# ON THE TRACE FIELDS OF HYPERBOLIC DEHN FILLINGS

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Abstract. Assuming Lehmer's conjecture, we estimate the degree of the trace field $K(M_{p/q})$ of a hyperbolic Dehn-filling $M_{p/q}$ of a 1-cusped hyperbolic 3-manifold $M$ by

$$\frac{1}{C}(\max\{|p|,|q|\}) \leq \deg K(M_{p/q}) \leq C(\max\{|p|,|q|\})$$

where $C = C_M$ is a constant that depends on $M$.

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## 1. Introduction

1.1. **Main result.** An important arithmetic invariant of a complete hyperbolic 3-manifold $M$ of finite volume is its trace field $K(M) = \mathbb{Q}(\text{tr } \rho(g) \mid g \in \pi_1(M))$ generated by the traces $\rho(g)$ of the elements of the fundamental group $\pi_1(M)$ of the geometric representation $\rho : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$. Mostow rigidity implies that $\rho$ is rigid, hence it can be conjugated to lie in $\mathrm{PSL}_2(\overline{\mathbb{Q}})$, and as a result, it follows that $K(M)$ is a number field.

Given a 1-cusped hyperbolic 3-manifold $M$, Thurston showed that the manifolds $M_{p/q}$ obtained by $p/q$ Dehn-filling on $M$ are hyperbolic for all but finitely many pairs of coprime integers $(p,q)$ [Thu77]. Thus, it is natural to ask how the trace field $K(M_{p/q})$ depends on the Dehn-filling parameters. In [Hod], C. Hodgson proved the following.

**Theorem 1.1** (Hodgson)**.** *Let $M$ be a 1-cusped hyperbolic 3-manifold. Then there are only finitely many hyperbolic Dehn fillings of $M$ of bounded trace field degree.*

*Date*: 1 March 2021.

*Key words and phrases: cusped hyperbolic 3-manifolds, Dehn-filling, trace field, A-polynomial, Newton polygon, Mahler's measure, Lehmer's Conjecture, fewnomials.*

Additional proofs of this theorem were given by Long–Reid [LR03, Thm.3.2] and by the second author in his thesis [Jeo13]. A simple invariant of a number field is its degree, and a natural question is to ask for the behavior of the degree of the trace field of a cusped hyperbolic 3-manifold under Dehn filling. In this paper, we give a partial and conditional answer to the question as follows:

**Theorem 1.2.** *Let $M$ be a 1-cusped hyperbolic 3-manifold and $M_{p/q}$ be a hyperbolic $p/q$-Dehn filling. Assuming Lehmer's conjecture, there exists $C = C_M$ depending on $M$ such that*

$$\frac{1}{C}(\max\{|p|,|q|\}) \leq \deg K(M_{p/q}) \leq C(\max\{|p|,|q|\}) \tag{1}$$

Note that the upper bound in (1) follows from Bezóut's theorem and so what really matters in (1) is its lower bound.

1.2. **Key Observation.** The proof of the above theorem uses the fact that the geometric representation of $M_{p/q}$ lies in the geometric component of the $\mathrm{PSL}_2(\mathbb{C})$-character variety of $M$ (known as Thurston's hyperbolic Dehn-surgery theorem [Thu77]). The latter defines an algebraic curve in $\mathbb{C}^* \times \mathbb{C}^*$ (the so-called $A$-polynomial curve [CCG+94]) in meridian-longitude coordinates and a suitable point in its intersection with $m^p\ell^q = 1$ determines the geometric representation of $M_{p/q}$. Thus, the determination of the degree of $K(M_{p/q})$ is reduced to a bound for the irreducible components of the Dehn-filling polynomial, a polynomial whose coefficients are independent of $(p,q)$ for large $|p|+|q|$.

To explain the key idea further in detail, as a toy model, let us assume the A-polynomial of a 1-cusped hyperbolic 3-manifold $M$ is simply given as

$$m\Big(l-\frac{1}{l}\Big)+1+\frac{1}{m}\Big(\frac{1}{l}-l\Big)=0. \tag{2}$$

Then finding the intersection between (2) and $m^p\ell^q = 1$ is equivalent to solving

$$t^{-q}(t^p - t^{-p}) + 1 + t^q(t^{-p} - t^p) = 0,$$

which is normalized as

$$t^{2p+2q} - t^{2p} - t^{p+q} - t^{2q} + 1 = 0 \tag{3}$$

under $0 < q < p$. If $t_0$ is a root of (3) with $|t_0| > 1$, then

$$|t_0|^{2p+2q} < |t_0|^{2p} + |t_0|^{p+q} + |t_0|^{2q} + 1 < 4|t_0|^{2p} \Longrightarrow |t_0|^{2q} < 4.$$

Taking logarithms, we get

$$|t_0| < 1 + \frac{1}{q}.$$

If $C_1$ be an arbitrary constant with $\frac{p}{q} < C_1$, clearly

$$|t_0| < 1 + \frac{C_1}{p} \tag{4}$$

and, assuming Lehmer's conjecture, one further finds $C_1'$ depending only on $C_1$ satisfying[1]

$$C_1' p < \deg\ t_0. \tag{5}$$

[1] In this case, one may further obtain (5) from (4) unconditionally, thanks to Dimitrov's recent proof of the Schinzel-Zassenhaus conjecture. See Theorem 2.8 and Theorem 4.1.

Note that (5) partially verifies Theorem 1.2 for the given example over the following restricted domain of $p$ and $q$:

$$\Big\{(p,q)\in\mathbb{Z}^2 \;:\; 1<\frac{p}{q}<C_1\Big\}, \tag{6}$$

and $C_1'$, as a function of $C_1$, goes to 0 as $C_1$ approaches $\infty$.

Now suppose $\frac{p}{q}>C_1$. Let $t_0$ (with $|t_0|>1$) be a root of

$$t^{2p+2q}-t^{2p}-t^{p+q}-t^{2q}+1=t^{2p}(t^{2q}-1)-t^{p+q}-(t^{2q}-1)=0 \tag{7}$$

and $C_{p,q}$ be

$$\min\big\{|t_0^{2q}-1| \mid t_0 \text{ is a root of (7) with } |t_0|>1\big\}. \tag{8}$$

Then

$$\begin{gathered} C_{p,q}|t_0^{2p}|(<|t_0^{2p}||t_0^{2q}-1|)<|t_0^{p+q}|+|t_0^{2q}|+1<3|t_0^{p+q}| \\ \Longrightarrow |t_0^{p-q}|<\frac{3}{C_{p,q}} \Longrightarrow |t_0^{(1-\frac{1}{C_1})p}|<\frac{3}{C_{p,q}} \Longrightarrow |t_0|<1+\frac{C'_{p,q}}{p} \end{gathered} \tag{9}$$

for some $C'_{p,q}$ depending on $C_1$ and $C_{p,q}$. Hence, combining with Lehmer's conjecture, it follows that

$$C''_{p,q}p<\deg\ t_0 \tag{10}$$

with $C''_{p,q}$ depending on $C'_{p,q}$. Note that both $C'_{p,q}$ and $C''_{p,q}$ depend only on $C_{p,q}$ for $C_1$ sufficiently large, and

$$\lim_{C_{p,q}\to 0} C'_{p,q}=\lim_{C_{p,q}\to 0} C''_{p,q}=0.$$

Consequently, if

$$\inf_{\substack{(p,q)\in\mathbb{Z}^2\\ 0<q<p}} C_{p,q}>0, \tag{11}$$

the statement of Theorem 1.2 holds over

$$\big\{(p,q)\in\mathbb{Z}^2 \;:\; 0<q<p\big\} \tag{12}$$

for the above example.

However, (11) is not true in general and, in fact, the following holds:

$$\inf_{\substack{(p,q)\in\mathbb{Z}^2\\ 0<q<p}} C_{p,q}=0.$$

Moreover, for any sufficiently large $C_1$, we always find $(p,q)$ in (12) and roots of (7) whose absolute values are bigger than $1+\frac{C_1}{p}$, which means the above heuristic argument is indeed not enough to induce the claim of Theorem 1.2. This is the very non-trivial point of the problem.

To resolve this issue, in Lemmas 3.2-3.3, we fix some sufficiently large $C_1$ and count the number of roots of (7) whose absolute values are larger than $1+\frac{C_1}{p}$. We further get the lower and upper bounds of those roots, and study their distribution between the two bounds. It is then shown that the product of the moduli of the roots is not that big and is actually small enough to derive the desired result from Lehmer's conjecture.

Proving Lemmas 3.2-3.3 is the key technical heart of the paper and this requires a detailed study of the local geometry of an analytic curve. By using only elementary means such as trigonometry and basic analysis, we analyze the desired properties of a curve in the proofs.

We encourage a reader to compare the discussion in this subsection to the statements of Lemma 3.3.

Finally remark that, once the claim of Theorem 1.2 is proven over the domain in (12), the rest will follow by symmetric properties of the A-polynomial as well as a change of variables on it.

## 2. Preliminaries

**2.1. The $A$-polynomial.** For a given 1-cusped hyperbolic 3-manifold $M$, the A-polynomial of $M$ is a polynomial with 2-variables introduced by Cooper–Culler–Gillett–Long–Shalen in [CCG+94]. More precisely, let $T$ be a torus cross-section of the cusp and $\mu, \lambda$ be the chosen meridian and longitude of $T$. Then the $A$-polynomial of $M$ is the quotient space of

$$\mathrm{Hom}\,\big(\pi_1(M), \mathrm{SL}_2\mathbb{C}\big) \tag{13}$$

by conjugation, parametrized by the derivatives of the holonomies $m, \ell$ of $\mu, \lambda$, respectively.

The following properties are well-known:

**Theorem 2.1.** [CCG+94] *Let $A(m,\ell) \in \mathbb{Z}[m,\ell]$ be the $A$-polynomial of a 1-cusped hyperbolic 3-manifold $M$.*

1. *$A(m,\ell) = \pm A(m^{-1}, \ell^{-1})$ up to powers of $m$ and $l$.*
2. *$(m,\ell) = (1,1)$ is a point on $A(m,\ell) = 0$, which gives rise to a discrete faithful representation of $\pi_1(M)$.*

Writing

$$A(m,\ell) = \sum_{i,j} c_{i,j} m^i \ell^j, \tag{14}$$

the Newton polygon $\mathcal{N}(A)$ of $A(m,\ell)$ is defined as the convex hull in the plane of the set $\{(i,j) \,:\, c_{i,j} \neq 0\}$.

**Theorem 2.2.** [CCG+94][CL97] *Let $M, A(m,\ell) = 0$ and $\mathcal{N}(A)$ be the same as above. Suppose $A(m,\ell)$ is normalized so that the greatest common divisor of the coefficients is 1.*

1. *If $(i,j)$ is a corner of $\mathcal{N}(A)$, then $c_{i,j} = \pm 1$.*
2. *For each edge of $\mathcal{N}(A)$, the corresponding edge-polynomial is a product of cyclotomic polynomials.*

**2.2. The Dehn filling polynomial.** Fix a 1-cusped hyperbolic 3-manifold $M$ with a meridian and longitude $\mu$ and $\lambda$. Let $M_{p/q}$ denote its $p/q$-Dehn filling, where here and throughout, $(p,q)$ denotes a pair of coprime integers. Thurston's hyperbolic Dehn-surgery theorem [Thu77] implies that $M_{p/q}$ is a hyperbolic manifold for all but finitely many pairs $(p,q)$, or equivalently for almost all pairs $(p,q)$, or yet equivalently, for all pairs $(p,q)$ with $|p|+|q|$ large. It follows by the Seifert-Van Kampen theorem that $\pi_1(M_{p/q}) = \pi_1(M)/(\mu^p\lambda^q = 1)$. Hence if the A-polynomial of $M$ is given by $A(m,\ell) = 0$, a discrete faithful representation $\phi : \pi_1(M_{p/q}) \longrightarrow \mathrm{PSL}_2(\mathbb{C})$ is obtained by finding a point on

$$\big(A(m,\ell) = 0\big) \cap \big(m^p \ell^q = 1\big), \tag{15}$$

which is equivalent to an equation $A_{p,q}(t) = 0$ for a one-variable Dehn-filling polynomial $A_{p,q}(t) = A(t^{-q}, t^p)$. More precisely, let us assume (14) is normalized as

$$A(m, \ell) = \sum_{j=0}^{n} \Big( \sum_{i=a_j}^{b_j} c_{i,j} m^i \Big) \ell^j \tag{16}$$

where $a_j, b_j$ and $c_{i,j}$ are integers with $c_{a_j,j}, c_{b_j,j} \neq 0$ for all $j$, then the Dehn-filling polynomial is given by

$$A_{p,q}(t) = \sum_{j=0}^{n} \Big( \sum_{i=a_j}^{b_j} c_{i,j} t^{-qi} \Big) t^{pj}. \tag{17}$$

We now discuss some elementary properties of the Dehn-filling polynomial. Let

$$s_A := \max_{0 \le j \le n-1} \left\{ \frac{a_n - a_j}{n - j} \right\} \tag{18}$$

denote the largest slope of the Newton polygon of the $A$-polynomial.

**Lemma 2.3.** When $|p| + |q|$ are large,

(a) no two terms in (17) are equal,
(b) further, if $q > 0$, the leading term of $A_{p,q}(t)$ is of the form $c_{a_k,k} t^{-a_k q + kp}$ for some $0 \le k \le n$. Moreover, $k = n$ for $p/q > s_A$ and $q > 0$.

It follows that for almost all pairs $(p, q)$, $A_{p,q}(t)$ has degree piece-wise linear in $(p, q)$, and leading coefficient (due to Theorem 2.2) $\pm 1$.

*Proof.* Observe that $t^{-qi+pj} = t^{-qi'+pj'}$ implies that $p/q = (i' - i)/(j' - j)$ which takes finitely many values according to (16). It follows that when $|q| + |q|$ is sufficiently large, then no two terms in (17) are equal, and hence the Newton polygon of $A_{p,q}(t)$ is the image of the Newton polygon of $A(m, \ell)$ under the linear map $\mathbb{R}^2 \to \mathbb{R}$ which sends $(i, j)$ to $-qi + pj$. This proves that the leading term of $A_{p,q}(t)$ is the image under the above map of a corner of $A(m, \ell)$. When $p/q > s_A$ and $q > 0$, it is easy to see that this corner of $A(m, \ell)$ is $(a_n, n)$. $\square$

We now relate the trace field $K(M_{p/q})$ with the point $(m, \ell) = (t_0^{-q}, t_0^p)$ on (15) that gives rise to the discrete faithful representation of $\pi_1(M_{p/q})$. (Of course, $t_0$ depends on $p$ and $q$.)

**Theorem 2.4.** *With the above notation notation, there exist constants $C_1, C_2$ depending only on $M$ such that*

$$C_1 \deg t_0 \le \deg K(M_{p/q}) \le C_2 \deg t_0. \tag{19}$$

*Proof.* It is well-known (see for instance [MR03]) that $K(M)$ is generated by the traces of products of at most three generators of $\pi_1(M)$. Considering each such trace as a function over (13), as $m$ and $\ell$ parametrize (13), it follows that each trace is an algebraic function of $m$ and $\ell$. Since $\pi_1(M)$ is finitely generated, the result follows. $\square$

Thus, the main theorem 1.2 follows from the following.

**Theorem 2.5.** *Fix $M$ as in Theorem 1.2. Assuming Lehmer's conjecture, there exist $C_1, C_2$ depending only on $M$ such that, for any non-cyclotomic integer irreducible factor $g(t)$ of $A_{p,q}(t)$,*

$$C_1 \max\{|p|, |q|\} \le \deg g(t) \le C_2 \max\{|p|, |q|\}. \tag{20}$$

The upper bound follows trivially from (17), and can even be replaced by a piece-wise linear function of $(p,q)$, as was commented after Lemma 2.3. The difficult part is the lower bound, and this requires an analysis of the roots of $A_{p,q}(t)$ near 1, along with careful estimates, as well as Lehmer's conjecture.

We end with a remark regarding the factorization of the Dehn-surgery polynomials.

**Remark 2.6.** It is a well-known result (for example, see [GR07]) in number theory that $A_{p,q}(t)$ has a bounded number of cyclotomic factors for any $p$ and $q$ (although of unknown multiplicity). Assuming Lehmer's Conjecture, the above theorem implies that $A_{p,q}(t)$ has a bounded number of non-cyclotomic factors with degree bounded below by $\max\{|p|,|q|\}$ for $|p|+|q|$ large. In some special circumstances, such as Dehn-filling on one component of the Whitehead link $W$, it is possible to prove unconditionally that the degree of $K(W_{1/q})$ is given by an explicit piece-wise linear function of $q$ (see Hoste–Shanahan [HS01]). More generally, one can prove that the degree of $K(W_{p/q})$ for fixed $p$ and large $q$ is given by an explicit piece-wise linear function of $q$ [FGH].

2.3. **Mahler measure and Lehmer's conjecture.** The Mahler measure $\mathscr{M}(f)$ and length $\mathscr{L}(f)$ of an integer polynomial

$$f(x) = a_n x^n + \cdots + a_1 x + a_0 = a_n(x-\alpha_1)\cdots(x-\alpha_n) \in \mathbb{Z}[x]$$

are defined by

$$\mathscr{M}(f) = |a_n| \prod_{i=1}^{n} \max(|\alpha_i|, 1),$$
$$\mathscr{L}(f) = |a_0| + \cdots + |a_n|$$

respectively. Then the following properties are standard [Mah76]:

(1) $\mathscr{M}(f_1 f_2) = \mathscr{M}(f_1)\mathscr{M}(f_2)$
(2) $\mathscr{M}(f) \leq \mathscr{L}(f)$

where $f_1$ and $f_2$ are two integer polynomials. One of the well-known unsolved problems in number theory is the following conjecture proposed by D. Lehmer in 1930s:

**Conjecture 2.7** (D. Lehmer)**.** There exists a constant $c > 1$ such that the Mahler measure of any non-cyclotomic irreducible integer polynomial $f$ satisfies

$$\mathscr{M}(f) \geq c.$$

It was further conjectured that the constant $c$ is achieved by the following polynomial:

$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$$

for which the Mahler measure is the smallest Salem number $1.176280818\ldots$

2.4. **Schinzel-Zassenhaus conjecture.** Recently there has been some progress towards Lehmer's conjecture. In [Dim], V. Dimitrov proved the following theorem:

**Theorem 2.8** (Schinzel-Zassenhous conjecture)**.** *Let $f(t)$ be a monic integer irreducible polynomial of degree $n > 1$. If $f(t)$ is not cyclotomic, then*

$$\max_{t_0\,:\, f(t_0)=0} |t_0| \geq 2^{\frac{1}{4n}} = 1 + \frac{\log 2}{4n} + O(\frac{1}{n^2}).$$

As mentioned in Section 1.2, once we restrict our attention to a small domain like (6), we obtain the conclusion of Theorem 1.2 unconditionally (i.e. without Lehmer) over the domain simply from the above theorem (see Theorem 4.1). However we do not believe Theorem 2.8 in general provides an unconditional proof of Theorem 1.2 over the entire domain. See also Remark 3.5.

## 3. Bounds for the roots of the Dehn-filling polynomial

In this section we study the roots of the Dehn-filling polynomial $A_{p,q}(t)$ for sufficiently large $|p|+|q|$. Recall the constant $s_A$ from (18). The next lemma shows that when $p/q > s_A$ with $p$ and $q$ sufficiently large, the roots of $A_{p,q}(t)$ are near the closed unit disk in the complex plane.

**Lemma 3.1.** (a) Fix a positive constant $C_1$ with $C_1 > s_A$. There exists $D > 0$ that depends on $C_1$ such that for any coprime pair $(p, q) \in \mathbb{N}^2$ satisfying $\frac{p}{q} > C_1$, and for any root $t_0$ of $A_{p,q}(t)$, we have

$$|t_0| < 1 + \frac{D}{q}\,.$$

(b) If in addition $C_2 > C_1$, $C_1 < \frac{p}{q} < C_2$, and $t_0$ is any root of $A_{p,q}(t)$, then

$$|t_0| < 1 + \frac{D'}{p}$$

for some $D'$ that depends on $C_1$ and $C_2$ only.

*Proof.* For $C_1 > s_A$, the leading term of $A_{p,q}(t)$ is $c_{a_n,n}t^{-a_nq+np}$ and $c_{a_n,n} = \pm 1$ by Lemma 2.3. Thus

$$|t_0^{-a_nq+np}| < CL|t_0^{-aq+bp}| \Longrightarrow |t_0^{(n-b)p+(a-a_n)q}| < CL \tag{21}$$

where $L$ is the number of terms of $A_{p,q}(t)$ and $C$ is the maximum among all the coefficients of $A_{p,q}(t)$ and $-aq + bp$ is the second largest exponent of $A_{p,q}(t)$. It follows by Lemma 2.3 that $L$ and $C$ are independent of $p, q$ for almost all $(p, q)$.

We now consider two cases: $b = n$ and $b < n$. If $b = n$, then $|t_0^q| < CL$, implying

$$|t_0| < 1 + \frac{D}{q} \tag{22}$$

for some constant $D$ depending on $C, L$.

If $b < n$, since $p, q > 0$, Lemma 2.3 implies that $(a, b) = (a_j, j)$ for some $0 \le j \le n-1$. Thus

$$CL > \left|t_0^{(n-j)p+(a_i-a_n)q}\right| = \left|t_0^{(n-j)\left(p-\frac{a_n-a_j}{n-j}q\right)}\right| > \left|t_0^{(n-j)\left(C_1-\frac{a_n-a_j}{n-j}\right)q}\right|$$

by (21). By the assumption, since $C_1$ is strictly larger than $\frac{a_n-a_j}{n-j}$ for every $i$, we get

$$|t_0^q| < D$$

for some constant $D$ that depends on $C_1$. Part (b) follows easily from part (a). $\square$

The next lemma is a model for the roots of the Dehn-filling polynomial outside the unit circle. Indeed, as we will see later (in the proof of Lemma 3.3), after a change of variables, we will bring the equation $A_{p,q}(t) = 0$ in the form $z^q \phi(z)^p = 1$ where $\phi$ is an analytic function at $z = 0$ with $\Phi(0) = 1$. Hence, the lemma below is the key technical tool which is used to bound the roots of the Dehn-filling polynomial. In a simplified form, note that the equation $z^q(1+z)^p = 1$ has $p+q$ solutions in the complex plane for $p, q > 0$. On the other hand, only a fraction of them are near zero whereas at the same time $1+z$ is outside the unit circle, and moreover, we have a bound for the size of such solutions.

**Lemma 3.2.** Let $w = \phi(z)$ be an analytic function defined near $(z, w) = (0, 1)$ and $\epsilon$ be some sufficiently small number. Then there exists $C(\epsilon) > 0$ such that, for every coprime pair $(p, q) \in \mathbb{N}^2$ with $p/q > \frac{1}{\epsilon}$, the number of $(z, w)$ satisfying

$$z^q w^p = 1, \quad |w| > 1, \quad |w - 1| < \epsilon \tag{23}$$

is at most $2\left(\left\lceil \frac{C(\epsilon)p}{2\pi q} \right\rceil + 1\right)q$. Moreover, $C(\epsilon) \to 0$ as $\epsilon \to 0$ and, for each $h$ $\left(1 \le h \le \left\lceil \frac{C(\epsilon)p}{2\pi q} \right\rceil + 1\right)$, the product of the moduli of the first $2hq$ largest $w$ satisfying (23) is bounded above by

$$\prod_{l=1}^{h} \left(1 + \frac{d \log \frac{p/q}{l}}{p/q}\right)^{2q} \tag{24}$$

where $d$ is some constant depending only on $\phi$.

*Proof.* (1) We first prove the lemma in the special case of $w = \phi(z) = 1 + z$. The first equation in (23) is equivalent to

$$(1+z)^{p/q} = \frac{e^{2\pi i k/q}}{z} \tag{25}$$

where $0 \le k \le q-1$. Considering $0, 1, 1+z$ as three vertices of a triangle $\triangle(z)$ in the complex plane, we denote $1+z$ and $z$ by

$$ae^{i\theta_1} := 1 + z, \quad be^{i\theta_2} := z \tag{26}$$

respectively where $a, b > 0$ and $-\pi < \theta_1, \theta_2 < \pi$.[2] Then (25) is equivalent to

$$(ae^{i\theta_1})^{p/q} = \frac{e^{2\pi i k/q}}{be^{i\theta_2}},$$

which is further reduced to

$$a^{p/q} = b^{-1}, \quad \frac{\theta_1 p}{q} \equiv \frac{2\pi k}{q} - \theta_2 \pmod{2\pi}. \tag{27}$$

By trigonometry, we have:

$$b^2 = a^2 + 1 - 2a\cos\theta_1, \quad a^2 = b^2 + 1 + 2b\cos\theta_2. \tag{28}$$

[2] Note that if $\theta_1 > 0$ (resp. $\theta_1 < 0$), then $\theta_2 > 0$ (resp. $\theta_2 < 0$).

Since $b = a^{-p/q}$ by (27), it follows that

$$\begin{aligned}\cos\theta_1 &= \frac{1}{2a}(a^2+1-a^{-\frac{2p}{q}}) = \frac{1}{2}\Big(a+\frac{1}{a}-\frac{1}{a^{\frac{2p}{q}+1}}\Big),\\ \cos\theta_2 = \frac{1}{2b}(a^2-b^2-1) &= \frac{a^{\frac{p}{q}}}{2}(a^2-a^{\frac{-2p}{q}}-1) = \frac{1}{2}\Big(a^{\frac{p}{q}+2}-a^{\frac{-p}{q}}-a^{\frac{p}{q}}\Big).\end{aligned} \tag{29}$$

Let[3]

$$r := p/q \quad \text{and} \quad x := r(a-1), \tag{30}$$

and rewrite the equations in (29) as

$$\begin{aligned}\cos\theta_1 = \frac{1}{2}\left(1+\frac{x}{r}+\frac{1}{1+\frac{x}{r}}-\frac{1}{\left(1+\frac{x}{r}\right)^{2r+1}}\right) &= \frac{1}{2}\left(2+\frac{x}{r}-\frac{\frac{x}{r}}{1+\frac{x}{r}}-\frac{1}{\left(1+\frac{x}{r}\right)^{2r+1}}\right),\\ \cos\theta_2 = \frac{1}{2}\left(\left(1+\frac{x}{r}\right)^{r+2}-\frac{1}{\left(1+\frac{x}{r}\right)^{r}}-\left(1+\frac{x}{r}\right)^{r}\right)&,\end{aligned} \tag{31}$$

which implies

$$\begin{aligned}\sin^2\theta_1 = 1-\cos^2\theta_1 &= 1-\left(1+\frac{1}{2}\left(\frac{x}{r}-\frac{\frac{x}{r}}{1+\frac{x}{r}}-\frac{1}{\left(1+\frac{x}{r}\right)^{2r+1}}\right)\right)^2\\ &= -\left(\frac{x}{r}-\frac{\frac{x}{r}}{1+\frac{x}{r}}-\frac{1}{\left(1+\frac{x}{r}\right)^{2r+1}}\right)-\frac{1}{4}\left(\frac{x}{r}-\frac{\frac{x}{r}}{1+\frac{x}{r}}-\frac{1}{\left(1+\frac{x}{r}\right)^{2r+1}}\right)^2\\ \Longrightarrow \sin\theta_1 &= \pm\sqrt{\left(\frac{1}{\left(1+\frac{x}{r}\right)^{2r+1}}+\frac{\frac{x}{r}}{1+\frac{x}{r}}-\frac{x}{r}\right)-\frac{1}{4}\left(\frac{1}{\left(1+\frac{x}{r}\right)^{2r+1}}+\frac{\frac{x}{r}}{1+\frac{x}{r}}-\frac{x}{r}\right)^2}.\end{aligned}$$

Considering $\theta_1 = \theta_1(x)$ as a function of $x$ and assuming $\theta_1(x) \geq 0$ for the sake of simplicity, we first set up the domain of $x$ satisfying $|ae^{i\theta_1(x)}-1| = |z| < \epsilon$ and then list all the solutions to the equation $z^q(1+z)^p = 1$ over the domain found.

Note that the following

$$\sqrt{\frac{1}{\left(1+\frac{x}{r}\right)^{2r+1}}+\frac{\frac{x}{r}}{1+\frac{x}{r}}-\frac{x}{r}} = \sqrt{\frac{1}{\left(1+\frac{x}{r}\right)^{2r+1}}-\frac{1}{\frac{r}{x}\left(\frac{r}{x}+1\right)}} = \sqrt{\frac{1}{\left(1+\frac{x}{r}\right)^{2r+1}}-\frac{x^2}{r(x+r)}} \tag{32}$$

is a decreasing function of $x$ and is 1 when $x = 0$. Further, if

$$\frac{1}{(1+\frac{x}{r})^{2r+1}} = \frac{x^2}{r(x+r)},$$

then

$$\left(\frac{r}{x+r}\right)^{2r+1} = \frac{x^2}{r(x+r)} \Rightarrow x^2 = \frac{r^{2r+2}}{(x+r)^{2r}} \Rightarrow x = \frac{r^{r+1}}{(x+r)^r} \Rightarrow r = x\left(1+\frac{x}{r}\right)^r, \tag{33}$$

[3]Since $|w| = a > 1$ and $|z| = |ae^{i\theta_1}-1| < \epsilon$ by the assumption, we assume $x > 0$ and $\frac{x}{r}$ is sufficiently small.

and, as $\left(1+\dfrac{x}{r}\right)^{\frac{r}{x}}$ is approximately equal to $e$ (simply say $\left(1+\dfrac{x}{r}\right)^{\frac{r}{x}} \approx e$) for $\dfrac{x}{r}$ sufficiently small, the root of (33) is very close to the root of the Lambert equation $r = xe^x$, studied in great detail in [CGH+96]. Hence, for $x = \phi(r)$ satisfying (33), it follows that

$$\log r - \log\log r < \phi(r) < \log r - \log\log r + \log\log\log r. \tag{34}$$

Remark that if $x = 0$, then $\theta_1(0) = \frac{\pi}{3}, \theta_2(0) = \frac{2\pi}{3}$ and thus $\triangle(z = e^{\frac{2\pi}{3}i})$ is the unit equilateral triangle. As $x$ increases, both $\theta_1(x)$ and $\theta_2(x)$ decrease, and, finally when $x = \phi(r)$, $\theta_1(\phi(r)) = \theta_2(\phi(r)) = 0$ and $\triangle\left(z = \frac{1}{\left(1+\frac{\phi(r)}{r}\right)^r}\right)$ becomes a flat triangle.

(a) By the assumption,

$$|z| = |ae^{i\theta_1(x)} - 1| < \epsilon$$

and, as

$$|ae^{i\theta_1(x)} - 1| = \left|\left(1+\frac{x}{r}\right)e^{i\theta_1(x)} - 1\right| = \left(1+\frac{x}{r}\right)^2(2 - 2\cos\theta_1(x))$$

(with $\cos\theta_1(x)$ in (31)) is a decreasing function of $x$, there exists $c = c(\epsilon)$ depending on $\epsilon$ such that

$$\left|\left(1+\frac{x}{r}\right)e^{i\theta_1(x)} - 1\right| < \epsilon \Longleftrightarrow c < x \le \phi(r).$$

In conclusion, $(c, \phi(r)]$ a desired domain for $\theta_1(x)$ (and $\theta_2(x)$) with the required property.

(b) Now we list the solutions of $z^q(1+z)^p = 1$. First the second equation in (27) is reduced to

$$r\theta_1(x) + \theta_2(x) - \frac{2\pi k}{q} \in 2\pi\mathbb{Z} \tag{35}$$

and

$$-\frac{2\pi k}{q} \le r\theta_1(x) + \theta_2(x) - \frac{2\pi k}{q} \le r\theta_1(c) + \theta_2(c) - \frac{2\pi k}{q}$$

for $c \le x \le \phi(r)$. Since $r\theta_1(x)+\theta_2(x)$ is a decreasing function of $x$ and $\theta_2(c) \le \frac{2\pi}{3}$, for each $k$ $(0 \le k \le q-1)$, the number of $x$ satisfying (35) is at most

$$\left\lfloor \frac{r\theta_1(c) + \theta_2(c)}{2\pi} - \frac{2\pi k}{q} \right\rfloor + 1 = \left\lceil \frac{r\theta_1(c)}{2\pi} \right\rceil + 1.$$

Let $x_l^k \in (c, \phi(r)]$ a number satisfying

$$\frac{r\theta_1(x_l^k) + \theta_2(x_l^k) - \frac{2\pi k}{q}}{2\pi} = l$$

where $0 \le l \le \left\lceil \dfrac{r\theta_1(c)}{2\pi} \right\rceil$ and $l \in \mathbb{Z}$. Then

$$\theta_1(x_l^k) = \frac{2\pi l - \theta_2(x_l^k) + \frac{2\pi k}{q}}{r} \ge \frac{2\pi(l - \frac{1}{3})}{r} \ge \frac{l+1}{r}$$

for every $k$ $(0 \le k \le q-1)$ and $l \ge 1$. Using the fact that $\theta_1(x)$ is a decreasing function, one further gets

$$x_l^k \le \log \frac{r}{l+1}$$

and so

$$1 + \frac{x_l^k}{r} \le 1 + \frac{\log \frac{r}{l+1}}{r} \quad (l \ge 1).$$

Clearly $x_0^k \le \phi(r) \le \log r$ (by (34)) for $l = 0$ and $k$ $(0 \le k \le q-1)$. Consequently, for each $h$ $(1 \le h \le \left\lceil \frac{r\theta_1(c)}{2\pi} \right\rceil + 1)$, the product of the moduli of the first $hq$ largest $w$ satisfying $\theta_1(x), \theta_2(x) \ge 0$ and (35) is bounded above by[4]

$$\prod_{k=0}^{q-1} \prod_{l=0}^{h-1} \left(1 + \frac{x_l^k}{r}\right) \le \prod_{l=0}^{h-1} \left(1 + \frac{\log \frac{p/q}{l+1}}{p/q}\right)^q = \prod_{l=1}^{h} \left(1 + \frac{\log \frac{p/q}{l}}{p/q}\right)^q.$$

Similarly, counting $x$ with $\theta_1(x), \theta_2(x) \le 0$, one finally gets the desired result. This concludes the proof of the lemma for the special case of $w = 1 + z$.

(2) To prove the lemma for the general case, suppose $w = \phi(z)$ is given as $\phi(z) = 1 + \sum_{i=1}^{\infty} c_i z^i$ and let

$$ae^{i\theta_1} := 1 + \sum_{i=1}^{\infty} c_i z^i, \quad be^{i\theta_2} := \sum_{i=1}^{\infty} c_i z^i, \quad ce^{i\theta_3} := z.$$

We consider $b$ (resp. $\theta_2$) as an analytic function of $c$ (resp. $\theta_3$). Thus, for $z$ sufficiently small, $b$ (resp. $\theta_1$) is approximately very close to $|c_m|c^m$ (resp. $\arg c_m + m\theta_3$) where $c_m$ is the coefficient of the first non-zero (and non-constant) term of $\phi$. Now (23) is

$$(1 + \phi(z))^{p/q} = \frac{e^{2\pi i k/q}}{z} \quad (0 \le k \le q-1)$$

and this is equivalent to

$$c = a^{-r}, \quad r\theta_1 + \theta_3 \equiv \frac{2\pi i k}{q} \quad (\mod 2\pi i)$$

where $r = p/q$. Similar to the previous case, provided $x := r(a-1)$, we get $\frac{1}{c} = a^r = \left(1 + \frac{x}{r}\right)^r$ and

$$\begin{aligned} \sin\theta_1 &= \pm\sqrt{-\left(\frac{x}{r} - \frac{\frac{x}{r}}{1+\frac{x}{r}} - \frac{b^2}{a}\right) - \frac{1}{4}\left(\frac{x}{r} - \frac{\frac{x}{r}}{1+\frac{x}{r}} - \frac{b^2}{a}\right)^2} \\ &= \pm\sqrt{\left(\frac{b^2}{a} - \frac{x^2}{r(x+r)}\right) - \frac{1}{4}\left(\frac{b^2}{a} - \frac{x^2}{r(x+r)}\right)^2}. \end{aligned}$$

[4] Recall from (26) and (30) that $|w| = |1+z| = |a| = 1 + \frac{x}{r}$.

Since $c = a^{-r}$ and $b \approx |c_m|c^m$ for $z$ small, $b \approx |c_m|a^{-rm}$ and so

$$\begin{aligned}\sin\theta_1 &\approx \pm\sqrt{\left(\frac{|c_m|^2}{a^{2rm+1}} - \frac{x^2}{r(x+r)}\right) - \frac{1}{4}\left(\frac{|c_m|^2}{a^{2rm+1}} - \frac{x^2}{r(x+r)}\right)^2} \\ &= \pm\sqrt{\left(\frac{|c_m|^2}{\left(1+\frac{x}{r}\right)^{2rm+1}} - \frac{x^2}{r(x+r)}\right) - \frac{1}{4}\left(\frac{|c_m|^2}{\left(1+\frac{x}{r}\right)^{2rm+1}} - \frac{x^2}{r(x+r)}\right)^2}\end{aligned} \tag{36}$$

for $z$ sufficiently small. Let $\phi(r)$ is the number satisfying

$$\frac{|c_m|}{\left(1+\frac{x}{r}\right)^{rm}} = \frac{x}{r}. \tag{37}$$

As $\left(1+\frac{x}{r}\right)^r \approx e^x$ for $\frac{x}{r}$ small, $\phi(r)$ is very close to the root of $|c_m|r = xe^{xm}$ and

$$d_1 \log r < \phi(r) < d_2 \log r.$$

for some $d_1, d_2 \in \mathbb{Q}$. Further, one can check (36) is a decreasing function of $x$ over $0 \leq x \leq \phi(r)$. Applying similar methods used in the proof of the previous special case, one gets the desired result.

□

In the next lemma, consider $M$ and $M_{p/q}$ and the Dehn-filling polynomial $A_{p,q}(t)$ as usual. As remarked earlier, the motivation for each statement of Lemma 3.3 was already explained in Section 1.2.

**Lemma 3.3.** For every $\epsilon > 0$ sufficiently small, there exist $C_1(\epsilon)$ and $C_2(\epsilon)$ such that for any coprime pair $(p,q) \in \mathbb{N}^2$ with $p/q > \frac{1}{\epsilon}$, the following hold.

(1) If $t_0$ is a root of $A_{p,q}(t)$ such that $1 < |t_0|^p < \frac{1}{\epsilon}$, then $|t_0| < 1 + \frac{C_1(\epsilon)}{p}$.

(2) If $t_0$ is a root of $A_{p,q}(t)$ such that $|t_0| > 1$ and $|t_0^q - \zeta| > \epsilon$ for every $\zeta$ satisfying $\sum_{i=a_n}^{b_n} c_{i,n}\zeta^{-i} = 0$, then $|t_0| < 1 + \frac{C_1(\epsilon)}{p}$.

(3) There are at most $2\left\lceil C_2(\epsilon)p/q\right\rceil q$ roots of $A_{p,q}(t) = 0$ whose moduli are bigger than $1 + \frac{C_1(\epsilon)}{p}$. Further, for each $h$ ($1 \leq h \leq \left\lceil\frac{C_2(\epsilon)p}{q}\right\rceil + 1$), the product of the moduli of the first $2hq$ largest roots of $A_{p,q}(t)$ is bounded above by[5]

$$\prod_{l=1}^{h}\left(1 + \frac{d\log\frac{p/q}{l}}{p/q}\right)^2 \tag{38}$$

where $d$ is some constant depending only on $M$.

(4) $C_1(\epsilon) \to \infty$ and $C_2(\epsilon) \to 0$ as $\epsilon \to 0$.

*Proof.* For (1), if $|t_0|^p < \frac{1}{\epsilon}$ with $p$ sufficiently large, taking logarithms, there exists $C_1(\epsilon)$ such that $|t_0| < 1 + \frac{C_1(\epsilon)}{p}$.

[5] Note that the exponent 2 in (38) is differed from the exponent $2q$ given in (24).

For (2), if $|t_0^q-\zeta|>\epsilon$ for every $\zeta$ satisfying $\sum_{i=a_n}^{b_n} c_{i,n}\zeta^{-i}=0$, then there exists $C'(\epsilon)>0$ such that $\Big|\sum_{i=a_n}^{b_n} c_{i,n}t_0^{-iq}\Big|>C'(\epsilon)$ and so

$$C'(\epsilon)|t_0^{np}|<\Big|\sum_{i=a_n}^{b_n} c_{i,n}t_0^{-iq}\Big||t_0^{np}|<nC|t_0^{(n-1)p}|\Longrightarrow |t_0^p|<\frac{nC}{C'(\epsilon)} \tag{39}$$

where $C$ is some number bigger than $\max_{0\leq j\leq n-1}\Big|\sum_{i=a_j}^{b_j} c_{i,j}t_0^{-iq}\Big|$. By (39), there exists some constant $C_1(\epsilon)$ depending on $\epsilon$ such that $|t_0|<1+\frac{C_1(\epsilon)}{p}$.

For (3), suppose $t_0$ is a root of $A_{p,q}(t)=0$ such that $|t_0^q-\zeta|<\epsilon$ where $\sum_{i=a_n}^{b_n} c_{i,n}\zeta^{-i}=0$ and $|t_0|^p>\frac{1}{\epsilon}$ for some sufficiently small $\epsilon$. Note that $\zeta$ is a root of unity by Theorem 2.2 and, without loss of generality, we further assume $\zeta=1$. If $A(m,\ell)=0$ is the A-polynomial of $M$ as given in (16), one can view $(m,\ell)=(t_0^{-q},t_0^p)$ as an intersection point between $A(m,\ell)=0$ and $m^p\ell^q=1$. If we let $m':=\frac{1}{m}$ and $f(m',\ell):=A(\frac{1}{m'},\ell)$, $(m',\ell)=(t_0^q,t_0^p)$ is a point lying over $f(m',\ell)=0$ and $\ell^q=(m')^p$. For the sake of simplicity, we consider the projective closure of $f(m',\ell)=0$ and work with a different affine chart of it. More precisely, let $h(x',y',z')=0$ be the homogeneous polynomial representing the projective closure of $f(m',\ell)=0$ with $m'=\frac{x'}{z'},\ell=\frac{y'}{z'}$. That is, $h(x',y',z')=0$ is obtained from $f\Big(\frac{x'}{z'},\frac{y'}{z'}\Big)=0$ by multiplying by a power of $z'$ if necessary. Further, if $x:=\frac{x'}{y'},z:=\frac{z'}{y'}$ and $k(x,z):=h(x,1,z)$, since $x=\frac{m'}{\ell},z=\frac{1}{\ell}$ and

$$\ell^q=(m')^p\Rightarrow\Big(\frac{y'}{z'}\Big)^q=\Big(\frac{x'}{z'}\Big)^p\Rightarrow z^{p-q}=x^p, \tag{40}$$

$(x,z)=(t_0^{-p+q},t_0^{-p})$ is an intersection point between $k(x,z)$ and $z^{p-q}=x^p$. As $(t_0^{-p+q},t_0^{-p})$ is sufficiently close to $(0,0)$, it follows that $k(0,0)=0$ and thus $x$ is represented as an analytic function $\varphi(z)$ of $z$ near $(0,0)$ (i.e. $k(\varphi(z),z)=0$) with $(t_0^{-p+q},t_0^{-p})\in(\varphi(z),z)$. Moreover, since $\frac{t_0^{-p+q}}{t_0^{-p}}=t_0^q$ is close enough to 1, $\varphi(z)$ is of the following form:

$$x=\varphi(z)=z(1+\sum_{i=1}^{\infty}c_iz^i). \tag{41}$$

Let $\phi(z):=1+\sum_{i=1}^{\infty}c_iz^i$. Then

$$z^{p-q}=x^p\Rightarrow z^{p-q}=z^p\phi(z)^p\Rightarrow z^q\phi(z)^p=1. \tag{42}$$

In conclusion, $(t_0^q,t_0^{-p})$ is a point on $(\frac{x}{z},z)=(\phi(z),z)$ satisfying (42). Let $w:=\phi(z)$. Since $|t_0|>1$ and $|t_0^q-1|<\epsilon$ by the assumption, we get the desired result by Lemma 3.2.

Finally, (4) is clear by the construction of $C_1(\epsilon)$ and Lemma 3.2. □

**Remark 3.4.** Switching $p$ and $q$, one obtains an analogous result for a coprime pair $(p,q)$ with $q/p$ is sufficiently large.

**Remark 3.5.** Note that if $t_0$ (where $|t_0| > 1$) is a root of $A_{p,q}(t)$ giving rise to the discrete faithful representation of $M_{p/q}$, for $|p| + |q|$ sufficiently large, $|t_0|$ is asymptotic to $1 + \frac{2\pi \mathrm{Im}\, \tau}{|p+\tau q|^2}$ where $\tau$ is a complex number depending only on $M$ satisfying Im $\tau \neq 0$ ($\tau$ is called the cusp shape of $M$) [NZ85]. This, combining with the above remark, implies that, for $|p| + |q|$ sufficiently large, $t_0$ is not the root of its minimal polynomial that appears in Theorem 2.8.

## 4. Proof of the main theorem

Now we prove the main theorem of the paper. Recall that Theorem 1.2 follows from Theorem 2.5. We first prove a special case of Theorem 2.5 over a restricted domain of $p$ and $q$. In particular, on this restricted domain, the statement of Theorem 2.5 holds unconditionally (i.e. without Lehmer's conjecture) thanks to Theorem 2.8. We then expand the domain further and prove Theorem 2.5 over the extended domain. We shall use Lemma 3.3 and assume Lehmer's conjecture from Theorem 4.2. Once Theorem 4.2 is proven, the rest of the proof of Theorem 2.5 will follow by symmetric properties of the A-polynomial as well as a monomial change of variables on it.

Finally let us remark again that, as the upper bound in Theorem 2.5 follows as a corollary of Bezóut's theorem, only the lower bound in (20) is proved.

**Theorem 4.1.** *(without Lehmer) Let $M, M_{p/q}$ be as usual and $A_{p,q}(t), C_1, C_2$ be the same as in Lemma 3.1. Then there exists $C_3$ depending on $C_1$ and $C_2$ such that, for any coprime $(p, q) \in \mathbb{N}^2$ satisfying $C_1 < \frac{p}{q} < C_2$ and any non-cyclotomic irreducible integer factor $g(t)$ of $A_{p,q}(t)$,*

$$C_3 \max\{p, q\} \leq \deg g(t).$$

*Proof.* By Lemma 3.1 (b), there exists $D$ depending on $C_1$ and $C_2$ such that $|t_0| < 1 + \frac{D}{p}$ for any root $t_0$ of $A_{p,q}(t) = 0$. By Theorem 2.8, the result follows. □

**Theorem 4.2.** *(with Lehmer) Let $M, M_{p/q}$ be as usual and $A_{p,q}(t), C_1$ be the same as in Lemma 3.1. Assuming Lehmer's conjecture, there exists $C_2$ depending on $C_1$ such that, for any coprime $(p, q) \in \mathbb{N}^2$ satisfying $\frac{p}{q} > C_1$ and any non-cyclotomic irreducible integer factor $g(t)$ of $A_{p,q}(t)$,*

$$C_2 \max\{p, q\} \leq \deg g(t). \tag{43}$$

*Proof.* Let $\epsilon$ be some sufficiently small number such that $C_1 < \frac{1}{\epsilon}$. By Theorem 4.1, we find $C_2'$ depending on $C_1$ and $\epsilon$ satisfying

$$C_2' \max\{p, q\} \leq \deg g(t).$$

for any coprime $(p, q) \in \mathbb{N}^2$ with $C_1 < \frac{p}{q} < \frac{1}{\epsilon}$ and any non-cyclotomic irreducible integer factor $g(t)$ of $A_{p,q}(t)$.

Now suppose there are sequences of positive real numbers $\{c_i\}_{i\in\mathbb{N}}$ and coprime pairs $\{(p_i, q_i)\}_{i\in\mathbb{N}}$ satisfying $\lim_{i\to\infty} c_i = 0$ and $\frac{p_i}{q_i} > \frac{1}{\epsilon}$. We further assume, for each $i$, there exists a non-cyclotomic irreducible integer factor $g_i(t)$ of $A_{p_i,q_i}(t)$ such that

$$\deg g_i(t) \leq c_i \max\{p_i, q_i\}. \tag{44}$$

We claim

$$\varlimsup_{i\to\infty} \mathscr{M}(g_i(t)) < c \tag{45}$$

Since the Mahler measure of any non-cyclotomic polynomial is strictly bigger than $c > 1$ (with $c$ given in Conjecture 2.7) by Lehmer's conjecture, this is a contradiction and thus there exists a constant $C_2''$ depending on $\epsilon$ such that

$$C_2'' \max\{p, q\} \leq \deg g(t)$$

for any coprime $(p, q) \in \mathbb{N}^2$ satisfying $\frac{p}{q} > \frac{1}{\epsilon}$ and any non-cyclotomic irreducible integer factor $g(t)$ of $A_{p,q}(t)$. Since $\epsilon$ is arbitrarily, we may choose $C_2$ satisfying the statement of Theorem 4.2. $\square$

We now give the proof of (45). By Theorem 2.2, Lemma 2.3 and multiplying by a power of $t$ if necessary, we assume both $A_{p_i,q_i}(t)$ and its irreducible factor $g_i(t)$ are monic integer polynomials.

As $c_i \to 0$, for $i$ sufficiently large, the product of the moduli of the first $\lfloor c_i p_i \rfloor$ largest roots of $A(t^{-q_i}, t^{p_i}) = 0$ (and so $\mathscr{M}(g_i(t))$ as well) is bounded above by

$$\prod_{l=1}^{\lceil \frac{c_i p_i}{q_i} \rceil} \left(1 + \frac{d \log \frac{p_i/q_i}{l}}{p_i/q_i}\right)^2 \tag{46}$$

where $d$ is some constant depending only on $M$ by Lemma 3.3. We show (46) is strictly less than $c$ as $i \to \infty$. To simplify notation, let $r_i := p_i/q_i$. Taking logarithms to $\mathscr{M}(g_i(t))$ and (46),

$$\log \mathscr{M}(g_i(t)) \leq 2 \sum_{l=1}^{\lceil c_i r_i \rceil} \log\Big(1 + \frac{d \log \frac{r_i}{l}}{r_i}\Big).$$

Since

$$2 \sum_{l=1}^{\lceil c_i r_i \rceil} \log\Big(1 + \frac{d \log \frac{r_i}{l}}{r_i}\Big) \leq 2 \sum_{l=1}^{\lceil c_i r_i \rceil} \frac{d \log \frac{r_i}{l}}{r_i}$$

and

$$\lceil c_i r_i \rceil \log \lceil c_i r_i \rceil - \lceil c_i r_i \rceil < \log \lceil c_i r_i \rceil !,$$

it follows that

$$\begin{aligned}
\log \mathscr{M}(g_i(t)) &\leq 2 \sum_{l=1}^{\lceil c_i r_i \rceil} \frac{d \log \dfrac{r_i}{l}}{r_i} = \frac{2d \log \dfrac{r_i^{\lceil c_i r_i \rceil}}{\lceil c_i r_i \rceil !}}{r_i} = \frac{2d \lceil c_i r_i \rceil \log r_i - 2d \log \lceil c_i r_i \rceil !}{r_i} \\
&< \frac{2d \lceil c_i r_i \rceil \log r_i - 2d\big(\lceil c_i r_i \rceil \log \lceil c_i r_i \rceil - \lceil c_i r_i \rceil\big)}{r_i} < \frac{2d \lceil c_i r_i \rceil \log \dfrac{r_i}{\lceil c_i r_i \rceil} + 2d \lceil c_i r_i \rceil}{r_i}.
\end{aligned} \tag{47}$$

We now consider two cases.

**Case 1.** If $c_i r_i \geq 1$, then (47) is bounded above by

$$\frac{4 d c_i r_i \log \frac{1}{c_i} + 4 d c_i r_i}{r_i} = -4 d c_i \log c_i + 4 d c_i. \tag{48}$$

Since $\lim_{i\to\infty} c_i = 0$, (48) $\big($resp. (46)$\big)$ converges to 0 (resp. 1) as $i \to \infty$.

**Case 2.** If $c_i r_i < 1$, then (47) is bounded above by

$$\frac{4d\log r_i + 4d}{r_i}. \tag{49}$$

As $r_i(= p_i/q_i) > \frac{1}{\epsilon}$ and $d$ depends only on $M$, (49) is strictly less than $\log c$ provided $\epsilon$ is sufficiently small. This completes the proof of (45). □

**Remark 4.3.** For $p > 0$ and $q < 0$, one analogously gets the following result. Suppose $A_{p,q}(t)$ is given as in (17). Let $C_1$ be some positive constant such that $C_1 > \frac{b_j - b_n}{n-j}$ for every $j$ $(0 \le j \le n-1)$. Assuming Lehmer's conjecture, there exists $C_2$ depending on $C_1$ such that, for any coprime pair $(p,q)$ with $p > 0, q < 0, \frac{p}{|q|} > C_1$ and any non-cyclic irreducible factor $g(t)$ of $A_{p,q}(t)$,

$$C_2 \max\{p, |q|\} \le \deg g(t).$$

We now use the $\mathrm{SL}_2(\mathbb{Z})$ action on the lattice $\mathbb{Z}^2$ which amounts to a monomial change of variables on $A(m,\ell)$ and hence on $(p,q)$ and on $A_{p,q}(t)$. Combining this action with Theorem 4.2 and Remark 4.3, we obtain the following.

**Theorem 4.4.** *(with Lehmer) Let $M, M_{p/q}$ and $A_{p,q}(t)$ be as in Theorem 4.2. Assuming Lehmer's conjecture, the following statements hold.*

(1) *If $\frac{a_n - a_j}{n-j} < 0$ for all $0 \le j \le n-1$, then there exists $C_2$ depending on $M$ such that, for any coprime pair $(p,q) \in \mathbb{N}^2$ and any non-cyclic irreducible factor $g(t)$ of $A_{p,q}(t)$,*

$$C_2 \max\{p, q\} \le \deg g(t).$$

(2) *If $\frac{a_n - a_j}{n-j} > 0$ for some $0 \le j \le n-1$, then there exists $C_2$ depending on $M$ such that, for any coprime pair $(p,q) \in \mathbb{N}^2$ with $\frac{p}{q} > s_A$ where $s_A$ is the one given in (18) and any non-cyclic irreducible factor $g(t)$ of $A_{p,q}(t)$,*

$$C_2 \max\{p, q\} \le \deg g(t).$$

*Proof.* For (1), if $\frac{a_n - a_j}{n-j} < 0$ for all $0 \le j \le n-1$, equivalently, it means $c_{a_n,n} t^{-a_n q + np}$ is the leading term of $A_{p,q}(t)$ for every coprime pair $(p,q) \in \mathbb{N}^2$ with $p+q$ sufficiently large. Provided $C_1$ is chosen to be some sufficiently small $\epsilon$, the claim is true for any coprime pair $(p,q) \in \mathbb{N}^2$ with $\frac{p}{q} > \epsilon$ by Theorem 4.2. If $\frac{p}{q} < \epsilon$ (or, equivalently, $\frac{q}{p} > \frac{1}{\epsilon}$), by switching $p$ and $q$, one gets the desired result by an analogue of Lemma 3.3 (see Remark 3.4) and similar arguments given in the proof of Theorem 4.2.

For (2), suppose $\frac{a_n - a_j}{n-j} > 0$ for some $j$ $(0 \le j \le n-1)$ and let $s_A$ be as in (18). Let $C_1 := s_A + \epsilon$ where $\epsilon$ some sufficiently small number. If $(p,q) \in \mathbb{N}^2$ with $\frac{p}{q} > C_1$, the result follows by Theorem 4.2. For $(p,q) \in \mathbb{N}^2$ satisfying $s_A < \frac{p}{q} < C_1$, first let $(a,b)$ $\big($resp. $(r,s)\big)$ be a coprime pair such that $\frac{a}{b} = s_A$ (resp. $bs + ar = 1$). We further assume $a, b > 0$. Since $s_A(= \frac{a}{b}) < \frac{p}{q} < C_1(= C + \epsilon)$,

$$\Big|\frac{p}{q} - \frac{a}{b}\Big| < \epsilon \Longrightarrow \Big|\frac{bq}{aq - bp}\Big| > \frac{1}{\epsilon}$$

and so

$$\frac{rp+sq}{aq-bp} = \frac{-\frac{r}{b}(aq-bp)+\frac{bs+ar}{b}q}{aq-bp} = -\frac{r}{b}+\frac{1}{b}\frac{q}{aq-bp}, \tag{50}$$

implying

$$\Big|\frac{rp+sq}{aq-bp}\Big| > \frac{1}{b^2\epsilon}-\frac{r}{b}. \tag{51}$$

As given in Section 2.1, let $\mu, \lambda$ be a chosen meridian-longitude pair of $T$, a torus cross section of the cusp of $M$. By setting $\mu' := \mu^a\lambda^b, \lambda' := \mu^{-s}\lambda^r$, we change basis of $T$ from $\mu, \lambda$ to $\mu', \lambda'$. Also let $A'(m', \ell') = 0$ be the A-polynomial of $M$ obtained from the new basis. Since $\mu = (\mu')^r(\lambda')^{-b}, \lambda = (\mu')^s(\lambda')^a$ and $\mu^p\lambda^q = (\mu')^{rp+sq}(\lambda')^{-bp+aq}$, $\frac{p}{q}$-Dehn filling of $M$ under the original basis corresponds to $\big(\frac{rp+sq}{-bp+aq}\big)$-Dehn filling of $M$ under the new basis. Let $p' := rp+sq, q' := -bp+aq$. Note that

$$p' > 0, \quad q' < 0, \quad |p'/q'| > \frac{1}{b^2\epsilon}-\frac{r}{b} \tag{52}$$

by (50) and (51). Since $\frac{1}{b^2\epsilon}-\frac{r}{b}$ is sufficiently big, by Remark 4.3, there is $C_1'$ such that, for any $(p', q')$ satisfying (52) and any non-cyclotomic irreducible factor of $A'(t^{-q'}, t^{p'}) = 0$,

$$C_1'\, p' \leq \deg g(t).$$

Equivalently, this means the degree of any non-cyclotomic irreducible factor of $A_{p,q}(t) = 0$ is bounded above and below by constant multiples of $rp+sq$. This completes the proof. ☐

Now we are ready to complete the proof of Theorem 2.5.

*Proof of Theorem 2.5.* We prove the theorem only for $p, q > 0$. The rest of the cases can be treated similarly. Let $A_{p,q}(t)$ be normalized as (17). For $\frac{a_n-a_j}{n-j} < 0$ for every $j$ ($0 \leq j \leq n-1$), the result follows by Theorem 4.4 (1) and so it is assumed $\frac{a_n-a_j}{n-j} > 0$ for some $j$ ($0 \leq j \leq n-1$). Let $s_A$ be given as in (18). For a coprime pair $(p, q)$ satisfying $p/q > s_A$, since the claim was also proved in Theorem 4.4 (2), we further suppose $p/q < s_A$ and let

$$n_2 := \min_{0\leq j\leq n-1}\Big\{j \mid s_A = \frac{a_n-a_j}{n-j}\Big\}.$$

Note that $(a_{n_2}, n_2)$ is a corner of $\mathcal{N}(A)$ and $s_A$ is the slope of the edge $E_{s_A}$ connecting two corners $(a_n, n)$ and $(a_{n_2}, n_2)$ of $\mathcal{N}(A)$.

We distinguish two cases.

**Case 1.** If $n_2 = 0$ or $\frac{a_{n_2}-a_j}{n_2-j} < 0$ for every $0 \leq j \leq n_2-1$, then $c_{a_{n_2},n_2}t^{-a_{n_2}q+n_2p}$ is the leading term of $f_{p,q}(t)$ for any $(p, q) \in \mathbb{N}^2$ with $\frac{p}{q} < s_A$ and $p+q$ sufficiently large. By interchanging $p$ and $q$, one gets the desired result following similar steps shown in the proof of Theorem 4.4 (2).

**Case 2.** Now suppose $n_2 \neq 0$ and $\frac{a_{n_2}-a_j}{n_2-j} > 0$ for some $0 \leq j \leq n_2-1$. Let

$$s_{A_2} := \max_{0\leq j\leq n_2-1}\Big\{\frac{a_{n_2}-a_j}{n_2-j}\Big\}.$$

Then $s_A > s_{A_2}$ and $s_{A_2}$ is the slope of the edge $E_{s_{A_2}}$ of $\mathcal{N}(f)$ adjacent to $E_{s_A}$. To show the claim for $(p, q) \in \mathbb{N}^2$ satisfying $s_{A_2} < \frac{p}{q} < s_A$, let $T, \mu, \lambda, \mu'(= \mu^a\lambda^b), \lambda'(= \mu^{-s}\lambda^r)$ be the same as in the proof of Theorem 4.4 (2). Also we denote the A-polynomial of $M$ obtained from $\mu', \lambda'$ by $A'(m', \ell') = 0$ and assume its $p'/q'$-Dehn filling equation $A'_{p',q'}(t) := A'(t^{-q'}, t^{p'})$ is given as

$$\sum_{j=0}^{n'} \Big( \sum_{i=a'_j}^{b'_j} c'_{i,j} t^{-q'i} \Big) t^{p'j}$$

where $a'_j, b'_j \in \mathbb{Z}$. Under $p' = rp + sq$ and $q' = -bp + aq$, the set of coprime pairs $(p, q)$ satisfying $s_{A_2} < \frac{p}{q} < s_A$ is transformed to the set of $(p', q')$ satisfying $\frac{p'}{q'} > s_{A'}$ where

$$s_{A'} := \max_{0 \le j \le n'-1} \frac{a'_{n'} - a'_j}{n' - j}.$$

Since the conclusion holds for any $A'_{p',q'}(t)$ with $(p', q') \in \mathbb{N}^2, \frac{p'}{q'} > s_{A'}$ by Theorem 4.4, equivalently, it holds for $A_{p,q}(t)$ with $(p, q) \in \mathbb{N}^2, s_{A_2} < \frac{p}{q} < s_A$ as well.

Analogously, using the same ideas, one can also prove the statement for $(p, q) \in \mathbb{N}^2$ satisfying $s_{A_3} < \frac{p}{q} < s_{A_2}$ where $s_{A_3}$ is the slope of the edge $E_{s_{A_3}} (\neq E_A)$ of $\mathcal{N}(A)$ adjacent to $E_{s_{A_2}}$. Since $\mathcal{N}(A)$ has only finitely many edges, we eventually get the desired result. □

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