SUPER-REPRESENTATIONS OF 3-MANIFOLDS AND TORSION POLYNOMIALS

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ABSTRACT. Torsion polynomials connect the genus of a hyperbolic knot (a topological invariant) with the discrete faithful representation (a geometric invariant). Using a new combinatorial structure of an ideal triangulation of a 3-manifold that involves edges as well as faces, we associate a polynomial to a cusped hyperbolic manifold that conjecturally agrees with the $\mathbb{C}^2$-torsion polynomial, which conjecturally detects the genus of the knot. The new combinatorics is motivated by super-geometry in dimension 3, and more precisely by super-Ptolemy assignments of ideally triangulated 3-manifolds and their $\mathrm{OSp}_{2|1}(\mathbb{C})$-representations.

CONTENTS

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1. Introduction

1.1. Overview. A well-known topic in geometry and topology is the study of representations of surface groups into simple Lie groups. Recently, this topic has been extended by replacing simple Lie groups (such as $\mathrm{SL}_2(\mathbb{C})$) with super-Lie groups, most notably by the orthosymplectic group $\mathrm{OSp}_{2|1}(\mathbb{C})$. These representations have been studied from at least three different points of view: as character varieties, as cluster algebras, and as super-Teichmüller space. See, for instance, [PZ19, IPZ18, MOZ, She] as well as [Wit].

In our paper, we extend this study on surfaces to the context of 3-manifolds equipped with an ideal triangulation. Explicitly,

- We introduce super-Ptolemy coordinates for 3-dimensional triangulations that parametrize $\mathrm{OSp}_{2|1}(\mathbb{C})$-representations of 3-manifold groups (see Section 2).
- We introduce a new 1-loop polynomial, using defining equations of super-Ptolemy coordinates (see Section 3).

Regarding our 1-loop polynomial,

- We show that it is a topological invariant (Theorem 3.4).
- We show that an $\mathrm{SL}_2(\mathbb{C})$-representation lifts to an $\mathrm{OSp}_{2|1}(\mathbb{C})$-representation if and only if the 1-loop polynomial evaluated at $t = 1$ vanishes (Theorem 3.6).
- We conjecture that our 1-loop polynomial coincides with the $\mathbb{C}^2$-torsion polynomial (Conjecture 3.7).

Both the 1-loop and the $\mathbb{C}^2$-torsion polynomials have coefficients in the trace field and can be exactly computed. Doing so, we will confirm our conjecture for the 4_1 knot.

1.2. Torsion polynomials: a Thurstonian connection. The complement $M = S^3 \setminus K$ of a hyperbolic knot K in S^3 has two interesting invariants, both defined by Thurston

- the genus, i.e., the minimal genus of all spanning surfaces of K [Thu86],
- the discrete faithful representation $\pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$ [Thu97].

These two invariants, one topological and the other geometric, are beautifully linked to each other via torsion polynomials, revealing a "remarkable Thurstonian connection between the topology and geometry of 3-manifolds", to quote Agol–Dunfield [AD20]. Torsion polynomials are twisted versions of the Alexander polynomial, where one twists the homology of the infinite cyclic cover of M using an $\mathrm{SL}_2(\mathbb{C})$-lift ρ of the geometric representation, or a symmetric power $\mathrm{Sym}^{n-1}(\rho)$ of it, the corresponding polynomial being denoted by $\tau_{M,\rho,n}(t)$. These geometric invariants are Laurent polynomials in t with coefficients in the trace field of M, and a key feature is that their degrees give bounds for the genus of the knot. More precisely, one has

$$2 \cdot \mathrm{genus}(K) - 1 \geq \frac{1}{n} \deg \tau_{M,\rho,n}(t) \tag{1}$$

for all $n \geq 2$. When $n = 3$, examples show that the above bound is not sharp, but when $n = 2$, it was conjectured in [DFJ12], for reasons that are not entirely clear, and proven in several families that the inequality in (1) becomes an equality [AD20]. As Agol–Dunfield state, this is a remarkable Thurstonian connection between the topology and geometry of 3-manifolds.

1.3. **1-loop polynomial from super-Ptolemy coordinates.** Our paper is motivated by the following problem, which asks about the relation of torsion polynomials with ideal triangulations, another development by Thurston [Thu97].

Problem 1. Can one compute the $\mathbb{C}^n$-torsion polynomial $\tau_{M,\rho,n}(t)$ from an ideal triangulation $\mathcal{T}$ of M?

In this paper, we address the problem for $n = 2$, which is the most interesting case as explained in the previous section. Our answer begins by introducing super-Ptolemy assignments for $\mathcal{T}$, which parametrize $\mathrm{OSp}_{2|1}(\mathbb{C})$-representations of $\pi_1(M)$. It may seem unrelated to the problem, but we bulid an unexpected connection as follows. Super-Ptolemy assignments lead new combinatorics of an ideal triangulation beyond the well-known Neumann–Zagier matrices. This newly found combinatorics involves linear equations associated to faces and tetrahedra. Using these linear equations, we introduce a new 1-loop polynomial and conjecture that our 1-loop polynomial equals to the $\mathbb{C}^2$-torsion polynomial, thereby giving a (conjectural) solution to the problem for $n = 2$. In subsequent work with Dunfield, this conjecture is proven for all fibered manifolds and numerically verified for thousands of non-fibered manifolds [DGY].

Before delving into the details, we briefly outline the construction of our 1-loop polynomial from super-Ptolemy assignments in dimension 3, postponing precise definitions, notations and properties for later.

A Ptolemy assignment c assigns a nonzero complex number to each edge of $\mathcal{T}$ satisfying the equation

$$c_{01}c_{23} - c_{02}c_{13} + c_{03}c_{12} = 0 \tag{2}$$

for each tetrahedron, where $c_{ij} := c(e_{ij})$ and e_{ij} is the (i, j)-edge of a tetrahedron [GTZ15]. On the other hand, a super-Ptolemy assignment is a pair of assignments (c, θ) that assign an invertible even element of a Grassmann algebra to each edge and an odd element to each face. Instead of Equation (2), a super-Ptolemy assignment satisfies one equation

$$c_{01}c_{23} - c_{02}c_{13} + c_{03}c_{12} + c_{01}c_{03}c_{12}c_{13}c_{23}\theta_0\theta_2 = 0 \tag{3}$$

for each tetrahedron as well as one equation for each face

$$\begin{aligned} c_{12}\theta_0 - c_{02}\theta_1 + c_{01}\theta_2 &= 0 \qquad & c_{13}\theta_0 - c_{03}\theta_1 + c_{01}\theta_3 &= 0 \\ c_{23}\theta_0 - c_{03}\theta_2 + c_{02}\theta_3 &= 0 \qquad & c_{23}\theta_1 - c_{13}\theta_2 + c_{12}\theta_3 &= 0 \end{aligned} \tag{4}$$

of each tetrahedron. Here $c_{ij} := c(e_{ij})$ and $\theta_k := \theta(f_k)$ where e_{ij} is the edge (i, j) and f_k is the face opposite to the vertex k as in Figure 3. Super-Ptolemy assignments lead to a fundamental correspondence described by a pair of bijections

$$\left\{ \begin{matrix} \text{Generically decorated} \\ (\mathrm{OSp}_{2|1}(\mathbb{C}), N)\text{-reps on } M \end{matrix} \right\} \xleftrightarrow{1-1} P_{2|1}(\mathcal{T}) \xleftrightarrow{1-1} \left\{ \begin{matrix} \text{Natural } (\mathrm{OSp}_{2|1}(\mathbb{C}), N)\text{-} \\ \text{cocycles on } \mathring{\mathcal{T}} \end{matrix} \right\} \tag{5}$$

which are given explicitly in Figures 1 and 2; see Section 2 for details.

As written in (4), a super-Ptolemy assignment (c, θ) satisfies linear equations in θ. It turns out that these linear equations can be written in a matrix form

$$F_c\,\theta = 0 \tag{6}$$

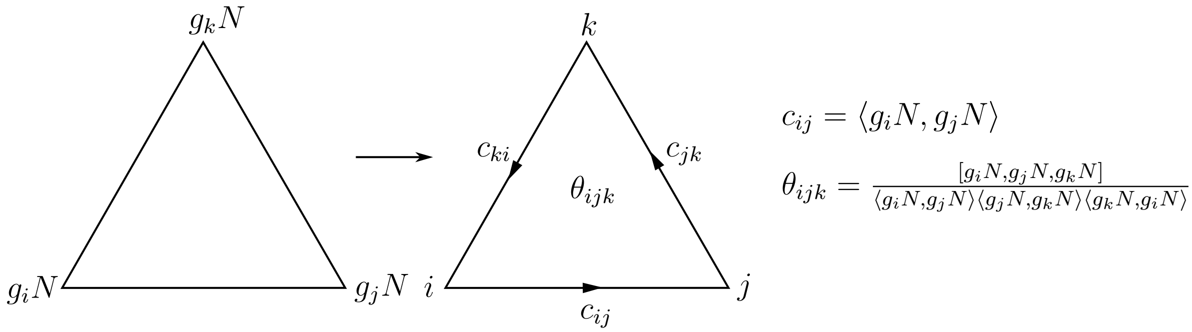

FIGURE 1. From decorated representations to Ptolemy assignments, with the bilinear and trilinear forms as in (26) and (27).

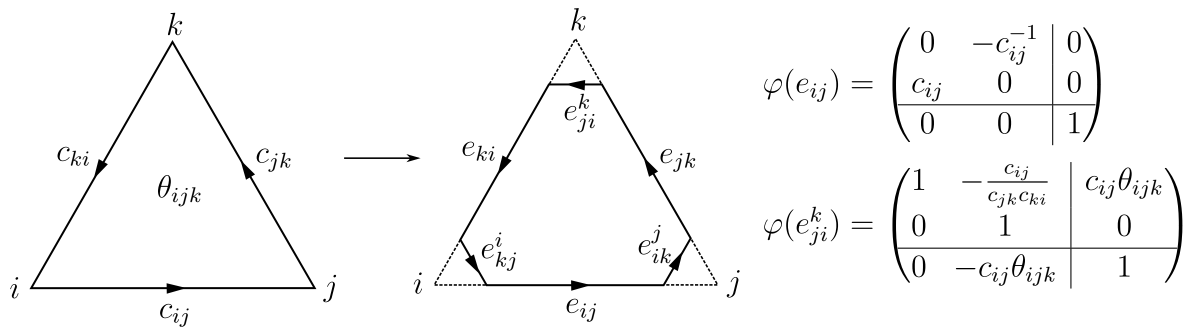

FIGURE 2. From Ptolemy assignments to natural cocycles.

where F_c is a sqaure matrix whose entries are given by the Ptolemy variable c with some signs. Obviously, we are interested in the case of F_c being singular; otherwise, θ should be trivial. This motivates the definition of a 1-loop invariant

$$\delta_{\mathcal{T},c,2} = \left(\prod_e \frac{1}{c(e)} \prod_\Delta \frac{1}{c(e_\Delta)}\right) \det F_c \tag{7}$$

given in terms of the determinant of F_c with some normalizations multified. What's more, it motivates the definition of a 1-loop polynomial

$$\delta_{\mathcal{T},c,2}(t) = \left(\prod_e \frac{1}{c(e)} \prod_\Delta \frac{1}{c(e_\Delta)}\right) \det F_c(t) \tag{8}$$

given in terms of the determinant of a t-twisted version $F_c(t)$ of F_c; see Section 3 for details. Regarding the 1-loop polynomial $\delta_{\mathcal{T},c,2}(t)$, we will prove two important features:

- it is unchanged under Pachner 2–3 moves (see Theorem 3.4);
- its value $\delta_{\mathcal{T},c,2}(1)$ at $t = 1$ determines whether the $\mathrm{SL}_2(\mathbb{C})$-representation of $\pi_1(M)$ corresponding to the Ptolemy assignment c admits an $\mathrm{OSp}_{2|1}(\mathbb{C})$-lift or not (see Theorem 3.6).

Furthermore, based on the analogy of our previous work [GY23], we conjecture that the 1-loop polynomial $\delta_{\mathcal{T},c,2}(t)$ equals to the $\mathbb{C}^2$-torsion polynomial $\tau_{M,\rho,2}(t)$ (see Conjecture 3.7).

2. $\mathrm{OSp}_{2|1}(\mathbb{C})$-REPRESENTATIONS OF 3-MANIFOLDS

2.1. **The orthosymplectic group.** In this section, we recall the definition of the orthosymplectic group $\mathrm{OSp}_{2|1}(\mathbb{C})$. For a detailed description of super-manifolds and super-Lie groups we refer the reader to [Ber87, Man91, CR88].

Let $\mathrm{G}(\mathbb{C})$ be the Grassmann algebra over the complex numbers with unit 1 generated by ϵ_i for $i \in \mathbb{N}$:

$$\mathrm{G}(\mathbb{C}) = \mathbb{C}\langle 1, \epsilon_1, \epsilon_2, \dots \mid 1\epsilon_i = \epsilon_i = \epsilon_i 1,\ \epsilon_i\epsilon_j = -\epsilon_j\epsilon_i \text{ for all } i, j \in \mathbb{N}\rangle\,. \tag{9}$$

It is a $\mathbb{Z}/2\mathbb{Z}$-graded algebra with the unit having degree 0 and each ϵ_i having degree 1. We denote by $\mathrm{G}_0(\mathbb{C})$ and $\mathrm{G}_1(\mathbb{C})$ its even and odd part, respectively, and by $\mathrm{G}_0^*(\mathbb{C})$ the set of invertible elements in $\mathrm{G}_0(\mathbb{C})$. There is an algebra epimorphism $\sharp : \mathrm{G}(\mathbb{C}) \to \mathbb{C}$ sending all ϵ_i to 0, hence an element $e \in \mathrm{G}(\mathbb{C})$ is invertible if and only if $\sharp(e) \neq 0$. We write $\sharp(e)$ simply as $e^\sharp$ and call it the body of e.

An even $n|m \times n|m$-matrix g is of the form

$$g = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \tag{10}$$

where $A \in M_{n,n}(\mathrm{G}_0(\mathbb{C}))$, $B \in M_{n,m}(\mathrm{G}_1(\mathbb{C}))$, $C \in M_{m,n}(\mathrm{G}_1(\mathbb{C}))$, and $D \in M_{m,m}(\mathrm{G}_0(\mathbb{C}))$. The super-transpose of g is given by

$$g^{\mathrm{st}} = \left(\begin{array}{c|c} A^t & C^t \\ \hline -B^t & D^t \end{array}\right) \tag{11}$$

and the Berezinian (or super-determinant) of g is defined as

$$\mathrm{Ber}(g) = \det(A - BD^{-1}C)\det(D)^{-1} \tag{12}$$

provided that A and D are invertible.

The orthosymplectic group $\mathrm{OSp}_{2|1}(\mathbb{C})$ is the group of even $2|1 \times 2|1$-matrices g satisfying

$$g^{st}\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & -1 \end{array}\right) g = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & -1 \end{array}\right) \tag{13}$$

and $\mathrm{Ber}(g) = 1$. Writing an even $2|1 \times 2|1$-matrix explicitly as

$$g = \left(\begin{array}{cc|c} a & b & \alpha \\ c & d & \beta \\ \hline \gamma & \delta & e \end{array}\right) \quad \text{for} \quad a, b, c, d, e \in \mathrm{G}_0(\mathbb{C}), \quad \alpha, \beta, \gamma, \delta \in \mathrm{G}_1(\mathbb{C}), \tag{14}$$

the defining equations (13) of $\mathrm{OSp}_{2|1}(\mathbb{C})$ are

$$ad - bc - \gamma\delta = e^2 + 2\alpha\beta = 1, \qquad a\beta - c\alpha - e\gamma = b\beta - d\alpha - e\delta = 0 \tag{15}$$

together with

$$\mathrm{Ber}(g) = (ad - bc)(1 - 2\alpha\beta e^{-2})e^{-1} = 1\,. \tag{16}$$

Note that these equations imply that $e^{\pm 1} = 1 \mp \gamma\delta$; in particular, $e^\sharp = 1$. Note also that the inverse of g as in (14) is given by

$$g^{-1} = \left(\begin{array}{cc|c} d & -b & \delta \\ -c & a & -\gamma \\ \hline -\beta & \alpha & e \end{array}\right). \tag{17}$$

The special linear group $\mathrm{SL}_2(\mathbb{C})$ embeds in $\mathrm{OSp}_{2|1}(\mathbb{C})$ in an obvious way:

$$\mathrm{SL}_2(\mathbb{C}) \hookrightarrow \mathrm{OSp}_{2|1}(\mathbb{C}), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(\begin{array}{cc|c} a & b & 0 \\ c & d & 0 \\ \hline 0 & 0 & 1 \end{array}\right). \tag{18}$$

Conversely, applying the epimorphism $\sharp$ entrywise, we obtain an epimorphism

$$\mathrm{OSp}_{2|1}(\mathbb{C}) \twoheadrightarrow \mathrm{SL}_2(\mathbb{C}), \qquad \left(\begin{array}{cc|c} a & b & \alpha \\ c & d & \beta \\ \hline \gamma & \delta & e \end{array}\right) \mapsto \begin{pmatrix} a^\sharp & b^\sharp \\ c^\sharp & d^\sharp \end{pmatrix}. \tag{19}$$

Abusing notation, we also denote by $\sharp$ the above epimorphism and refer to $\sharp(g) = g^\sharp$ as the body of $g \in \mathrm{OSp}_{2|1}(\mathbb{C})$. It follows from (18) and (19) that for any group G the epimorphism $\sharp$ induces a surjective map

$$\mathrm{Hom}(G, \mathrm{OSp}_{2|1}(\mathbb{C}))/_\sim \overset{\sharp}{\twoheadrightarrow} \mathrm{Hom}(G, \mathrm{SL}_2(\mathbb{C}))/_\sim \tag{20}$$

where the quotient $\sim$ is given by conjugation.

Remark 2.1. For full generality we use the Grassmann algebra with infinitely many generators as in (9), but one may use one with finitely many generators. In particular, if we use the Grassmann algebra with one odd generator

$$\mathbb{C}\langle 1, \epsilon \,|\, 1\epsilon = \epsilon = \epsilon 1,\, \epsilon^2 = 0\rangle \;= \mathbb{C}[\epsilon]/(\epsilon^2), \tag{21}$$

then its even and odd parts are $\mathbb{C}$ and $\mathbb{C}\epsilon$, respectively, and the orthosymplectic group $\mathrm{OSp}_{2|1}(\mathbb{C})$ reduces to the special affine transformation group $\mathrm{SL}_2(\mathbb{C}) \ltimes \mathbb{C}^2$. Indeed, for the case of one odd generator, the map

$$\mathrm{OSp}_{2|1}(\mathbb{C}) \to \mathrm{SL}_2(\mathbb{C}) \ltimes \mathbb{C}^2, \qquad \left(\begin{array}{cc|c} a & b & \alpha\epsilon \\ c & d & \beta\epsilon \\ \hline \gamma\epsilon & \delta\epsilon & 1 \end{array}\right) \mapsto \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right) \tag{22}$$

is an isomorphism. This may seem to simplify things too much but, in fact, will be sufficient for our 1-loop invariants–see Section 3 below.

2.2. **The unipotent subgroup and pairings.** Since the body $g^\sharp$ of $g \in \mathrm{OSp}_{2|1}(\mathbb{C})$ is in $\mathrm{SL}_2(\mathbb{C})$, the natural action of $\mathrm{OSp}_{2|1}(\mathbb{C})$ on $\mathrm{G}_0(\mathbb{C})^2 \oplus \mathrm{G}_1(\mathbb{C})$ restricts to an action on $A^{2|1}$, the pre-image of $\mathbb{C}^2 \setminus \{(0,0)^t\}$ under the map

$$\mathrm{G}_0(\mathbb{C})^2 \oplus \mathrm{G}_1(\mathbb{C}) \to \mathbb{C}^2, \quad (a, b, \alpha)^t \mapsto (a^\sharp, b^\sharp)^t. \tag{23}$$

The induced action is transitive, and the stabilizer group at $(1,0,0)^t \in A^{2|1}$ is the unipotent subgroup of $\mathrm{OSp}_{2|1}(\mathbb{C})$:

$$N = \left\{ \left(\begin{array}{cc|c} 1 & b & \alpha \\ 0 & 1 & 0 \\ \hline 0 & -\alpha & 1 \end{array}\right) \;\middle|\; b \in \mathrm{G}_0(\mathbb{C}),\, \alpha \in \mathrm{G}_1(\mathbb{C}) \right\} . \tag{24}$$

This induces a bijection

$$\mathrm{OSp}_{2|1}(\mathbb{C})/N \;\leftrightarrow\; A^{2|1}, \qquad gN \;\leftrightarrow\; \text{left column of } g\,. \tag{25}$$

The space $A^{2|1} \simeq \mathrm{OSp}_{2|1}(\mathbb{C})/N$ comes equipped with an even-valued bilinear pairing

$$\langle \cdot,\cdot \rangle : A^{2|1} \times A^{2|1} \to \mathrm{G}_0(\mathbb{C}), \qquad \left\langle \begin{pmatrix} a \\ b \\ \alpha \end{pmatrix}, \begin{pmatrix} c \\ d \\ \beta \end{pmatrix} \right\rangle := ad - bc - \alpha\beta \tag{26}$$

and an odd-valued trilinear pairing

$$[\cdot,\cdot,\cdot] : A^{2|1} \times A^{2|1} \times A^{2|1} \to \mathrm{G}_1(\mathbb{C}), \quad \left[\begin{pmatrix} a \\ b \\ \alpha \end{pmatrix}, \begin{pmatrix} b \\ c \\ \beta \end{pmatrix}, \begin{pmatrix} e \\ f \\ \gamma \end{pmatrix} \right] := \det \begin{pmatrix} a & b & e \\ b & c & f \\ \alpha & \beta & \gamma \end{pmatrix} - 2\alpha\beta\gamma\,. \tag{27}$$

Both pairings are skew-symmetric and $\mathrm{OSp}_{2|1}(\mathbb{C})$-invariant

$$\langle v, w \rangle = \langle gv, gw \rangle, \qquad [u,v,w] = [gu,gv,gw], \qquad g \in \mathrm{OSp}_{2|1}(\mathbb{C})\,. \tag{28}$$

2.3. **Super-Ptolemy assignments.** Let M be a compact 3-manifold with non-empty boundary and $\mathcal{T}$ be an ideal triangulation of its interior. We denote by $\mathcal{T}^1$ and $\mathcal{T}^2$ the sets of oriented edges and unoriented faces of $\mathcal{T}$, respectively. We first consider the case when $\mathcal{T}$ is ordered, i.e., each tetrahedron of $\mathcal{T}$ has a vertex-ordering respecting the face-gluing. This condition will be relaxed in Section 2.8.

Definition 2.2. A *super-Ptolemy assignment* on $\mathcal{T}$ is a pair of maps

$$c : \mathcal{T}^1 \to \mathrm{G}_0^*(\mathbb{C}), \qquad \theta : \mathcal{T}^2 \to \mathrm{G}_1(\mathbb{C}) \tag{29}$$

satisfying $c(-e) = -c(e)$ for all $e \in \mathcal{T}^1$ and

$$c_{01}c_{23} - c_{02}c_{13} + c_{03}c_{12} + c_{01}c_{03}c_{12}c_{13}c_{23}\theta_0\theta_2 = 0 \tag{30}$$

as well as

$$\begin{array}{llll} E_{\Delta,f_3}: & c_{12}\theta_0 - c_{02}\theta_1 + c_{01}\theta_2 = 0 & \qquad E_{\Delta,f_2}: & c_{13}\theta_0 - c_{03}\theta_1 + c_{01}\theta_3 = 0 \\ E_{\Delta,f_1}: & c_{23}\theta_0 - c_{03}\theta_2 + c_{02}\theta_3 = 0 & \qquad E_{\Delta,f_0}: & c_{23}\theta_1 - c_{13}\theta_2 + c_{12}\theta_3 = 0 \end{array} \tag{31}$$

for each tetrahedron Δ of $\mathcal{T}$. Here $c_{ij} = c(e_{ij})$ where e_{ij} is the oriented edge $[i,j]$ of Δ and $\theta_k = \theta(f_k)$ where f_k is the face of Δ opposite to the vertex k as in Figure 3.

Lemma 2.3. If any two of (31) together with (30) are satisfied, then so are the other two.

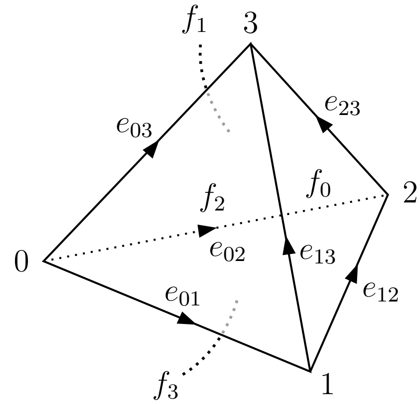

FIGURE 3. Edge and face labels for a tetrahedron.

Proof. Multiplying the first equation in (31) by θ_0 implies that $c_{01}\theta_0\theta_2 = c_{02}\theta_0\theta_1$. Similarly, we deduce that $c_{ij}^{-1}\theta_i\theta_j$ does not depend on a choice of $i \neq j$. It follows that Equation (30) is equivalent to

$$c_{01}c_{23} - c_{02}c_{13} + c_{03}c_{12} + c_{01}c_{02}c_{03}c_{12}c_{13}c_{23}\, c_{ij}^{-1}\theta_i\theta_j = 0 \quad \text{for } i \neq j. \tag{32}$$

This implies that $(c_{01}c_{23} - c_{02}c_{13} + c_{03}c_{12})\theta_i = 0$ for all i. Then one easily checks that any three out of the four equations in (31) are linearly dependent. For instance,

$$c_{01}E_{\Delta,f_1} - c_{02}E_{\Delta,f_2} + c_{03}E_{\Delta,f_3} = 0\,. \tag{33}$$

This completes the proof. □

Remark 2.4. Each equation in (31) corresponds to a face of a tetrahedron Δ. It follows that

$$\begin{aligned} E_{\Delta,f_0} &= 0 \\ -E_{\Delta,f_1} &= 0 \\ E_{\Delta,f_2} &= 0 \\ -E_{\Delta,f_3} &= 0 \end{aligned} \;\Leftrightarrow\; F_{\Delta,c}\begin{pmatrix}\theta_0\\ \theta_1\\ \theta_2\\ \theta_3\end{pmatrix} = 0 \tag{34}$$

where $F_{\Delta,c}$ is a 4×4 matrix whose rows and columns are indexed by the faces of Δ, given explicitly by

$$F_{\Delta,c} = \begin{pmatrix} 0 & c_{23} & -c_{13} & c_{12} \\ -c_{23} & 0 & c_{03} & -c_{02} \\ c_{13} & -c_{03} & 0 & c_{01} \\ -c_{12} & c_{02} & -c_{01} & 0 \end{pmatrix}. \tag{35}$$

Note that $F_{\Delta,c}$ is a skew-symmetric matrix whose (i,j)-entry for $i \neq j$ is, up to a sign, the Ptolemy variable of the edge $f_i \cap f_j$. Note also that $F_{\Delta,c}$ has rank 3, whereas its body $F^\sharp_{\Delta,c}$ (the matrix obtained by applying the epimorphism $\sharp$ to all entries of $F_{\Delta,c}$) has rank 2.

Let $P_{2|1}(\mathcal{T})$ be the set of all super-Ptolemy assignments on $\mathcal{T}$. Composing the epimorphism $\sharp : \mathrm{G}(\mathbb{C}) \to \mathbb{C}$ with a super-Ptolemy assignment (c,θ), we obtain a Ptolemy assignment $c^\sharp : \mathcal{T}^1 \to \mathbb{C}^*$ (note that θ vanishes if we apply $\sharp$). That is, $c^\sharp$ satisfies

$$c^\sharp_{01}c^\sharp_{23} - c^\sharp_{02}c^\sharp_{13} + c^\sharp_{03}c^\sharp_{12} = 0 \tag{36}$$

for each tetrahedron of $\mathcal{T}$ [GTZ15]. On the other hand, any Ptolemy assignment on $\mathcal{T}$ forms a super-Ptolemy assignment with the trivial map $\mathcal{T}^2 \to \mathrm{G}_1(\mathbb{C})$, assigning 0 to all faces. Therefore, the epimorphism $\sharp$ induces a surjective map

$$P_{2|1}(\mathcal{T}) \overset{\sharp}{\twoheadrightarrow} P_2(\mathcal{T}) \tag{37}$$

where $P_2(\mathcal{T})$ is the set of all Ptolemy assignments on $\mathcal{T}$.

2.4. **Natural cocycles.** After truncating the ideal tetrahedra of $\mathcal{T}$, one obtains a cell decomposition $\mathring{\mathcal{T}}$ of M; see Figure 4. We denote by $\mathring{\mathcal{T}}^1$ and $\mathring{\mathcal{T}}^2$ the sets of oriented edges and unoriented faces of $\mathring{\mathcal{T}}$, respectively. An edge of $\mathring{\mathcal{T}}$ is either long or short, and a face is either hexagal or triangular; short edges are ones in the boundary of a triangular face. The triangles of $\mathring{\mathcal{T}}$ form a triangulation of the boundary ∂M of M, and the 2-skeleton of $\mathring{\mathcal{T}}$ defines a natural groupoid associated to $\mathcal{T}$, whose generators are the edges of $\mathring{\mathcal{T}}$ and relations are the faces of $\mathring{\mathcal{T}}$.

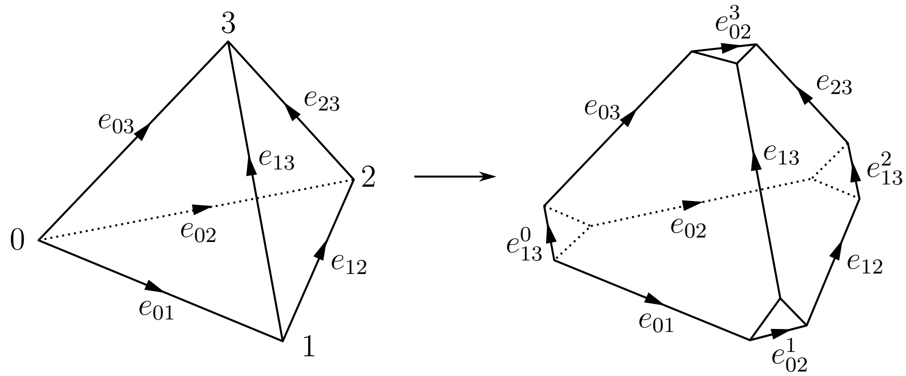

FIGURE 4. Truncating an ideal tetrahedron.

Definition 2.5. A *natural* $(\mathrm{OSp}_{2|1}(\mathbb{C}), N)$*-cocycle*, or simply *natural cocycle*, on $\mathring{\mathcal{T}}$ is a map $\varphi : \mathring{\mathcal{T}}^1 \to \mathrm{OSp}_{2|1}(\mathbb{C})$ of the form

$$\varphi(\text{short}) = \left(\begin{array}{cc|c} 1 & a & \theta \\ 0 & 1 & 0 \\ \hline 0 & -\theta & 1 \end{array}\right), \qquad \varphi(\text{long}) = \left(\begin{array}{cc|c} 0 & -b^{-1} & 0 \\ b & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \tag{38}$$

that maps the hexagons and the triangles to the identity. In other words, a natural cocycle is an $\mathrm{OSp}_{2|1}(\mathbb{C})$-representation of the groupoid of $\mathcal{T}$ whose generators have the form (38).

Given a natural cocycle φ, let us denote

$$\begin{aligned} &\varphi_0(\text{short}) := \varphi(\text{short})_{1,2} \in \mathrm{G}_0(\mathbb{C}), \qquad \varphi_0(\text{long}) := \varphi(\text{long})_{2,1} \in \mathrm{G}_0^*(\mathbb{C}), \\ &\varphi_1(\text{short}) := \varphi(\text{short})_{1,3} \in \mathrm{G}_1(\mathbb{C})\,. \end{aligned}$$

We now express the cocycle condition for φ explicitly in terms of φ_0 and φ_1. Note that $\varphi(-e) = \varphi(e)^{-1}$ for $e \in \mathring{\mathcal{T}}^1$ if and only if $\varphi_i(-e) = -\varphi_i(e)$ for $i = 0, 1$; see Equation (17).

Lemma 2.6. φ satisfies the cocycle condition for a hexagon if and only if

$$\varphi_0(e_{ji}^k) = -\frac{\varphi_0(e_{ij})}{\varphi_0(e_{jk})\varphi_0(e_{ki})} \tag{39}$$

for all cyclic permutations (i,j,k) of $(0,1,2)$ and

$$\frac{\varphi_1(e_{10}^2)}{\varphi_0(e_{01})} = \frac{\varphi_1(e_{21}^0)}{\varphi_0(e_{12})} = \frac{\varphi_1(e_{02}^1)}{\varphi_0(e_{20})} \tag{40}$$

where e_{ij} and e_{ij}^k denote the edges of the hexagon as in Figure 5.

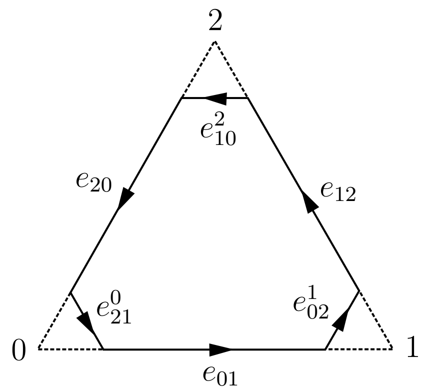

FIGURE 5. A hexagonal face.

Proof. The proof follows from a straightforward computation for the cocycle condition, i.e., comparing the entries of $\varphi(e_{21}^0)\varphi(e_{01})\varphi(e_{02}^1) = \varphi(e_{20})^{-1}\varphi(e_{10}^2)^{-1}\varphi(e_{12})^{-1}$. □

Equation (40) is an equality of odd elements, and we denote its value by $\theta \in \mathrm{G}_1(\mathbb{C})$. Lemma 2.6 shows that the φ_0 and φ_1-values on the short edges are determined by the φ_0-values on the long edges, together with θ. Precisely, we have

$$\varphi(e_{ji}^k) = \left(\begin{array}{cc|c} 1 & -\frac{\varphi_0(e_{ij})}{\varphi_0(e_{jk})\varphi_0(e_{ki})} & \varphi_0(e_{ij})\theta \\ 0 & 1 & 0 \\ \hline 0 & -\varphi_0(e_{ij})\theta & 1 \end{array}\right) \tag{41}$$

for any cyclic permutations (i,j,k) of $(0,1,2)$. Identifying each long edge of $\mathring{\mathcal{T}}$ with an edge of $\mathcal{T}$ and placing the odd element θ to the corresponding hexagonal face of $\mathring{\mathcal{T}}$, or equivalently, to the corresponding face of $\mathcal{T}$, we deduce that a natural cocycle φ is determined by two maps

$$c : \mathcal{T}^1 \to \mathrm{G}_0^*(\mathbb{C}), \quad \theta : \mathcal{T}^2 \to \mathrm{G}_1(\mathbb{C})$$

where c is the restriction of φ_0 to the long edges.

Lemma 2.7. φ satisfies the cocycle condition for all triangular faces of $\mathring{\mathcal{T}}$ if and only if the pair (c,θ) defined above is a super-Ptolemy assignment, i.e., satisfies (30) and (31) for all tetrahedra of $\mathcal{T}$.

Proof. Labeling the vertices of a tetrahedron with $\{0,1,2,3\}$ and using the same notation as in Lemma 2.6, the cocycle condition for the triangular faces has the form $\varphi(e^i_{jl}) = \varphi(e^i_{jk})\varphi(e^i_{kl})$. For instance, the triangular face near the vertex 0 gives $\varphi(e^0_{31}) = \varphi(e^0_{32})\varphi(e^0_{21})$:

$$\left(\begin{array}{cc|c} 1 & -\frac{c_{13}}{c_{30}c_{01}} & c_{13}\theta_2 \\ 0 & 1 & 0 \\ \hline 0 & -c_{13}\theta_2 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 1 & -\frac{c_{23}}{c_{30}c_{02}} & c_{23}\theta_1 \\ 0 & 1 & 0 \\ \hline 0 & -c_{23}\theta_1 & 1 \end{array}\right) \left(\begin{array}{cc|c} 1 & -\frac{c_{12}}{c_{20}c_{01}} & c_{12}\theta_3 \\ 0 & 1 & 0 \\ \hline 0 & -c_{12}\theta_3 & 1 \end{array}\right). \tag{42}$$

Comparing the entries of the above equation, we obtain (30) and the first equation in (31). We obtain the other three equations of (31) similarly from the other triangular faces. $\square$

Lemmas 2.6 and 2.7 imply a one-to-one correspondence

$$P_{2|1}(\mathcal{T}) \xleftrightarrow{1-1} \left\{\begin{matrix}\text{Natural } (\mathrm{OSp}_{2|1}(\mathbb{C}), N)\text{-} \\ \text{cocycles on } \mathring{\mathcal{T}}\end{matrix}\right\}, \tag{43}$$

the second bijection in the fundamental correspondence (5), where its explicit formula is summarized in Figure 2.

Remark 2.8. Applying the epimorphism $\sharp : \mathrm{OSp}_{2|1}(\mathbb{C}) \to \mathrm{SL}_2(\mathbb{C})$ to both sides of (43), the correspondence (43) reduces to the bijection between $P_2(\mathcal{T})$ and the set of natural $(\mathrm{SL}_2(\mathbb{C}), N_2)$-cocycles [GGZ15b, Sec.1.2]. Here N_2 is the set of unipotent matrices in $\mathrm{SL}_2(\mathbb{C})$. Namely, there is a commutative diagram

$$\begin{tikzcd} P_{2|1}(\mathcal{T}) \arrow[r, leftrightarrow, "1-1"] \arrow[d, two heads, "\sharp"] & \left\{\begin{matrix}\text{Natural } (\mathrm{OSp}_{2|1}(\mathbb{C}), N)\text{-} \\ \text{cocycles on } \mathring{\mathcal{T}}\end{matrix}\right\} \arrow[d, two heads, "\sharp"] \\ P_2(\mathcal{T}) \arrow[r, leftrightarrow, "1-1"] & \left\{\begin{matrix}\text{Natural } (\mathrm{SL}_2(\mathbb{C}), N_2)\text{-} \\ \text{cocycles on } \mathring{\mathcal{T}}\end{matrix}\right\} \end{tikzcd} \tag{44}$$

2.5. **Decorations.** Let $\widetilde{M}$ be the universal cover of M and $\widetilde{\mathcal{T}}$ be the ideal triangulation of its interior induced from $\mathcal{T}$. We denote by $\widetilde{\mathcal{T}}^0$ the set of (ideal) vertices of $\widetilde{\mathcal{T}}$ and use similar notations for $\mathring{\mathcal{T}}$.

Definition 2.9. (a) An $(\mathrm{OSp}_{2|1}(\mathbb{C}), N)$-*representation* is an $\mathrm{OSp}_{2|1}(\mathbb{C})$-representation ρ of $\pi_1(M)$ such that $\rho(\pi_1(\partial M))$ lies in the unipotent subgroup N up to conjugation.
(b) A *decoration* of an $(\mathrm{OSp}_{2|1}(\mathbb{C}), N)$-representation ρ is a map

$$D : \widetilde{\mathcal{T}}^0 \to \mathrm{OSp}_{2|1}(\mathbb{C})/N \tag{45}$$

such that $D(\gamma \cdot v) = \rho(\gamma)D(v)$ for $\gamma \in \pi_1(M)$ and $v \in \widetilde{\mathcal{T}}^0$. We say that a decoration is *generic* if for all vertices v_0 and v_1 joined by an edge of $\widetilde{\mathcal{T}}$, we have

$$\langle D(v_0), D(v_1)\rangle^\sharp \neq 0\,. \tag{46}$$

Here we use the identification (25) and the bilinear pairing (26).

In what follows, by a generically decorated representation we mean an $(\mathrm{OSp}_{2|1}(\mathbb{C}), N)$-representation with a generic decoration. For simplicity we identify a generically decorated representation (ρ, D) with $(g\rho g^{-1}, gD)$ for all $g \in \mathrm{OSp}_{2|1}(\mathbb{C})$. Note that if D is a generic decoration of ρ, then gD is a generic decoration of $g\rho g^{-1}$.

Lemma 2.10. For N-cosets gN and hN with $\langle gN, hN\rangle^\sharp \neq 0$, there is a unique pair of coset-representatives $g' \in gN$ and $h' \in hN$ such that $(g')^{-1}h'$ is of the form

$$(g')^{-1}h' = \left(\begin{array}{cc|c} 0 & -c^{-1} & 0 \\ c & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) . \tag{47}$$

Moreover, $c = \langle gN, hN\rangle$.

Proof. We may assume that $gN = N$ and hN corresponds to $(a, c, \gamma)^t \in A^{2|1}$ with $c^\sharp \neq 0$. Then a straightforward computation

$$\left(\begin{array}{cc|c} 1 & f & \epsilon \\ 0 & 1 & 0 \\ \hline 0 & -\epsilon & 1 \end{array}\right)^{-1} \left(\begin{array}{cc|c} a & b & \alpha \\ c & d & \beta \\ \hline \gamma & \delta & e \end{array}\right) = \left(\begin{array}{cc|c} a - cf - \epsilon\gamma & b - df - \epsilon\delta & \alpha - e\epsilon \\ c & d & \beta \\ \hline c\epsilon + \gamma & d\epsilon + \delta & \epsilon\beta + e \end{array}\right) \tag{48}$$

shows that the right-hand side has the form (47) only if $\epsilon = -\gamma/c$, $f = a/c$ and $d = \beta = 0$. Then it follows from the defining equations of $\mathrm{OSp}_{2|1}(\mathbb{C})$ that $\delta = 0$, $e = 1$, $b = -1/c$ and $\alpha = -\gamma/c$. This proves that the desired pair of coset-representatives exists and is unique. □

For a generically decorated representation (ρ, D), Lemma 2.10 implies that there is a unique map

$$\psi : \tilde{\mathring{\mathcal{T}}}^0 \to \mathrm{OSp}_{2|1}(\mathbb{C}) \tag{49}$$

such that

- $\psi(v) \in D(w)$ if v is in the boundary component of $\widetilde{M}$ corresponding to $w \in \widetilde{\mathcal{T}}^0$;
- $\psi(v_0)^{-1}\psi(v_1)$ is a matrix of the form (47) if v_0 and v_1 are joined by a long edge.

From the definition of a decoration, we have $\psi(\gamma \cdot v) = \rho(\gamma)\psi(v)$ for $\gamma \in \pi_1(M)$, hence

$$\psi(\gamma \cdot v_0)^{-1}\psi(\gamma \cdot v_1) = \psi(v_0)^{-1}\psi(v_1) \tag{50}$$

for any vertices v_0 and v_1. Therefore, if we define

$$\varphi : \mathring{\mathcal{T}}^1 \to \mathrm{OSp}_{2|1}(\mathbb{C}), \quad \varphi(e) := \psi(v_0)^{-1}\psi(v_1) \tag{51}$$

for any lift $[v_0, v_1]$ of an edge $e \in \mathring{\mathcal{T}}^1$, then φ is well-defined and by definition is a natural cocycle. This construction induces a one-to-one correspondence

$$\left\{\begin{array}{c} \text{Natural } (\mathrm{OSp}_{2|1}(\mathbb{C}), N)\text{-} \\ \text{cocycles on } \mathring{\mathcal{T}} \end{array}\right\} \xleftrightarrow{1-1} \left\{\begin{array}{c} \text{Generically decorated} \\ (\mathrm{OSp}_{2|1}(\mathbb{C}), N)\text{-reps on } M \end{array}\right\} . \tag{52}$$

Remark 2.11. Applying the epimoprhism $\sharp : \mathrm{OSp}_{2|1}(\mathbb{C}) \to \mathrm{SL}_2(\mathbb{C})$ to both sides of (52), the correspondence (52) reduces to the bijection between natural $(\mathrm{SL}_2(\mathbb{C}), N_2)$-cocycles and generically decorated $(\mathrm{SL}_2(\mathbb{C}), N_2)$-representations; see [GGZ15b, Sec 1.2].

Combining the correspondences (43) and (52), we obtain

$$\left\{\begin{matrix}\text{Generically decorated}\\ (\mathrm{OSp}_{2|1}(\mathbb{C}),N)\text{-reps on } M\end{matrix}\right\} \xleftrightarrow{1-1} P_{2|1}(\mathcal{T}) \ , \tag{53}$$

the first bijection in the fundamental correspondence (5), where its explicit formula is summarized in Figure 1:

$$c(e_{ij}) = \langle g_iN, g_jN\rangle, \qquad \theta(f_{ijk}) = \frac{[g_iN, g_jN, g_kN]}{\langle g_iN, g_jN\rangle\langle g_jN, g_kN\rangle\langle g_kN, g_iN\rangle}\,. \tag{54}$$

Here we use the identification (25) and the bilinear and trilinear pairings (26) and (27).

The next theorem is a direct consequence of the fundamental correspondence (5) (see also the diagram (44)).

Theorem 2.12. *There is a map* $P_{2|1}(\mathcal{T}) \to \mathrm{Hom}(\pi_1(M), \mathrm{OSp}_{2|1}(\mathbb{C}))/_\sim$ *which fits into a commutative diagram*

$$\begin{tikzcd} P_{2|1}(\mathcal{T}) \arrow[r] \arrow[d, "\sharp"] & \mathrm{Hom}(\pi_1(M), \mathrm{OSp}_{2|1}(\mathbb{C}))/_\sim \arrow[d, "\sharp"] \\ P_2(\mathcal{T}) \arrow[r] & \mathrm{Hom}(\pi_1(M), \mathrm{SL}_2(\mathbb{C}))/_\sim \end{tikzcd} \tag{55}$$

and whose image is the set of all conjugacy classes of $(\mathrm{OSp}_{2|1}(\mathbb{C}), N)$*-representations admitting a generic decoration.*

2.6. **Action on** $P_{2|1}(\mathcal{T})$**.** Let h be the the number of (ideal) vertices of $\mathcal{T}$. There is an action of $\mathrm{G}_0^*(\mathbb{C})^h$ on $P_{2|1}(\mathcal{T})$

$$\mathrm{G}_0^*(\mathbb{C})^h \times P_{2|1}(\mathcal{T}) \to P_{2|1}(\mathcal{T}), \quad (x,(c,\theta)) \mapsto x\cdot(c,\theta) = (x\cdot c,\ x\cdot\theta) \tag{56}$$

where $x\cdot c$ and $x\cdot\theta$ are defined as follows. Regarding that $x=(x_1,\dots,x_h)$ is assigned to the vertices of $\mathcal{T}$,

$$x\cdot c : \mathcal{T}^1 \to \mathrm{G}_0^*(\mathbb{C}), \quad e \mapsto x_ix_jc(e) \tag{57}$$

where x_i and x_j are assigned to the vertices of e, and

$$x\cdot\theta : \mathcal{T}^2 \to \mathrm{G}_1(\mathbb{C}), \quad f \mapsto (x_ix_jx_k)^{-1}\theta(f) \tag{58}$$

where x_i, x_j, and x_k are assigned to the vertices of f. One easily checks that $x\cdot(c,\theta)$ satisfies Equations (30) and (31), i.e., $x\cdot(c,\theta)\in P_{2|1}(\mathcal{T})$. This action reduces to the $(\mathbb{C}^*)^h$-action on $P_2(\mathcal{T})$ described in [GGZ15b, §4] if we forget θ and restrict x to $(\mathbb{C}^*)^h$.

Theorem 2.13. *The super-Ptolemy assignments* (c,θ) *and* $x\cdot(c,\theta)$ *determine up to conjugation the same representation.*

Proof. Let (ρ, D) be a generically decorated representation corresponding to $(c,\theta)\in P_{2|1}(\mathcal{T})$. Regarding $x=(x_1,\dots,x_h)$ is assigned to the vertices of $\mathcal{T}$, we define

$$x\cdot D : \widetilde{\mathcal{T}}^0 \to \mathrm{OSp}_{2|1}(\mathbb{C})/N, \quad v \mapsto x_iD(v) \tag{59}$$

if v is a lift of the i-th vertex of $\mathcal{T}$. Here we use the identification $\mathrm{OSp}_{2|1}(\mathbb{C})/N \simeq A^{2|1}$, hence the scalar multiplication is well-defined. Then $x \cdot D$ is also a generic decoration of ρ, and Equation (54) implies that $(\rho, x \cdot D)$ corresponds to $(x \cdot c, x \cdot \theta)$. $\square$

Remark 2.14. For $k \in \mathrm{G}_0^*(\mathbb{C})$ we have $(k, \dots, k) \cdot (c, \theta) = (k^2 c, k^{-3}\theta)$. In particular, (c, θ) and $(c, -\theta)$ determine the same representation, up to conjugation.

2.7. (m, l)**-deformation.** One can generalize super-Ptolemy assignments and all previous arguments to $\mathrm{OSp}_{2|1}(\mathbb{C})$-representations that may not $(\mathrm{OSp}_{2|1}(\mathbb{C}), N)$. This can be done by considering the natural action of $\mathrm{OSp}_{2|1}(\mathbb{C})$ on $A^{2|1}/\mathrm{G}_0^*(\mathbb{C})$, instead of $A^{2|1}$, where the quotient is given by identifying $(a, b, \alpha)^t$ and $c(a, b, \alpha)^t$ for all $c \in \mathrm{G}_0^*(\mathbb{C})$. Note that the stabilizer group of this action at $[(1, 0, 0)^t]$ is

$$B = \left\{ \left(\begin{array}{cc|c} a & b & \alpha \\ 0 & a^{-1} & 0 \\ \hline 0 & -a^{-1}\alpha & 1 \end{array}\right) \;\middle|\; a \in \mathrm{G}_0^*(\mathbb{C}),\, b \in \mathrm{G}_0(\mathbb{C}),\, \alpha \in \mathrm{G}_1(\mathbb{C}) \right\}. \tag{60}$$

To be preicse, we fix a cocycle σ that assigns an element of $\mathrm{G}_0^*(\mathbb{C})$ to each short edge of $\mathring{\mathcal{T}}$ and consider representations $\rho : \pi_1(M) \to \mathrm{OSp}_{2|1}(\mathbb{C})$ such that up to conjugation

$$\rho(\gamma) = \left(\begin{array}{cc|c} \sigma(\gamma) & * & * \\ 0 & \sigma(\gamma)^{-1} & 0 \\ \hline 0 & * & 1 \end{array}\right) \qquad \text{for all } \gamma \in \pi_1(\partial M)\,. \tag{61}$$

Here we use the same notation σ for the cocycle and for the morphism $\pi_1(\partial M) \to \mathrm{G}_0^*(\mathbb{C})$ induced from it; hopefully this will cause no confusion. As a generalization of Definitions 2.2, 2.5, and 2.9, we define:

Definition 2.15. A *σ-deformed super-Ptolemy assignment* on $\mathcal{T}$ is a pair of maps

$$c : \mathcal{T}^1 \to \mathrm{G}_0^*(\mathbb{C}), \qquad \theta : \mathcal{T}^2 \to \mathrm{G}_1(\mathbb{C}) \tag{62}$$

satisfying $c(-e) = -c(e)$ for all $e \in \mathcal{T}^1$ and

$$c_{01}c_{23} - \frac{\sigma_{03}^2\sigma_{12}^3}{\sigma_{03}^1\sigma_{12}^0} c_{02}c_{13} + \frac{\sigma_{02}^3\sigma_{13}^2}{\sigma_{02}^1\sigma_{13}^0} c_{03}c_{12} + \frac{\sigma_{02}^3}{\sigma_{02}^1} c_{01}c_{03}c_{12}c_{13}c_{23}\theta_0\theta_2 = 0 \tag{63}$$

as well as

$$\begin{aligned} E_{\Delta,f_3} &: \frac{\sigma_{23}^1}{\sigma_{23}^0} c_{12}\theta_0 - c_{02}\theta_1 + \frac{\sigma_{12}^0}{\sigma_{12}^3} c_{01}\theta_2 = 0 \qquad & E_{\Delta,f_2} &: c_{13}\theta_0 - \frac{\sigma_{01}^3}{\sigma_{01}^2} c_{03}\theta_1 + \frac{\sigma_{12}^0}{\sigma_{12}^3} c_{01}\theta_3 = 0 \\ E_{\Delta,f_1} &: \frac{\sigma_{03}^1}{\sigma_{03}^2} c_{23}\theta_0 - \frac{\sigma_{01}^3}{\sigma_{01}^2} c_{03}\theta_2 + c_{02}\theta_3 = 0 \qquad & E_{\Delta,f_0} &: \frac{\sigma_{03}^1}{\sigma_{03}^2} c_{23}\theta_1 - c_{13}\theta_2 + \frac{\sigma_{23}^1}{\sigma_{23}^0} c_{12}\theta_3 = 0 \end{aligned} \tag{64}$$

for each tetrahedron Δ of $\mathcal{T}$. Here $\sigma_{jk}^i \in \mathrm{G}_0^*(\mathbb{C})$ is the element assigned by σ at the short edge that is near to the vertex i and parallel to the edge $[j, k]$; see Figure 5.

Definition 2.16. A *σ-deformed natural cocycle* on $\mathring{\mathcal{T}}$ is map $\varphi : \mathring{\mathcal{T}}^1 \to \mathrm{OSp}_{2|1}(\mathbb{C})$ of the form

$$\varphi(\text{short}) = \left(\begin{array}{cc|c} \sigma(\text{short}) & a & \theta \\ 0 & \sigma(\text{short})^{-1} & 0 \\ \hline 0 & -\sigma(\text{short})^{-1}\theta & 1 \end{array}\right), \qquad \varphi(\text{long}) = \left(\begin{array}{cc|c} 0 & -b^{-1} & 0 \\ b & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \tag{65}$$

that maps the hexagons and the triangles to the identity. In other words, a natural cocycle is an $\mathrm{OSp}_{2|1}(\mathbb{C})$-representation of the groupoid of $\mathcal{T}$ whose generators have the form (65).

Definition 2.17. (a) An $(\mathrm{OSp}_{2|1}(\mathbb{C}), B)$-*representation* is an $\mathrm{OSp}_{2|1}(\mathbb{C})$-representation ρ of $\pi_1(M)$ such that $\rho(\pi_1(\partial M))$ lies in the stabilizer group B up to conjugation.
(b) A *decoration* of an $(\mathrm{OSp}_{2|1}(\mathbb{C}), B)$-representation ρ is a map

$$D : \widetilde{\mathcal{T}}^0 \to \mathrm{OSp}_{2|1}(\mathbb{C})/B \tag{66}$$

such that $D(\gamma \cdot v) = \rho(\gamma)D(v)$ for $\gamma \in \pi_1(M)$ and $v \in \widetilde{\mathcal{T}}^0$. We say that a decoration is *generic* if for all vertices v_0 and v_1 joined by an edge of $\widetilde{\mathcal{T}}$, we have

$$\langle D(v_0), D(v_1)\rangle^\sharp \neq 0 \tag{67}$$

Note that this condition makes sense, even though $\langle D(v_0), D(v_1)\rangle$ can be only defined up to $\mathrm{G}_0^*(\mathbb{C})$.

Repeating the same arguments in Sections 2.3–2.5 (see also [Yoo19, §2]), we obtain

$$\left\{\begin{matrix}\text{Generically decorated}\\ (\mathrm{OSp}_{2|1}(\mathbb{C}), B)\text{-reps on } M\\ \text{satisfying (61)}\end{matrix}\right\} \xleftrightarrow{1-1} P^\sigma_{2|1}(\mathcal{T}) \xleftrightarrow{1-1} \left\{\begin{matrix}\sigma\text{-deformed natural}\\ \text{cocycles on } \mathring{\mathcal{T}}\end{matrix}\right\} \tag{68}$$

where $P^\sigma_{2|1}(\mathcal{T})$ is the set of all σ-deformed super-Ptolemy assignments on $\mathcal{T}$. We note that:
1. The same argument used in Lemma 2.3 shows that any three of (64) are linearly dependent.
2. Composing the epimorphism $\sharp : \mathrm{G}(\mathbb{C}) \to \mathbb{C}$ with $(c, \theta) \in P^\sigma_{2|1}(\mathcal{T})$, we obtain a $\sigma^\sharp$-deformed Ptolemy assignment $c^\sharp : \mathcal{T}^1 \to \mathbb{C}^*$. That is, $c^\sharp$ satisfies

$$c^\sharp_{01}c^\sharp_{23} - \frac{(\sigma^2_{03})^\sharp(\sigma^3_{12})^\sharp}{(\sigma^1_{03})^\sharp(\sigma^0_{12})^\sharp}c^\sharp_{02}c^\sharp_{13} + \frac{(\sigma^3_{02})^\sharp(\sigma^2_{13})^\sharp}{(\sigma^1_{02})^\sharp(\sigma^0_{13})^\sharp}c^\sharp_{03}c_{12}c^\sharp_{01} = 0 \tag{69}$$

for each tetrahedron of $\mathcal{T}$ [Yoo19]. This defines a map

$$P^\sigma_{2|1}(\mathcal{T}) \xrightarrow{\sharp} P^{\sigma^\sharp}_2(\mathcal{T}) \tag{70}$$

where $P^{\sigma^\sharp}_2(\mathcal{T})$ is the set of all $\sigma^\sharp$-deformed Ptolemy assignments on $\mathcal{T}$. This map is surjective if $\sigma^\sharp = \sigma$, i.e. σ takes values in $\mathbb{C}^*$.
3. The σ-deformed natural cocycle φ corresponding to $(c, \theta) \in P^\sigma_{2|1}(\mathcal{T})$ is explicitly given by

$$\varphi(e^k_{ji}) = \left(\begin{array}{cc|c} \sigma^k_{ji} & -\frac{\sigma^j_{ik}}{\sigma^i_{kj}}\frac{c_{ij}}{c_{jk}c_{ki}} & c_{ij}\theta/\sigma^i_{kj} \\ 0 & 1/\sigma^k_{ji} & 0 \\ \hline 0 & -c_{ij}\theta/(\sigma^i_{kj}\sigma^k_{ji}) & 1 \end{array}\right), \quad \varphi(e_{ij}) = \left(\begin{array}{cc|c} 0 & -c^{-1}_{ij} & 0 \\ c_{ij} & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \tag{71}$$

for Figure 5 where (i, j, k) is a cyclic permutation of $(0, 1, 2)$.
4. There is a $\mathrm{G}_0^*(\mathbb{C})^h$-action on $P^\sigma_{2|1}(\mathcal{T})$ defined in the same way as that on $P_{2|1}(\mathcal{T})$. Moreover, Theorem 2.13 also holds for $(c, \theta) \in P^\sigma_{2|1}(\mathcal{T})$.

The next theorem is a direct consequence of the correspondence (68).

Theorem 2.18. *There is a map $P^{\sigma}_{2|1}(\mathcal{T}) \to \mathrm{Hom}(\pi_1(M), \mathrm{OSp}_{2|1}(\mathbb{C}))/_{\sim}$ which fits into*

$$\begin{tikzcd} P^{\sigma}_{2|1}(\mathcal{T}) \arrow[r] \arrow[d, "\sharp"] & \mathrm{Hom}(\pi_1(M), \mathrm{OSp}_{2|1}(\mathbb{C}))/_{\sim} \arrow[d, two heads, "\sharp"] \\ P^{\sigma^\sharp}_{2}(\mathcal{T}) \arrow[r] & \mathrm{Hom}(\pi_1(M), \mathrm{SL}_2(\mathbb{C}))/_{\sim} \end{tikzcd} \tag{72}$$

and whose image is the set of all conjugacy classes of $(\mathrm{OSp}_{2|1}(\mathbb{C}), B)$*-representations admitting a generic decoration and satisfying* (61).

2.8. **Concrete triangulations.** In Section 2.3, we defined the super-Ptolemy assignments for ordered triangulations. In this section, we discuss how to define these assignments for concrete triangulations, that is, for triangulations where each tetrahedron comes with a bijection of its vertices with those of the standard 3-simplex. Concrete triangulations were used in [GGZ15a] to define the gluing equations of $\mathrm{PGL}_n(\mathbb{C})$ representations of $\pi_1(M)$, as well as in [GGZ15b]. Note that all the triangulations in `SnapPy` and `Regina` are concrete [CDW, Bur].

Recall that a super-Ptolemy assignment consists of two assignments: one assigns an invertible even element to each oriented edge and the other assigns an odd element to each unoriented face. The former deals with every edge with both orientations, satisfying that reversing the orientation of an edge reverses the value of the assigned element. Therefore, the former has no issues for concrete triangulations, where edges could be identified in an orientation-reversed way. However, the latter may cause a problem, as it only deals with unoriented faces. To prevent this problem, we simply consider both sides of a face with each side having one odd element. Namely, we assign one odd element to each oriented face. This seems to double the number of odd elements, but in fact, two odd elements θ and θ' assigned to the front and back sides of a face (as in Figure 6) should be related as follows.

Recall from Equation (71) that for any cyclic permutation (i,j,k) of $(0,1,2)$, we have

$$\varphi(e^k_{ji}) = \left(\begin{array}{cc|c} \sigma^k_{ji} & -\frac{\sigma^j_{ik}}{\sigma^i_{kj}}\frac{c_{ij}}{c_{jk}c_{ki}} & c_{ij}\theta/\sigma^i_{kj} \\ 0 & 1/\sigma^k_{ji} & 0 \\ \hline 0 & -c_{ij}\theta/(\sigma^i_{kj}\sigma^k_{ji}) & 1 \end{array}\right). \tag{73}$$

Applying the same formula to the back side of the face, we obtain

$$\varphi(e^k_{ij}) = \left(\begin{array}{cc|c} \sigma^k_{ij} & -\frac{\sigma^i_{jk}}{\sigma^j_{ki}}\frac{c_{ji}}{c_{ik}c_{kj}} & c_{ji}\theta'/\sigma^j_{ki} \\ 0 & 1/\sigma^k_{ij} & 0 \\ \hline 0 & -c_{ji}\theta'/(\sigma^j_{ki}\sigma^k_{ij}) & 1 \end{array}\right). \tag{74}$$

Note that $c_{ij} = -c_{ji}$ and $\sigma^i_{jk} = 1/\sigma^i_{kj}$. Then a straightforward computation shows that $\varphi(e^k_{ji})\varphi(e^k_{ij}) = I$ if and only if

$$\theta' = \sigma^k_{ij}\sigma^i_{jk}\sigma^j_{ki}\theta\,. \tag{75}$$

This shows that super-Ptolemy assignments on a concrete triangulation are described by the same equations (63) and (64) but some of θ_i may be replaced by θ'_i, where $\theta_0,\dots,\theta_3$ in (64) are assigned to the sides of $f_0,\dots,f_3$ that face front. Note that if a face-pairing preserves the orientation of the faces induced from the vertex-orderings, then only one of θ_i or θ'_i appears in the face equations, and otherwise, both θ_i and θ'_i appear.

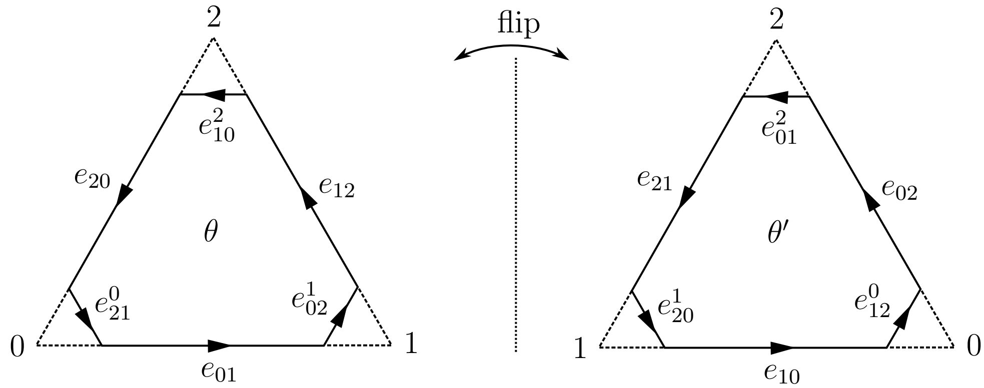

FIGURE 6. Front/back sides of a face

2.9. **Example: the 4_1 knot.** Let $\mathcal{T}$ be the standard ideal triangulation of the knot complement of 4_1 obtained by the face-pairings of two ordered ideal tetrahedra Δ_1 and Δ_2 with edges e_1 and e_2 and with faces $f_1, \ldots, f_4$. See Figure 7. We choose a cocycle σ on the short edges for $m, \ell \in \mathrm{G}^*_0(\mathbb{C})$ as follows (see [Yoo19, Ex.2.8]):

$$\sigma(s_2) = \sigma(s_5) = \sigma(s_8) = \sigma(s_{11}) = m, \quad \sigma(s_6) = \sigma(s_9) = \sigma(s_{12}) = m^{-1} \tag{76}$$

$$\sigma(s_4) = \sigma(s_7) = \sigma(s_{10}) = 1, \quad \sigma(s_1) = \ell^{-1}m^{-2}, \quad \sigma(s_3) = \ell m \tag{77}$$

where $s_1, \ldots, s_{12}$ are the short edges of $\mathring{\mathcal{T}}$ as in Figure 7. Note that the morphism induced by σ sends the meridian and canonical longitude of the knot to m and ℓ, respectively.

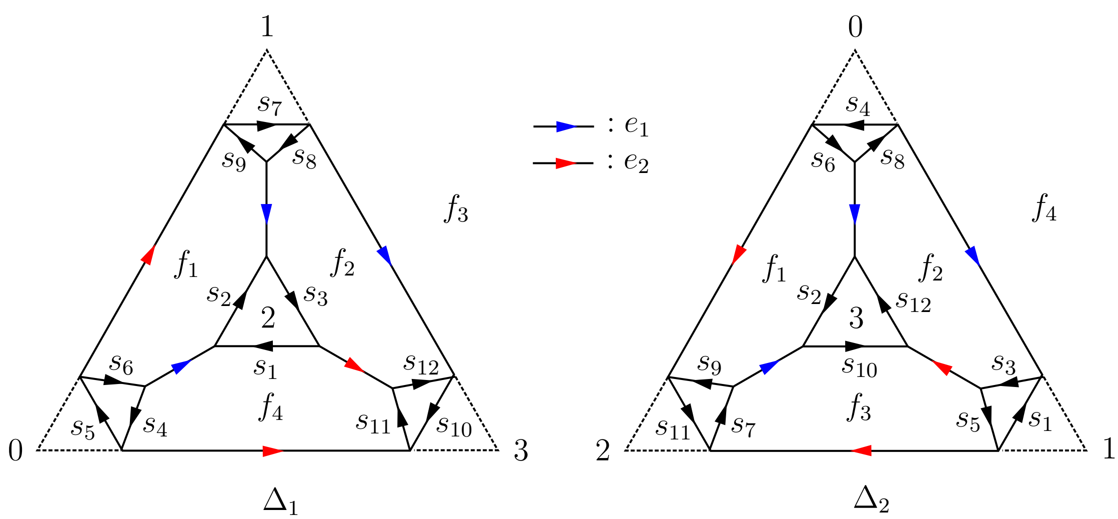

FIGURE 7. The knot complement of 4_1.

A σ-deformed super-Ptolemy assignments is a pair of maps $c : \{e_1, e_2\} \to \mathrm{G}_0^*(\mathbb{C})$ and $\theta : \{f_1, \dots, f_4\} \to \mathrm{G}_1(\mathbb{C})$ satisfying

$$\begin{aligned} c_2^2 - \ell m^4 c_1^2 + lm^2 c_1 c_2 + m^2 c_1^3 c_2^2 \theta_2 \theta_3 = 0 \\ c_1^2 - \ell^{-1} c_2^2 + \ell^{-1} c_1 c_2 + \ell^{-1} m^{-1} c_1^3 c_2^2 \theta_3 \theta_2 = 0 \end{aligned} \tag{78}$$

and

$$\begin{aligned} E_{\Delta_1, f_4} : &\quad \ell^{-1} m^{-2} c_2 \theta_2 - m^{-1} c_2 \theta_3 + c_1 \theta_1 = 0 \\ E_{\Delta_1, f_3} : &\quad c_1 \theta_2 - m^{-1} c_2 \theta_4 + m^{-2} c_2 \theta_1 = 0 \\ E_{\Delta_2, f_4} : &\quad c_2 \theta_3 - c_2 \theta_1 + c_1 \theta_2 = 0 \\ E_{\Delta_2, f_3} : &\quad \ell c_1 \theta_1 - c_2 \theta_2 + c_2 \theta_4 = 0 \end{aligned} \tag{79}$$

where $c_i := c(e_i)$ and $\theta_i := \theta(f_i)$. Writing the equations in (79) in a matrix form, we have

$$\begin{pmatrix} m^{-2} c_2 & c_1 & 0 & -m^{-1} c_2 \\ c_1 & \ell^{-1} m^{-2} c_2 & -m^{-1} c_2 & 0 \\ -c_2 & c_1 & c_2 & 0 \\ \ell c_1 & -c_2 & 0 & c_2 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} . \tag{80}$$

We are interested in the case of the above 4×4-matrix F being singular, as all θ_i should be zero, otherwise. One computes that if $\det F = 0$, then the kernel F is a free $\mathrm{G}_1(\mathbb{C})$-module of rank 1:

$$(\theta_1, \theta_2, \theta_3, \theta_4) = \eta \left(\frac{c_1 + \frac{1}{\ell m} c_2}{m c_1 - c_2}, \, 1, \, -\frac{m c_1^2 + \frac{1}{\ell m} c_2^2}{c_2 (m c_1 - c_2)}, \, \frac{\ell c_1^2 + (m + \frac{1}{m}) c_1 c_2 - c_2^2}{c_2 (m c_1 - c_2)} \right), \quad \eta \in \mathrm{G}_1(\mathbb{C}).$$

It follows that either $\det F = 0$ or not, we have $\theta_i \theta_j = 0$ for any i, j and thus Equation (78) is simplified to

$$\begin{aligned} c_2^2 - \ell m^4 c_1^2 + lm^2 c_1 c_2 = 0 \\ c_1^2 - \ell^{-1} c_2^2 + \ell^{-1} c_1 c_2 = 0 \end{aligned} \tag{81}$$

with

$$\det F = 2 c_1^1 c_2^3 \, m^{-2} (m + m^{-1} - 1) \,. \tag{82}$$

This shows that a σ-deformed Ptolemy assignment (c, θ) with $\theta \neq 0$ exists if and only if $m + m^{-1} - 1 = 0$. For instance, we restrict m and ℓ to complex numbers, then we have

$$m = \frac{1 \pm \sqrt{-3}}{2}, \quad \ell = -1, \quad (c_1, c_2) = k \left(\frac{1 \mp \sqrt{-3}}{2}, 1 \right) \quad \text{for } k \in \mathrm{G}_0^*(\mathbb{C}) \,. \tag{83}$$

Note that the $\mathbb{C}^2$-torsion $2(m + m^{-1} - 1)$ of the knot 4_1 appears as a factor of Equation (82).

3. 1-loop and $\mathbb{C}^2$-torsion polynomials

In this section, we define the 1-loop invariant, the 1-loop polynomial, and their (m, l)-deformed version from an ideal triangulation.

3.1. **The face-matrix of an ideal triangulation.** As mentioned in Remark 2.1, henceforth, we use the Grassmann algebra with one odd generator. Its even and odd part are both isomorphic to $\mathbb{C}$, and a product of any two odd elements is zero. In particular, Equation (30) reduces to the ordinary Ptolemy equation

$$c_{01}c_{23} - c_{02}c_{13} + c_{03}c_{12} = 0\,. \tag{84}$$

Therefore, a super-Ptolemy assignment (c, θ) on an ideal triangulation $\mathcal{T}$ is given by a pair of a Ptolemy assignment $c : \mathcal{T}^1 \to \mathbb{C}^*$ with a map $\theta : \mathcal{T}^2 \to \mathbb{C}$ satisfying Equation (31) for each tetrahedron of $\mathcal{T}$.

Suppose $\mathcal{T}$ has N tetrahedra. Then it has N edges and $2N$ faces. Hence a super-Ptolemy assignment is represented by a tuple $c = (c_1, \ldots, c_N)$ of non-zero complex numbers satisfying the Ptolemy equation (84) for each tetrahedron and a tuple $\theta = (\theta_1, \ldots, \theta_{2N})^t$ of complex numbers satisfying four linear equations (31) for each tetrahedron. We call these linear equations face-equations and write them in matrix form as

$$\begin{pmatrix} F_c^0 \\ F_c^1 \\ F_c^2 \\ F_c^3 \end{pmatrix} \theta = 0, \qquad \theta = (\theta_1, \ldots, \theta_{2N})^t \tag{85}$$

where F_c^k for $k = 0, 1, 2, 3$ are $N \times 2N$ matrices whose rows and columns are indexed by the tetrahedra and the faces of $\mathcal{T}$, respectively. However, it was shown in Lemma 2.3 that at each tetrahedron any three of the linear equations in (31) are dependent. Thus we choose two equations from each tetrahedron Δ. Such a choice can be represented by an edge of Δ, as each equation in (31) corresponds to a face of Δ. More precisely, we choose an edge e_Δ for each tetrahedron Δ of $\mathcal{T}$ and use two equations in (31) that correspond to the two faces adjacent to e_Δ. This creates a $2N \times 2N$ matrix F_c, called a face-matrix, so that the face-equations for θ take the form

$$F_c\,\theta = 0\,. \tag{86}$$

Note that F_c is a trimmed version of the $4N \times 2N$ matrix of Equation (85) and that entries of F_c are linear forms on c (in fact, the nonzero entries at up to sign, equal the value of c on an edge, see Equation (35)).

3.2. **1-loop invariant.** We now have all the ingredients to define the 1-loop invariant.

Definition 3.1. For a Ptolemy assignment c on $\mathcal{T}$ we define the 1-loop invariant by

$$\delta_{\mathcal{T},c,2} := \left(\prod_e \frac{1}{c(e)} \prod_\Delta \frac{1}{c(e_\Delta)} \right) \det F_c \tag{87}$$

where the products are taken over all edges e and all tetrahedra Δ of $\mathcal{T}$, respectively.

Note that the 1-loop invariant $\delta_{\mathcal{T},c,2}$ has degree 0 in c, i.e., is invariant under scaling c to kc for all $k \in \mathbb{C}^*$. It turns out that the 1-loop invariant does not depend on the choice of edges e_Δ and is invariant under 2–3 Pachner moves. This follows from the specialization of Lemma 3.3 and Theorem 3.4 at $t = 1$ below. In addition, we conjecture that the 1-loop invariant $\delta_{\mathcal{T},c,2}$ is equal to the $\mathbb{C}^2$-torsion $\tau_{M,\rho,2}$ up to sign, where $\rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ is a representation associated to the Ptolemy assignment c. This is the specialization at $t=1$ of Conjecture 3.7 below.

3.3. **1-loop polynomial.** In this section, we upgrade our 1-loop invariant to the 1-loop polynomial.

Suppose M has an infinite cyclic cover $\widetilde{M}$ and let $\widetilde{\mathcal{T}}$ be the ideal triangulation of $\widetilde{M}$ induced from $\mathcal{T}$. We identify the deck transformation group of $\widetilde{M}$ with $\{t^k \,|\, k \in \mathbb{Z}\}$ for a formal variable t and fix a lift of every cell of $\mathcal{T}$ to $\widetilde{\mathcal{T}}$. Then a cell of $\widetilde{\mathcal{T}}$ is uniquely represented by a cell of $\mathcal{T}$ with a monomial in t. For instance, a face of $\widetilde{\mathcal{T}}$ is represented by $t^k \cdot f$ for a face f of $\mathcal{T}$ and $k \in \mathbb{Z}$.

Recall that a face-equation is of the form

$$c_\alpha \, \theta(f_0) + c_\beta \, \theta(f_1) + c_\gamma \, \theta(f_2) = 0 \tag{88}$$

where f_0, f_1, and f_2 are three faces of a tetrahedron Δ. Since the lift of Δ has three faces that are represented by $t^{k_i} \cdot f_i$ for some $k_i \in \mathbb{Z}$, we can formally modify Equation (88) as

$$c_\alpha t^{k_0} \, \theta(f_0) + c_\beta t^{k_1} \, \theta(f_1) + c_\gamma \, t^{k_2} \, \theta(f_2) = 0 \,. \tag{89}$$

The effect of this insertion of monomials in t leads to a t-twisted version $F_c(t)$ of F_c.

Definition 3.2. For a Ptolemy assignment c on $\mathcal{T}$ we define the 1-loop polynomial by

$$\delta_{\mathcal{T},c,2}(t) := \left(\prod_e \frac{1}{c(e)} \prod_\Delta \frac{1}{c(e_\Delta)} \right) \det F_c(t) \tag{90}$$

where the products are taken over all edges e and all tetrahedra Δ of $\mathcal{T}$, respectively.

It is clear from $F_c = F_c(1)$ that the 1-loop invariant $\delta_{\mathcal{T},c,2}$ is the specialization $\delta_{\mathcal{T},c,2}(1)$ at $t=1$. In addition, the 1-loop polynomial $\delta_{\mathcal{T},c,2}(t)$ determines the 1-loop invariant $\delta_{\mathcal{T}^{(n)},c,2}$ of all cyclic n-covers $M^{(n)}$ of M. This follows by arguments similar to the ones presented in [GY23] (for the 1-loop polynomial $\delta_{\mathcal{T},c,3}(t)$) and will not be repeated here.

Lemma 3.3. The 1-loop polynomial $\delta_{\mathcal{T},c,2}(t)$ does not depend on the choice of edge e_Δ.

Proof. It suffices to compare two different edge-choices for one tetrahedron Δ. Comparing $e_\Delta = [0,1]$ and $e_\Delta = [0,2]$, we have from Equation (33)

$$\begin{pmatrix} c_{02} & -c_{03} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_{13} & -c_{03} & 0 & c_{01} \\ c_{12} & -c_{02} & c_{01} & 0 \end{pmatrix} = \begin{pmatrix} c_{01} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_{23} & 0 & -c_{03} & c_{02} \\ c_{12} & -c_{02} & c_{01} & 0 \end{pmatrix} . \tag{91}$$

It implies that the 1-loop invariant $\delta_{\mathcal{T},c,2}$ is unchanged even if we change $e_\Delta = [0,1]$ to $[0,2]$. The insertion of monomials in t affects on both sides of (91) by multiplying the same diagonal matrix (with diagonal in monomials in t) on the right. Therefore, the 1-loop polynomial $\delta_{\mathcal{T},c,2}(t)$ is also unchanged. ∎

Theorem 3.4. *The 1-loop polynomial $\delta_{\mathcal{T},c,2}(t)$ is invariant under 2–3 Pachner moves.*

Proof. Suppose that $\mathcal{T}$ has two tetrahedra $[0,2,3,4]$ and $[1,2,3,4]$ with a common face $[2,3,4]$ as in Figure 8. Let $\mathcal{T}'$ denote the ideal triangulation obtained by replacing these two tetrahedra by $[0,1,2,3]$, $[0,1,3,4]$, and $[0,1,2,4]$.

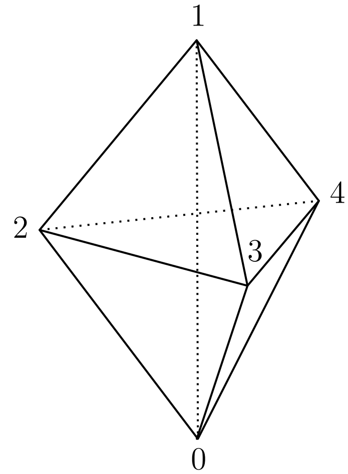

FIGURE 8. A 2–3 Pachner move.

Recall that the face equation of a face $[i, j, k]$ of a tetrahedron $[i, j, k, l]$ with $i < j < k$ is given by

$$E^l_{ijk} : c_{ij}\theta_{ijl} - c_{ik}\theta_{ikl} + c_{jk}\theta_{jkl} = 0\,. \tag{92}$$

For $\Delta = [0, 2, 3, 4]$ and $[1, 2, 3, 4]$ we choose $e_\Delta = [2, 4]$. Then the face-matrix $F_{\mathcal{T},c}$ of $\mathcal{T}$ contains a submatrix

	θ_{234}	others
E^3_{024}	c_{24}	R^3_{024}
E^0_{234}		R^0_{234}
E^3_{124}	c_{24}	R^3_{124}
E^1_{234}		R^1_{234}

(93)

where R^l_{ijk} is the row of E^l_{ijk} except the θ_{234}-entry. Using an elementary row operation, we can modify the face-matrix without changing its determinant as

	θ_{234}	others
E^3_{024}		$R^3_{024} - R^3_{124}$
E^0_{234}		R^0_{234}
E^3_{124}	c_{24}	R^3_{124}
E^1_{234}		R^1_{234}

(94)

For $\Delta = [0,1,2,3]$ (resp., $[0,1,3,4]$ and $[0,1,2,4]$) we choose $e_\Delta = [2,3]$ (resp., $[3,4]$ and $[2,4]$). Then the face-matrix $F_{\mathcal{T}',c}$ of $\mathcal{T}'$ contains a submatrix

$$\begin{array}{c|c|c|c|c|}
 & \theta_{012} & \theta_{013} & \theta_{014} & \text{others} \\ \hline
E^0_{123} & c_{12} & -c_{13} & & R^0_{123} \\
E^1_{023} & c_{02} & -c_{03} & & R^1_{023} \\
E^0_{134} & & c_{13} & -c_{14} & R^0_{134} \\
E^1_{034} & & c_{03} & -c_{04} & R^1_{034} \\
E^0_{124} & c_{12} & & -c_{14} & R^0_{124} \\
E^1_{024} & c_{02} & & -c_{04} & R^1_{024}
\end{array} \tag{95}$$

Preserving the determinant, we apply elementary row operations to obtain:

$$\begin{array}{c|c|c|c|c|}
 & \theta_{012} & \theta_{013} & \theta_{014} & \text{other 6 faces} \\ \hline
E^0_{123} & & & & R^0_{123} - \frac{c_{12}}{c_{02}} R^1_{023} - \frac{c_{23}}{c_{02}}\frac{c_{04}}{c_{34}} R^0_{134} + \frac{c_{23}}{c_{02}}\frac{c_{14}}{c_{34}} R^1_{034} \\
E^1_{023} & c_{02} & -c_{03} & & R^1_{023} \\
E^0_{134} & & -\frac{c_{34}}{c_{04}} c_{01} & & R^0_{134} - \frac{c_{14}}{c_{04}} R^1_{034} \\
E^1_{034} & & c_{03} & -c_{04} & R^1_{034} \\
E^0_{124} & & & & R^0_{124} - R^0_{134} - R^0_{123} \\
E^1_{024} & & & & R^1_{024} - R^1_{034} - R^1_{023}
\end{array} \tag{96}$$

On the other hand, one computes that

$$\begin{aligned}
R^0_{234} &= R^0_{124} - R^0_{134} - R^0_{123} \\
R^1_{234} &= R^1_{024} - R^1_{034} - R^1_{023} \\
R^3_{024} - R^3_{124} &= \frac{c_{02}}{c_{23}} R^0_{123} - \frac{c_{12}}{c_{23}} R^1_{023} - \frac{c_{04}}{c_{34}} R^0_{134} + \frac{c_{14}}{c_{34}} R^1_{034}
\end{aligned} \tag{97}$$

It follows that (96) is equal to

$$\begin{array}{c|c|c|c|c|}
 & \theta_{012} & \theta_{013} & \theta_{014} & \text{others} \\ \hline
E^0_{123} & & & & \frac{c_{23}}{c_{02}}(R^3_{024} - R^3_{124}) \\
E^1_{023} & c_{02} & -c_{03} & & R^1_{023} \\
E^0_{134} & & -\frac{c_{34}}{c_{04}} c_{01} & & R^0_{134} - \frac{c_{14}}{c_{04}} R^1_{034} \\
E^1_{034} & & c_{03} & -c_{04} & R^1_{034} \\
E^0_{124} & & & & R^0_{234} \\
E^1_{024} & & & & R^1_{234}
\end{array} \tag{98}$$

Comparing (94) and (98), we have

$$\det F_{\mathcal{T},c} = \frac{c_{24}}{c_{01}c_{23}c_{34}} \det F_{\mathcal{T}',c} \,. \tag{99}$$

The monomial factor in the right-hand side agrees with the difference coming from the monomial term $\prod c(e) \prod c(e_\Delta)$ in (87). This proves that $\delta_{\mathcal{T},c,2} = \delta_{\mathcal{T}',c,2}$. As the effect of the insertion of monomials in t is separated from the above computation, this also proves that $\delta_{\mathcal{T},c,2}(t) = \delta_{\mathcal{T}',c,2}(t)$. $\square$

Remark 3.5. The proof of the above theorem contains the behavior of the Ptolemy variety under Pachner 2–3 moves. This can be used to show that the determinant of the Jacobian of the Ptolemy variety, suitable normalized, is invariant under 2–3 Pachner moves, and conjecturally equal to the 1-loop invariant defined in [DG13]; see [Yoo].

Theorem 3.6. *A Ptolemy assignment c on $\mathcal{T}$ lifts to a super-Ptolemy assignment (c, θ) with $\theta \neq 0$ if and only if $\delta_{\mathcal{T},c,2}(1) = 0$.*

Proof. It is clear that if $\delta_{\mathcal{T},c,2}(1) \neq 0$, or equivalently, if $F_c = F_c(1)$ is non-singular, then θ should be zero. Conversely, if $\delta_{\mathcal{T},c,2}(1) = 0$, then there is a nonzero vector $v \in \mathbb{C}^{2N}$ with $F_c\, v = 0$, and for any $\eta \neq 0 \in \mathbb{C}$ the pair $(c, \eta v)$ is a super-Ptolemy assignment. $\square$

Conjecture 3.7. The 1-loop polynomial is equal to the $\mathbb{C}^2$-torsion polynomial

$$\delta_{\mathcal{T},c,2}(t) = \tau_{M,\rho,2}(t) \tag{100}$$

up to multiplying signs and monomials in t. Here $\rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ is a representation associated to the Ptolemy assignment c.

3.4. (m, l)**-deformation.** In this section, we deform the 1-loop invariant as well as the 1-loop polynomial.

We fix a cocycle σ that assigns a non-zero complex number to each short edge of $\mathring{\mathcal{T}}$. Recall Equation (64) that the face-equations in (31) admit a deformation according to σ. As in Section 3.1, we choose two face-equations in (64) by choosing an edge e_Δ for each tetrahedron Δ of $\mathcal{T}$. This creates a $2N \times 2N$ matrix F_c^σ, so that the chosen face-equations take the form

$$F_c^\sigma\, \theta = 0 \tag{101}$$

as well as its t-twisted version $F_c^\sigma(t)$, as explained in Section 3.3.

Definition 3.8. For a σ-deformed Ptolemy assignment c on $\mathcal{T}$ we define the 1-loop invariant by

$$\delta_{\mathcal{T},c,2} := \left(\prod_e \frac{1}{c(e)} \prod_\Delta \frac{1}{c^\sigma(e_\Delta)} \right) \det F_c^\sigma \tag{102}$$

and the 1-loop polynomial by

$$\delta_{\mathcal{T},c,2}(t) := \left(\prod_e \frac{1}{c(e)} \prod_\Delta \frac{1}{c^\sigma(e_\Delta)} \right) \det F_c^\sigma(t) \tag{103}$$

where $c^\sigma(e_\Delta)$ is the value of c on the edge e_Δ times its σ-coefficient in (64):

e_Δ	[0,1]	[0,2]	[0,3]	[1,2]	[1,3]	[2,3]
$c^\sigma(e_\Delta)$	$\frac{\sigma_{21}^0}{\sigma_{12}^3} c_{01}$	c_{02}	$\frac{\sigma_{01}^3}{\sigma_{01}^2} c_{03}$	$\frac{\sigma_{23}^1}{\sigma_{23}^0} c_{12}$	c_{13}	$\frac{\sigma_{03}^1}{\sigma_{03}^2} c_{23}$

(104)

Repeating the same computation given in Sections 3.2 and 3.3, one can prove that (a) a σ-deformed Ptolemy assignment c on $\mathcal{T}$ lifts to a super-Ptolemy assignment (c, θ) with $\theta \neq 0$ if and only if $\delta_{\mathcal{T},c,2} = 0$; (b) the 1-loop polynomial $\delta_{\mathcal{T},c,2}(t)$ does not depend on the choice of edge e_Δ and is invariant under 2–3 Pachner moves up to scalar multiplication by non-zero complex numbers. In addition, we proposed the same conjecture (100) for σ-deformed ones, but up to multiplying non-zero complex numbers and monomials in t. Note that this scalar-multiplication ambiguity is given by a product of complex numbers that σ assigns to short edges. In particular, if σ takes values in $\{\pm 1\}$ (for instance, if M is hyperbolic and ρ is a lift of the geometric representation), then we can replace this scalar-multiplication ambiguity by sign-ambiguity as in Conjecture 3.7.

3.5. **Example: the 4_1 knot continued.** In this section we verify our conjecture for the 4_1 knot. With the notation of Section 2.9, we have $e_{\Delta_1} = [0,3]$ and $e_{\Delta_2} = [1,2]$, hence

$$c^\sigma(e_{\Delta_1}) = \frac{\sigma(s_{10})^{-1}}{\sigma(s_2)} c_2 = \frac{c_2}{m}, \qquad c^\sigma(e_{\Delta_2}) = \frac{\sigma(s_5)^{-1}}{\sigma(s_6)} c_2 = c_2 \tag{105}$$

As explained in Section 3.3 and [GY23, Sec.3.1], we insert monomials in t to the matrix in Equation (80) to obtain

$$F^\sigma_c(t) = \begin{pmatrix} m^{-2}c_2 & c_1 & 0 & -m^{-1}c_2 \\ c_1 & \ell^{-1}m^{-2}c_2 & -m^{-1}c_2 & 0 \\ -c_2 & c_1 & c_2 t & 0 \\ \ell c_1 & -c_2 & 0 & c_2 t \end{pmatrix}. \tag{106}$$

Then, we have

$$\delta_{\mathcal{T},c,2}(t) = \frac{1}{c_1 c_2}\frac{m}{c_2^2} \det F^\sigma_c(t) = m^{-1}(t^2 - 2(m+m^{-1})t + 1) \tag{107}$$

which agrees with the $\mathbb{C}^2$-torsion polynomial of 4_1 up to m^{-1}.
In particular, for $m = 1$ (and $l = -1$), we obtain the $\mathbb{C}^2$-torsion polynomial

$$t^2 - 4t + 1 \tag{108}$$

for an $\mathrm{SL}_2(\mathbb{C})$-lift of the geometric representation of the 4_1 knot, in agreement with `SnapPy`

```
snappy.Manifold('4_1').hyperbolic_SLN_torsion(2)
a^2 - 4.000000000000000000000000000*a + 0.9999999999999999999999999999
```

4. Further discussion

In this paper and in our prior work [GY23], we introduced 1-loop polynomials $\delta_{\mathcal{T},2}(t)$ and $\delta_{\mathcal{T},3}(t)$ determined, respectively, by the twisted face-matrix and twisted NZ-matrix of an ideal triangulation $\mathcal{T}$ of a 3-manifold M, and conjectured to be equal to the torsion polynomials $\tau_{M,2}(t)$ and $\tau_{M,3}(t)$. In this section we explain briefly how to derive 1-loop polynomials that conjecturally equal to the torsion polynomials $\tau_{M,n}(t)$ for all $n \geq 2$.

The torsion polynomials are closely related to the adjoint reprensentation of $\mathrm{PGL}_n(\mathbb{C})$ which decomposes

$$\mathrm{Ad}(\mathrm{PGL}_n(\mathbb{C})) = \oplus_{i=1}^{n-1} \mathbb{C}^{2i+1} \tag{109}$$

into *odd* dimensional representations of $\mathrm{SL}_2(\mathbb{C})$. This decomposition is not special to $\mathrm{PGL}_n(\mathbb{C})$, indeed every complex semisimple Lie group G has a canonical principal $\mathrm{SL}_2(\mathbb{C})$ subgroup, and decomposing the adjoint representation of G as an $\mathrm{SL}_2(\mathbb{C})$-representation

$$\mathrm{Ad}(G) = \oplus_{i=1}^{r} \mathbb{C}^{2e_i+1} \tag{110}$$

one obtains only *odd* dimensional irreducible representations of $\mathrm{SL}_2(\mathbb{C})$, where e_i are the exponents of G [Kos59]. Using the above decomposition (109), one can define for each $n \geq 2$ the product

$$\prod_{i=1}^{n-1} \tau_{M,2i+1}(t) \tag{111}$$

from which one can extract the odd torsion polynomials $\tau_{M,\mathrm{odd}}(t)$, see e.g. [Por, Sec.5].

Using the fact that $\mathrm{PGL}_n(\mathbb{C})$-representations can be described by gluing equations associated to $\mathrm{PGL}_n(\mathbb{C})$-type Neumann–Zagier matrices [GGZ15a], if one twists these matrices by considering their lifts to an infinite cyclic cover as was done in [GY23], one can define a 1-loop polynomial $\delta_{\mathcal{T},\mathrm{PGL}_n}(t)$ which would factor as $\prod_{i=1}^{n-1} \delta_{M,2i+1}(t)$ and would conjecturally equal to the polynomial (111). Doing so, one can obtain the odd 1-loop polynomials that conjecturally equal to the corresponding odd torsion polynomials.

Likewise, an extension of the decomposition (110) to high-dimensional orthosymplectic groups, together with a construction of Neumann–Zagier matrices that describe representations of 3-manifold groups to orthosymplectic groups, along with their twisted version would determine even 1-loop polynomials that conjecturally equal to the corresponding even torsion polynomials.

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References

[AD20] Ian Agol and Nathan Dunfield, *Certifying the Thurston norm via* $\mathrm{SL}(2,\mathbb{C})$*-twisted homology*, What's next?—the mathematical legacy of William P. Thurston, Ann. of Math. Stud., vol. 205, Princeton Univ. Press, Princeton, NJ, 2020, pp. 1–20.

[Ber87] Felix Alexandrovich Berezin, *Introduction to superanalysis*, Mathematical Physics and Applied Mathematics, vol. 9, D. Reidel Publishing Co., Dordrecht, 1987, Edited and with a foreword by A. A. Kirillov, With an appendix by V. I. Ogievetsky, Translated from the Russian by J. Niederle and R. Kotecký, Translation edited by Dimitri Leĭtes.

[Bur] Benjamin Burton, *Regina: Normal surface and 3-manifold topology software*, `http://regina.sourceforge.net`.

[CDW] Marc Culler, Nathan Dunfield, and Jeffrey Weeks, *SnapPy, a computer program for studying the topology of* 3*-manifolds*, Available at `http://snappy.computop.org` (30/01/2015).

[CR88] Louis Crane and Jeffrey Rabin, *Super Riemann surfaces: uniformization and Teichmüller theory*, Comm. Math. Phys. **113** (1988), no. 4, 601–623.

[DFJ12] Nathan Dunfield, Stefan Friedl, and Nicholas Jackson, *Twisted Alexander polynomials of hyperbolic knots*, Exp. Math. **21** (2012), no. 4, 329–352.

[DG13] Tudor Dimofte and Stavros Garoufalidis, *The quantum content of the gluing equations*, Geom. Topol. **17** (2013), no. 3, 1253–1315.

[DGY] Nahtan Dunfield, Stavros Garoufalidis, and Seokbeom Yoon, *1-loop equals torsion for fibered 3-manifolds*, Preprint 2023, arXiv:2304.00469.

[GGZ15a] Stavros Garoufalidis, Matthias Goerner, and Christian Zickert, *Gluing equations for* $\mathrm{PGL}(n,\mathbb{C})$*-representations of 3-manifolds*, Algebr. Geom. Topol. **15** (2015), no. 1, 565–622.

[GGZ15b] ———, *The Ptolemy field of 3-manifold representations*, Algebr. Geom. Topol. **15** (2015), no. 1, 371–397.

[GTZ15] Stavros Garoufalidis, Dylan Thurston, and Christian Zickert, *The complex volume of* $\mathrm{SL}(n,\mathbb{C})$*-representations of 3-manifolds*, Duke Math. J. **164** (2015), no. 11, 2099–2160.

[GY23] Stavros Garoufalidis and Seokbeom Yoon, *Twisted Neumann–Zagier matrices*, Research in the Mathematical Sciences **10** (2023), no. 4, 37.

[IPZ18] Ivan Ip, Robert Penner, and Anton Zeitlin, $\mathcal{N} = 2$ *super-Teichmüller theory*, Adv. Math. **336** (2018), 409–454.

[Kos59] Bertram Kostant, *The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group*, Amer. J. Math. **81** (1959), 973–1032.

[Man91] Yuri Manin, *Topics in noncommutative geometry*, M. B. Porter Lectures, Princeton University Press, Princeton, NJ, 1991.

[MOZ] Gregg Musiker, Nicholas Ovenhouse, and Sylvester Zhang, *Matrix formulae for decorated super Teichmüller spaces*, Preprint 2022, arXiv:2206.12072.

[Por] Joan Porti, *Reidemeister torsion, hyperbolic three-manifolds, and character varieties*, Preprint 2015, arXiv:1511.00400.

[PZ19] Robert Penner and Anton Zeitlin, *Decorated super-Teichmüller space*, J. Differential Geom. **111** (2019), no. 3, 527–566.

[She] Ekaterina Shemyakova, *On super cluster algebras based on super Plucker and super Ptolemy relations*, Preprint 2022, arXiv:2206.12072.

[Thu86] William Thurston, *A norm for the homology of* 3*-manifolds*, Mem. Amer. Math. Soc. **59** (1986), no. 339, i–vi and 99–130.

[Thu97] ———, *Three-dimensional geometry and topology. Vol. 1*, Princeton Mathematical Series, vol. 35, Princeton University Press, Princeton, NJ, 1997, Edited by Silvio Levy, `http://msri.org/publications/books/gt3m`.

[Wit] Edward Witten, *Notes on Super Riemann Surfaces and their Moduli*, Preprint 2012, arXiv:1209.2459.

[Yoo] Seokbeom Yoon, *The twisted 1-loop invariant and the Jacobian of Ptolemy coordinates*, Preprint 2021, arXiv:2110.11003.

[Yoo19] ———, *The volume and Chern-Simons invariant of a Dehn-filled manifold*, Topology Appl. **256** (2019), 208–227.

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