THE (TWISTED/L^2)-ALEXANDER POLYNOMIAL OF IDEALLY TRIANGULATED 3-MANIFOLDS

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ABSTRACT. We establish a connection between the Alexander polynomial of a knot and its twisted and L^2-versions with the triangulations that appear in 3-dimensional hyperbolic geometry. Specifically, we introduce twisted Neumann–Zagier matrices of ordered ideal triangulations and use them to provide formulas for the Alexander polynomial and its variants, the twisted Alexander polynomial and the L^2-Alexander torsion.

CONTENTS

1. INTRODUCTION

The Alexander polynomial is a fundamental invariant of knots that dates back to the origins of algebraic topology [Ale28]. It has been studied time and again from various points of view that include twisting by a representation [Wad94, Lin01], or considering L^2-versions [LÖ2, DFL15]. There are numerous results and surveys to this subject that the reader may consult that include [FV11, DFJ12, Kit15].

The goal of the paper is to establish a connection of this classical topological invariant and its variants with the triangulations that appear in 3-dimensional hyperbolic geometry [Thu77]. These triangulations involve ideal tetrahedra, which one can think of as tetrahedra with their vertices removed, whose faces are identified in pairs so as to obtain the interior of

Date: 3 January 2024.

Key words and phrases. Alexander polynomial, twisted Alexander polynomial, L^2-Alexander torsion, Neumann–Zagier matrices, ordered ideal triangulation, Mahler measure, Fuglede-Kadison determinant.

a compact 3-manifold. Under such an identification, an edge can lie in more than one tetrahedron, or said differently, going around an edge, one traverses several tetrahedra, possibly with repetition. Keeping track of the total number a tetrahedron winds around an edge (in each of its possible three ways) gives rise to a pair of Neumann–Zagier matrices [NZ85]. An ideal triangulation of such a manifold lifts to an ideal triangulation of its universal cover, and this gives rise to twisted Neumann–Zagier matrices; see Section 2 for details.

Our main theorems provide explicit relations of the twisted Neumann–Zagier matrices with the Alexander polynomial and its variants, the twisted Alexander polynomial and the L^2-Alexander torsion (Theorems 3.1–3.3). These relations follow from a connection between the twisted Neumann-Zagier and Fox calculus [Fox53] which we will discuss in Section 4.

The paper is organized as follows. In Section 2, we briefly recall Neumann–Zagier matrices and introduce their twisted version. In Section 3, we present our main theorems and their corollaries. In Section 4, we show that twisted Neumann–Zagier matrices can be obtained from Fox calculus and prove our main theorems. We give an explicit computation for the figure-eight knot and verify our theorems in Section 5.

2. Twisted Neumann–Zagier matrices

In this section we briefly recall ideal triangulations of 3-manifolds, their gluing equation and Neumann–Zagier matrices following [Thu77, NZ85], and introduce their twisted versions. Fix a compact 3-manifold M with torus boundary and $\mathcal{T}$ an ideal triangulation of the interior of M. We denote the edges and the tetrahedra of $\mathcal{T}$ by e_i and by Δ_j, respectively, for $1 \leq i, j \leq N$. Note that the number of edges is equal to that of tetrahedra. Every tetrahedron Δ_j is equipped with shape parameters, i.e. each edge of Δ_j is assigned to one shape parameter among z_j, z_j' and z_j'' with opposite edges having same parameters as in Figure 1. If $\mathcal{T}$ is *ordered*, i.e. if every tetrahedron has vertices labeled with $\{0, 1, 2, 3\}$ and every face-pairing respects the vertex-order, then we assign the edges (01) and (23) of each tetrahedron Δ_j to the shape parameter z_j.

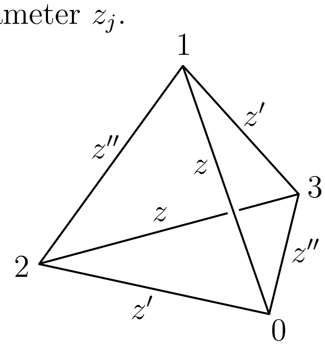

Figure 1. A tetrahedron with shape parameters.

The *gluing equation matrices* G, G' and G'' of $\mathcal{T}$ are $N \times N$ integer matrices whose rows and columns are indexed by the edges and by the tetrahedra of $\mathcal{T}$, respectively. The (i, j)-entry of $G^\square$ for $\square \in \{\ ,', ''\}$ is the number of edges of Δ_j assigned to the shape parameter $z_j^\square$ and identified with the edge e_i in $\mathcal{T}$. The *Neumann–Zagier matrices* of $\mathcal{T}$ are defined as the differences of the gluing equation matrices:

$$A := G - G', \qquad B := G'' - G' \in M_{N\times N}(\mathbb{Z})\,. \tag{1}$$

We now define a twisted version of the above matrices. These are essentially the Neumann–Zagier matrices of the ideal triangulation $\widetilde{\mathcal{T}}$ of the universal cover of M obtained by pulling back $\mathcal{T}$. We choose a lift $\widetilde{e}_i$ of e_i and $\widetilde{\Delta}_j$ of Δ_j for all $1 \leq i, j \leq N$ so that every edge and tetrahedron of $\widetilde{\mathcal{T}}$ is expressed as $\gamma \cdot \widetilde{e}_i$ or $\gamma \cdot \widetilde{\Delta}_j$ for $\gamma \in \pi := \pi_1(M)$. Analogous to the gluing equation matrices, let $G_\gamma^\square$ for $\square \in \{ \;, ', '' \}$ and $\gamma \in \pi$ denote $N \times N$ integer matrices whose (i, j)-entry is the number of edges of $\gamma \cdot \widetilde{\Delta}_j$ assigned to the shape parameter $z_j^\square$ and identified with the edge $\widetilde{e}_i$ in $\widetilde{\mathcal{T}}$. We define the *twisted gluing equation matrices* of $\mathcal{T}$ by

$$\mathbf{G}^\square := \sum_{\gamma \in \pi} G_\gamma^\square \otimes \gamma \in M_{N \times N}(\mathbb{Z}[\pi]) \tag{2}$$

and the *twisted Neumann–Zagier matrices* of $\mathcal{T}$ by

$$\mathbf{A} := \mathbf{G} - \mathbf{G}', \qquad \mathbf{B} := \mathbf{G}'' - \mathbf{G}' \in M_{N \times N}(\mathbb{Z}[\pi]) \,. \tag{3}$$

The above notation differs slightly from the one used in [GY23]; hopefully this will not cause any confusion. Note that $G_\gamma^\square$ is the zero matrix for all but finitely many γ, hence the sum in (2) is finite. Since the above matrices are well-defined after fixing lifts of each edge and tetrahedron of $\mathcal{T}$, a different choice of lifts changes $\mathbf{G}^\square$, $\mathbf{A}$ and $\mathbf{B}$ by multiplication from the left or right by the same diagonal matrix with entries in π. This ambiguity propagates to any invariant constructed using these matrices.

The Neumann–Zagier matrices of an ideal triangulation satisfy a key symplectic property [NZ85] which has been the source of many invariants in quantum topology. In particular, it follows that AB^T is a symmetric matrix. This property generalizes for twisted Neumann–Zagier matrices

$$\mathbf{A}\,\mathbf{B}^* = \mathbf{B}\,\mathbf{A}^* \tag{4}$$

where the adjoint X^* of a matrix $X \in M_{N \times N}(\mathbb{Z}[\pi])$ is given by the transpose followed by the involution of $\mathbb{Z}[\pi]$ defined by $\gamma \mapsto \gamma^{-1}$ for all $\gamma \in \pi$. The above equation can be proved by repeating the same argument as in the proof of [GY23, Theorem 1.2] or [Cho06].

One important aspect of our results is the use of ordered ideal triangulations. It is known that every 3-manifold with nonempty boundary has such a triangulation [BP97]. The choice of an ordered triangulation breaks the symmetry between the two Neumann–Zagier matrices, and distinguishes the $\mathbf{B}$ among the two.

3. Alexander invariants from twisted NZ matrices

In this section we express the Alexander polynomial and its twisted and L^2-versions in terms of the twisted Neumann–Zagier matrix $\mathbf{B}$. Throughout the section, we fix

(†) a compact 3-manifold M with torus boundary, an *ordered* ideal triangulation $\mathcal{T}$ of the interior of M and a group homomorphism $\alpha : \pi \to \mathbb{Z}$.

3.1. Alexander polynomial.

The homomorphism α in (†) gives rise to a homomorphism $\alpha : \mathbb{Z}[\pi] \to \mathbb{Z}[\mathbb{Z}] \simeq \mathbb{Z}[t^{\pm 1}]$ of group rings, and we define

$$\mathbf{A}_\alpha(t) := \alpha(\mathbf{A}), \qquad \mathbf{B}_\alpha(t) := \alpha(\mathbf{B}) \in M_{N \times N}(\mathbb{Z}[t^{\pm 1}]) \,. \tag{5}$$

Our first theorem relates the determinant of one of these matrices with the Alexander polynomial $\Delta_\alpha(t)$ associated with α, assuming that this is well-defined, that is, the (cellular) chain complex of M with local coefficient twisted by α is acyclic. A typical case is M being the complement of a knot in a homology sphere with α being the abelianization map. Note that the determinants of $\mathbf{A}_\alpha(t)$ and $\mathbf{B}_\alpha(t)$ as well as $\Delta_\alpha(t)$ are well-defined up to multiplication by $\pm t^k$, $k \in \mathbb{Z}$. Below, we denote by $\doteq$ the equality of Laurent polynomials (or functions of t) up to multiplication by $\pm t^k$, $k \in \mathbb{Z}$.

Theorem 3.1. *Fix M,$\mathcal{T}$ and α as in (†). Then either* $\det \mathbf{B}_\alpha(t) = 0$ *or*

$$\det \mathbf{B}_\alpha(t) \doteq \frac{\Delta_\alpha(t)}{t-1}\,(t^n - 1)^m \tag{6}$$

for some $n \geq 0$ and $m \geq 1$.

The matrix $\mathbf{A}_\alpha(t)$ also satisfies a similar equation, but only modulo 2. See Remark 4.5 for details.

3.2. **Twisted Alexander polynomial.** The homomorphism α in Theorem 3.1 can be replaced by $\alpha \otimes \rho$ for any representation $\rho : \pi \to \mathrm{SL}_n(\mathbb{C})$, provided that the twisted Alexander polynomial $\Delta_{\alpha\otimes\rho}(t)$ associated with $\alpha \otimes \rho$ is defined. This happens when the (cellular) chain complex of M with local coefficient twisted by $\alpha \otimes \rho$ is acyclic. A typical case is M being the complement of a hyperbolic knot in a homology sphere with $\rho : \pi \to \mathrm{SL}_2(\mathbb{C})$ being a lift of the geometric representation.

Theorem 3.2. *Fix M,$\mathcal{T}$ and α as in (†) and a representation $\rho : \pi \to \mathrm{SL}_n(\mathbb{C})$. Then either* $\det \mathbf{B}_{\alpha\otimes\rho}(t) = 0$ *or*

$$\det \mathbf{B}_{\alpha\otimes\rho}(t) \doteq \Delta_{\alpha\otimes\rho}(t)\, \det(\rho(\gamma)\, t^{\alpha(\gamma)} - I_n)^m \tag{7}$$

for some peripheral curve γ and $m \geq 1$ where I_n is the identity matrix of rank n.

Note that if ρ is the trivial 1-dimensional representation, we have $\mathbf{B}_{\alpha\otimes\rho}(t) = \mathbf{B}_\alpha(t)$ and $\Delta_\alpha(t)/(t-1) = \Delta_{\alpha\otimes\rho}(t)$ [Wad94]. Hence Theorem 3.1 is a special case of Theorem 3.2.

3.3. **L^2-Alexander torsion.** In [DFL15] Dubois–Friedl–Lück introduced the L^2-Alexander torsion as an L^2-version of the Alexander polynomial

$$\tau^{(2)}(M,\alpha) : \mathbb{R}^+ \to [0,\infty), \qquad t \mapsto \tau^{(2)}(M,\alpha)(t)\,. \tag{8}$$

As the Alexander polynomial, $\tau^{(2)}(M,\alpha)$ is well-defined up to multiplication by a function $t \mapsto t^r$ for $r \in \mathbb{R}$. We will write $f \doteq g$ for functions f and $g : \mathbb{R}^+ \to [0,\infty)$ if $f(t) = t^r g(t)$ for some $r \in \mathbb{R}$. Briefly, for fixed $t > 0$, $\tau^{(2)}(M,\alpha)(t)$ is defined to be the L^2-torsion of the chain complex of $\mathbb{R}[\pi]$-modules

$$\mathbb{R}[\pi] \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{M};\mathbb{Z}) \tag{9}$$

where $\widetilde{M}$ is the universal cover of M and $\mathbb{R}[\pi]$ is viewed as a $\mathbb{Z}[\pi]$-module using the homomorphism

$$\alpha_t : \mathbb{Z}[\pi] \to \mathbb{R}[\pi], \quad g \mapsto t^{\alpha(g)} g\,. \tag{10}$$

The L^2-torsion of the above complex is defined in terms of the Fulgede-Kadison determinant of matrices with entries in $\mathbb{R}[\pi]$. Roughly speaking, the Fulgede-Kadison determinant of a matrix X is defined in terms of the spectral density function of X, viewed as a map between direct sums of the Hilbert space $\ell^2(\pi)$ of squared-summable formal sums over π. We refer to [LÖ2, DFL15] for the precise definition. However, we will not use the definition, but only some basic properties for square matrices, such as

$$\begin{aligned} \det^r_{\mathcal{N}(\pi)}(XY) &= \det^r_{\mathcal{N}(\pi)}(X)\det^r_{\mathcal{N}(\pi)}(Y)\,, \\ \det^r_{\mathcal{N}(\pi)}\begin{pmatrix} X & 0 \\ Z & Y \end{pmatrix} &= \det^r_{\mathcal{N}(\pi)}(X)\det^r_{\mathcal{N}(\pi)}(Y)\,. \end{aligned} \tag{11}$$

Here X and Y are square matrices with entries in $\mathbb{R}[\pi]$, and $\det^r_{\mathcal{N}(\pi)}(X)$ denotes the regular Fuglede-Kadison determinant of X, which equals to the Fuglede-Kadison determinant of X if X has full rank, and zero otherwise.

We now consider the Fuglede–Kadison determinant of the twisted Neumann–Zagier matrices and relate it with the L^2-Alexander torsion. Recall that the twisted Neumann–Zagier matrices are square matrices with entries in the group ring $\mathbb{Z}[\pi]$. We define a function

$$\det(\mathbf{B},\alpha) : \mathbb{R}^+ \to [0,\infty), \qquad t \mapsto \det^r_{\mathcal{N}(\pi)}(\alpha_t(\mathbf{B})) \tag{12}$$

where $\alpha_t : \mathbb{Z}[\pi] \to \mathbb{R}[\pi]$ is the homomorphism given in (10).

Theorem 3.3. *Fix M,$\mathcal{T}$ and α as in (†). Suppose that every component of the Z-curves of $\mathcal{T}$ has infinite order in π (see Section 4.2 for the definition of Z-curves). Then we have*

$$\det(\mathbf{B},\alpha) \doteq \tau^{(2)}(M,\alpha)\max\{1,t^n\} \tag{13}$$

for some $n \in \mathbb{Z}$.

4. Fox calculus and twisted NZ matrices

In this section, we discuss a connection between twisted Neumann–Zagier matrices and Fox calculus, and prove Theorems 3.1–3.3.

4.1. **Fox calculus.** Let M be a compact 3-manifold with torus boundary and $\mathcal{T}$ an ideal triangulation of the interior of M with N tetrahedra. The dual complex $\mathcal{D}$ of $\mathcal{T}$ is a 2-dimensional cell complex with $2N$ edges and N faces. We choose an orientation of each edge and let $\mathcal{F}_\mathcal{D}$ be the free group generated by the edges of $\mathcal{D}$; if $\mathcal{T}$ is ordered, we choose the orientation by the one induced from the vertex-order.

The faces of $\mathcal{D}$ correspond to words $r_1,\ldots,r_N \in \mathcal{F}_\mathcal{D}$ well-defined up to conjugation. Two consecutive letters of r_i $(1 \le i \le N)$ correspond to two adjacent face pairings of $\mathcal{T}$, hence there is a shape parameter lying in between. Here we regard that the first and the last letter of r_i are also consecutive. Inserting such shape parameters between the letters of r_i, we obtain a word R_i whose length is two times that of r_i. More precisely, let $\mathcal{F}_{\hat{z}}$ be the free group generated by $\hat{z}_j^\square$ for $1 \le j \le N$ and $\square \in \{\ ,', ''\}$ (hence $\mathcal{F}_{\hat{z}}$ has $3N$ generators) where $\hat{z}_j^\square$ is a formal variable corresponding to a shape parameter $z_j^\square$. Then we define a word $R_i \in \mathcal{F}_\mathcal{D} * \mathcal{F}_{\hat{z}}$ by its $2k$-th letter to be the k-th letter of r_i and its $(2k-1)$-st letter to be a generator of $\mathcal{F}_{\hat{z}}$ corresponding to the shape parameter lying between the $(k-1)$-st and the k-th letters of r_i. Here $k \ge 1$ and the 0-th letter of r_i means the last letter of r_i.

We choose $N-1$ generators of $\mathcal{F}_\mathcal{D}$ forming a spanning tree in $\mathcal{D}$ and define a map

$$p : \mathcal{F}_\mathcal{D} * \mathcal{F}_{\hat{z}} \to \pi \tag{14}$$

by eliminating those $N-1$ generators of $\mathcal{F}_\mathcal{D}$ and all generators $\hat{z}_j^\square$ of $\mathcal{F}_{\hat{z}}$. Note that the rest $N+1$ generators of $\mathcal{F}_\mathcal{D}$ with N relators $p(r_1),\dots,p(r_N)$ give a presentation of $\pi=\pi_1(M)$, hence the map p is well-defined.

Proposition 4.1. The twisted gluing equation matrices $\mathbf{G}^\square$ of $\mathcal{T}$ agree with

$$\begin{pmatrix} p\left(\dfrac{\partial R_1}{\partial \hat{z}_1^\square}\right) & \cdots & p\left(\dfrac{\partial R_1}{\partial \hat{z}_N^\square}\right) \\ \vdots & & \vdots \\ p\left(\dfrac{\partial R_N}{\partial \hat{z}_1^\square}\right) & \cdots & p\left(\dfrac{\partial R_N}{\partial \hat{z}_N^\square}\right) \end{pmatrix} \in M_{N\times N}(\mathbb{Z}[\pi]) \tag{15}$$

up to left multiplication by a diagonal matrix with entries in π.

Proof. Let $\widetilde{\mathcal{T}}$ be the ideal triangulation of the universal cover of M induced from $\mathcal{T}$. For two tetahedra Δ and Δ' of $\widetilde{\mathcal{T}}$ let $d(\Delta,\Delta')\in\mathcal{F}_\mathcal{D}$ be a word representing an oriented curve that starts at Δ and ends at Δ'. We choose a lift $\widetilde{\Delta}_j$ of each tetrahedron Δ_j of $\mathcal{T}$ such that

$$p\left(d(\widetilde{\Delta}_{j_0},\widetilde{\Delta}_{j_1})\right)=1 \tag{16}$$

for all $1\le j_0,j_1\le N$. We also choose any lift $\widetilde{e}_i$ of each edge e_i of $\mathcal{T}$ so that the twisted gluing equation matrices $\mathbf{G}^\square$ are determined. Precisely, the (i,j)-entry of $\mathbf{G}^\square$ is given by

$$\sum_\Delta p\left(d(\widetilde{\Delta}_1,\Delta)\right)\in\mathbb{Z}[\pi] \tag{17}$$

where the sum is taken over all tetrahedra Δ of $\widetilde{\mathcal{T}}$ contributing $z_j^\square$ to $\widetilde{e}_i$. The index of $\widetilde{\Delta}_1$ can be replaced by any $1\le j\le N$ due to Equation (16).

On the other hand, there is an initial tetrahedron, say $\hat{\Delta}_i$, around $\widetilde{e}_i$ such that the word $r_i\in\mathcal{F}_\mathcal{D}$ is obtained by winding around the edge $\widetilde{e}_i$ starting from $\hat{\Delta}_i$. Then it follows from the definition of R_i that

$$p\left(\frac{\partial R_i}{\partial \hat{z}_j^\square}\right)=\sum_\Delta p\left(d(\hat{\Delta}_i,\Delta)\right)\in\mathbb{Z}[\pi] \tag{18}$$

where the sum is taken over all tetrahedra Δ of $\widetilde{\mathcal{T}}$ contributing $z_j^\square$ to $\widetilde{e}_i$. Since

$$p\left(d(\widetilde{\Delta}_1,\Delta)\right)=p\left(d(\widetilde{\Delta}_1,\hat{\Delta}_i)\right)\,p\left(d(\hat{\Delta}_i,\Delta)\right) \tag{19}$$

for any Δ, we deduce from (17) and (18) that the matrix (15) agrees with $\mathbf{G}^\square$ up to left multiplication by a diagonal matrix with entries in π. □

4.2. **Curves in triangulations.** The 1-skeleton $\mathcal{D}^{(1)}$ of the dual complex $\mathcal{D}$ intersects with a tetrahedron in four points. Hence there are three ways of smoothing it in each tetrahedron as in Figure 2. Each smoothing makes two curves in a tetrahedron winding two edges with the same shape parameter. We thus refer to it as Z, Z', or Z''-smoothing accordingly. Applying Z-smoothing to $\mathcal{D}^{(1)}$ for all tetrahedra, we obtain finitely many loops, which we call Z-curves of $\mathcal{T}$. We define Z' and Z''-curves of $\mathcal{T}$ similarly.

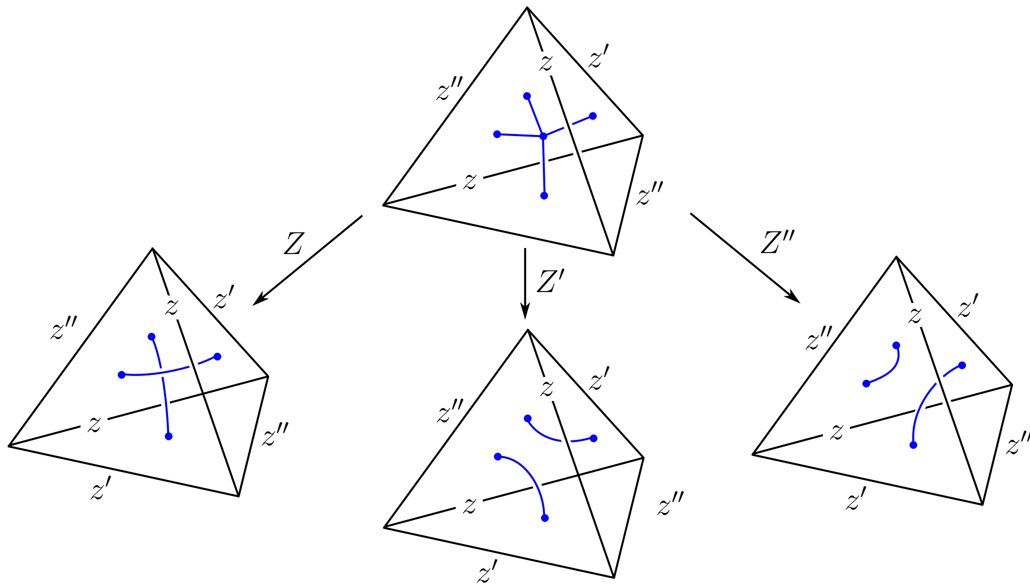

FIGURE 2. Three ways of smoothing $\mathcal{D}^{(1)}$.

Proposition 4.2. If $\mathcal{T}$ is ordered, the Z-curves homotope to disjoint peripheral curves.

Proof. For ordered $\mathcal{T}$, each face of $\mathcal{T}$ has a "middle" vertex, the one whose label is neither greatest nor smallest among the three vertices of the face. Recall that the Z-curves intersect with each face f of $\mathcal{T}$ in a point. We push the intersection point toward the middle vertex of f. Doing so for all faces of $\mathcal{T}$, the Z-curves homotope to disjoint peripheral curves. Note that then the Z-curves make two small curves in each tetrahedron lying in a neighborhood of the vertices 1 and 2 as in Figure 3. □

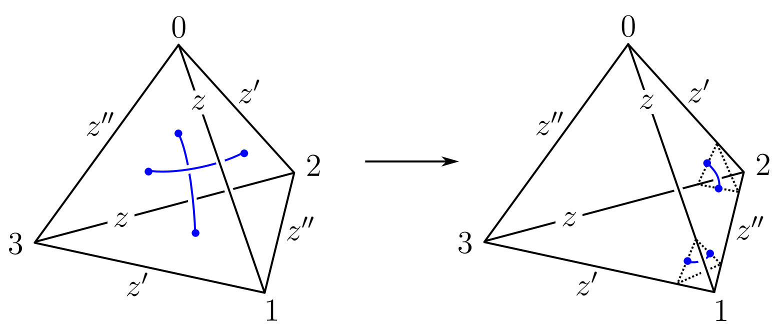

FIGURE 3. Homotope Z-curves to peripheral curves.

We now fix a tetrahedron Δ_j of $\mathcal{T}$. Recall that the free group $\mathcal{F}_{\mathcal{D}}$ has $2N$ generators, say $g_1, \dots, g_{2N}$, and that a face f of Δ_j corresponds to one generator g_i, oriented either inward or outward to Δ_j. We define a column vector $v_f \in \mathbb{Z}[\pi]^{2N}$

$$v_f = \begin{cases} p(g_i)\, e_i & \text{if } g_i \text{ is inward to } \Delta_j \\ -e_i & \text{if } g_i \text{ is outward to } \Delta_j \end{cases} \tag{20}$$

where $(e_1, \dots, e_{2N})$ is the standard basis of $\mathbb{Z}^{2N}$. We say that two faces of Δ_j are Z-*adjacent* if they are joined by one of two curves in Δ_j obtained from Z-smoothing (see Figure 2). Note that Δ_j has two pairs of Z-adjacent faces.

Proposition 4.3. If $\mathcal{T}$ is ordered, the column vector

$$\begin{pmatrix} p\left(\dfrac{\partial r_1}{\partial g_1}\right) & \cdots & p\left(\dfrac{\partial r_1}{\partial g_{2N}}\right) \\ \vdots & & \vdots \\ p\left(\dfrac{\partial r_N}{\partial g_1}\right) & \cdots & p\left(\dfrac{\partial r_N}{\partial g_{2N}}\right) \end{pmatrix} (v_{f_0} + v_{f_1}) \in \mathbb{Z}[\pi]^N \tag{21}$$

is equal to the j-th column of

$$\begin{pmatrix} p\left(\dfrac{\partial R_1}{\partial \hat{z}''_1}\right) & \cdots & p\left(\dfrac{\partial R_1}{\partial \hat{z}''_N}\right) \\ \vdots & & \vdots \\ p\left(\dfrac{\partial R_N}{\partial \hat{z}''_1}\right) & \cdots & p\left(\dfrac{\partial R_N}{\partial \hat{z}''_N}\right) \end{pmatrix} - \begin{pmatrix} p\left(\dfrac{\partial R_1}{\partial \hat{z}'_1}\right) & \cdots & p\left(\dfrac{\partial R_1}{\partial \hat{z}'_N}\right) \\ \vdots & & \vdots \\ p\left(\dfrac{\partial R_N}{\partial \hat{z}'_1}\right) & \cdots & p\left(\dfrac{\partial R_N}{\partial \hat{z}'_N}\right) \end{pmatrix} \tag{22}$$

up to sign where f_0 and f_1 are Z-adjacent faces of Δ_j.

Proof. Two faces of Δ_j are Z-adjacent if and only if they are adjacent to either the edge (01) or (23). We first consider two faces adjacent to the edge (01). One of the two faces is oriented inward to Δ_j, and the other is oriented outward. Let f_0 and f_1 be the former and the latter, respectively, as in Figure 4. Note that the orientation of every edge of f_0 and f_1 is determined, regardless of the vertices 2 and 3 of Δ_j.

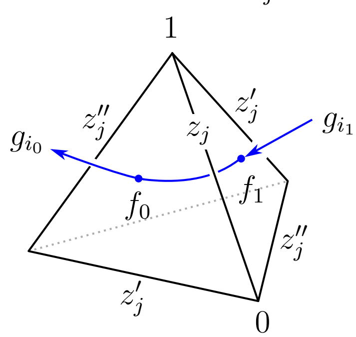

FIGURE 4. Two generators joined by B-smoothing.

From the edges of f_0 and f_1, we deduce that the generators g_{i_0} and g_{i_1} corresponding to f_0 and f_1 respectively appear in the words $R_1, \dots, R_N$ as follows.

$$\begin{aligned} &\cdots g_{i_1} \hat{z}_j \, g_{i_0} \cdots \\ &\cdots \hat{z}''_j \, g_{i_0} \cdots \\ &\cdots g_{i_0}^{-1} \, \hat{z}'_j \cdots \\ &\cdots g_{i_1} \, \hat{z}'_j \cdots \\ &\cdots \hat{z}''_j \, g_{i_1}^{-1} \cdots \end{aligned} \tag{23}$$

We stress that g_{i_0} and g_{i_1} do not appear elsewhere other than listed above, and neither do $\hat{z}'_j$ and $\hat{z}''_j$. It follows that for all $1 \le k \le N$

$$p\left(\frac{\partial r_k}{\partial g_{i_0}} - \frac{\partial r_k}{\partial g_{i_1}} g_{i_1}\right) = p\left(\frac{\partial R_k}{\partial \hat{z}''_j} - \frac{\partial R_k}{\partial \hat{z}'_j}\right). \tag{24}$$

Writing the above equation in a matrix form, we obtain the proposition. We prove similarly for two faces adjacent to the edge (23), in which case the left-hand side of (24) is equal to negative of the right-hand side. □

Remark 4.4. One can deduce similar equations for Z' and Z''-adjacent faces, but the equations only hold modulo 2. Precisely, for Z'-adjacent faces f_0 and f_1 of Δ_j, the column vector (21) and the j-th column of

$$\begin{pmatrix} p\left(\frac{\partial R_1}{\partial \hat{z}_1}\right) & \cdots & p\left(\frac{\partial R_1}{\partial \hat{z}_N}\right) \\ \vdots & & \vdots \\ p\left(\frac{\partial R_N}{\partial \hat{z}_1}\right) & \cdots & p\left(\frac{\partial R_N}{\partial \hat{z}_N}\right) \end{pmatrix} - \begin{pmatrix} p\left(\frac{\partial R_1}{\partial \hat{z}''_1}\right) & \cdots & p\left(\frac{\partial R_1}{\partial \hat{z}''_N}\right) \\ \vdots & & \vdots \\ p\left(\frac{\partial R_N}{\partial \hat{z}''_1}\right) & \cdots & p\left(\frac{\partial R_N}{\partial \hat{z}''_N}\right) \end{pmatrix} \tag{25}$$

are congruent modulo 2, i.e. they induce the same vector over $(\mathbb{Z}/2\mathbb{Z})[\pi]$. Similarly, for Z''-adjacent faces f_0 and f_1 of Δ_j, the column vector (21) and the j-th column of

$$\begin{pmatrix} p\left(\frac{\partial R_1}{\partial \hat{z}_1}\right) & \cdots & p\left(\frac{\partial R_1}{\partial \hat{z}_N}\right) \\ \vdots & & \vdots \\ p\left(\frac{\partial R_N}{\partial \hat{z}_1}\right) & \cdots & p\left(\frac{\partial R_N}{\partial \hat{z}_N}\right) \end{pmatrix} - \begin{pmatrix} p\left(\frac{\partial R_1}{\partial \hat{z}'_1}\right) & \cdots & p\left(\frac{\partial R_1}{\partial \hat{z}'_N}\right) \\ \vdots & & \vdots \\ p\left(\frac{\partial R_N}{\partial \hat{z}'_1}\right) & \cdots & p\left(\frac{\partial R_N}{\partial \hat{z}'_N}\right) \end{pmatrix} \tag{26}$$

are congruent modulo 2

4.3. **Determinants of NZ matrices and the (twisted) Alexander polynomial.** Combining Propositions 4.1–4.3, we obtain Theorems 3.1 and 3.2. We present details here.

Proof of Theorems 3.1 and 3.2. Let $\mathcal{D}$ be the dual cell complex of $\mathcal{T}$ and consider the cellular chain complex of $\mathcal{D}$ with local coefficient $\mathbb{Z}[t^{\pm 1}]$ twisted by $\alpha : \pi \to \mathbb{Z} \simeq t^{\mathbb{Z}}$:

$$0 \longrightarrow C_2(\mathcal{D}; \mathbb{Z}[t^{\pm1}]_\alpha) \xrightarrow{\partial_2} C_1(\mathcal{D}; \mathbb{Z}[t^{\pm1}]_\alpha) \xrightarrow{\partial_1} C_0(\mathcal{D}; \mathbb{Z}[t^{\pm1}]_\alpha) \longrightarrow 0\,. \tag{27}$$

Here $C_i(\mathcal{D}; \mathbb{Z}[t^{\pm1}]_\alpha) := C_i(\widetilde{\mathcal{D}}; \mathbb{Z}) \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[t^{\pm1}]$, where $\widetilde{\mathcal{D}}$ is the universal cover of $\mathcal{D}$, is a free $\mathbb{Z}[t^{\pm1}]$-module of rank N for $i = 0, 2$ and of rank $2N$ for $i = 1$.

We choose a spanning tree of $\mathcal{D}$, hence $N-1$ edges of $\mathcal{D}$. Lifting the tree to $\widetilde{\mathcal{D}}$, we obtain a basis of $C_i(\mathcal{D}; \mathbb{Z}[t^{\pm1}]_\alpha)$. It is well-known that the boundary map ∂_2 in (27) is given by the Fox derivative

$$\partial_2 = \begin{pmatrix} \alpha(p(\frac{\partial r_1}{\partial g_1})) & \cdots & \alpha(p(\frac{\partial r_1}{\partial g_{2N}})) \\ \vdots & & \vdots \\ \alpha(p(\frac{\partial r_N}{\partial g_1})) & \cdots & \alpha(p(\frac{\partial r_N}{\partial g_{2N}})) \end{pmatrix}^T \in M_{2N\times N}(\mathbb{Z}[t^{\pm1}]) \tag{28}$$

where p is the map eliminating all generators in the tree. Also, the boundary map ∂_1 can be expressed in terms of the vector described in (20). Precisely, the j-th row of ∂_1 is

$$\alpha(v_{f_0}^T) + \cdots + \alpha(v_{f_3}^T) \in \mathbb{Z}[t^{\pm 1}]^{2N} \tag{29}$$

where $f_0, \ldots, f_3$ are the faces of Δ_j. Recall that v_f is a column vector, hence its transpose v_f^T is a row vector. Since Z-smoothing couples the faces $f_0, \ldots, f_3$ of Δ_j into pairs, we can decompose ∂_1 into

$$\partial_1 = \partial_{1,B} + (\partial_1 - \partial_{1,B}) \tag{30}$$

where the j-th rows of both $\partial_{1,B}$ and $\partial_1 - \partial_{1,B}$ are of the form $\alpha(v_f^T) + \alpha(v_{f'}^T)$ for Z-adjacent faces f and f' of Δ_j. Then Propositions 4.1 and 4.3 imply that

$$\partial_2^T \, \partial_{1,B}^T = D\mathbf{B}_\alpha(t) \tag{31}$$

where D is a diagonal matrix with entries in $\{\pm t^k \,|\, k \in \mathbb{Z}\}$. It follows that for any N-tuple $b = (b_1, \ldots, b_N)$ of column vectors in $C_1(\mathcal{D}; \mathbb{Z}[t^{\pm 1}]_\alpha)$, we have

$$\left(\frac{\partial_2^T}{b^T}\right) \left(\partial_{1,B}^T \;\middle|\; \partial_1^T \right) = \left(\begin{array}{c|c} D\mathbf{B}_\alpha(t) & 0 \\ \hline \partial_{1,B}(b)^T & \partial_1(b)^T \end{array}\right) \tag{32}$$

and thus

$$\frac{\det \left(\partial_2 \;\middle|\; b \right)}{\det \partial_1(b)} \det \left(\frac{\partial_{1,B}}{\partial_1}\right) \doteq \det \mathbf{B}_\alpha(t) \tag{33}$$

provided that $\det \partial_1(b) \neq 0$.

The first term of the left-hand side of (33) is by definition $\Delta_\alpha(t)/(t-1)$ where $\Delta_\alpha(t)$ is the Alexander polynomial associated with α. The second term obviously satisfies

$$\det \left(\frac{\partial_{1,B}}{\partial_1}\right) = \det \left(\frac{\partial_{1,B}}{\partial_1 - \partial_{1,B}}\right). \tag{34}$$

Recall that each row of $\partial_{1,B}$ and $\partial_1 - \partial_{1,B}$ is of the form $v_f^T + v_{f'}^T$ for some faces f and f' and that each column of ∂_1 has at most two non-trivial entries. It follows that each row and column of the matrix in the right-hand side of (34) has at most two non-trivial entries. Such a matrix after changing some rows and columns can be expressed as a direct sum of matrices of the form

$$\begin{pmatrix} x_1 & -y_1 & & & \\ & x_2 & -y_2 & & \\ & & \ddots & \ddots & \\ & & & x_{n-1} & -y_{n-1} \\ -y_n & & & & x_n \end{pmatrix} \tag{35}$$

whose determinant is $x_1 \cdots x_n - y_1 \cdots y_n$. In our case, expressing the matrix in the right-hand side of (34) as in the form (35) is carried out by following the Z-curves. In particular, all x_i are of the form $t^{\alpha(g_i)}$ and all y_i are 1. It follows that the right-hand side of (34) equals to $\prod(t^{\alpha(Z_i)} - 1)$ where the product is over all components Z_i of the Z-curves. Therefore, we obtain

$$\det \mathbf{B}_\alpha(t) \doteq \frac{\Delta_\alpha(t)}{t-1} \prod_i (t^{\alpha(Z_i)} - 1). \tag{36}$$

On the other hand, Proposition 4.2 says that the Z-curves homotope to disjoint peripheral curves. If one component is homotopically trivial, we have $\det \mathbf{B}_\alpha(t) = 0$ from Equation (36). Otherwise, the Z-curves are m-parallel copies of a peripheral curve γ for $m \geq 1$, hence Equation (6) holds for $n = \alpha(\gamma)$. This completes the proof of Theorem 3.1.

We obtain Theorem 3.2 by simply replacing α in the above proof of Theorem 3.1 by $\alpha \otimes \rho$. We omit details, as this is indeed a repetition with only obvious variants. For instance, the coefficient of the chain complex (27) is replaced by $(\mathbb{Z}[t^{\pm 1}] \otimes \mathbb{C}^n)_{\alpha\otimes\rho}$, the matrix (35) should be viewed as a block matrix, and Equation (36) is replaced by

$$\det \mathbf{B}_{\alpha\otimes\rho}(t) \doteq \Delta_{\alpha\otimes\rho}(t) \prod_i \det(\rho(Z_i)\, t^{\alpha(Z_i)} - I_n) \tag{37}$$

where I_n is the identity matrix of rank n. □

Remark 4.5. Applying the same argument as in the proof of Theorem 3.1, we deduce equations in $(\mathbb{Z}/2\mathbb{Z})[t^{\pm 1}]$ from Remark 4.4, analogous to Equation (36):

$$\det \mathbf{A}_\alpha(t) \equiv \frac{\Delta_\alpha(t)}{t-1} \prod_i (t^{\alpha(Z''_i)} - 1) \qquad (\mathrm{mod}\ 2)\,, \tag{38}$$

$$\det(\mathbf{A}_\alpha(t) - \mathbf{B}_\alpha(t)) \equiv \frac{\Delta_\alpha(t)}{t-1} \prod_i (t^{\alpha(Z'_i)} - 1) \qquad (\mathrm{mod}\ 2)\,. \tag{39}$$

Here Z'_i and Z''_i are the Z' and Z''-curves of $\mathcal{T}$, respectively. These equations usually fail in $\mathbb{Z}[t^{\pm 1}]$; see Section 5 for an example.

Remark 4.6. The palindromicity of $\det \mathbf{B}_\alpha(t)$ follows from the palindromicity of the Alexander polynomial $\Delta_\alpha(t)$ [Mil62] together with Theorem 3.1. Here we call a Laurent polynomial $p(t)$ palindromic if $p(t) \doteq p(t^{-1})$. Equation (4) specialized to (5) implies that if $\mathbf{B}_\alpha(t)$ is non-singular, then $\mathbf{B}_\alpha(t)^{-1}\mathbf{A}_\alpha(t)$ is invariant under the transpose followed the involution $t \mapsto t^{-1}$. Hence $\det \mathbf{B}_\alpha(t)^{-1}\mathbf{A}_\alpha(t)$ is palindromic, and so is $\det \mathbf{A}_\alpha(t)$.

4.4. **FK determinants of NZ matrices and the L^2-Alexander torsion.** Imitating the proof of Theorem 3.1 with the Fuglede-Kadison determinant, we obtain Theorem 3.3. We present details here.

Proof of Theorem 3.3. Let $\mathcal{D}$ be the dual cell complex of $\mathcal{T}$. The universal cover $\widetilde{\mathcal{D}}$ of $\mathcal{D}$ has the cellular chain complex of left $\mathbb{Z}[\pi]$-modules

$$0 \longrightarrow C_2(\widetilde{\mathcal{D}};\mathbb{Z}) \xrightarrow{\partial_2} C_1(\widetilde{\mathcal{D}};\mathbb{Z}) \xrightarrow{\partial_1} C_0(\widetilde{\mathcal{D}};\mathbb{Z}) \longrightarrow 0 \tag{40}$$

where $C_i := C_i(\widetilde{\mathcal{D}};\mathbb{Z})$ has rank N for $i = 0, 2$ and rank $2N$ for $i = 1$. The boundary maps $\partial_i : C_i \to C_{i-1}$ act on the right, i.e., we have

$$\partial_2 \in M_{N,2N}(\mathbb{Z}[\pi]), \quad \partial_1 \in M_{N,2N}(\mathbb{Z}[\pi])\,. \tag{41}$$

As in the proof of Theorem 3.1, we decompose ∂_1 as $\partial_1 = \partial_{1,B} + (\partial_1 - \partial_{1,B})$ where the j-th columns of both $\partial_{1,B}$ and $\partial_1 - \partial_{1,B}$ are of the form $v_f + v_{f'}$ for Z-adjacent faces f and f' of Δ_j. Then Propositions 4.1 and 4.3 imply that

$$\partial_2\, \partial_{1,B} = D\mathbf{B} \tag{42}$$

where D is a diagonal matrix with entries in $\pm\pi$.

We now fix $t \in \mathbb{R}^+$ and twist the coefficient of C_i by using the homomorphism α_t, i.e. consider the chain complex $C'_i := \mathbb{R}[\pi] \otimes_{\mathbb{Z}[\pi]} C_i$ where $\mathbb{R}[\pi]$ is viewed as a $\mathbb{Z}[\pi]$-module using the homomorphism α_t. Note that the boundary maps of C'_i are given by $\partial'_i = \alpha_t(\partial_i)$. It follows from Equation (42) that for any N-tuple $b = (b_1, \dots, b_N)$ of (row) vectors, we have

$$\left(\frac{\partial'_2}{b}\right) \left(\partial'_{1,B} \mid \partial'_1 \right) = \left(\begin{array}{c|c} \alpha_t(D\mathbf{B}) & 0 \\ \hline \partial'_{1,B}(b) & \partial'_1(b) \end{array}\right) \tag{43}$$

where $\partial'_{1,B} = \alpha_t(\partial_{1,B})$. Therefore, we obtain

$$\frac{\det^r_{\mathcal{N}(\pi)}\left(\frac{\partial'_2}{b}\right)}{\det^r_{\mathcal{N}(\pi)}(\partial_1(b))} \det^r_{\mathcal{N}(\pi)} \left(\partial'_{1,B} \mid \partial'_1 \right) = t^k \det^r_{\mathcal{N}(\pi)}(\alpha_t(\mathbf{B})) \tag{44}$$

for fixed $k \in \mathbb{Z}$, provided that $\det^r_{\mathcal{N}(\pi)}(\partial_1(b)) \neq 0$. The first term of the left-hand side of (44) is $\tau^{(2)}(M, \alpha)(t)$ (see [DFL15, Lemma 3.1]), and the second term satisfies

$$\det^r_{\mathcal{N}(\pi)} \left(\partial'_{1,B} \mid \partial'_1 \right) = \det^r_{\mathcal{N}(\pi)} \left(\partial'_{1,B} \mid \partial'_1 - \partial'_{1,B} \right) . \tag{45}$$

Recall that the matrix $(\partial_{1,B} \,|\, \partial_1 - \partial_{1,B})$ after changing some rows and columns is the direct sum of matrices of the form (35) with $x_i \in \pi$ and $y_i = 1$. Such matrices decompose into

$$\begin{pmatrix} x_1 & & & & \\ & x_2 & & & \\ & & \ddots & & \\ & & & x_{n-1} & \\ & & & & x_n \end{pmatrix} \begin{pmatrix} 1 & -x_1^{-1} & & & \\ & 1 & -x_2^{-1} & & \\ & & \ddots & \ddots & \\ & & & 1 & -x_{n-1}^{-1} \\ & & & & 1 \end{pmatrix} \begin{pmatrix} 1 - x_1^{-1} \cdots x_n^{-1} & & & & \\ -x_2^{-1} \cdots x_n^{-1} & 1 & & & \\ \vdots & & \ddots & & \\ -x_{n-1}^{-1} x_n^{-1} & & & 1 & \\ -x_n^{-1} & & & & 1 \end{pmatrix},$$

hence we deduce that

$$\det^r_{\mathcal{N}(\pi)} \left(\partial'_{1,B} \mid \partial'_1 - \partial'_{1,B} \right) = \prod_i \det^r_{\mathcal{N}(\pi)}(1 - \alpha_t(Z_i^{-1})) = \prod_i \det^r_{\mathcal{N}(\pi)}(1 - t^{-\alpha(Z_i)} Z_i^{-1}) \tag{46}$$

where the products are over all the components Z_i of the Z-curves of $\mathcal{T}$. Since we assumed that each component Z_i has infinite order in π, $\det^r_{\mathcal{N}(\pi)}(1 - t^{-\alpha(Z_i)} Z_i^{-1})$ is the Mahler measure of $Z_i - t^{-\alpha(Z_i)}$, viewed as a polynoimal in Z_i, which equals to $\max\{1, t^{-\alpha(Z_i)}\}$. It follows that

$$\det^r_{\mathcal{N}(\pi)}(\alpha_t(\mathbf{B})) = t^{-k} \, \tau^{(2)}(M, \alpha)(t) \prod_i \max\{1, t^{-\alpha(Z_i)}\} \tag{47}$$

for fixed $k \in \mathbb{Z}$. Since each component Z_i is of infinite order and, in particular, non-trivial, Proposition 4.2 implies that all $\alpha(Z_i)$ should be the same up to sign. Thus Equation (47) implies Theorem 3.3. □

5. Example

As is customary in hyperbolic geometry, in this section we give an example of a cusped hyperbolic 3-manifold M, the complement of the knot 4_1 in S^3. The default `SnapPy` triangulation $\mathcal{T}$ of M consists of two ideal tetrahedra Δ_1 and Δ_2, and is orderable with the ordering shown in Figure 5 [CDGW]. It has two edges e_1 and e_2; $(01), (03), (23)$ of Δ_1 and

$(02),(12),(13)$ of Δ_2 are identified with e_1; $(02),(12),(13)$ of Δ_1 and $(01),(03),(23)$ of Δ_2 are identified swith e_2.

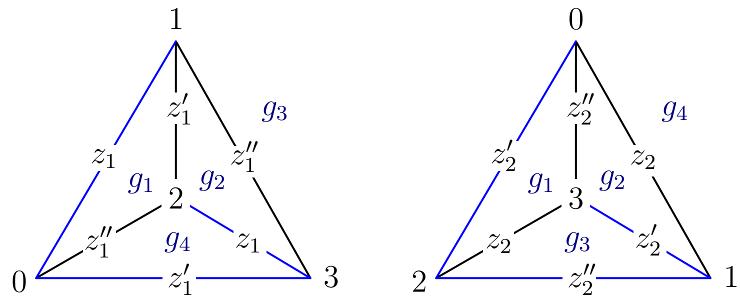

FIGURE 5. An ordered ideal triangulation of 4_1.

The dual cell complex of $\mathcal{T}$ has 4 edges and 2 faces, hence we have two words r_1 and r_2 in four generators $g_1,\dots,g_4$ Note that g_1 and g_4 (resp., g_2 and g_3) are oriented inward to Δ_1 (resp., Δ_2) and that the words r_1 and r_2 are obtained from winding around the edges of $\mathcal{T}$:

$$\begin{aligned} e_1 : &\quad g_1 \xrightarrow{z_1} g_3 \xrightarrow{z_2''} g_4 \xrightarrow{z_1} g_2 \xrightarrow{z_2'} g_3^{-1} \xrightarrow{z_1'} g_4^{-1} \xrightarrow{z_2'} g_1\,, \\ e_2 : &\quad g_1 \xrightarrow{z_1'} g_2 \xrightarrow{z_2} g_4 \xrightarrow{z_1''} g_1^{-1} \xrightarrow{z_2''} g_2^{-1} \xrightarrow{z_1''} g_3 \xrightarrow{z_2} g_1\,. \end{aligned} \tag{48}$$

Precisely, $r_1 = g_3g_4g_2g_3^{-1}g_4^{-1}g_1$ and $r_2 = g_2g_4g_1^{-1}g_2^{-1}g_3g_1$. Eliminating one generator, say g_1, we obtain a presentation of $\pi = \pi_1(M)$:

$$\pi = \langle g_2, g_3, g_4 \,|\, g_3g_4g_2g_3^{-1}g_4^{-1},\ g_2g_4g_2^{-1}g_3\rangle\,. \tag{49}$$

Note that g_4 is a meridian of the knot.

As in Section 4, we define a word R_i for $i = 1, 2$ by inserting shape parameters to the word r_i (c.f. (48)):

$$\begin{aligned} R_1 &= \hat{z}_1\, g_3\, \hat{z}_2''\, g_4\, \hat{z}_1\, g_2\, \hat{z}_2'\, g_3^{-1}\, \hat{z}_1'\, g_4^{-1}\, \hat{z}_2'\, g_1\,, \\ R_2 &= \hat{z}_1'\, g_2\, \hat{z}_2\, g_4\, \hat{z}_1''\, g_1^{-1}\, \hat{z}_2''\, g_2^{-1}\, \hat{z}_1''\, g_3\, \hat{z}_2\, g_1\,. \end{aligned}$$

Due to Proposition 4.1, the twisted gluing equation matrices $\mathbf{G}^\square$ of $\mathcal{T}$ are equal to $(\partial R_i/\partial \hat{z}_j^\square)$ followed by eliminating g_1 and all $\hat{z}_j^\square$. Explicitly, we have

$$\begin{aligned} \mathbf{G} &= \begin{pmatrix} 1 + g_3g_4 & 0 \\ 0 & g_2 + g_2g_4g_2^{-1}g_3 \end{pmatrix}, \\ \mathbf{G}' &= \begin{pmatrix} g_3g_4g_2g_3^{-1} & g_3g_4g_2 + g_3g_4g_2g_3^{-1}g_4^{-1} \\ 1 & 0 \end{pmatrix}, \\ \mathbf{G}'' &= \begin{pmatrix} 0 & g_3 \\ g_2g_4 + g_2g_4g_2^{-1} & g_2g_4 \end{pmatrix} \end{aligned}$$

and thus the twisted Neumann–Zagier matrices of $\mathcal{T}$ are given as

$$\mathbf{A} = \begin{pmatrix} 1 + g_3g_4 - g_3g_4g_2g_3^{-1} & -g_3g_4g_2 - g_3g_4g_2g_3^{-1}g_4^{-1} \\ -1 & g_2 + g_2g_4g_2^{-1}g_3 \end{pmatrix}, \tag{50}$$

$$\mathbf{B} = \begin{pmatrix} -g_3g_4g_2g_3^{-1} & g_3 - g_3g_4g_2 - g_3g_4g_2g_3^{-1}g_4^{-1} \\ g_2g_4 + g_2g_4g_2^{-1} - 1 & g_2g_4 \end{pmatrix}. \tag{51}$$

On the other hand, the abelianization map $\alpha : \pi \to \mathbb{Z}$ is given by $\alpha(g_2) = 0$, $\alpha(g_3) = -1$ and $\alpha(g_4) = 1$. Applying α to the twisted Neumann–Zagier matrices, we obtain

$$\mathbf{A}_\alpha(t) = \begin{pmatrix} 2-t & -2 \\ -1 & 2 \end{pmatrix}, \qquad \mathbf{B}_\alpha(t) = \begin{pmatrix} -t & t^{-1}-2 \\ 2t-1 & t \end{pmatrix}. \tag{52}$$

One easily computes that

$$\det \mathbf{B}_\alpha(t) \doteq (t-1)(t^2 - 3t + 1) \tag{53}$$

which verifies Theorem 3.1 as well as Equation (36). Note that the Alexander polynomial of 4_1 is $t^2 - 3t + 1$ and that $\mathcal{T}$ has two Z-curves $Z_1 = g_1g_3$ and $Z_2 = g_4g_2$ with $\alpha(Z_1) = -1$, $\alpha(Z_2) = 1$. We also check that

$$\det \mathbf{A}_\alpha(t) \equiv \det(\mathbf{A}_\alpha(t) - \mathbf{B}_\alpha(t)) \equiv 0 \pmod 2 \tag{54}$$

which verifies Remark 4.5. Note that $\mathcal{T}$ has one Z'-curve $Z'_1 = g_1g_2g_3^{-1}g_4^{-1}$ with $\alpha(Z'_1) = 0$ and one Z''-curve $Z''_1 = g_2^{-1}g_3g_4g_1^{-1}$ with $\alpha(Z''_1) = 0$.

We now compute a (positive) lift $\rho : \pi \to \mathrm{SL}_2(\mathbb{C})$ of the geometric representation of M. Since g_4 is a meridian of the knot, we may let (see [Ril84, Lemma 1])

$$\rho(g_4) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho(g_2) = \begin{pmatrix} n & 0 \\ u & 1/n \end{pmatrix}. \tag{55}$$

A straightforward computation shows that the above assignment induces a representation ρ of π if and only if $u = -(1-4n^2+n^4)/(3n+3n^3)$ and $1-3n+5n^2-3n^3+n^4 = 0$. Applying $\alpha \otimes \rho$ to Equation (51), one computes that

$$\det \mathbf{B}_{\alpha\otimes\rho}(t) \doteq (t-1)^4(t^2 - 4t + 1)/t^2. \tag{56}$$

This verifies Theorem 3.2 as well as Equation (37). Note that the twisted Alexander polynomial of 4_1 associated with $\alpha \otimes \rho$ is $t^2 - 4t + 1$.

Remark 5.1. For ordered ideal triangulations, $\det \mathbf{A}_\alpha(t)$ is often a multiple of 2 and thus vanishes in $(\mathbb{Z}/2\mathbb{Z})[t^{\pm 1}]$. One example which is not the case is the knot 8_2. Its default `SnapPy` triangulation is orderable, and Philip Choi's program computes that

$$\mathbf{A}_\alpha(t) = \begin{pmatrix} t^{-4}+1 & 1 & -t^{-4} & t^{-4} & 0 & 0 \\ -t^{-2}-1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & t & 1 & -1 \\ 0 & -t^{-1} & -t & -t^{-4} & -t & t^{-5} \\ 0 & 1 & 0 & 0 & 0 & t \\ 0 & 0 & 1 & -1 & 0 & -1 \end{pmatrix},$$

$$\mathbf{B}_\alpha(t) = \begin{pmatrix} t^{-2} & 0 & -t^{-4} & 0 & 0 & 0 \\ -t^{-2}-1 & t^{-2} & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & t & t-1 \\ 0 & 1-t^{-1} & -t & t^{-5}-t^{-4} & -t & 0 \\ 0 & 0 & t^2 & t^2 & -t & t \\ 0 & 0 & 0 & -1 & 1 & -1 \end{pmatrix}$$

with

$$\det \mathbf{A}_\alpha(t) = (t-1)(t^{12}+t^7-2t^6+t^5+1)\,,$$
$$\det \mathbf{B}_\alpha(t) = (t-1)(t^6-3t^5+3t^4-3t^3+3t^2-3t+1)\,.$$

Note that the Alexander polynomial of the knot 8_2 is the second factor of $\det \mathbf{B}_\alpha(t)$ (hence this verifies Theorem 3.1) and that

$$\det \mathbf{A}_\alpha(t) \equiv (t-1)^2(t^4+t^3+t^2+t+1)(t^6-3t^5+3t^4-3t^3+3t^2-3t+1) \pmod 2\,.$$

Acknowledgments. The authors wish to thank Thang Le for enlightening conversations. S.Y. wishes to thank Philip Choi for his help with computer calculations.

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