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Universality and asymptotics of graph counting problems in non-orientable surfaces ☆

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ABSTRACT

Bender–Canfield showed that a plethora of graph counting problems in orientable/non-orientable surfaces involve two constants t_g and p_g for the orientable and the non-orientable case, respectively. T.T.Q. Le and the authors recently discovered a hidden relation between the sequence t_g and a formal power series solution $u(z)$ of the Painlevé I equation which, among other things, allows to give exact asymptotic expansion of t_g to all orders in $1/g$ for large g. The paper introduces a formal power series solution $v(z)$ of a Riccati equation, gives a non-linear recursion for its coefficients and an exact asymptotic expansion to all orders in g for large g, using the theory of Borel transforms. In addition, we conjecture a precise relation between the sequence p_g and $v(z)$. Our conjecture is motivated by the enumerative aspects of a quartic matrix model for real symmetric matrices, and the analytic properties of its double scaling limit. In particular, the matrix model provides a computation of the number of rooted quadrangulations in the 2-dimensional projective plane. Our conjecture implies analyticity of the O(N)- and Sp(N)-types of free energy of an arbitrary closed 3-manifold in a neighborhood of zero. Finally, we give a matrix model calculation of the Stokes constants, pose several problems that can be answered by the Riemann–Hilbert approach, and provide ample numerical evidence for our results.

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Contents

1. Introduction

1.1. Counting rooted maps in orientable surfaces

This paper introduces and studies the asymptotics of a sequence of rational numbers (v_n) and a conjecture relating them to a sequence of constants (p_g) that appear in a rooted graph counting problem of [4]. The problem of counting the number of graphs with a fixed number of edges that can be embedded in a surface of genus g has a long history. For planar graphs, the problem was solved by Tutte [54]. In their seminal paper [4], Bender–Canfield consider the number $T_g(n)$ of *rooted maps* (G, S), that is embeddings of a graph G with n edges in a closed connected orientable surface S of genus g, such that every component of $S \setminus G$ is a disk, and such that an edge, an orientation and a side of it is chosen. Bender–Canfield gave an inductive computation of the natural number $T_g(n)$ and also proved that for fixed g and large n, $T_g(n)$ is asymptotic to

$$T_g(n) \sim t_g n^{\gamma(g-1)} \lambda^n \tag{1}$$

where

$$\gamma = \frac{5}{2}, \qquad \lambda = 12 \tag{2}$$

and the constants t_g are computable by some complicated non-linear recursion that depends on an auxiliary partition; see [4, Eq. (4.2)] for a non-linear recursion for $\hat{\phi}_g^{(t)}(I, \alpha)$ that computes t_g via [4, Eq. (4.1)]. The recursion of Bender–Canfield, although cumbersome, gives exact answers for t_g that do not depend on any unknown constants. In particular, Bender–Canfield obtained the first three values

$$t_0 = \frac{2}{\sqrt{\pi}}, \qquad t_1 = \frac{1}{24}, \qquad t_2 = \frac{7}{4320\sqrt{\pi}}.$$

In the early nineties, it was realized in combinatorial enumeration that several classes of counting problems (such as cubic maps, quartic maps) also lead to an asymptotic expansion of the form (1); see for example [31,32].

1.2. Counting ribbon graphs in orientable surfaces

In the physics literature, graphs appear often as Feynman diagrams of a perturbative quantum field theory. In the case of gauge theories with gauge group $U(N)$, the Feynman diagrams are *ribbon graphs*, i.e., graphs with a cyclic order of flags around each vertex. Perturbative gauge theory counts ribbon graphs with a fixed genus, number of edges and prescribed valency, and with weight being the inverse of the size of their automorphism group. This is precisely the content of matrix models and was discussed extensively in the eighties, see [9,6]. One of the results found in this period in the matrix model community was that the generating functions counting ribbon graphs at fixed genus are analytic functions with a finite radius of convergence, which is the same for all genera. In the late eighties matrix model were studied in the so-called double scaling limit [10,23,37], where roughly speaking one considers the generating functions near their singularity and extracts the coefficients u_g of the leading poles for different genera. It was realized that these coefficients are *universal*, i.e., they do not depend on the details of the matrix model potential. It was also realized that the ribbon counting problem for arbitrary potential gives rise to a generating series in infinitely many variables that satisfies some *universal* (e.g. KdV or KP) *hierarchy*. In addition, the double scaling limit of the matrix model satisfies a non-linear differential equation which in the simplest case is the famous *Painlevé I* equation; see for example [22,55] for a survey of these developments in the physics literature.

Two years ago, Goulden–Jackson proved that a similar generating series that counts maps is a solution to the KP hierarchy, and in particular deduced a quadratic non-linear recursion for $T_g(n)$ in terms of $T_{g'}(n')$ for suitable $(g', n') < (g, n)$; see [35, Eq. (45)]. Using the quadratic recursion of $T_g(n)$, in [5] Bender–Gao–Richmond give a quadratic recursion relation for the constant t_g that involves only $t_{g'}$ for $g' < g$. This allowed them to prove that $t_g/g!^2$ grows exponentially with a numerical non-zero constant [5]. Recently, it was realized in [30] that the quadratic recursion relation for t_g is, in disguise, the recursion characterizing a formal power series solution to Painlevé I. Although this (as well as Goulden–Jackson's work [35]) came as a surprise to the enumerative combinatorics community, it is hardly a surprise from the physics point of view. The fact that the t_g enter into different map counting problems [31] is, from the matrix model point of view, a manifestation of universality.

The analytic structure of solutions to Painlevé I is well-known, mostly though the Riemann–Hilbert approach (see for example [29]), and allows one to give the full exact asymptotic expansion of t_g in inverse powers of g; see for example [30, Appendix A].

1.3. The case of non-orientable surfaces

The above discussion focused on the counting problems of rooted maps and ribbon graphs in orientable surfaces. Although the problems are different, their view from a distance (in the double-scaling limit) is the same, described by a universal non-linear differential equation, Painlevé I. On the combinatorial side, Bender–Canfield had also developed a theory of counting rooted maps in non-orientable surfaces. Let $P_g(n)$ be the number of n-edged rooted maps on a non-orientable surface of type g, where

$$g = 1 - \frac{1}{2}\chi. \tag{3}$$

Notice that, in the non-orientable case, g can be integer or half-integer. Then, one has the asymptotics [4]

$$P_g(n) \sim p_g n^{\gamma(g-1)} \lambda^n, \quad g > 0. \tag{4}$$

This defines a sequence of constants p_g. On the physics side, the Feynman diagrams for $O(N)$ or $Sp(N)$ theories are ribbon graphs with *crosscaps* that are embedded in *non-orientable surfaces*. This is discussed in detail in Section 3. If one believes in a matching between the counting problems of combinatorics and matrix models, the constants p_g ought to be able to be computed and asymptotically analyzed by the double scaling limit of an $O(N)$ or $Sp(N)$ matrix model. The latter is a pair of functions that satisfies a coupled system of two non-linear differential equations, as was explained in detail by Brézin et al. [11] and Harris and Martinec [39].

Before we get into details, let us mention that the constants p_g are notoriously hard to compute. In [4, Eq. (3.6)] Bender–Canfield give a non-linear recursion for $\phi_g^{(t)}(I,\alpha)$ that computes p_g via [4, Eq. (3.3)]. As in the case of t_g, the recursion is rather cumbersome and gives exact answers for p_g that do not depend on any unknown constants. In particular, Bender–Canfield obtained the first three values [4, p. 245]

$$p_{1/2} = -\frac{2\sqrt{6}}{\Gamma(-1/4)}, \qquad p_1 = \frac{1}{2}, \qquad p_{3/2} = \frac{\sqrt{6}}{3\Gamma(1/4)}. \tag{5}$$

The purpose of the present paper is to introduce a sequence (v_n) that is easy to compute via a quadratic non-linear recursion relation, and whose asymptotic is easy to analyze, and conjecturally agrees with p_g when $g = (n+1)/2$. Using the recursion of Bender–Canfield, Gao was able to match our conjecture for p_g for the first six values of g. Beyond that, nothing is known. The motivation for the sequence (v_n) comes from a study of the double scaling limit of $\mathrm{O}(N)$ matrix models following Brézin et al. [11] and Harris and Martinec [39]. This is discussed at leisure in Section 6.

1.4. The sequence (u_n)

Since the counting problems of non-orientable surfaces mix with those of the orientable ones (studied in [30]), in this section we give a brief review of Appendix A of [30]. Consider a function $u(z)$ that satisfies the Painlevé I differential equation

$$u^2 - \frac{1}{6}u'' = z. \tag{6}$$

Consider the unique formal power series solution to (6) asymptotic to $z^{1/2}$ for large $z > 0$

$$u(z) = z^{1/2}\sum_{n=0}^{\infty} u_n z^{-5n/2}. \tag{7}$$

It follows that the sequence (u_n) satisfies the following recursion relation

$$u_n = \frac{25(n-1)^2-1}{48}u_{n-1} - \frac{1}{2}\sum_{k=1}^{n-1} u_k u_{n-k}, \qquad u_0 = 1. \tag{8}$$

It was observed that the sequences (u_g) are (t_g) are related by

$$t_g = -\frac{1}{2^{g-2}\Gamma(\frac{5g-1}{2})}u_g. \tag{9}$$

To find the asymptotics of (u_n) one uses a *trans-series* solution of the differential equation (6). For a detailed discussion, see Section 5 below. In our context, trans-series are mild generalizations of formal power series and can be automatically computed much like the sequence (u_n) itself. The computation involves a finite number of unknown Stokes constants (i.e., adiabatic invariants) S. In our case, we have that the sequence (u_n) has an asymptotic expansion of the form

$$u_n \sim A^{-2n+\frac{1}{2}}\Gamma\left(2n-\frac{1}{2}\right)\frac{S}{2\pi\mathrm{i}}\left\{1 + \sum_{l=1}^{\infty}\frac{\mu_l A^l}{\prod_{m=1}^{l}(2n-1/2-m)}\right\}, \tag{10}$$

where the so-called *instanton value* A and the *Stokes constant* S are given by

$$A = \frac{8\sqrt{3}}{5}, \qquad S = -\mathrm{i}\frac{3^{\frac{1}{4}}}{\sqrt{\pi}}, \tag{11}$$

and the μ_l are defined by the recursion relation

$$\mu_l = \frac{5}{16\sqrt{3}l}\left\{\frac{192}{25}\sum_{k=0}^{l-1}\mu_k u_{(l-k+1)/2} - \left(l-\frac{9}{10}\right)\left(l-\frac{1}{10}\right)\mu_{l-1}\right\}, \qquad \mu_0 = 1 \tag{12}$$

with the understanding that $u_{n/2}=0$ if n is odd. To better understand the recursion relation (12), and to write it in a more compact form that relates to the trans-series of (6), consider the generating series

$$u_1(z)=z^{1/2}z^{-5/8}\mathrm{e}^{-Az^{5/4}}\sum_{n=0}^{\infty}\mu_n z^{-5n/4}. \tag{13}$$

Then (12) is equivalent to the following linearized version of (6)

$$u_1''-12uu_1=0 \tag{14}$$

where $u(z)$ is given by (6). Hopefully, the reader will not confuse the first term u_1 of the sequence (u_n) with the function $u_1(z)$ in Eq. (14).

1.5. The sequence (v_n)

In this section $u(z)$ will denote a function that satisfies the differential Eq. (6). Consider a function $v(z)$ that satisfies the differential equation

$$2v'-v^2+3u=0. \tag{15}$$

Consider the unique formal power series solution to (15) asymptotic to $z^{1/4}$ for large $z>0$

$$v(z)=z^{1/4}\sum_{n=0}^{\infty}v_n z^{-5n/4}. \tag{16}$$

It follows that the sequence (v_n) satisfies the following recursion relation

$$v_n=\frac{1}{2\sqrt{3}}\left(-3u_{n/2}+\frac{5n-6}{2}v_{n-1}+\sum_{k=1}^{n-1}v_k v_{n-k}\right),\qquad v_0=-\sqrt{3} \tag{17}$$

where (u_n) is given by (8), with the understanding that $u_{n/2}=0$ if n is not even. Our theorem concerns the asymptotics of (v_n) for large n.

Theorem 1. *The sequence (v_n) has an asymptotic expansion of the form*

$$v_n\sim (A/2)^{-n}\Gamma(n)\frac{S'}{2\pi\mathrm{i}}\left\{1+\sum_{l=1}^{\infty}\frac{\nu_l(A/2)^l}{\prod_{m=1}^{l}(n-m)}\right\} \tag{18}$$

where A is given in (11), $S'\neq 0$ is some non-zero Stokes constant, and the sequence (ν_n) is defined by the recursion relation

$$\nu_n=-\frac{4}{5n}\sum_{k=0}^{n-1}v_{n+1-k}\nu_k,\qquad \nu_0=1. \tag{19}$$

An even more compact form of the recursion relation (19) can be given by introducing the generating series

$$v_1(z)=z^{1/4}\sum_{n=0}^{\infty}\nu_n z^{-5n/4}. \tag{20}$$

Then the recursion relation (19) is equivalent to the following linearized form of (15)

$$v_1'=vv_1 \tag{21}$$

where $v(z)$ is given by (16).

1.6. Two conjectures

We can now formulate two conjectures which are motivated by Section 6 below. Our first conjecture links the sequence (v_n) with the sequence p_g.

Conjecture 1. *For all* $n = 0, 1, 2, \ldots$ *we have*

$$p_{\frac{n+1}{2}} = \frac{1}{2^{\frac{n-3}{2}}\Gamma(\frac{5n-1}{4})} v_n. \tag{22}$$

This conjecture reproduces the first three values (5) obtained in [4] and predicts for the next few values

$$p_2 = \frac{5}{36\sqrt{\pi}}, \qquad p_{5/2} = \frac{1033}{1024\sqrt{6}\Gamma(\frac{19}{4})}, \qquad p_3 = \frac{3149}{442\,368},$$
$$p_{7/2} = \frac{1\,599\,895}{294\,912\sqrt{6}\Gamma(\frac{29}{4})}, \qquad p_4 = \frac{484\,667}{560\,431\,872\sqrt{\pi}}, \qquad \ldots. \tag{23}$$

Conjecture 2. *The Stokes constant* S' *is given by*:

$$S' = \mathrm{i}\sqrt{6}. \tag{24}$$

Remark 1.1. Theorem 1 and Conjecture 1 reveal a single Stokes constant S' associated with the asymptotics of the sequence (v_n). A second Stokes constant is needed for the asymptotics of the k-instanton expansion of (v_n); see Theorem 2 and Remark 8.1.

2. Future directions

2.1. Approaches to Conjectures 1 and 2

In this section we discuss some potential approaches to Conjectures 1 and 2. Recall that Goulden–Jackson construct a solution to the KP hierarchy for a generating series associated to the $\mathrm{U}(N)$ gauge group.

Problem 1. Construct a version of the KP hierarchy corresponding to the generating series of the $\mathrm{O}(N)$ gauge group.

Perhaps the theory of zonal polynomials will play a role analogous to the Schur functions in generalizing the work of [35,49]. This integrable hierarchy without doubt will imply, as in [35, Eq. (45)] a quadratic recursion relation for the number of rooted maps $P_g(n)$ in a non-orientable surface of Euler characteristic $2 - 2g$ for a half-integer g.

Problem 2. State and prove a quadratic recursion relation for $P_g(n)$ in terms of $P_{g'}(n')$ for $(g', n') < (g, n)$.

Given this recursion relation, one may obtain a combinatorial proof of Conjecture 1 along the line of thought of [5]. A solution to Problem 2 can be obtained by identifying $P_g(n)$ with an expectation value of an $\mathrm{O}(N)$ matrix model and deduce the quadratic relation from the so-called *pre-string equation* of the matrix model; see for example [47].

Let us point out that the pair of functions (u, v) have a Lax pair and a Riemann–Hilbert problem, as was explained in [11]. Moreover, it is well-known that every solution to the Riemann–Hilbert problem with rational jump functions is meromorphic, see [48]. In addition, the Stokes constants are exactly calculable from a Riemann–Hilbert problem. For numerous instances of this calculation, that includes the case of the Painlevé equations, see [29] and also [42].

Problem 3. Give a solution to the Riemann–Hilbert problem of the pair (u, v) and compute the Stokes constant S' confirming Conjecture 2.

2.2. Relation to algebraic geometry

Let $\overline{M}_{g,n}$ be the Deligne–Mumford moduli space of Riemann surfaces of genus g with n punctures, and let ψ_i, $i = 1, \dots, n$ be the two-cohomology class defined by

$$\psi_i = c_1(\mathcal{L}_i), \quad i = 1, \dots, n, \tag{25}$$

where $\mathcal{L}_i$ is the bundle over $\overline{M}_{g,n}$ whose fiber at $\Sigma_g \in \overline{M}_{g,n}$ is the cotangent space $T^*\Sigma_g$ at the i-th puncture. In his seminal paper [55], Witten explains how to package the enumerative intersection theory of the moduli space of curves into a generating function that ought to satisfy the KdV equation and some initial conditions. The conjecture was subsequently proven by Kontsevich [43]. Using the Witten–Kontsevich theorem [55,43] together with the results of [40] it is possible to show that the intersection number

$$\langle \sigma_2^{3g-3} \rangle_g = \int_{\overline{M}_{g,3g-3}} \psi_1^2 \wedge \cdots \wedge \psi_{3g-3}^2, \quad g \geqslant 1, \tag{26}$$

is related to the coefficients u_n defined in (8) as follows

$$\frac{\langle \sigma_2^{3g-3} \rangle_g}{(3g-3)!} = -\frac{4^g}{(5g-5)(5g-3)} u_g. \tag{27}$$

A proof of Eq. (27) is given in [40, Section 6] using the ansatz [40, Eq. (5.22)]. A proof of this ansatz is given in [36, Theorem 3.1] and also in [24].

Through (9) one finds an algebro-geometric interpretation of t_g in terms of intersection numbers on $\overline{M}_{g,n}$. This relation motivates the following problem.

Problem 4. Give an enumerative algebro-geometric definition of the numbers v_n (or, equivalently, p_g).

3. Analyticity of the O(*N*) and Sp(*N*) free energy of a closed 3-manifold

An initial motivation for our work is the problem of the free energy of a closed 3-manifold M. In [30] it was shown that the U(N) version of the free energy $F_M^U(\tau, \hbar)$ of M has the form

$$F_M^U(\tau, \hbar) = \hbar^{-2} \sum_{g=0}^{\infty} \hbar^{2g} F_{M,g}^U(\tau)$$

where $\tau = N\hbar$ and $F_{M,g}^U(\tau) \in \mathbb{Q}[\![\tau]\!]$ are formal power series analytic in a disk D_M independent of g that contains 0 and depends on M. F_M^U is defined to be the logarithm of the LMO invariant, evaluated under the U(N) weight system. One may also define the O(N) (resp. Sp(N)) free energy of a closed 3-manifold M by

$$F_M^O(\tau, \hbar) = W_{\mathfrak{o}_N}\big(\log(Z_M)\big), \qquad F_M^{Sp}(\tau, \hbar) = W_{\mathfrak{sp}_N}\big(\log(Z_M)\big) \tag{28}$$

where Z_M is the LMO invariant of M and $W_{\mathfrak{g}}$ denotes the weight system of a metrized Lie algebra $\mathfrak{g}$; see [2]. The O(N) (and also the Sp(N)) free energy of a closed 3-manifold M can be written in the form

$$F_M^O(\tau, \hbar) = \hbar^{-2} \sum_{g=0}^{\infty} \hbar^{g} F_{M,g}^O(\tau) \tag{29}$$

where $\tau = N\hbar$ and $F_{M,g}^O(\tau) \in \mathbb{Q}[\![\tau]\!]$ for all g. In [30, Remark 4.2] it was observed that if p_g satisfies an asymptotic expansion similar to t_g, then the O(N) and Sp(N) free energy of M enjoys the same analyticity property as the U(N)-free energy. Thus, Conjecture 1 implies that the O(N) and Sp(N) free energy of M is analytic in the above sense.

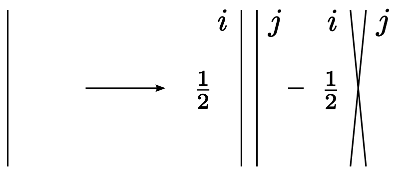
Let us discuss in detail the weight system for the Lie algebra $\mathfrak{o}_N$ of the orthogonal group O(N). This explains the appearance of graphs in non-orientable surfaces. Let e_{ij} denote the square matrix of size N with ij entry equal to 1 and all other entries zero. A basis for $\mathfrak{o}_N$ is $N_{ij} = e_{ij} - e_{ij}$ for $1 \leqslant i < j \leqslant N$. Since $e_{ij} e_{kl} = \delta_{j,k} e_{il}$ (where $\delta_{a,b} = 0$ (resp. 1) for $a = b$ (resp. $a \neq 1$)), and $\mathrm{Tr}(e_{ij}) = \delta_{i,j}$, (where $\mathrm{Tr}(M)$ denotes the trace of a square matrix M), it follows that the *Killing form* on $\mathfrak{o}_N$ is given by

$$(N_{ij}, N_{kl}) = 2(\delta_{jk}\delta_{il} - \delta_{jl}\delta_{ik}) \tag{30}$$

and its inverse (the so-called *propagator*) is given by

$$\langle N_{ij}, N_{kl} \rangle = \frac{1}{2}(\delta_{jk}\delta_{il} - \delta_{jl}\delta_{ik}). \tag{31}$$

This translates to the following diagrammatic way for the $\mathfrak{o}_N$ weight system of a vertex-oriented cubic graph (see for example [2] and compare also with [30, Eq. (18)]):



The resulting graphs are cubic ribbon graphs with cross-caps, and give rise exactly to embedded trivalent graphs in non-orientable surfaces.

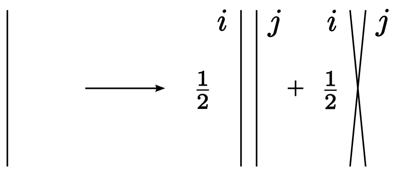
In Section 6 we will study the constants p_g using not the adjoint, but the symmetric representation of $\mathfrak{o}_N$. To explain this, observe that if $\mathbb{C}^N$ denotes the fundamental representation of $\mathfrak{o}_N$, then $\bigwedge^2(\mathbb{C}^N)$ is the adjoint representation of $\mathfrak{o}_N$ and $\mathrm{Sym}^2(\mathbb{C}^N)$ can be identified with the set of symmetric matrices of size N, with a basis given by

$$M_{ij} = e_{ij} + e_{ji}, \quad \text{for } 1 \leqslant i \leqslant j \leqslant N.$$

In that case, the symmetric matrix model (63) of Section 6 when $\beta = 1/2$ has propagator given by

$$\langle M_{ij}, M_{kl} \rangle = \frac{1}{2}(\delta_{jk}\delta_{il} + \delta_{jl}\delta_{ik}) \tag{32}$$

and leads to the following diagrammatic way



4. The method of Borel transform

The aim of this section is to prove the existence of an effective asymptotic expansion for the coefficients of a non-linear Euler-type differential equation using the theory of Borel transforms; see Theorem 2 below. This will provide a proof of Theorem 1. In fact, Theorem 1 requires only a small portion of the theory, since the function v of Theorem 1 satisfies a Riccati-type equation (15) that is well-studied in an excellent exposition of [52] and [7, Section 5]. Nonetheless, the method of Borel transform is rather general and perhaps not as widely known. For the benefit of the reader, we will give a short introduction to this beautiful theory in the case of the *non-resonant Euler-type* differential equation

$$y' = -\lambda y - \frac{1}{x}\beta y + g(x, y) \tag{33}$$

where $g(x, y)$ is analytic at $(\infty, 0)$, $g(x, y) = O(y^2, x^{-2})$, and the non-resonance condition is $\lambda \neq 0$. We will call λ and β the *eigenvalue* and the *exponent* of (33). λ and β are precisely the coefficients of the linearized equation

$$y' = -\lambda y - \frac{1}{x}\beta. \tag{34}$$

The Euler-type differential equation includes as a special case the *Riccati-type* differential equation

$$y' = -\lambda y - \frac{1}{x}\beta y + g_0(x) y + g_1(x) y^2 \tag{35}$$

where g_0 and g_1 are analytic at $x = \infty$ and $g_0(x) = O(x^{-2})$.

It is easy to see that a non-resonant Euler-type differential equation (33) has a unique formal power series solution $f_0(x) \in \mathbb{C}[[1/x]]$. In fact, Eq. (33) has a unique *trans-series* solution of the form $\hat{f}(x)$ where

$$\hat{f}(x) = \sum_{k=0}^{\infty} C^k f_k(x), \quad f_k(x) = x^{-\beta k} \mathrm{e}^{-\lambda k x} \sum_{n=0}^{\infty} \mu_{n,k} \frac{1}{x^n}, \ \mu_{1,0} = 1 \tag{36}$$

is obtained by substituting the above expression in (33) and equating the coefficient of every power of C to zero. This leads to a hierarchy of differential equations for $f_k(x)$ which is non-linear for $k = 0$, linear homogeneous for $k = 1$, and linear inhomogeneous for $k \geqslant 2$. For example, we have:

$$f_1' = -\lambda f_1 + g_y(x, f_0) f_1. \tag{37}$$

The above hierarchy of differential equations was first discovered by Écalle and was studied in [12, p. 54], [15], [50, pp. 11–12] and [51, Section 8]. An excellent presentation for the special case of the *Riccati equation* was given in [7, Section 5] and [52]. This special case is of interest to us since $v(z)$ satisfies the Riccati equation (15). It is important to realize that the coefficients $\mu_{n,k}$ are automatically computed from (33) by means of a non-linear recursion relation that involves $\mu_{n',k'}$ for $(n', k') < (n, k)$.

The next theorem, presumably well-known to the experts but absent from the literature, links the asymptotics of the trans-series coefficients $(\mu_{n,k})$ in terms of the neighboring coefficients $(\mu_{l,k\pm1})$ and two Stokes constants $S_{\pm1}$. In the physics literature, $f_k(x)$ often occurs as the perturbation theory of the k-instanton solution, in which case Theorem 2 states that the asymptotics of the coefficients of the k-instanton solution can be computed by the $(k \pm 1)$-instanton solutions and two adiabatic invariants $S_{\pm1}$.

Theorem 2. *Consider the differential equation* (33) *with* $\lambda \neq 0$ *and the coefficients* $(\mu_{n,k})$ *of the unique trans-series solution* (36)*. Then, for every* $k = 0, 1, 2, \ldots$ *and* n *large we have an asymptotic expansion*

$$\mu_{n,k} \sim \lambda^{-n+\beta}(k+1)\frac{S_1}{2\pi \mathrm{i}}\Gamma(n-\beta)\left\{\mu_{0,k+1} + \sum_{l=1}^{\infty} \frac{\mu_{l,k+1}\lambda^l}{\prod_{m=1}^{l}(n-\beta-m)}\right\}$$
$$+ (-\lambda)^{-n-\beta}(k-1)\frac{S_{-1}}{2\pi \mathrm{i}}\Gamma(n+\beta)\left\{\mu_{0,k-1} + \sum_{l=1}^{\infty} \frac{\mu_{l,k-1}(-\lambda)^l}{\prod_{m=1}^{l}(n+\beta-m)}\right\} \tag{38}$$

with the understanding that $\mu_{n,k} = 0$ *for* $k < 0$*.* $S_{\pm1}$ *are two Stokes constants, defined below.*

Proof. We will give the proof in three steps. At first, let us suppose that $k = 0$ and let us denote $\mu_{n,0} = a_n$. In other words, we have

$$f_0(x) = \sum_{n=1}^{\infty} a_n \frac{1}{x^n} \in \mathbb{C}[[x^{-1}]] \tag{39}$$

is the unique formal power series solution of (33). The first step provides the existence of an asymptotic expansion of (a_n).

Step 1. With the above assumptions, the sequence (a_n) has an asymptotic expansion of the form

$$a_n \sim \lambda^{-n+\beta} \frac{S_1}{2\pi \mathrm{i}} \Gamma(n-\beta) \left\{ 1 + \sum_{l=1}^{\infty} \frac{\mu_l \lambda^l}{\prod_{m=1}^{l}(n-\beta-m)} \right\} \tag{40}$$

where S_1 is an unspecified Stokes constant.

A proof of Step 1 is given in [19, Lemma 1.1]. The proof of Lemma 1.1 of [19] starts by writing down a non-linear recursion relation for the rescaled coefficients $b_n = \lambda^n a_n / n!$ of a formal power series solution in the form

$$b_n = b_{n-1} + R_n(b_0, \ldots, b_{n-1})$$

where $R_n(b_0, \ldots, b_{n-1})$ is a non-linear term, a polynomial in the variables $b_0, \ldots, b_{n-1}$. Then, one obtains an estimate of the non-linear term of the form $R_n(b_0, \ldots, b_n) = O(1/n^2)$. This implies that

$$b_n = b_{n-1} + O\left(\frac{1}{n^2}\right).$$

It follows that (b_n) is increasing and bounded above. Thus, the limit of b_n exists. Using this as input, one bootstraps and obtains, order by order in powers of $1/n$ an asymptotic expansion of (b_n).

The next step identifies the coefficients μ_l of the asymptotic expansion of (a_n) with the coefficients of the first trans-series $f_1(x)$ of (33).

Step 2. With the assumption of Step 1, we have

$$\mu_n = \mu_{n,1} \tag{41}$$

for all n. This proves Theorem 2 when $k = 0$.

A proof of Step 2 appears implicitly in [20, p. 1939] and also in [12,7,52,50,51]. Let us take this opportunity to sketch the argument for completeness and also to fix several typographical errors of [20, p. 1939]. Recall the notion of *Borel transform* which sends power series in $1/x$ to power series in p:

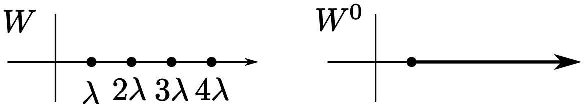
$$\frac{1}{x^\alpha} \mapsto \frac{p^{\alpha-1}}{\Gamma(\alpha)}.$$

Consider the differential Eq. (33) with unique formal power series solution $f_0(x)$, and let $\phi_0(p)$ denote the Borel transform of $f_0(x)$ given by

$$f_0(x) = \sum_{n=0}^{\infty} a_n \frac{1}{x^n}, \qquad \phi_0(p) = \sum_{n=0}^{\infty} \frac{a_{n+1}}{n!} p^n. \tag{42}$$

Step 1 implies that $\phi_0(p)$ is analytic at $p = 0$. Consider the punctured plane $W = \mathbb{C} \setminus \mathcal{L}$ where $\mathcal{L} = \lambda\mathbb{N}^+$ is a discrete subset of punctures that lies in a ray $\lambda[1, \infty)$. Consider the universal cover $\widetilde{W}$ which is identified with the set of homotopy classes (rel. boundary) of paths that begin at 0 and lie in W. Of importance is the portion $W^0 \cup W^1$ of W where

(a) W^0 is the first Riemann sheet of W given by $W^0 = \mathbb{C} \setminus \lambda[1, \infty)$. In other words, W^0 is the plane minus one cut.



(b) W^1 is the second Riemann sheet of W given by all paths in W that cross the cut $\lambda[1, \infty)$ at most once.

In [19, Theorem A.1] it is shown that ϕ_0 admits analytic continuation as an analytic function in $W^0 \cup W^1$ and $\phi_0(p)$ has local expansion around the singularity $p = \lambda$ of the form

$$\Delta_\lambda \phi_0 = S_1 \phi_1 \tag{43}$$

where $S_1 \in \mathbb{C}$ and

$$\phi_1(p) = p^{\beta-1} \sum_{l=0}^{\infty} \frac{\mu_{l,1}}{\Gamma(\beta + l)} p^l \tag{44}$$

is the Borel transform of the trans-series solution $f_1(x)$ and

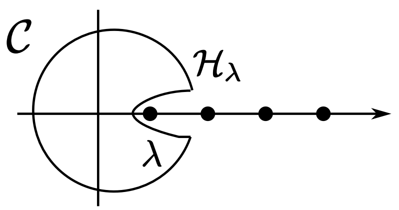
$$\Delta_\mu g(p) = \lim_{\epsilon \to 0^+} g\big(\mu + p\mathrm{e}^{\mathrm{i}\epsilon}\big) - g\big(\mu + p\mathrm{e}^{-\mathrm{i}\epsilon}\big) \tag{45}$$

denotes the *variation* of a multivalued analytic germ g near a singularity $p = \mu$; the so-called *alien derivative* in Écalle's language [28].

A well-known application of Cauchy's theorem in Borel plane (see [20, p. 1939] and also [17, Section 7]) now implies the asymptotic expansion (40) of (a_n). Let us sketch the argument, fixing some typographical errors from [20, p. 1939], and giving an exact formula, well-known to the physics community. Cauchy's theorem for $\phi_0(p)$ together with (42) and the analyticity of $\phi_0(p)$ at $p = 0$ implies that

$$\frac{a_{n+1}}{n!} = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{\phi_0(p)}{p^{n+1}} \mathrm{d}p$$

where γ is a small circle around $p = 0$. Now enlarge the contour γ to a contour $\mathcal{C} \cup \mathcal{H}_\lambda$ in the first sheet W^0, where $\mathcal{C}$ is a circle of radius $|\lambda| + \epsilon$, minus an arc, and $\mathcal{H}_\lambda$ is a Hankel contour centered at λ, for some fixed ϵ:



Thus,

$$\frac{a_{n+1}}{n!} = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}} \frac{\phi_0(p)}{p^{n+1}} \mathrm{d}p + \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{H}_\lambda} \frac{\phi_0(p)}{p^{n+1}} \mathrm{d}p.$$

The first integral can be estimated by $O(n!(|\lambda| + \epsilon)^{-n})$. For the second integral, make a change of variables $p = \lambda(1 + z)$ and let $\mathcal{H}$ denote a Hankel contour centered around 0. Then, we have

$$\int_{\mathcal{H}_\lambda} \frac{\phi_0(p)}{p^{n+1}} \mathrm{d}p = \lambda^{-n} \int_{\mathcal{H}_0} \frac{\phi_0(\lambda(1+z))}{(1+z)^{n+1}} \mathrm{d}z = \lambda^{-n} \int_0^\epsilon \frac{\Delta_\lambda \phi_0(\lambda z)}{(1+z)^{n+1}} \mathrm{d}z.$$

Now, Eqs. (43) and (44) imply that

$$\Delta_\lambda \phi_0(\lambda z) = S_1 \lambda^{\beta-1} \sum_{l=0}^{\infty} \frac{\mu_{l,1}\lambda^l}{\Gamma(\beta+l)} z^{\beta+l-1}$$

and the series on the right-hand side, after multiplication by $p^{-\beta+1}$, is analytic at $p=0$. A useful *Beta-integral* calculation gives that

$$\int_0^{\infty} \frac{z^{\gamma-1}}{(1+z)^{n+1}} \mathrm{d}z = \frac{\Gamma(\gamma)\Gamma(n+1-\gamma)}{\Gamma(n+1)},$$

and therefore

$$\int_0^{\epsilon} \frac{z^{\gamma-1}}{(1+z)^{n+1}} \mathrm{d}z = \frac{\Gamma(\gamma)\Gamma(n+1-\gamma)}{\Gamma(n+1)} \left(1 + O\left(\left(|\lambda|+\epsilon\right)^{-n}\right)\right).$$

Interchanging summation and integration by applying Watson's lemma, (see [49]) it follows that

$$a_{n+1} \sim \lambda^{-n+\beta-1} \frac{S_1}{2\pi \mathrm{i}} \sum_{l=0}^{\infty} \Gamma(n-\beta-l+1)\mu_{l,1}\lambda^l.$$

Comparing the above with Eq. (40) concludes Step 2. Strictly speaking, the above analysis works only when $\Re(\beta) > -1$. This is a local integrability assumption of the Beta-integral. The asymptotic expansion (40) remains valid as stated even when $\Re(\beta) \leqslant -1$ as follows by first integrating $f(x)$ a sufficient number of times, and then applying the analysis. This is exactly what was done in [20] and [17, Section 7].

Let

$$\phi_k(p) = p^{-k\beta-1} \sum_{n=0}^{\infty} \frac{\mu_{n,k}}{\Gamma(k\beta+n)} p^n$$

denote the Borel transform of $f_k(x)$ (given by (36)) for $k=0,1,2,\ldots$.

Step 3. For $k=0,1,2,\ldots$, $\phi_k(p)$ has analytic continuation as a multivalued analytic function in $\mathbb{C} \setminus (-k\lambda + \mathbb{N}^+\lambda)$. Moreover, we have

$$\Delta_{l\lambda}\phi_k = (k+l)S_l\phi_{k+l} \tag{46}$$

for $l = 1, -1, -2, -3, \ldots$, $k = 0,1,2,3,\ldots$, $k+l = 0,1,2,\ldots$ where S_l are constants. Moreover [7, Proposition 5.4], in the case of the Riccati equation (35), we have $S_l = 0$ for $l \leqslant -2$. Eq. (46) is the so-called *Bridge Equation* of Écalle, and appears in [12, p. 63] and also in [50, pp. 11–12]. For a detailed discussion, see also [7, Proposition 5.4] and [52,51].

Said differently, the Bridge Equation implies that $\phi_0(p)$ admits analytic continuation as an analytic function in W and the local expansion of every branch of $\phi_0(p)$ at each singularity is of the form (43) for a suitable Stokes constant S_j. This is the notion of *resurgence* coined and studied systematically by Écalle in the eighties. Unfortunately, Écalle's work remains unpublished, and appears in three Orsay preprints that the interested reader is welcomed to read, [28]. Fortunately for our purposes, the Bridge Equation for Euler-type differential equations has been established in print in the above mentioned references.

The Bridge Equation implies that the nearest non-zero singularity of $\phi_k(p)$ appears at $p = \pm\lambda$ (resp. $p=\lambda$) when $k \geqslant 2$ (resp. $k=0,1$), and that the variation of $\phi_k(p+\lambda)$ (resp. $\phi_k(p-\lambda)$) is proportional to $\phi_{k+1}(p)$ (resp. $\phi_{k-1}(p)$). Theorem 2 follows for all k by applying Cauchy's formula to $\phi_k(p)$, and deforming the contour of integration as in the case of $k=0$. $\square$

Let us make some remarks.

Remark 4.1. As stated, Theorem 2 requires analyticity of the coefficient

$$g(x, y) = \sum_{n=-1}^{\infty} g_n(x) y^{n+1}$$

of (33) at $(\infty, 0)$. In fact, the key Eq. (43) and Theorem 2 remains true when $k = 0$ and the Borel transform $\gamma_n(p)$ of $g_n(x)$ has analytic continuation as a multivalued analytic function in $\mathbb{C}$ minus a discrete set of points, and $\gamma_n(p)$ is analytic in the disk $|p| < |\lambda| + \epsilon$ for all $n \in \mathbb{N}$. This follows from the proof of the resurgence of $\phi_k(p)$ in Borel plane; see [12,50,51,7,52]. In particular, Eq. (43) and Theorem 2 holds for $k = 0$ when the $\gamma_n(p)$ is analytic multivalued in $\mathbb{C} \setminus 2\mathbb{Z}^*\lambda$ for all $n \in \mathbb{N}$, where $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$. Theorem 2 will be applied in this form to give a proof of Theorem 1.

Remark 4.2. Starting from a sequence $(\mu_{n,0})$ with an asymptotic expansion (38), one can consider the finitely many sequences $(\mu_{n,1})$ that arise, and repeat this process. This may be continued for ever, arriving at a notion of resurgence for sequences $(\mu_{n,0})$ that may not come from differential equations; see for example [17,18,34].

Remark 4.3. The constants $S_{\pm 1}$ in Eq. (38), if non-zero, can be numerically approximated rigorously and efficiently. For examples, see [16,41,5]. On the other hand, deciding whether $S_{\pm 1} = 0$ is not algorithmically known. In general the Stokes constants are transcendental invariants of the differential equation, sometimes known by the name of *adiabatic invariants*.

Remark 4.4. Theorem 2 is valid for systems of first order non-linear differential equations of rank 1 under a non-resonance assumption, see [15] for Steps 1 and 2 and a comment on [51, Section 13]. In that case the non-linear equation is

$$\mathbf{y}' = -\Lambda \mathbf{y} - \frac{1}{x} B\mathbf{y} + \mathbf{g}(x, \mathbf{y}) \tag{47}$$

and the linearized equation is

$$\mathbf{y}' = -\Lambda \mathbf{y} - \frac{1}{x} B\mathbf{y}$$

with *eigenvalue matrix* $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_r)$ and *exponent matrix* $B = (\beta_1, \ldots, \beta_r)$, where $\operatorname{diag}(d_1, \ldots, d_r)$ denotes a diagonal matrix with diagonal entries $d_1, \ldots, d_r$. The Borel transform $\boldsymbol{\phi_0}$ of the unique formal power series solution $\mathbf{f_0}$ to (47) is a multivalued analytic function on $W = \mathbb{C} \setminus \mathcal{L}$ where $\mathcal{L} = \lambda_1 \mathbb{N}^+ + \cdots + \lambda_r \mathbb{N}^+$. Moreover, $\boldsymbol{\phi_0}$ has analytic continuation in the principal sheet $W^0 = \mathbb{C} \setminus \bigcup_{j=1}^{r} \lambda_j [1, \infty)$ which is a plane cut by r rays. The asymptotic expansion of the coefficients of a component of a formal power series solution $\mathbf{f_0}$ of the non-linear differential equation has an asymptotic expansion of the form

$$a_n \sim \sum_{j \in J} \lambda_j^{-n+\beta_j} \frac{S_j}{2\pi \mathrm{i}} \Gamma(n - \beta_j) \left\{ 1 + \sum_{l=1}^{\infty} \frac{\mu_{l,j} \lambda_j^l}{\prod_{m=1}^{l} (n - \beta_j - m)} \right\} \tag{48}$$

where $J = \{j \in \{1, \ldots, r\} \mid |\lambda_j| = \min\{|\lambda_s| \mid s = 1, \ldots, r\}\}$.

The analogue of Eq. (38) does not seem to exist in the literature, especially in the resonant case. For some partial results regarding the Painlevé I equation, see [19].

5. Proof of Theorem 1

5.1. Theorem 2 implies Theorem 1

In this section we give a proof of Theorem 1 using Theorem 2. Consider the differential equation (15) where $u(z)$ satisfies (6). The change of variables

$$u(z) = z^{1/2}\big(1 + U(x)\big), \qquad v(z) = z^{1/4}\big(-\sqrt{3} + V(x)\big), \quad x = z^{5/4} \tag{49}$$

converts (6) to the following rank-1 differential equation

$$U''(x)+\frac{U'(x)}{x}-\frac{96U(x)^2}{25}-\frac{4U(x)}{25x^2}-\frac{4}{25x^2}=0 \tag{50}$$

which can be written as a first order rank-1 differential equation in the form

$$\mathbf{U}'=-\hat{\Lambda}\mathbf{U}-\frac{1}{x}\hat{B}\mathbf{U}+\hat{\mathbf{g}}(1/x,\mathbf{U}) \tag{51}$$

where

$$\mathbf{U}=\begin{pmatrix} U \\ U' \end{pmatrix}, \qquad \hat{\Lambda}=\begin{pmatrix} 0 & -1 \\ -\frac{192}{25} & 0 \end{pmatrix}, \qquad \hat{B}=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{52}$$

and

$$\hat{\mathbf{g}}(1/x,\mathbf{U})=\begin{pmatrix} 0 \\ \frac{96U(x)^2}{25}+\frac{4U(x)}{25x^2}+\frac{4}{25x^2} \end{pmatrix}.$$

A gauge transformation $\mathbf{U}\mapsto\mathbf{U}G$ converts (51) into the normalized resonant rank-1 differential equation

$$\mathbf{U}'=-\Lambda\mathbf{U}-\frac{1}{x}B\mathbf{U}+\mathbf{g}(1/x,\mathbf{U}) \tag{53}$$

where

$$\Lambda=\begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}, \qquad B=\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \tag{54}$$

and $A=8\sqrt{3}/5$ as in (11). Observe that the differential equation (53) is resonant. Nonetheless, Theorem 1 involves the function $v(z)$ which satisfies a Riccati equation (15). The substitution (49) converts (15) to the following rank 1 differential equation

$$V'(x)+\frac{4\sqrt{3}}{5}V(x)-\frac{\sqrt{3}}{5x}+\frac{6U(x)}{5}+\frac{V(x)}{5x}-\frac{2V(x)^2}{5}=0. \tag{55}$$

This is a non-resonant Riccati differential equation of the form (35) with eigenvalue $A/2=\frac{4\sqrt{3}}{5}$ and exponent $\beta=0$. Even though the coefficients of (55) are not analytic, their Borel transform is a multivalued analytic function in $\mathbb{C}\setminus\mathbb{Z}^*A$ and Remark 4.1 applies. The trans-series solution of (15) is of the form

$$v(z)=\sum_{l=0}^{\infty}C^l v_l(z) \tag{56}$$

where $v_0(z)$ is given in (16) and

$$v_k(z)=z^{1/4}\mathrm{e}^{-\frac{A}{2}z^{5/4}l}\sum_{n=0}^{\infty}v_{n,k}z^{-5n/4}$$

and we normalize $v_{0,1}=1$. Theorem 1 uses only $v_1(z)$ which satisfies the linearized version

$$v_1'-v_0v_1=0$$

of (15), as stated in (21). Theorem 1 follows from Theorem 2 when $k=0$ applied to the Riccati equation (55). This Riccati equation does not have analytic coefficients in a neighborhood of $x=\infty$, however Remark 4.1 applies. A rigorous numerical calculation of the Stokes constant S' is possible, following standard arguments that appear for example in [16,41] to conclude that $S'\neq 0$. This concludes the proof of Theorem 1. □

Remark 5.1. One can study in detail the full trans-series solution (56). The v_k satisfy the differential equations:

$$v_k' - \frac{1}{2}\sum_{i=0}^{k} v_i v_{k-i} = 0. \tag{57}$$

In principle, in writing the equation for the formal trans-series solution to v, we have to include in Eq. (15) a full formal trans-series solution for $u(z)$, but it is easy to see that the only way to find a solution for the v_k is to set this trans-series to zero. From (57) one easily deduces the following recursion relation for $v_{n,l}$:

$$v_{n+1,k} = -\frac{1}{\sqrt{3}(k-1)}\left\{\frac{5n}{4}v_{n,k} + \sum_{l=2}^{n+1} v_{n+1-l,k}v_l + \frac{1}{2}\sum_{i=1}^{k-1}\sum_{l=0}^{n+1} v_{l,i}v_{n+1-l,k-i}\right\}, \quad k \geqslant 2. \tag{58}$$

One finds, for example,

$$v_{0,k} = (-1)^{k-1}(2\sqrt{3})^{1-k}, \quad k \geqslant 1. \tag{59}$$

For equations of the Riccati type, it was shown in [7] that the full trans-series expansion can be written in terms of three formal power series, as follows. Let us denote

$$\hat{v}_0(x) = \sum_{n=2}^{\infty} v_n x^{-n}, \qquad \hat{v}_k(x) = \sum_{n=0}^{\infty} v_{n,k}x^{-n}, \quad k \geqslant 1. \tag{60}$$

Then, there are formal power series $v_\pm(x)$ such that

$$\hat{v}_k = (-1)^{k-1}v_+^{k-1}v_-^k(1 - v_+\hat{v}_0), \quad k \geqslant 1. \tag{61}$$

In our case we have

$$\begin{aligned} v_+(x) &= \frac{1}{2\sqrt{3}} + \frac{5}{192\sqrt{3}}\frac{1}{x^2} - \frac{25}{1152}\frac{1}{x^3} + \frac{3149}{36\,864\sqrt{3}}\frac{1}{x^4} - \frac{15\,995}{110\,592}\frac{1}{x^5} + \mathcal{O}\big(x^{-6}\big), \\ v_-(x) &= 1 - \frac{1}{4\sqrt{3}}\frac{1}{x} - \frac{1}{24}\frac{1}{x^2} - \frac{1459}{11\,520\sqrt{3}}\frac{1}{x^3} - \frac{5429}{34\,560}\frac{1}{x^4} - \frac{114\,343}{138\,240\sqrt{3}}\frac{1}{x^5} + \mathcal{O}\big(x^{-6}\big). \end{aligned} \tag{62}$$

5.2. *A brief discussion of the Riemann–Hilbert approach*

The Riemann–Hilbert method is an alternative way of proving and computing asymptotic expansions of the form (48). For an example relevant to the results of the paper, see [42, Appendix A] where Kapaev computes the asymptotic expansion of the sequence (u_n). The Riemann–Hilbert method uses the fact that all solutions of the Riemann–Hilbert problem (such as the function $u(z)$ and conjecturally also $v(z)$) of isomonodromy are *meromorphic* functions in the z-plane, with prescribed behavior at various sectors. Applying the Cauchy formula and a deformation of the contour argument as in Claim 2 above, allows one to deduce the asymptotic expansion of the coefficients (a_n) of a formal solution; see for example [42, Appendix A]. In addition, the Riemann–Hilbert approach computes exactly the corresponding Stokes constants S_j in (48). Whether the Riemann–Hilbert method computes the coefficients μ_n in Eq. (40) via the recursion (12) is unknown to us.

The Riemann–Hilbert approach overlaps with (but is neither a subset or a superset of) the theory of Borel transform. On the one hand, the differential equations that are amenable to the Riemann–Hilbert method are very closely linked to the notion of *integrability*, whereas the Borel transform method can deal with generic non-linear differential equations. On the other hand, the Riemann–Hilbert method can deal with variational problems that do not come from differential equations.

It was recently realized that the method of Borel transform is useful in studying problems of physical origin (such as those originating in 3-dimensional Quantum Topology) that do not come from differential/difference equations. See for example the results of [17,18] and the survey paper [34].

Without doubt, the Riemann–Hilbert method is complementary and overlapping with the method of Borel transform, and the combination of the two methods can give powerful results which have yet to be witnessed.

An alternative method to asymptotic expansions, that include exact computation of the Stokes constants is available from physics. The part of this theory, relevant to our problem, will be reviewed in Section 7.

6. Matrix models, orthogonal ensembles and non-orientable graphs

In this section we will give our motivation for Conjecture 1 following the study of the *symmetric quartic matrix model* studied by Brézin et al. [11] and Harris and Martinec [39].

After reduction to eigenvalues, the so-called *partition function* of a matrix model can be written as

$$Z_\beta = \frac{1}{(N!)^\beta (2\pi)^{\beta N}} \int \prod_{i=1}^{N} \mathrm{d}\lambda_i \, \left|\Delta(\lambda)\right|^{2\beta} \mathrm{e}^{-\frac{\beta}{g_s}\sum_{i=1}^{N} V(\lambda_i)}, \tag{63}$$

where

$$\Delta(\lambda_i) = \prod_{i<j} (\lambda_i - \lambda_j) \tag{64}$$

is the Vandermonde determinant of the eigenvalues and $V(\lambda)$ is a polynomial called the *potential* of the matrix model. The index β takes the values $1, 1/2, 2$ for Hermitian, real symmetric and symplectic matrices, respectively. The *Gaussian* matrix model is obtained for a potential of the form

$$V(\lambda) = \frac{1}{2}\lambda^2. \tag{65}$$

In order to obtain generating functions of maps, one finds an asymptotic expansion of Z around $N = \infty$. One also sets

$$g_s = \frac{t}{N} \tag{66}$$

where t is the so-called *'t Hooft parameter* and is kept finite in the expansion. This asymptotic expansion is called the large N expansion. Since equivalently we are doing the expansion around $g_s = 0$, we see from (63) that the large N expansion is a generalization of the standard asymptotic expansions of integrals depending on parameters. For the case $\beta = 1/2$, which is the one considered in [11,39], one finds that $F = \log Z$, the so-called *free energy*, can be written in the form

$$F(g_s, t) = \frac{1}{2} F_{\mathrm{o}}(g_s, t) + F_{\mathrm{u}}(g_s, t) \tag{67}$$

where $F_{\mathrm{o}}(g_s, t)$ and $F_{\mathrm{u}}(g_s, t)$ have the asymptotic expansions

$$F_{\mathrm{o}}(g_s, t) = \sum_{g=0}^{\infty} g_s^{2g-2} F_g^{\mathrm{o}}(t), \qquad F_{\mathrm{u}}(g_s, t) = \sum_{r=1/2}^{\infty} g_s^{2r-2} F_r^{\mathrm{u}}(t), \tag{68}$$

where in the second sum r takes both *integer* and *half-integer* values. The generating functions $F_g^{\mathrm{o}}(t)$ count maps on an orientable surface of genus g, while $F_r^{\mathrm{u}}(t)$ count maps on a non-orientable surface with $2r$ crosscaps. These generating functions can be computed in closed form by using elegant methods started in [9,6] and which culminated in the algebro-geometric formulation of Eynard and collaborators [25,13,27]. It turns out that all these quantities can be calculated by computing residues of meromorphic forms on an algebraic curve of the form

$$y(x) = M(x)\sqrt{\sigma(x)}, \quad \sigma(x) = (x-a)(x-b). \tag{69}$$

If $V(x)$ is a polynomial of degree d, $M(x)$ is also a polynomial, of degree $d-2$. This curve is called the *spectral curve* of the matrix model, and its detailed form depends on the potential $V(z)$ used in (63).

Notice that this curve has a branch cut on $[a,b]$, and the branch points a,b, as well as $M(x)$ are easy to compute once the potential is given (see for example [22,46] for reviews and examples). For a *quartic potential*

$$V(z)=\frac{z^2}{2}+\lambda z^4 \tag{70}$$

one has

$$b=-a=2\alpha,\quad \alpha^2=\frac{1}{24\lambda}\left(-1+\sqrt{1+48\lambda t}\right), \tag{71}$$

and

$$M(x)=1+8\lambda\alpha^2+4\lambda x^2. \tag{72}$$

If we set $t=1$ for simplicity, it is easy to see from this description that the generating functions $F_g^{\mathrm{o}}(\lambda)$, $F_r^{\mathrm{u}}(\lambda)$ are analytic at $\lambda=0$ and that the nearest singularity is at

$$\lambda_c=-\frac{1}{48}. \tag{73}$$

In terms of the variable y defined by

$$y=g_s^{-4/5}\left(1-\frac{\lambda}{\lambda_c}\right) \tag{74}$$

one finds that the generating functions behave as

$$g_s^{2g-2}F_g^{\mathrm{o}}(\lambda)\sim c_g^{\mathrm{o}}y^{-5(g-1)/2},\qquad g_s^{2r-2}F_g^{\mathrm{u}}(\lambda)\sim c_r^{\mathrm{u}}y^{-5(r-1)/2}, \tag{75}$$

for $g\neq 1$. For $g=1$ we rather have

$$F_1^{\mathrm{o}}(\lambda)\sim c_1^{\mathrm{o}}\log y. \tag{76}$$

The *double-scaled* generating functions $F_{\mathrm{ds}}^{\mathrm{o}}(y)$, $F_{\mathrm{ds}}^{\mathrm{u}}(y)$ are then defined by

$$\begin{aligned} F_{\mathrm{ds}}^{\mathrm{o}}(y)&=c_0^{\mathrm{o}}y^{5/2}+c_1^{\mathrm{o}}\log y+\sum_{g\geqslant 2}c_g^{\mathrm{o}}y^{-5(g-1)/2},\\ F_{\mathrm{ds}}^{\mathrm{u}}(y)&=\sum_{r\geqslant 1/2}c_r^{\mathrm{u}}y^{-5(r-1)/2}. \end{aligned} \tag{77}$$

It turns out that these asymptotic expansions can be obtained as solutions to ordinary differential equations. For the orientable part, it was shown already in [10,23,37] that

$$f(y)=-\left(F_{\mathrm{ds}}^{\mathrm{o}}\right)''(y) \tag{78}$$

(also called the double-scaled *specific heat*) satisfies the Painlevé I equation

$$f^2-\frac{1}{3}f''=y. \tag{79}$$

In order to compute the coefficients c_r^{o} one considers the formal power series for f

$$f(y)=\sum_{n=0}^{\infty}f_n y^{-(5n-1)/2}. \tag{80}$$

On the other hand, it was shown in [11,39] that the first derivative

$$g(y)=-2\left(F_{\mathrm{ds}}^{\mathrm{u}}\right)' \tag{81}$$

satisfies the differential equation

$$g^3-6gg'+4g''-6gf+6f'=0. \tag{82}$$

In order to compute the coefficients c_r^{u} one considers the formal power series for g

$$g(y)=\sum_{n=0}^{\infty} g_n y^{-(5n-1)/4}.$$

The differential equation (82) can be integrated out once and gives [11, Eq. (3.25)]

$$f=-\frac{2}{3}g'+\frac{1}{6}g^2+c\exp\left(\int_1^y \mathrm{d}y'\, g\bigl(y'\bigr)\right). \tag{83}$$

Brézin–Neuberger argue that the appropriate boundary conditions fix $c=0$, and it follows that (f,g) satisfy the pair of equations

$$f^2-\frac{1}{3}f''=y, \qquad f=-\frac{2}{3}g'+\frac{1}{6}g^2. \tag{84}$$

Notice that the second equation is a Riccati-type equation for g with f known.

To bring these two equations into the form (6), (15) we must make the change of variables

$$y=2^{\frac{2}{5}}z, \qquad f=2^{\frac{1}{5}}u. \tag{85}$$

The change of variables (85) forces the following change of variables

$$g=2^{\frac{3}{5}}v, \qquad \mathcal{F}_{\mathrm{ds}}(z)=2F_{\mathrm{ds}}(y). \tag{86}$$

With these change of variables, the differential equations (84) for the pair of functions (u,v) become (6) and (15). In the new variables, the specific heat of the matrix model

$$-\mathcal{F}_{\mathrm{ds}}''(z)=-\frac{1}{2}\bigl(\mathcal{F}_{\mathrm{ds}}^{\mathrm{o}}\bigr)''(z)-\bigl(\mathcal{F}_{\mathrm{ds}}^{\mathrm{u}}\bigr)''(z) \tag{87}$$

is given by

$$u(z)+v'(z)=\sum_{n=0}^{\infty} u_n z^{-(5n-1)/2}-\sum_{n=0}^{\infty}\frac{5n-1}{4}v_n z^{-(5n+3)/2} \tag{88}$$

up to an overall factor 1/2. Notice that in the second sum, the index n is related to the half-genus g of [4, p. 244] by

$$n=2g-1.$$

Thus, the total specific heat (88) can be written as

$$u(z)+v'(z)=\sum_g\left(u_g-\frac{5g-3}{2}v_{2g-1}\right)z^{-(5g-1)/2} \tag{89}$$

where g runs now through the natural numbers and the half natural numbers. On the other hand, the total heat for this matrix model should behave the same way as the generating series

$$\sum_g (t_g+p_g)x^g$$

of [4]. The relationship (9) between t_g and u_g forces us to predict that

$$p_g=\frac{1}{2^{g-2}\Gamma(\frac{5g-1}{2})}\frac{5g-3}{2}v_{2g-1}=\frac{1}{2^{g-2}\Gamma(\frac{5g-3}{2})}v_{2g-1}$$

which is exactly Conjecture 1.

We emphasize that this conjecture is a consequence of the matching between counting problems in combinatorics and matrix models. Although this relation can be rigorously established in certain cases (see for example [8]), the physics techniques to extract the coefficients t_g, p_g seem to be more efficient than current mathematical techniques, as they lead to simple non-linear differential equations.

7. A physics derivation of the Stokes constants

This section is of independent interest and is included for completeness. It offers an alternative exact computation (including the Stokes constant S' of Eq. (24)) of the asymptotic expansion of sequences of enumerative interest, and although it is not rigorous, it is a motivation of Conjecture 2.

In many quantum field theories, the standard perturbative expansion in powers of the coupling constant g has non-perturbative instanton corrections which behave as $\mathrm{e}^{-kA/g}$, where A is the instanton action and k is the instanton number. Perturbation theory in powers of g at a fixed instanton number k is in principle possible, and when successful, it leads to a computation of the coefficients $a_{k,n}$ of series of the form

$$\mathrm{e}^{-kA/g} g^{\gamma} \sum_{n=0}^{\infty} a_{k,n} g^n. \tag{90}$$

In the physics community it is believed that, at least in some quantum field theories, the asymptotic expansion of $(a_{k,n})$ for fixed k and large n is exactly computable in terms of the coefficients $(a_{k',n})$ for k' nearby to k much like Eq. (38) of Theorem 2. This belief can be mathematically formulated as a resurgence conjecture for the power series that appear in perturbative quantum field theory. For an example of this principle for a 3-dimensional quantum field theory, see [34]. One important aspect of the above belief is the ability to compute exactly all constants (including the Stokes constants) in the asymptotic expansions, including the Stokes constants, in some class of quantum field theories. This is a reflection of integrability of these theories.

The $1/N$ expansion of gauge theories with gauge group $\mathrm{U}(N)$ is in this respect very similar to the standard perturbative expansion in powers of the coupling constant. The $1/N$ expansion also has non-perturbative k-instanton corrections which behave as e^{-kAN} [44,53] and have the structure

$$\mathrm{e}^{-kNA} N^{\gamma} \sum_{n=0}^{\infty} \frac{a_{k,n}}{N^n}. \tag{91}$$

In some cases these series should also have resurgence properties and the corresponding Stokes constants should be exactly calculable.

The field theory relevant to the current paper is a matrix model, where resurgence properties are certainly expected. We can study the behavior of the instanton corrections (90) in this matrix model near the singularity (73) and extract as before the most singular part of each coefficient $a_{k,n}$ (which in this case will be functions of t, λ). One finds that, in this double-scaling limit, the k-instanton correction (90), leads to the k-th term in the trans-series expansion of the partition function. Moreover, one can show (at the physics level of rigor) that the one-instanton amplitude gives the discontinuity of the first term of the trans-series along the Stokes line (see for example [14] for a detailed exposition), and this makes possible to calculate the Stokes constant. In the case of matrix models, this was shown in an important paper by F. David [21], who calculated explicitly the Stokes constant S in (11) by using matrix model techniques. His calculation was further clarified in [38] and extended in [47]. The connection between instanton calculus in matrix models and trans-series of differential equations of the Painlevé type is further discussed in [45]. In this section, we will compute the Stokes constants S and S' by adapting the matrix model technology of [47] to the case of matrix ensembles with arbitrary values of β. Needless to say, the computations in this section are not rigorous.

In computing the asymptotic expansion of the partition function (63) we have expanded around the saddle-point $\lambda_i = 0$ for all the eigenvalues $i = 1, \ldots, N$. The first instanton correction corresponds to a saddle-point expansion in which one of the eigenvalue integrations is around a non-trivial saddle-point $\lambda = x_0$, and can be written as

$$Z_{\beta}^{(1)}(N) = \frac{N}{(N!)^{\beta}(2\pi)^{\beta N}} \int_{x\in\mathcal{I}} \mathrm{d}x\, \mathrm{e}^{-\frac{\beta}{g_s}V(x)} \int_{\lambda\in\mathcal{I}_0} \prod_{i=1}^{N-1} \mathrm{d}\lambda_i \left|\Delta(x, \lambda_1, \ldots, \lambda_{N-1})\right|^{2\beta} \mathrm{e}^{-\frac{\beta}{g_s}\sum_{i=1}^{N-1} V(\lambda_i)}, \tag{92}$$

where the first integral in x is over the saddle-point contour around the non-trivial saddle-point, which we have denoted by $x \in \mathcal{I}$, while the rest of the $N-1$ eigenvalues are integrated over the saddle-point contour $\mathcal{I}_0$ around the trivial saddle-point. The overall factor of N is due to the fact that there are N choices for x among the N eigenvalues. We have

$$Z_\beta^{(1)}(N) = \frac{N}{(2\pi N)^\beta} Z_\beta^{(0)}(N-1) \int_{x\in\mathcal{I}} \mathrm{d}x \left\langle \det|x\mathbf{1} - \Lambda_{N-1}|^{2\beta}\right\rangle_{N-1}^{(0)} \mathrm{e}^{-\frac{\beta}{2g_s}V(x)}$$
$$\equiv \frac{N}{(2\pi N)^\beta} Z_\beta^{(0)}(N-1) \int_{x\in\mathcal{I}} \mathrm{d}x\, f(x). \tag{93}$$

The notation in this equation is as follows. $Z_\beta^{(0)}(N)$ is the partition function (63) evaluated around the trivial saddle-point $\lambda_i = 0$. Λ_{N-1} is the diagonal $(N-1)\times(N-1)$ matrix given by $\mathrm{diag}(\lambda_1, \dots, \lambda_{N-1})$. $\langle \mathcal{O}\rangle_N^{(0)}$ is the normalized average of any symmetric polynomial $\mathcal{O}(\lambda_i)$ in the eigenvalues λ_i, computed by a saddle-point calculation around the standard saddle-point,

$$\langle \mathcal{O}\rangle_N^{(0)} = \frac{\int_{\lambda\in\mathcal{I}_0} \prod_{i=1}^N \mathrm{d}\lambda_i\, |\Delta(\lambda)|^{2\beta} \mathcal{O}(\lambda) \mathrm{e}^{-\frac{\beta}{2g_s}\sum_{i=1}^N V(\lambda_i)}}{\int_{\lambda\in\mathcal{I}_0} \prod_{i=1}^N \mathrm{d}\lambda_i\, |\Delta(\lambda)|^{2\beta} \mathrm{e}^{-\frac{\beta}{2g_s}\sum_{i=1}^N V(\lambda_i)}}. \tag{94}$$

Finally, we have also defined

$$f(x) = \left\langle \det|x\mathbf{1} - \Lambda_{N-1}|^{2\beta}\right\rangle_{N-1}^{(0)} \mathrm{e}^{-\frac{\beta}{2g_s}V(x)}. \tag{95}$$

The total partition function will be written as a trans-series expansion

$$Z_\beta = Z_\beta^{(0)} + Z_\beta^{(1)} + \cdots \tag{96}$$

since, as we will see in a moment, $Z_\beta^{(1)}$ is exponentially small, and it is proportional to the small parameter $\mathrm{e}^{-\mathcal{A}/g_s}$ (where $\mathcal{A}$ will be calculated shortly). It follows that the trans-series expansion of the free energy will be given by

$$F = F^{(0)} + F^{(1)} + \cdots \tag{97}$$

where

$$F^{(1)} = \frac{Z_\beta^{(1)}(N)}{Z_\beta^{(0)}(N)} = \frac{N}{(2\pi N)^\beta} \frac{Z_\beta^{(0)}(N-1)}{Z_\beta^{(0)}(N)} \int_{x\in\mathcal{I}} \mathrm{d}x\, f(x). \tag{98}$$

The calculation of this quantity is very similar to the one performed in [47], and we will just present the main intermediate steps. The result involves the connected averages of the matrix model,

$$W_h(p_1, \dots, p_h) = \beta^{h-1} \left\langle \mathrm{tr}\,\frac{1}{p_1 - \Lambda_N} \cdots \mathrm{tr}\,\frac{1}{p_h - \Lambda_N}\right\rangle_{(\mathrm{c})}, \tag{99}$$

where the subscript (c) stands for connected. These averages have an asymptotic g_s expansion of the form

$$W_h(p_1, \dots, p_h) = \sum_{r=0}^\infty g_s^{2r+h-2} W_{r,h}(p_1, \dots, p_h; t), \tag{100}$$

where r runs over non-negative integers and half-integers. We will also need the integrated version of these averages,

$$A_{r,h}(x;t) = \int^{x_1} \mathrm{d}p_1 \cdots \int^{x_h} \mathrm{d}p_h\, W_{r,h}(p_1, \dots, p_h)\bigg|_{x_1=\cdots=x_h=x}, \tag{101}$$

where the integration constant can be fixed by imposing appropriate boundary conditions at $x \to \infty$ (see [47] for details). Another quantity that enters the computation is the *effective potential*

$$V_{\mathrm{eff}}(x;t) = V(x) - 2A_{0,1}(x;t) \tag{102}$$

which satisfies

$$\frac{\mathrm{d}V_{\mathrm{eff}}(x;t)}{\mathrm{d}x} = y(x). \tag{103}$$

We can now compute the integral over $f(x)$ at leading order in g_s in terms of these quantities. It is easy to see that, due to (103), the non-trivial saddle-points are given by the zeros of the moment function $M(x)$. Let x_0 be such a zero. We obtain

$$\int\limits_{x\in\mathcal{I}} \mathrm{d}x\, f(x) = \sqrt{\frac{2\pi g_s}{\beta V''_{\mathrm{eff}}(x_0)}} \exp\left(-\frac{\beta}{g_s} V_{\mathrm{eff}}(x_0) + \Phi(x_0)\right)\left(1 + \mathcal{O}(g_s)\right), \tag{104}$$

where

$$\Phi(x) = \beta\left(2A_{0,2}(x;t) + \partial_t V_{\mathrm{eff}}(x;t)\right) + A_{1/2,1}(x;t). \tag{105}$$

The quotient appearing in (98) can be computed as

$$\frac{Z_\beta^{(0)}(N-1)}{Z_\beta^{(0)}(N)} = \frac{\Gamma(1+\beta)}{\beta^\beta (4\pi^2 t)^{\frac{1-\beta}{2}}} \exp\left[-\frac{\beta}{g_s}\partial_t F_0 + \frac{\beta}{2}\partial_t^2 F_0 - \partial_t F_{1/2} + \mathcal{O}(g_s)\right]. \tag{106}$$

Putting everything together, we obtain the contribution of x_0 to $F^{(1)}$ at leading order in the g_s expansion,

$$F_{x_0}^{(1)} = \frac{N}{(2\pi N)^\beta} \frac{\Gamma(1+\beta)}{\beta^\beta (4\pi^2 t)^{\frac{1-\beta}{2}}} \sqrt{\frac{2\pi g_s}{\beta V''_{\mathrm{eff}}(x_0)}} \exp\left(-\frac{\beta}{g_s}\mathcal{A}\right) \times \exp\left[\Phi(x_0) + \frac{\beta}{2}\partial_t^2 F_0 - \partial_t F_{1/2}\right] \tag{107}$$

where [47]

$$\mathcal{A} = V_{\mathrm{eff}}(x_0) + \partial_t F_0 = \int_b^{x_0} y(x)\,\mathrm{d}x \tag{108}$$

is the instanton action of the matrix model corresponding to the non-trivial saddle-point at x_0. Given a matrix model potential, the leading contribution to $F^{(1)}$ is given by the sum of the contributions of the saddle-points with lowest instanton action (in absolute value).

All the quantities appearing in (107) which are needed when $\beta = 1$ have been already computed explicitly in [47]. Using the results of [13] we can also compute all the quantities needed for $\beta \neq 1$, in terms of data of the spectral curve (69). Since $M(z)$ is a polynomial, it can be written as

$$M(z) = c\prod_{i=1}^{d-2}(z - z_i). \tag{109}$$

We find for example

$$W_{1/2,1}(p) = \left(1-\beta^{-1}\right)\left\{\frac{d-1}{2\sqrt{\sigma(p)}} - \frac{1}{4}\frac{2p-a-b}{(p-a)(p-b)} - \frac{1}{2\sqrt{\sigma(p)}}\sum_{i=1}^{d-2}\left(\frac{\sqrt{\sigma(p)}-\sqrt{\sigma(z_i)}}{p-z_i}\right)\right\}, \tag{110}$$

and

$$\partial_t F_{1/2} = -(1-\beta)\left\{\frac{1}{2}\log\left[\frac{1}{t}\left(\frac{b-a}{4}\right)^2\right] + \log c + \sum_{i=1}^{d-2}\log\left[\frac{1}{2}\left(z_i - \frac{a+b}{2} + \sqrt{\sigma(z_i)}\right)\right]\right\}. \tag{111}$$

In the expression for $\partial_t F_{1/2}$ we have subtracted $\partial_t F^G_{1/2}$, which is the $r = 1/2$ free energy of the Gaussian model.

Let us specialize these results for the quartic matrix model considered in [11,39]. From (72) we see that $M(x)$ has two zeros at $\pm x_0$,

$$x_0^2 = -\frac{1+8\lambda\alpha^2}{4\lambda}. \tag{112}$$

Setting $\beta = 1/2$ one finds by integration of (110) that

$$\begin{aligned} A_{1/2,1}(p) = {} & \frac{1}{4}\log\left(p^2 - 4\alpha^2\right) - \frac{3}{2}\log\left[p + \sqrt{p^2 - 4\alpha^2}\,\right] \\ & + \frac{1}{2}\log\left[4\alpha^2\left(x_0^2 + p^2\right) - 2x_0^2 p^2 - 2x_0^2 p^2\sqrt{\left(1 - 4\alpha^2/p^2\right)\left(1 - 4\alpha^2/x_0^2\right)}\,\right] \\ & + \frac{3}{2}\log 2 - \frac{1}{2}\log\left[4\alpha^2 - 2x_0^2 - 2x_0^2\sqrt{\left(1 - 4\alpha^2/x_0^2\right)}\,\right] \end{aligned} \tag{113}$$

and from (111) that

$$\partial_t F_{1/2} = -\frac{1}{4}\log\frac{\alpha^2}{t} - \log\left[\frac{1}{2}\left(\sqrt{1+8\lambda\alpha^2} + \sqrt{1+24\lambda\alpha^2}\,\right)\right]. \tag{114}$$

Before proceeding with the calculation of (107), we present a test of these expressions. Based on general arguments relating matrix integrals and enumeration of maps [26,8] it follows that

$$\left\langle \operatorname{tr} \Lambda_N^4 \right\rangle_{\mathbb{RP}^2} = \operatorname{Res}_{p=0} p^4 W_{1/2,1}(p) \tag{115}$$

is a generating functional for the number of rooted quadrangulations c_n of the projective plane with n vertices,

$$\left\langle \operatorname{tr} \Lambda_N^4 \right\rangle_{\mathbb{RP}^2} = t^2 \sum_{n=1}^{\infty} c_n (-4\lambda t)^{n-1}. \tag{116}$$

This generating functional can be explicitly obtained from (113) in terms of α and x_0,

$$\left\langle \operatorname{tr} \Lambda_N^4 \right\rangle_{\mathbb{RP}^2} = x_0^4 - \alpha^4 - x_0^2\left(x_0^2 + 2\alpha^2\right)\sqrt{1 - \frac{4\alpha^2}{x_0^2}}. \tag{117}$$

One finds for the c_n sequence the values

$$5, 38, 331, 3098, 30\,330, 306\,276, 3\,163\,737, \ldots. \tag{118}$$

The first ones can be tested with the results of [1]. It should be also possible to obtain the generating functional for these numbers from the general combinatorial results of [33].

The zeros (112) of $M(x)$ give two saddle-points with the same value of $\mathcal{A}$ and the same $F^{(1)}_{\pm x_0}$, therefore they both contribute to the instanton amplitude. In order to make contact with the trans-series solution of the differential equation, we must analyze the behavior of (107) near the singular point (73) (we set again $t = 1$). Using the variable z introduced in (85) and taking into account the factor of 2 relating F and $\mathcal{F}$ in (86), one obtains

$$\mathcal{F}^{(1)}_{\beta=1}(z) = \frac{\mathrm{i}}{8\cdot 3^{\frac{3}{4}}\pi^{\frac{1}{2}}} z^{-5/8}\exp\left(-\frac{8\sqrt{3}}{5}z^{5/4}\right)+\cdots,$$
$$\mathcal{F}^{(1)}_{\beta=\frac{1}{2}}(z) = \frac{\mathrm{i}}{\sqrt{2}}\exp\left(-\frac{4\sqrt{3}}{5}z^{5/4}\right)+\cdots. \tag{119}$$

From here, and taking into account the relations between $\mathcal{F}(z)$ and $u(z)$, $v(z)$, we find

$$u_1(z) = -\left(\mathcal{F}^{(1)}_{\beta=1}\right)''(z) = -\mathrm{i}\frac{3^{\frac{1}{4}}}{2\sqrt{\pi}}z^{-5/8}\exp\left(-\frac{8\sqrt{3}}{5}z^{5/4}\right)\left(1+\mathcal{O}\left(z^{-5/4}\right)\right) \tag{120}$$

and

$$v_1(z) = -2\left(\mathcal{F}^{(1)}_{\beta=\frac{1}{2}}\right)'(z) = \mathrm{i}\sqrt{6}z^{\frac{1}{4}}\exp\left(-\frac{4\sqrt{3}}{5}z^{5/4}\right)\left(1+\mathcal{O}\left(z^{-5/4}\right)\right). \tag{121}$$

We now recall that this computation computes the discontinuity across the Stokes line, therefore the overall constants appearing in these two trans-series solutions are the Stokes constants S, S' that we introduced before. Let us end this section with a problem.

Problem 5. Compute rigorously the k-instanton expansion of a matrix model (with polynomial potential) using the Riemann–Hilbert method or the method of Borel transforms.

8. Numerics

We now give numerical evidence for Conjecture 2. To do this, we study the sequence

$$s_n = \frac{2\pi(A/2)^n}{\Gamma(n)}v_n \tag{122}$$

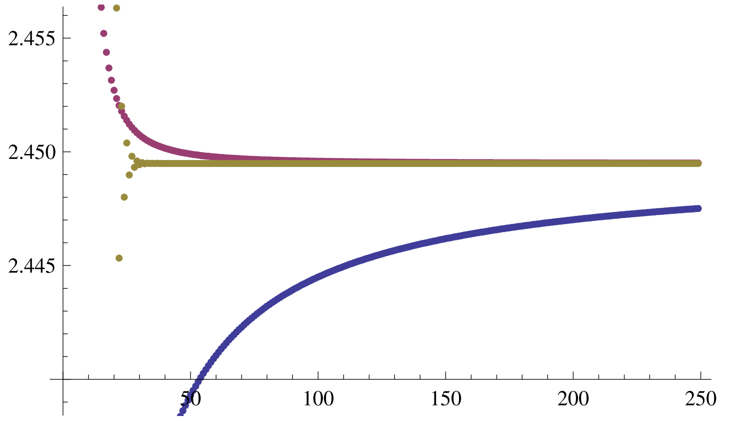
which according to Theorem 1 has the following asymptotic behavior

$$s_n \sim \sum_{k=0}^{\infty}\frac{a_k}{n^k}, \quad a_0 = -\mathrm{i}S',\ a_1 = -\frac{\mathrm{i}}{2}S'A\nu_1,\ \ldots. \tag{123}$$

In order to test Theorem 1 and Conjecture 2, it is useful to have a precise numerical method to determine the numbers a_k. The method of *Richardson transforms* (see for example [3, p. 375] for an exposition) is very well-suited for sequences of the type (123) and it is easy to implement. Given a sequence of the form (123), its N-th Richardson transform is defined by

$$s_n^{(N)} = \sum_{k=0}^{N}\frac{s_{n+k}(g+k)^N(-1)^{k+N}}{k!(N-k)!}. \tag{124}$$

The effect of this transformation is to remove subleading tails in (123). The values $s_n^{(N)}$ give numerical approximations to a_0, and these approximations become better as N, n increase. As an example of this procedure, we plot the values of the sequences $S_n^{(N)}$, for $N = 0, 1, 5$ and $n = 1, \ldots, 250$. The bottom curve corresponds to $N = 0$, the top curve corresponds to $N = 1$, and the intermediate curve is $N = 5$.



We can see that the Richardson transforms provide a very fast convergence to the expected value

$$\sqrt{6} = 2.449489742783178098197284074 71\ldots. \tag{125}$$

Numerically, we find that $n = 250$ and $N = 20$, 30 provide an approximation to the expected value with 28 (respectively, 30) significant digits,

$$\begin{aligned} s_{250}^{(20)} &= 2.44948974278317809819728407459\ldots, \\ s_{250}^{(30)} &= 2.44948974278317809819728407471\ldots. \end{aligned} \tag{126}$$

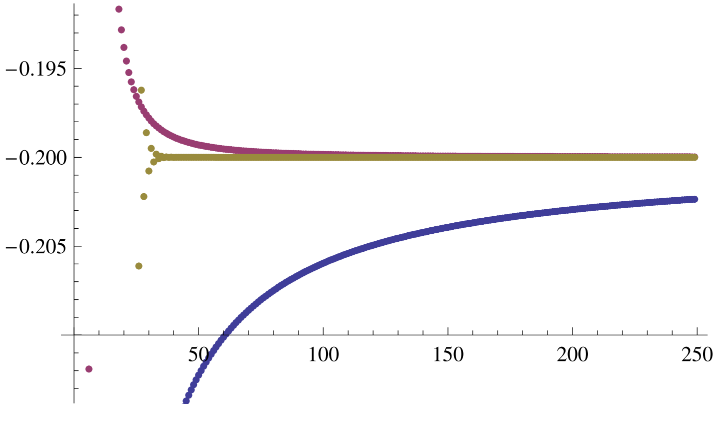
We can verify in a similar way the value of a_1 by considering the auxiliary sequence

$$r_n = n\left(\mathrm{i}\frac{s_n}{S'} - 1\right) \tag{127}$$

with the asymptotic behavior

$$r_n \sim \frac{1}{2}\nu_1 A + \mathcal{O}(1/n), \quad n \to \infty. \tag{128}$$

We plot the values of the sequences $r_n^{(N)}$, for $N = 0$, 1, 5 and $n = 1, \ldots, 250$.



Like before, we have very fast convergence to the expected value

$$\frac{1}{2}\nu_1 A = -\frac{1}{5}. \tag{129}$$

One finds numerically

$$\begin{aligned} r_{250}^{(20)} &= -0.200000000000000000000000001520\ldots, \\ r_{250}^{(30)} &= -0.200000000000000000000000000002\ldots. \end{aligned} \tag{130}$$

Remark 8.1. We can also study the asymptotics of the coefficients $v_{n,k}$ appearing in the trans-series solution (56). This asymptotics is governed by Eq. (38), where $\lambda = A/2$, $\beta = 0$, and $S_1 = S'$. Based on our numerical results, we conjecture that

$$S_{-1} = -\mathrm{i}\frac{\sqrt{6}}{12}. \tag{131}$$

Finally, we end with the following expectation of Conjecture 1:

$$p_{\frac{41}{2}} = \frac{1238878081129358302459331398309144842472024202171957968278854904568087551305256373}{1098603008254895032115743533344988955141157683 2\sqrt{6}\,\Gamma(\frac{199}{4})}.$$

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