SKEIN THEORY FOR THE LINKS–GOULD POLYNOMIAL

STAVROS GAROUFALIDIS, MATTHEW HARPER, RINAT KASHAEV, BEN-MICHAEL KOHLI, JIEBO SONG, AND GUILLAUME TAHAR

Abstract. Building further on work of Marin and Wagner, we give a braid-type skein theory of the Links–Gould polynomial invariant of oriented links, prove that it can be used to evaluate any oriented link and prove that it is also shared by the V_1-polynomial defined by two of the authors, deducing the equality of the two link polynomials. This implies specialization properties of the V_1-polynomial to the Alexander polynomial and to the ADO_3-invariant, the fact that it is a Vassiliev power series invariant, as well as a Seifert genus bound for knots.

Contents

1. Introduction

1.1. **Multivariable knot polynomials.** Recently, a systematic way to define and effectively compute multivariable knot polynomials was introduced in [GK], using as input a finite dimensional Nichols algebra (or a finite-dimensional Drienfeld–Yetter module of it)

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with an automorphism. From such an algebra, one can define a rigid R-matrix and construct a state-sum invariant of long knots by applying the well-known Reshetikhin–Turaev functor.

Nichols algebras are easy to describe, and in the simplest case of rank 1, such an algebra is uniquely determined by the data

$$\mathrm{basis}(V) = \{x\}, \qquad \Delta(x) = x \otimes 1 + 1 \otimes x, \qquad \tau(x \otimes x) = qx \otimes x, \qquad \phi(x) = tx\,. \tag{1}$$

In [GK], it was shown that the corresponding invariants are the ADO invariants of a knot [ADO92], and the colored Jones polynomials of a knot [RT90, Tur88].

The next case of a Nichols algebra of rank 2 leads to a family of 2-variable polynomials $\Lambda_\omega(t_0, t_1)$ at each root of unity ω and a sequence $V_n(t, q)$ of 2-variable polynomials, where $n \geqslant 1$ is an integer.

The above polynomials are defined for oriented long knots, but the construction can often be extended to polynomial invariants of framed, oriented links in 3-space. Whereas a general theorem is not known for all finite dimensional Nichols algebras with automorphisms, it was shown in [GHK+] that the long knot V_1, Λ_1, and Λ_{-1} polynomials do extend to framed, oriented links (and that they are independent of the framing), and two of them were identified with the Alexander polynomial and the $\mathfrak{sl}_3$-link polynomial of [Har]

$$\Lambda_{1,L}(t_0, t_1) = \Delta_L(t_0)\, \Delta_L(t_1), \qquad \Lambda_{-1,L}(t^{-2}, s^{-2}) = \Delta_{\mathfrak{sl}_3,L}(t, s) \tag{2}$$

as was conjectured in [GK]. Our goal is to identify the V_1 polynomial of [GK] with the Links–Gould polynomial [LG92], as was conjectured in [GK].

1.2. V_1 = LG **via skein theory.** Throughout the paper, all links will be oriented and considered up to ambient isotopy in 3-space.

Theorem 1.1. *For all links L we have:*

$$V_{1,L}(t_0, t_1) = \mathrm{LG}_L(t_0, t_1) \in \mathbb{Z}[t_0^{\pm 1}, t_1^{\pm 1}]\,. \tag{3}$$

Whereas both polynomial invariants V_1 and LG are defined by 4-dimensional R-matrices, we were unable to show that these are conjugate or braid-conjugate (borrowing terminology from [GHK+]), and hence we could not use the methods of [GHK+] to deduce the above theorem.

Instead, we prove the above theorem by showing that both invariants satisfy a common skein theory that uniquely determines them, hence equality follows. This common skein theory that we shortly discuss does not describe a presentation of the braided monoidal category of representations of $U_q(\mathfrak{sl}(2|1))$, but instead is tailored to relations on the braid-group representations of these invariants, and ultimately to polynomial equations satisfied by the R-matrices of both the V_1 and the the LG invariants.

To describe these rather complicated skein relations, we choose to present braids algebraically rather than pictorially, as words in the standard generators s_i and their inverses $\overline{s}_i$ for $i = 1, \ldots, n-1$ of the Artin braid group B_n [Art47] shown in Figure 1.

There is a natural inclusion of $B_n \to B_{n+1}$ obtained by adding a vertical strand on the right, and as is customary in the literature (see e.g., [FM12] and references therein), we denote by s_i the corresponding elements in B_n and in B_{n+1}.

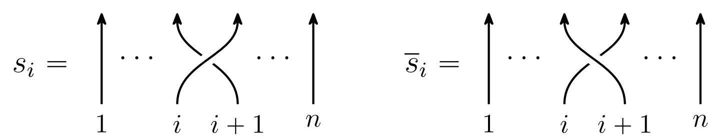


FIGURE 1. The standard generators s_i of the braid group B_n and their inverses $\overline{s}_i$ for $i = 1, \dots, n-1$.

Lemma 1.2. Both V_1 and LG satisfy the skein relations in B_n and $i, j, k = 1, \dots, n-1$:

$$s_i^2 + (1 - t_0 - t_1)\, s_i + (t_0 t_1 - t_0 - t_1)\, 1 + (t_0 t_1)\, \overline{s}_i = 0\,, \tag{R_1}$$

$$\overline{s}_i s_j s_i - s_i s_j \overline{s}_i - \overline{s}_i \overline{s}_j s_i + s_i \overline{s}_j \overline{s}_i = s_i s_j - s_i \overline{s}_j - \overline{s}_i s_j + \overline{s}_i \overline{s}_j - s_j s_i + s_j \overline{s}_i + \overline{s}_j s_i - \overline{s}_j \overline{s}_i \tag{R_2}$$

for $|i - j| = 1$,

$$s_i \overline{s}_k s_j \overline{s}_k - \overline{s}_k s_j \overline{s}_k s_i = \sum_{\ell=1}^{78} a_\ell w_\ell \tag{R_3}$$

for $k - 2 = j - 1 = i$, where w is

$$\begin{aligned} w =(& s_i s_j, s_i \overline{s}_j, s_j s_i, \overline{s}_j s_i, s_k s_j, \overline{s}_k s_j, s_k \overline{s}_j, \overline{s}_k \overline{s}_j, s_j s_k, \overline{s}_j s_k, s_j \overline{s}_k, \overline{s}_j \overline{s}_k, s_i s_k s_j, s_i s_k \overline{s}_j, s_i \overline{s}_k s_j, \\ & s_i \overline{s}_k \overline{s}_j, \overline{s}_i s_k s_j, \overline{s}_i s_k \overline{s}_j, \overline{s}_i \overline{s}_k s_j, \overline{s}_i \overline{s}_k \overline{s}_j, s_i s_j s_k, s_i s_j \overline{s}_k, s_i \overline{s}_j s_k, s_i \overline{s}_j \overline{s}_k, \overline{s}_i s_j s_k, \overline{s}_i s_j \overline{s}_k, \overline{s}_i \overline{s}_j s_k, \\ & \overline{s}_i \overline{s}_j \overline{s}_k, s_j s_i s_k, s_j s_i \overline{s}_k, s_j \overline{s}_i s_k, s_j \overline{s}_i \overline{s}_k, \overline{s}_j s_i s_k, \overline{s}_j s_i \overline{s}_k, \overline{s}_j \overline{s}_i s_k, \overline{s}_j \overline{s}_i \overline{s}_k, s_k s_j s_i, s_k s_j \overline{s}_i, s_k \overline{s}_j s_i, \\ & s_k \overline{s}_j \overline{s}_i, \overline{s}_k s_j s_i, \overline{s}_k s_j \overline{s}_i, \overline{s}_k \overline{s}_j s_i, \overline{s}_k \overline{s}_j \overline{s}_i, s_k \overline{s}_i \overline{s}_j \overline{s}_i, \overline{s}_k \overline{s}_i \overline{s}_j \overline{s}_i, s_k \overline{s}_i \overline{s}_j s_i, \overline{s}_k \overline{s}_i \overline{s}_j s_i, s_k \overline{s}_i s_j \overline{s}_i, \overline{s}_k \overline{s}_i s_j \overline{s}_i, \\ & s_k s_i \overline{s}_j \overline{s}_i, \overline{s}_k s_i \overline{s}_j \overline{s}_i, s_k s_i \overline{s}_j s_i, \overline{s}_k s_i \overline{s}_j s_i, s_k s_i s_j \overline{s}_i, \overline{s}_k s_i s_j \overline{s}_i, \overline{s}_i \overline{s}_j \overline{s}_i s_k, \overline{s}_i \overline{s}_j \overline{s}_i \overline{s}_k, \overline{s}_i \overline{s}_j s_i s_k, \overline{s}_i \overline{s}_j s_i \overline{s}_k, \\ & \overline{s}_i s_j \overline{s}_i s_k, \overline{s}_i s_j \overline{s}_i \overline{s}_k, s_i \overline{s}_j \overline{s}_i s_k, s_i \overline{s}_j \overline{s}_i \overline{s}_k, s_i \overline{s}_j s_i s_k, s_i \overline{s}_j s_i \overline{s}_k, s_i s_j \overline{s}_i s_k, s_i s_j \overline{s}_i \overline{s}_k, s_i \overline{s}_j \overline{s}_k s_j, s_i \overline{s}_j s_k \overline{s}_j, \\ & s_i s_j \overline{s}_k \overline{s}_j, s_i s_j \overline{s}_k s_j, s_i s_j s_k \overline{s}_j, \overline{s}_j \overline{s}_k s_j s_i, \overline{s}_j s_k \overline{s}_j s_i, s_j \overline{s}_k \overline{s}_j s_i, s_j \overline{s}_k s_j s_i, s_j s_k \overline{s}_j s_i) \end{aligned} \tag{4}$$

and the coefficients $a_l \in \mathbb{Q}(t_0, t_1)$ for $l = 1, \dots, 78$ are given explicitly in Appendix B.

The relation (R_1) is derived from the R-matrix R_{LG} given in Appendix A whose minimal polynomial is cubic with roots $1, t_0, t_1$. The relation (R_2) was discovered by Ishii [Ish04b]. The existence of a relation (R_3) was proven by Marin–Wagner [MW13, Sec.6.2, Sec.6.3] with no explicit description. Since the support of (R_3) is important in the reduction algorithm of Theorem 1.3 below, we give its coefficients explicitly, and explain in Section (2.4) how it was found.

Note that (R_1), (R_2) and (R_3) come from relations in the braid groups B_2, B_3, and B_4 involving 2, 3, and 4 braid strands respectively.

The proof of the above lemma follows from the fact that the R-matrices for V_1 and LG given in the appendix satisfy the polynomial identities (R_1), (R_2) and (R_3), a fact certified by a computer calculation.

An important complement of the above skein relations is their completeness, that is they allow the computation of the invariant for every link. This is achieved by an effective reduction algorithm given in Theorem 1.3 below. To phrase it, consider the quotient

$$C_n = \mathbb{Q}(t_0, t_1)[B_n]/(R_1, R_2, R_3) \tag{5}$$

of the group-algebra $\mathbb{Q}(t_0, t_1)[B_n]$ of the braid group B_n by the 2-sided ideal (R_1, R_2, R_3). C_n is an associative, non-commutative unital algebra over the field $\mathbb{Q}(t_0, t_1)$.

Note that if $c \in C_m$ for $m < n$ and and $\beta \in B_n$, then $\beta c, c\beta \in C_n$. In particular, there is a natural map $C_m \to C_n$ obtained from the braid group inclusion $B_m \to B_n$.

Theorem 1.3. *For every $n \geqslant 3$, there is a reduction algorithm that implies an equality*

$$C_n = C_{n-1} + C_{n-1}s_{n-1}C_{n-1} + C_{n-1}\overline{s}_{n-1}C_{n-1} + C_{n-2}\overline{s}_{n-1}s_{n-2}\overline{s}_{n-1} \tag{6}$$

of $\mathbb{Q}(t_0, t_1)$-vector spaces.

In other words, the theorem above asserts that for $n \geqslant 3$, every braid $\beta \in B_n$ can be reduced to an element of the right hand side of Equation (6).

Theorem 1.3 is an effective version of Theorem 5.4 (ii) and Theorem 6.1 (after fixing a typo) of Marin–Wagner [MW13].

The next remark is important for specialization of this skein theory, e.g., to the case of ADO_ω.

Remark 1.4. Although C_n is a $\mathbb{Q}(t_0, t_1)$-algebra and the relations (R_1), (R_2) and (R_3) as well as the proof of Theorem 1.3 involve denominators, the statement and the proof are valid if we replace the field $\mathbb{Q}(t_0, t_1)$ with the ring

$$\mathbb{Z}[t_0, t_1, \delta(t_0, t_1)^{-1}] \tag{7}$$

where

$$\delta(t_0, t_1) = t_0t_1(t_0+t_1)(t_0t_1+1)(t_0t_1-1)(1+t_0)(1+t_1)(t_0+t_1-1)(1+t_0t_1+t_0^2t_1+t_0t_1^2)\,. \tag{8}$$

The next remark concerns the dimension of C_n.

Remark 1.5. A corollary of Theorem 1.3 is that C_n is a finite-dimensional $\mathbb{Q}(t_0, t_1)$-vector space. In fact, it is conjectured in [MW13] and in analogous algebras studied in [Ang21], that

$$\dim(C_n) = \frac{(2n-2)!(2n-1)!}{((n-1)!n!)^2} \tag{9}$$

with the first few values for $n = 3, \dots, 10$ given by

$$3, 20, 175, 1764, 19404, 226512, 2760615, 34763300\,. \tag{10}$$

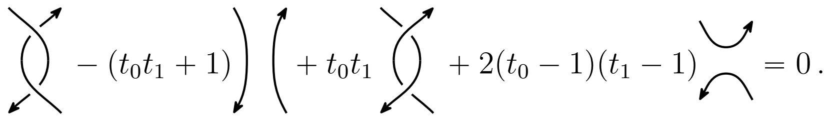
Unfortunately, Equation (6) for $n = 3$ gives only the bound $\dim(C_3) \leqslant 22$, and in this case it can be improved to an explicit spanning set of 20 elements which is linearly independent, hence deducing $\dim(C_3) = 20$; see Corollary 2.5 below. But beyond that, although Theorem 1.3 constructs explicit spanning sets for C_n, it does not give sharp bounds for $\dim(C_n)$ for $n > 3$.

A straightforward consequence of Theorem 1.3 is the following.

Corollary 1.6. A link invariant that satisfies the skein relations (R_1), (R_2) and (R_3) and vanishes on split links is uniquely determined by its value on the unknot.

This corollary combined with Lemma 1.2 and Lemma 3.2 below implies Theorem 1.1.

Corollary 1.6 has an alternative formulation that swaps the global condition of vanishing on diagrams of split links for an extra local skein relation that involves tangles (as opposed to braids). Consider the relation (S_2) introduced by Ishii [Ish04a]:



(S_2)

In [Ish04a, Prop.3.3], Ishii shows that a link invariant that satisfies (R_1) and (S_2) vanishes on split links. Therefore Corollary 1.6 implies the following result.

Corollary 1.7. A link invariant that satisfies the skein relations (R_1), (R_2), (S_2) and (R_3) is uniquely determined by its value on the unknot.

Another consequence of Corollary 1.6 is that a rank 1 Nichols algebra invariant, namely ADO_ω, is equal to a specialization of a rank 2 Nichols algebra invariant, namely the LG invariant, as conjectured by Geer and Patureau-Mirand [GPM08, Conj.4.7] and by [GK].

Theorem 1.8. *For every link L we have*

$$\mathrm{ADO}_{\omega,L}(t) = \mathrm{LG}_L(t^2, \omega^2 t^{-2}), \tag{11}$$

where $\omega = e^{2\pi i/6}$.

This follows from the fact that the R-matrix for ADO_ω satisfies the $(t_0, t_1) = (t^2, \omega^2 t^{-2})$ specialization of (R_1), (R_2) and (R_3) and Remark 1.4.

A partial case of the above theorem for links that come from closures of 5-strand braids was given by Takenov [Tak].

Note that the multi-color version of the Geer–Patureau-Mirand conjecture remains open.

Interestingly, Theorem 1.8 gives an example of two R matrices on a 3 and a 4-dimensional vector space with the same knot polynomial invariant. Any connection between these two R-matrices remains to be investigated.

Remark 1.9. The effective proof of Theorem 1.3, which leads to an effective computation of the $V_1 = \mathrm{LG}$-polynomials, is by no means comparable in speed to the tangle computation of these invariants given in [GL]. Indeed, skein theory computations have apparent exponential complexity whereas tangle computations tend to have polynomial complexity.

1.3. **Specialization, Vassiliev invariants and genus bounds for V_1.** We now discuss some applications of our main Theorem 1.1.

Using the variables (t_0, t_1), the equality of LG and V_1 and previously known results for LG [DWIL05, Ish06, Koh16, KPM17] imply the following corollaries conjectured in [GK].

Corollary 1.10. The V_1 polynomial of a link L satisfies the specializations

$$V_{1,L}(t_0, t_0^{-1}) = \Delta_L(t_0)^2, \qquad V_{1,L}(t_0, -t_0^{-1}) = \Delta_L(t_0^2), \qquad V_{1,L}(t_0, 1) = V_{1,L}(1, t_1) = 1 \tag{12}$$

The genus bounds for LG from [KT] imply the following.

Corollary 1.11. For a knot K, we have the bound

$$\deg_t V_{1,K}(t,q) \leqslant 4\,\mathrm{genus}(K)\,, \tag{13}$$

where $\deg_t$ of a Laurent polynomial in t is the difference between the highest and the lowest power of t and $\mathrm{genus}(K)$ is the minimal genus of an embedded oriented surface spanning K.

Vassiliev power series invariants were introduced in [BNG96]. Geer [Gee05] showed that the Links–Gould polynomial is obtained from the evaluation of the Kontsevich integral under the $\mathfrak{sl}(2|1)$ weight-system.

Corollary 1.12. The oriented link polynomial V_1 is a Vassiliev power series invariant.

Remark 1.13. Note that there are three sets of variables used in the literature, namely (t_0, t_1) introduced by Ishii [Ish06], (q^α, q) used in the context of representation theory e.g. to study LG [LG92, KT] and $(t, \tilde{q})$ used in [GK]. This is a point that leads to much confusion. The relations between these different sets of variables are

$$(t_0, t_1) = (t\tilde{q}^{-n/2}, t^{-1}\tilde{q}^{-n/2}), \qquad (q^\alpha, q) = (t^{-1/2}\tilde{q}^{1/4}, \tilde{q}^{-1/2})\,. \tag{14}$$

In this work we mostly use the known properties of LG to deduce similar results for the V_1-polynomial, but we can also do the converse. A corollary of Theorem 1.1 is the following nontrivial symmetry of the LG-polynomial due to Ishii.

Corollary 1.14 ([Ish06, Thm.1])**.** For any link L, we have

$$\mathrm{LG}_L(t_0, t_1) \in \mathbb{Z}[t_0^{\pm 1}, t_1^{\pm 1}]\,. \tag{15}$$

We end this section with a caution regarding deducing statements by specialization of skein-theory.

Remark 1.15. Although the Alexander polynomial comes from an enhanced R-matrix, the skein theory approach that proves Theorem 1.1 cannot be used to provide novel proofs of either of the known specializations

$$\mathrm{LG}_L(t_0, t_0^{-1}) = \Delta_L(t_0)^2, \qquad \mathrm{LG}_L(t_0, -t_0^{-1}) = \Delta_L(t_0^2) \tag{16}$$

of [DWIL05, Ish06, Koh16, KPM17]. This is because the left side of the (R_3) relation vanishes under these specializations and thus, the reduction algorithm does not apply. However, the symmetry of the relations (R_1), (R_2), (R_3) in t_0 and t_1 give an alternative proof of the fact

$$\mathrm{LG}_L(t_0, t_1) = \mathrm{LG}_L(t_1, t_0)\,. \tag{17}$$

1.4. **Organization of the paper.** In Section 2 we introduce the main properties of skein relations based on the braid group and use them to prove Theorem 1.3. In Section 3 we recall briefly the definition and common properties of the three link polynomials LG, V_1 and ADO_ω that are the focus of our paper. We also explain how we found the explicit relation (R_3).

In Appendix A we give the R-matrices of the three polynomial invariants that we study, and in Appendix B we give the lengthy coefficients of the skein relation (R_3). Appendices C and D are dedicated to the proof of two technical results used in the proof of Theorem 1.3.

Acknowledgements. The authors would like to thank Ivan Marin and Emmanuel Wagner for enlightening conversations.

2. Skein theory for the LG polynomial

In this section we prove Theorem 1.3 with a reduction algorithm. Recall that the braid group B_n has standard generators s_i with inverses $\overline{s}_i$ for $i = 1, \dots, n-1$ shown in Figure 1. They satisfy the relations

$$s_i s_j s_i = s_j s_i s_j \text{ for } |i-j| = 1, \qquad s_i s_j = s_j s_i \text{ for } |i-j| > 1\,. \tag{18}$$

Fix a braid $\beta \in B_n$ for $n \geqslant 2$. We will prove that it can be reduced to belong to the $\mathbb{Q}(t_0, t_1)$-vector space on the right hand side of Equation (6).

Note that if $n = 2$, it follows from (R_1) that every braid in B_2 can be reduced to a linear combination of $1, s_1, \overline{s}_1$.

We will first prove Theorem 1.3 for $n = 3, 4$, which are the hardest cases. Then we prove the result by induction on $n \geqslant 5$.

2.1. The $n = 3$ case.

Let $\beta \in B_3$. By applying (R_1), reduce it to the case

$$\beta = s_1^{\varepsilon_1} s_2^{\varepsilon_2} s_1^{\varepsilon_3} \dots \text{ or } \beta = s_2^{\varepsilon_1} s_1^{\varepsilon_2} s_2^{\varepsilon_3} \dots\ ,\ \varepsilon_i = \pm 1. \tag{19}$$

Lemma 2.1 (Equivalent formulation of (R_2))**.** The following relation is equivalent to relation (R_2) modulo relation (R_1). For $1 \leqslant i \leqslant n-2$ and $j = i+1$:

$$\begin{aligned} s_j \overline{s}_i s_j = {} & (t_0 t_1 + 1 - t_0 - t_1)\, s_i + (t_0 + t_1 - t_0 t_1 - 1)\, \overline{s}_i \\ & - (t_0 + t_1 - 1)\, s_i s_j + (t_0 + t_1 - 1)\, \overline{s}_i s_j + t_0 t_1\, s_i \overline{s}_j - t_0 t_1\, \overline{s}_i \overline{s}_j \\ & + s_j s_i - s_j \overline{s}_i - (t_0 + t_1 - t_0 t_1)\, \overline{s}_j s_i + (t_0 + t_1 - t_0 t_1)\, \overline{s}_j \overline{s}_i \\ & - (t_0 + t_1)\, s_i s_j \overline{s}_i + (t_0 + t_1)\, s_i \overline{s}_j \overline{s}_i + s_i s_j s_i - t_0 t_1\, \overline{s}_i \overline{s}_j \overline{s}_i + t_0 t_1\, \overline{s}_j s_i \overline{s}_j\,. \end{aligned} \tag{20}$$

Proof. We start by writing $(R_2) \cdot s_j$:

$$\begin{aligned} (s_j \overline{s}_i) s_j = {} & (\overline{s}_i s_j s_i) s_j - (s_i s_j \overline{s}_i) s_j - (\overline{s}_i \overline{s}_j s_i) s_j + (s_i \overline{s}_j \overline{s}_i) s_j - (s_i s_j) s_j + (s_i \overline{s}_j) s_j \\ & + (\overline{s}_i s_j) s_j - (\overline{s}_i \overline{s}_j) s_j + (s_j s_i) s_j - (\overline{s}_j s_i) s_j + (\overline{s}_j \overline{s}_i) s_j \\ = {} & \overline{s}_i (s_i s_j s_i) - \overline{s}_j s_i s_j s_j - s_j \overline{s}_i \overline{s}_j s_j + s_i s_i \overline{s}_j \overline{s}_i - s_i ((t_0 + t_1 - 1)\, s_j \\ & + (t_0 + t_1 - t_0 t_1)\, 1 - t_0 t_1\, \overline{s}_j) + s_i \\ & + \overline{s}_i ((t_0 + t_1 - 1)\, s_j + (t_0 + t_1 - t_0 t_1)\, 1 - t_0 t_1\, \overline{s}_j) - \overline{s}_i + s_i s_j s_i - s_i s_j \overline{s}_i + s_i \overline{s}_j \overline{s}_i \\ = {} & s_j s_i - \overline{s}_j s_i ((t_0 + t_1 - 1)\, s_j + (t_0 + t_1 - t_0 t_1)\, 1 - t_0 t_1\, \overline{s}_j) - s_j \overline{s}_i \\ & + ((t_0 + t_1 - 1)\, s_i + (t_0 + t_1 - t_0 t_1)\, 1 - t_0 t_1\, \overline{s}_i) \overline{s}_j \overline{s}_i \\ & - (t_0 + t_1 - 1)\, s_i s_j - (t_0 + t_1 - t_0 t_1)\, s_i + t_0 t_1\, s_i \overline{s}_j + s_i \\ & + (t_0 + t_1 - 1)\, \overline{s}_i s_j + (t_0 + t_1 - t_0 t_1)\, \overline{s}_i - t_0 t_1\, \overline{s}_i \overline{s}_j - \overline{s}_i + s_i s_j s_i - s_i s_j \overline{s}_i + s_i \overline{s}_j \overline{s}_i \\ = {} & s_j s_i - (t_0 + t_1 - 1)\, s_i s_j \overline{s}_i - (t_0 + t_1 - t_0 t_1)\, \overline{s}_j s_i + t_0 t_1\, \overline{s}_j s_i \overline{s}_j \\ & - s_j \overline{s}_i + (t_0 + t_1 - 1)\, s_i \overline{s}_j \overline{s}_i + (t_0 + t_1 - t_0 t_1)\, \overline{s}_j \overline{s}_i - t_0 t_1\, \overline{s}_i \overline{s}_j \overline{s}_i \\ & - (t_0 + t_1 - 1)\, s_i s_j - (t_0 + t_1 - t_0 t_1)\, s_i + t_0 t_1\, s_i \overline{s}_j + s_i \\ & + (t_0 + t_1 - 1)\, \overline{s}_i s_j + (t_0 + t_1 - t_0 t_1)\, \overline{s}_i - t_0 t_1\, \overline{s}_i \overline{s}_j - \overline{s}_i + s_i s_j s_i - s_i s_j \overline{s}_i + s_i \overline{s}_j \overline{s}_i \\ = {} & (t_0 t_1 + 1 - t_0 - t_1)\, s_i + (t_0 + t_1 - t_0 t_1 - 1)\, \overline{s}_i - (t_0 + t_1 - 1)\, s_i s_j \\ & + (t_0 + t_1 - 1)\, \overline{s}_i s_j + t_0 t_1\, s_i \overline{s}_j - t_0 t_1\, \overline{s}_i \overline{s}_j + s_j s_i - s_j \overline{s}_i - (t_0 + t_1 - t_0 t_1)\, \overline{s}_j s_i \end{aligned}$$

$$+ (t_0 + t_1 - t_0t_1)\, \overline{s}_j\overline{s}_i - (t_0 + t_1)\, s_is_j\overline{s}_i + (t_0 + t_1)\, s_i\overline{s}_j\overline{s}_i + s_is_js_i - t_0t_1\, \overline{s}_i\overline{s}_j\overline{s}_i$$
$$+ t_0t_1\, \overline{s}_js_i\overline{s}_j\,.$$

□

We now consider the inner automorphism $b \mapsto \check{b}$ of the braid group B_n defined by

$$\check{b_n} = g_n^{-1} b g_n \tag{21}$$

where $g_n = (s_1s_2 \dots s_{n-1}) \dots (s_1s_2s_3)(s_1s_2)(s_1)$ is a half twist in n strands. Topologically, $\check{b}$ is braid b "looked at from behind". It is easy to show the following.

Lemma 2.2. The following hold true:

- for all $k = 1, \dots, n-1$, we have $\check{s_k} = s_{n-k}$,
- for all $b, c \in B_n$, $\widecheck{bc} = \check{b}\check{c}$,
- for all $b \in B_n$, braids b and $\check{b}$ have the same link closure.

This automorphism will be used in the proof of Lemma 2.3 below as well as in numerous locations in the Appendix C and D. In addition, applying this automorphism shows that (20) of Lemma 2.1 is also true if you exchange the roles of i and j, even though they do not play symmetric roles in the relation.

Lemma 2.3. The following relations hold in C_n. In each ℓ-letter relation, we assume $1 \leqslant i \leqslant n - \ell$, $j = i + 1$, and $k = i + 2$.

1-letter relations

- Inverse relation:
$$s_i\overline{s}_i = \overline{s}_is_i = 1\,, \tag{22}$$
- Relation (R_1) and its equivalent version:
$$\begin{aligned} s_i^2 &= (t_0 + t_1 - 1)\, s_i + (t_0 + t_1 - t_0t_1)\, 1 - (t_0t_1)\, \overline{s}_i\,,\\ \overline{s}_i^2 &= (t_0^{-1} + t_1^{-1} - 1)\, \overline{s}_i + (t_0^{-1} + t_1^{-1} - t_0^{-1}t_1^{-1})\, 1 - (t_0^{-1}t_1^{-1})\, s_i\,. \end{aligned} \tag{23}$$

2-letter relations

- Far commutativity:
$$s_l^{\pm1}s_m^{\pm1} = s_m^{\pm1}s_l^{\pm1} \text{ for } |l - m| \geqslant 2\,, \tag{24}$$
- The braid relation and its equivalent formulations:
$$\begin{gathered} s_is_js_i = s_js_is_j\,,\\ (s_j^as_i^a)s_j^b = s_i^b(s_j^as_i^a) \text{ and } (s_i^as_j^a)s_i^b = s_j^b(s_i^as_j^a) \quad \text{for } a, b = \pm1, \end{gathered} \tag{25}$$
for $|i - j| = 1$.
- Relation (R_2) and its equivalent version:
$$\begin{aligned} &\overline{s}_is_js_i - s_is_j\overline{s}_i - \overline{s}_i\overline{s}_js_i + s_i\overline{s}_j\overline{s}_i =\\ &\qquad s_is_j - s_i\overline{s}_j - \overline{s}_is_j + \overline{s}_i\overline{s}_j - s_js_i + s_j\overline{s}_i + \overline{s}_js_i - \overline{s}_j\overline{s}_i\,, \end{aligned} \tag{26}$$
$$\begin{aligned} &\overline{s}_js_is_j - s_js_i\overline{s}_j - \overline{s}_j\overline{s}_is_j + s_j\overline{s}_i\overline{s}_j =\\ &\qquad s_js_i - s_j\overline{s}_i - \overline{s}_js_i + \overline{s}_j\overline{s}_i - s_is_j + s_i\overline{s}_j + \overline{s}_is_j - \overline{s}_i\overline{s}_j\,. \end{aligned} \tag{27}$$

3-letter relations

- Relations implied by (R_3):

$$
\begin{aligned}
&s_i^{\pm 1}\overline{s}_k s_j \overline{s}_k = \overline{s}_k s_j \overline{s}_k s_i^{\pm 1} + \alpha\,, \quad && s_i^{\pm 1} s_k \overline{s}_j s_k = s_k \overline{s}_j s_k s_i^{\pm 1} + \gamma\,,\\
&s_k^{\pm 1}\overline{s}_i s_j \overline{s}_i = \overline{s}_i s_j \overline{s}_i s_k^{\pm 1} + \delta\,, && s_k^{\pm 1} s_i \overline{s}_j s_i = s_i \overline{s}_j s_i s_k^{\pm 1} + \eta\,,
\end{aligned}
\tag{28}
$$

where α, γ are $\mathbb{Q}(t_0,t_1)$-linear combinations of braid words with at most 4 letters and at most one $s_k^{\pm 1}$, and δ, η are $\mathbb{Q}(t_0,t_1)$-linear combinations of braid words with at most 4 letters and at most one $s_i^{\pm 1}$.

Proof. The only non-obvious relations are the 3-letter relations. The first of these four relations is (R_3) in the case of the positive exponents. In the case of the negative exponents, the first relation is obtained from (R_3) by writing $s_i^{-1}\cdot(R_3)\cdot s_i^{-1}$. The two versions of the second relation follow from the two versions of the first relation and from Lemma 2.1:

$$
s_i^{\pm 1}(s_k \overline{s}_j s_k) = t_0 t_1\, s_i^{\pm 1}(\overline{s}_k s_j \overline{s}_k) \;+\; \text{words with at most 4 letters and at most one } s_k^{\pm 1}\,, \tag{29}
$$

and

$$
(s_k \overline{s}_j s_k) s_i^{\pm 1} = t_0 t_1\, (\overline{s}_k s_j \overline{s}_k) s_i^{\pm 1} \;+\; \text{words with at most 4 letters and at most one } s_k^{\pm 1}\,. \tag{30}
$$

Replacing $s_i^{\pm 1}(\overline{s}_k s_j \overline{s}_k)$ and $(\overline{s}_k s_j \overline{s}_k)s_i^{\pm 1}$ in the different versions of (R_3) using these two identities, we get both versions of the second 3 letter identity of Lemma 2.3. Finally, the third and fourth 3-letter relations are obtained from the first two relations by applying the inner automorphism defined through Equation (21). ☐

Lemma 2.4. In C_3, any word $\beta \in B_3$ is a linear combination of at most 3 letter words, each of which:

- either has at most one $s_2^{\pm 1}$,
- or is $\overline{s}_2 s_1 \overline{s}_2$.

We prove Lemma 2.4 in Appendix C. This implies Theorem 1.3 in the three strand case $n = 3$. Actually it implies an enhancement of Equation (6) for $n = 3$ given in the next corollary.

Corollary 2.5. The following set is a $\mathbb{Q}(t_0,t_1)$-linear basis of C_3:

$$
\begin{aligned}
\{1, s_1, \overline{s}_1, s_2, \overline{s}_2, s_2 s_1, s_2\overline{s}_1, \overline{s}_2 s_1, \overline{s}_2\overline{s}_1, s_1 s_2, \overline{s}_1 s_2, s_1\overline{s}_2, \overline{s}_1\overline{s}_2, s_1 s_2 s_1, s_1 s_2 \overline{s}_1,\\
s_1\overline{s}_2\overline{s}_1, \overline{s}_1\overline{s}_2 s_1, \overline{s}_1\overline{s}_2\overline{s}_1, \overline{s}_1 s_2 \overline{s}_1 \text{ (or } s_1\overline{s}_2 s_1), \overline{s}_2 s_1 \overline{s}_2\}\,,
\end{aligned}
\tag{31}
$$

hence the dimension of C_3 is 20.

Proof. Using Lemma 2.4, the following set is a $\mathbb{Q}(t_0,t_1)$-generating set of C_3:

$$
\begin{aligned}
\{1, s_1, \overline{s}_1, s_2, \overline{s}_2, s_2 s_1, s_2\overline{s}_1, \overline{s}_2 s_1, \overline{s}_2\overline{s}_1, s_1 s_2, \overline{s}_1 s_2, s_1\overline{s}_2, \overline{s}_1\overline{s}_2, s_1 s_2 s_1, s_1 s_2 \overline{s}_1,\\
\overline{s}_1 s_2 s_1, \overline{s}_1 s_2 \overline{s}_1, s_1\overline{s}_2 s_1, s_1\overline{s}_2\overline{s}_1, \overline{s}_1\overline{s}_2 s_1, \overline{s}_1\overline{s}_2\overline{s}_1, \overline{s}_2 s_1 \overline{s}_2\}\,.
\end{aligned}
\tag{32}
$$

Relation (R_2) shows that the vector $\overline{s}_1 s_2 s_1$ can be expressed in terms of the other vectors of the family. Once that vector is removed, (20) shows that $s_1\overline{s}_2 s_1$ (or $\overline{s}_1 s_2 \overline{s}_1$) can also be removed. Thus we have a generating set for C_3 with 20 elements.

Just as Marin–Wagner did in [MW13, Thm.1.1], we can prove that this set of 20 elements is linearly independent over $\mathbb{Q}(t_0, t_1)$ by using the map $\rho_{\mathrm{LG}} : C_3 \to \mathrm{End}(V^{\otimes 3})$ on a 4-dimensional vector space V and check that the system of $4^6 = 4096$ linear equations in 20 unknowns with coefficients in the field $\mathbb{Q}(t_0, t_1)$ has a unique solution, namely zero. ☐

2.2. **The $n = 4$ case.** We prove Theorem 1.3 for words $\beta \in B_4$ inductively on $\#_3(\beta)$, where $\#_3(\beta)$ is the sum of the number of s_3 and of the number of $\overline{s}_3$ that appear in the expression of β in terms Artin generators. Moreover, like in the three strand case, because of (R_1), we need only to consider

$$\beta = s_{i_1}^{\varepsilon_1} s_{i_2}^{\varepsilon_2} s_{i_3}^{\varepsilon_3} \dots \ , \varepsilon_i = \pm 1. \tag{33}$$

The base case of the induction – where the result is obviously true – is when $\#_3(\beta) \leqslant 1$.

Now consider a word β such that $\#_3(\beta) \geqslant 2$. We can write $\beta = x s_3^{\pm 1} w s_3^{\pm 1} y$ with $w, y \in \langle s_1, s_2 \rangle \cong B_3$ and $x \in B_4$. Since $w \in B_3$, we may express it in the generating set of Lemma 2.4:

- either w has at most one $s_2^{\pm 1}$,
- or $w = \overline{s}_2 s_1 \overline{s}_2$.

$\boxed{\text{Case 1: } \#_2(w) = 0.}$ In this case, w is a power of s_1 and $\beta = x s_3^{\pm 1} w s_3^{\pm 1} y = x w s_3^{\pm 1} s_3^{\pm 1} y$. The number of $s_3^{\pm 1}$ used to write β decreases thanks to (R_2) or a straightforward simplification of inverses. So we have the result in this case inductively.

$\boxed{\text{Case 2: } \#_2(w) = 1.}$ We can write $w = s_1^{\varepsilon} s_2^{\pm 1} s_1^{\gamma}$ for some $\varepsilon, \gamma \in \{-1, 0, 1\}$. Then $\beta = x' s_3^{\pm 1} s_2^{\pm 1} s_3^{\pm 1} y'$ with $y' \in \langle s_1, s_2 \rangle$ and $x' \in B_4$. We apply Lemma 2.4 to $s_3^{\pm 1} s_2^{\pm 1} s_3^{\pm 1} \in \langle s_2, s_3 \rangle \cong B_3$, then modulo terms with fewer $s_3^{\pm 1}$,

$$\beta = x' \overline{s}_3 s_2 \overline{s}_3 y'' \quad \text{for some } y'' \in \langle s_1, s_2 \rangle. \tag{34}$$

If the left-most letter in y'' is $s_1^{\pm 1}$, then that letter commutes with $\overline{s}_3 s_2 \overline{s}_3$ modulo words with fewer $s_3^{\pm 1}$ by applications of (R_3). If the left-most letter is $s_2^{\pm 1}$, then we may apply Lemma 2.4 once again. The sub-word $\overline{s}_3 s_2 \overline{s}_3 s_2^{\pm 1}$ reduces in $\langle s_2, s_3 \rangle \cong B_3$ to $\overline{s}_3 s_2 \overline{s}_3$ modulo words with fewer $s_3^{\pm 1}$ and we have removed the left-most letter from y'. Thus, an inductive argument on length of y' shows

$$\beta = x'' \overline{s}_3 s_2 \overline{s}_3 \quad \text{for some } x'' \in B_4 \tag{35}$$

modulo words with fewer $s_3^{\pm 1}$. Note that

$$\#_3(x'' \overline{s}_3 s_2 \overline{s}_3) = \#_3(x \overline{s}_3 w \overline{s}_3 y) \quad \text{and} \quad \#_3(x'') < \#_3(x \overline{s}_3 w \overline{s}_3 y). \tag{36}$$

Thus, using the inductive hypothesis on x'', it can be written as a linear combination of words as described in Theorem 1.3. The different elements in the sum can be considered independently.

<u>Sub-case 2.1:</u> Suppose $x'' \in \langle s_1, s_2 \rangle \cong B_3$. In this instance, x'' can itself be reduced using Lemma 2.4. If $x'' = s_1^{\varepsilon_1} s_2^{\varepsilon_2} s_1^{\varepsilon_3}$ for $\varepsilon_i \in \{-1, 0, 1\}$, then modulo words with fewer $s_3^{\pm 1}$ and up to a scalar,

$$\begin{aligned} \beta &= s_1^{\varepsilon_1} s_2^{\varepsilon_2} \big(s_1^{\varepsilon_3} \overline{s}_3 s_2 \overline{s}_3\big) \overset{(R_3)}{=} s_1^{\varepsilon_1} \big(s_2^{\varepsilon_2} \overline{s}_3 s_2 \overline{s}_3\big) s_1^{\varepsilon_3} \\ &\overset{2.4}{=} s_1^{\varepsilon_1} \big(\overline{s}_3 s_2 \overline{s}_3\big) s_1^{\varepsilon_3} = s_1^{\varepsilon_1} \big(\overline{s}_3 s_2 \overline{s}_3 s_1^{\varepsilon_3}\big) \overset{(R_3)}{=} s_1^{\varepsilon_1 + \varepsilon_3} \overline{s}_3 s_2 \overline{s}_3. \end{aligned} \tag{37}$$

Now β is expressed in the desired form.
If on the other hand $x'' = \overline{s}_2 s_1 \overline{s}_2$, then modulo words with fewer $s_3^{\pm 1}$ and up to a scalar,

$$\beta = \overline{s}_2 s_1 (\overline{s}_2 \overline{s}_3 s_2 \overline{s}_3) \overset{2.4}{=} \overline{s}_2 s_1 (\overline{s}_3 s_2 \overline{s}_3), \tag{38}$$

which now expresses $x''(\overline{s}_3 s_2 \overline{s}_3)$ in the form $s_1^0 s_2^{-1} s_1^1 (\overline{s}_3 s_2 \overline{s}_3)$ considered previously.
Sub-case 2.2: Suppose $\#_3(x'') = 1$. This time $x'' = u s_3^{\pm 1} v$, with $u, v \in \langle s_1, s_2 \rangle$. Modulo terms with fewer $s_3^{\pm 1}$ and up to a scalar we write:

$$\begin{aligned} \beta = u s_3^{\pm 1} (v \overline{s}_3 s_2 \overline{s}_3) &\overset{\text{Sub-case 2.1}}{=} u s_3^{\pm 1} s_1^{\varepsilon} \overline{s}_3 s_2 \overline{s}_3 = u s_1^{\varepsilon} s_3^{\pm 1} \overline{s}_3 s_2 \overline{s}_3 \\ = u s_1^{\varepsilon} (s_3^{\pm 1} \overline{s}_3 s_2 \overline{s}_3) &\overset{2.4}{=} u s_1^{\varepsilon} \overline{s}_3 s_2 \overline{s}_3 \overset{\text{Sub-case 2.1}}{=} s_1^{\gamma} \overline{s}_3 s_2 \overline{s}_3. \end{aligned} \tag{39}$$

The expression for β completes the proof in this sub-case.
Sub-case 2.3: Next assume $x'' = s_1^{\delta} \overline{s}_3 s_2 \overline{s}_3$ with $\delta \in \{-1, 0, 1\}$. Again, modulo terms with fewer $s_3^{\pm 1}$ we have, up to a scalar:

$$\beta = s_1^{\delta} (\overline{s}_3 s_2 \overline{s}_3 \overline{s}_3 s_2 \overline{s}_3) \overset{2.4}{=} s_1^{\delta} \overline{s}_3 s_2 \overline{s}_3. \tag{40}$$

And we get the result in this case by structural induction.
$\boxed{\text{Case 3: } w = \overline{s}_2 s_1 \overline{s}_2.}$ Then $\beta = x s_3^{\pm 1} \overline{s}_2 s_1 \overline{s}_2 s_3^{\pm 1} y$ with $y \in \langle s_1, s_2 \rangle \cong B_3$. We need to be able to reduce $s_3^{\pm 1} \overline{s}_2 s_1 \overline{s}_2 s_3^{\pm 1}$ in order to conclude in this case. The following lemma is proven in Appendix D and does just that.

Lemma 2.6. In C_4, the four words $s_3^{\pm 1} \overline{s}_2 s_1 \overline{s}_2 s_3^{\pm 1} \in B_4$ can be reduced to linear combinations of words of one of the following types:

- words with at most one $s_3^{\pm 1}$,
- $\overline{s}_3 s_2 \overline{s}_3$, $s_1 \overline{s}_3 s_2 \overline{s}_3$ or $\overline{s}_1 \overline{s}_3 s_2 \overline{s}_3$.

Using Lemma 2.6, β reduces modulo words with fewer $s_3^{\pm 1}$ and up to a scalar to:

$$\beta \overset{2.6}{=} x s_1^{\varepsilon} \overline{s}_3 s_2 \overline{s}_3 y. \tag{41}$$

Here we have β given in the form of Case 2 (34). Case 3 is now proven in the same way. This proves Theorem 1.3 in the four strand case $n = 4$.

2.3. **The $n \geqslant 5$ case.** We suppose that Theorem 1.3 is true for some $n \geqslant 5$. Let us prove that then it is also true for B_{n+1}. Because of (R_1), we need only to prove the result for $\beta \in B_{n+1}$ that can be written:

$$\beta = s_{i_1}^{\varepsilon_1} s_{i_2}^{\varepsilon_2} s_{i_3}^{\varepsilon_3} \ldots \ , \varepsilon_i = \pm 1. \tag{42}$$

Like in the four strand case, we prove the result for words by structural induction on $\#_n(\beta)$.

The result is clearly true when $\#_n(\beta) \leqslant 1$. For β such that $\#_n(\beta) \geqslant 2$, we can write $\beta = x s_n^{\pm 1} w s_n^{\pm 1} y$ with $w, y \in \langle s_1, s_2, \ldots, s_{n-1} \rangle \cong B_n$. Using the induction hypothesis, w is a $\mathbb{Q}(t_0, t_1)$-linear combination of words in the form prescribed by Theorem 1.3. In the following we assume that w is given in the form of one of these spanning words.
$\boxed{\text{Case 1: } \#_n(w) \leqslant 1.}$ Like in the four strand case, we can write $w = a s_{n-1}^{\varepsilon} b$ with $\varepsilon \in \{-1, 0, 1\}$ and $a, b \in \langle s_1, ..., s_{n-2} \rangle$. Therefore

$$\beta = x' s_n^{\pm 1} s_{n-1}^{\varepsilon} s_n^{\pm 1} y' \quad \text{with } y' \in \langle s_1, s_2, \ldots, s_{n-1} \rangle \text{ and } x' \in B_{n+1}. \tag{43}$$

Again following the ideas of the B_4 case, this word reduces to match the description of it given in Theorem 1.3. The proof is completely similar because $s_1, s_2, \ldots, s_{n-3}$ commute with s_{n-1} and s_n.

$\boxed{\text{Case 2: } w = u\overline{s}_{n-1}s_{n-2}\overline{s}_{n-1},\ u \in \langle s_1, \ldots, s_{n-3}\rangle.}$ In this case we can write:

$$\beta = x' s_n^{\pm 1} \overline{s}_{n-1} s_{n-2} \overline{s}_{n-1} s_n^{\pm 1} y \quad \text{for some } x' \in \langle s_1, s_2, \ldots, s_{n-1}\rangle. \tag{44}$$

The conclusion follows like in the B_4 case. This concludes the proof of Theorem 1.3. $\square$

2.4. **How (R_3) was found.** Once (R_3) is found, Lemma 1.2 follows by a machine computation. But this also gives a method to find an (R_3). Namely, (R_3) is derived from the realization of $s_1\overline{s}_3s_2\overline{s}_3 - \overline{s}_3s_2\overline{s}_3s_1 \in C_4$ as a $\mathbb{Q}(t_0, t_1)$-linear combination of the 175 words in a spanning set for a version of C_4 given in [MW13]. We can use the explicit R-matrices of LG or V_1 and the corresponding $\mathbb{Q}(t_0, t_1)$-linear map $C_4 \to \mathrm{End}(V^{\otimes 4})$ for a 4-dimensional vector space V to reduce this to a linear algebra question over $\mathbb{Q}(t_0, t_1)$. This results in solving a system of $4^8 = 65536$ sparse linear equations in 175 variables.

To reduce the complexity of the task we used the R-matrix R_V for V_1 rather than R_{LG} since the former appeared to have simpler coefficients. Then, we solved the sparse linear system of equations for a sample of 20 different values of the pair (t_0, t_1). Doing this, we found that only 78 of the 175 unknown variables are rational functions with nonzero specializations, which reflects the sparsity of the system, and moreover, the set of these 78 variables was the same for all attempted specializations. We thus reduced the system of unknowns from 175 to the 78 ones found above, and then by the use of a computer and some by-hand eliminations, we found the unique solution given in the appendix.

This produced a potential 80-term (R_3) skein relation that we then checked was satisfied for both R-matrices involved.

A `Mathematica` program that includes the R-matrices of the LG, V_1 and ADO_ω-polynomials and checks that they satisfy the skein relations is given in [Gar].

3. Basics of the LG, V_1 and ADO_ω link invariants

The three polynomial invariants of links that we study in our paper, namely the LG, V_1 and ADO_ω come from the well-known Reshetikhin–Turaev construction [RT90, Tur94] applied to enhanced matrices given below.

Instead of repeating definitions and notations from previous works and arguments that we will not use, we comment briefly how these invariants are defined following [GHK+] and references therein.

A rigid R-matrix leads to invariants of long knots [Kas23] and Nichols algebras with automorphisms (or suitable finite dimensional quotients thereof) produce rigid R-matrices and hence invariants of long knots [GK].

All three polynomial invariants come from rigid R-matrices, and in fact from enhanced ones, in the sense of Ohtsuki and Turaev [Oht02, Tur88]. For a precise definition see [GHK+, Sec.2], where an extension to tangles is given, and a comparison of the various definitions is also discussed.

A common feature of the invariants of tangles given by the three polynomials that we study is that they vanish when evaluated to closed links, since in a sense all three are fermionic invariants.

To overcome this problem and define a nontrivial invariant of oriented links, one cuts one component to obtain a $(1,1)$-tangle, and then shows that the invariant of $(1,1)$-tangles is a scalar (so-called property (P_1) in [GHK$^+$, Sec.1.1]), and then that an invariant of a $(2,2)$-tangle is unchanged if we close it on one or the other side (property (P_2)).

All three tangle invariants satisfy properties (P_1) and (P_2) and consequently give well-defined invariants of oriented links. Although these link invariants are highly nontrivial, they do vanish on split links.

Note that in dealing with the Links–Gould invariant of links, we stick to conventions used by Ishii for example in [Ish06]. In doing so, $\mathrm{LG}_L(p^{-2}, p^2q^2)$ with $p = q^\alpha$ coincides with the Links-Gould invariant from [DWLK99].

Having discussed the basic properties of the three invariants of interest, we give their R-matrices in Appendix A, and their enhancements here:

$$\begin{aligned} h_{\mathrm{LG}} &= \mathrm{diag}(t_0^{-1}, -t_1, -t_0^{-1}, t_1) \in \mathrm{End}(W) \\ h_V &= \mathrm{diag}(-1, 1, 1, -1) \in \mathrm{End}(V) \\ h_{\mathrm{ADO}} &= \mathrm{diag}(t^2, \omega^2 t^2, \omega^4 t^2) \in \mathrm{End}(X)\,. \end{aligned} \tag{45}$$

Lemma 3.1. $(R_{\mathrm{LG}}, h_{\mathrm{LG}})$, (R_V, h_V) and $(R_{\mathrm{ADO}}, h_{\mathrm{ADO}})$ are enhanced R-matrices and satisfy properties (P_1) and (P_2) of [GHK$^+$, Sec.1.1].

The statement about enhancement follows by an explicit computation. Regarding properties (P_1) and (P_2), (R_V, h_V) satisfies them as was shown in [GHK$^+$, Sec.3]. So do $(R_{\mathrm{LG}}, h_{\mathrm{LG}})$ and $(R_{\mathrm{ADO}}, h_{\mathrm{ADO}})$ since they are defined representation theoretically via a ribbon category and the tangle invariants are colored by a simple ambidextrous object in the sense of [GPMT09].

The link invariants can be computed in terms of a braid presentation $\beta \in B_n$ of an oriented link L as stated in [GHK$^+$, Rem.2.3] and for the convenience of the reader, we reproduce here:

$$\begin{aligned} F_{L^{\mathrm{cut}}} &= \mathrm{tr}_{2,\dots,n}\left((\mathrm{id}_V \otimes h^{\otimes(n-1)}) \circ \rho_R(\beta)\right) \in \mathrm{End}(V) \\ \langle F_{L^{\mathrm{cut}}} \rangle &= \frac{1}{\dim(V)} \mathrm{tr}\left((\mathrm{id}_V \otimes h^{\otimes(n-1)}) \circ \rho_R(\beta)\right)\,. \end{aligned} \tag{46}$$

As mentioned before, all three link invariants thus defined have the following common feature.

Lemma 3.2. The LG, V_1 and ADO_ω polynomials vanish on split links, and are equal to 1 on the unknot.

Proof. The vanishing on split links follows from the definition of the invariants and the fact that the diagonal matrices h_{LG}, h_V and h_{ADO} have trace zero. The value of the unknot, whose long version is a single vertical strand, is obvious. $\square$

Appendix A. The R-matrices for the Links–Gould, ADO_ω and V_1 polynomials

In this appendix we write the three R-matrices that we need in the paper.

Since all three R-matrices are sparse, we present them in the following way. Suppose V is a vector space over a field k with an ordered basis $(v_1,\dots,v_n)$ and an R matrix $R \in \mathrm{End}(V\otimes V)$. Abbreviating $v_{ij} = v_i \otimes v_j$ for $i,j = 1,\dots n$, the $n^2 \times n^2$ matrix R can be presented as an $n \times n$ matrix $\mathsf{R} := (R(x_{ij}))_{1\leqslant i,j\leqslant n}$ whose entries are k-linear combinations of v_{ij}.

We now give the three R-matrices, beginning with the Links–Gould polynomial whose R-matrix is defined as follows. Consider a 4-dimensional $\mathbb{Q}(t_0,t_1)$-vector space W with ordered basis (w_1,w_2,w_3,w_4). Abbreviating $w_{ij} = w_i \otimes w_j$ for $i,j = 1,\dots 4$, the R-matrix $\mathsf{R}_{\mathrm{LG}} := ((R_{\mathrm{LG}})(w_{ij}))_{1\leqslant i,j\leqslant 4}$ is given by

$$\mathsf{R}_{\mathrm{LG}} = \begin{pmatrix} t_0 w_{11} & t_0^{1/2} w_{21} & t_0^{1/2} w_{31} & w_{41} \\ t_0^{1/2} w_{12} + (t_0-1)w_{21} & -w_{22} & (t_0t_1-1)w_{23} - t_0^{1/2}t_1^{1/2}w_{32} - t_0^{1/2}t_1^{1/2} Y w_{41} & t_1^{1/2} w_{42} \\ t_0^{1/2} w_{13} + (t_0-1)w_{31} & -t_0^{1/2}t_1^{1/2} w_{23} + Y w_{41} & -w_{33} & t_1^{1/2} w_{43} \\ w_{14} - t_0^{1/2}t_1^{1/2} Y w_{23} + Y w_{32} + Y^2 w_{41} & t_1^{1/2} w_{24} + (t_1-1) w_{42} & t_1^{1/2} w_{34} + (t_1-1)w_{43} & t_1 w_{44} \end{pmatrix}$$

with $Y = \sqrt{(t_0-1)(1-t_1)}$. Actually, the entries in the above matrix are in the quadratic extension $\mathbb{Q}(t_0,t_1)[Y]$ of the field $\mathbb{Q}(t_0,t_1)$ but this plays no important role in our arguments.

Next we give the R-matrix of the V_1-polynomial whose explicit computation was discussed in [GK] and further studied in [GHK+]. We consider a 4-dimensional $\mathbb{Q}(t_0,t_1)$-vector space V with ordered basis (v_1,v_2,v_3,v_4). As before, with $v_{ij} = v_i \otimes v_j$, the R-matrix $\mathsf{R}_{V,r} := (R_{V,r}(v_{ij}))_{1\leqslant i,j\leqslant 4}$ is given by

$$\mathsf{R}_{V,r} = \begin{pmatrix} -v_{11} & -t_0 v_{21} & -t_1 v_{31} & -t_0 t_1 v_{41} \\ -v_{12} + (t_0-1)v_{21} & t_0 v_{22} & -r t_1 v_{32} + (t_0-1)t_1 v_{41} & r t_0 t_1 v_{42} \\ -v_{13} + (t_1-1)v_{31} & -r^{-1}t_1^{-1}v_{23} + r^{-1}(1-t_0)v_{41} & t_1 v_{33} & r^{-1} v_{43} \\ \begin{bmatrix} -v_{14} + (t_1^{-1}-1)v_{23} \\ +r(t_1-1)v_{32} + (t_0+t_1-2)v_{41} \end{bmatrix} & r^{-1}t_1^{-1} v_{24} + (t_0-1)v_{42} & r t_1 v_{34} + (t_1-1)v_{43} & -v_{44} \end{pmatrix}.$$

Taking $r = 1$, the R-matrix $R_V := R_{V,1}$ has an enhancement as was explained in [GHK+, Sec.3].

Lastly, we give the R-matrix used to define the ADO_ω invariant of links. With $\omega = e^{2\pi i/6}$, consider a 3-dimensional $\mathbb{Q}(\omega,t)$-vector space X with an ordered basis (x_0,x_1,x_2). With $x_{ij} = x_i \otimes x_j$, the 3×3 R-matrix $\mathsf{R}_{\mathrm{ADO}} := (R_{\mathrm{ADO}}(x_{ij}))_{1\leqslant i,j\leqslant 3}$ is given by

$$\mathsf{R}_{\mathrm{ADO}} = \begin{pmatrix} t^2 x_{00} & (t^2-1)x_{01} + t x_{10} & (t^2-1)(1-\omega^2 t^{-2})x_{02} + (t^{-1}+\omega t)x_{11} + x_{20} \\ t x_{01} & (t-t^{-1})x_{02} + \omega^2 x_{11} & (\omega^2 t^{-2} - 1)x_{12} + -\omega t^{-1} x_{21} \\ x_{02} & -\omega t^{-1} x_{12} & \omega^2 t^{-2} x_{22} \end{pmatrix}.$$

Appendix B. The coefficients of the (R_3) skein relation

In this section we give the coefficients of the (R_3)-skein relation.

$$a_1 = -\frac{(t_1-1)(t_0-1)(-t_1-t_0-2t_1t_0-t_1^2t_0-t_1t_0^2+t_1^3t_0^2+t_1^2t_0^3)}{t_1t_0(t_1+t_0)(t_1t_0-1)(1+t_1t_0)},$$

$$a_2 = -\frac{(t_1-1)(t_0-1)(t_1+t_0+2t_1t_0)}{(t_1+t_0)(t_1t_0-1)(1+t_1t_0)},$$

$$a_3 = \frac{(t_1-1)(t_0-1)(-t_1-t_0-2t_1t_0-t_1^2t_0-t_1t_0^2+t_1^3t_0^2+t_1^2t_0^3)}{t_1t_0(t_1+t_0)(t_1t_0-1)(1+t_1t_0)},$$
$$a_4 = \frac{(t_1-1)(t_0-1)(t_1+t_0+2t_1t_0)}{(t_1+t_0)(t_1t_0-1)(1+t_1t_0)},$$
$$a_5 = \frac{1+t_1t_0+t_1^2t_0+t_1t_0^2}{t_1(1+t_1)t_0(1+t_0)(1+t_1t_0)},$$
$$a_6 = -\frac{t_1+t_0+t_1t_0+t_1^2t_0^2}{t_1(1+t_1)t_0(1+t_0)(1+t_1t_0)},$$
$$a_7 = -\frac{(1+t_1t_0+t_1^2t_0+t_1t_0^2)(t_1-t_1^2+t_0-t_1t_0+t_1^2t_0-t_0^2+t_1t_0^2+t_1^2t_0^2)}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(1+t_1t_0)},$$
$$a_8 = \frac{(t_1+t_0+t_1t_0+t_1^2t_0^2)(t_1-t_1^2+t_0-t_1t_0+t_1^2t_0-t_0^2+t_1t_0^2+t_1^2t_0^2)}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(1+t_1t_0)},$$
$$a_9 = -\frac{1+t_1t_0+t_1^2t_0+t_1t_0^2}{t_1(1+t_1)t_0(1+t_0)(1+t_1t_0)},$$
$$a_{10} = \frac{(1+t_1t_0+t_1^2t_0+t_1t_0^2)(t_1-t_1^2+t_0-t_1t_0+t_1^2t_0-t_0^2+t_1t_0^2+t_1^2t_0^2)}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(1+t_1t_0)},$$
$$a_{11} = \frac{t_1+t_0+t_1t_0+t_1^2t_0^2}{t_1(1+t_1)t_0(1+t_0)(1+t_1t_0)},$$
$$a_{12} = -\frac{(t_1+t_0+t_1t_0+t_1^2t_0^2)(t_1-t_1^2+t_0-t_1t_0+t_1^2t_0-t_0^2+t_1t_0^2+t_1^2t_0^2)}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(1+t_1t_0)},$$
$$a_{13} = \frac{-t_1^2-t_1t_0-t_1^2t_0-t_0^2-t_1t_0^2+t_1^2t_0^2}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(1+t_1t_0)},$$
$$a_{14} = \frac{-t_1+t_1^2-t_0+t_1t_0+t_1^2t_0+t_0^2+t_1t_0^2-t_1^2t_0^2+2t_1^3t_0^2+t_1^4t_0^2+2t_1^2t_0^3+2t_1^3t_0^3+t_1^4t_0^3+t_1^2t_0^4+t_1^3t_0^4}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(1+t_1t_0)},$$
$$a_{15} = -\frac{-t_1^3-t_1t_0-t_1^4t_0-t_1^2t_0^2-t_0^3+t_1^4t_0^3-t_1t_0^4+t_1^3t_0^4}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(1+t_1t_0)},$$
$$a_{16} = -\frac{-t_1^2+t_1^3-t_1t_0+t_1^2t_0+t_1^3t_0-t_0^2+t_1t_0^2+3t_1^2t_0^2+t_1^4t_0^2+t_0^3+t_1t_0^3+2t_1^3t_0^3+t_1^4t_0^3+t_1^2t_0^4+t_1^3t_0^4}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(1+t_1t_0)},$$
$$a_{17} = -\frac{1+t_1t_0+t_1^2t_0+t_1t_0^2}{(1+t_1)(1+t_0)(t_1+t_0)(1+t_1t_0)},$$
$$a_{18} = -\frac{(t_1+t_0-1)(1+t_1t_0+t_1^2t_0+t_1t_0^2)}{(1+t_1)(1+t_0)(t_1+t_0)(1+t_1t_0)},$$
$$a_{19} = \frac{t_1+t_0+t_1t_0+t_1^2t_0^2}{(1+t_1)(1+t_0)(t_1+t_0)(1+t_1t_0)},$$
$$a_{20} = \frac{(t_1+t_0-1)(t_1+t_0+t_1t_0+t_1^2t_0^2)}{(1+t_1)(1+t_0)(t_1+t_0)(1+t_1t_0)},$$
$$a_{21} = -\frac{t_1+t_0+t_1t_0+t_1^2t_0^2}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(1+t_1t_0)},$$
$$a_{22} = -\frac{-t_1^2-t_1t_0-t_1^4t_0-t_0^2+t_1^2t_0^2-t_1^3t_0^2-t_1^2t_0^3+t_1^4t_0^3-t_1t_0^4+t_1^3t_0^4}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(1+t_1t_0)},$$
$$a_{23} = \frac{t_1-t_1^2+t_0-t_1t_0-t_1^4t_0-t_0^2+t_1^2t_0^2-2t_1^3t_0^2-2t_1^2t_0^3+t_1^4t_0^3-t_1t_0^4+t_1^3t_0^4}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(1+t_1t_0)},$$
$$a_{24} = -\frac{(t_1+t_0-1)(-t_1^2-t_1t_0-t_1^2t_0-t_0^2-t_1t_0^2+t_1^2t_0^2)}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(1+t_1t_0)},$$
$$a_{25} = \frac{1+t_1t_0+t_1^2t_0+t_1t_0^2}{(1+t_1)(1+t_0)(t_1+t_0)(1+t_1t_0)},$$
$$a_{26} = -\frac{t_1+t_0+t_1t_0+t_1^2t_0^2}{(1+t_1)(1+t_0)(t_1+t_0)(1+t_1t_0)},$$
$$a_{27} = \frac{(t_1+t_0-1)(1+t_1t_0+t_1^2t_0+t_1t_0^2)}{(1+t_1)(1+t_0)(t_1+t_0)(1+t_1t_0)},$$

$$a_{28} = -\frac{(t_1+t_0-1)(t_1+t_0+t_1t_0+t_1^2t_0^2)}{(1+t_1)(1+t_0)(t_1+t_0)(1+t_1t_0)}\,,$$
$$a_{29} = \frac{(1+t_1)(1+t_0)}{(t_1+t_0)(t_1t_0-1)(1+t_1t_0)}\,,$$
$$a_{30} = \frac{-1-2t_1+t_1^2-2t_0+t_1t_0-t_1^3t_0+t_0^2-2t_1^2t_0^2+t_1^3t_0^2-t_1t_0^3+t_1^2t_0^3}{(t_1+t_0)(t_1t_0-1)(1+t_1t_0)}\,,$$
$$a_{31} = \frac{(1+t_1t_0+t_1^2t_0+t_1t_0^2)(-t_1^2-t_1t_0-t_1^2t_0-t_0^2-t_1t_0^2+t_1^2t_0^2)}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(t_1t_0-1)(1+t_1t_0)}\,,$$
$$a_{32} = -\frac{(t_1+t_0+t_1t_0+t_1^2t_0^2)(-t_1^2-t_1t_0-t_1^2t_0-t_0^2-t_1t_0^2+t_1^2t_0^2)}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(t_1t_0-1)(1+t_1t_0)}\,,$$
$$a_{33} = -\frac{t_1+t_0+t_1t_0+t_1^2t_0+t_1^3t_0+t_1t_0^2+4t_1^2t_0^2-t_1^4t_0^2+t_1t_0^3-t_1^3t_0^3+t_1^4t_0^3+t_1^5t_0^3-t_1^2t_0^4+t_1^3t_0^4+2t_1^4t_0^4+t_1^5t_0^4+t_1^3t_0^5+t_1^4t_0^5}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(t_1t_0-1)(1+t_1t_0)}\,,$$
$$a_{34} = \frac{t_1^2+3t_1t_0+2t_1^2t_0-t_1^3t_0+t_0^2+2t_1t_0^2+t_1^3t_0^2-t_1t_0^3+t_1^2t_0^3+t_1^3t_0^3+t_1^5t_0^3+2t_1^4t_0^4+t_1^5t_0^4+t_1^3t_0^5+t_1^4t_0^5}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(t_1t_0-1)(1+t_1t_0)}\,,$$
$$a_{35} = \frac{(1+t_1t_0+t_1^2t_0+t_1t_0^2)(t_1^2+t_1t_0+t_0^2-t_1^2t_0^2+t_1^3t_0^2+t_1^2t_0^3)}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(t_1t_0-1)(1+t_1t_0)}\,,$$
$$a_{36} = -\frac{(t_1+t_0+t_1t_0+t_1^2t_0^2)(t_1^2+t_1t_0+t_0^2-t_1^2t_0^2+t_1^3t_0^2+t_1^2t_0^3)}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(t_1t_0-1)(1+t_1t_0)}\,,$$
$$a_{37} = -\frac{t_1+t_0+t_1t_0+t_1^2t_0^2}{t_1t_0(t_1+t_0)(t_1t_0-1)(1+t_1t_0)}\,,$$
$$a_{38} = -\frac{(1+t_1t_0+t_1^2t_0+t_1t_0^2)(-t_1^2-t_1t_0-t_1^2t_0-t_0^2-t_1t_0^2+t_1^2t_0^2)}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(t_1t_0-1)(1+t_1t_0)}\,,$$
$$a_{39} = \frac{t_1+t_0+t_1t_0+2t_1^2t_0+t_1^3t_0-t_1^4t_0+2t_1t_0^2+4t_1^2t_0^2-t_1^3t_0^2+t_1^5t_0^2+t_1t_0^3-t_1^2t_0^3+t_1^3t_0^3+3t_1^4t_0^3-t_1t_0^4+3t_1^3t_0^4-t_1^5t_0^4+t_1^2t_0^5-t_1^4t_0^5}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(t_1t_0-1)(1+t_1t_0)}\,,$$
$$a_{40} = -\frac{(1+t_1t_0+t_1^2t_0+t_1t_0^2)(t_1^2+t_1t_0+t_0^2-t_1^2t_0^2+t_1^3t_0^2+t_1^2t_0^3)}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(t_1t_0-1)(1+t_1t_0)}\,,$$
$$a_{41} = \frac{t_1^2+3t_1t_0-t_1^2t_0+t_0^2-t_1t_0^2+t_1^2t_0^2+t_1^3t_0^2-t_1^4t_0^2+t_1^2t_0^3-2t_1^3t_0^3+t_1^4t_0^3-t_1^2t_0^4+t_1^3t_0^4}{t_1t_0(t_1+t_0)(t_1t_0-1)(1+t_1t_0)}\,,$$
$$a_{42} = \frac{(t_1+t_0+t_1t_0+t_1^2t_0^2)(-t_1^2-t_1t_0-t_1^2t_0-t_0^2-t_1t_0^2+t_1^2t_0^2)}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(t_1t_0-1)(1+t_1t_0)}\,,$$
$$a_{43} = -\frac{t_1^2+3t_1t_0+2t_1^2t_0-t_1^3t_0+t_0^2+2t_1t_0^2+2t_1^3t_0^2+t_1^4t_0^2-t_1t_0^3+2t_1^2t_0^3+3t_1^3t_0^3+t_1^2t_0^4}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(t_1t_0-1)(1+t_1t_0)}\,,$$
$$a_{44} = \frac{(t_1+t_0+t_1t_0+t_1^2t_0^2)(t_1^2+t_1t_0+t_0^2-t_1^2t_0^2+t_1^3t_0^2+t_1^2t_0^3)}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(t_1t_0-1)(1+t_1t_0)}\,,$$
$$a_{45} = \frac{(1+t_1t_0+t_1^2t_0+t_1t_0^2)(t_1+t_0+t_1t_0+t_1^2t_0^2)}{(1+t_1)(1+t_0)(t_1+t_0)(t_1t_0-1)(1+t_1t_0)}\,,$$
$$a_{46} = -\frac{(t_1+t_0+t_1t_0+t_1^2t_0^2)^2}{(1+t_1)(1+t_0)(t_1+t_0)(t_1t_0-1)(1+t_1t_0)}\,,$$
$$a_{47} = \frac{1+t_1t_0+t_1^2t_0+t_1t_0^2}{(1+t_1)(1+t_0)(t_1+t_0)(1+t_1t_0)}\,,$$
$$a_{48} = -\frac{t_1+t_0+t_1t_0+t_1^2t_0^2}{(1+t_1)(1+t_0)(t_1+t_0)(1+t_1t_0)}\,,$$
$$a_{49} = -\frac{(t_1+t_0+2t_1t_0)(1+t_1t_0+t_1^2t_0+t_1t_0^2)}{(1+t_1)(1+t_0)(t_1+t_0)(t_1t_0-1)(1+t_1t_0)}\,,$$
$$a_{50} = \frac{(t_1+t_0+2t_1t_0)(t_1+t_0+t_1t_0+t_1^2t_0^2)}{(1+t_1)(1+t_0)(t_1+t_0)(t_1t_0-1)(1+t_1t_0)}\,,$$
$$a_{51} = \frac{(1+t_1t_0+t_1^2t_0+t_1t_0^2)(-t_1^2-t_1t_0-t_1^2t_0-t_0^2-t_1t_0^2+t_1^2t_0^2)}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(t_1t_0-1)(1+t_1t_0)}\,,$$
$$a_{52} = -\frac{(t_1+t_0+t_1t_0+t_1^2t_0^2)(-t_1^2-t_1t_0-t_1^2t_0-t_0^2-t_1t_0^2+t_1^2t_0^2)}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(t_1t_0-1)(1+t_1t_0)}\,,$$

$$a_{53} = -\frac{(t_1+t_0+t_1t_0)(1+t_1t_0+t_1^2t_0+t_1t_0^2)}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(1+t_1t_0)},$$
$$a_{54} = \frac{(t_1+t_0+t_1t_0)(t_1+t_0+t_1t_0+t_1^2t_0^2)}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(1+t_1t_0)},$$
$$a_{55} = \frac{(1+t_1t_0+t_1^2t_0+t_1t_0^2)(t_1^2+t_1^2t_0+t_0^2+t_1t_0^2)}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(t_1t_0-1)(1+t_1t_0)},$$
$$a_{56} = -\frac{(t_1^2+t_1^2t_0+t_0^2+t_1t_0^2)(t_1+t_0+t_1t_0+t_1^2t_0^2)}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(t_1t_0-1)(1+t_1t_0)},$$
$$a_{57} = -\frac{(1+t_1t_0+t_1^2t_0+t_1t_0^2)(t_1+t_0+t_1t_0+t_1^2t_0^2)}{(1+t_1)(1+t_0)(t_1+t_0)(t_1t_0-1)(1+t_1t_0)},$$
$$a_{58} = \frac{(t_1+t_0+t_1t_0+t_1^2t_0^2)^2}{(1+t_1)(1+t_0)(t_1+t_0)(t_1t_0-1)(1+t_1t_0)},$$
$$a_{59} = -\frac{1+t_1t_0+t_1^2t_0+t_1t_0^2}{(1+t_1)(1+t_0)(t_1+t_0)(1+t_1t_0)},$$
$$a_{60} = \frac{t_1+t_0+t_1t_0+t_1^2t_0^2}{(1+t_1)(1+t_0)(t_1+t_0)(1+t_1t_0)},$$
$$a_{61} = \frac{(t_1+t_0+2t_1t_0)(1+t_1t_0+t_1^2t_0+t_1t_0^2)}{(1+t_1)(1+t_0)(t_1+t_0)(t_1t_0-1)(1+t_1t_0)},$$
$$a_{62} = -\frac{(t_1+t_0+2t_1t_0)(t_1+t_0+t_1t_0+t_1^2t_0^2)}{(1+t_1)(1+t_0)(t_1+t_0)(t_1t_0-1)(1+t_1t_0)},$$
$$a_{63} = -\frac{(1+t_1t_0+t_1^2t_0+t_1t_0^2)(-t_1^2-t_1t_0-t_1^2t_0-t_0^2-t_1t_0^2+t_1^2t_0^2)}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(t_1t_0-1)(1+t_1t_0)},$$
$$a_{64} = \frac{(t_1+t_0+t_1t_0+t_1^2t_0^2)(-t_1^2-t_1t_0-t_1^2t_0-t_0^2-t_1t_0^2+t_1^2t_0^2)}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(t_1t_0-1)(1+t_1t_0)},$$
$$a_{65} = \frac{(t_1+t_0+t_1t_0)(1+t_1t_0+t_1^2t_0+t_1t_0^2)}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(1+t_1t_0)},$$
$$a_{66} = -\frac{(t_1+t_0+t_1t_0)(t_1+t_0+t_1t_0+t_1^2t_0^2)}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(1+t_1t_0)},$$
$$a_{67} = -\frac{(1+t_1t_0+t_1^2t_0+t_1t_0^2)(t_1^2+t_1^2t_0+t_0^2+t_1t_0^2)}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(t_1t_0-1)(1+t_1t_0)},$$
$$a_{68} = \frac{(t_1^2+t_1^2t_0+t_0^2+t_1t_0^2)(t_1+t_0+t_1t_0+t_1^2t_0^2)}{t_1(1+t_1)t_0(1+t_0)(t_1+t_0)(t_1t_0-1)(1+t_1t_0)},$$
$$a_{69} = \frac{1}{t_1+t_0},$$
$$a_{70} = \frac{t_1t_0}{1+t_1t_0},$$
$$a_{71} = \frac{(t_1-1)(1+t_1)(t_0-1)(1+t_0)}{(t_1+t_0)(t_1t_0-1)(1+t_1t_0)},$$
$$a_{72} = -\frac{1}{1+t_1t_0},$$
$$a_{73} = \frac{t_1^2+t_0^2-2}{(t_1+t_0)(t_1t_0-1)(1+t_1t_0)},$$
$$a_{74} = -\frac{1}{t_1+t_0},$$
$$a_{75} = -\frac{t_1t_0}{1+t_1t_0},$$
$$a_{76} = -\frac{(t_1-1)(1+t_1)(t_0-1)(1+t_0)}{(t_1+t_0)(t_1t_0-1)(1+t_1t_0)},$$
$$a_{77} = \frac{1}{1+t_1t_0},$$

$$a_{78} = -\frac{t_1^2 + t_0^2 - 2}{(t_1 + t_0)(t_1 t_0 - 1)(1 + t_1 t_0)}\,.$$

Appendix C. Proof of Lemma 2.4

Here we prove Lemma 2.4. Note that it is enough to prove the result when the word has at most 4 letters. The general case follows by induction on the length of the word. Now, recall that for $\beta \in B_3$, because of (R_1), we need only to consider

$$\beta = s_1^{\varepsilon_1} s_2^{\varepsilon_2} s_1^{\varepsilon_3} \dots \text{ or } \beta = s_2^{\varepsilon_1} s_1^{\varepsilon_2} s_2^{\varepsilon_3} \dots\ ,\ \varepsilon_i = \pm 1.$$

So, taking all reductions into account, it is enough to prove the statement for the following list of $16 + 16 + 8 = 40$ words:

$$s_1^{\pm 1} s_2^{\pm 1} s_1^{\pm 1} s_2^{\pm 1}\,;\ s_2^{\pm 1} s_1^{\pm 1} s_2^{\pm 1} s_1^{\pm 1}\,;\ s_2^{\pm 1} s_1^{\pm 1} s_2^{\pm 1}.$$

We will express these words as linear combinations of:

- words of length at most 3 with at most one $s_2^{\pm 1}$,
- $\overline{s}_2 s_1 \overline{s}_2$.

To do that we can use the one and two letter relations from Lemmas 2.1 and 2.3. Starting with the 32 four letter words, let us see which ones reduce obviously to a linear combination of words with at most one $s_2^{\pm 1}$ and fewer letter words.

$s_1(s_2 s_1 s_2)$: reduces modulo braid relation and (R_1),
$(s_1 s_2 s_1)\overline{s}_2$: reduces modulo braid relation,
$(s_1 s_2 \overline{s}_1) s_2$: reduces modulo braid relation and (R_1),
$(s_1 s_2 \overline{s}_1)\overline{s}_2$: reduces modulo braid relation,
$s_1(\overline{s}_2 s_1 s_2)$: reduces modulo braid relation and (R_1),
$\boxed{s_1 \overline{s}_2 s_1 \overline{s}_2}$: harder, must be studied separately,
$(s_1 \overline{s}_2 \overline{s}_1) s_2$: reduces modulo braid relation and (R_1),
$(s_1 \overline{s}_2 \overline{s}_1)\overline{s}_2$: reduces modulo braid relation,
$(\overline{s}_1 s_2 s_1) s_2$: reduces modulo braid relation,
$(\overline{s}_1 s_2 s_1)\overline{s}_2$: reduces modulo braid relation and (R_1),
$\boxed{\overline{s}_1 s_2 \overline{s}_1 s_2}$: harder, must be studied separately,
$\overline{s}_1(s_2 \overline{s}_1 \overline{s}_2)$: reduces modulo braid relation and (R_1),
$\overline{s}_1(\overline{s}_2 s_1 s_2)$: reduces modulo braid relation,
$(\overline{s}_1 \overline{s}_2 s_1)\overline{s}_2$: reduces modulo braid relation and (R_1),
$(\overline{s}_1 \overline{s}_2 \overline{s}_1) s_2$: reduces modulo braid relation,
$(\overline{s}_1 \overline{s}_2 \overline{s}_1)\overline{s}_2$: reduces modulo braid relation and (R_1),
$(s_2 s_1 s_2) s_1$: reduces modulo braid relation and (R_1),
$s_2(s_1 s_2 \overline{s}_1)$: reduces modulo braid relation,
$(s_2 s_1 \overline{s}_2) s_1$: reduces modulo braid relation and (R_1),
$s_2(s_1 \overline{s}_2 \overline{s}_1)$: reduces modulo braid relation,
$s_2(\overline{s}_1 s_2 s_1)$: reduces modulo braid relation and (R_1),
$\boxed{s_2 \overline{s}_1 s_2 \overline{s}_1}$: harder, must be studied separately,
$(s_2 \overline{s}_1 \overline{s}_2) s_1$: reduces modulo braid relation and (R_1),
$(s_2 \overline{s}_1 \overline{s}_2)\overline{s}_1$: reduces modulo braid relation,
$(\overline{s}_2 s_1 s_2) s_1$: reduces modulo braid relation,
$(\overline{s}_2 s_1 s_2)\overline{s}_1$: reduces modulo braid relation and (R_1),
$\boxed{\overline{s}_2 s_1 \overline{s}_2 s_1}$: harder, must be studied separately,
$\overline{s}_2(s_1 \overline{s}_2 \overline{s}_1)$: reduces modulo braid relation and (R_1),
$\overline{s}_2(\overline{s}_1 s_2 s_1)$: reduces modulo braid relation,
$(\overline{s}_2 \overline{s}_1 s_2)\overline{s}_1$: reduces modulo braid relation and (R_1),
$\overline{s}_2(\overline{s}_1 \overline{s}_2 s_1)$: reduces modulo braid relation,
$(\overline{s}_2 \overline{s}_1 \overline{s}_2)\overline{s}_1$: reduces modulo braid relation and (R_1).

For the 8 three letter words, let us similarly see which ones reduce directly.

$s_2 s_1 s_2$: reduces modulo braid relation,
$s_2 s_1 \overline{s}_2$: reduces modulo braid relation,
$s_2 \overline{s}_1 s_2$: reduces using the equivalent version of (R_2) expressed in Lemma 2.1,
$s_2 \overline{s}_1 \overline{s}_2$: reduces modulo braid relation,

$\overline{s}_2 s_1 s_2$: reduces modulo braid relation,
$\overline{s}_2 s_1 \overline{s}_2$: reduced already,
$\overline{s}_2 \overline{s}_1 s_2$: reduces modulo braid relation,
$\overline{s}_2 \overline{s}_1 \overline{s}_2$: reduces modulo braid relation.

So 4 words remain to be studied more precisely:

$$s_1\overline{s}_2s_1\overline{s}_2\,;\ \overline{s}_1s_2\overline{s}_1s_2\,;\ s_2\overline{s}_1s_2\overline{s}_1\,;\ \overline{s}_2s_1\overline{s}_2s_1.$$

- $\boxed{\text{Case of } s_1\overline{s}_2s_1\overline{s}_2.}$ Let us compute $s_1\overline{s}_2 \cdot (R_2)$:

$$\begin{aligned}
s_1\overline{s}_2(s_1\overline{s}_2) &= -s_1\overline{s}_2(\overline{s}_1s_2s_1) + s_1\overline{s}_2(s_1s_2\overline{s}_1) + s_1\overline{s}_2(s_1s_2) - s_1\overline{s}_2(\overline{s}_1s_2)\\
&\quad + s_1\overline{s}_2(\overline{s}_1\overline{s}_2) - s_1\overline{s}_2(s_2s_1) + s_1\overline{s}_2(s_2\overline{s}_1) + s_1\overline{s}_2(\overline{s}_2s_1) - s_1\overline{s}_2(\overline{s}_2\overline{s}_1)\\
&\quad + s_1\overline{s}_2(\overline{s}_1\overline{s}_2s_1) - s_1\overline{s}_2(s_1\overline{s}_2\overline{s}_1)\\
&= -(s_1s_1)\overline{s}_2(\overline{s}_1s_1) + s_1(\overline{s}_2\overline{s}_2)s_1s_2 + (s_1s_1)s_2\overline{s}_1 - (s_1s_1)\overline{s}_2\overline{s}_1\\
&\quad + (s_1\overline{s}_1)\overline{s}_2\overline{s}_1 - (s_1s_1) + 1\\
&\quad + s_1((t_0^{-1}+t_1^{-1}-1)\,\overline{s}_2 + (t_0^{-1}+t_1^{-1}-t_0^{-1}t_1^{-1})\,1 - t_0^{-1}t_1^{-1}\,s_2)\,s_1\\
&\quad - s_1((t_0^{-1}+t_1^{-1}-1)\,\overline{s}_2 + (t_0^{-1}+t_1^{-1}-t_0^{-1}t_1^{-1})\,1 - t_0^{-1}t_1^{-1}\,s_2)\,\overline{s}_1\\
&\quad + (s_1\overline{s}_1)\overline{s}_2(\overline{s}_1s_1) - s_1(\overline{s}_2\overline{s}_2)\overline{s}_1s_2\\
&= -((t_0+t_1-1)\,s_1 + (t_0+t_1-t_0t_1)\,1 - t_0t_1\,\overline{s}_1)\,\overline{s}_2\\
&\quad + s_1((t_0^{-1}+t_1^{-1}-1)\,\overline{s}_2 + (t_0^{-1}+t_1^{-1}-t_0^{-1}t_1^{-1})\,1 - t_0^{-1}t_1^{-1}\,s_2)\,s_1s_2\\
&\quad + ((t_0+t_1-1)\,s_1 + (t_0+t_1-t_0t_1)\,1 - t_0t_1\,\overline{s}_1)\,s_2\overline{s}_1\\
&\quad - ((t_0+t_1-1)\,s_1 + (t_0+t_1-t_0t_1)\,1 - t_0t_1\,\overline{s}_1)\,\overline{s}_2\overline{s}_1\\
&\quad + \overline{s}_2\overline{s}_1 - ((t_0+t_1-1)\,s_1 + (t_0+t_1-t_0t_1)\,1 - t_0t_1\,\overline{s}_1)\\
&\quad + 1 + (t_0^{-1}+t_1^{-1}-1)\,s_1\overline{s}_2s_1 + (t_0^{-1}+t_1^{-1}-t_0^{-1}t_1^{-1})\,s_1s_1 - t_0^{-1}t_1^{-1}\,s_1s_2s_1\\
&\quad - (t_0^{-1}+t_1^{-1}-1)\,s_1\overline{s}_2\overline{s}_1 - (t_0^{-1}+t_1^{-1}-t_0^{-1}t_1^{-1})\,1 + t_0^{-1}t_1^{-1}\,s_1s_2\overline{s}_1\\
&\quad - \overline{s}_2 - s_1((t_0^{-1}+t_1^{-1}-1)\,\overline{s}_2 + (t_0^{-1}+t_1^{-1}-t_0^{-1}t_1^{-1})\,1 - t_0^{-1}t_1^{-1}\,s_2)\,\overline{s}_1s_2\\
&= -(t_0+t_1-1)\,s_1\overline{s}_2 - (t_0+t_1-t_0t_1)\,\overline{s}_2 + t_0t_1\,\overline{s}_1\overline{s}_2\\
&\quad + (t_0^{-1}+t_1^{-1}-1)\,s_1(\overline{s}_2s_1s_2) + (t_0^{-1}+t_1^{-1}-t_0^{-1}t_1^{-1})\,(s_1s_1)s_2 - t_0^{-1}t_1^{-1}\,s_1(s_2s_1s_2)\\
&\quad + (t_0+t_1-1)\,s_1s_2\overline{s}_1 + (t_0+t_1-t_0t_1)\,s_2\overline{s}_1 - t_0t_1\,\overline{s}_1s_2\overline{s}_1\\
&\quad - (t_0+t_1-1)\,s_1\overline{s}_2\overline{s}_1 - (t_0+t_1-t_0t_1)\,\overline{s}_2\overline{s}_1 + t_0t_1\,\overline{s}_1\overline{s}_2\overline{s}_1 + \overline{s}_2\overline{s}_1\\
&\quad - (t_0+t_1-1)\,s_1 - (t_0+t_1-t_0t_1)\,1 + t_0t_1\,\overline{s}_1 + 1 + (t_0^{-1}+t_1^{-1}-1)\,s_1\overline{s}_2s_1\\
&\quad + (t_0^{-1}+t_1^{-1}-t_0^{-1}t_1^{-1})((t_0+t_1-1)\,s_1 + (t_0+t_1-t_0t_1)\,1 - t_0t_1\,\overline{s}_1)\\
&\quad - t_0^{-1}t_1^{-1}\,s_1s_2s_1 - (t_0^{-1}+t_1^{-1}-1)\,s_1\overline{s}_2\overline{s}_1 - (t_0^{-1}+t_1^{-1}-t_0^{-1}t_1^{-1})\,1\\
&\quad + t_0^{-1}t_1^{-1}\,s_1s_2\overline{s}_1 + \overline{s}_2 - (t_0^{-1}+t_1^{-1}-1)\,s_1(\overline{s}_2\overline{s}_1s_2)\\
&\quad - (t_0^{-1}+t_1^{-1}-t_0^{-1}t_1^{-1})\,s_2 + t_0^{-1}t_1^{-1}\,(s_1s_2\overline{s}_1)s_2\,.
\end{aligned}$$

If we then expand all the terms in the sum and group together those that are multiples of the same braid word, we find that:

$$\begin{aligned} s_1\overline{s}_2(s_1\overline{s}_2) = {} & (t_0+t_1-t_0t_1-1)(t_0^{-1}+t_1^{-1}-t_0^{-1}t_1^{-1}-1)\,1 \\ & + (t_0+t_1-1)(t_0^{-1}+t_1^{-1}-t_0^{-1}t_1^{-1}-1)\,s_1 + (t_0+t_1-t_0t_1-1)(t_0^{-1}+t_1^{-1}-t_0^{-1}t_1^{-1})\,s_2 \\ & - (t_0+t_1-t_0t_1-1)\,\overline{s}_1 - (t_0+t_1-t_0t_1-1)\,\overline{s}_2 \\ & + (t_0+t_1-1)(t_0^{-1}+t_1^{-1}-t_0^{-1}t_1^{-1})\,s_1s_2 - (t_0+t_1-1)\,s_1\overline{s}_2 - (t_0+t_1-1)\,\overline{s}_1s_2 \\ & + t_0t_1\,\overline{s}_1\overline{s}_2 - (t_0^{-1}+t_1^{-1}-1)\,s_2s_1 + (t_0^{-1}+t_1^{-1})(t_0+t_1-t_0t_1)\,s_2\overline{s}_1 \\ & + (t_0^{-1}+t_1^{-1}-1)\,\overline{s}_2s_1 + (t_0+t_1-t_0t_1^{-1}-t_0^{-1}t_1-1)\,\overline{s}_2\overline{s}_1 - (t_0^{-1}+t_1^{-1})\,s_1s_2s_1 \\ & + (t_0+t_1)(t_0^{-1}+t_1^{-1})\,s_1s_2\overline{s}_1 + (t_0^{-1}+t_1^{-1}-1)\,s_1\overline{s}_2s_1 - (t_0^{-1}t_1+t_0t_1^{-1}+1)\,s_1\overline{s}_2\overline{s}_1 \\ & + \overline{s}_1s_2s_1 - (t_0+t_1)\,\overline{s}_1s_2\overline{s}_1 + (t_0+t_1)\,\overline{s}_1\overline{s}_2\overline{s}_1 - \overline{s}_2s_1\overline{s}_2\,. \end{aligned}$$

- $\boxed{\text{Case of } \overline{s}_1s_2\overline{s}_1s_2.}$ Like in the previous case, one can compute $(R_2)\cdot s_1s_2$ to find that:

$$\begin{aligned} \overline{s}_1s_2\overline{s}_1s_2 = {} & (t_0+t_1-t_0t_1-1)(t_0^{-1}+t_1^{-1}-t_0^{-1}t_1^{-1}-1)\,1 + (1-t_0^{-1}t_1^{-1})(t_0+t_1-t_0t_1-1)\,s_1 \\ & + (t_0^{-1}+t_1^{-1}-2)(t_0+t_1-t_0t_1-1)\,\overline{s}_1 + (t_0^{-1}+t_1^{-1}-t_0^{-1}t_1^{-1}-1)(t_0+t_1-1)\,s_2 \\ & - (t_0+t_1-t_0t_1-1)\,\overline{s}_2 + (1-t_0^{-1}t_1^{-1})(t_0+t_1-1)\,s_1s_2 + (1-t_0t_1)\,s_1\overline{s}_2 \\ & + (t_0^{-1}+t_1^{-1}-2)(t_0+t_1-1)\,\overline{s}_1s_2 - (t_0+t_1-2t_0t_1)\,\overline{s}_1\overline{s}_2 + (t_0+t_1-t_0t_1)\,\overline{s}_2s_1 \\ & - (t_0+t_1-t_0t_1-1)\overline{s}_2\overline{s}_1 - s_1s_2s_1 + (t_0+t_1)\,s_1s_2\overline{s}_1 - (t_0+t_1-1)\,s_1\overline{s}_2\overline{s}_1 \\ & + \overline{s}_1s_2s_1 - \overline{s}_1s_2\overline{s}_1 + t_0t_1\,\overline{s}_1\overline{s}_2\overline{s}_1 - t_0t_1\,\overline{s}_2s_1\overline{s}_2\,. \end{aligned}$$

- $\boxed{\text{Case of } s_2\overline{s}_1s_2\overline{s}_1.}$ If we simplify $s_2\overline{s}_1\cdot(R_2)$ we get the following expression:

$$\begin{aligned} s_2\overline{s}_1s_2\overline{s}_1 = {} & (t_0+t_1-t_0t_1-1)(t_0^{-1}+t_1^{-1}-t_0^{-1}t_1^{-1}-1)\,1 + (1-t_0^{-1}t_1^{-1})(t_0+t_1-t_0t_1-1)\,s_1 \\ & + (t_0^{-1}+t_1^{-1}-2)(t_0+t_1-t_0t_1-1)\,\overline{s}_1 + (t_0^{-1}+t_1^{-1}-t_0^{-1}t_1^{-1}-1)(t_0+t_1-1)\,s_2 \\ & - (t_0+t_1-t_0t_1-1)\,\overline{s}_2 - (t_0^{-1}+t_1^{-1}-1)(t_0+t_1)\,s_1s_2 \\ & + (t_0^{-1}+t_1^{-1}+1)(t_0+t_1-t_0t_1)\,s_1\overline{s}_2 + (t_0^{-1}+t_1^{-1}-1)(t_0+t_1)\,\overline{s}_1s_2 \\ & + (1-(t_0^{-1}+t_1^{-1}+1)(t_0+t_1-t_0t_1))\,\overline{s}_1\overline{s}_2 + (1+t_0^{-1}t_1+t_0t_1^{-1}-t_0^{-1}-t_1^{-1}+t_0^{-1}t_1^{-1})\,s_2s_1 \\ & - (t_0+t_0^{-1}+t_1+t_1^{-1}-2)\,s_2\overline{s}_1 - (1+t_0^{-1}t_1+t_0t_1^{-1}-t_0-t_1+t_0t_1)\,\overline{s}_2s_1 \\ & + t_0t_1(1+(t_0^{-1}+t_1^{-1}-1)^2)\overline{s}_2\overline{s}_1 - s_1s_2s_1 + (1-(t_0^{-1}+t_1^{-1}-1)(t_0+t_1))\,s_1s_2\overline{s}_1 \\ & + (t_0^{-1}+t_1^{-1}-1)(t_0+t_1)\,s_1\overline{s}_2\overline{s}_1 + (t_0^{-1}+t_1^{-1})(t_0+t_1)\,\overline{s}_1s_2s_1 - \overline{s}_1s_2\overline{s}_1 \\ & - (t_0^{-1}t_1+t_0t_1^{-1}+1)\overline{s}_1\overline{s}_2s_1 + t_0t_1\,\overline{s}_1\overline{s}_2\overline{s}_1 - t_0t_1\,\overline{s}_2s_1\overline{s}_2\,. \end{aligned}$$

- $\boxed{\text{Case of } \overline{s}_2s_1\overline{s}_2s_1.}$ Finally, if we write $\overline{s}_2s_1\cdot(R_2)$ we get:

$$\begin{aligned} \overline{s}_2s_1\overline{s}_2s_1 = {} & (t_0+t_1-t_0t_1-1)(t_0^{-1}+t_1^{-1}-t_0^{-1}t_1^{-1}-1)\,1 + (t_0+t_1-1)(t_0^{-1}+t_1^{-1}-t_0^{-1}t_1^{-1}-1)\,s_1 \\ & - (t_0+t_1-t_0t_1-1)\,\overline{s}_1 - (t_0^{-1}+t_1^{-1}-t_0^{-1}t_1^{-1}-1)\,s_2 \\ & + (t_0+t_1-t_0t_1-1)(t_0^{-1}+t_1^{-1}-1)\,\overline{s}_2 + s_1s_2 - s_1\overline{s}_2 + \overline{s}_1\overline{s}_2 \\ & - (t_0^{-1}+t_1^{-1}-t_0^{-1}t_1^{-1})\,s_2s_1 + s_2\overline{s}_1 + (t_0+t_1-1)(t_0^{-1}+t_1^{-1}-1)\,\overline{s}_2s_1 \end{aligned}$$

$$- (t_0 + t_1 - t_0t_1)\overline{s}_2\overline{s}_1 + s_1s_2\overline{s}_1 - s_1\overline{s}_2s_1 + \overline{s}_1\overline{s}_2s_1 - \overline{s}_2s_1\overline{s}_2\,.$$

This ends the proof of Lemma 2.4.

Appendix D. Proof of Lemma 2.6

Let us show that the following four words

$$s_3\overline{s}_2s_1\overline{s}_2s_3\ ,\ s_3\overline{s}_2s_1\overline{s}_2\overline{s}_3\ ,\ \overline{s}_3\overline{s}_2s_1\overline{s}_2s_3 \text{ and } \overline{s}_3\overline{s}_2s_1\overline{s}_2\overline{s}_3 \in B_4$$

can be reduced to linear combinations of words of one of the following types:

- [Type 1] words with at most one $s_3^{\pm 1}$,
- [Type 2] $\overline{s}_3s_2\overline{s}_3$, $s_1\overline{s}_3s_2\overline{s}_3$ or $\overline{s}_1\overline{s}_3s_2\overline{s}_3$.

To do so we write some equations that will be useful eventually.

Proposition D.1. The following equations are true in C_4, modulo terms of Types 1 and 2:

$$s_3\overline{s}_2s_1\overline{s}_2s_3 = \overline{s}_3\overline{s}_2s_1\overline{s}_2s_3 - s_2s_3\overline{s}_2s_1\overline{s}_2s_3 + s_2\overline{s}_3\overline{s}_2s_1\overline{s}_2s_3, \tag{47}$$

$$\overline{s}_3\overline{s}_2s_1\overline{s}_2\overline{s}_3 = s_2s_3\overline{s}_2s_1\overline{s}_2\overline{s}_3 - s_2\overline{s}_3\overline{s}_2s_1\overline{s}_2\overline{s}_3 + s_3\overline{s}_2s_1\overline{s}_2\overline{s}_3. \tag{48}$$

Proof. We use relation (R_2) on the first two letters of certain words to find Equations (47) and (48). The following equalities are written modulo Type 1 and Type 2 terms.

$$\begin{aligned}(s_3\overline{s}_2)s_1\overline{s}_2s_3 = &-(\overline{s}_3s_2s_3)s_1\overline{s}_2s_3 + (s_3s_2\overline{s}_3)s_1\overline{s}_2s_3 + (s_3s_2)s_1\overline{s}_2s_3\\ &- (\overline{s}_3s_2)s_1\overline{s}_2s_3 + (\overline{s}_3\overline{s}_2)s_1\overline{s}_2s_3 - (s_2s_3)s_1\overline{s}_2s_3\\ &+ (s_2\overline{s}_3)s_1\overline{s}_2s_3 + (\overline{s}_2s_3)s_1\overline{s}_2s_3 - (\overline{s}_2\overline{s}_3)s_1\overline{s}_2s_3\\ &+ (\overline{s}_3\overline{s}_2s_3)s_1\overline{s}_2s_3 - (s_3\overline{s}_2\overline{s}_3)s_1\overline{s}_2s_3\,.\end{aligned}$$

Some of the terms can be simplified modulo Type 1 and Type 2 terms:

$$\begin{aligned}(s_3s_2\overline{s}_3)s_1\overline{s}_2s_3 &= \overline{s}_2s_3(s_2s_1\overline{s}_2)s_3 = \overline{s}_2s_3\overline{s}_1s_2s_1s_3 = \overline{s}_2\overline{s}_1(s_3s_2s_3)s_1 = \overline{s}_2\overline{s}_1s_2s_3s_2s_1\,,\\ s_3(s_2s_2\overline{s}_2)s_3 &= s_3\overline{s}_1s_2s_1s_3 = \overline{s}_1(s_3s_2s_3)s_1 = \overline{s}_1s_2s_3s_2s_1\,,\\ \overline{s}_3(s_2s_1\overline{s}_2)s_3 &= \overline{s}_3\overline{s}_1s_2s_1s_3 = \overline{s}_1\overline{s}_3s_2s_3s_1 = \overline{s}_1s_2s_3\overline{s}_2s_1\,,\\ s_2s_3s_1\overline{s}_2s_3 &= s_2(s_1s_3\overline{s}_2s_3) \overset{(R_3)}{=} (s_2s_3\overline{s}_2s_3)s_1 \overset{\text{reduction in } \langle s_2,s_3\rangle}{=} \overline{s}_3s_2\overline{s}_3s_1 \overset{(R_3)}{=} s_1\overline{s}_3s_2\overline{s}_3\,,\\ s_2\overline{s}_3s_1\overline{s}_2s_3 &= s_2s_1(\overline{s}_3\overline{s}_2s_3) = s_2s_1s_2\overline{s}_3\overline{s}_2\,,\\ \overline{s}_2s_3s_1\overline{s}_2s_3 &= \overline{s}_2(s_1s_3\overline{s}_2s_3) \overset{(R_3)}{=} (\overline{s}_2s_3\overline{s}_2s_3)s_1 \overset{\text{reduction in } \langle s_2,s_3\rangle}{=} \overline{s}_3s_2\overline{s}_3s_1 \overset{(R_3)}{=} s_1\overline{s}_3s_2\overline{s}_3\,,\\ \overline{s}_2\overline{s}_3s_1\overline{s}_2s_3 &= \overline{s}_2s_1(\overline{s}_3\overline{s}_2s_3) = \overline{s}_2s_1s_2\overline{s}_3\overline{s}_2\,,\\ (s_3\overline{s}_2\overline{s}_3)s_1\overline{s}_2s_3 &= \overline{s}_2\overline{s}_3(s_2s_1\overline{s}_2)s_3 = \overline{s}_2\overline{s}_3\overline{s}_1s_2s_1s_3 = \overline{s}_2\overline{s}_1(\overline{s}_3s_2s_3)s_1 = \overline{s}_2\overline{s}_1s_2s_3\overline{s}_2s_1\,.\end{aligned}$$

So

$$\begin{aligned}s_3\overline{s}_2s_1\overline{s}_2s_3 &= -(\overline{s}_3s_2s_3)s_1\overline{s}_2s_3 + \overline{s}_3\overline{s}_2s_1\overline{s}_2s_3 + (\overline{s}_3\overline{s}_2s_3)s_1\overline{s}_2s_3\\ &= -s_2s_3\overline{s}_2s_1\overline{s}_2s_3 + \overline{s}_3\overline{s}_2s_1\overline{s}_2s_3 + s_2\overline{s}_3\overline{s}_2s_1\overline{s}_2s_3\,.\end{aligned}$$

This proves Equation (47).

$$\begin{aligned}(\overline{s}_3\overline{s}_2)s_1\overline{s}_2\overline{s}_3 = &-(\overline{s}_2s_3s_2)s_1\overline{s}_2\overline{s}_3 + (s_2s_3\overline{s}_2)s_1\overline{s}_2\overline{s}_3 + (s_2s_3)s_1\overline{s}_2\overline{s}_3\\ &- (s_2\overline{s}_3)s_1\overline{s}_2\overline{s}_3 - (\overline{s}_2s_3)s_1\overline{s}_2\overline{s}_3 + (\overline{s}_2\overline{s}_3)s_1\overline{s}_2\overline{s}_3\end{aligned}$$

$$- (s_3 s_2) s_1 \overline{s}_2 \overline{s}_3 + (s_3 \overline{s}_2) s_1 \overline{s}_2 \overline{s}_3 + (\overline{s}_3 s_2) s_1 \overline{s}_2 \overline{s}_3$$
$$+ (\overline{s}_2 \overline{s}_3 s_2) s_1 \overline{s}_2 \overline{s}_3 - (s_2 \overline{s}_3 \overline{s}_2) s_1 \overline{s}_2 \overline{s}_3 \,.$$

Like in the previous case, most terms can be reduced to linear combinations of Type 1 and Type 2 quantities: $\overline{s}_2 s_3 s_2 s_1 \overline{s}_2 \overline{s}_3$, $s_2 s_3 s_1 \overline{s}_2 \overline{s}_3$, $s_2 \overline{s}_3 s_1 \overline{s}_2 \overline{s}_3$, $\overline{s}_2 s_3 s_1 \overline{s}_2 \overline{s}_3$, $\overline{s}_2 \overline{s}_3 s_1 \overline{s}_2 \overline{s}_3$, $s_3 s_2 s_1 \overline{s}_2 \overline{s}_3$, $\overline{s}_3 s_2 s_1 \overline{s}_2 \overline{s}_3$ and $\overline{s}_2 \overline{s}_3 s_2 s_1 \overline{s}_2 \overline{s}_3$. And we can write:

$$\overline{s}_3 \overline{s}_2 s_1 \overline{s}_2 \overline{s}_3 = s_2 s_3 \overline{s}_2 s_1 \overline{s}_2 \overline{s}_3 + s_3 \overline{s}_2 s_1 \overline{s}_2 \overline{s}_3 - s_2 \overline{s}_3 \overline{s}_2 s_1 \overline{s}_2 \overline{s}_3 \,.$$

So Equation (48) holds. □

Proposition D.2. The words $s_3 \overline{s}_2 s_1 \overline{s}_2 \overline{s}_3$ and $\overline{s}_3 \overline{s}_2 s_1 \overline{s}_2 s_3$ reduce.

Proof. Let us start by writing an identity that will be useful subsequently. All equalities here are to be understood modulo Type 1 and Type 2 terms.

$$\begin{aligned} s_2 s_3 \overline{s}_2 s_1 (\overline{s}_2 s_3) \overset{(R_2)}{=} & -s_2 s_3 \overline{s}_2 s_1 (\overline{s}_2 s_3 s_2) + s_2 s_3 \overline{s}_2 s_1 (s_2 s_3 \overline{s}_2) + s_2 s_3 \overline{s}_2 s_1 (s_2 s_3) \\ & - s_2 s_3 \overline{s}_2 s_1 (s_2 \overline{s}_3) + s_2 s_3 \overline{s}_2 s_1 (\overline{s}_2 \overline{s}_3) - s_2 s_3 \overline{s}_2 s_1 (s_3 s_2) \\ & + s_2 s_3 \overline{s}_2 s_1 (s_3 \overline{s}_2) + s_2 s_3 \overline{s}_2 s_1 (\overline{s}_3 s_2) - s_2 s_3 \overline{s}_2 s_1 (\overline{s}_3 \overline{s}_2) \\ & + s_2 s_3 \overline{s}_2 s_1 (\overline{s}_2 \overline{s}_3 s_2) - s_2 s_3 \overline{s}_2 s_1 (s_2 \overline{s}_3 \overline{s}_2) \,. \end{aligned} \tag{49}$$

Some of the terms in the previous equality are linear combinations of Type 1 and Type 2 terms:

$$\begin{aligned} & s_2 s_3 (\overline{s}_2 s_1 s_2) s_3 \overline{s}_2 = s_2 s_3 s_1 s_2 \overline{s}_1 s_3 \overline{s}_2 = s_2 s_1 (s_3 s_2 s_3) \overline{s}_1 \overline{s}_2 = s_2 s_1 s_2 s_3 s_2 \overline{s}_1 \overline{s}_2 \,, \\ & s_2 s_3 (\overline{s}_2 s_1 s_2) s_3 = s_2 s_3 s_1 s_2 \overline{s}_1 s_3 = s_2 s_1 (s_3 s_2 s_3) \overline{s}_1 = s_2 s_1 s_2 s_3 s_2 \overline{s}_1 \,, \\ & s_2 s_3 (\overline{s}_2 s_1 s_2) \overline{s}_3 = s_2 s_3 s_1 s_2 \overline{s}_1 \overline{s}_3 = s_2 s_1 (s_3 s_2 \overline{s}_3) \overline{s}_1 = s_2 s_1 \overline{s}_2 s_3 s_2 \overline{s}_1 \,, \\ & s_2 s_3 \overline{s}_2 s_1 s_3 s_2 = s_2 (s_3 \overline{s}_2 s_3 s_1) s_2 \overset{(R_3)}{=} s_2 s_1 (s_3 \overline{s}_2 s_3 s_2) \overset{\text{reduction in } \langle s_2, s_3 \rangle}{=} s_2 (s_1 \overline{s}_3 s_3 s_2 \overline{s}_3) \\ & \qquad \overset{(R_3)}{=} (s_2 \overline{s}_3 s_2 \overline{s}_3) s_1 \overset{\text{reduction in } \langle s_2, s_3 \rangle}{=} \overline{s}_3 s_2 \overline{s}_3 s_1 \overset{(R_3)}{=} s_1 \overline{s}_3 s_2 \overline{s}_3 \,, \\ & s_2 s_3 \overline{s}_2 s_1 s_3 \overline{s}_2 = s_2 (s_3 \overline{s}_2 s_3 s_1) \overline{s}_2 \overset{(R_3)}{=} s_2 s_1 (s_3 \overline{s}_2 s_3 \overline{s}_2) \overset{\text{reduction in } \langle s_2, s_3 \rangle}{=} s_2 (s_1 \overline{s}_3 s_3 s_2 \overline{s}_3) \\ & \qquad \overset{(R_3)}{=} (s_2 \overline{s}_3 s_2 \overline{s}_3) s_1 \overset{\text{reduction in } \langle s_2, s_3 \rangle}{=} \overline{s}_3 s_2 \overline{s}_3 s_1 \overset{(R_3)}{=} s_1 \overline{s}_3 s_2 \overline{s}_3 \,, \\ & s_2 s_3 \overline{s}_2 s_1 \overline{s}_3 s_2 = s_2 (s_3 \overline{s}_2 \overline{s}_3) s_1 s_2 = s_2 \overline{s}_2 \overline{s}_3 s_2 s_1 s_2 \,, \\ & s_2 s_3 \overline{s}_2 s_1 \overline{s}_3 \overline{s}_2 = s_2 (s_3 \overline{s}_2 \overline{s}_3) s_1 \overline{s}_2 = s_2 \overline{s}_2 \overline{s}_3 s_2 s_1 \overline{s}_2 \,, \\ & s_2 s_3 (\overline{s}_2 s_1 s_2) \overline{s}_3 \overline{s}_2 = s_2 s_3 s_1 s_2 \overline{s}_1 \overline{s}_3 \overline{s}_2 = s_2 s_1 (s_3 s_2 \overline{s}_3) \overline{s}_1 \overline{s}_2 = s_2 s_1 \overline{s}_2 s_3 s_2 \overline{s}_1 \overline{s}_2 \,. \end{aligned}$$

So Equation (49) can be written in a simpler way.

$$s_2 s_3 \overline{s}_2 s_1 \overline{s}_2 s_3 = -s_2 s_3 \overline{s}_2 s_1 \overline{s}_2 s_3 s_2 + s_2 s_3 \overline{s}_2 s_1 \overline{s}_2 \overline{s}_3 + s_2 s_3 \overline{s}_2 s_1 \overline{s}_2 \overline{s}_3 s_2 \,.$$

Let us reduce each of the three terms on the right hand side of the previous equality.

$$\begin{aligned} (s_2 s_3 \overline{s}_2) s_1 \overline{s}_2 s_3 s_2 &= \overline{s}_3 s_2 s_3 s_1 \overline{s}_2 s_3 s_2 = \overline{s}_3 s_2 s_1 s_3 (\overline{s}_2 s_3 s_2) = \overline{s}_3 s_2 s_1 (s_3 s_3) s_2 \overline{s}_3 \\ &= (t_0 + t_1 - 1)\, \overline{s}_3 s_2 s_1 (s_3 s_2 \overline{s}_3) + (t_0 + t_1 - t_0 t_1)\, \overline{s}_3 (s_2 s_1 s_2) \overline{s}_3 - t_0 t_1\, \overline{s}_3 s_2 s_1 \overline{s}_3 s_2 \overline{s}_3 \\ &= (t_0 + t_1 - 1)\, \overline{s}_3 (s_2 s_1 \overline{s}_2) s_3 s_2 + (t_0 + t_1 - t_0 t_1)\, \overline{s}_3 s_1 s_2 s_1 \overline{s}_3 - t_0 t_1\, \overline{s}_3 s_2 s_1 \overline{s}_3 s_2 \overline{s}_3 \\ &= (t_0 + t_1 - 1)\, \overline{s}_3 \overline{s}_1 s_2 s_1 s_3 s_2 + (t_0 + t_1 - t_0 t_1)\, s_1 \overline{s}_3 s_2 \overline{s}_3 s_1 - t_0 t_1\, \overline{s}_3 s_2 s_1 \overline{s}_3 s_2 \overline{s}_3 \end{aligned}$$

$$\begin{aligned}
&= (t_0+t_1-1)\,\overline{s}_1\overline{s}_3s_2s_3s_1s_2 + (t_0+t_1-t_0t_1)\,s_1s_1\overline{s}_3s_2\overline{s}_3 - t_0t_1\,\overline{s}_3s_2s_1\overline{s}_3s_2\overline{s}_3\\
&= (t_0+t_1-1)\,\overline{s}_1s_2s_3\overline{s}_2s_1s_2 + (t_0+t_1-t_0t_1)(s_1s_1)\overline{s}_3s_2\overline{s}_3 - t_0t_1\,\overline{s}_3s_2s_1\overline{s}_3s_2\overline{s}_3\\
&= -t_0t_1\,\overline{s}_3s_2s_1\overline{s}_3s_2\overline{s}_3\,.
\end{aligned}$$

Now continuing down the rabbit hole, we can use Lemma 2.1's version of (R_2) to write that:

$$\begin{aligned}
s_2s_3\overline{s}_2s_1\overline{s}_2s_3s_2 &= -t_0t_1\,\overline{s}_3s_2s_1(\overline{s}_3s_2\overline{s}_3) \overset{(R_2)}{=} -(t_0t_1)(t_0^{-1}t_1^{-1})\,\overline{s}_3s_2s_1(s_3\overline{s}_2s_3) + \lambda\\
&= -(\overline{s}_3s_2s_3)s_1\overline{s}_2s_3 + \lambda = -s_2s_3\overline{s}_2s_1\overline{s}_2s_3 + \lambda\,,
\end{aligned}$$

where λ is a linear combination of the following words, all of which reduce:

$$\begin{aligned}
\overline{s}_3s_2s_1(\overline{s}_2\overline{s}_3\overline{s}_2) &= \overline{s}_3(s_2s_1\overline{s}_2)\overline{s}_3\overline{s}_2 = \overline{s}_3(\overline{s}_1s_2s_1)\overline{s}_3\overline{s}_2\\
&= \overline{s}_1(\overline{s}_3s_2\overline{s}_3s_1)\overline{s}_2 \overset{(R_3)}{=} (\overline{s}_1s_1)(\overline{s}_3s_2\overline{s}_3\overline{s}_2) \overset{\text{reduction in } \langle s_2,s_3\rangle}{=} \overline{s}_3s_2\overline{s}_3\,,\\
\overline{s}_3s_2s_1(s_2s_3s_2) &= \overline{s}_3(s_2s_1s_2)s_3s_2 = \overline{s}_3s_1s_2s_1s_3s_2 = s_1(\overline{s}_3s_2s_3)s_1s_2 = s_1s_2s_3\overline{s}_2s_1s_2\,,\\
\overline{s}_3s_2s_1(s_2\overline{s}_3\overline{s}_2) &= \overline{s}_3(s_2s_1s_2)\overline{s}_3\overline{s}_2 = \overline{s}_3s_1s_2s_1\overline{s}_3\overline{s}_2\\
&= s_1(\overline{s}_3s_2\overline{s}_3s_1)\overline{s}_2 \overset{(R_3)}{=} s_1s_1(\overline{s}_3s_2\overline{s}_3\overline{s}_2) \overset{\text{reduction in } \langle s_2,s_3\rangle}{=} s_1s_1(\overline{s}_3s_2\overline{s}_3)\,,\\
\overline{s}_3s_2s_1(s_2s_3\overline{s}_2) &= \overline{s}_3(s_2s_1s_2)s_3\overline{s}_2 = \overline{s}_3(s_1s_2s_1)s_3\overline{s}_2 = s_1(\overline{s}_3s_2s_3)s_1\overline{s}_2 = s_1s_2s_3\overline{s}_2s_1\overline{s}_2\,,\\
\overline{s}_3s_2s_1(\overline{s}_3\overline{s}_2) &= (\overline{s}_3s_2\overline{s}_3s_1)\overline{s}_2 \overset{(R_3)}{=} s_1(\overline{s}_3s_2\overline{s}_3\overline{s}_2) \overset{\text{reduction in } \langle s_2,s_3\rangle}{=} s_1\overline{s}_3s_2\overline{s}_3\,,\\
\overline{s}_3s_2s_1(\overline{s}_3s_2) &= (\overline{s}_3s_2\overline{s}_3s_1)s_2 \overset{(R_3)}{=} s_1(\overline{s}_3s_2\overline{s}_3s_2) \overset{\text{reduction in } \langle s_2,s_3\rangle}{=} s_1\overline{s}_3s_2\overline{s}_3\,,\\
\overline{s}_3s_2s_1(s_3\overline{s}_2) &= \overline{s}_3s_2s_3s_1\overline{s}_2 = (\overline{s}_3s_2s_3)s_1\overline{s}_2 = s_2s_3\overline{s}_2s_1\overline{s}_2\,,\\
\overline{s}_3s_2s_1(s_3s_2) &= \overline{s}_3s_2s_3s_1s_2 = (\overline{s}_3s_2s_3)s_1s_2 = s_2s_3\overline{s}_2s_1s_2\,,\\
\overline{s}_3s_2s_1(\overline{s}_2\overline{s}_3) &= \overline{s}_3(s_2s_1\overline{s}_2)\overline{s}_3 = \overline{s}_3\overline{s}_1s_2s_1\overline{s}_3 = \overline{s}_1(\overline{s}_3s_2\overline{s}_3s_1) \overset{(R_3)}{=} \overline{s}_1s_1\overline{s}_3s_2\overline{s}_3 = \overline{s}_3s_2\overline{s}_3\,,\\
\overline{s}_3s_2s_1(s_2\overline{s}_3) &= \overline{s}_3(s_2s_1s_2)\overline{s}_3 = \overline{s}_3s_1s_2s_1\overline{s}_3 = s_1(\overline{s}_3s_2\overline{s}_3s_1) \overset{(R_3)}{=} (s_1s_1)\overline{s}_3s_2\overline{s}_3\,,\\
\overline{s}_3s_2s_1(\overline{s}_2s_3) &= \overline{s}_3(s_2s_1\overline{s}_2)s_3 = \overline{s}_3\overline{s}_1s_2s_1s_3 = \overline{s}_1(\overline{s}_3s_2s_3)s_1 = \overline{s}_1s_2s_3\overline{s}_2s_1\,,\\
\overline{s}_3s_2s_1(s_2s_3) &= \overline{s}_3(s_2s_1s_2)s_3 = \overline{s}_3s_1s_2s_1s_3 = s_1(\overline{s}_3s_2s_3)s_1 = s_1s_2s_3\overline{s}_2s_1\,,\\
\overline{s}_3s_2s_1(\overline{s}_2) &= \overline{s}_3s_2s_1\overline{s}_2\,,\\
\overline{s}_3s_2s_1(s_2) &= \overline{s}_3s_2s_1s_2\,.
\end{aligned}$$

This means that modulo Type 1 and Type 2 terms:

$$s_2s_3\overline{s}_2s_1\overline{s}_2s_3s_2 = -s_2s_3\overline{s}_2s_1\overline{s}_2s_3\,,$$

and Equation (49) can be further simplified:

$$\cancel{s_2s_3\overline{s}_2s_1\overline{s}_2s_3} = \cancel{-s_2s_3\overline{s}_2s_1\overline{s}_2s_3s_2} + s_2s_3\overline{s}_2s_1\overline{s}_2\overline{s}_3 + s_2s_3\overline{s}_2s_1\overline{s}_2\overline{s}_3s_2\,.$$

Also:

$$\begin{aligned}
s_2s_3\overline{s}_2s_1(\overline{s}_2\overline{s}_3s_2) &= s_2s_3\overline{s}_2s_1s_3\overline{s}_2\overline{s}_3 = (s_2s_3\overline{s}_2)s_3s_1\overline{s}_2\overline{s}_3 = \overline{s}_3s_2(s_3s_3)s_1\overline{s}_2\overline{s}_3\\
&\overset{(R_1)}{=} (t_0+t_1-1)\,(\overline{s}_3s_2s_3)s_1\overline{s}_2\overline{s}_3 + (t_0+t_1-t_0t_1)\,\overline{s}_3(s_2s_1\overline{s}_2)\overline{s}_3 - t_0t_1\,\overline{s}_3s_2\overline{s}_3s_1\overline{s}_2\overline{s}_3\\
&= (t_0+t_1-1)\,s_2s_3\overline{s}_2s_1\overline{s}_2\overline{s}_3 + (t_0+t_1-t_0t_1)\,\overline{s}_3\overline{s}_1s_2s_1\overline{s}_3 - t_0t_1\,\overline{s}_3s_2s_1(\overline{s}_3\overline{s}_2\overline{s}_3)
\end{aligned}$$

$$
\begin{aligned}
&= (t_0+t_1-1)\, s_2s_3\overline{s}_2s_1\overline{s}_2\overline{s}_3 + (t_0+t_1-t_0t_1)\, \overline{s}_1(\overline{s}_3s_2\overline{s}_3s_1) - t_0t_1\, \overline{s}_3(s_2s_1\overline{s}_2)\overline{s}_3\overline{s}_2 \\
&\overset{(R_3)}{=} (t_0+t_1-1)\, s_2s_3\overline{s}_2s_1\overline{s}_2\overline{s}_3 + (t_0+t_1-t_0t_1)\, (\overline{s}_1s_1)\overline{s}_3s_2\overline{s}_3 - t_0t_1\, \overline{s}_3\overline{s}_1s_2s_1\overline{s}_3\overline{s}_2 \\
&= (t_0+t_1-1)\, s_2s_3\overline{s}_2s_1\overline{s}_2\overline{s}_3 + (t_0+t_1-t_0t_1)\, \overline{s}_3s_2\overline{s}_3 - t_0t_1\, \overline{s}_1(\overline{s}_3s_2\overline{s}_3s_1)\overline{s}_2 \\
&\overset{(R_3)}{=} (t_0+t_1-1)\, s_2s_3\overline{s}_2s_1\overline{s}_2\overline{s}_3 - t_0t_1\, \overline{s}_1s_1(\overline{s}_3s_2\overline{s}_3\overline{s}_2) \\
&= (t_0+t_1-1)\, s_2s_3\overline{s}_2s_1\overline{s}_2\overline{s}_3 - t_0t_1\, \overline{s}_3s_2\overline{s}_3\overline{s}_2 \\
&= (t_0+t_1-1)\, s_2s_3\overline{s}_2s_1\overline{s}_2\overline{s}_3 \text{ (reduction in } \langle s_2,s_3\rangle) .
\end{aligned}
$$

So Equation (49) can now be written:

$$
0 = s_2s_3\overline{s}_2s_1\overline{s}_2\overline{s}_3 + (t_0+t_1-1)\, s_2s_3\overline{s}_2s_1\overline{s}_2\overline{s}_3 \text{ , i.e. } 0 = (t_0+t_1)\, s_2s_3\overline{s}_2s_1\overline{s}_2\overline{s}_3 .
$$

So $s_2s_3\overline{s}_2s_1\overline{s}_2\overline{s}_3$ reduces. And multiplying by $\overline{s}_2$ from the left, we deduce that $s_3\overline{s}_2s_1\overline{s}_2\overline{s}_3$ reduces as well. Indeed:

$$
\begin{aligned}
s_2s_3\overline{s}_2s_1\overline{s}_2\overline{s}_3 = {} & a(s_1,s_2) + b(s_1,s_2)\, s_3\, c(s_1,s_2) + d(s_1,s_2)\, \overline{s}_3\, e(s_1,s_2) \\
& + \lambda\, \overline{s}_3s_2\overline{s}_3 + \mu\, s_1\overline{s}_3s_2\overline{s}_3 + \nu\, \overline{s}_1\overline{s}_3s_2\overline{s}_3 ,
\end{aligned}
$$

so

$$
\begin{aligned}
s_3\overline{s}_2s_1\overline{s}_2\overline{s}_3 &= \overline{s}_2\, a(s_1,s_2) + \overline{s}_2\, b(s_1,s_2)\, s_3\, c(s_1,s_2) + \overline{s}_2\, d(s_1,s_2)\, \overline{s}_3\, e(s_1,s_2) \\
&\quad + \lambda\, \overline{s}_2\overline{s}_3s_2\overline{s}_3 + \mu\, \overline{s}_2s_1\overline{s}_3s_2\overline{s}_3 + \nu\, \overline{s}_2\overline{s}_1\overline{s}_3s_2\overline{s}_3 \\
&= \tilde{a}(s_1,s_2) + \tilde{b}(s_1,s_2)\, s_3\, c(s_1,s_2) + \tilde{d}(s_1,s_2)\, \overline{s}_3\, e(s_1,s_2) \\
&\quad + \lambda\, \overline{s}_3s_2\overline{s}_3 + \mu\, s_1(\overline{s}_2\overline{s}_3s_2\overline{s}_3) + \nu\, \overline{s}_2(\overline{s}_1\overline{s}_3s_2\overline{s}_3) \text{ (reduction in } \langle s_2,s_3\rangle) \\
&= \tilde{a}(s_1,s_2) + \tilde{b}(s_1,s_2)\, s_3\, c(s_1,s_2) + \tilde{d}(s_1,s_2)\, \overline{s}_3\, e(s_1,s_2) \\
&\quad + \lambda\, \overline{s}_3s_2\overline{s}_3 + \mu\, s_1\overline{s}_3s_2\overline{s}_3 + \nu\, (\overline{s}_2\overline{s}_3s_2\overline{s}_3)\overline{s}_1 \text{ (reduction in } \langle s_2,s_3\rangle + (R_3)) \\
&= \tilde{a}(s_1,s_2) + \tilde{b}(s_1,s_2)\, s_3\, c(s_1,s_2) + \tilde{d}(s_1,s_2)\, \overline{s}_3\, e(s_1,s_2) \\
&\quad + \lambda\, \overline{s}_3s_2\overline{s}_3 + \mu\, s_1\overline{s}_3s_2\overline{s}_3 + \nu\, \overline{s}_3s_2\overline{s}_3\overline{s}_1 \text{ (reduction in } \langle s_2,s_3\rangle) \\
&= \tilde{a}(s_1,s_2) + \tilde{b}(s_1,s_2)\, s_3\, c(s_1,s_2) + \tilde{d}(s_1,s_2)\, \overline{s}_3\, e(s_1,s_2) \\
&\quad + \lambda\, \overline{s}_3s_2\overline{s}_3 + \mu\, s_1\overline{s}_3s_2\overline{s}_3 + \nu\, \overline{s}_1\overline{s}_3s_2\overline{s}_3 \text{ } ((R_3)) \\
&= 0 .
\end{aligned}
$$

Since $s_3\overline{s}_2s_1\overline{s}_2\overline{s}_3$ reduces, if we start the previous computation again writing everything from right to left when it was written from left to right, we find in the same way that $\overline{s}_3\overline{s}_2s_1\overline{s}_2s_3$ reduces. ☐

Proposition D.3. The words $s_3\overline{s}_2s_1\overline{s}_2s_3$ and $\overline{s}_3\overline{s}_2s_1\overline{s}_2\overline{s}_3$ reduce.

Proof. Using Proposition D.2, Equations (47) and (48) can be written in a simpler way:

$$
(47) : s_3\overline{s}_2s_1\overline{s}_2s_3 + s_2s_3\overline{s}_2s_1\overline{s}_2s_3 = 0 ,
$$

$$
(48) : \overline{s}_3\overline{s}_2s_1\overline{s}_2\overline{s}_3 + s_2\overline{s}_3\overline{s}_2s_1\overline{s}_2\overline{s}_3 = 0 .
$$

To prove e.g. that $s_3\overline{s}_2s_1\overline{s}_2s_3$ reduces, one way is to find a different equation involving $s_3\overline{s}_2s_1\overline{s}_2s_3$ and $s_2s_3\overline{s}_2s_1\overline{s}_2s_3$. To do so, we write down $(R_3)\cdot\overline{s}_2s_3$ explicitly and we find that modulo Type 1 and Type 2 terms, it can be expressed as follows:

$$a_{78}\,(s_2s_3\overline{s}_2s_1)\overline{s}_2s_3 + (a_{39}-a_{75}-a_{53}-a_{47})\,s_3\overline{s}_2s_1\overline{s}_2s_3 = 0\,. \tag{50}$$

Therefore the system

$$(\Sigma):\begin{cases} s_3\overline{s}_2s_1\overline{s}_2s_3 + s_2s_3\overline{s}_2s_1\overline{s}_2s_3 = 0 \\ a_{78}\,(s_2s_3\overline{s}_2s_1)\overline{s}_2s_3 + (a_{39}-a_{75}-a_{53}-a_{47})\,s_3\overline{s}_2s_1\overline{s}_2s_3 = 0 \end{cases}$$

has the following determinant:

$$\begin{aligned} \det(\Sigma) &= \begin{vmatrix} a_{39}-a_{75}-a_{53}-a_{47} & 1 \\ a_{78} & 1 \end{vmatrix} = -a_{78}+a_{39}-a_{75}-a_{53}-a_{47} \\ &= \frac{(t_0+t_1-1)(1+t_0t_1+t_0^2t_1+t_0t_1^2)}{(t_0+t_1)(t_0t_1+1)(t_0t_1-1)} \neq 0\,. \end{aligned}$$

So (Σ) is an invertible system. Thus $s_3\overline{s}_2s_1\overline{s}_2s_3$ (and $s_2s_3\overline{s}_2s_1\overline{s}_2s_3$) can be reduced.

Similarly, we can prove that $\overline{s}_3\overline{s}_2s_1\overline{s}_2\overline{s}_3$ (and $s_2\overline{s}_3\overline{s}_2s_1\overline{s}_2\overline{s}_3$) reduce by considering the system comprised of the reduced version of Equation (48) and $(R_3)\cdot\overline{s}_2\overline{s}_3$. □

Summing up, Lemma 2.6 is now proved.

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International Center for Mathematics, Department of Mathematics, Southern University of Science and Technology, Shenzhen, China
http://people.mpim-bonn.mpg.de/stavros
Email address: stavros@mpim-bonn.mpg.de

Michigan State University, East Lansing, Michigan, USA
Email address: mrhmath@proton.me

Section de Mathématiques, Université de Genève, 2-4 rue du Lièvre, Case Postale 64, 1211 Genève 4, Switzerland
http://www.unige.ch/math/folks/kashaev
Email address: Rinat.Kashaev@unige.ch

Yau Mathematical Sciences Center, Jingzhai, Tsinghua University, Haidian District, Beijing, China
Email address: bm.kohli@protonmail.ch

Beijing Institute of Mathematical Sciences and Applications, Huairou District, Beijing, China
Email address: songjiebo@bimsa.cn

Beijing Institute of Mathematical Sciences and Applications, Huairou District, Beijing, China
Email address: guillaume.tahar@bimsa.cn