NAHM SUMS, STABILITY AND THE COLORED JONES POLYNOMIAL

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Abstract. Nahm sums are q-series of a special hypergeometric type that appear in character formulas in Conformal Field Theory, and give rise to elements of the Bloch group, and have interesting modularity properties. In our paper, we show how Nahm sums arise naturally in Quantum Knot Theory, namely we prove the stability of the coefficients of the colored Jones polynomial of an alternating link and present a Nahm sum formula for the resulting power series, defined in terms of a reduced diagram of the alternating link. The Nahm sum formula comes with a computer implementation, illustrated in numerous examples of proven or conjectural identities among q-series.

Contents

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1. Introduction

The colored Jones polynomial of a link is a sequence of Laurent polynomials in one variable with integer coefficients. We prove in full a conjecture concerning the stability of the colored Jones polynomial for all alternating links.

A weaker form of stability (0-stability, defined below) for the colored Jones polynomial of an alternating knot was conjectured by Dasbach and Lin. The 0-stability is also proven independently by Armond for all adequate links [Arm13], which include alternating links and closures of positive braids, see also [Arm14]. The advantage of our approach is that it proves stability to all orders and gives explicit formulas (in the form of generalized Nahm sums) for the limiting series, which in particular implies convergence in the open unit disk in the q-plane and allow for the study of their redial asymptotics.

Stability was observed in some examples by Zagier, and conjectured by the first author to hold for all knots, assuming that we restrict the sequence of colored Jones polynomials to suitable arithmetic progressions, dictated by the quasi-polynomial nature of its q-degree [Gar11b, Gar11a]. Zagier asked about modular and asymptotic properties of the limiting q-series. In a similar direction, Habiro asked about 0-stability of the cyclotomic function of alternating links in [Hab10].

Our generalized Nahm sum formula comes with a computer implementation (using as input a planar diagram of a link), and allows the study of its asymptotics when q approaches radially a root of unity. Our Nahm sum formula is reminiscent to the cohomological Hall algebra of motivic Donaldson-Thomas invariants of Kontsevich-Soibelman [KS11], and complement recent work of Witten [Wit12] and Dimofte-Gaiotto-Gukov [DGG].

1.1. **Nahm sums.** Recall the *quantum factorial* and *quantum Pochhammer symbol* defined by [And76]:

$$(x;q)_n = \prod_{k=0}^{n-1}(1-xq^k), \qquad (x;q)_\infty = \prod_{k=0}^{\infty}(1-xq^k)$$

We will abbreviate $(x;q)_n$ by $(x)_n$.

In [NRT93] Nahm studied q-hypergeometric series $f(q) \in \mathbb{Z}[[q]]$ of the form

$$f(q) = \sum_{n_1,\dots,n_r \geq 0} \frac{q^{\frac{1}{2}n^t \cdot A \cdot n + b \cdot n}}{(q)_{n_1} \dots (q)_{n_r}}$$

where A is a positive definite even integral symmetric matrix and $b \in \mathbb{Z}^r$.

Nahm sums appear in character formulas in Conformal Field Theory, and define analytic functions in the complex unit disk $|q| < 1$ with interesting asymptotics at complex roots of unity, and with sometimes modular behavior. Examples of Nahm sums are the seven famous, mysterious q-series of Ramanujan that are nearly modular (in modern terms, mock modular). For a detailed discussion, see [Zag09]. Nahm sums give rise to elements of the

Bloch group, which governs the leading radial asymptotics of $f(q)$ as q approaches a complex root of unity. Nahm's Conjecture concerns the modularity of a Nahm sum $f(q)$, and was studied extensively by Zagier, Vlasenko-Zwegers and others [VZ11, Zag07].

The limit of the colored Jones function of an alternating link leads us to consider generalized Nahm sums of the form

$$\Phi(q) = \sum_{n \in C \cap \mathbb{N}^r} (-1)^{a \cdot n} \frac{q^{\frac{1}{2} n^t \cdot A \cdot n + b \cdot n}}{(q)_{n_1} \dots (q)_{n_r}} \tag{1}$$

where C is a rational polyhedral cone in $\mathbb{R}^r$, $b, a \in \mathbb{Z}^r$ and A is a symmetric (possibly indefinite) symmetric matrix. We will say that the generalized Nahm sum (1) is *regular* if the function

$$n \in C \cap \mathbb{N}^r \mapsto \frac{1}{2} n^t \cdot A \cdot n + b \cdot n$$

is proper and bounded below. Regularity ensures that the series (1) is a well-defined element of the Novikov ring

$$\mathbb{Z}((q)) = \{\sum_{n \in \mathbb{Z}} a_n q^n \,|\, a_n = 0, \, n \ll 0\}$$

of power series in q with integer coefficients and bounded below minimum degree. In the remaining of the paper, by Nahm sum we will mean a generalized Nahm sum. The paper is concerned with a new source of Nahm sums that originate in Quantum Knot Theory.

1.2. **Stability of a sequence of polynomials.** For $f(q) = \sum a_j q^j \in \mathbb{Z}((q))$ let $\mathrm{mindeg}_q f(q)$ denote the smallest j such that $a_j \neq 0$ and let $\mathrm{coeff}(f(q), q^j) = a_j$ denote the coefficient of q^j in $f(q)$.

Definition 1.1. Suppose $f_n(q), f(q) \in \mathbb{Z}((q))$. We write that

$$\lim_{n \to \infty} f_n(q) = f(q)$$

if

- there exists C such that $\mathrm{mindeg}_q(f_n(q)) \geq C$ for all n, and
- for every $j \in \mathbb{Z}$,

$$\lim_{n \to \infty} \mathrm{coeff}(f_n(q), q^j) = \mathrm{coeff}(f(q), q^j). \tag{2}$$

Since Equation (2) involves a limit of integers, the above definition implies that for each j, there exists N_j such that

$$f_n(q) - f(q) \in q^j \mathbb{Z}[[q]]$$

(and in particular, $\mathrm{coeff}(f_n(q), q^j) = \mathrm{coeff}(f(q), q^j)$) for all $n > N_j$.

Remark 1.2. Although for every integer j we have $\lim_{n \to \infty} \mathrm{coeff}(q^{-n^2}, q^j) = 0$, it is not true that $\lim_{n \to \infty} q^{-n^2} = 0$.

Definition 1.3. A sequence $f_n(q) \in \mathbb{Z}[[q]]$ is *k-stable* if there exist $\Phi_j(q) \in \mathbb{Z}((q))$ for $j = 0, \dots, k$ such that

$$\lim_{n \to \infty} q^{-k(n+1)} \left(f_n(q) - \sum_{j=0}^{k} \Phi_j(q) q^{j(n+1)} \right) = 0 \tag{3}$$

We say that $(f_n(q))$ is *stable* if it is k-stable for all k. Notice that if $f_n(q)$ is k-stable, then it is k'-stable for all $k' < k$ and moreover $\Phi_j(q)$ for $j = 0, \dots, k$ is uniquely determined by $f_n(q)$. We call $\Phi_k(q)$ the k-limit of $(f_n(q))$. For a stable sequence $(f_n(q))$, its associated series is given by

$$F_f(x,q) = \sum_{k=0}^{\infty} \Phi_k(q) x^k \in \mathbb{Z}((q))[[x]] \,.$$

It is easy to see that the pointwise sum and product of k-stable sequences are k-stable.

1.3. **Stability of the colored Jones function for alternating links.** Given a link K, let $J_{K,n}(q) \in \mathbb{Z}[q^{\pm 1/2}]$ denote its colored Jones polynomial (see e.g. [Oht02, Tur88]) with each component colored by the $(n+1)$-dimensional irreducible representation of $\mathfrak{sl}_2$ and normalized by

$$J_{\text{Unknot},n}(q) = (q^{(n+1)/2} - q^{-(n+1)/2})/(q^{1/2} - q^{-1/2}) \,.$$

When K is an *alternating* link, the lowest degree of $J_{K,n}(q)$ is known and the lowest coefficient is ± 1 (see [Lê06, Lic97] and Section 7). We divide $J_{K,n}(q)$ by its lowest monomial to obtain $\hat{J}_{K,n}(q)$. Although $J_{K,n}(q) \in \mathbb{Z}[q^{1/2}]$, we have $\hat{J}_{K,n}(q) \in \mathbb{Z}[q]$; see [Le00].

Our main results link the colored Jones polynomial and its stability with Nahm sums. The first part of the result, with proof given in Section 9, is the following.

Theorem 1.4. *For every alternating link K, the sequence $(\hat{J}_{K,n}(q))$ is stable and its associated k-limit $\Phi_{K,k}(q)$ and series $F_K(x,q)$ can be effectively computed from any reduced, alternating diagram D of K.*

Let us give some remarks regarding Theorem 1.4.

Remark 1.5. If one uses the new normalization where with $J_{\text{Unknot},n}(q) = 1$, the above theorem still holds. The new $F_K(x,q)$ is equal to the old one times $(1-q)/(1-x)$.

Remark 1.6. If $\bar{K}$ is the mirror image of K, then $J_{\bar{K},n}(q^{-1}) = J_{K,n}(q)$. If K is alternating, then so is $\bar{K}$. Hence, applying Theorem 1.4 to $\bar{K}$, we see that similar stability result holds for the head of the colored Jones polynomial of alternating link.

Remark 1.7. The weaker 0-stability (conjectured by Dasbach and Lin) is proven independently by Armond [Arm14]. In [Arm14], 0-stability is proved for all A-adequate links, which include all alternating links, but no stability in full is proven there, nor any formula for the 0-limit is given. As we will see, the proof of stability in full is more complicated than that of 0-stability and occupies the more difficult part of our paper, given in Sections 8-10.

Remark 1.8. A sharp estimate regarding the rate of convergence of the stable sequence $\hat{J}_{K,n}(q)$ is given in Theorem 1.16.

1.4. **Explicit Nahm sum formulas for the 0-limit and 1-limit.** Throughout this subsection D is a reduced diagram of a *non-split* alternating link K with c crossings.

1.4.1. *Laplacian of a graph.* In this paper a graph is a finite one-dimensional CW-complex. *A plane graph* is a graph Γ (with loops and multiple edges allowed) together with an embedding of Γ into $\mathbb{R}^2 \subset S^2$. A plane graph Γ gives rise to a polygonal complex structure of S^2, and its set of *vertices*, set of *edges*, and set of *polygons* are denoted respectively by $\mathcal{V}(\Gamma)$, $\mathcal{E}(\Gamma)$ and $\mathcal{P}(\Gamma)$.

The *adjacency matrix* $\mathrm{Adj}(\Gamma)$ is the $\mathcal{V}(\Gamma) \times \mathcal{V}(\Gamma)$ matrix defined such that $\mathrm{Adj}(\Gamma)(v, v')$ is the number of edges connecting v and v'. Let $\mathrm{Deg}(G)$ be the diagonal $\mathcal{V}(\Gamma) \times \mathcal{V}(\Gamma)$ matrix such that $\mathrm{Deg}(\Gamma)(v, v)$ is the degree of the vertex v, i.e. the number of edges incident to v, with the convention that each loop edge at v is counted twice.

The *Laplacian* $\mathcal{L}(\Gamma) := -\mathrm{Deg}(\Gamma) + \mathrm{Adj}(\Gamma)$ plays an important role in graph theory.

1.4.2. *Graphs associated to a reduced alternating non-split link diagram D.* The diagram D gives rise to a polygonal complex of $S^2 = \mathbb{R}^2 \cup \infty$ with c vertices, $2c$ edges, and $c+2$ polygons. Since D is alternating, there is a way to assign a color A or B to each polygon such that in a neighborhood of each crossing the colors are as in the following figure, see e.g. [Tur87, p.217].

Figure 1. A checkerboard coloring of alternating planar projections

This is the usual checkerboard coloring of the regions of an alternating link diagram, used already by Tait. When we rotate the overcrossing arc at a crossing counterclockwise (resp. clockwise), we swap a A-type (resp. B-type) angle. Note that orientation dose not take part in the definition of A-angles and B-angles.

Let D^* be the dual of the plane graph D. Since D has a checkerboard coloring of its faces, it follows that D^* has a coloring of its vertices by A or B. Thus, $\mathcal{V}(D^*) = \mathcal{V}_A \sqcup \mathcal{V}_B$ give a *bipartite* structure on $\mathcal{V}$ where $\mathcal{V}_A$ and $\mathcal{V}_B$ are the sets of A-colored and B-colored vertices of $\mathcal{V}$.

Since the degree of each vertex of D is 4, each polygon of D^* is a quadrilateral, having 4 vertices, two of which are A-vertices and two are B-vertices. Moreover, the vertices of each quadrilateral alternate in color. Connect the two B-vertices of each quadrilateral of D^* by a diagonal inside that quadrilateral, and call it a $\mathcal{T}$-edge. *The Tait graph of D* is defined to be the plane graph $\mathcal{T}$ whose set of vertices is $\mathcal{V}_B$, and whose set of edges is the set of $\mathcal{T}$-egdes. The plane graph Tait graph totally determines the alternating link K up to orientation. The graph $\mathcal{T}$ can be defined for any link diagram, and is studied extensively, see e.g. [FKP11, Oza11, Thi88].

Note that for a vertex $v \in \mathcal{V}_B$, its degrees in D^* and in $\mathcal{T}$ are the same.

1.4.3. *The lattice and the cone.* Fix an A-vertex of D^* and call it v_∞. We will focus on $\Lambda := \mathbb{Z}[\mathcal{V}(D^*)]$, the $\mathbb{Z}$-lattice of rank $c+2$ freely spanned by the vertices of D^*. Let $\Lambda_0 = \mathbb{Z}[\mathcal{V}(D^*) \setminus \{v_\infty\}]$, a sublattice of Λ of rank $c+1$.

For an edge $e \in \mathcal{E}(D^*)$, define the $\mathbb{Z}$-linear map $e : \Lambda \to \mathbb{Z}$ by

$$e(v) = \begin{cases} 1 & \text{if } v \text{ is a vertex of } e, \\ 0 & \text{otherwise.} \end{cases}$$

An element $x \in \Lambda_0$ is *admissible* if $e(x) \geq 0$ for every edge $e \in \mathcal{E}(D^*)$. The set $\mathrm{Adm} \subset \Lambda_0$ of all admissible elements is the intersection of Λ_0 with a rational convex cone in $\Lambda \otimes \mathbb{R}$.

Define the $\mathbb{Z}$-linear map $L : \Lambda \to \frac{1}{2}\mathbb{Z}$ by

$$L(v) = \begin{cases} 1 & \text{if } v \in \mathcal{V}_B \\ \frac{\deg(v)}{2} - 1 & \text{if } v \in \mathcal{V}_A. \end{cases}$$

Let $\mathcal{Q}$ be the symmetric $\mathcal{V}(D^*) \times \mathcal{V}(D^*)$ matrix defined by

$$\mathcal{Q} := \mathrm{Deg}(D^*) + \mathrm{Adj}(D^*) + \mathcal{L}(\mathcal{T}). \tag{4}$$

Note that a priori $\mathcal{L}(\mathcal{T})$ is a $\mathcal{V}_B \times \mathcal{V}_B$ matrix, and is considered as a $\mathcal{V}(D^*) \times \mathcal{V}(D^*)$ matrix in the right hand side of (4) by the trivial extension, i.e. in the extension, any entry outside the block $\mathcal{V}_B \times \mathcal{V}_B$ is 0.

The symmetric matrix $\mathcal{Q}$ defines a symmetric bilinear form $\mathcal{Q}(x, y) : \Lambda \otimes_{\mathbb{Z}} \Lambda \to \mathbb{Z}$. Let $Q : \Lambda \to \frac{1}{2}\mathbb{Z}$ be the corresponding quadratic form, i.e.

$$Q(\lambda) := \frac{1}{2}\mathcal{Q}(\lambda, \lambda).$$

Remark 1.9. Although $Q(\lambda)$ and $L(\lambda)$ take value in $\frac{1}{2}\mathbb{Z}$, we later show that $Q(\lambda)+L(\lambda) \in \mathbb{Z}$. While Q, L depend only on D, the set Adm depends on the choice of an A-vertex v_∞.

Examples that illustrate the above definitions are given in Section 1.6.

1.4.4. *Nahm sum for the* 0*-limit.* The next theorem is proven in Section 7.

Theorem 1.10. *Suppose D is a reduced alternating diagram of a non-split link K. Fix any choice of v_∞. Then the* 0*-limit of $\hat{J}_{K,n}(q)$ is equal to*

$$\Phi_{K,0}(q) = (q)_\infty^c \sum_{\lambda \in \mathrm{Adm}} (-1)^{2L(\lambda)} \frac{q^{Q(\lambda)+L(\lambda)}}{\prod_{e \in \mathcal{E}(D^*)} (q)_{e(\lambda)}}. \tag{5}$$

The generalized Nahm sum on the right hand side is regular and belongs to $\mathbb{Z}[[q]]$.

A categorification of the above theorem was given recently by Rozansky [Roz12]. Here are two consequences of this explicit formula. The next corollary is proven in Section 7.4.

Corollary 1.11. For every alternating link K, $\Phi_{K,0}(q) \in \mathbb{Z}[[q]]$ is analytic in the unit disk $|q| < 1$.

The next corollary is shown in Section 13.

Corollary 1.12. If the reduced Tait graphs of two alternating links K_1, K_2 are isomorphic as abstract graphs, then they have the same 0-limit, $\Phi_{K_1,0}(q) = \Phi_{K_2,0}(q)$.

Here the reduced Tait graph $\mathcal{T}'$ is obtained from $\mathcal{T}$ by replacing every set of parallel edges by an edge; and two edges are parallel if they connect the same two vertices. This corollary had been proven by Armond and Dasbach: in [AD11], it is proved that if two alternating links have the same reduced Tait graph, and the 0-limit of the first link exists, then the 0-limit of the second one exists and is equal to that of the first one. In section 13 we will derive Corollary 1.12 from the explicit formula of Theorem 1.10.

We end this section with a remark on normalizations.

Remark 1.13. The colored Jones polynomial $J_{K,n}(q)$ (and consequently, its shifted version $\hat{J}_{K,n}(q) \in 1 + q\mathbb{Z}[q]$) is independent of the orientation of the components of a link K [Tur94]. With our normalization we have

$$\begin{aligned}\Phi_{\mathrm{Unknot},0}(q) &= \frac{1}{1-q}, \qquad F_{\mathrm{Unknot}}(x,q) = \frac{1-x}{1-q} \\ \Phi_{K_1 \sqcup K_2,0}(q) &= \Phi_{K_1,0}(q)\Phi_{K_2,0}(q) \\ \Phi_{K_1 \sharp K_2,0}(q) &= (1-q)\Phi_{K_1,0}(q)\Phi_{K_2,0}(q)\end{aligned}$$

where $\sqcup$ and $\sharp$ denotes the disjoint union and the connected sum respectively.

1.4.5. *The* 1*-limit.* For a quadrilateral p of D^*, define a $\mathbb{Z}$-linear map $p : \Lambda \to \mathbb{Z}$ by

$$p(v) = \begin{cases} 1 & \text{if } v \text{ is one of the four vertices of } p \\ 0 & \text{otherwise.} \end{cases}$$

The next theorem is proven in Section 12.2.

Theorem 1.14. *Suppose D is a reduced alternating diagram of a non-split link L. Fix any choice of v_∞. The 1-limit of $\hat{J}_{K,n}(q)$ is*

$$(6)\ \Phi_{K,1}(q) = \frac{(q)_\infty^c}{1-q}\left(\sum_{\lambda\in\mathrm{Adm}}(-1)^{2L(\lambda)}\frac{q^{Q(\lambda)+L(\lambda)}}{\prod_{e\in\mathcal{E}(D^*)}(q)_{e(\lambda)}}\left(\sum_{e\in\mathcal{E}(D^*)}q^{-e(\lambda)}-\sum_{p\in\mathcal{P}(D^*)}q^{-p(\lambda)}\right)\right.$$
$$\left.-\sum_{v\in\mathcal{V}_B}\frac{1}{(q)_\infty^{\deg(v)}}\sum_{\lambda\in\mathrm{Adm}_v}(-1)^{2L(\lambda)}\ \frac{q^{Q(\lambda)+L(\lambda)}}{\prod_{e\in\mathcal{E}(D^*)}(q)_{e(\lambda)}}\right).$$

where Adm_v is the set of all admissible x such that $p(x) = 0$ for every $p \in \mathcal{P}(D^*)$ incident to v.

Remark 1.15. The fact that the series (6) is convergent is not obvious. It follows from the fact that we can separate the sum over admissible states λ to those which are not 1-bounded and those which are 1-bounded. Here, a state λ is 1-unbounded if $Q(\lambda) + L(\lambda) > 4/3\max\{p(\lambda) \,|\, p \in \mathcal{P}(D^*)\}$; see Definition 8.1. It is easy to see that the contribution of the 1-unbounded states in (6) forms a convergent series. For the 1-bounded states s, one uses a decomposition theorem $s = ms_P + s'$ discussed in Example 8.8. Then, the contribution of such a state to (6) comes with minimum degree $Q(s') + L(s') + m$. This implies that the contribution of the 1-bounded states in (6) forms a convergent series too.

For an example illustrating Theorems 1.10 and 1.14, see Section 1.6.

1.5. ***q*-holonomicity.** Recall the notion of a q-holonomic sequence sequence and series from [Zei90, PWZ96]. We say that $f_n(q)$, belonging to a $\mathbb{Z}[q^{\pm 1}]$-module for $n = 1, 2, \dots,$ is q-*holonomic* if it satisfies a linear recursion of the form

$$\sum_{j=0}^{d} c_j(q^n, q) f_{n+j}(q) = 0\,, \tag{7}$$

for all $n \in \mathbb{N}$ where $c_j(u, v) \in \mathbb{Z}[u, v]$ for all j and $c_d \neq 0$.

The next theorem (proven in Section 11) shows the q-holonomicity of $\Phi_{K,n}(q)$ for an alternating link, and gives a sharp improvement of the rate of convergence in the definition of stability.

Theorem 1.16. (a) For every alternating link K, $\Phi_{K,n}(q)$ is q-holonomic.
(b) Moreover, there exist constants C and C' such that

$$\mathrm{mindeg}_q(\Phi_{K,k}(q)) \geq -Ck^2 - C' \tag{8}$$

for all k and

$$\left(f_n(q) - \sum_{j=0}^{k} \Phi_k(q) q^{j(n+1)} \right) q^{-k(n+1)} \in q^{n+1-C(k+1)^2-C'}\mathbb{Z}[[q]] \tag{9}$$

for all k when n is sufficiently large (depending on k).

The lowest q exponent in Equation (9) is a quadratic function of k. This result is sharp when $K = 4_1$ knot [GZa, GZb].

Question 1.17. Does $F_{K,n}(x, q)$ uniquely determine the sequence $(\hat{J}_{K,n}(q))$ for the case of knots?

1.6. **Applications: *q*-series identities.** In this section we illustrate Theorem 1.10 explicitly for the 4_1 knot. Consider the planar projection D of 4_1 given in Figure 2. This planar projection is A-infinite.

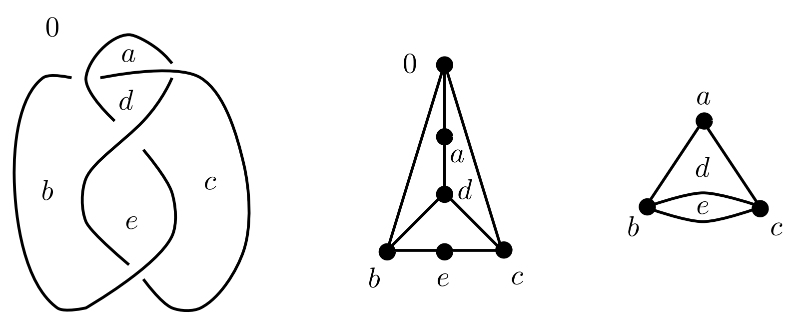


Figure 2. A planar projection D of the 4_1 knot on the left, the dual graph D^* in the middle and the Tait graph $\mathcal{T}$ on the right.

To compute $\Phi_{4_1,0}(q)$, proceed as follows:

- Checkerboard color the regions of D with A or B with the unbounded region colored by A.

- Assign variables a, b, c to the three B-regions and e, f to the two bounded A-regions, and assign 0 to the unbounded A-region. Let $\lambda = (a, b, c, d, e)^T$.
- Color each arc of the diagram D with the sum of the colors of its two neighboring regions. λ is admissible if the color of each arc is a nonnegative integer number, i.e., $\lambda \in \mathbb{Z}^5$ satisfies

$$a,\ b,\ c,\ a+d,\ b+d,\ c+d,\ b+e,\ c+e \geq 0\,.$$

- Construct a square matrix (and a corresponding quadratic form $Q(\lambda)$) which consists of four blocks: BB-block, AB-block, BA-block and AA block. On the BB, AB and BA blocks we place the adjacency matrix of the corresponding regions: the adjacency number between two distinct B-regions is the number of common vertices, whereas the adjacency number between an A-region and a B-region is the number of common edges. In the case when two regions share common vertices, the adjacency number is the number of common vertices. On the AA-block we place the diagonal matrix whose diagonal entries are the number of sides of each A-region.
- We construct a linear form $L(\lambda)$ in λ where the coefficient of each B-variable a, b, c is one, and the coefficient of each A-variable d, e is half the number of the sides of the corresponding region minus 1.

Explicitly, with the conventions of Figure 2 we have

$$Q(\lambda) = \frac{1}{2}\lambda^T \left(\begin{array}{ccc|cc} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 & 1 \\ 1 & 2 & 0 & 1 & 1 \\ \hline 1 & 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 & 2 \end{array}\right)\lambda, \qquad L(\lambda) = (1, 1, 1, \frac{1}{2}, 0)\lambda\ .$$

Then,

$$\Phi_{4_1,0}(q) = (q)_\infty^4 \sum_{\lambda \in \mathrm{Adm}} (-1)^d \frac{q^{Q(\lambda)+L(\lambda)}}{(q)_a(q)_b(q)_c(q)_{a+d}(q)_{b+d}(q)_{c+d}(q)_{b+e}(q)_{c+e}}\,.$$

Alternative formulas for the colored Jones polynomial of 4_1 lead to identities among q-series. For instance, the Habiro formula for 4_1 [Hab08] combined with the above formula for $\Phi_{4_1,0}(q)$ leads to the following identity:

$$\frac{1}{(1-q)(q)_\infty^3} = \sum_{\lambda \in \mathrm{Adm}} (-1)^d \frac{q^{Q(\lambda)+L(\lambda)}}{(q)_a(q)_b(q)_c(q)_{a+d}(q)_{b+d}(q)_{c+d}(q)_{b+e}(q)_{c+e}}\,. \tag{10}$$

The above identity has been proven by Armond-Dasbach. A detailed list of identities for knots with knots with at most 8 crossings is given in Appendix D.

1.7. **Extensions of stability.** The methods that prove Theorem 1.4 are general and apply to several other circumstances of q-holonomic sequences that appear in Quantum Topology. We will list two results here.

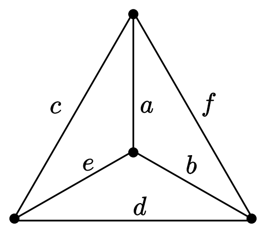
Theorem 1.18. *If K is a positive link, then $\hat{J}_{K,n}(q)$ is stable and the corresponding limit $F_K(x,q)$ is obtained by a Nahm sum associated to a positive downwards diagram of K. Moreover, for every $k \in \mathbb{N}$ we have $\Phi_{K,k}(q) \in \mathbb{Z}[q^{\pm 1}]$.*

The proof of the above theorem is easier than that of Theorem 1.4 since it does not require to center the states of the R-matrix state sum of a positive link. An example that illustrates the above theorem is taken from [HL05, Sec.1.1.4]: for the right handed trefoil 3_1, its associated series is

$$F_{3_1}(x,q) = \frac{1-x}{1-q} \sum_{k=0}^{\infty} x^k \left(1 - \frac{x}{q}\right) \dots \left(1 - \frac{x}{q^k}\right) \in \mathbb{Z}[q^{\pm 1}][[x]] .$$

Some results related to the 0-stability of a class of positive knots are obtained in [CK13].

Next we discuss an extension of Theorem 1.4 to evaluations of quantum spin networks. For a detailed discussion of those, we refer the reader to [Cos14, KL94, GvdV12]. Using the notation of [GvdV12], let $\gamma = (a,b,c,d,e,f)$ be an admissible coloring of the edges of the standard tetrahedron



Consider the standard spin network evaluation $J_{\triangle, n\gamma}(q) \in \mathbb{Z}[q^{\pm 1}]$ [GvdV12, Cos14].

Theorem 1.19. *For every admissible γ, the sequence $\hat{J}_{\triangle, n\gamma}(q)$ is stable, and its limit is given by a Nahm sum.*

For example, if $\gamma = (2,2,2,2,2,2)$, then

$$\hat{J}_{\triangle, n\gamma}(q) = \frac{1}{1-q} \sum_{k=0}^{n} (-1)^k \frac{q^{\frac{3}{2}k^2 + \frac{1}{2}k}}{(q)_k^3} \frac{(q)_{4n+1-k}}{(q)_k^3 (q)_{n-k}^4} ,$$

and

$$F_{\triangle}(x,q) = \frac{1}{(1-q)(q)_\infty^3} \sum_{k=0}^{\infty} (-1)^k \frac{q^{\frac{3}{2}k^2 + \frac{1}{2}k}}{(q)_k^3} \frac{(xq^{-k})_\infty^4}{(x^4 q^{-k+1})_\infty} \in \mathbb{Z}((q))[[x]] ,$$

where $x = q^{n+1}$. In particular,

$$\Phi_{\triangle, 0}(q) = \frac{1}{(1-q)(q)_\infty^3} \sum_{k=0}^{\infty} (-1)^k \frac{q^{\frac{3}{2}k^2 + \frac{1}{2}k}}{(q)_k^3} \in \mathbb{Z}[[q]] .$$

The proof of the above theorem follows easily from the fact that the quantum 6j-symbol is given by a 1-dimensional sum of a q-proper hypergeometric summand, and the sum is already centered. The analytic and arithmetic properties of the corresponding Nahm sum will be discussed in forthcoming work [GZa, GZb].

1.8. **Plan of the proof.** The strategy to prove Theorems 1.10 and 1.4 is the following.

We begin with the R-matrix state sum for the colored Jones polynomial, reviewed in Sections 2.2-2.4.

We center the downward diagram, its corresponding states and their weights in Section 4.1.

We factorize the weights of the centered states as the product of a monomial and an element of $\mathbb{Z}_{\succ}[q]$ in Section 4.1. The advantage of using centered states is that the lowest q-degree of their weights is the sum of a quadratic function $Q(s)$ of s with a quadratic function of n.

Although $Q(s)$ is not a positive definite quadratic form, in Section 5.3 we show that $Q(s)$ is copositive on the cone of the centered states. The proof uses the combinatorics of alternating downward diagrams, and their centered states, reminiscent to the Kauffman bracket.

Section 7 we prove the 0-stability Theorem 1.10.

If $Q(s)$ were positive definite, then it would be easy to deduce Theorem 1.4. Unfortunately, $Q(s)$ is never positive definite, and it always has directions of linear growth in the cone of centered states. In Sections 8 we state a partition of the set of k-bounded states, and prove stability away from the region of linear growth. In Section 9 deal with stability in the region of linear growth.

Section 10 is rather technical, and gives a proof of the key Proposition 8.7 that partitions the set of k-bounded states.

Section 11 deduces the q-holonomicity of the sequence $\Phi_{K,k}(q)$ of an alternating link from the q-holonomicity of the corresponding colored Jones polynomial. As a result, we obtain sharp quadratic lower bounds for the minimum degree of $\Phi_{K,k}(q)$ and sharp bounds for the convergence of the colored Jones polynomial stated in Theorem 1.16.

In Section 12 we give an algorithm for computing $\Phi_{K,k}(q)$ from a reduced alternating planar projection.

In Section 13 we prove that $\Phi_{K,0}(q)$ is determined by the reduced Tait graph of an alternating link K.

In Section 14 we give some illustrations of Theorems 1.10 and 1.4.

1.9. **Follow-up work.** The topic of stability (affectionately called *head/tail* by Dasbach-Lin [DL06]) has recently attracted a lot of attention. After the papearance of our paper on the `arxiv` in the late 2011, a number of papers have since been posted. Among them, Hajij gives a skein-theory proof of 0-stability for alternating links and some quantum spin networks [Haj14a, Haj14b]. Motivated by the q-series of Nahm type, Andrews proves some Rogers-Ramanujan type identities [And13]. Vuong and the first author give efficient algorithms to compute the 0-limit of an alternating knot [GV13]. Vuong-Norin and the first author identify the coefficients of q^k of the 0-limit for $k = 0, \dots, 3$ in terms of graph countings of induced plane subgraphs of the reduced Tait graph of an alternating link [GNV13]. Norin and the first author prove that each coefficient of q^k in the 0-limit is a polynomial of induced plane subgraphs of the reduced Tait graph of an alternating link [GN14]. Finally, in another direction, q-series of Nahm type were studied by Beem-Dimofte-Pasquetti and Kashaev and the first author; see [BDP12, GK].

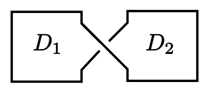
Acknowledgment. An early version of the paper was presented in talks of the first author to a Spring School in Geometry and Quantum Topology in the Diablerets 2011, and in the Mathematische Arbeitstagung in Bonn, 2011. The authors wish to thank the organizers of the above conferences for their hospitality, C. Armond and O. Dasbach for explaining to us their beautiful work. The first named author wishes to thank T. Dimofte and D. Zagier for their interest, encouragement and for the generous sharing of their ideas.

2. The R-matrix state-sum of the colored Jones polynomial

In this section we review the R-matrix state sum of the colored Jones function, discussed in detail in [Tur88, Tur94, Oht02]. We will use the following standard notation in q-calculus.

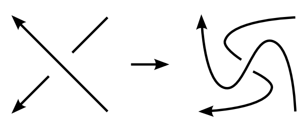
$$\binom{a}{b}_q = \frac{(q;q)_a}{(q;q)_b(q;q)_{a-b}}, \qquad \text{for } a,b \in \mathbb{N}, b \le a.$$

2.1. **Downward link diagram.** Recall that a link diagram $D \subset \mathbb{R}^2$ is *alternating* if walking along it, the sequence of crossings alternates from overcrossings to undercrossings. A diagram D is *reduced* if it is not of the form



where D_1 and D_2 are diagrams with at least one crossing.

A *downward link diagrams of links* is an oriented link diagram in the standard plane in general position (with its height function) such that at every crossing the orientation of both strands of the link is downward. A usual link diagram may not satisfy the downward requirement on the orientation at a crossing. However, it is easy to convert a link diagram into a downward one by rotating the non-downward crossings as follows:



2.2. **Link diagrams and states.** Fix a downward link diagram D of an oriented link K with c_D crossing. Considering D as a 4-valent graph, it has $2c_D$ edges. A *state* of D is a map

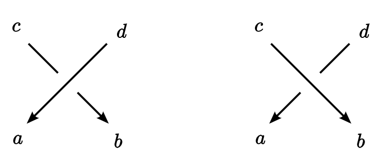
$$r : \{\text{edges of } D\} \to \mathbb{R}$$

such that at every crossing we have

$$a + b = c + d,$$

where a, b, c, d are the values of s of the edges incident to the crossing as in the following figure

(11)



The set $S_{D,\mathbb{R}}$ of all states of D is a vector space. For a state $r \in S_{D,\mathbb{R}}$ and a crossing v of D define

$$r(v) = \text{sign}(v)\,(a - d)\,,$$

where as usual the sign of the crossing on the left hand side of (11) is positive and the sign of the one on the right hand side is negative. For a positive integer n, a state $r \in S_{D,\mathbb{R}}$ is called *n-admissible* if the values of r are integers in $[0, n]$ and $r(v) \ge 0$ for every crossing v. Let $S_{D,n}$ be the set of all n-admissible states.

Remark 2.1. Later we will prove that $\dim S_{D,\mathbb{R}} = c_D + 1$. By definition, $S_{D,n}$ in 1-1 correspondence with the set $nP_D \cap \mathbb{Z}^{2c_D}$ of lattice points of nP_D for a lattice polytope P_D in $\mathbb{R}^{2c_D}$ where c_D is the number of crossings of D.

2.3. **Winding number and its local weight.** Suppose α is an oriented simple closed curve in the standard plane. By the winding number $W(\alpha)$ we mean the winding number of α with respect to a point in the region bounded by α. Observe that $W(\alpha) = 1$ if α is counterclockwise, -1 if otherwise.

The winding number $W(\alpha)$ can be calculated by a local weight sum as follows. A *local part* of α is a small neighborhood of a local maximum or minmum. For a local part X define $W(X) = 1/2$ if X is winding counterclockwise, $-1/2$ if otherwise. In other words, we have

W () $=W$ () $= 1/2$, W () $=W$ () $= -1/2$.

The next lemma is elementary.

Lemma 2.2. For every simple closed curve α,

$$W(\alpha) = \sum_X W(X)\,, \tag{12}$$

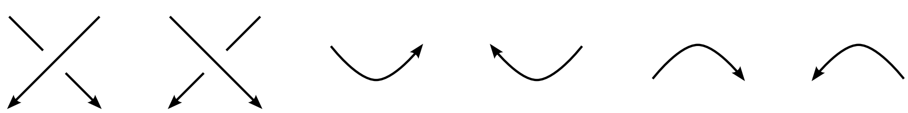
where the sum is over all the local parts of α.

2.4. **Local weights, the colored Jones polynomial, and their factorization.** Consider the monoid

$$\mathbb{Z}_\succ[q] = 1 + q\mathbb{Z}[q]\,.$$

Fix a natural number $n \geq 1$ and a downward link diagram D.

A local part of D is a small neighborhood of a crossing or a local extreme of D. There are six types of local parts of D: two types of crossings (positive or negative) and four types of local extrema (minima or maxima, oriented clockwise, or counterclockwise):

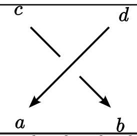
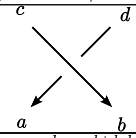
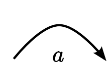
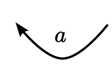
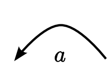
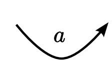
(13) 

For an n-admissible state r and a local part X, the weight $w(X,r)$ is defined by

$$w(X,r) = w_{\mathrm{lt}}(X,r)w_\succ(X,r)\,,$$

where $w_{\mathrm{lt}}(X,r) \in \{\pm q^{m/4} \mid m \in \mathbb{Z}\}$ is a monomial, $w_\succ(X,r) \in \mathbb{Z}_\succ[q]$, and $w_{\mathrm{lt}}(X,r)$ and $w_\succ(X,r)$ are given by Table 1.

Table 1. The local weights w_{lt} and $w_\succ$ of a state.

						
w_{lt}	$q^{(n+nd+nb-ab-dc)/2}$	$(-1)^{b-c}q^{(-n-nb-nd+bd+ac-b+c)/2}$	$q^{-(2a-n)/4}$	$q^{-(2a-n)/4}$	$q^{(2a-n)/4}$	$q^{(2a-n)/4}$
$\omega_\succ$	$(q;q)_{c-b}\binom{n-d}{a-d}_q\binom{c}{c-b}_q$	$(q;q)_{b-c}\binom{n-c}{b-c}_q\binom{d}{d-a}_q$	1	1	1	1

For a local extreme point X with the value of the state a, we have the convenient formula

$$w(X,a) = q^{W(X)(2a-n)/2}\,.$$

Let the weight of a state be defined by

$$w(r) = \prod_X w(X,r)\,,$$

where the product is over all the local parts of D. Then the unframed version of the colored Jones polynomial of the link K, each component of which is colored by the $n+1$-dimensional sl_2-module, is given by

$$J_{K,n}(q) = \sum_{r\in S_{D,n}} w(r)\,, \tag{14}$$

where $S_{D,n}$ is the set of all n-admissible states of D. For example, the value of the unknot is

$$J_{\mathrm{Unknot},n}(q) = [n+1] := \frac{q^{(n+1)/2} - q^{-(n+1)/2}}{q^{1/2} - q^{-1/2}}\,.$$

Note that $J_{K,0}(q) = 1$ for all links and $J_{K,1}(q^{-1})/J_{\mathrm{Unknot},1}(q^{-1})$ is the Jones polynomial of K [Jon87]. Since we could not find a reference for the state sum formula (14) in the literature, we will give a proof in the Appendix.

3. Alternating link diagrams and centered states

In this section we will discuss the combinatorics of alternating diagrams.

3.1. **Alternating link diagrams and A-infinite type.** Recall that a link diagram D gives rise to a polygonal complex structure of $S^2 = \mathbb{R}^2 \cup \{\infty\}$, and if D is alternating and connected, then the checkerboard coloring with colors A and B at each crossing looks like Figure 1.

If K is non-split, then D is a connected graph. If K is split, then D has several connected components. We will say that an alternating diagram D is A-*infinite* if the point $\infty \in S^2$ is contained in an A-polygon of every connected subgraph of D. It is clear that by moving the connected components of D around in S^2, we can assume that D is A-infinite. This will make the colors of different connected components compatible.

We will use the following obvious property of an A-infinite alternating link diagram: all the B-polygons are finite, i.e. in $\mathbb{R}^2 = S^2 \setminus \{\infty\}$.

For example, the left-handed trefoil given by the standard closure of the braid s_1^{-3} is A-infinite, whereas the right-handed trefoil given by the standard closure of the braid s_1^3 is not. Here s_1 is the standard generator of the braid group in two strands.

3.2. **The digraph $\vec{D}$ of an alternating diagram D.** Let D be an oriented link diagram. Recall that we consider D also as a graph whose edges are oriented. We say that an edge of D is *of type* O if it begins as an overpass, and *of type* U if it begins as an underpass. If D is alternating and one travel along the the link the edges alternate from type U to type O and vice-versa.

For a link diagram D let $\vec{D}$ be the directed graph on $\mathbb{R}^2$ obtained from the projection of D on the plane by reversing the orientation of all edges of type O. For example, a planar projection D and the corresponding digraph $\vec{D}$ of the 4_1 knot is shown in Figure 3.

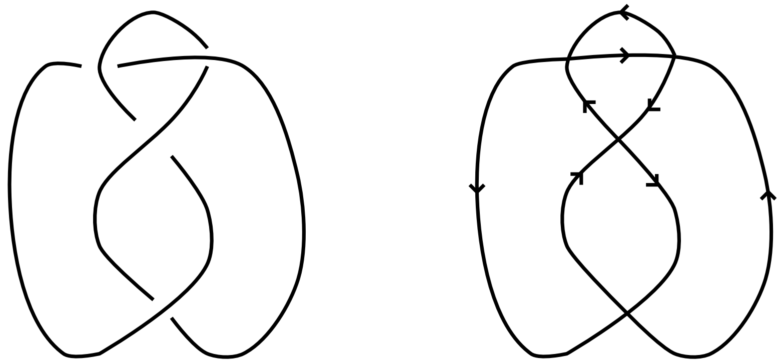
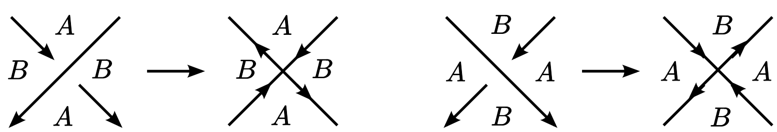


Figure 3. A planar projection D of the 4_1 knot on the left and the corresponding digraph $\vec{D}$ on the right. If we checkerboard color the faces of $\vec{D}$ with the unbounded one being white, then all black faces are oriented counter clockwise and all bounded white faces are oriented clockwise.

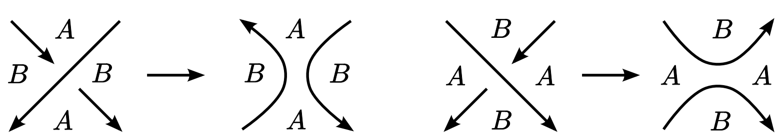
If D is *downward alternating*, it is easy to see that $\vec{D}$ is obtained from D by the following changing of orientations near a crossing point,

(15)



i.e., if the crossing is a positive one, then the two left edges incident to it get orientation reversed, and if the crossing is negative, then the two right edges incident to it get orientation reversed. We will retain the markings A and B for angles and regions of complements of $\vec{D}$.

At every vertex of $\vec{D}$ (or a crossing of D) there are two ways to smoothen the diagram. Following Kauffman [Kau87b] we call the A-*smoothening* (resp., B-*smoothening*) the one where the two A-regions (resp., B-regions) get connected. See the following figure for two examples of an A-smoothening.



Note that after either type of smoothening, the orientation of the edges of $\vec{D}$ is still well-defined.

Remark 3.1. Doing an A-resolution (resp. B-resolution) on a vertex of $\vec{D}$ is the same as doing an A-resolution (resp. B-resolution) on the original diagram D in the sense of Kauffman [Kau87a]. The advantage here, with directed graph $\vec{D}$ for the case of alternating links, is that the resulting graph of any resolution is still oriented.

Part (b) of the following lemma is where A-infinity is used in an essential way.

Lemma 3.2. Suppose D is an alternating link diagram.

(a) To the right of every oriented edge of $\vec{D}$ is an A-polygon, and to the left of every oriented edge of $\vec{D}$ is a B-polygon.

(b) Suppose D is A-infinite, then every circle obtained from $\vec{D}$ by after doing A-resolution at every vertex of $\vec{D}$ bounds a polygonal region of type B. Moreover every such circle is winding counterclockwise, i.e., it has winding number 1.

(c) If D is reduced, then each circle in (b) does not self-touch, i.e., the two arcs resulting from the A-resolution at one vertex do not belong to the same circle.

Proof. (a) follows easily by inspecting the directions of the edges and the markings of the regions at the two types of vertices of $\vec{D}$.

(b) The boundaries of the B-polygons are exactly the circles obtained from D after doing A-resolution at every vertex of D. Since the infinity region is not a B-type region, every circle does bound a B-type region in the plane $\mathbb{R}^2$. From part (a) it follows that each circle, which is the boundary of a polygonal region of type B, is counterclockwise.

(c) This is a well-known fact. A link diagram having the property that no circle obtained after doing A-resolution at every crossing has a self-touching point is known as an *A-adequate diagram*. In [Lic97, Prop.5.3] it was proved that every reduced alternating link diagram is A-adequate. □

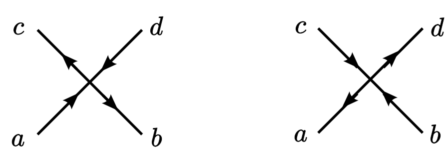
3.3. **Centered states.** Fix an alternating downward diagram D with c_D crossings and its directed graph $\vec{D}$. Recall that $\mathcal{E}(\vec{D})$ and $\mathcal{V}(\vec{D})$ denote respectively the set of oriented edges of $\vec{D}$ and the set of vertices of $\vec{D}$.

A *centered state* of $\vec{D}$ is a map $s : \mathcal{E}(\vec{D}) \to \mathbb{R}$ such that at every vertex v, we have

$$a + d = b + c \,, \tag{16}$$

with the convention that a, b, c, d are the values of s as indicated in the following figure

(17)



For the above vertex we define

$$s(v) = a + d = b + c \,, \tag{18}$$

thus extending s to a map $s : \mathcal{E}(\vec{D}) \cup \mathcal{V}(\vec{D})) \to \mathbb{R}$.

Let $S_{\vec{D},\mathbb{R}}$ and $S_{\vec{D},\mathbb{N}}$ denote the sets of all centered states of $\vec{D}$ with values respectively in $\mathbb{R}$ and in $\mathbb{N}$. For a fixed positive integer n, define a map

$$S_{D,\mathbb{R}} \longrightarrow S_{\vec{D},\mathbb{R}} \qquad r \mapsto \hat{r} \tag{19}$$

by

$$\hat{r}(e) = \begin{cases} n - r(e) & \text{if the edge } e \text{ is of type } O \\ r(e) & \text{if } e \text{ is of type } U. \end{cases}$$

It is easy to see that the map (19) is a vector space isomorphism. If $r \in S_{D,n}$, i.e., r is n-admissible, then $\hat{r}$ is called *n-admissible*. Let $S_{\vec{D},n}$ be the set of all n-admissible centered states.

To characterize n-admissible centered states let us introduce the following norm for $s \in S_{\vec{D},\mathbb{N}}$:

$$|s| = \max_{v \in \mathcal{V}(\vec{D})} s(v) = \max_{u \in \mathcal{V}(\vec{D}) \cup \mathcal{E}(\vec{D})} s(u)$$

The following is a reformulation of n-admissibility in terms of centered states.

Lemma 3.3. A centered state s is n-admissible if and only if $s \in S_{\vec{D},\mathbb{N}}$ and $|s| \leq n$. In other words,

$$S_{\vec{D},n} = \{s \in S_{\vec{D},\mathbb{N}} : |s| \leq n\}$$

Proof. This follows immediately from the definition, since for any state r and for every vertex v of $\vec{D}$ we have $r(v) = n - \hat{r}(v)$. □

It follows that if a centered state is n-admissible, then it is $(n+1)$-admissible.

4. Local weights in terms of centered states

In this section we will give an explicit formula for the weight of a centered state. It turns out that the state sum of the colored Jones polynomial in terms of centered states has the important property of separation of variables needed in the proof of the stability. See Remark 4.4.

4.1. **Local weights of centered states and their factorization.** For an n-admissible centered state $s = \hat{r}$, let us define $w(s) := w(r)$. From the state sum of $w(r)$ we get the following state sum for $w(s)$

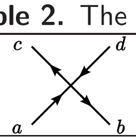
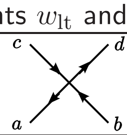
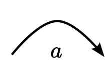
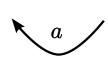
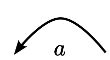
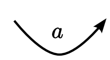
$$w(s) = \sum_X w(X, s)\,, \tag{20}$$

where the sum is over all the local parts X of $\vec{D}$. Here a local parts of $\vec{D}$ is a neighborhood of either a vertex or an extreme point of $\vec{D}$, and the value of

$$w(X, s) = w_{\mathrm{lt}}(X, s) w_{\succ}(X, s)$$

is obtained by replacing Table (1) with Table (2),

Table 2. The local weights w_{lt} and $w_{\succ}$ of centered states

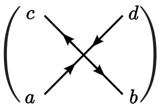
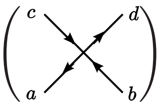
						
w_{lt}	$q^{\frac{n+ab+cd}{2}}$	$(-1)^{n-a-d} q^{\frac{-n^2-2n+ac+bd+b+c}{2}}$	$q^{-(2a-n)/4}$	$q^{-(2a-n)/4}$	$q^{(2a-n)/4}$	$q^{(2a-n)/4}$
$\omega_{\succ}$	$w_{\succ}(X)^{c,d}_{a,b}$	$w_{\succ}(X)^{c,d}_{a,b}$	1	1	1	1

where

$$w_{\succ}(X)^{c,d}_{a,b} = (q;q)_{n-a-d} \binom{n-d}{n-a-d}_q \binom{n-c}{n-c-b}_q \tag{21}$$

Note that $w_{\succ}(X,s)$ is independent of the sign of the local crossing, and takes the same value 1 at all local extrema. Hence, we use the notation $w_{\succ}(v,s)$ for the right hand side of (21), where $v \in \mathcal{V}$ is the involved vertex. The following is a convenient way rewrite the value of $w_{\succ}(v,s)$.

Lemma 4.1. For a vertex v in (17) and $x = q^{n+1}$, we have

(22) $w_{\succ}\Big($  $\Big) = w_{\succ}\Big($  $\Big) = \dfrac{(x\,q^{-a-d})_\infty}{(x\,q^{-d})_\infty (x\,q^{-c})_\infty} \dfrac{(q)_\infty}{(q)_a (q)_b}\,.$

Proof. The identity follows from Equation (21), and the following (easy to check) identities

$$\binom{n}{k}_q = \frac{(q^{n-k+1})_\infty}{(q)_k\,(q^{n+1})_\infty}$$

$$(q)_k = \frac{(q)_\infty}{(q^{k+1})_\infty}.$$

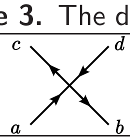
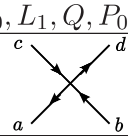
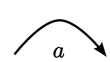
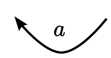
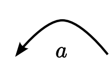
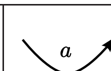
□

Remark 4.2. The right hand side of Equation (22) can also be written in the following form:

$$\frac{(x\,q^{-a-d})_\infty}{(x\,q^{-d})_\infty (x\,q^{-c})_\infty} \frac{(q)_\infty}{(q)_a (q)_b} = \frac{(q)_\infty}{(q)_a (q)_b} \frac{(x\,q^{-s(v)})_a\,(x\,q^{-s(v)})_b}{(x\,q^{-s(v)})_\infty} = \frac{(q)_\infty}{(q)_a (q)_b} \frac{(x\,q^{-s(v)})_b}{(x\,q^{-d})_\infty}\,. \tag{23}$$

4.2. **The functionals** P_0, P_1, Q, L_0, L_1**.** To study the power of q in Table (2), let us introduce the following functionals P_0, P_1, Q, L_0, L_1 on centered states, defined by local weights as in Table 3.

Table 3. The definition of L_0, L_1, Q, P_0 and P_1.

						
L_1	0	$(b+c)/2 = (a+b+c+d)/4$	$-a/2$	$-a/2$	$a/2$	$a/2$
L_0	0	$a+d=b+c$	0	0	0	0
Q	$(ab+cd)/2$	$(ac+bd)/2$	0	0	0	0
P_0	0	n	0	0	0	0
P_1	$n/2$	$-n^2/2-n$	$n/4$	$n/4$	$-n/4$	$-n/4$

If F is one of the functionals P_1, P_2, Q, L_0, L_1, and s is a centered state, then we define

$$F(s) = \sum_X F(X,s)\,,$$

where the sum is over all local parts X, with the value of F at a local part is given in Table (3). These functionals are introduced so that for a local part X with centered state s we have

$$w(X,s) = (-1)^{P_0(X,s)+L_0(X,s)}\,q^{P_1(X,s)+Q(X,s)+L_1(X,s)}\,w_{\succ}(X,s)\,.$$

From Equation (20) we have

$$w(s) = (-1)^{P_0(s)+L_0(s)}\, q^{P_1(s)+Q(s)+L_1(s)}\, w_{\succ}(s)\,. \tag{24}$$

Here $w_{\succ}(s) = \prod_X w_{\succ}(X,s)$, where the product is over all local parts of D. Note that $w_{\succ}(X,s) \in \mathbb{Z}_{\succ}[q]$. The functionals L_0, L_1 are linear forms on $S_{\vec{D},\mathbb{R}}$ and do not depend on n in the sense that the value of each of L_0, L_1 will be the same if we consider s as an $(n+1)$-admissible centered state instead of an n-state. The functional $Q_2 = Q + L_1$ is a quadratic form on $S_{\vec{D},\mathbb{R}}$ not depending on n. The two functionals P_0, P_1 depend only on n, i.e., if s, s' are n-admissible centered states, then $P_i(s) = P_i(s')$. Hence we will also write $P_i(n)$ instead of $P_i(s)$, for $i = 0, 1$.

Lemma 4.3. We have

$$J_{K,n}(q) = (-1)^{P_0(n)}\, q^{P_1(n)} \sum_{s \in S_{\vec{D},n}} F(q^{n+1}, q, s), \tag{25}$$

where

$$F(x,q,s) = (q)_\infty^{c_D}\, (-1)^{L_0(s)}\, \frac{q^{Q_2(s)}}{\prod_{e\in\mathcal{E}(\vec{D})} (q)_{s(e)}} \, \frac{\prod_{v\in\mathcal{V}(\vec{D})} \left(xq^{-s(v)}\right)_\infty}{\prod_{e\in\mathcal{E}(\vec{D})} \left(xq^{-s(e)}\right)_\infty}\,. \tag{26}$$

Proof. By (22) we have

$$\prod_{v\in\mathcal{V}(\vec{D})} w_{\succ}(v,s) = \frac{(q)_\infty^{c_D}}{\prod_{v\in\mathcal{V}(\vec{D})} (q)_a (q)_b}\, \frac{\prod_{v\in\mathcal{V}(\vec{D})} \left(q^{n+1-s(v)}\right)_\infty}{\prod_{v\in\mathcal{V}(\vec{D})} (q^{n+1-d})_\infty\, (q^{n+1-c})_\infty} \in \mathbb{Z}_{\succ}[q]\,.$$

Here a and b (respectively c and d) are the s-values of the two lower (respectively upper) edges incident to v. When v runs the set $\mathcal{V}$ of vertices, the two lower edges of v run the set $\mathcal{E}$ of all edges, as do the two upper edges of v. Hence

$$\prod_{v\in\mathcal{V}(\vec{D})} w_{\succ}(v,s) = \frac{(q)_\infty^{c_D}}{\prod_{e\in\mathcal{E}(\vec{D})} (q)_{s(e)}}\, \frac{\prod_{v\in\mathcal{V}(\vec{D})} \left(q^{n+1-s(v)}\right)_\infty}{\prod_{e\in\mathcal{E}(\vec{D})} \left(q^{n+1-s(e)}\right)_\infty}\,. \tag{27}$$

From Equation (24) and $J_{K,n}(q) = \sum_{s\in S_{\vec{D},n}} w(s)$, we have

$$J_{K,n}(q) = (-1)^{P_0(n)}\, q^{P_1(n)} \sum_{s\in S_{\vec{D},n}} (-1)^{L_0(s)}\, q^{Q_2(s)} \prod_{v\in\mathcal{V}(\vec{D})} w_{\succ}(v,s)\,,$$

which is equal to the right hand side of (25) by identity (27) and the definition of $F(x,q,s)$. $\square$

Remark 4.4. (a) It is important for the stability that there is no mixing between n and s in the formulas of the functionals P_0, P_1, Q, L_0, L_1. In the states-sum using states in D, mixing occurs, and this is the reason why we introduce centered states.

(b) The quadratic form Q has the following simple description. Suppose α is an A-angle of the digraph $\vec{D}$, and the s-values of the two edges of α are a and b. Define $Q(\alpha, s) = ab/2$. Then

$$Q(s) = \sum_{\alpha} Q(\alpha, s)\,, \tag{28}$$

where the sum is over all A-angles α.

5. Positivity of Q_2 and the lowest degree of the colored Jones polynomial

In this section we prove the copositivity of $Q_2 := Q + L_1$ on the cone $S_{\vec{D},\mathbb{N}}$ and derive a formula for the lowest degree of the colored Jones polynomial. As before, we fix a reduced, alternating, A-infinite downward diagram D with c_D crossings.

5.1. **A Hilbert basis for $S_{\vec{D},\mathbb{N}}$: elementary centered states.** From its very definition, the set $S_{\vec{D},\mathbb{N}}$ of $\mathbb{N}$-valued centered states of $\vec{D}$ can be identified with the set of lattice points of a lattice cone in $\mathbb{R}^{2c_D}$. In general, the set of lattice points of a rational cone is a monoid, and a generating set is called a *Hilbert basis* which plays an important role in *integer programming*; see for instance [Stu96, Sec.13] and also [Sch86, Sec.16.4]. Note that every element of a finitely generated additive monoid is an $\mathbb{N}$-linear combination of a Hilbert basis. Although the natural number coefficients are not unique, this is not a problem for applications.

The goal of this section is to describe a useful Hilbert basis for $S_{\vec{D},\mathbb{N}}$.

Recall that $\vec{D}$ is a directed graph. Suppose γ is a directed cycle of $\vec{D}$, i.e., closed path consisting of a sequence of distinct edges $e_1, \dots e_n$ of D such that the ending point of e_j is the starting points of e_{j+1} (index is taken modulo n) and there is no repeated vertex along the path except for the obvious case where the first vertex is also the last vertex. An example of a cycle of $\vec{D}$ is the boundary of a polygon in the complement of $\vec{D}$.

Definition 5.1. For a directed cycle γ of $\vec{D}$ let s_γ be the function on the set of edges of $\vec{D}$ which assigns 1 to every edge of γ and 0 to every other edge. Such a centered state is called *elementary*, and γ is called its support. Let $\mathcal{B}$ denote the (finite set) of all elementary centered states of $\vec{D}$.

For a polygon $p \in \mathcal{P}(\vec{D})$ the boundary ∂p is a directed cycle of $\vec{D}$, and we will use the notation $s_p := s_{\partial p}$.

From Lemma 3.3 we see that s_γ is an n-admissible centered state for every $n \geq 1$.

Lemma 5.2. $\mathcal{B}$ is a Hilbert basis of $S_{\vec{D},\mathbb{N}}$.

Proof. Let s be a $\mathbb{N}$-valued centered state of $\vec{D}$. Suppose e is an oriented edge such that $s(e) > 0$. At the ending vertex v of e let e' and e'' be the two edges which are perpendicular to e. Inspection of Figure (15) shows that v is the starting vertex for both e' and e''. Equation (16) shows that $s(e') + s(e'') \geq s(e)$. Hence one of them, say $s(e') > 0$. This means if e is an edge with $s(e) > 0$, we can continue e to another edge e' for which $s(e') > 0$. Repeating this process we can construct a cycle γ of $\vec{D}$ such that the value of s is positive on any edge of γ. This means $s - s_\gamma$ is an $\mathbb{N}$-valued centered state. Induction completes the proof of the lemma. □

Remark 5.3. It is easy to see that any $s \in \mathcal{B}$ is not a $\mathbb{N}$-linear combination of the other elements in $\mathcal{B}$. Thus there is no redundant element in $\mathcal{B}$. Of course $\mathcal{B}$ is linearly dependent over $\mathbb{R}$ (or over $\mathbb{Z}$), and we will extract a $\mathbb{R}$-basis from the set $\mathcal{B}$ later.

5.2. **Values of L_1 and Q on elementary centered states.** Suppose γ is a directed cycle of $\vec{D}$ and v is a vertex of γ. Among the four edges of $\vec{D}$ incident to v, the two edges of γ are not two opposite edges because of the orientation constraint, see (15). In other words, at each vertex v, γ is an angle. We say that a vertex v of γ is of type A or B according as the two edges of γ at v form an angle of type A or B. Let $N_{\gamma,A}$ be the number of vertices of γ of type A. The fact that D is reduced is used in the proof of part (b) of the next lemma.

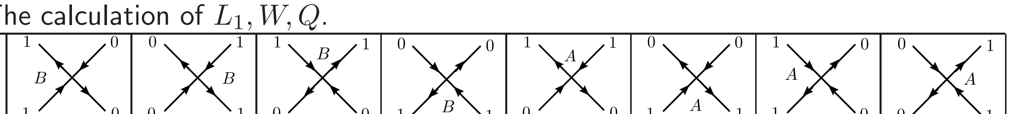
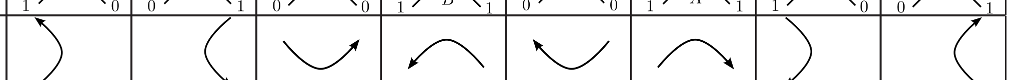
Lemma 5.4. Suppose $s = s_\gamma \in \mathcal{B}$ is an elementary centered state.
(a) We have

$$L_1(s) = W(\gamma) + \frac{1}{2} N_{\gamma,A} \tag{29}$$

(b) Moreover, $L_1(s) \geq 0$, and $L_1(s) = 0$ if and only if γ is clockwise and has exactly two vertices of type A.

Proof. (a) For a local part X of $\vec{D}$, let $\gamma_X = \gamma \cap X$. Clearly $L_1(X, s) = 0$ if $\gamma_X = \emptyset$. If X is a small neighborhood of a vertex of $\vec{D}$, then γ_X is two sides of an angle of γ, and we will smoothen γ_X at the corner to get an oriented smoothed arc. See Row 1 and Row 2 of Table 4 for various X and smoothened γ_X. In the table, X is a small neighborhood of a vextex. The two edges incident to the vertex with label 1 belong to γ. The marking A or B at one of the angles of X indicates the type of the vertex, which appear in Row 3. In Row 3 we also indicate the sign of the crossing of X (as it appeared originally in D); this makes the computation of L_1 easier.

Table 4. The calculation of L_1, W, Q.

X								
γ_X								
vertex type	$B,+$	$B,+$	$B,-$	$B,-$	$A,+$	$A,+$	$A,-$	$A,-$
$L_1(X, s_\gamma)$	0	0	1/2	1/2	0	0	1/2	1/2
$W(\gamma_X)$	0	0	1/2	1/2	$-1/2$	$-1/2$	0	0
$Q(X)$	0	0	0	0	1/2	1/2	1/2	1/2

We define $W(\gamma_X)$ to be its local winding number if γ_X contains a local extreme point, 0 otherwise. Rows 4 and 5 of Table 4 gives the values of $L_1(X, s_\gamma)$ and $W(\gamma_X)$. From the table, together with the obvious case when X is a neighborhood of a local extreme point of $\vec{D}$, we have

$$L_1(X, s_\gamma) = \begin{cases} W(\gamma_X) + \frac{1}{2} & \text{if } X \text{ is a vertex of } \gamma \text{ of type } A \\ W(\gamma_X) & \text{otherwise.} \end{cases}$$

Summing up the above identity over all the local parts X and using (12), we get (29).

(b) Case 1: γ is counterclockwise. From (29) we have $L_1(s) \ge W(\gamma) = 1 > 0$. In this case L_1 is strictly positive.

Case 2: γ is clockwise. Then $L_1(s) = -1 + N_{\gamma,A}/2$. We will show that $N_{\gamma,A} \ge 2$.

If $N_{\gamma,A} = 0$, then γ is one of the circles obtained from $\vec{D}$ by doing A-resolution at every vertex. By part (b) of Lemma 3.2, γ is counterclockwise. Thus $N_{\gamma,A} \ne 0$ if γ is clockwise.

Suppose $N_{\gamma,A} = 1$, i.e. γ has exactly one vertex of type A, say v; all other vertices of γ are of type B. If one does A-resolution at every vertex of $\vec{D}$, then $\gamma \setminus \{v\}$ is part of one of the resulting circles, and this circle has a self-touching point at v. This is impossible if the diagram D is reduced, see part (c) of Lemma 3.2. Thus $N_{\gamma,A} \ne 1$.

We have shown that if γ is clockwise then $N_{\gamma,A} \ge 2$. Hence $L_1(s) = -1 + N_{\gamma,A}/2 \ge 0$, and equality happens if and only $N_{\gamma,A} = 2$. □

Remark 5.5. We see that for the proof of part (b), we need only the fact that D is A-adequate.

Lemma 5.6. (a) For all $\mathbb{N}$-valued centered states s and s' we have

$$Q(s+s') \ge Q(s) + Q(s') \tag{30}$$

(b) Suppose $s = s_\gamma \in \mathcal{B}$ is elementary centered state. Then

$$Q(s) = \frac{N_{\gamma,A}}{2}. \tag{31}$$

It follows that $Q(s) \ge 0$, with equality if and only if γ is the boundary of a polygonal region of type B.

Proof. (a) Since Q is defined by an expression with positive coefficients, we have $Q(s+s') \ge Q(s) + Q(s')$.

(b) Row 6 of Table 4 shows that every vertex of type A of γ contributes $1/2$ to the value of Q, while others contribute 0. Hence $Q(s) = \frac{N_{\gamma,A}}{2}$. □

5.3. Copositivity of Q_2.

Recall that $Q_2 = Q + L_1$.

Proposition 5.7. (a) For $s, s' \in S_{\vec{D},\mathbb{N}}$, we have

$$Q_2(s+s') \ge Q_2(s) + Q_2(s').$$

(b) If $s = \sum_{j=1}^{l} m_j s_j$, where $s_j \in \mathcal{B}$ and $m_j \in \mathbb{N}$, then

$$Q_2(s) \ge \sum_j m_j \ge |s|. \tag{32}$$

In particular, Q_2 is copositive in the cone $S_{\vec{D},\mathbb{N}}$, i.e., for every $s \in S_{\vec{D},\mathbb{N}}$, $Q_2(s) \ge 0$ and equality happens if and if only $s = 0$.

Proof. (a) follows immediately form Lemma 5.6(a), noting that $L_1(s+s') = L_1(s) + L_1(s')$.

(b) The second inequality of (32) follows immediately from the definition.

From part (a) one needs only to prove the first inequality of (32) for $s \in \mathcal{B}$ an elementary centered state with support γ. By (29) and (31),

$$Q_2(s) = W(\gamma) + N_{\gamma,A}.$$

In particular, $Q_2(s)$ is an integer.

By Lemmas 5.4(b) and 5.6(b), we have $L_1(s)+Q(s) \geq 0$, and equality happens only when $L_1(s) = Q(s) = 0$. However, if $L_1(s) = 0$, then by Lemma 5.4(b), $N_{\gamma,A} = 2$, and then $Q(s) = N_{\gamma,A}/2 = 1 > 0$. Thus, we have proved that if s is an elementary centered state, then $Q(s) > 0$. Since $Q(s) \in \mathbb{Z}$, we have $Q(s) \geq 1$. □

Remark 5.8. In general $Q(s), L_1(s) \in \frac{1}{2}\mathbb{Z}$. In the proof we show that $Q_2(s) = Q(s)+L_1(s) \in \mathbb{Z}$ for any elementary centered state s. One can also show that $Q_2(s) \in \mathbb{Z}$ for all $s \in S_{\vec{D},\mathbb{N}}$. This can be deduced from the fact that $\hat{J}_{K,n}(q) \in \mathbb{Z}[q]$, see the discussion on fractional powers of $J_{K,n}$ in [Le00].

5.4. **The lowest degree of the colored Jones polynomial.** In the next proposition K is an alternating link.

Proposition 5.9. (a) The minimal degree of q in of $J_{K,n}(q)$ is

$$P_1(n) = \frac{n}{2}c_+ - \frac{n^2+2n}{2}c_- - \frac{n}{2}\sum_M W(M)\,,$$

where the last sum is over all the local extreme points of D.

(b) With $F(x,q,s)$ defined by (26), we have

$$\hat{J}_{K,n}(q) = \sum_{s\in S_{\vec{D},n}} F(q^{n+1},q,s). \tag{33}$$

Proof. (a) By (25), the minimal degree of q in $w(s)$ is $P_1(n)+Q_2(s)$. When $s \neq 0$, Proposition 5.7 implies that $Q_2(s) > 0$. Hence the smallest degree of $J_{K,n}(q)$ is $P_1(n)$. From the values of $P_1(X,s)$ in Table 3 we see that $P_1(n) = \frac{n}{2}c_+ - \frac{n^2+2n}{2}c_- - \frac{n}{2}\sum_M W(M)$.

(b) follows easily from part (a) and (27). □

The value of P_0 in Table 4, Equation 25, and Proposition 5.9 imply the following.

Corollary 5.10. We have

$$J_{K,n}(q) = (-1)^{nc_-}\, q^{P_1(n)}\, \hat{J}_{K,n}(q). \tag{34}$$

Remark 5.11. The minimal degree of the colored Jones polynomial $J_{K,n}(q)$ of an alternating link had been calculated using the Kauffman bracket skein module, and is given by $P_1'(n) := \frac{n}{2}c_+ - \frac{n^2+n}{2}c_- - \frac{n}{2}s_A$, where s_A is the number of circles obtained from $\vec{D}$ by doing A-resolution at every vertex; see [Lê06, Proposition 2.1] and keep in mind that the framing of K in [Lê06] is different from the one in the current paper. Our result implies that $P_1(n) = P_1'(n)$. We will give a direct proof of this identity in the Appendix. Note also that $s_A - c_+ = \sigma + 1$ (see [Tur87, Mur87]), where σ is the signature of the link. Hence the lowest degree of q is given by $-\frac{n^2+n}{2}c_- - \frac{n}{2}(\sigma+1)$.

6. From $\vec{D}$ to the dual graph D^*

In this section we give a correspondence between the centered states on $\vec{D}$ with the admissible colorings of the dual graph D^*. The main idea is summarized in the following figure. If a crossing has coloring a, b, c, d at the four regions of it counterclockwise, then there is a coloring of the four arcs such that the sum of the colors of the two overarcs is equal to the sum of the colors of the two underarcs:

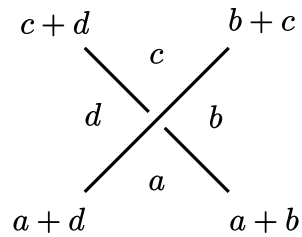


Figure 4. From a coloring of the regions to a coloring of the arcs.

An example of this correspondence for the 4_1 knot is shown in Figure 5.

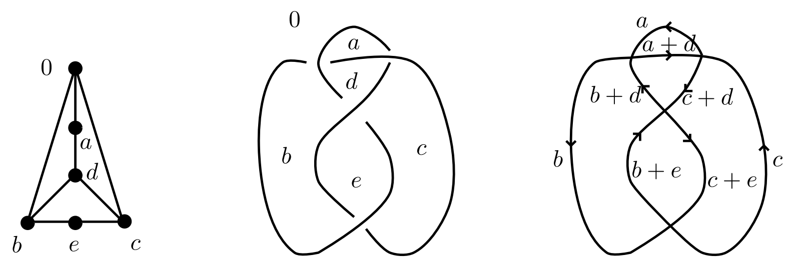


Figure 5. An admissible coloring of D^* on the left, the coloring of the polygons of D in the middle and the corresponding centered state on $\vec{D}$ on the right for the 4_1 knot.

Recall from Section 1.4 that D^* is the dual graph of $\vec{D}$, considered as an unoriented graph. We have defined the Tait graph $\mathcal{T}$, the lattice Λ_0 with its subsets $\mathrm{Adm}, \mathrm{Adm}(n)$, and functions L, Q on Λ_0. Note that one does not need to bring D to a downward position by twisting in small neighborhood of crossing points in order to construct D^*.

For $\lambda \in \Lambda_0$ let $\tau(\lambda) : \mathcal{E}(\vec{D}) \to \mathbb{R}$ be the linear map defined by

$$\tau(\lambda)(e) = e^*(\lambda),$$

where $e^* \in \mathcal{E}(\Delta^*)$ is the dual edge of e. Then $\tau : \Lambda_0 \otimes \mathbb{R} \to S_{\vec{D},\mathbb{R}}$ is a $\mathbb{R}$-linear map.

Proposition 6.1. (a) The map $\tau : \Lambda_0 \otimes \mathbb{R} \to S_{\vec{D},\mathbb{R}}$ is vector space isomorphism.
(b) τ maps Adm and $\mathrm{Adm}(n)$ isomorphically onto respectively $S_{\vec{D},\mathbb{N}}$ and $S_{\vec{D},n}$.
(c) We have

$$L_1(\tau(\lambda)) = L(\lambda) \tag{35}$$

$$Q(\tau(\lambda)) = Q(\lambda) \tag{36}$$

(d) For every centered state s, we have

$$L_0(\tau(s)) \equiv 2L(s) \pmod 2. \tag{37}$$

Proof. (a) Fix $s \in S_{\vec{D},\mathbb{R}}$. We will show that the equation

$$\tau(\lambda) = s \tag{38}$$

has one and exactly one solution $\lambda \in \Lambda_0$. This will prove the bijectivity of τ.

With the basis $\mathbf{b} := \mathcal{V}(D^*) \setminus \{v_\infty\}$ of Λ_0, every $\lambda \in \Lambda_0$ has a unique presentation $\lambda = \sum_{v \in \mathcal{V}(D^*)} k_v v$ with $k_v = 0$ for $v = v_\infty$. We need to solve for $k_v, v \in \mathbf{b}$ from Equation (38).

Equation (38) is the same as the following linear system of $2c_D$ equations: For every edge $e^* \in \mathcal{E}(D^*)$ whose end points are v and v',

$$k_v + k_{v'} = s(e)\,. \tag{39}$$

If k_v is known, and v' is connected to v by an edge, then there is only one possible value for $k_{v'}$, namely $k_{v'} = s(e) - k_v$. We call such $k_{v'}$ the extension of the value k_v at v along the edge e^*. Since the graph D^* is connected, and $k_{v_\infty} = 0$, we see that there is at most one solution $\lambda \in \Lambda_0$ of (38).

Now let us look at the existence of solution of (38). Given $v \in \mathcal{V}(D^*)$, $a \in \mathbb{R}$, and a path α of the graph Δ^* connecting v to $v' \in \mathcal{V}(\Delta^*)$, there is only one way to extend $k_v = a$ at v to v' along the path α. Denote by $\lambda_{\alpha,a}(v')$ the value at v' of this extension. When α is a closed path, i.e. $v' = v$, let $\Delta(\alpha, a) = \lambda_{\alpha,a}(v') - a$. We will show that $\Delta(\alpha, a) = 0$ for any closed path α. This will prove the existence of the solution.

On $\mathbb{R}^2$, the closed path α encloses a region R. When the region is just a polygon of D^* (which must be a quadrilateral), the fact that $\Delta(\alpha, a) = 0$ follows easily from (16). For general closed path α, since $\Delta(\alpha, a)$ is the sum of Δ_{α_j,a_j}, where α_j's are the boundaries of all the polygons of D^* in R, we also have $\Delta(\alpha, a) = 0$.

The above fact shows that if we begin with $k_{v_\infty} = 0$, we can uniquely extend k_v to all vertices of D^*, and obtain in this way an inverse of s.

The proof actually shows that τ is a $\mathbb{Z}$-isomorphism between Λ_0 and $S_{\vec{D}\mathbb{Z}}$.

(b) Because $\tau(\lambda)(e) = \lambda(e^*)$, this follows easily from the definitions.

(c) To prove (35), it is enough to consider the case $\lambda = v \in \mathbf{b} = \mathcal{V}(D^*) \setminus \{v_\infty\}$, a basis vector. Let $p = v^* \in \mathcal{P}(D)$ be the dual polygon. From the definition we have $\tau(v) = s_p$, where s_p is the elementary centered state with support the boundary of p. Now the identity $L_1(\tau(v)) = L(v)$ follows from the value of L_1 given in Lemma 5.4 and the definition of L. Actually, the definition of L was built so that (35) holds.

Let us turn to (36). To show that two quadratic forms on a vector space are the same it is enough to show that they agree on the set $v + v'$, where v, v' are elements in a basis of the vector space. A basis of Λ_0 is $\mathcal{V}(D^*) \setminus \{v_\infty\}$. Hence we need to check that if $v_1, v_2 \in \mathcal{V}(D^*) \setminus \{v_\infty\}$,

$$Q(\tau(v_1 + v_2)) = Q(v_1 + v_2)\,. \tag{40}$$

There are three cases to consider: both v_1, v_2 are A-vertices, both are B-vertices, and exactly one of them is an A-vertex. In each case, the identity (40) can be verified easily. Actually, the matrix $\mathcal{Q}$ in Equation (4) was defined so that Equation (40) holds.

(d) We only need to check (37) for $s = \tau(v), v \in \mathbf{b}$. Let $p = v^* \in \mathcal{V}(\vec{D})$ be the dual polygon. We already saw that $\tau(v) = s_p$. From the definition of L_0 given by Table 3, we have that $L_0(s_p)$ is the number of negative vertices of p. Here a vertex is negative if it is negative as a crossing of the link diagram D.

There are two cases.

Case 1: p is a B-polygon. Suppose or is an arbitrary orientation on edges of p. A vertex v of p is or-incompatible if the orientations of the two edges incident to v are incompatible, i.e. the two incident edges are both going out from v or both coming in to v. Let $f(\mathrm{or})$ be the number of all or-incompatible vertices. It is easy to see that if or$'$ is obtained from or by changing the orientation at exactly one edge, then $f(\mathrm{or}) = f(\mathrm{or}') \pmod 2$. It follows that $f(\mathrm{or}) = 0 \pmod 2$ for any orientation or, since if we orient all the edges counterclockwise then $f = 0$.

Let the orientation of D on the edges of p be denoted by or_D. By inspection of Figure (15) one sees that a vertex v of p is a negative crossing if and only v is or_D-incompatible. Thus $L_0(s_p) = f(\mathrm{or}_D)$, which is even by the above argument. On the other hand, $2L_1(s_p) = 2$ by Lemma 5.4.

Case 2: p is an A-polygon. By inspection of Figure (15) one sees that a vertex v of p is a positive crossing if and only v is or_D-incompatible. This means $L_0(s_p) = \deg(v) - f(\mathrm{or}_D) \equiv \deg(v) \pmod 2$, where $\deg(v)$ is the number of vertices of p, which is equal to the degree of v in the graph D^*. By Lemma 5.4, $2L_1(s_p) = -2 + \deg(v)$. Hence we also have (37). ☐

Example 6.2. As an illustration of Equation (36), consider the coloring on D^* and the corresponding state on $\vec{D}$ from Figure 5. We have

$$\begin{aligned} Q(\tau(\lambda)) &= \frac{1}{2}\left(3d^3 + 2e^2 + ab + ac + 2bc + ad + bd + be + cd + ce\right) \\ Q(\lambda) &= \frac{1}{2}\left((ba + ac + cb) + ((a+d)(b+d) + (b+d)(c+d) + (c+d)(a+d)) \right. \\ &\quad \left. + ((b+e)(c+e) + (c+e)(b+e))\right) \end{aligned}$$

The reader can vefiry the equaility (36).

Corollary 6.3. The dimension of $S_{D,\mathbb{R}}$ (or $S_{\vec{D},\mathbb{R}}$) is $c_D + \ell$, where ℓ is the number of connected components of the graph D.

Remark 6.4. One can show that the integer-valued admissible colorings of D^* are the lattice points in a $2c_D$ dimensional cone with $2c_D$ independent rays.

7. 0-stability

In this section we give a proof of the 0-stability of the colored Jones polynomial of an alternating link and Theorem 1.10, which describes the 0-limit as a generalized Nahm sum.

7.1. Expansion of F and adequate series.

Definition 7.1. We say that a series $G(x,q) = \sum_{m=0}^{\infty} a_m(q) x^m \in \mathbb{Z}((q))[[x]]$ is *x-adequate* of order $\le t$ if $G(xq^t, q) \in \mathbb{Z}[[q]][[x]]$, i.e. for every m, we have

$$\operatorname{mindeg}_q(a_m(q)) \ge -mt\,.$$

Lemma 7.2. (a) For every $t \in \mathbb{N}$, the set of x-adequate series of order $\leq t$ is a subring of $\mathbb{Z}((q))[[x]]$.
(b) If $G(x,q)$ is x-adequate of order $\leq t$, then it is x-adequate of order $\leq t'$ for any $t' \geq t$.
(c) If $G(x,q)$ is x-adequate of order $\leq t$, then the series $f_n(q) = G(q^n, q)$ converges in the q-adic topology and defines an element in $\mathbb{Z}[[q]]$ for every $n > t$.
(d) The sequence $(f_n(q))$ is stable and its associated series $F_f(x,q)$ satisfies $F_f(x,q) = G(x,q)$.

Proof. Parts (a), (b) and (c) follow easily from the definition of an x-adequate series.

For (d), let $G(x,q) = \sum_{m=0}^{\infty} a_m(q) x^m \in \mathbb{Z}((q))[[x]]$ and define $\Phi_k(q) = a_k(q)$ for all $k \in \mathbb{N}$. Then, we have for $n > t+1$

$$q^{-k(n+1)} \left(f_n(q) - \sum_{j=0}^{k} \Phi_j(q) q^{(n+1)j} \right) = q^{-k(n+1)} \sum_{m=k+1}^{\infty} a_m(q) q^{(n+1)m} .$$

The minimum degree of the summand is bounded below by

$$f(m) = -k(n+1) - mt + (n+1)m .$$

Since $f(m)$ is a linear function of m and the coefficient of m in $f(m)$ is $n+1-t > 0$, it follows that

$$f(m) \geq f(1) = n+1 - t(k+1) .$$

Thus,

$$q^{-k(n+1)} \left(f_n(q) - \sum_{j=0}^{k} \Phi_j(q) q^{(n+1)j} \right) \in q^{n+1-(k+1)t} \mathbb{Z}[[q]]$$

which implies (d). □

Recall that for a centered state s, $F(x,q,s)$ defined by (26), satisfies

(41)
$$\begin{aligned} F(x,q,s) &= q^{Q_2(s)} \tilde{F}(x,q,s) \\ \tilde{F}(x,q,s) &:= \frac{(-1)^{L_0(s)} (q)_\infty^{c_D}}{\prod_{e \in \mathcal{E}(\vec{D})} (q)_{s(e)}} \frac{\prod_{v \in \mathcal{V}(\vec{D})} \left(xq^{-s(v)}\right)_\infty}{\prod_{e \in \mathcal{E}(\vec{D})} \left(xq^{-s(e)}\right)_\infty} . \end{aligned}$$

Using the well-known identities (see e.g [KC02])

(42)
$$(x)_\infty = \sum_{j=0}^{\infty} \frac{(-1)^j \, q^{\binom{j}{2}}}{(q)_j} x^j, \qquad \frac{1}{(x)_\infty} = \sum_{j=0}^{\infty} \frac{x^j}{(q)_j}$$

we can expand $\tilde{F}$ into power series in x,

(43)
$$\tilde{F}(x,q,s) = \sum_{m=0}^{\infty} a_m(q,s) x^m \in \mathbb{Z}((q))[[x]] .$$

The negative powers of q in $a_m(q,s)$ come from the negative powers $q^{-s(v)}, q^{-s(e)}$ that appear in the expression of $\tilde{F}$. Since $|s| \geq \max(s(v), s(e))$, we have the following.

Lemma 7.3. For every $s \in S_{\vec{D},\mathbb{N}}$, $\tilde{F}(x,q,s)$ is x-adequate of order $\leq |s|$.

7.2. **Proof of 0-stability.** Now we show that $\hat{J}_{K,n}(q)$ is 0-stable, and identify its 0-limit. Recall $F(x,q,s)$ given by (26) or (41). By (33)

$$\hat{J}_{K,n}(q) = \sum_{s \in S_{\vec{D},n}} F(x,q,s)\big|_{x=q^{n+1}} .$$

Hence we expect that the 0-limit is

$$\Phi_0(q) := \sum_{s \in S_{\vec{D},\mathbb{N}}} F(0,q,s) . \tag{44}$$

We have

$$F(0,q,s) = (q)_\infty^{c_D} (-1)^{L_0(s)} \frac{q^{Q_2(s)}}{\prod_{e \in \mathcal{E}(\vec{D})} (q)_{s(e)}}. \tag{45}$$

Part (b) of Proposition 5.7 shows that the right hand side of (44) is regular, and defines an element in $\mathbb{Z}[[q]]$. We will show that

$$\hat{J}_{K,n}(q) - \Phi_0(q) \in q^{n+1}\mathbb{Z}[[q]] \tag{46}$$

for all n. This certainly implies that the 0-limit of $\hat{J}_{K,n}(q)$ exists and is equal to $\Phi_0(q)$. We have

$$\hat{J}_{K,n}(q) - \Phi_0(q) = \sum_{s:|s|\leq n} \left[F(q^{n+1},q,s) - F(0,q,s)\right] - \sum_{s:n<|s|} F(0,q,s) . \tag{47}$$

By part (b) of Proposition 5.7, $Q_2(s) \geq |s|$. Then (45) implies that $F(0,q,s) \in q^{|s|}\mathbb{Z}[[q]]$, and hence the second sum on the right hand side of (47) is in $q^{n+1}\mathbb{Z}[[q]]$.

Let us look at the first term. Using the expansion (43), we have

$$\begin{aligned} F(q^{n+1},q,s) - F(0,q,s) &= q^{Q_2(s)} \sum_{m=1}^{\infty} a_m(q,s) q^{m(n+1)} \\ &= \sum_{m=1}^{\infty} \left[a_m(q,s)\, q^{m|s|}\right] \, q^{f(m)}, \end{aligned} \tag{48}$$

where $f(m) = Q_2(s) + m(n+1) - m|s| = m(n+1-|s|) + Q_2(s)$, which is linear in m. Since $n \geq |s|$, $f(m)$ achieves minimum when $m = 1$:

$$f(m) \geq f(1) = n + 1 - |s| + Q_2(s) \geq n + 1.$$

By Lemma 7.3, $a_m(q,s)\, q^{m|s|}$ has only non-negative powers of q. It follows that the right hand side of (48) belongs to $q^{n+1}\mathbb{Z}[[q]]$. This completes the proof of Equation (46). □

Remark 7.4. Equation (46) is stronger than 0-stability, and implies that for every $m \in \mathbb{N}$, the coefficient of q^m in $\hat{J}_{K,n}(q)$ is independent of n for all $n > m$.

7.3. **End of the proof of Theorem 1.10.** To complete the proof of Theorem 1.10, it remains to prove that the right hand side of (44) is equal to that of (5). This follows from Proposition 6.1. □

Remark 7.5. The fact that D is reduced is used only in the proof of Lemma 5.4. As seen in Remark 5.5, Lemma 5.4 holds if D is A-adequate, hence Theorem 1.10 holds if D is not necessarily reduced, but A-adequate.

7.4. **Proof of Corollary 1.11.** Fix a complex number q with $|q| = a < 1$. We only need to show that the sum on the right hand side of (44) is absolutely convergent.

Choose $0 < \varepsilon < 1 - a$ such that

$$a < (a+\varepsilon)^{2c_D}. \tag{49}$$

This is possible by continuity since if $\varepsilon = 1 - a$, the right hand side of the above inequality is 1 and $a < 1$. Since $\lim_{j\to\infty}(1-q)^j = 1$, it follows that $|1 - q^j| > a + \varepsilon$ for j big enough. Thus, there is constant $C_1 > 0$ such that for every n,

$$|(q)_n| > C_1(a+\varepsilon)^n\,. \tag{50}$$

Since $Q_2(s) \geq |s| \geq s(e)$ for every $e \in \mathcal{E}(\vec{D})$, we have

$$\sum_{e\in\mathcal{E}(\vec{D})} s(e) \leq 2c_D\, Q_2(s)\,. \tag{51}$$

We have

$$\begin{aligned}\left|\frac{q^{Q_2(s)}}{\prod_{e\in\mathcal{E}(\vec{D})}(q)_{s(e)}}\right| &< \frac{a^{Q_2(s)}}{\prod_{e\in\mathcal{E}(\vec{D})} C_1\,(a+\varepsilon)^{s(e)}} \quad \text{by (50)}\\ &< (C_1)^{-2c_D}\frac{a^{Q_2(s)}}{(a+\varepsilon)^{2c_D Q_2(s)}} \quad \text{by (51)}\\ &= (C_1)^{-2c_D}\left(\frac{a}{(a+\varepsilon)^{2c_D}}\right)^{Q_2(s)}.\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{s\in S_{\vec{D},\mathbb{N}}}\left|\frac{q^{Q_2(s)}}{\prod_{e\in\mathcal{E}(\vec{D})}(q)_{s(e)}}\right| &< (C_1)^{-2c_D}\sum_{s\in S_{\vec{D},\mathbb{N}}}\left(\frac{q}{(a+\varepsilon)^{2c_D}}\right)^{Q_2(s)}\\ &= (C_1)^{-2c_D}\sum_{m=0}^{\infty} g(m)\left(\frac{a}{(a+\varepsilon)^{2c_D}}\right)^{m},\end{aligned} \tag{52}$$

where $g(m)$ is the number of $s \in S_{\vec{D},\mathbb{N}}$ such that $Q_2(s) = m$. Because $Q_2(s)$ is quadratic and co-positive in $S_{\vec{D},\mathbb{N}}$, $g(m)$ is bounded above by a quadratic function of m for large enough m. From Equation (49) it follows that the right hand side of (52) is absolutely convergent. This completes the proof of Corollary 1.11.

8. Linearly bounded states

In this section we will introduce a partition of the set of linearly bounded centered states, which will be key to the k-stability of the colored Jones polynomial. Throughout this section we fix a reduced, alternating, A-infinite downward alternating link diagram D with c_D crossings. Let $\mathcal{S} := S_{\vec{D},\mathbb{N}}$. Recall that for a polygon $p \in \mathcal{P}(\vec{D})$, s_p is the elementary centered state with support the boundary of p.

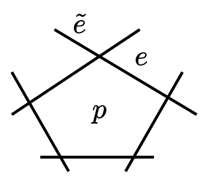
If $Q : S_{\vec{D},\mathbb{R}} \to \mathbb{R}$ were positive definite, it would be easy to prove the stability of $\hat{J}_{K,n}(q)$. Unfortunately, Q is not positive definite, and the summation cone $S_{\vec{D},[0,\infty)}$ *always* contains directions where $Q_2(s) = Q(s) + L_1(s)$ grows linearly, and not quadratically. For instance, if p is a B-polygon, then $Q_2(ns_p) = n$ is a linear function of n.

Definition 8.1. We say that a centered state $s \in \mathcal{S}$ is *k-bounded* for a natural number k if

$$Q_2(s) \leq (k + 1/3)|s|$$

For a subset $\mathcal{M} \subset \mathcal{S}$ let $\mathcal{M}^{(k)}$ denote the set of k-bounded centered states in $\mathcal{M}$.

8.1. **Balanced states at B-polygons.** Suppose e is an edge of a B-polygon p. In this section we always use the orientation on e coming from the directed graph $\vec{D}$. The orientation of e is counterclockwise with respect to the interior of p. Incident to the ending vertex of e are four edges of $\vec{D}$, and let $\tilde{e}$ be the one opposite to e, i.e. in a small neighborhood of the vertex, $\vec{D}$ looks like a cross, and e and $\tilde{e}$ are on a line as in the following figure



Suppose $s \in \mathcal{S}$ is a centered state. Recall that s is a function on the set of edges of $\vec{D}$ and that we already extended s to the vertices of $\vec{D}$; see Equation (18). Now we further extend s to the set of B-polygons of $\vec{D}$. Suppose p is a B-polygon of $\vec{D}$. Let

$$s(p) = \sum_{e \in \mathcal{E}(p)} s(\tilde{e}). \tag{53}$$

Definition 8.2. We will say that a state $s \in \mathcal{S}$ is *balanced* at a B-polygon p if $s(v) \leq s(p)$ for every vertex v of p, and equality holds for at least one vertex.

8.2. **Seeds.** In this section we introduce seeds, their partial ordering, and relative seeds.

We say that two B-polygons are *disjoint* if they do not have a common vertex. Suppose Π is a collection of disjoint B-polygons. Let $\mathrm{nbd}(\Pi)$ be the set of all edges of $\vec{D}$ incident to a vertex of a polygon in Π. Observe that every edge of a polygon in Π is in $\mathrm{nbd}(\Pi)$.

Definition 8.3. (a) A seed $\theta = (\Pi, \sigma)$ consists of a collection Π of disjoint B-polygons and a map $\sigma : \mathrm{nbd}(\Pi) \to \mathbb{N}$ such that σ can be extended to a centered state $s \in \mathcal{S}$ which is balanced at every polygon in Π. Such s is called an extension of σ, and the set of all extensions of σ is denoted by $\mathcal{S}_\theta$. If σ is a seed and s an extension of it, then for all polygons p in Π and vertices v of p we can define $\sigma(v) = s(v)$ and $\sigma(p) = s(p)$ independent of s. In

particular, σ is balanced at all polygons of Π.
(b) The *B-norm* $\|\theta\|_B$ of θ is the number of B-polygons in Π.

Note that for the empty seed $\theta = \emptyset$, we have $\mathcal{S}_\theta = \mathcal{S}$. Next we define a partial order on the set of seeds.

Definition 8.4. Suppose $\theta = (\Pi, \sigma)$ and $\theta' = (\Pi', \sigma')$ are seeds. Then $\theta \leq \theta'$ if $\Pi \subset \Pi'$ and σ is the restriction of σ'. We write $\theta < \theta'$ if $\theta \leq \theta'$ and $\theta \neq \theta'$.

Observe that $\emptyset \leq \theta$ for any seed θ. Moreover, if $\theta < \theta'$, then $\|\theta\|_B < \|\theta'\|_B$. Since the number of B-polygons is finite, we have the following simple but important fact.

Lemma 8.5. Every strictly increasing sequence of seeds is finite.

We now introduce relative seeds.

Definition 8.6. Suppose $\theta = (\Pi, \sigma)$ and $\theta' = (\Pi', \sigma')$ are seeds with $\theta < \theta'$.
(a) Let $|\theta' \setminus \theta| := \max_{p \in \Pi' \setminus \Pi} \sigma'(p) = \max_{v \in \mathcal{V}(\Pi' \setminus \Pi)} \sigma'(v)$, where the last equality follows from the fact that seeds are balanced.
(b) Let $\mathcal{S}_{\theta < \theta'}$ be the set of all $s \in \mathcal{S}_\theta$ of the form

$$s = s' + \sum_{p \in (\Pi' \setminus \Pi)} (m - \sigma'(p))\, s_p \tag{54}$$

where

$$s' \in \mathcal{S}_{\theta'} \quad \text{and } |s'| < m. \tag{55}$$

8.3. **A partition of the set of k-bounded states.** In this section we give a partition of the set of k-bounded states $\mathcal{S}^{(k)} = \mathcal{S}^{(k)}_\emptyset$ and more generally, the set $\mathcal{S}^{(k)}_\theta$ of k-bounded states with seed θ. The next proposition will be proven in Section 10.

Proposition 8.7. For every non-negative integer k and every seed θ there exists a constant $C > 0$ such that if $|s| > Ck^2$ and $s \in \mathcal{S}^{(k)}_\theta$, then $s \in \mathcal{S}_{\theta < \theta'}$ for a unique seed $\theta' > \theta$ with $|\theta' \setminus \theta| < k$. In other words, up to elements s with $|s| \leq Ck^2$, we have the following finite partition of the set $\mathcal{S}^{(k)}_\theta$ of k-bounded states:

$$\mathcal{S}^{(k)}_\theta = \bigsqcup_{\theta' > \theta,\, |\theta' \setminus \theta| < k} \mathcal{S}^{(k)}_{\theta < \theta'}\,.$$

Example 8.8. Observe that if $\theta = \emptyset$ and $|\theta'| < 1$ then $\sigma = 0$. Proposition 8.7 implies that if s is a 1-bounded state of sufficiently large $|s|$, then

$$s = m s_P + s'$$

where $m = |s| > |s'|$, P is a B-polygon and the support of s' is disjoint from P.

On the other hand, if $|\theta'| = 1$ then the restriction of σ on P looks like Figure 6. In other words, the support of σ is an incoming edge of P, followed by a counterclockwise path along P and finally by an outgoing edge of P. Proposition 8.7 implies that if s is a 2-bounded state of sufficiently large $|s|$, then either

$$s = m s_{P_1} + m s_{P_2} + s'$$

where $m = |s| > |s'|$, P_1, P_2 are disjoint B-polygon and the support of s' is disjoint from $P_1 \cup P_2$, or

$$s = (m-1)s_P + s'$$

where $m = |s| > |s'|$, P is a B-polygon as in Figure 6 and s' is balanced on P.

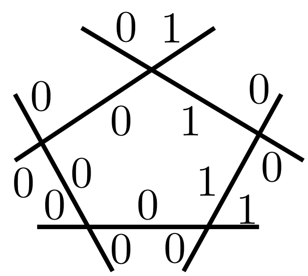


Figure 6. A seed θ' with $|\theta'| = 1$.

When θ is maximal Proposition 8.7 implies the following.

Corollary 8.9. For every non-negative integer k and every maximal seed θ, $\mathcal{S}_\theta^{(k)}$ is a finite set.

The next proposition will also be proven in Section 10.

Proposition 8.10. (a) Suppose $s \in S_{\theta<\theta'}$ with presentation (54) as in Definition 8.6. One has $|s| = m$ and

$$Q_2(s) - Q_2(s') = \sum_{p \in (\Pi \setminus \Pi')} (m - \sigma(p))(\sigma(p)+1), \tag{56}$$

$$L_0(s) \equiv L_0(s') \pmod 2. \tag{57}$$

(b) Fixing $\theta < \theta'$, the presentation of $s \in \mathcal{S}_{\theta<\theta'}$ given by Equation (54), with (m, s') satisfying (55) is unique. In other words, the map $(m, s') \mapsto s$ given by (54) is a bijection between the set of pairs (m, s') satisfying (55) and $\mathcal{S}_{\theta<\theta'}$.

8.4. **The weight of k-bounded states.** In this section we express $F(x, q, s)$ in terms of $F(x, q, s')$ for centered states s, s' related by Equation (54).

Definition 8.11. We say that a series $G(x, y, q) = \sum_{i,j=0}^{\infty} G_{i,j}(q) x^i y^j \in \mathbb{Z}((q))[[x, y]]$ is *weakly x-adequate* of order less $\leq t$ if $G(xq^t, y, q) \in q^{-C}\mathbb{Z}[[q]][[x, y]]$ for some constant C depending on G, i.e.

$$\mathrm{mindeg}_q(G_{i,j}(q)) > -ti - C$$

for every $i, j \geq 0$.

The next lemma is elementary.

Lemma 8.12. (a) If $G(x, y, q)$ is weakly x-adequate of order $\leq t$, then $G(q^k, q^l, q) \in \mathbb{Z}((q))$ for every $k \geq t+1$, $l \geq 0$.
(b) If $G(x, y, q) \in q^{-C}\mathbb{Z}[[q]][[x, y]]$ is weakly x-adequate of order $\leq t$, then for every $l \in \mathbb{N}$,

$q^{-C}G(x,q^l,q) \in \mathbb{Z}[[q]][[x]]$ is x-adequate of order t.
(c) The set of weakly x-adequate series of order less $\leq t$ is closed under addition and multiplication, i.e. it is a $\mathbb{Z}$-subalgebra of $\mathbb{Z}((q))[[x,y]]$.
(d) If $G(x,y,q)$ is weakly x-adequate of order $\leq t$, then it is weakly x-adequate of order $\leq t'$ for every $t' \geq t$.

The next lemma uses the notation of Definition 8.6.

Lemma 8.13. Given $s \in \mathcal{S}_{\theta<\theta'}$ with presentation (54) and let $\ell = \|\theta'\|_B - \|\theta\|_B$. Then there exists a weakly x-adequate series $G_{\theta<\theta'}(x,y,q) \in y^\ell \mathbb{Z}((q))[[x,y]]$ of order $\leq |\theta' \setminus \theta|$ such that for $n \geq |s|$ we have

$$F(q^{n+1},q,s) = G_{\theta<\theta'}(q^{n+1},q^{|s|},q)\, F(q^{n+1},q,s').$$

Moreover, $G_{\theta<\theta'}(q^{n+1},q^{|s|},q) \in \mathbb{Z}[[q]]$.

Proof. Let $x = q^{n+1}$ and $y = q^{|s|}$. By Proposition 8.10, $m = |s|$. For convenience we write $\theta' \setminus \theta$ for $\Pi' \setminus \Pi$, and σ for $\sigma(\theta')$. We have the following relations, followed directly from the definition.

$$\sigma(v) = \sigma(e) + \sigma(\tilde{e}) \tag{58}$$

$$\sigma(p) = \sum_{e \in \mathcal{E}(p)} \sigma(\tilde{e}) \tag{59}$$

$$s(e) = m - \sigma(p) + \sigma(e) \tag{60}$$

$$s(v) = m - \sigma(p) + \sigma(v). \tag{61}$$

Here e is an edge and v is a vertex of a B-polygon p in $\theta' \setminus \theta$, and $\tilde{e}$ is defined as in Section 8.1. Besides, in (58), v is the ending vertex of the edge e. Besides, each of $\sigma(p), \sigma(v) = \sigma(e) + \sigma(\tilde{e})$ is bounded from above by $|\theta' \setminus \theta|$, by definition.

From the definition (26) and Proposition 8.10, we have

$$\begin{aligned}
\frac{F(q^{n+1},q,s)}{F(q^{n+1},q,s')} &= q^{\sum_{p \in (\theta'\setminus\theta)} (m-\sigma(p))(\sigma(p)+1)} \prod_{e \in \mathcal{E}(\theta'\setminus\theta)} \frac{(q)_{\sigma(e)}\, (xq^{-\sigma(e)})_\infty}{(q)_{s(e)}\, (xq^{-s(e)})_\infty} \prod_{v \in \mathcal{V}(\theta'\setminus\theta)} \frac{(xq^{-s(v)})_\infty}{(xq^{-\sigma(v)})_\infty} \\
&= \prod_{p \in (\theta'\setminus\theta)} \left\{ \frac{y}{q^{\sigma(p)}} \prod_{e \in \mathcal{E}(p)} \left[\frac{(q)_{\sigma(e)}}{(xq^{-\sigma(e)-\sigma(\tilde{e})})_{\sigma(\tilde{e})}} \right] \left[\frac{(yq^{\sigma(e)+1-\sigma(p)})_\infty}{(q)_\infty} \right] \right. \\
&\qquad \left. \left[\left(\frac{y}{q^{\sigma(p)}} \right)^{\sigma(\tilde{e})} \left(\frac{q^{\sigma(p)}}{y} x\, q^{-\sigma(e)-\sigma(\tilde{e})} \right)_{\sigma(\tilde{e})} \right] \right\}
\end{aligned}$$

where the second identity follows from a simplification of q-factorial using relations (58)–(61). Let us look at the factors in square brackets.

Since $\sigma(e)+\sigma(\tilde{e}) \leq |\theta' \setminus \theta|$, the first square bracket factor is x-adequate with order $\leq |\theta' \setminus \theta|$.

The second square bracket factor is in $q^{-C}\,\mathbb{Z}[[q]][[y]]$, where $C = |\theta' \setminus \theta|(|\theta' \setminus \theta| + 1)/2$.

The third square bracket factor is a polynomial in x, y with coefficients in $\mathbb{Z}[q^{\pm 1}]$, and it is x-adequate with order $\leq |\theta' \setminus \theta|$. $\square$

8.5. **Stability away from the region of linear growth.** In this section we show the stability for the k-unbounded centered states.

Proposition 8.14. Fix $k, l \in \mathbb{N}$. Suppose $\theta < \theta'$ are seeds and $G(x, y, s) \in \mathbb{Z}((q))[[x, y]]$ is weakly x-adequate of order $\leq l + |\theta' \setminus \theta|$. Then

$$B_n(q) \;\; := \sum_{s:\, |s| \leq n-l\, , s \in \mathcal{S}_{\theta<\theta'} \setminus \mathcal{S}^{(k)}_{\theta<\theta'}} F(q^{n+1}, q, s)\, G(q^{n+1}, q^{|s|}, q)$$

is k-stable.

Proof. Recall $\widetilde{F}(x, q, s)$ from Equation (43). Expand

$$\widetilde{F}(x, q, s) G(x, q^{|s|}, q) = \sum_{m=0}^{\infty} a_m(q, s) x^m$$

into a power series in x and define

$$\Phi_j(q) = \sum_{s:\, s \notin S^{(k)}} q^{Q_2(s)}\, a_j(q, s)$$

for $j \leq k$. The weak x-adequate condition on G and adequate condition on F (from Lemma 7.3) imply that for all but finitely many s and for $j \leq k$ we have

$$Q_2(s) - \mathrm{mindeg}_q(a_j(q, s)) > (k + 1/3)|s| - \mathrm{mindeg}_q(a_j(q, s)) \geq |s|/3 - C\,,$$

where $C \in \mathbb{Z}$ is such that $G(x, y, s) \in q^{-C}\mathbb{Z}[[q]][[x, y]]$. It follows that $\Phi_j(q) \in \mathbb{Z}((q))$ is convergent. Let $f_n(q) = \sum_{s:\, |s| \leq n, s \notin S^{(k)}} F(q^{n+1}, q, s)$. We now follow the proof of part (d) of Lemma 7.2. We have:

$$\left(f_n(q) - \sum_{j=0}^{k} \Phi_j(q) q^{j(n+1)} \right) q^{-k(n+1)} = \Sigma_{1,n} - \Sigma_{2,n}$$

where

$$\Sigma_{1,n} \;=\; \sum_{s:\, |s| \leq n\, , s \notin S^{(k)}} \sum_{j=k+1}^{\infty} q^{Q_2(s)}\, a_j(q, s) q^{(j-k)(n+1)}$$

$$\Sigma_{2,n} \;=\; \sum_{s:\, |s| > n\, , s \notin S^{(k)}} \sum_{j=0}^{k} q^{Q_2(s)}\, a_j(q, s) q^{(j-k)(n+1)}$$

For $\Sigma_{1,n}$ we use the x-adequacy of order $\leq |s|$ to obtain

$$Q_2(s) + \mathrm{mindeg}_q(a_j(q, s)) + (j-k)(n+1) \geq Q_2(s) - j|s| + (j-k)(n+1)$$

Since the coefficient of j in the above expression is $n+1-|s| > 0$, it follows that its minimum as a function of j is attained at $j = k+1$, i.e.,

$$Q_2(s) - j|s| + (j-k)(n+1) \geq Q_2(s) - (k+1)|s| + n + 1$$

Since $s \notin S^{(k)}$ and $|s| \leq n$ it follows

$$Q_2(s) - (k+1)|s| + n + 1 \geq (k + 1/3)|s| - (k+1)|s| + n + 1 = -2|s|/3 + n + 1 > n/3\,.$$

For $\Sigma_{2,n}$ since $|s| > n$ we the fact that s is not k-bounded to obtain

$$Q_2(s) - j|s| + (j-k)(n+1) \geq Q_2(s) - k|s| \geq |s|/3 > n/3\,.$$

Thus,

$$\left(f_n(q) - \sum_{j=0}^{k} \Phi_j(q) q^{j(n+1)} \right) q^{-k(n+1)} \in q^{n/3}\mathbb{Z}[[q]]\,.$$

This completes the proof of the proposition. ☐

9. Stability in the region of linear growth

Theorem 9.1. *Suppose θ is a seed and $G(x,y,q) \in \mathbb{Z}((q))[[x,y]]$ is weakly x-adequate of order $\leq |\theta| + l$, where $l \in \mathbb{N}$. Then the sequence*

$$H_n(q) = \sum_{s:\,|s| \leq n-l\,, s \in \mathcal{S}_\theta} F(q^{n+1}, q, s)\, G(q^{n+1}, q^{|s|}, q)$$

is stable.

Remark 9.2. In particular, the above theorem holds when $\theta = \emptyset$, $l = 0$ and $G = 1$. In that case, Proposition 5.9 implies that $H_n(q) = \hat{J}_{K,n}(q)$ and we conclude the stability of the colored Jones polynomial of an alternating link K.

Proof. Fix a natural number k. We will prove that $H_n(q)$ is k-stable. Subtracting the k-unbounded part from $H_n(q)$ and using Proposition 8.14, it is enough to show that

$$H'_n(q) = \sum_{s:\,|s| \leq n-l\,, s \in \mathcal{S}^{(k)}_\theta} \mathcal{F}_n(q,s)$$

is k-stable. We proceed by downwards induction, starting from the case when θ is maximal. This case follows from Corollary 8.9, which states that $\mathcal{S}^{(k)}_\theta$ is a finite set, and Lemma 9.3.

Assume that the statement holds for all θ' strictly greater than θ. We will show that the statement holds for θ. Then Lemma 8.5 implies that the statement holds for any seed θ.

Using the partition of $\mathcal{S}^{(k)}_\theta$ described in Proposition 8.7, and n sufficiently large, we obtain that

$$\sum_{s:\,|s| \leq n-l\,, s \in \mathcal{S}^{(k)}_\theta} \mathcal{F}_n(q,s) = \sum_{\theta' > \theta,\, |\theta' \setminus \theta| \leq k} \left(\sum_{s:\,|s| \leq n-l\,, s \in \mathcal{S}^{(k)}_{\theta < \theta'}} \mathcal{F}_n(q,s) \right) + \mathrm{Err}, \tag{62}$$

where Err is a finite alternating sum of terms of the form $\mathcal{F}_n(q,s)$ for some $s \in \mathcal{S}_\theta$. By Lemma 9.3, Err is stable. Because the outer sum on the right hand side of (62) is finite, it is enough to prove k-stability for each inner sum

$$H''_n(q) := \sum_{s:\,|s| \leq n-l\,, s \in \mathcal{S}^{(k)}_{\theta < \theta'}} \mathcal{F}_n(q,s)\,.$$

Adding back the k-unbounded part (using Proposition 8.14), it is enough to show that

$$H_n'''(q) := \sum_{s:\,|s|\leq n-l\,,s\in\mathcal{S}_{\theta<\theta'}} \mathcal{F}_n(q,s)$$

is k-stable. Using the decomposition of Lemma 8.13, we have

$$\begin{aligned}\mathcal{F}_n(q,s) &= G(q^{n+1},q^m,q)\,G_{\theta<\theta'}(q^{n+1},q^m,q)\,F(q^{n+1},q,s') \\ &= G'(q^{n+1},q^m,q)\;F(q^{n+1},q,s')\,,\end{aligned} \tag{63}$$

where $G'(x,y,q)=G(x,y,q)\,G_{\theta<\theta'}(x,y,q)$, and $s'\in\mathcal{S}_{\theta'}$. $G(x,y,q)$ is weakly x-adequate of order $\leq|\theta|+l$ and $|\theta|+l\leq|\theta'+l|$. Moreover, $G_{\theta<\theta'}(x,y,q)$ is weakly x-adequate of order $\leq|\theta'\setminus\theta|$ and $|\theta'\setminus\theta|\leq|\theta'|\leq|\theta'|+l$. Lemma 8.12 implies that $G'(x,y,q)$ is weakly x-adequate of order $\leq|\theta'|+l$.

By part (b) of Proposition 8.10, $\mathcal{S}_{\theta<\theta'}$ is parametrized by pairs (m,s') with $s'\in\mathcal{S}_{\theta'}$ with $|s'|<m$. We have

$$\begin{aligned}H_n'''(q) &= \sum_{s:\,|s|\leq n-l\,,s\in\mathcal{S}_{\theta<\theta'}} \mathcal{F}_n(q,s) \\ &= \sum_{m=1}^{n-l} G'(q^{n+1},q^m,q) \sum_{s':\,|s'|<m\,,s'\in\mathcal{S}_{\theta'}} F(q^{n+1},q,s') \\ &= \sum_{s:\,|s|\leq n-l-1\,,s\in\mathcal{S}_{\theta'}} F(q^{n+1},q,s) \sum_{m=|s|+1}^{n-l} G'(q^{n+1},q^m,q) \\ &= \sum_{s:\,|s|\leq n-l-1\,,\,s\in\mathcal{S}_{\theta'}} F(q^{n+1},q,s)\,G''(q^{n+1},q^{|s|},q)\,,\end{aligned}$$

where the second identity follows from (63) and the above mentioned parametrization of $\mathcal{S}_{\theta<\theta'}$, the third identity follows by changing notation s' to s and exchanging the two summations, and the fourth identity follows from Lemma 9.4 below, with $G''(x,y,q)$ a weakly x-adequate series of order $\leq|\theta'|+l$. By induction hypothesis, the last sum of the above identity is k-stable. This completes the proof of Theorem 9.1. $\square$

Lemma 9.3. For a fixed $s\in\mathcal{S}_\theta$, and $G(x,y,q)$ weakly x-adequate of order $\leq t$, the sequence $\mathcal{F}_n(q,s):=F(q^{n+1},q,s)\,G(q^{n+1},q^{|s|},q)$ is stable.

Proof. Lemma 8.12 implies that $q^{-C}G(x,q^{|s|},q)$ is x-adequate and part (a) of Lemma 7.2 implies that $q^{-C}F(x,q,s)G(x,q^{|s|},q)$ is x-adequate, too. The result follows from part (d) of Lemma 7.2. $\square$

The next lemma is reminiscent to the notion of a q-Laplace transform.

Lemma 9.4. Suppose $l,t\in\mathbb{N}$, and $G(x,y,q)\in q^{-C}\mathbb{Z}[[q]][[x,y]]$ is weakly x-adequate of order $\leq l+t$. Then there exists a weakly x-adequate series $H(x,y,q)\in q^{-C}\mathbb{Z}[[q]][[x,y]]$ of order $\leq l+t$, such that for every $a, n \in \mathbb{N}$ with $n \geq l+t+1$ and $n \geq l+a+1$,

$$\sum_{m=a+1}^{n-l} G(q^{n+1}, q^m, q) = H(q^{n+1}, q^a, q)\,. \tag{64}$$

Proof. Let $G(x,y,q) = \sum G_{i,j}(q) x^i y^j \in \mathbb{Z}((q))[[x,y]]$. We have:

$$\sum_{m=a+1}^{n-l} q^{mj} = \frac{q^{j(a+1)} - q^{(n+1-l)j}}{1-q^j} = \frac{y^j q^j - x^j q^{-lj}}{1-q^j}\Big|_{x=q^{n+1}, y=q^a} \tag{65}$$

Hence if we define

$$H(x,y,q) = \sum_{i,j} G_{i,j}(q) x^i \frac{y^j q^j - x^j q^{-lj}}{1-q^j}\,,$$

then (64) holds. It is easy to see that H is weakly x-adequate of order $\leq l+t$. □

10. Partition of the set of k-bounded states

In this section, we will prove Propositions 8.7 and Proposition 8.10. We will fix an A-infinite alternating, diagram D with c_D crossings. We assume that D represent a non-trivial link, hence $c_D \geq 2$.

10.1. **Some lemmas regarding k-centered states.** Suppose p is a B-polygon of $\vec{D}$. Recall that the orientation of every edge of p is counterclockwise. Incident to the ending vertex of en edge $e \in \mathcal{E}(p)$ there are two edges of $\vec{D}$ no belonging to p; one of them is $\tilde{e}$ defined in Section 8.1, and let $\breve{e}$ be the other edge as in Figure 7.

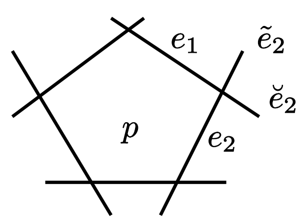


Figure 7. A vertex of a B-polygon p and its neighboring edges.

Adding up Equations (16) for all vertices of p, and using the definition of $s(p)$ from Equation (53) it follows that

$$s(p) = \sum_{e \in \mathcal{E}(p)} s(\widetilde{e}) = \sum_{e \in \mathcal{E}(p)} s(\breve{e})\,. \tag{66}$$

Note that if we think of s as a flow on $\vec{D}$, then a B-polygon p is oriented counterclockwise and $s(p)$ measures the amount that flows towards p. Equation (66) states that $s(p)$ also equals to the amount that flows away from p. The next lemma motivates the definition of $s(p)$.

Lemma 10.1. Suppose $p \in \mathcal{P}(\vec{D})$ is a B-polygon, s' is a centered state, $l \in \mathbb{N}$, and

$$s = l s_p + s' .$$

Then

$$Q_2(s) = Q_2(s') + l(s'(p) + 1) .$$

Proof. Recall that for a centered state s,

$$Q(s) = \frac{1}{2} \sum_{\alpha} ab, \tag{67}$$

where the sum is over all angles α of type A, and a and b are the s-values of the two edges forming the angle α.

Note that $s(e) = s'(e)$ except when e is an edge of p. Hence

$$Q(s) - Q(s') = \frac{1}{2} \sum_{\alpha} \big(s(e)s(f) - s'(e)s'(f) \big) , \tag{68}$$

where the sum is over all A-angles α whose vertex is a vertex of p. Each vertex v has two A-angles, and each such A-angle has one edge in p, denoted by e in (68), and one edge not belonging to p, denoted by f in (68). Then $s(f) = s'(f)$ and $s(e) - s'(s) = l$, hence from (68)

$$Q(s) - Q(s') = l \sum_{\alpha} s'(f)/2 = l s'(p) .$$

Since L_1 is linear we have have $L_1(s) - L_1(s') = L_1(l s_p) = l$, where the last identity comes from (29). Hence

$$Q_2(s) - Q_2(s') = Q(s) - Q_2(s') + L_1(s) - L_1(s') = l(s'(p) + 1) . \tag{69}$$

□

Lemma 10.2. Suppose p is a B-polygon, s a centered state, and $m = \max_{v \in \mathcal{V}(p)} s(v)$. Then for $s(e) \geq m - s(p)$ for every $e \in \mathcal{E}(p)$.

Proof. Suppose $m = s(v)$, where v is the ending vertex of the edge e_1. Assume that $e_1, e_2, \dots e_t$ are all edges of p, counting clockwise, as in Figure 7. By identity (16) at the ending vertex of e_j, we have $s(e_j) - s(e_{j-1}) = s(\breve{e}_j) - s(\tilde{e}_j)$. Hence

$$s(e_j) - s(e_{j-1}) \geq -s(\tilde{e}_j) . \tag{70}$$

Summing the above inequalities with j from 2 to n, together with the identity $s(e_1) = m - s(\tilde{e}_1)$, we have

$$s(e_n) \geq m - \sum_{j=1}^{n} s(\tilde{e}_j) \geq m - s(p) .$$

□

10.2. A decomposition of k-bounded states.

Definition 10.3. For a centered state s and a positive integer k, a polygon $p \in \mathcal{P}(\vec{D})$ is (k,s)-*big* if s achieves the maximal value $|s|$ at one of the the vertices of p and $s(p) < k$.

It is obvious that every centered state s has some B-polygons such that s achieves the maximum value at a vertex of those polygons. On the other hand, is it not true that every state s has (k,s)-big polygons (for some k), which are always B-polygons and always disjoint. However, this is true for k-bounded states. This is the content of the following lemma. Its proof reveals a close connection between the notions of a k-bounded state (given in Definition 8.1) and balanced polygons of B-type (given in Definition 8.2).

Lemma 10.4. Suppose s is a k-bounded centered state satisfying

$$|s| > 12k(2k+1)c_D. \tag{71}$$

(a) Any (k,s)-big polygon is a B-polygon and any two (k,s)-big polygons are disjoint.
(b) Suppose s achieves maximum at a vertex v, i.e. $s(v) = |s|$. Then exactly one of the two B-polygons incident to v is (k,s)-big.

Proof. (a) If two edges $e, f \in \mathcal{E}(\vec{D})$ form an A-angle, then from (67) we have $Q(s) \geq s(e)s(f)/2$. Hence if s is k-bounded we have

$$(k+1/3)|s| \geq \frac{s(e)s(f)}{2}. \tag{72}$$

If p is an A-polygon, then any two consecutive edges of p form an A-angle. Suppose p is (k,s)-big. Then $s(p) < k$, and by Lemma 10.2, $s(e) > |s| - k$ for every edge e of p. Also, from (71) it is clear that $k < |s|/2$. It follows from (72) that

$$(k+1/3)|s| \geq (|s|-k)^2/2 > |s|^2/8. \tag{73}$$

Hence $|s| < 8(k+1/3)$, which contradicts (71).

Now suppose two (k,s)-big polygons share a common vertex v. Then for any A-angle at v the s-value of any edge is $\geq |s| - k$. We again lead to (73), which is a contradiction.

(b) To prove part (b) we first prove a few claims.

Claim 1. Suppose p is a B-polygon of $\vec{D}$. Assume that $s(e) > |s| - 4kc_D$ for an edge e of p. Then $s(e') \geq s(e) - 2k$ for any edge e' of p incident to e.

Proof of Claim 1. Assume the contrary that $s(e') < s(e) - 2k$. Suppose v is the common vertex of e, e and f, f' are the two remaining edges incident to v such that f is opposite to e as in the following Figure 8.

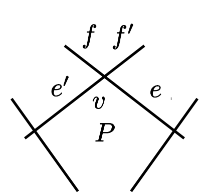


Figure 8. A vertex v of B-polygon and its neighboring edges.

Since $s(f') - s(f) = s(e) - s(e') > 2k$, we have $s(f') \geq 2k+1$. Since the angle between e and f' is of type A,

$$Q_2(s) \geq s(e)s(f')/2 \geq (2k+1)(|s| - 4kc_D)/2 > |s|(k+1/3),$$

where the last inequality follows from (71). The above inequality contradicts the assumption that s is k-bounded. □

Let v be a vertex of $\vec{D}$ where $s(v) = |s|$. One of the four edges incident to v, say e, has s-value $\geq |s|/2$. Let p be the unique B-polygon of $\vec{D}$ having e as an edge on the boundary. We will prove that p is (k,s)-big. For this, we need to show that $s(p) < k$.

Claim 2. The s-value of every edge of p is at least $|s| - 4kc_D$.

Proof of Claim 2. Besides e, suppose e' is the other edge incident to v which is also an edge of p, and f, f' are the other two edges which are not edges of p, as in Figure (8). Note that the number of edges of p is less than $2c_D$, the total number of edges of $\vec{D}$.

By (72) and $s(e) \geq |s|/2$, we have

$$(k+1/3)|s| \geq s(e)s(f')/2 \geq |s|s(f')/4\,.$$

It follows that $s(f') \leq 4(k+1/3) \leq 4k+2$, and hence by Equations (16) we have $s(e') \geq |s| - 4k - 2$.

If g is an edge of p, then there is a path from e' to g consisting of at most $c_D - 1$ edges. It follows from Claim 1 that

$$s(g) \geq s(e') - 2k(c_D - 1) \geq |s| - 4k - 2 - 2k(c_D-1) = |s| - (2k+2+2kc_D) \geq |s| - 4kc_D.$$

In the last inequality we used the fact that $k \geq 1$ and $c_D \geq 2$. □

Now we can finish the proof of part (b) of Lemma 10.4. It remains to prove that $s(p) < k$. Assume the contrary, i.e., that $s(p) \geq k$. Let $s' = s - (|s| - 4kc_D)s_p$. By Claim 1, s' takes non-negative value at every edge of $\vec{D}$, hence s' is a centered state, and $Q_2(s') \geq 0$. Moreover, $s(p) = s'(p)$, since s and s' agree on any edge not belonging to p. Since $s = s' + (|s| - 4kc_D)s_p$, Lemma 10.1 implies that

$$Q_2(s) = Q_2(s') + (|s| - 4kc_D)(s(p)+1) \geq (|s| - 4kc_D)(k+1) > (k+1/2)|s|\,,$$

which contradicts the k-boundedness of s. This completes the proof of Lemma 10.4. □

10.3. **Proof of Proposition 8.10.** Part (a). Suppose $s \in \mathcal{S}^{(k)}_{\theta<\theta'}$ has the presentation (54)

$$s = s' + \sum_{p \in (\Pi' \setminus \Pi)} (m - s(p))s_p \tag{74}$$

with $s' \in S_{\theta'}$ and $|s'| < m$. The $s'(e) = s(e)$ for every edge e outside $\Pi' \setminus \Pi$. Hence if v is is not a vertex of any $p \in (\Pi' \setminus \Pi)$, then $s(v) = s'(v) < m$.

On the other hand if v is a vertex of $p \in (\Pi' \setminus \Pi)$, then

$$s(v) = s'(v) + (m - s'(p)) \leq m\,,$$

where the inequality follows from the fact that s' is balanced at p. But there is a vertex of p such that $s'(v) = s'(p)$, and for which $s(v) = m$. It follows that the maximum of $s(v)$ is m, or $|s| = m$.

Identity (56) follows right away from Lemma 10.1. Identity (57) follows that the fact that L_0 is a linear map, $L_0(s_p) =\equiv 2L_1(s_p) \equiv 2 \pmod 2$, by Lemmas 6.1 and 5.4.

Part(b). We have to show that s' and m are uniquely determined by s. In fact, by part (a), $m = |s|$. Then (74) shows that s' is determined by s and m. This completes the proof of Proposition 8.10. □

10.4. **Proof of Proposition 8.7.** Suppose $\theta = (\Pi, \sigma)$ is a seed and consider a k-bounded centered state $s \in \mathcal{S}_\theta^{(k)}$. Recall that $|\theta| = \max_{v\in\mathcal{V}(\Pi)} \sigma(v)$. Assume that

$$|s| > \max(12k(2k+1)c_D, |\theta| + k). \tag{75}$$

Will show that if $s \in \mathcal{S}_\theta^{(k)}$ satisfying the lower bound (75), then there is a unique $\theta' > \theta$ with $|\theta' \setminus \theta| < k$ such that $s \in \mathcal{S}_{\theta<\theta'}^{(k)}$. This will prove Proposition 8.7.

Uniqueness. Assume that $s \in \mathcal{S}_{\theta<\theta'}^{(k)}$ with $|\theta' \setminus \theta| < k$. Then s has a presentation given by (74). By Proposition 8.10(a), $m = |s|$ is uniquely determined by s. In the proof of Proposition 8.10(a) in Section 10.3 we showed that if $p \in (\Pi' \setminus \Pi)$ then there is a vertex v of p such that $s(v) = |s|$. We also have that $s(p) \le |\theta' \setminus \theta| < k$. Thus every $p \in \Pi' \setminus \Pi$ is (k,s)-big.

Conversely, suppose p is a (k,s)-big polygon. Then there is a vertex v of p such that $s(v) = |s|$. The proof of Proposition 8.10(a) showed that v is a vertex of a polygon $p' \in \Pi' \setminus \Pi$. Both p and p' are incident to v and both are (k,s)-big. By Proposition 10.4(a), $p = p'$.

Thus $\Pi' \setminus \Pi$ is the set of all (k,s)-big polygons. This determines Π' uniquely. Then (74) shows that s' is uniquely determined by s, and hence σ', which is the restriction of s' on $\mathrm{nbd}(\Pi')$ is uniquely determined by s. This completes the proof of uniqueness.

Existence. The proof of the uniqueness already shows us how to construct a presentation (74) for $s \in \mathcal{S}_\theta^{(k)}$.

Let Ψ be the set of all (k,s)-big polygons. If p is (k,s)-big, then by Lemma 10.2 and (75), $s(v) > |s| - k > |\theta|$ for every vertex $v \in \mathcal{V}(p)$. This implies if p is disjoint from any polygon in Π. In particular, $\Pi \cap \Psi = \emptyset$. Let $\Pi' = \Pi \cup \Psi$.

By Lemma 10.2, for any edge e of a B-polygon $p \in \Psi$, $s(e) \ge |s| - \sigma(p)$. Then

$$s' := s - \sum_{p\in\Psi} (|\sigma| - s(p)) s_p \tag{76}$$

takes non-negative integer value at every edge of $\vec{D}$, and hence is a centered state. Note that $s(p) = s'(p)$ for any $p \in \Psi$ since s and s' agree on any edge outside Ψ. We will show that (76) gives us the presentation (74).

If v is any vertex of $\vec{D}$ for which $s(v) = |s|$, then Lemma 10.4(b) shows that v is a vertex of some polygon $p \in \Psi$. Hence $s'(v) = s(v) - (|s| - s(p)) < s(v)$. This means $|s'| < |s|$.

If v is vertex of $p \in \Psi$, then

$$s'(v) = s(v) - (|s| - s(p)) = (s(v) - |s|) - s(p) \le s(p) = s'(p)\,.$$

On the other hand, if v is a vertex of $p \in \Psi$ for which $s(v) = |s|$, then the above identity shows that $s'(v) = s'(p)$. This means s' is balanced at every $p \in \Psi$. Since $s' = \sigma$ in $\mathrm{nbd}(\Pi)$, it is balanced at every $p \in \Pi$. Thus s' is balanced at every $p \in \Pi' = \Pi \cup \Psi$.

Let σ' be the restriction of s' on $\mathrm{nbd}(\Pi')$ and $\theta' = (\Pi', \sigma')$. Then $s' \in S_{\theta'}$, and (76) gives us the presentation (74), and we have $s \in \mathcal{S}^{(k)}_{\theta<\theta'}$.

Let us estimate $|\theta' \setminus \theta|$. By definition 8.6,

$$|\theta' \setminus \theta| = \max_{v \in \mathcal{V}(\Psi)} s'(v) = \max_{p \in \Psi} s'(p) < k.$$

Thus we conclude that every $s \in \mathcal{S}^{(k)}_{\theta}$ satisfying (75) is an element of $\mathcal{S}^{(k)}_{\theta<\theta'}$ for some $\theta' > \theta$ with $|\theta' \setminus \theta| < k$. This concludes the proof of the existence, and whence Proposition 8.7. □

11. Proof of Theorem 1.16

In this section we prove Theorem 1.16. It is well-known that pointwise sums and products of q-holonomic sequences are q-holonomic (see [PWZ96, Zei90]). Moreover, the colored Jones polynomial $(J_{K,n}(q))$ of every link is q-holonomic [GL05]. Using (34) we deduce that $(\hat{J}_{K,n}(q))$ is q-holonomic for every alternating link K. Using a recursion relation (7) for $f_n(q) = \hat{J}_{K,n}(q)$ and the stability Theorem 1.4, and collecting powers of q and q^n, it follows that $\Phi_{K,k}(q)$ is q-holonomic.

Using a linear recursion for $\Phi_{K,k}(q)$, it is easy to see that $\mathrm{mindeg}_q(\Phi_{K,k}(q))$ is bounded below by a quadratic function of k; see for example [GL11, Thm.10.3]. A stronger statement is known [Gar11a], namely $\mathrm{mindeg}_q(\Phi_{K,k}(q))$ is a quadratic quasi-polynomial of k. This proves Equation (8).

Equation (9) follows from Equation (8) using Lemma 11.1 below. This concludes the proof of Theorem 1.16. □

Lemma 11.1. Fix $f_n(q) \in \mathbb{Z}((q))$ and $\Phi_k(q) \in \mathbb{Z}((q))$ and let

$$R_{k,n}(q) = \left(f_n(q) - \sum_{j=0}^{k} \Phi_k(q) q^{j(n+1)} \right) q^{-k(n+1)}$$

Assume that $\lim_{n\to\infty} R_{k,n}(q) = 0$ for all k. Then the following are equivalent:

(a) $\mathrm{mindeg}_q(\Phi_k(q)) \geq -C_1 k^2 - C_2$ for all k.

(b) $\mathrm{mindeg}_q(R_{k,n}(q)) \geq n + 1 - C_1(k+1)^2 - C_2$ for all k and all n large enough.

Proof. Let $v = \mathrm{mindeg}_q$. The assumption on $R_{k,n}(q)$ implies that

$$\lim_{n\to\infty} \mathrm{mindeg}_q(R_{k,n}(q)) = +\infty\,. \tag{77}$$

It is easy to see that for all k and n we have

$$\Phi_k(q) = R_{k,n}(q) - q^{-n-1} R_{k-1,n}(q)\,. \tag{78}$$

It follows that

$$-n - 1 + v(R_{k-1,n}(q)) \geq \min\{v(R_{k,n}(q)), v(\Phi_k(q))\} \tag{79}$$

and

$$v(\Phi_k(q)) \geq \min\{v(R_{k,n}(q)), -n - 1 + v(R_{k-1,n}(q))\} \tag{80}$$

Now, (a) implies (b) by Equations (77) and (80) and (b) implies (a) by Equations (77) and (79). □

12. An algorithm for the computation of $\Phi_{K,k}(q)$

12.1. A parametrization of 1-bounded states. In this section we will compute explicitly the series $\Phi_{K,1}(q)$ of an alternating knot in terms of a planar projection as in Theorems 1.10 and 1.14. We begin with a corollary of Proposition 8.7 for $k=1$ and $\theta=\emptyset$. See also Example 8.8.

Corollary 12.1. Suppose that s is a 1-bounded centered state and $|s|>6$. Then, there exists a B-polygon p and a state s' such that

$$s=|s|s_P+s' \tag{81}$$

and $s(e)=0$ if e is an edge of $\vec{D}$ which contains a vertex of p. Moreover, (P,s') are uniquely determined by s.

12.2. The computation of $\Phi_{K,1}(q)$ in terms of a planar diagram. We start with the state-sum of $\hat{J}_{K,n}(q)$ over the set of states s with $|s|\leq n$ and separate it in two different sums: $Q_2(s)>4|s|/3$ or $Q_2(s)\leq 4|s|/3$. Then we have

$$\hat{J}_{K,n}(q)=f_n^{(1)}(q)+f_n^{(2)}(q)\,,$$

where

$$f_n^{(1)}(q)=\sum_{s:\,|s|\leq n\,,Q_2(s)>4|s|/3}F(q^{n+1},q,s),\qquad f_n^{(2)}(q)=\sum_{s:\,|s|\leq n\,,Q_2(s)\leq 4|s|/3}F(q^{n+1},q,s)\,.$$

We will show that $f_n^{(i)}(q)$ are 1-stable for $i=1,2$ and compute their 1-stable limit. For the stability of $f_n^{(1)}(q)$, write

$$F(x,q,s)=q^{Q_2(s)}\sum_{k=0}^{\infty}a_k(q,s)x^k\,,$$

where $a_k(q,s)\in\mathbb{Z}((q))$ satisfy $\mathrm{mindeg}_q(a_k(q,s))\geq -k|s|$. Define

$$\Phi_k^{(1)}(q)=\sum_{s:Q_2(s)>4|s|/3}q^{Q_2(s)}a_k(q,s)$$

for $k=0,1$. Using Equation (43) we see that

$$\begin{aligned}
a_0(s) &= (q)_\infty^{c_D}\frac{(-1)^{L_0(s)}}{\prod_{e\in\mathcal{E}}(q)_{s(e)}}\\
a_1(s) &= \frac{(q)_\infty^{c_D}}{1-q}\frac{(-1)^{L_0(s)}}{\prod_{e\in\mathcal{E}}(q)_{s(e)}}\left(\sum_{e\in\mathcal{E}}q^{-s(e)}-\sum_{v\in\mathcal{V}}q^{-s(v)}\right)
\end{aligned}$$

where $\mathcal{V}$ and $\mathcal{E}$ are the vertices and the edges of $\vec{D}$.

The series $\Phi_k^{(1)}(q)$ for $k=0,1$ are convergent since $Q_2(s)-k|s|>4|s|/3-k|s|\geq |s|/3$ for $k=0,1$. Moreover,

$$(f_n^{(1)}(q)-\Phi_0^{(1)}(q)-q^{n+1}\Phi_1^{(1)}(q))q^{-n-1}=\Sigma_{1,n}(q)-\Sigma_{2,n}(q)\,,$$

where

$$\Sigma_{1,n}(q) = \sum_{Q_2(s)>4|s|/3,\, |s|\leq n} q^{Q_2(s)} \mathrm{ev}_n\left(\sum_{k=2}^{\infty} a_k(q,s)x^{k-1}\right)$$

$$\Sigma_{2,n}(q) = \sum_{Q_2(s)>4|s|/3,\, |s|>n} q^{Q_2(s)} \mathrm{ev}_n\left(\sum_{k=0}^{1} a_k(q,s)x^{k-1}\right).$$

Here, $\mathrm{ev}_n(f(x)) = f(q^{n+1})$. For the first sum, it suffices to consider $k=2$ and then

$$Q_2(s)+n+1-2|s| > 4|s|/3+n+1-2|s| = |s|/3+n+1-|s|$$

Now,

$$\min_{|s|\leq n}(|s|/3+n+1-|s|) = (|s|/3+n+1-|s|)|_{|s|=n} = n/3+1$$

thus $\lim \Sigma_{1,n}(q)=0$. For the second sum, we have:

$$Q_2(s)-n-1 \geq 4|s|/3-n-1 > 4n/3-n-1 \geq n/3-1$$

thus $\lim \Sigma_{2,n}(q)=0$.

For the 1-stability of $f^{(2)}_n(q)$, use Corollary 12.1 to write $s = ms_P + s'$ where $|s|=m$ and p is a B-polygon with $\kappa(P)$ edges. It follows that

$$F(x,q,s) = \frac{q^m (q^{m+1})_\infty^{\kappa(P)}}{(q)_\infty^{\kappa(P)}} F(x,q,s').$$

It follows that

$$f^{(2)}_\infty(q) = \sum_{s\colon Q_2(s)\leq 4|s|/3} F(0,q,s)$$

exists and $\lim_{n\to\infty} f^{(2)}_n(q) = f^{(2)}_\infty(q)$. Moreover,

$$q^{-(n+1)}(f^{(2)}_n(q)-f^{(2)}_\infty(q)) = q^{-(n+1)} \sum_{s\colon |s|>n\,, Q_2(s)\leq 4|s|/3} F(0,q,s)+\epsilon_n(q),$$

where $\lim_{n\to\infty}\epsilon_n(q)=0$. Since $|s|=m>n$, we change variables to $m=n+1+l$. Then,

$$q^{-(n+1)}(f^{(2)}_n(q)-f^{(2)}_\infty(q))-\epsilon_n(q) = (q)_\infty^{c_D}\sum_P\sum_{l,s'}(-1)^{L_0(s')}\frac{q^l(q^{n+2+l})_\infty^{\kappa(P)}}{(q)_\infty^{\kappa(P)}}\;\frac{q^{Q_2(s)}}{\prod_{e\in\mathcal{E}_P}(q)_{s(e)}}.$$

It follows that the limit is obtained by setting $q^n=0$ in the above expression and summing over l, we obtain that

$$\lim_{n\to\infty} q^{-(n+1)} f^{(2)}_n(q) = \Phi^{(2)}_1(q)$$

where

$$\Phi^{(2)}_1(q) = \frac{(q)_\infty^{c_D}}{1-q}\sum_P\sum_{s'}(-1)^{L_0(s')}\frac{1}{(q)_\infty^{\kappa(P)}}\;\frac{q^{Q_2(s)}}{\prod_{e\in\mathcal{E}_P}(q)_{s(e)}}.$$

Setting $\Phi^{(2)}_0(q)=f^{(2)}_\infty$, it follows that

$$\lim_{n\to\infty}(f^{(2)}_n(q)-\Phi^{(2)}_0(q)-q^{n+1}\Phi^{(2)}_1(q))q^{-n-1}=0.$$

Using Section 6 we can convert the above formula for $\Phi_{K,1}(q)$ in terms of the Tait graph of an alternating planar projection of K. This concludes the proof of Theorem 1.14. $\square$

The above algorithm can be used to give a formula for $\Phi_{K,k}(q)$ as follows. Separate the state-sum of Equation (33) in two regions:

- s is not k-bounded.
- s is k-bounded.

In the first region, use Proposition 8.14 to compute the k-stable limit. In the second region, use Proposition 8.7 to write

$$s = s^{(1)} + s', \qquad s = \sum_{j=1}^{t} (m - k_j) s_{P_j} \,.$$

Observe that $t \leq k$. If $t = k$ we stop. Else, replace (s, k) by $(s', k - t)$ in the above step and and run it again. Keep going. Since each step requires at least one new polygon of B-type which is vertex-disjoint from the previous ones, this algorithm terminates in finitely many steps.

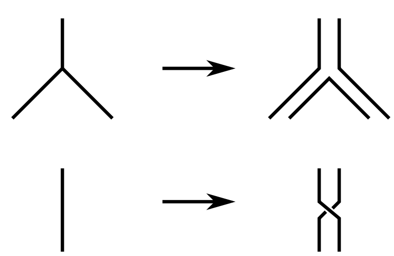
Remark 12.2. Using the parametrization of 2-bounded states from Example 8.8 and the above algorithm, the reader may obtain a formula for $\Phi_{K,2}(q)$.

13. Φ_0 is determined by the reduced Tait graph

In this Section we prove Corollary 1.12. Throughout we use the following convention on graphs: a graph is a finite 1-dimensional CW-complex without loop edge.

Recall that a plane graph is a pair $\gamma = (\Gamma, f)$, where Γ is a finite connected planar graph and $f : \mathcal{T} \to \mathbb{R}^2 \subset S^2$ is an embedding. For example, if D is an alternating nonsplit link diagram, then the Tait graph $\gamma(D) = (\mathcal{T}, f)$ is a plane graph. One can recover K from $\mathcal{T}$ up to orientation.

13.1. **From plane graph to non-oriented alternating link.** For a plane graph $\gamma = (\Gamma, f)$ with $f(\Gamma) \subset \mathbb{R}^2$ define an alternating A-infinite link diagram $D(\gamma)$ as follows. If we replace $f(\Gamma)$ by a small normal neighborhood in $\mathbb{R}^2$ and twist each edge as indicated below, then the boundary of the resulting surface is $D(\gamma)$.



Note that $D(\gamma)$ is a non-oriented alternating A-infinite link diagram. The resulting $D(\gamma)$, although alternating, maybe reducible. If D is an alternating link diagram, and $\mathcal{T}$ be its Tait graph, then $D(\mathcal{T}) = D$.

Exercise 13.1. Show that $D(\gamma)$ is reducible if and only if Γ contains a cut edge, i.e. an edge e such that removing in interior of e make Γ disconnected.

For a plane graph γ let $K(\gamma)$ be the non-oriented alternating link whose diagram is $D(\gamma)$. Even when $D(\gamma)$ is reducible, it is still A-adequate. Hence we can use $D(\gamma)$ to calculate $\Phi_{K(\gamma),0}$, as in Theorem 1.10, see Remark 7.5. This means

$$\Phi_{\gamma,0}(q) = \Phi_{K(\gamma),0}(q), \tag{82}$$

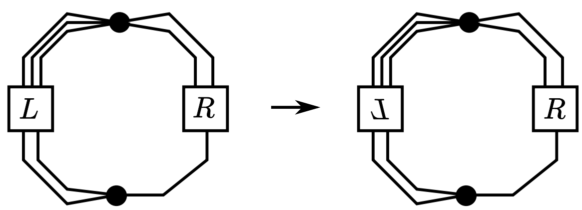
where $\Phi_{\gamma,0}$ is given by the the right hand side of (5) with $D = D(\gamma), \mathcal{T} = \gamma$.

The dual $D^*(\gamma)$ (in S^2) of $D(\gamma)$ can be constructed directly from γ as follows: in each region p of γ choose a point u_p and connect u_p to all the vertices of p by edges inside the region p so that the edges do not intersect except at u_p. Then $D^*(\gamma)$ is the plane graph whose vertex set is $\{u_p, p \in \mathcal{P}(\gamma)\} \cup \mathcal{V}(\gamma)$ and whose edges are all the edges just constructed. The edges of γ are not edges of D^*.

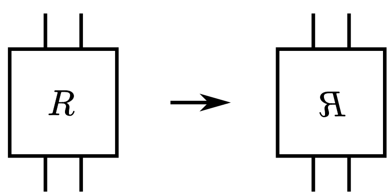
13.2. ***k*-edge-connected graphs.** Recall that a vertex v of a graph Γ is a cut vertex if Γ is the union of two proper subgraphs Γ_1 and Γ_2 so that $\Gamma_1 \cap \Gamma_2 = \{v\}$. A graph is 2-connected if it is connected and has no cut vertex.

A pair (u, v) of vertices of Γ is a *cut pair* if Γ is the union of two proper subgraphs Γ_1 and Γ_2, neither of which is an edge, so that $\Gamma_1 \cap \Gamma_2 = \{u, v\}$.

Suppose u, v are a cut pair for a 2-connected plane graph γ. A *Whitney flip* is the operation that replaces a plane graph γ by a plane graph γ' as follows



From the definition it is clear that $K(\gamma')$ is then obtained from $K(\gamma)$ by a Conway mutation, described in the following figure:



By Whitney's theorem [Whi33], two planar embeddings of a 2-connected planar graph are related by a sequence of Whitney flips, composed with a homeomorphism of S^2. Since Conway mutation does not change the colored Jones polynomial [MT88], from (82) we have the following.

Lemma 13.2. If γ_1 and γ_2 are two planar embeddings of the same 2-connected graph, then $\Phi_{\gamma_1,0} = \Phi_{\gamma_2,0}$.

13.3. **Planar collapsing of a bigon.** Suppose $\gamma = (\Gamma, f)$ is a plane graph, and among the regions of $\mathbb{R}^2 \setminus f(\Gamma)$ there is a bigon u with vertices v_1, v_2 and edges e_1, e_2. Let β be the plane graph obtained from γ by squeezing the bigon into one edge, called e, so that the bigon disappears and both e_1 and e_2 becomes e. We call $\gamma \to \beta$ a planar collapsing.

Lemma 13.3. If $\gamma \to \beta$ is a planar collapsing, then $\Phi_{\gamma,0} = \Phi_{\gamma',0}$.

Proof. The bigon contributes an A-vertex v_u to the set of vertices of $D^*(\gamma)$. Then $\deg(v_u) = 2$, and hence $L(v_u) = 0$. By isolating the factors in the formula (5) of $\Phi_{\gamma,0}$ involving the vertex v_u we have

$$\Phi_{\gamma,0} = \Phi_{\beta,0}\left[(q)_\infty \sum_{a:a+b_1\geq 0, a+b_2\geq 0} \frac{q^{(a+b_1)(a+b_2)}}{(q)_{a+b_1}(q)_{a+b_2}}\right].$$

Here a, b_1, b_2 are the coordinates of λ at respectively v_u, v_1, v_2; and b_1 and b_2 are fixed in the sum. By the well-known Durfee's identity (see [And71, Eqn.2.6]), the factor in the square bracket is equal to 1. □

Remark 13.4. Suppose K_1 and K_2 are alternating links such that after several planar collapsings from $\mathcal{T}(K_1)$ and and $\mathcal{T}(K_2)$ one gets the same plane graph, then the above lemma says that $\Phi_{K_1,0} = \Phi_{K_2,0}$. This was proved in [AD11] by another method.

13.4. **Abstract collapsing.** Suppose Γ_1 is an abstract graph with a pair of parallel edges e_1, e_2. Removing the interior of e_1, from Γ_1 we get a graph Γ_2. We say that the move $\Gamma_1 \to \Gamma_1$ a collapsing. Note that if Γ_1 is 2-connected then Γ_2 is also 2-connected.

Lemma 13.5. Suppose $\gamma_1 = (\Gamma_1, f_1)$ is a plane graph, and Γ_2 is obtained from Γ_1 by collapsing a pair of parallel edges e_1, e_2. Then there is a planar embedding γ_2 of Γ_2, $\gamma_2 = (\Gamma_2, f_2)$, such that $\Phi_{\gamma_1,0} = \Phi_{\gamma_2,0}$.

Proof. In the planar embedding $f_1(\Gamma_1) \subset \mathbb{R}^2$, e_1 and e_2 bound a region which may contain a subgraph Γ_0 of Γ_1. Note that the common vertices v_1 and v_2 of e and e_2 form a cut pair for Γ_1. By flipping $f_1(e_2) \cup f_2(\Gamma_0)$ through v_1 and v_2, from γ_1 we get a new plane graph $\gamma_3 = (\Gamma_1, f_3)$ in which $f_3(e_1)$ and $f_3(e_2)$ form a bigon, and the result of planar collapsing this bigon is denoted by γ_2. By Lemmas 13.3 and 13.2, we have $\Phi_{\gamma_1,0} = \Phi_{\gamma_3,0} = \Phi_{\gamma_2,0}$. □

13.5. **Proof of Corollary 1.12.** We will first prove the following statement.

Lemma 13.6. Suppose $\gamma_i = (\Gamma_i, f_i)$ for $i = 1, 2$ are 2-connected graphs such that $\Gamma'_1 = \Gamma'_2$ as abstract graphs. Then $\Phi_{\gamma_1,0} = \Phi_{\gamma_2,0}$.

Proof. Case 1: both Γ_1 and Γ_2 do not have multiple edges. Then $\Gamma'_i = \Gamma_i$, hence $\Gamma_1 = \Gamma_2$, and γ_1 and γ_2 are planar embeddings of the same 2-connect graph. Lemma 13.2 tells us that $\Phi_{\gamma_1,0} = \Phi_{\gamma_2,0}$.

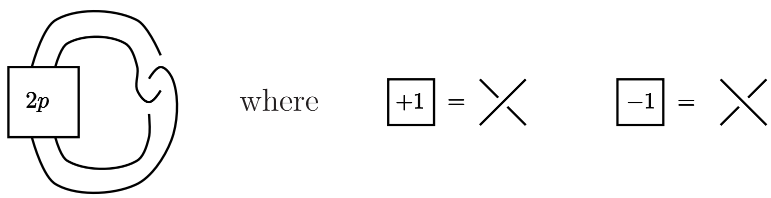
Case 2: General case. This case is reduced to Case 1 by induction on the number of total pairs of parallel edges in Γ_1 and Γ_2. If there is no pair of parallel edges, this is Case 1. Suppose Γ_1 has a pair of parallel edges, and let Γ_3 be the result of abstract collapsing this pair of parallel edges. By Lemma 13.5, there is a planar embedding γ_3 of Γ_3 such that $\Phi_{\gamma_1,0} = \Phi_{\gamma_3,0}$. Note that $\Gamma'_3 = \Gamma'_1 = \Gamma'_2$. By induction, we have $\Phi_{\gamma_3,0} = \Phi_{\gamma_2,0}$. This proves $\Phi_{\gamma_1,0} = \Phi_{\gamma_2,0}$. □

Let us proceed to the proof of Corollary 1.12. Suppose K_1 and K_2 are alternating links such that $\mathcal{T}(K_1)$ is isomorphic to $\mathcal{T}(K_2)$ as abstract graphs. We can assume that both K_1 and K_2 are non-split. For $i = 1, 2$ let $\gamma_i = (\mathcal{T}(D_i), f_i)$ be the plane Tait graph of a reduced A-infinite alternating link diagram of K_i. Note that $\mathcal{T}(D_i)$ is connected since K_i is non-split. Moreover $\mathcal{T}(D_i)$ does not have a cut vertex since D_i is reduced. That is, $\mathcal{T}(D_i)$ is 2-connected. From

Lemma 13.6 and Theorem 1.10 we have $\Phi_{K_1,0} = \Phi_{K_2,0}$. This completes the proof of Corollary 1.12.

14. Examples

In this section we give a formula for the q-series $\Phi_{K,0}(q)$ for all twist knots and their mirrors, taken from unpublished work of the first author and D. Zagier. In some caes, similar formulas have also been obtained by Armond-Dasbach. Recall the family of *twist knots* K_p for an integer p depicted as follows:



The planar projection of K_p has $2|p| + 2$ crossings, $2|p|$ of which come from the full twists, and 2 come from the negative *clasp*. For small p, the twist knots appear in Rolfsen's table [Rol90] as follows:

Twist knot	K_{-4}	K_{-3}	K_{-2}	K_{-1}	K_1	K_2	K_3	K_4
Rolfsen notation	10_1	8_1	6_1	4_1	3_1	5_2	7_2	9_2

Recall that $\mathrm{sgn}(n) = +1, 0, -1$ when $n < 0, n = 0, n > 0$ respectively.

Theorem 14.1. *For $p < 0$ we have:*

$$\Phi_{K_p,0}(q) = (q;q), \qquad \Phi_{-K_p,0}(q) = \frac{(q;q)}{(q^2;q^{2|p|+1})(q^3;q^{2|p|+1})\dots(q^{2|p|-1};q^{2|p|+1})}\,. \tag{83}$$

For $p > 0$ we have

$$\Phi_{K_p,0}(q) = \sum_{n=0}^{\infty} q^{pn^2+(p-1)n} - \sum_{n=0}^{\infty} q^{pn^2+(p+1)n+1} = 1 + \sum_{n\in\mathbb{Z}} \mathrm{sgn}(n) q^{pn^2+(p-1)n} \tag{84}$$

$$\Phi_{-K_p,0}(q) = (q;q)\,. \tag{85}$$

Equation (83) implies that for $p < 0$, $\Phi_{\pm K_p,0}(q)$ are modular forms [BvdGHZ08]. On the other hand, when $p > 1$, $\Phi_{K_p,0}(q)$ is *not* modular of any weight, according to K. Ono. This disproves any conjectured modularity properties of $\Phi_0(q)$, even for 5_2. On the other hand, $\Phi_{\pm K_p,0}(q)$ is a *false theta series* of Rogers.

The modular form $\Phi_{K_p,0}(q)$ for $p > 0$ is a beautiful theta series, with a factorization

$$\frac{(q;q)}{\prod_{k=2}^{2b-1}(q^k;q^{2b+1})} = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{2b+1}{2}n^2 + \frac{2b-1}{2}n} \tag{86}$$

for all natural numbers b. It was pointed out to us by D. Zagier that the above identity follows immediately from the Jacobi triple product identity, discussed in detail in [BvdGHZ08].

Appendix A. Proof of the state-sum formula for the colored Jones function

In this section we give a proof of Equation (14) which we could not find in the literature. We begin by recalling the definition of the colored Jones polynomial using R-matrix.

A.1. **Link invariant associated to a ribbon algebra.**

Quantum link invariants can be defined using a ribbon Hopf algebra. We recall the formula for the invariant here. For further details, see [RT90] or [Oht02].

A ribbon Hopf algebra $\mathcal{U}$ over a ground field $\mathcal{F}$ has an R-matrix $R \in \mathcal{U} \otimes \mathcal{U}$ and a group-like element $g \in \mathcal{U}$ satisfying

$$S^2(x) = gxg^{-1} \quad \forall x \in \mathcal{U},$$

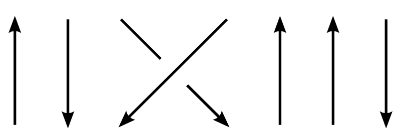
where S is the antipode of the $\mathcal{U}$.

Suppose V is a $\mathcal{U}$-module, and K is a *framed* link with a downward planar diagram D, where the framing is the blackboard framing. The dual space V^* has a natural structure of a $\mathcal{U}$-module. Fix a basis $\{e_j\}$ of V and a dual basis $\{e_j^*\}$ of V^*.

The quantum invariant $\hat{J}_K(V)$ is defined through tangle operator invariants as follows.

The six tangle diagrams in Figure (13) are called *elementary tangle diagrams. An extension* of an elementary tangle diagram is the result of adding some (maybe none) vertical lines to the left and to the right of an elementary tangle diagram, with arbitrary orientations on the added lines, as in the following figure

(87)



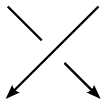
Suppose D is a downward link diagram which is in general position. Using horizontal lines we cut D into tangles, each is an extension of an elementary tangle diagram. Let T be one of the resulting tangles. On the bottom boundary of T assign V to each endpoint of T where T is oriented down, and the dual object V^* to each endpoint where T is oriented up. Tensoring from left to right, this gives the boundary object $\partial_- T$. One defines similarly $\partial_+ T$, using the top boundary endpoints instead of the bottom boundary ones. By convention, the empty product is the the ground field $\mathcal{F}$. It is clear that if T' is the tangle right above T, then $\partial_-(T') = \partial_+(T)$. For example, for the tangle T of Figure (87) we have

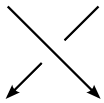
$$\partial_- T = \partial_+ T = V^* \otimes V \otimes V \otimes V \otimes V^* \otimes V^* \otimes V$$

For each tangle T as above we will define an operator

$$\check{J}_T : \partial_- T \to \partial_+ T$$

as follows. First if T is one of the elementary tangles, then $\check{J}_T$ is given by

(88) $T =$  $\check{J}_T : V \otimes V \to V \otimes V$ given by $\check{J}_T = \mathbf{b} := \sigma\, R$

(89) $T =$  $\check{J}_T : V \otimes V \to V \otimes V$ given by $\check{J}_T = \mathbf{b}^{-1} = R^{-1}\,\sigma$

(90) $T =$  $\check{J}_T : V^* \otimes V \to \mathcal{F}$ given by $f \otimes x \to f(x)$

(91) $T =$  $\check{J}_T : V \otimes V^* \to \mathcal{F}$ given by $x \otimes f \to f(gx)$

(92) $T =$  $\check{J}_T : \mathcal{F} \to V \otimes V^*$ given by $1 \to \sum_j e_j \otimes e_j^*$

(93) $T =$  $\check{J}_T : \mathcal{F} \to V^* \otimes V$ given by $1 \to \sum_j e_j^* \otimes g^{-1}(e_j).$

Here $\sigma : V \otimes V$ is the permutation, $\sigma(x \otimes y) = y \otimes x$. If T is an extension of an elementary tangle E, say T is the result of adding m vertical lines to the left and n vertical lines to right of E, then define

$$\check{J}_T = \mathrm{id}^{\otimes m} \otimes \check{J}_E \otimes \mathrm{id}^{\otimes n}.$$

Finally, if $T_1, \dots, T_m$ are the tangles in the decomposition of the downward diagram D of the link K, counting from top to bottom, then

$$\check{J}_K := \check{J}_{T_1} \dots \check{J}_{T_m},$$

is an element of $\mathrm{Hom}_{\mathcal{U}}(\mathcal{F}, \mathcal{F})$ which one identifies with $\mathcal{F}$.

A.2. **The case $\mathcal{U} = U_h(\mathfrak{sl}_2)$.** The colored Jones polynomial is the quantum link invariant corresponding to the ribbon Hopf algebra $\mathcal{U} := U_h(\mathfrak{sl}_2)$, the quantized enveloping algebra of $\mathfrak{sl}_2$. There are two versions of $\mathcal{U}$ in the literature, we will use here the version used in [Kas95, Oht02], which has the opposite co-product structure of the one used in [Jan96, Lus93]. The ground ring is $\mathbb{Q}[[h]]$ is not a field, but the theory carries over without changes.

Recall that $\mathcal{U} = U_h(sl_2)$ is the h-adically completed $\mathbb{Q}[[h]]$-algebra generated by H, E, F subject to the relations

$$HE = E(H+2), \quad HF = F(H-2), \quad EF - FE = \frac{K - K^{-1}}{v - v^{-1}}.$$

where

$$v = \exp(h/2) = q^{1/2}$$

and $K = \exp(hH/2)$. The group like element is $g = K$. Recall the balanced quantum integer, and the corresponding balanced quantum factorials and binomials defined by

$$[a] = \frac{v^a - v^{-a}}{v - v^{-1}}, \quad [a]! = \prod_{k=1}^{a} [k] \quad \text{for } a \in \mathbb{N}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{[a]!}{[b]![a-b]!} \quad \text{for } a, b \in \mathbb{N}, b \le a.$$

The R-matrix is an element of $\mathcal{U}\hat{\otimes}\mathcal{U}$, the completed tensor product of $\mathcal{U}$ and $\mathcal{U}$, given by

$$R = D \sum_{k=0}^{\infty} \frac{v^{k(k-1)/2}\,(v-v^{-1})^k}{[k]!} E^n \otimes F^n$$

where $D = \exp(hH \otimes H/4)$, which is called the diagonal part.

The inverse of R is

$$R^{-1} = \left(\sum_{k=0}^{\infty} \frac{(-1)^k v^{-k(k-1)/2}\,(v-v^{-1})^k}{[k]!} E^n \otimes F^n \right) D^{-1}$$

For each positive integer n there is a unique $n+1$-dimensional $\mathcal{U}$-module V_n such that there is an element $e_0 \in V_n$ satisfying

$$H(e_0) = ne_0, \quad E(e_0) = 0$$

The module V_n is freely spanned by $F^j(e_0), j = 0, 1, \dots, n$. The basis

$$\{u_j = F^j(e_0)/[j]! \mid j = 0, \dots, n\}$$

is known as the canonical basis of V_n.

For a framed link K let $\check{J}_{K,n}$ be the invariant of K with color V_n. It is known that if one increases the framing of a component of K by one, then $J_{K,n}$ gets multiplied by $v^{(n^2+2n)/2}$.

Define $\tilde{J}_{K,n}$ in the same way as in the definition of $\check{J}_{K,n}$, with $\mathbf{b}$ replaced by $\tilde{\mathbf{b}} := v^{-\frac{n^2}{2}-n}\mathbf{b}$ and $\mathbf{b}^{-1}$ replaced by $\tilde{\mathbf{b}}^{-1}$. Then $\tilde{J}_{K,n}$ is an invariant of unframed links. Since $\tilde{J}_{K,n} = q^a\,\check{J}_{K,n}$ for some $a \in \frac{1}{4}\mathbb{Z}$, when dividing by the smallest monomial, both $\tilde{J}_{K,n}$ and $\check{J}_{K,n}$ are the same.

A.3. **R-matrix in the canonical basis.** In this section we calculate the matrix of $\tilde{\mathbf{b}}$ in the product of the canonical basis. The action of H, E^k and F^k on the canonical basis is given by

$$F^k(u_a) = \frac{[a+k]!}{[a]!}\,u_{a+k}, \quad E^k(u_a) = \frac{[n+k-a]!}{[n-a]!}\,u_{a-k}, \quad H(u_j) = (n-2j)e_j$$

where we assume $u_j = 0$ if $j < 0$ or $j > n$. From here one can easily calculate the formula of $\tilde{\mathbf{b}}$ and $\tilde{\mathbf{b}}^{-1}$,

$$\tilde{\mathbf{b}}(u_a \otimes u_b) = \sum_k v^{-n-na-nb+2ab+2ak-2kb-\frac{3k^2+k}{2}} \{k\}! \begin{bmatrix} n+k-a \\ k \end{bmatrix} \begin{bmatrix} b+k \\ k \end{bmatrix} u_{b+k} \otimes u_{a-k} \tag{94}$$

$$\tilde{\mathbf{b}}^{-1}(u_a \otimes u_b) = \sum_k (-1)^k v^{n+nb+na-2ab-\binom{k}{2}} \{k\}! \begin{bmatrix} n+k-b \\ k \end{bmatrix} \begin{bmatrix} a+k \\ k \end{bmatrix} u_{b-k} \otimes u_{a+k} \tag{95}$$

Let us denote by $\tilde{\mathbf{b}}^{c,d}_{a,b}$ the matrix entry of $\tilde{\mathbf{b}}$, i.e.,

$$\tilde{\mathbf{b}}(u_a \otimes u_b) = \sum_{c,d} \tilde{\mathbf{b}}^{c,d}_{a,b}\, u_c \otimes u_d.$$

Then $\tilde{\mathbf{b}}^{c,d}_{a,b} = 0$ and $(\tilde{\mathbf{b}}^{-1})^{c,d}_{a,b} = 0$ unless the numbers a, b, c, d form an n-admissible state for the crossing, i.e., $a + b = c + d$, $\varepsilon(C)(a - d) \geq 0$ and $a, b, c, d \in [0, n] \cap \mathbb{Z}$. Here $\varepsilon(C)$ is the sign of the crossing C. If a, b, c, d form an n-admissible state for the crossing, then from (94) and (95) we have

$$(\tilde{\mathbf{b}})^{c,d}_{a,b} = v^{-n-nd-nb+ab+dc}(q^{-1};q^{-1})_k \binom{n-d}{a-d}_{q^{-1}} \binom{c}{c-b}_{q^{-1}} \tag{96}$$

$$(\tilde{\mathbf{b}}^{-1})^{c,d}_{a,b} = (-1)^k v^{n+nb+nd-bd-ac+b-c}\,(q^{-1};q^{-1})_k \binom{n-c}{b-c}_{q^{-1}} \binom{d}{d-a}_{q^{-1}} \tag{97}$$

Choose the following basis $\{f_0, \ldots, f_n\}$ for the dual V_n^* such that $f_j = v^{-(n-2j)/2} e_j^*$. Then

(98) $T =$  $\check{J}_T : V_n^* \otimes V_n \to \mathcal{F}$ given by $f_a \otimes e_b \to \delta_{ab} v^{-(n-2a)/2}$

(99) $T =$  $\check{J}_T : V_n \otimes V_n^* \to \mathcal{F}$ given by $e_a \otimes f_b \to \delta_{ab} v^{(n-2a)/2}$

(100) $T =$  $\check{J}_T : \mathcal{F} \to V_n \otimes V_n^*$ given by $1 \to \sum_a v^{(n-2a)/2} e_a \otimes f_a$

(101) $T =$  $\check{J}_T : \mathcal{F} \to V_n^* \otimes V_n$ given by $1 \to \sum_j v^{-(n-2a)/2} f_a \otimes e_a$.

From Equations (96)–(101) we see that

$$J_{K,n}(q) = \tilde{J}_{K,n}(q^{-1}),$$

where $J_{K,n}$ is the given in section 2.4.

Appendix B. The lowest degree of the colored Jones polynomial of an alternating link

We fix an A-infinite, reduced, alternating, downward diagram D of a link K. Corollary 5.9 shows that the minimal degree of the colored Jones polynomial $J_{K,n}$ is given by $P_1(n) = \frac{n}{2}c_+ - \frac{n^2+2n}{2}c_- - \frac{n}{2}\sum_M W(M)$, where the sum is over all the local extreme points of D. On the other hand, the Kauffman bracket approach gives the minimal degree as $\frac{n}{2}c_+ - \frac{n^2+n}{2}c_- - \frac{n}{2}s_A$, see [Tur87] and also [Lê06, Gar11b]. Here we show that the two results agree. Recall that s_A is the number of circles obtained from D after doing A-smoothenings at every vertex crossing of D. If D is a connected graph, then s_A is the number of A-vertices of Δ^*.

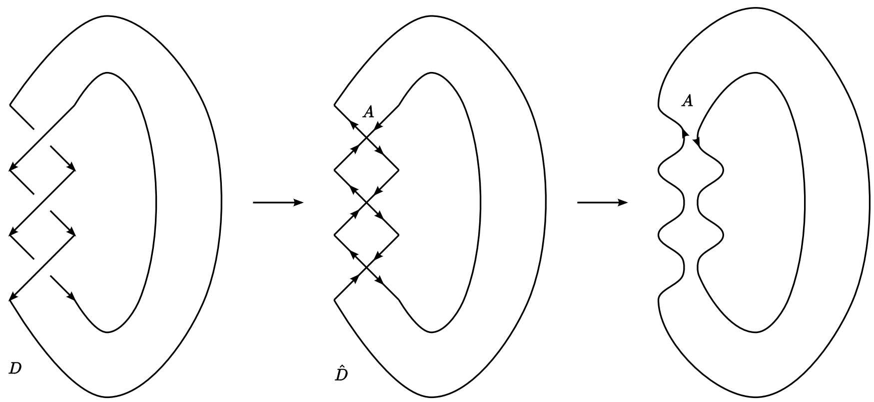
Lemma B.1. We have

$$c_- = s_A - \sum_M W(M) \tag{102}$$

Consequently,

$$P_1(n) = \frac{n}{2}c_+ - \frac{n^2+n}{2}c_- - \frac{n}{2}s_A. \tag{103}$$

Remark B.2. If D is not A-infinite, Equation (102) fails. For example, it fails for for the following diagram of the right handed trefoil



we have $c_- = 0$, $s_+ = 2$, and there are two clockwise local extrema and two counterclockwise local extrema giving $\sum_P \varepsilon(P) = 0$.

Proof. (of Lemma B.1) Suppose D' is the result of doing A-type resolution at every crossing of D. Since all crossings of D are downward, an A-type resolution at a positive crossing of $\vec{D}$ creates no local extreme point, while an A-type resolution at a negative crossing created 2 local extreme points, each has winding weight $1/2$.

It follows that

$$c_- + \sum_{M \in D} W(M) = \sum_{M \in D'} W(M)$$

By Lemma 3.2(b), D' consists of s_A circles, each having winding number 1. Hence $\sum_{M \in D'} W(M) = s_A$, and we get $c_- + \sum_{M \in D} W(M) = s_A$. $\square$

Appendix C. Regularity of Nahm sums

In this section we will give a necessary and sufficient criterion for regularity of a Nahm sum. This section is logically independent from the proof of Theorems 1.10 and 1.4, but we include it for completeness. Fix a pointed cone C in the Euclidean space $(\mathbb{R}^r, |\cdot|_2)$ with apex the origin that intersects the orthant $[0,\infty)^d$ other than in the origin and consider a polynomial function of degree d

$$f : \mathbb{R}^r \longrightarrow \mathbb{R}, \qquad f(s) = \sum_{i=0} f_i(s)$$

where f_i are homogeneous polynomials of degree i and f_d not identically zero. Assume that $f(\mathbb{Z}^r) \subset \mathbb{Z}$. Let $X = C \cap \mathbb{N}^d$, $X_{\mathbb{Q}} = C \cap \mathbb{Q}^d$ and $X_{\mathbb{R}} = C \cap [0,\infty)^r$.

Proposition C.1. The following are equivalent.

(a) $f : X \longrightarrow \mathbb{Z}$ is proper (i.e., the preimage of a finite set is finite) and bounded below.
(b) For $s \in X$ there exists i_0 such that $f_i(s) = 0$ for all $i < i_0$ and $f_{i_0}(s) \geq 0$.
(c) There exists $c > 0$ such that $F(s) \geq c|s|_2$ for all but finitely many $s \in C \cap \mathbb{N}^d$

Proof. (a) $\Longrightarrow$ (b) Fix $s \in C \cap \mathbb{N}^d$. By properness, it follows that the sequence $f(ns)$ is unbounded thus it has a subsequence that goes to infinity. Suppose that $f_i(s) = 0$ for $i < i_0$ and $f_{i_0}(s) \neq 0$. Then, $f(ns) = n^{i_0} f_{i_0}(s) + O(n^{i_0-1})$ goes to infinity. It follows that $f_{i_0}(s) > 0$.

(b) $\Longrightarrow$ (c) Indeed, (b) implies that for all $s \in X$ we have $f_d(s) \geq 0$, and if $f_d(s) = 0$ then (without loss of generality) we assume that $f_{d-1}(s) \geq 0$. Since f_i is homogeneous, it follows that for all $s \in X_{\mathbb{Q}}$ we have $f_d(s) \geq 0$, and if $f_d(s) = 0$ then $f_{d-1}(s) \geq 0$. Since f_i is continuous, it follows that for all $s \in X_{\mathbb{R}}$ we have $f_d(s) \geq 0$, and if $f_d(s) = 0$ then $f_{d-1}(s) \geq 0$. Let S denote the unit sphere in the Euclidean space $\mathbb{R}^r$ with the Euclidean norm $|\cdot|_2$. Then, $f_d : [0,\infty)^r \cap S \longrightarrow [0,\infty)$, and if $Z_d = \{x \in [0,\infty)^r \cap S \mid f_d(s) = 0\}$, then $f_{d-1}(Z_d) > 0$. By continuity, choose an open neighborhood U_d of Z_d in $[0,\infty)^r \cap S$ with closure $\bar{Z}_d$ such that $f_{d-1}(\bar{Z}_d) \subset (0,\infty)$. Then, $f_d(([0,\infty)^r \cap S) \setminus Z_d) \subset (0,\infty)$ and by compactness it follows that there exists $c' > 0$ such that $f_d(([0,\infty)^r \cap S) \setminus Z_d) \in [c',\infty)$ and $f_{d-1}(\bar{Z}_d) \subset [c',\infty)$. It follows that $(f_d + f_{d-1})([0,\infty)^r \cap S) \subset [c',\infty)$, and by homogeneity this implies that for all $s \in X_{\mathbb{R}}$, we have $(f_d + f_{d-1})(s) \geq c'|s|_2^{d-1}$. On the other hand, by homogeneity, we have $f_i(s) < c''|s|_2^i$ for $i < r-1$. Since $c'x^{r-1} - c''x^{r-2} \geq cx$ for some $c > 0$ and for all x sufficiently large, (c) follows.

(c) $\Longrightarrow$ (a) is immediate. $\square$

For example, the function Q_2 in Section 5 is proper and bounded from below. Actually, $Q_2 \geq 0$ on the cone $S_{\vec{D},\mathbb{N}}$.

Appendix D. Experimental formulas for knots with a low number of crossings

Theorem 1.4 gives an explicit Nahm sum formula for an alternating knot K. The first author programmed the above formula with input an alternating, reduced, A-infinite downward diagram of a knot, and with the help of D. Zagier computed the first 50 terms of the corresponding q-series for several examples, and then guessed the answer (in all but the case of 8_5 knot below). Every such guess is a q-series identity, whose proof is unknown to us. We thank D. Zagier for guidance and stimulating conversations. For an alternating knot K, let

$$\gamma(K) = (c_+, c_-, \sigma)$$

denote the triple of positive crossings, negative crossings and the signature of K. K has $c = c_+ + c_-$ crossings, and writhe $w = c^+ - c^-$. Let $\delta^*_K(n)$ and $\delta_K(n)$ denote the minimum and maximum degree of the colored Jones polynomial $J_{K,n}(q)$. Note that $\delta^*_K(n)$ and $\delta_K(n)$ are determined by $\gamma(K)$ by:

$$\delta^*_K(n) = -c_- \frac{n(n+1)}{2} - \frac{\sigma}{2} n - \frac{n}{2}, \qquad \delta_K(n) = c_+ \frac{n(n+1)}{2} - \frac{\sigma}{2} n + \frac{n}{2}$$

Let $\Phi^*_{K,0}(q) \in \mathbb{Z}[[q]]$ and $\Phi_{K,0}(q) \in \mathbb{Z}[[q]]$ denote the stable limit of the colored Jones polynomial from the left and from the right. The involution $K \mapsto -K$ given by the mirror image acts as follows:

$$c_\pm \mapsto c_\mp, \qquad \sigma \mapsto -\sigma, \qquad \delta \mapsto -\delta^*, \qquad \delta^* \mapsto -\delta, \qquad \Phi \mapsto \Phi^*, \qquad \Phi^* \mapsto \Phi$$

The formulas for $\Phi_0(q)$ presented below agree to the first 8 values with the `KnotAtlas` table of Bar-Natan, and also to 50 values with the Nahm sum formula of Theorem 1.4. The formulas are proven only for 3_1 and 4_1 knot, and remain conjectural for all others.

K_p is the pth *twist knot* obtained by $-1/p$ surgery on the Whitehead link for an integer p and $T(a,b)$ is the left-handed (a,b) torus knot. The results below are expressed in terms of the following series for a positive natural number b:

$$h_b(q) = \sum_{n\in\mathbb{Z}} (-1)^n q^{bn(n+1)/2-n}, \qquad h_b^*(q) = \sum_{n\in\mathbb{Z}} \varepsilon(n) q^{bn(n+1)/2-n} \tag{104}$$

Observe that

$$h_1(q) = 0, \qquad h_2^*(q) = 1, \qquad h_3(q) = (q)_\infty$$

K	c_-	c_+	σ	$\Phi^*_{K,0}(q)$	$\Phi_{K,0}(q)$
$3_1 = -K_1$	3	0	2	h_3	1
$4_1 = K_{-1}$	2	2	0	h_3	h_3
5_1	5	0	4	h_5	1
$5_2 = K_2$	0	5	-2	h_4^*	h_3
$6_1 = K_{-2}$	4	2	0	h_3	h_5
6_2	4	2	2	$h_3 h_4^*$	h_3
6_3	3	3	0	h_3^2	h_3^2
7_1	7	0	6	h_7	1
$7_2 = K_3$	0	7	-2	h_6^*	h_3
7_3	0	7	-4	h_4^*	h_5
7_4	0	7	-2	$(h_4^*)^2$	h_3
7_5	7	0	4	h_4^*	h_4^*
7_6	5	2	2	$h_3 h_4^*$	h_3^2
7_7	3	4	0	h_3^3	h_3^2
$8_1 = K_{-3}$	6	2	0	h_3	h_7
8_2	6	2	4	$h_3 h_6^*$	h_3
8_3	4	4	0	h_5	h_5
8_4	4	4	2	$h_4^* h_5$	h_3
8_5	2	6	-4	h_3	???
$K_p, p>0$	0	$2p+1$	-2	h_{2p}^*	h_3
$K_p, p<0$	$2\lvert p\rvert$	2	0	h_3	$h_{2\lvert p\rvert+1}$
$T(2,p), p>0$	$2p+1$	0	$2p$	h_{2p+1}	1

For 8_5, we have computed the first 100 terms using an 8-dim Nahm sum. The result slightly simplifies when divided by $h_3(q)$:

$\Phi_{8_5}(q)/h_3(q) = 1-q+q^2-q^4+q^5+q^6-q^8+2q^{10}+q^{11}+q^{12}-q^{13}-2q^{14}+2q^{16}+3q^{17}+2q^{18}+q^{19}-3q^{21}-2q^{22}+q^{23}+4q^{24}+4q^{25}+5q^{26}+3q^{27}+q^{28}-2q^{29}-3q^{30}-3q^{31}+5q^{33}+8q^{34}+8q^{35}+8q^{36}+6q^{37}+3q^{38}-2q^{39}-5q^{40}-6q^{41}-q^{42}+2q^{43}+9q^{44}+13q^{45}+17q^{46}+16q^{47}+14q^{48}+9q^{49}+4q^{50}-3q^{51}-8q^{52}-8q^{53}-5q^{54}+3q^{55}+14q^{56}+21q^{57}+27q^{58}+32q^{59}+33q^{60}+28q^{61}+21q^{62}+11q^{63}+q^{64}-9q^{65}-11q^{66}-11q^{67}-2q^{68}+9q^{69}+27q^{70}+40q^{71}+56q^{72}+60q^{73}+65q^{74}+62q^{75}+54q^{76}+$

$39q^{77}+23q^{78}+4q^{79}-9q^{80}-16q^{81}-14q^{82}-3q^{83}+16q^{84}+40q^{85}+67q^{86}+92q^{87}+114q^{88}+129q^{89}+135q^{90}+127q^{91}+115q^{92}+92q^{93}+66q^{94}+35q^{95}+9q^{96}-12q^{97}-14q^{98}-11q^{99}+13q^{100}+O(q)^{101}$

Let us summarize some observations.

- $\Phi_{K,0}(q)$ is not determined by $\gamma(K)$ alone: see for instance $(7_2, 7_4)$ and $(7_3, -7_5)$.
- In all knots above except 8_5, $\Phi_{K,0}(q)$ is a finite product of the form $\prod_i h_{a_i}^{b_i}(h_{a_i}^*)^{c_i}$ where $b_i, c_i \in \mathbb{N}$.
- The modularity properties of $\Phi_{8_5}(q)$ are completely unknown, and so is its behavior at $q = 1$ or at any complex root of unity.

References

[AD11] Cody Armond and Oliver Dasbach, *Rogers-Ramanujan type identities and the head and tail of the colored jones polynomial*, 2011, arXiv:1106.3948, Preprint.

[And71] George E. Andrews, *Generalizations of the Durfee square*, J. London Math. Soc. (2) **3** (1971), 563–570.

[And76] ———, *The theory of partitions*, Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1976, Encyclopedia of Mathematics and its Applications, Vol. 2.

[And13] George Andrews, *Knots and q-series*, 2013, Preprint.

[Arm13] Cody Armond, *The head and tail conjecture for alternating knots*, Algebr. Geom. Topol. **13** (2013), no. 5, 2809–2826.

[Arm14] Cody W. Armond, *Walks along braids and the colored Jones polynomial*, J. Knot Theory Ramifications **23** (2014), no. 2, 1450007, 15.

[BDP12] Christopher Beem, Tudor Dimofte, and Sara Pasquetti, *Holomorphic blocks in three dimensions*, arXiv/1211.1986, 2012.

[BvdGHZ08] Jan Hendrik Bruinier, Gerard van der Geer, Günter Harder, and Don Zagier, *The 1-2-3 of modular forms*, Universitext, Springer-Verlag, Berlin, 2008, Lectures from the Summer School on Modular Forms and their Applications held in Nordfjordeid, June 2004, Edited by Kristian Ranestad.

[CK13] Abhijit Champanerkar and Ilya Kofman, *On the tail of Jones polynomials of closed braids with a full twist*, Proc. Amer. Math. Soc. **141** (2013), no. 7, 2557–2567.

[Cos14] Francesco Costantino, *Integrality of Kauffman brackets of trivalent graphs*, Quantum Topol. **5** (2014), no. 2, 143–184.

[DGG] Tudor Dimofte, Davide Gaiotto, and Sergei Gukov, *3-manifolds and 3d indices*, arXiv:1112.5179, Preprint 2011.

[DL06] Oliver T. Dasbach and Xiao-Song Lin, *On the head and the tail of the colored Jones polynomial*, Compos. Math. **142** (2006), no. 5, 1332–1342.

[FKP11] David Futer, Efstratia Kalfagianni, and Jessica S. Purcell, *Jones polynomials, volume, and essential knot surfaces: a survey*, 2011, arXiv:1110.6388, Preprint.

[Gar11a] Stavros Garoufalidis, *The degree of a q-holonomic sequence is a quadratic quasi-polynomial*, Electron. J. Combin. **18** (2011), no. 2, Research Paper P4, 23.

[Gar11b] ———, *The Jones slopes of a knot*, Quantum Topol. **2** (2011), no. 1, 43–69.

[GK] Stavros Garoufalidis and Rinat Kashaev, *The q-dilogarithm, its state-integrals and their q-series*, Math. Res. Lett., arXiv:1304.2705.

[GL05] Stavros Garoufalidis and Thang T. Q. Lê, *The colored Jones function is q-holonomic*, Geom. Topol. **9** (2005), 1253–1293 (electronic).

[GL11] ———, *Asymptotics of the colored Jones function of a knot*, Geom. Topol. **15** (2011), no. 4, 2135–2180.

[GN14] Stavros Garoufalidis and Sergey Norin, *Graph counting and the stable coefficients of the jones polynomial*, 2014, Preprint.

[GNV13] Stavros Garoufalidis, Sergey Norin, and Thao Vuong, *Flag algebras and the stable coefficients of the jones polynomial*, 2013, arXiv:1309.5867, Preprint.

[GV13] Stavros Garoufalidis and Thao Vuong, *Alternating knots, planar graphs and q-series*, 2013, arXiv:1310.7143, Preprint.

[GvdV12] Stavros Garoufalidis and Roland van der Veen, *Asymptotics of quantum spin networks at a fixed root of unity*, Math. Ann. **352** (2012), no. 4, 987–1012.

[GZa] Stavros Garoufalidis and Don Zagier, *Asymptotics of quantum knot invariants*, Preprint 2013.

[GZb] ———, *Empirical relations between q-series and kashaev's invariant of knots*, Preprint 2013.

[Hab08] Kazuo Habiro, *A unified Witten-Reshetikhin-Turaev invariant for integral homology spheres*, Invent. Math. **171** (2008), no. 1, 1–81.

[Hab10] ———, *On certain limits of the reduced colored jones polynomials of knots*, 2010, Bedlewo, Poland, Talk.

[Haj14a] Mustafa Hajij, *The colored Kauffman skein relation and the head and tail of the colored jones polynomial*, 2014, arXiv:1401.4537, Preprint.

[Haj14b] ———, *The tail of a quantum spin network*, 2014, arXiv:1308.2369, Preprint.

[HL05] Vu Huynh and Thang T. Q. Le, *The colored Jones polynomial and the Kashaev invariant*, Fundam. Prikl. Mat. **11** (2005), no. 5, 57–78.

[Jan96] Jens Carsten Jantzen, *Lectures on quantum groups*, Graduate Studies in Mathematics, vol. 6, American Mathematical Society, Providence, RI, 1996.

[Jon87] V. F. R. Jones, *Hecke algebra representations of braid groups and link polynomials*, Ann. of Math. (2) **126** (1987), no. 2, 335–388.

[Kas95] Christian Kassel, *Quantum groups*, Graduate Texts in Mathematics, vol. 155, Springer-Verlag, New York, 1995.

[Kau87a] Louis H. Kauffman, *On knots*, Annals of Mathematics Studies, vol. 115, Princeton University Press, Princeton, NJ, 1987.

[Kau87b] ———, *State models and the Jones polynomial*, Topology **26** (1987), no. 3, 395–407.

[KC02] Victor Kac and Pokman Cheung, *Quantum calculus*, Universitext, Springer-Verlag, New York, 2002.

[KL94] Louis H. Kauffman and Sóstenes L. Lins, *Temperley-Lieb recoupling theory and invariants of 3-manifolds*, Annals of Mathematics Studies, vol. 134, Princeton University Press, Princeton, NJ, 1994.

[KS11] Maxim Kontsevich and Yan Soibelman, *Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants*, Commun. Number Theory Phys. **5** (2011), no. 2, 231–352.

[Le00] Thang T. Q. Le, *Integrality and symmetry of quantum link invariants*, Duke Math. J. **102** (2000), no. 2, 273–306.

[Lê06] Thang T. Q. Lê, *The colored Jones polynomial and the A-polynomial of knots*, Adv. Math. **207** (2006), no. 2, 782–804.

[Lic97] W. B. Raymond Lickorish, *An introduction to knot theory*, Graduate Texts in Mathematics, vol. 175, Springer-Verlag, New York, 1997.

[Lus93] George Lusztig, *Introduction to quantum groups*, Progress in Mathematics, vol. 110, Birkhäuser Boston Inc., Boston, MA, 1993.

[MT88] H. R. Morton and P. Traczyk, *The Jones polynomial of satellite links around mutants*, Braids (Santa Cruz, CA, 1986), Contemp. Math., vol. 78, Amer. Math. Soc., Providence, RI, 1988, pp. 587–592.

[Mur87] Kunio Murasugi, *Jones polynomials and classical conjectures in knot theory*, Topology **26** (1987), no. 2, 187–194.

[NRT93] W. Nahm, A. Recknagel, and M. Terhoeven, *Dilogarithm identities in conformal field theory*, Modern Phys. Lett. A **8** (1993), no. 19, 1835–1847.

[Oht02] Tomotada Ohtsuki, *Quantum invariants*, Series on Knots and Everything, vol. 29, World Scientific Publishing Co. Inc., River Edge, NJ, 2002, A study of knots, 3-manifolds, and their sets.

[Oza11] Makoto Ozawa, *Essential state surfaces for knots and links*, J. Aust. Math. Soc. **91** (2011), no. 3, 391–404.

[PWZ96] Marko Petkovšek, Herbert S. Wilf, and Doron Zeilberger, $A = B$, A K Peters Ltd., Wellesley, MA, 1996, With a foreword by Donald E. Knuth, With a separately available computer disk.

[Rol90] Dale Rolfsen, *Knots and links*, Mathematics Lecture Series, vol. 7, Publish or Perish Inc., Houston, TX, 1990, Corrected rEPRINT of the 1976 original.

[Roz12] Lev Rozansky, *Khovanov homology of a unicolored b-adequate link has a tail*, 2012, arXiv: 1203.5741, Preprint.

[RT90] N. Yu. Reshetikhin and V. G. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Comm. Math. Phys. **127** (1990), no. 1, 1–26.

[Sch86] Alexander Schrijver, *Theory of linear and integer programming*, Wiley-Interscience Series in Discrete Mathematics, John Wiley & Sons Ltd., Chichester, 1986, A Wiley-Interscience Publication.

[Stu96] Bernd Sturmfels, *Gröbner bases and convex polytopes*, University Lecture Series, vol. 8, American Mathematical Society, Providence, RI, 1996.

[Thi88] Morwen B. Thistlethwaite, *Kauffman's polynomial and alternating links*, Topology **27** (1988), no. 3, 311–318.

[Tur87] V. G. Turaev, *A simple proof of the Murasugi and Kauffman theorems on alternating links*, Enseign. Math. (2) **33** (1987), no. 3-4, 203–225.

[Tur88] ———, *The Yang-Baxter equation and invariants of links*, Invent. Math. **92** (1988), no. 3, 527–553.

[Tur94] ———, *Quantum invariants of knots and 3-manifolds*, de Gruyter Studies in Mathematics, vol. 18, Walter de Gruyter & Co., Berlin, 1994.

[VZ11] Masha Vlasenko and Sander Zwegers, *Nahm's conjecture: asymptotic computations and counterexamples*, Commun. Number Theory Phys. **5** (2011), no. 3, 617–642.

[Whi33] Hassler Whitney, *2-Isomorphic Graphs*, Amer. J. Math. **55** (1933), no. 1-4, 245–254.

[Wit12] Edward Witten, *Fivebranes and knots*, Quantum Topol. **3** (2012), no. 1, 1–137.

[Zag07] Don Zagier, *The dilogarithm function*, Frontiers in number theory, physics, and geometry. II, Springer, Berlin, 2007, pp. 3–65.

[Zag09] ———, *Ramanujan's mock theta functions and their applications (after Zwegers and Ono-Bringmann)*, Astérisque (2009), no. 326, Exp. No. 986, vii–viii, 143–164 (2010), Séminaire Bourbaki. Vol. 2007/2008.

[Zei90] Doron Zeilberger, *A holonomic systems approach to special functions identities*, J. Comput. Appl. Math. **32** (1990), no. 3, 321–368.

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