_2$, and is orderable with the ordering shown in Figure 5 [CDGW]. It has two edges $e_1$ and $e_2$; $(01), (03), (23)$ of $\Delta_1$ and $(02), (12), (13)$ of $\Delta_2$ are identified with $e_1$; $(02), (12), (13)$ of $\Delta_1$ and $(01), (03), (23)$ of $\Delta_2$ are identified swith $e_2$.

The dual cell complex of $\mathcal{T}$ has 4 edges and 2 faces, hence we have two words $r_1$ and $r_2$ in four generators $g_1, \ldots, g_4$ Note that $g_1$ and $g_4$ (resp., $g_2$ and $g_3$) are oriented inward to $\Delta_1$ (resp., $\Delta_2$) and that the words $r_1$ and $r_2$ are obtained from winding around the edges of $\mathcal{T}$:

$$\begin{aligned} e_1 : \quad & g_1 \xrightarrow{z_1} g_3 \xrightarrow{z''_2} g_4 \xrightarrow{z_1} g_2 \xrightarrow{z'_2} g_3^{-1} \xrightarrow{z'_1} g_4^{-1} \xrightarrow{z'_2} g_1, \\ e_2 : \quad & g_1 \xrightarrow{z'_1} g_2 \xrightarrow{z_2} g_4 \xrightarrow{z''_1} g_1^{-1} \xrightarrow{z''_2} g_2^{-1} \xrightarrow{z''_1} g_3 \xrightarrow{z_2} g_1. \end{aligned} \tag{44}$$

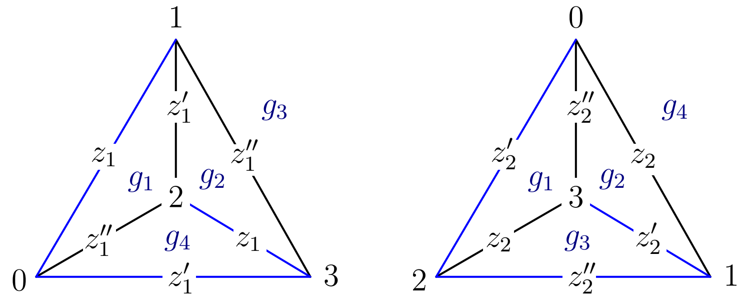


FIGURE 5. An ordered ideal triangulation of $4_1$.

Precisely, $r_1 = g_3g_4g_2g_3^{-1}g_4^{-1}g_1$ and $r_2 = g_2g_4g_1^{-1}g_2^{-1}g_3g_1$. Eliminating one generator, say $g_1$, we obtain a presentation of $\pi = \pi_1(M)$:

$$\pi = \langle g_2, g_3, g_4 \,|\, g_3g_4g_2g_3^{-1}g_4^{-1},\ \ g_2g_4g_2^{-1}g_3\rangle \,. \tag{45}$$

Note that $g_4$ is a meridian of the knot.

As in Section 3, we define a word $R_i$ for $i = 1, 2$ by inserting shape parameters to the word $r_i$ (c.f. (44)):

$$\begin{aligned} R_1 &= \hat{z}_1\, g_3\, \hat{z}_2''\, g_4\, \hat{z}_1\, g_2\, \hat{z}_2'\, g_3^{-1}\, \hat{z}_1'\, g_4^{-1}\, \hat{z}_2'\, g_1\,, \\ R_2 &= \hat{z}_1'\, g_2\, \hat{z}_2\, g_4\, \hat{z}_1''\, g_1^{-1}\, \hat{z}_2''\, g_2^{-1}\, \hat{z}_1''\, g_3\, \hat{z}_2\, g_1\,. \end{aligned}$$

Due to Theorem 3.1, the twisted gluing equation matrices $\mathbf{G}^{\square}$ of $\mathcal{T}$ are equal to $(\partial R_i/\partial \hat{z}_j^{\square})$ followed by eliminating $g_1$ and all $\hat{z}_j^{\square}$. Explicitly, we have

$$\begin{aligned} \mathbf{G} &= \begin{pmatrix} 1 + g_3g_4 & 0 \\ 0 & g_2 + g_2g_4g_2^{-1}g_3 \end{pmatrix}, \\ \mathbf{G}' &= \begin{pmatrix} g_3g_4g_2g_3^{-1} & g_3g_4g_2 + g_3g_4g_2g_3^{-1}g_4^{-1} \\ 1 & 0 \end{pmatrix}, \\ \mathbf{G}'' &= \begin{pmatrix} 0 & g_3 \\ g_2g_4 + g_2g_4g_2^{-1} & g_2g_4 \end{pmatrix} \end{aligned}$$

and thus the twisted Neumann–Zagier matrices of $\mathcal{T}$ are given as

$$\mathbf{A} = \begin{pmatrix} 1 + g_3g_4 - g_3g_4g_2g_3^{-1} & -g_3g_4g_2 - g_3g_4g_2g_3^{-1}g_4^{-1} \\ -1 & g_2 + g_2g_4g_2^{-1}g_3 \end{pmatrix}, \tag{46}$$

$$\mathbf{B} = \begin{pmatrix} -g_3g_4g_2g_3^{-1} & g_3 - g_3g_4g_2 - g_3g_4g_2g_3^{-1}g_4^{-1} \\ g_2g_4 + g_2g_4g_2^{-1} - 1 & g_2g_4 \end{pmatrix}. \tag{47}$$

On the other hand, the abelianization map $\alpha : \pi \to \mathbb{Z}$ is given by $\alpha(g_2) = 0$, $\alpha(g_3) = -1$ and $\alpha(g_4) = 1$. Applying $\alpha$ to the twisted Neumann–Zagier matrices, we obtain

$$\mathbf{A}_\alpha(t) = \begin{pmatrix} 2 - t & -2 \\ -1 & 2 \end{pmatrix}, \qquad \mathbf{B}_\alpha(t) = \begin{pmatrix} -t & t^{-1} - 2 \\ 2t - 1 & t \end{pmatrix}. \tag{48}$$

One easily verifies Theorem 4.1:

$$\det \mathbf{B}_\alpha(t) \doteq (t-1)(t^2 - 3t + 1)\,. \tag{49}$$

Note that the Alexander polynomial of $4_1$ is $t^2-3t+1$ and that $\mathcal{T}$ has two $Z$-curves $Z_1 = g_1g_3$ and $Z_2 = g_4g_2$ with $\alpha(Z_1) = -1$, $\alpha(Z_2) = 1$. One can also verify Remark 4.3:

$$\det \mathbf{A}_\alpha(t) \equiv \det(\mathbf{A}_\alpha(t) - \mathbf{B}_\alpha(t)) \equiv 0 \pmod 2 \ . \tag{50}$$

Note that $\mathcal{T}$ has one $Z'$-curve $Z'_1 = g_1g_2g_3^{-1}g_4^{-1}$ with $\alpha(Z'_1) = 0$ and one $Z''$-curve $Z''_1 = g_2^{-1}g_3g_4g_1^{-1}$ with $\alpha(Z''_1) = 0$.

We now compute a (positive) lift $\rho : \pi \to \mathrm{SL}_2(\mathbb{C})$ of the geometric representation of $M$. Since $g_4$ is a meridian of the knot, we may let (see [Ril84, Lemma 1])

$$\rho(g_4) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} , \quad \rho(g_2) = \begin{pmatrix} n & 0 \\ u & 1/n \end{pmatrix} . \tag{51}$$

A straightforward computation shows that the above assignment induces a representation $\rho$ of $\pi$ if and only if $u = -(1-4n^2+n^4)/(3n+3n^3)$ and $1-3n+5n^2-3n^3+n^4 = 0$. Applying $\alpha \otimes \rho$ to Equation (47), one verifies Theorem 4.4:

$$\det \mathbf{B}_{\alpha\otimes\rho}(t) \doteq (t-1)^4(t^2-4t+1)/t^2 . \tag{52}$$

Note that the twisted Alexander polynomial of $4_1$ associated with $\alpha \otimes \rho$ is $t^2 - 4t + 1$.

**Remark 5.1.** For ordered ideal triangulations, $\det \mathbf{A}_\alpha(t)$ is often a multiple of 2 and thus vanishes in $(\mathbb{Z}/2\mathbb{Z})[t^{\pm 1}]$. One example which is not the case is the knot $8_2$. Its default `SnapPy` triangulation is orderable, and Philip Choi's program computes that

$$\mathbf{A}_\alpha(t) = \begin{pmatrix} t^{-4}+1 & 1 & -t^{-4} & t^{-4} & 0 & 0 \\ -t^{-2}-1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & t & 1 & -1 \\ 0 & -t^{-1} & -t & -t^{-4} & -t & t^{-5} \\ 0 & 1 & 0 & 0 & 0 & t \\ 0 & 0 & 1 & -1 & 0 & -1 \end{pmatrix},$$

$$\mathbf{B}_\alpha(t) = \begin{pmatrix} t^{-2} & 0 & -t^{-4} & 0 & 0 & 0 \\ -t^{-2}-1 & t^{-2} & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & t & t-1 \\ 0 & 1-t^{-1} & -t & t^{-5}-t^{-4} & -t & 0 \\ 0 & 0 & t^2 & t^2 & -t & t \\ 0 & 0 & 0 & -1 & 1 & -1 \end{pmatrix}$$

with

$$\det \mathbf{A}_\alpha(t) = (t-1)(t^{12}+t^7-2t^6+t^5+1) ,$$
$$\det \mathbf{B}_\alpha(t) = (t-1)(t^6-3t^5+3t^4-3t^3+3t^2-3t+1) .$$

Note that the Alexander polynomial of the knot $8_2$ is the second factor of $\det \mathbf{B}_\alpha(t)$ (hence this verifies Theorem 4.1) and that

$$\det \mathbf{A}_\alpha(t) \equiv (t-1)^2(t^4+t^3+t^2+t+1)(t^6-3t^5+3t^4-3t^3+3t^2-3t+1) \pmod 2 .$$

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International Center for Mathematics, Department of Mathematics, Southern University of Science and Technology, Shenzhen, China
http://people.mpim-bonn.mpg.de/stavros
*Email address*: stavros@mpim-bonn.mpg.de

Department of Mathematics, Chonnam National University, Gwangju, South Korea
http://sites.google.com/view/seokbeom
*Email address*: sbyoon15@gmail.com