A REAPPEARENCE OF WHEELS

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Abstract. Recently, a number of authors [KS, Oh2, Ro] have independently shown that the universal finite type invariant of rational homology 3-spheres on the level of $\mathfrak{sl}_2$ can be recovered from the Reshetikhin-Turaev $\mathfrak{sl}_2$ invariant. An important role in Ohtsuki's proof [Oh3] plays a map j_1 (which joins the legs of 2-legged chinese characters) and its relation to a map α in terms of a power series (on the level of $\mathfrak{sl}_2$). The purpose of the present note is to give a universal formula of the map α in terms of a power series F of wheel chinese characters. The above formula is similar to universal formulas of wheel chinese characters considered in [BGRT1] and leads to a simple conceptual proof of the above mentioned relation between the maps j_1 and α on the level of $\mathfrak{sl}_2$.

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1. Introduction

1.1. **History.** Shortly after the introduction of finite type invariants of integral homology 3-spheres (see [Oh1], and [GO] for an extension to rational homology 3-spheres) and the construction of the universal such invariant by T.T.Q. Le, J. Murakami and T. Ohtsuki (see [LMO, L], and in addition [BGRT2, BGRT3]) it has been a folk conjecture that finite type invariants of rational homology 3-spheres should be related to the topological quantum field theory invariants. For a *statement* of the above mentioned folk conjecture on the level of the semisimple Lie algebras see [Oh2, conjecture 8.4]. The conjecture has been proven independently in [KS, Oh1, Ro, BGRT2] in the case of $\mathfrak{sl}_2$. A key ingredient in the construction of the universal 3-manifold invariant (often known as the LMO invariant) and in Ohtsuki's proof of the folk conjecture on the level of $\mathfrak{sl}_2$ is a map $j_1 : \mathcal{A}'(S^1) \to \mathcal{A}'(\phi)$

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(which joins the legs of 2-legged chinese characters) and its relation mod $\mathfrak{sl}_2$ to a map $\alpha : \mathcal{A}'(S^1) \to \mathcal{A}'(S^1)$ in terms of a power series $P_{\mathfrak{sl}_2}(\hbar) \in \mathbb{Q}[[\hbar]]$ with rational coefficients.

1.2. **Statement of the results.** We begin by reviewing some standard definitions and notation. $\mathcal{A}'(S^1)$ (resp. $\mathcal{A}'(\phi)$) is the vector space on chord diagrams on S^1 (resp. on the empty set) and $\mathcal{B}'$ is the vector space on chinese characters, [B-N]. Note that we *include* chord diagrams some of whose components are dashed trivalent graphs as well as chord diagrams that contain dashed circles; this is why we have chosen to put $'$ in $\mathcal{A}'(S^1), \mathcal{A}'(\phi)$ and $\mathcal{B}'$.

The purpose of the present note is to

- Give a universal expression for the map α in terms of wheels, see Proposition 1.3.
- To give an explicit formula for the power series $P_{\mathfrak{sl}_2}(\hbar)$.

For a chinese character C with n legs (i.e., univalent vertices), define an operator $\hat{C} : \mathcal{B}' \to \mathcal{A}'(\phi)$ as follows:

$$\hat{C}(C') = \begin{cases} 0 & \text{if } C' \text{ does not have exactly } n \text{ legs} \\ \text{the sum of all } n! \text{ ways of gluing} & \text{if } C' \text{ has exactly } n \text{ legs} \\ \text{all legs of } C \text{ to all legs of } C' & \end{cases}$$

Note that the above operator is a minor variation of the operator considered in [BGRT1, definition 1.1], and as such it can be linearly extended to the case of C being an infinite linear combination of chinese characters with an increasing number of legs. This defines a map $\mathcal{B}' \times \mathcal{B}' \to \mathcal{A}'(\phi)$ where the left copy of $\mathcal{B}'$ is equipped with the disjoint union multiplication of chinese characters. Following [LMO, Oh2], we define a map $j_1 : \mathcal{B}' \to \mathcal{A}'(\phi)$ by $j_1 = 1/2\hat{I}$, where I is the connected chinese character with no trivalent and 2 univalent vertices. Note that by definition j_1 closes a chinese character with 2 legs to a trivalent graph if the chinese character has 2 legs, and vanishes otherwise. The map j_1 is a key ingredient in the construction of the Le-Murakami-Ohtsuki invariant of 3-manifolds. On the level of $\mathfrak{sl}_2$, Ohtsuki expressed j_1 as a power series $P_{\mathfrak{sl}_2}$ on a map α and used this expression in his proof of the above mentioned folk conjecture on the level of $\mathfrak{sl}_2$. Recall [Oh2] that $\alpha : \mathcal{A}'(S^1) \to \mathcal{A}'(S^1)$ is defined (for every $C \in \mathcal{A}'(S^1)$) by:

$$\alpha(C) = (\text{replace the left} \bigcirc \text{by} \bigcirc) \circ \Delta_B(C) - \bigcirc \sqcup C$$

where $\mathcal{A}'(S^1 \sqcup S^1)$ is the vector space of chord diagrams on two ordered circles (called left and right circle), $\Delta_B : \mathcal{A}'(S^1) \to \mathcal{A}'(S^1 \sqcup S^1)$ is a comultiplication map which sends a chord diagram on S^1 to the sum of all possible ways of lifting the endpoints of its chords to either of the two ordered circles.

Though the map α is defined in a rather ad-hoc way, using the following principle[1]

Principle 1.1. *Replace chord diagrams in terms of chinese characters, via the (normalized) isomorphism* $\chi : \mathcal{B}' \to \mathcal{A}'(S^1)$ *of* [B-N, theorem 8]*. Expressed in terms of chinese characters, many maps simplify and their description often involves wheels, see Figure 1.*

[1]For other examples of the above principle, see [BGRT1, BGRT2, GH, KSA]

it turns out that the composite map $\alpha \circ \chi$ is much easier to describe in terms of wheels, thus explaining the title of the paper. Indeed, let

$$F = \sum_{p=1}^{\infty} \frac{w_{2p}}{(2p)!} \tag{1}$$

where w_{2n} is the $2n$-wheel, [BGRT1], see also figure 1.

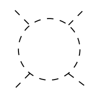


Figure 1. The wheel ω_4 with 4 legs. Its trivalent vertices are oriented clockwise.

Using the map $\epsilon : \mathcal{A}'(S^1) \to \mathcal{A}'(\phi)$ defined by

$$\epsilon(D) = \begin{cases} 0 & \text{if there are chords touching the solid } S^1 \\ \text{erase } S^1 & \text{otherwise} \end{cases}$$

we can express j_1 in terms of α as follows:

Proposition 1.2.

$$j_1 = \epsilon \circ P_{\mathfrak{sl}_2}(\alpha) \circ \chi mod\ \mathfrak{sl}_2 \tag{2}$$

where mod $\mathfrak{sl}_2$ is explained in section 2.2 and $P_{\mathfrak{sl}_2}$ is the formal power series defined by:

$$8P_{\mathfrak{sl}_2}(F_{\mathfrak{sl}_2}(\hbar)) = \hbar \text{ where } F_{\mathfrak{sl}_2}(\hbar) = 2cosh\sqrt{\frac{\hbar}{2}} - 2 \tag{3}$$

Moreover, $P_{\mathfrak{sl}_2}$ is the unique power series so that the above equation holds.

Furthermore, we can express $\epsilon \circ \alpha$ in terms of wheels as follows:

Proposition 1.3. *For all non-negative integers m we have:*

$$\epsilon \circ \alpha^m \circ \chi = \widehat{F^m} \tag{4}$$

It follows from equation (3) that the power series $P_{\mathfrak{sl}_2}(\hbar)$ can be calculated explicitly. T.T.Q Le and D. Thurston [LT] kindly inform us that their independent calculation implies that:

$$P_{\mathfrak{sl}_2}(\hbar) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\hbar^n}{2n^2\binom{2n}{n}} \tag{5}$$

1.3. **Some Questions.** It is a puzzle to explain the appearance of wheels in [BGRT1] and in the map α. The two power series (F considered here, and Ω considered in [BGRT1]) are very similar, with the only exception of the appearance of Bernoulli numbers in Ω but not in F.

It is an interesting question (related to a proof of the folk conjecture on the level of semisimple Lie algebras) to find an analogue of Proposition 1.2 for any semisimple Lie algebra.

1.4. **Acknowledgment.** The first idea of the present paper was conceived during a conversation with D. Bar-Natan at the January 1997 workshop in M.S.R.I. We wish to thank D. Bar-Natan for pointing out the relevance of wheels in the definition of the map α, as well as the $\mathfrak{sl}_2$ relation of Lemma 2.1. The power series $P_{\mathfrak{sl}_2}(\hbar)$ was calculated independently by T.T.Q. Le and D. Thurston, [LT]. We wish to thank them for sharing their result with us, and for correcting mistakes in an earlier version of the paper. Finally we wish to thank M. Hutchings for fruitful and clarifying conversations.

2. Proofs

2.1. **Proof of Proposition 1.3.**

Proof of Proposition 1.3. Throughout this proof we fix a chinese character C with l legs.

Then, we have the following equality:

where the notation is as follows:

- "Join" is all $l!$ ways of joining the l legs of C to l points in a solid circle.
- "Lift" is all ways of lifting some *non-empty* subset of the l points of the solid circle to the m ordered dashed circles.
- "Liftall" is all ways of lifting all the l points of a solid circle to m ordered dashed circles in such a way that each dashed circle has an even nonzero number of points.

The proof of the above identity follows easily from the definition of α together with the fact that a wheel with an odd number of legs vanishes.

If l is odd, this implies that $\epsilon \circ \alpha^m \circ \chi(C) = 0$ and also $\widehat{F^m}(C) = 0$ (since F^m is a linear combination of even legged diagrams and C has odd many leggs).

Assume from now on that $l = 2n$ is even. Notice that every element of "Liftall" gives rise to a partition of $2n$ into m ordered pieces, so that each piece has a positive even number associated to it. For $i = 1, \dots, m$, let j_i denote the number of pieces of size $2i$. Then we have,

$$\sum_{p=1}^{\infty} 2p j_p = 2n \text{ and } \sum_{p=1}^{\infty} j_p = m \tag{6}$$

where of course both are finite sums. Our goal is to replace each of the dashed ordered circles with a disjoint union of unordered wheels. There are:

$\binom{2n}{\underbrace{2,2}_{j_1},\underbrace{4,4}_{j_2}\ldots}$ ways of choosing a partition of the legs of C

$m!$ ways for ordering the partition

However, we must multiply by:

$\dfrac{1}{\prod_p j_p!}$ if we do not distinguish dashed circles with the same number of points

The above consideration together with the fact that $\binom{2n}{\underbrace{2,2}_{j_1},\underbrace{4,4}_{j_2}...} = \frac{(2n)!}{\prod_p (2p)!^{j_p}}$ imply that:

$$\epsilon \circ \alpha^m \circ \chi(C) = \frac{1}{(2n)!} \sum_j \frac{(2n)!m!}{j_p!((2p)!)^{j_p}} \prod_p \hat{w}_{2p}^{j_p}(C)$$

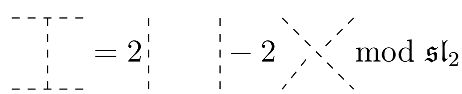
where the summation is over all $\{j\}$ that satisfy the conditions of equation (6). Using the disjoint union multiplication of wheels, together with the multibinomial theorem, it implies that:

$$\epsilon \circ \alpha^m \circ \chi(C) = (\sum_p \frac{\hat{w}_{2p}}{2p(2p)!})^m(C),$$

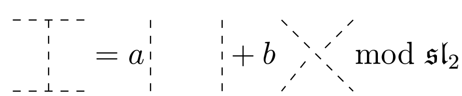
which concludes the proof of the Proposition. $\square$

2.2. **Proof of Proposition 1.2.** We begin by recalling some standard facts about weight systems, for a detailed discussion see [B-N, section 2.4]. Given a semisimple Lie algebra (such as $\mathfrak{sl}_2$) there is a map $W_{\mathfrak{sl}_2} : \mathcal{A}'(S^1) \to \mathbb{Q}[[\hbar]]$ defined by coloring the circle S^1 with the adjoint representation. This induces a map $\mathcal{B}' \to \mathbb{Q}[[\hbar]]$ (defined by composing with $\chi : \mathcal{B}' \to \mathcal{A}'(S^1)$) as well as a restriction map to $\mathcal{A}'(\phi)$ obtained by the inclusion $\mathcal{A}'(\phi) \hookrightarrow \mathcal{B}'$. Without loss of generality, we will denote the above three weight systems with the same name. We call two elements C_1, C_2 of $\mathcal{B}'$ $\mathfrak{sl}_2$-*equivalent* if $W_{\mathfrak{sl}_2}(C_1) = W_{\mathfrak{sl}_2}(C_2)$. Similarly, we call two operators $\hat{C}_1$ and $\hat{C}_2$ $\mathfrak{sl}_2$-*equivalent* if for every chinese character C, we have: $W_{\mathfrak{sl}_2}(\hat{C}_1(C)) = W_{\mathfrak{sl}_2}(\hat{C}_2(C)) \in \mathbb{Q}[[\hbar]]$.

We need the following classical indentity:[2]



with the understanding that the chinese characters outside the figure are equal. An alternative proof can be obtained as follows: From the theory of invariant $\mathfrak{sl}_2$ tensors it follows that



for some constants a, b. By AS it follows that $a + b = 0$. If we join the left two legs, using the fact that the value of the quadratic Casimir of $\mathfrak{sl}_2$ in the adjoint representation is 4, we obtain that: $4.3 = 9a + 3b$ from which it follows that $a = 2, b = -2$.

Lemma 2.1. *For every integer number $p \geq 1$, we have:*

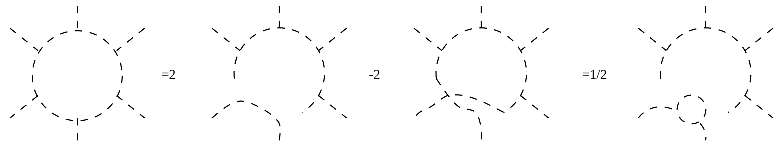
$$\hat{w}_{2p} = \frac{1}{2^{p-1}} \hat{w}_2^p mod\ \mathfrak{sl}_2,$$

which together with the notation of equation (3) *implies that:*

$$\hat{F} \equiv F_{\mathfrak{sl}_2}(\hat{w}_2) \bmod \mathfrak{sl}_2$$

[2] D. Bar-Natan informs us that it is usually written in the form: $A \times (B \times C) = (A \cdot B)C - (A \cdot C)B$, for A, B, C vectors in $\mathbb{R}^3$.

Proof. Modulo $\mathfrak{sl}_2$, the classical identity above implies that:



where the middle equality the right term is zero (since it is a tree chinese character) and third equality follows from the fact that w_2 is central, with value the quadratic Casimir in the adjoint representation of $\mathfrak{sl}_2$, i.e., 4. This shows that

$$\hat{w}_{2p} = \frac{1}{2}\widehat{w_{2p-2}w_2} \text{mod } \mathfrak{sl}_2$$

(where $w_{2p-2}w_2$ is the *disjoint* union multiplication of w_{2p-2} and w_2), from which the first part of the lemma follows by induction on p. The second part of the lemma follows from the first together with the mod $\mathfrak{sl}_2$ identity: $\hat{F} \equiv \sum_{p=1}^{\infty} \frac{\hat{w}_2^p}{2^{p-1}(2p)!} \equiv 2\cosh\sqrt{\frac{\hat{w}_2}{2}} - 2$ □

Proof of Proposition 1.2. We begin by fixing a power series $P(\hbar) = \sum_{m=1}^{\infty} c_m \hbar^m \in \mathbb{Q}[[\hbar]]$. Using the definition of j_1, together with the fact that $\hat{w}_2 = 4\hat{I}$, it follows that equation (2) is equivalent to:

$$\epsilon \circ \sum_{m=1}^{\infty} c_m \alpha^m \chi(C) = 1/8\hat{w}_2(C) \text{ mod } \mathfrak{sl}_2 \tag{7}$$

for all chinese characters C. On the other hand, Proposition 1.3 implies that $\epsilon\circ\alpha^m\circ\chi(C) = 0$ for $m > n \overset{\text{def}}{=}$ number of legs$(C)/2$, and together with Lemma 2.1, it follows that equation (7) is equivalent to:

$$\begin{aligned} 1/8\hat{w}_2(C) = & \ \epsilon \circ \textstyle\sum_{m=1}^{n} c_m \alpha^m \circ \chi(C) & \text{mod } \mathfrak{sl}_2 \\ = & \ \textstyle\sum_{m=1}^{n} c_m \hat{F}^m(C) & \text{mod } \mathfrak{sl}_2 \\ = & \ \textstyle\sum_{m=1}^{n} c_m F_{\mathfrak{sl}_2}(\hat{w}_2)^m(C) & \text{mod } \mathfrak{sl}_2 \\ = & \ P(F_{\mathfrak{sl}_2}(\hat{w}_2))(C) & \text{mod } \mathfrak{sl}_2 \end{aligned}$$

which in turn is equivalent to the identity of equation (3). This finishes the proof of Proposition 1.2. □

An explicit calculation of the coefficients of $P_{\mathfrak{sl}_2}(\hbar)$ up to arbitrary degree n can be obtained by running `Mathematica`:

```
In[1]:= P[h_,n_]:=1/8 InverseSeries[Sum[t^r/(2^(r-1) (2 r)! ), {r,1,n}]
+ O[t]^(n+1), h]
In[2]:= P[h,6]
                2    3       4       5         6
        h    h    h       h       h         h          7
Out[2]= - - -- + --- - ---- + ----- - ----- + O[h]
        4   48   360   2240   12600   66528
```

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