

# The Noncommutative $A$ -Polynomial of $(-2, 3, n)$ Pretzel Knots

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We study  $q$ -holonomic sequences that arise as the colored Jones polynomial of knots in 3-space. The minimal-order recurrence for such a sequence is called the (noncommutative)  $A$ -polynomial of a knot. Using the *method of guessing*, we obtain this polynomial explicitly for the  $K_p = (-2, 3, 3 + 2p)$  pretzel knots for  $p = -5, \dots, 5$ . This is a particularly interesting family, since the pairs  $(K_p, -K_{-p})$  are geometrically similar (in particular, scissors congruent) with similar character varieties. Our computation of the noncommutative  $A$ -polynomial complements the computation of the  $A$ -polynomial of the pretzel knots done by the first author and Mattman, supports the AJ conjecture for knots with reducible  $A$ -polynomial, and numerically computes the Kashaev invariant of pretzel knots in linear time. In a later publication, we will use the numerical computation of the Kashaev invariant to numerically verify the volume conjecture for the above-mentioned pretzel knots.

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## 1. THE COLORED JONES POLYNOMIAL: A $q$ -HOLONOMIC SEQUENCE OF NATURAL ORIGIN

### 1.1. Introduction

The colored Jones polynomial of a knot  $K$  in 3-space is a  $q$ -holonomic sequence of Laurent polynomials of natural origin in quantum topology [Garoufalidis and Lê Thang 05]. As a canonical recurrence relation for this sequence we choose the one with minimal order; this is the so-called noncommutative  $A$ -polynomial of a knot [Garoufalidis 04]. Using the computational *method of guessing* with undetermined coefficients [Kauers 09a, Kauers 09b] combined with a carefully chosen exponent set of monomials (given by a translate of the Newton polygon of the  $A$ -polynomial), we compute very plausible candidates for the noncommutative  $A$ -polynomial of the  $(-2, 3, 3 + 2p)$  pretzel knot family for  $p = -5, \dots, 5$ . Our computations of the noncommutative  $A$ -polynomial

- (a) complement the computation of the  $A$ -polynomial of the pretzel knots [Garoufalidis and Mattman 11],

2000 AMS Subject Classification: Primary 57N10; Secondary 57M25

Keywords: Colored Jones polynomial, knots, pretzel knots, noncommutative  $A$ -polynomial,  $q$ -holonomic sequences, recurrence ideal, quantum topology, volume conjecture, Kashaev invariant

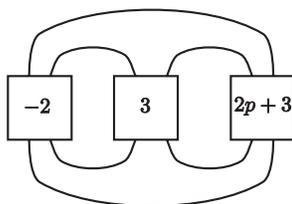
- (b) support the AJ conjecture of [Garoufalidis 04] (see also [Gelca 02]) for knots with reducible A-polynomial,
- (c) give an efficient linear-time algorithm for computing numerically the Kashaev invariant of the pretzel knots (with a fixed accuracy).

In [Garoufalidis and Zagier 12], we use the latter algorithm to numerically verify the volume conjecture of Kashaev [Kashaev 97, Murakami and Murakami 01] for the above-mentioned pretzel knots.

For an introduction to the polynomial invariants of knots that originate in quantum topology, see [Jones 87, Turaev 88, Turaev 94] and the book [Jantzen 96], where all the details of quantum group theory can be found. For up-to-date computer calculations of several polynomial invariants of knots, see [Bar-Natan 05]. For an introduction to  $q$ -holonomic sequences, see [Zeilberger 90, Wilf and Zeilberger 92, Petkovšek et al. 96]. For the appearance of  $q$ -holonomic sequences in quantum topology, see [Garoufalidis and Lê Thang 05, Garoufalidis and Sun 06, Garoufalidis and Sun 10] and also [Garoufalidis and Koutschan 11].

### 1.2. Fusion and the Colored Jones Polynomial of Pretzel Knots

Consider the 1-parameter family of *pretzel knots*  $K_p = (-2, 3, 3 + 2p)$  for an integer  $p$ :



where an integer  $m$  inside a box indicates the number  $|m|$  of half-twists, right-handed (if  $m > 0$ ) or left-handed (if  $m < 0$ ), according to the following figure:

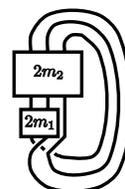


The pretzel knots  $K_p$  are interesting from many points of view, discussed in detail in [Culler et al. 87, Garoufalidis and Mattman 11, Garoufalidis 10a, Garoufalidis and Zagier 12]:

- In *hyperbolic geometry*,  $K_p$  is the torus knot  $5_1, 8_{19}, 10_{124}$  when  $p = -1, 0, 1$ , and  $K_p$  is a hyperbolic knot when  $p \neq -1, 0, 1$ .

- The pairs  $(K_p, -K_{-p})$  (where  $-K$  denotes the mirror of  $K$ ) are *geometrically similar* for  $p \geq 2$ . In particular, their complements are scissors congruent, with equal volume, and with Chern–Simons invariants differing by torsion [Garoufalidis 10a].
- The knots  $K_p$  appear in the study of *exceptional Dehn surgery* [Culler et al. 87].
- In *quantum topology*, the knots  $K_p$  have different Jones polynomial and different Kashaev invariants, which numerically verify the volume conjecture [Garoufalidis and Zagier 12].

Let  $J_{K,n}(q)$  denote the *colored Jones polynomial* of a knot  $K$  colored by the  $n$ -dimensional irreducible representation of  $\mathfrak{sl}_2$ , framed by zero and normalized to be 1 at the unknot [Turaev 88, Turaev 94]. So,  $J_{K,1}(q) = 1$  for all knots and  $J_{K,2}(q)$  is the Jones polynomial of  $K$  [Jones 87]. Our starting point is an explicit formula for the colored Jones polynomial  $J_{p,n}(q)$  of  $K_p$ . This comes from a theorem of [Garoufalidis 10b], which has two parts. The first part identifies the pretzel knots  $K_p$  with members of a 2-parameter family of *2-fusion knots*  $K(m_1, m_2)$  for integers  $m_1$  and  $m_2$ , drawn here



and discussed in detail in [Garoufalidis 10b]. The second part gives an explicit formula for the colored Jones polynomial of  $K(m_1, m_2)$ . To state it, we need to recall some notation. The *quantum integer*  $[n]$  and the *quantum factorial*  $[n]!$  of a natural number  $n$  are defined by

$$[n] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}, \quad [n]! = \prod_{k=1}^n [k]!,$$

with the convention that  $[0]! = 1$ . Let

$$\left[ \begin{matrix} a \\ a_1, a_2, \dots, a_r \end{matrix} \right] = \frac{[a]!}{[a_1]! \cdots [a_r]!}$$

denote the  $q$ -multinomial coefficient of natural numbers  $a_i$  such that  $a_1 + \cdots + a_r = a$ . We say that a triple  $(a, b, c)$  of natural numbers is *admissible* if  $a + b + c$  is even and the triangle inequalities hold. In the formulas below, we use the following basic trivalent graphs  $U, \Theta, \text{Tet}$  colored by one, three, and six natural numbers

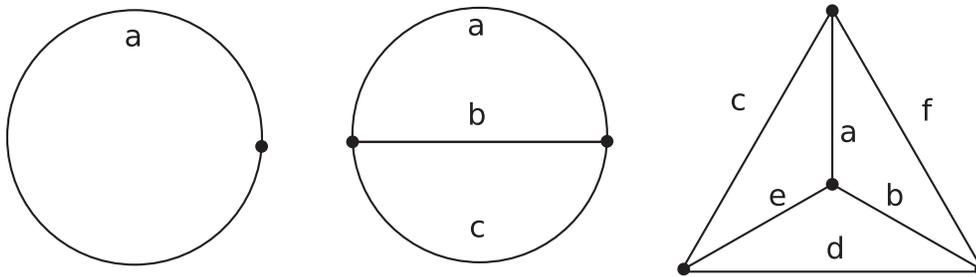


FIGURE 1. The  $U$ ,  $\Theta$ , and Tet graphs colored by an admissible coloring.

(one at each edge of the corresponding graph) such that the colors at every vertex form an admissible triple.

If a coloring of a graph is not admissible, its evaluation vanishes. When the colorings of the graphs in Figure 1 are admissible, their evaluations can be computed by the following functions:

$$\begin{aligned} \mu(a) &= (-1)^a q^{a(a+2)/4}, \\ \nu(c, a, b) &= (-1)^{(a+b-c)/2} q^{-(a(a+2)-b(b+2)+c(c+2))/8}, \\ U(a) &= (-1)^a [a + 1], \end{aligned}$$

$$\begin{aligned} \Theta(a, b, c) &= (-1)^{(a+b+c)/2} \left[ \frac{a+b+c}{2} + 1 \right] \\ &\quad \times \left[ \frac{a+b+c}{2}, \frac{a-b+c}{2}, \frac{a+b-c}{2} \right], \end{aligned}$$

$$\begin{aligned} \text{Tet}(a, b, c, d, e, f) &= \sum_{k=\max T_i}^{\min S_j} (-1)^k [k + 1] \\ &\quad \times \left[ S_1 - k, S_2 - k, S_3 - k, k - T_1, k - T_2, k - T_3, k - T_4 \right], \end{aligned}$$

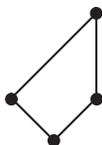
where

$$\begin{aligned} S_1 &= \frac{1}{2}(a + d + b + c), & S_2 &= \frac{1}{2}(a + d + e + f), \\ S_3 &= \frac{1}{2}(b + c + e + f), \end{aligned}$$

and

$$\begin{aligned} T_1 &= \frac{1}{2}(a + b + e), & T_2 &= \frac{1}{2}(a + c + f), \\ T_3 &= \frac{1}{2}(c + d + e), & T_4 &= \frac{1}{2}(b + d + f). \end{aligned}$$

An assembly of the five building blocks can compute the colored Jones function of any knot. Consider the rational convex planar polygon  $P$  with vertices  $\{(0, 0), (1/2, -1/2), (1, 0), (1, 1)\}$  in  $\mathbb{Q}^2$ :



**Theorem 1.1.** [Garoufalidis 10b]

- (a) For every integer  $p$ , we have  $K_p = K(p, 1)$ .
- (b) For every  $m_1, m_2 \in \mathbb{Z}$  and  $n \in \mathbb{N}$ , we have

$$\begin{aligned} J_{K(m_1, m_2), n+1}(1/q) &= \frac{\mu(n)^{-w(m_1, m_2)}}{U(n)} \\ &\quad \times \sum_{(k_1, k_2) \in nP \cap \mathbb{Z}^2} \nu(2k_1, n, n)^{2m_1+2m_2} \\ &\quad \times \nu(n + 2k_2, 2k_1, n)^{2m_2+1} \\ &\quad \times \frac{U(2k_1)U(n + 2k_2)}{\Theta(n, n, 2k_1)\Theta(n, 2k_1, n + 2k_2)} \\ &\quad \times \text{Tet}(n, 2k_1, 2k_1, n, n, n + 2k_2), \end{aligned} \tag{1-1}$$

where  $P$  is as above and the writhe of  $K(m_1, m_2)$  is given by  $w(m_1, m_2) = 2m_1 + 6m_2 + 2$ .

**1.3. Our Results**

Recall that a  $q$ -holonomic sequence  $(f_n(q))$  for  $n \in \mathbb{N}$  is a sequence (typically of rational functions  $f_n(q) \in \mathbb{Q}(q)$  in one variable  $q$ ) that satisfies a linear recurrence relation of the form

$$a_d(q^n, q)f_{n+d}(q) + \dots + a_0(q^n, q)f_n(q) = b(q^n, q) \tag{1-2}$$

for all  $n \in \mathbb{N}$ , where  $a_j(u, v) \in \mathbb{Q}[u, v]$  for all  $j = 0, \dots, d$  and  $b(u, v) \in \mathbb{Q}[u, v]$  [Zeilberger 90]. As is custom, one can phrase (1-2) in operator form, by considering the operators  $M$  and  $L$  that act on a sequence  $(f_n(q))$  by

$$(Lf)_n(q) = f_{n+1}(q), \quad (Mf)_n(q) = q^n f_n(q).$$

It is easy to see that the operators  $M$  and  $L$  satisfy the  $q$ -commutation relation

$$LM = qML.$$

Thus, we can write (1-2) in the form

$$Pf = b, \quad P = \sum_{j=0}^d a_j(M, q)L^j, \quad b = b(q^n, q)1.$$

We will call a  $q$ -holonomic bi-infinite sequence  $f_n(q)$  *palindromic* if either  $f_n(q) = f_{-n}(q)$  for all integers  $n$ , or  $f_n(q) = -f_{-n}(q)$  for all integers  $n$ . Given a palindromic sequence  $f_n(q)$ , we will call a recurrence relation (1-2) (and the corresponding operator  $(P, b)$ ) *palindromic* if (1-2) holds for all integers  $n$ .

With our normalizations, the colored Jones polynomial  $J_{K,n}(q)$  of a knot, defined for  $n \geq 1$ , extends to a palindromic sequence defined by  $J_{K,n}(q) = J_{K,-n}(q)$  for  $n < 0$  and  $J_{K,0}(q) = 1$ .

Let  $A_p(M, L) \in \mathbb{Q}[M^2, L]$  denote the  $A$ -polynomial of the pretzel knot  $K_p$ , given in [Garoufalidis and Mattman 11]. Let  $\epsilon_p(M) \in \mathbb{Q}[M]$  denote the  $M$ -factors given in Section 5. Let  $\Delta_p(t) \in \mathbb{Z}[t^{\pm 1}]$  denote the *Alexander polynomial* of  $K_p$  [Kauffman 87]. Then  $\Delta_p$  satisfies the linear recurrence relation

$$\Delta_{p+2} - (t + t^{-1})\Delta_{p+1} + \Delta_p = 0 \tag{1-3}$$

for all  $p \in \mathbb{Z}$ , with initial conditions

$$\begin{aligned} \Delta_0 &= \frac{1}{t^3} - \frac{1}{t^2} + 1 - t^2 + t^3, \\ \Delta_1 &= \frac{1}{t^4} - \frac{1}{t^3} + \frac{1}{t} - 1 + t - t^3 + t^4. \end{aligned}$$

**Theorem 1.2.**

(a) Consider the operators  $(A_p(M, L, q), b_p(M, q))$  of Section 5 for  $p = -5, \dots, 5$ . Then, we have

$$A_p(M, L, 1) = A_p(M^{1/2}, L)\epsilon_p(M)$$

in accordance with the AJ conjecture [Garoufalidis 04].

(b)  $(A_p(M, L, q), b_p(M, q))$  is palindromic.

(c) We also have

$$\frac{A_p(M, 1, 1)}{b_p(M, 1)} = \Delta_p(M) \tag{1-4}$$

for  $p \neq -3$ . When  $p = -3$  we have  $A_{-3}(M, 1, 1) = b_{-3}(M, 1) = 0$  and

$$\begin{aligned} \frac{A_{-3,q}(M, 1, 1)}{\Delta_{-3}(M)} - A_{-3,L}(M, 1, 1)M \frac{\Delta'_{-3}(M)}{\Delta_{-3}(M)^2} \\ = b_{-3,q}(M, 1), \end{aligned}$$

where primes indicate partial derivatives. This is in accordance with the loop expansion of the colored Jones polynomial [Garoufalidis 08]. For a definition of the loop expansion, see [Rozansky 98].

**Conjecture 1.3.** We conjecture that

$$A_p(M, L, q)J_{p,n}(q) = b_p(q^n, q) \tag{1-5}$$

for  $p = -5, \dots, 5$  and all  $n \in \mathbb{Z}$ .

**2. CONSISTENCY CHECKS**

Three consistency checks of Conjecture 1.3 were already mentioned in Theorem 1.2. In this section we discuss four independent consistency checks regarding Conjecture 1.3.

**2.1. Consistency with the Height**

In this section we discuss the height of (1-5). For fixed integers  $p$  and natural numbers  $n$ , both sides of (1-5) are Laurent polynomials in  $q$  with integer coefficients.

In general, the minimum and maximum degrees of a  $q$ -holonomic sequence of Laurent polynomials are quadratic quasipolynomials [Garoufalidis 11a]. In [Garoufalidis 10b], the first author studied the minimum and maximum degrees of the colored Jones polynomial of the 2-fusion knots  $K(m_1, m_2)$ . For the case of pretzel knots  $K_p$ , the maximum degree of  $J_{p,n}(q)$  is a quadratic quasipolynomial of  $n$ , and the minimum degree is a linear function of  $n$ . It follows that the terms on the left-hand side of (1-5) are polynomials in  $q$  of minimum degree a linear function of  $n$  and maximum degree a quasipolynomial quadratic function of  $n$ . Explicitly, for the case of  $K_2 = K(2, 1)$ , it was shown in [Garoufalidis 10b, Garoufalidis 11b] that  $J_{2,n}(q)$  is a polynomial in  $q$  of minimum degree  $\delta^*(n)$  and maximum degree  $\delta(n)$  given by

$$\begin{aligned} \delta(n) &= \left[ \frac{37}{8}n^2 + \frac{3}{4}n - \frac{31}{8} \right] = \frac{37}{8}n^2 + \frac{3}{4}n - \frac{31}{8} + \epsilon(n), \\ \delta^*(n) &= 5(n - 1), \end{aligned}$$

where  $\epsilon(n)$  is periodic of period 4 given by  $1/8, 0, 1/8, 1/2$  as  $n \equiv 0, 1, 2, 3 \pmod 4$  respectively.

Keep in mind that  $J_{K,n}(q)$  denotes the colored Jones polynomial of  $K$  colored by the  $n$ -dimensional irreducible representation of  $\mathfrak{sl}_2$ . It follows that for  $p = 2$ , the left-hand side of (1-5) is a polynomial in  $q$  of minimum degree  $60n + O(1)$  and maximum degree  $37/8n^2 + 9/4n + O(1)$ . We computed  $J_{2,n}(q)$  explicitly for  $1 \leq n \leq 70$  using Theorem 1.1. For instance,  $J_{2,70}(q)$  is a polynomial of

- minimum (respectively maximum) exponent 345 (respectively 22 606),
- maximum (in absolute value) coefficient 14 287 764 770 955, and

- sum of absolute values of its coefficients  
28 587 411 833 908 277.

It follows that that (1–5) for  $n = 70$  (which actually holds, by an explicit computation) involves the matching of about 22 500 powers of  $q$  with coefficients 14-digit integers. One can compare this to the modest size of  $(A_2(M, L, q), b_2(M, q))$  given in Section 4.

Of course, one can come up with operators that satisfy parts (a), (b), and (c) of Theorem 1.2 and (1–5) for a fixed natural number  $n$  (such as  $n = 70$ ). This is simply a problem of linear algebra with more unknowns than coefficients that in fact has infinitely many solutions. On the other hand, the operators given in Section 5 are of small height, given the height of the input, as was illustrated above. Note also that Theorem 1.2 could be proven rigorously if we knew a priori bounds for the  $M, L$  degrees of the operators involved and if we were able to compute enough values of the colored Jones polynomials, using the formula of Theorem 1.1.

### 2.2. Consistency with the Loop Expansion of the Colored Jones Polynomial

This section concerns the consistency of Conjecture 1.3 with the loop expansion of the colored Jones polynomial of  $K_p$ . The latter was introduced in [Rozansky 98] and has the form

$$J_{K,n}(q) = \sum_{k=0}^{\infty} \frac{P_{K,k}(q^n)}{\Delta_K(q^n)^{2k+1}} (q-1)^k \in \mathbb{Q}[[q-1]], \quad (2-1)$$

where  $P_k(t) \in \mathbb{Z}[t^{\pm 1}]$  are Laurent polynomials with  $P_{K,0} = 1$ , and  $\Delta_K(t) \in \mathbb{Z}[t^{\pm 1}]$  is the Alexander polynomial of  $K$ . With some effort, one can compute the  $k$ -loop polynomials  $P_{K,k}(M)$  of a knot for various small values of  $k$ . The loop expansion given by (2-1) *highly constrains* the coefficients of  $(A_K(M, L, q), b_K(M, q))$ , as was discussed in detail in [Garoufalidis 08]. The simplest constraint for the knots  $K_p$  is given in (1-4), which in fact is the first of a hierarchy of constraints. Each such constraint gives a consistency check for Conjecture 1.3 and offers a practical way to compute the loop expansion of the knots  $K_p$ . Consistency with the higher loop constraints has also been checked. We plan to discuss the details in a forthcoming publication.

### 2.3. Consistency with the AJ Conjecture

This section concerns the AJ conjecture of [Garoufalidis 04] for the knots  $K_{\pm 3}$  with reducible  $A$ -polynomial. The  $A$ -polynomial  $A_p(M, L)$  of the pretzel knots  $K_p$  was computed in [Garoufalidis and Mattman 11].

In [Mattman 02], it was shown that  $A_p(M, L)$  is irreducible if 3 does not divide  $p$  and otherwise it is the product of two irreducible factors when 3 divides  $p$ . Explicitly, we have

$$\begin{aligned} A_{-3}(M, L) = & -(-1 + L)(L^3 - L^4 - 5L^3M^2 + L^4M^2 - \\ & 2L^2M^4 - 2L^4M^4 - LM^6 - 4L^2M^6 + 3L^3M^6 + \\ & 2L^4M^6 - M^8 - 5LM^8 - 3L^3M^8 + L^4M^8 + LM^{10} - \\ & 3L^2M^{10} - 5L^4M^{10} - L^5M^{10} + 2LM^{12} + 3L^2M^{12} - \\ & 4L^3M^{12} - L^4M^{12} - 2LM^{14} - 2L^3M^{14} + LM^{16} - \\ & 5L^2M^{16} - LM^{18} + L^2M^{18}) \end{aligned}$$

and

$$\begin{aligned} A_3(M, L) = & (-1 + LM^{24})(-1 + LM^{16} - LM^{18} + \\ & 2LM^{20} - 5LM^{22} + LM^{24} + 5L^2M^{40} - 4L^2M^{42} + \\ & L^2M^{46} + L^3M^{62} + 3L^3M^{66} + 2L^3M^{68} - 2L^4M^{84} - \\ & 3L^4M^{86} + 3L^4M^{88} + 2L^4M^{90} - 2L^5M^{106} - 3L^5M^{108} - \\ & L^5M^{112} - L^6M^{128} + 4L^6M^{132} - 5L^6M^{134} - L^7M^{150} + \\ & 5L^7M^{152} - 2L^7M^{154} + L^7M^{156} - L^7M^{158} + L^8M^{174}). \end{aligned}$$

Theorem 1.2 matches exactly with the above values of the  $A$ -polynomials. The case of the knot  $K_{-3}$  is particularly interesting, since its  $SL_2(\mathbb{C})$  character variety has *three* components: the geometric one, the abelian  $L - 1$  component, and an additional  $L - 1$  component of nonabelian representations. Theorem 1.2 and Conjecture 1.3 support the idea that the AJ conjecture captures the multiplicity of the various components of the character variety.

### 2.4. Consistency with the Volume Conjecture

This section concerns the consistency of Conjecture 1.3 with the computation of the Kashaev invariant of the  $K_p$  knots. The  $N$ th *Kashaev invariant*  $\langle K \rangle_N$  of a knot  $K$  is defined by [Kashaev 97, Murakami and Murakami 01]

$$\langle K \rangle_N = J_{K,N}(e^{2\pi i/N}).$$

The volume conjecture of Kashaev states that if  $K$  is a hyperbolic knot, then

$$\lim_{N \rightarrow \infty} \frac{|\langle K \rangle_N|}{N} = \frac{\text{vol}(K)}{2\pi},$$

where  $\text{vol}(K)$  is the volume of the hyperbolic knot  $K$ . Since we are specializing to a root of unity, we might as well consider the remainder  $\tau_{K,N}(q)$  of  $J_{K,N-1}(q)$  by the  $N$ th cyclotomic polynomial  $\Phi_N(q)$ . In [Garoufalidis and Zagier 12], it was shown that given a recurrence relation for  $J_{K,N}(q)$ , there is a linear-time algorithm to numerically compute  $\langle K \rangle_N$ . Using the guessed recurrence relation for  $K_2$ , we compute  $\tau_{K_2,N}(q)$  for  $N = 1, \dots, 1000$ . Figure 2 shows a sample computation.

$$\begin{aligned} \tau_{K_2,100}(q) = & -1420771679897311607360 - 1402034476570732425908q - 1377764083694494707679q^2 - 1348056285420017550322q^3 \\ & - 1313028324854995190830q^4 - 1272818441358081463973q^5 - 1227585324968178744317q^6 - 1177507490130630983388q^7 \\ & - 1122782571182284245313q^8 - 1063626542375688303231q^9 + 420498814366636734411q^{10} + 469062907903390306537q^{11} \\ & + 515775824438145014436q^{12} + 560453209429428890901q^{13} + 602918741648741441924q^{14} + 643004829043136905736q^{15} \\ & + 680553270138355921566q^{16} + 715415878390451489264q^{17} + 747455067013913965248q^{18} + 77654439196778302155q^{19} \\ & - 618202628922511743188q^{20} - 576608139973286430388q^{21} - 532738042123286363977q^{22} - 486765470606610517117q^{23} \\ & - 438871858158259827294q^{24} - 389246218987652812332q^{25} - 338084402821172432280q^{26} - 285588321971646221647q^{27} \\ & - 231965154488540570326q^{28} - 177426526516296620808q^{29} + 1298584002796105745794q^{30} + 1335567867823634101034q^{31} \\ & + 1367280856639633305993q^{32} + 1393597812566394292363q^{33} + 1414414874600710903331q^{34} + 1429649887309469255114q^{35} \\ & + 1439242725058651352936q^{36} + 1443155529298983637839q^{37} + 1441372857979981026638q^{38} + 1433901746491878528487q^{39} \end{aligned}$$

FIGURE 2. Sample computation of  $\tau_{K_2,N}(q)$ .

Let

$$a_N = 2\pi \frac{\log |\langle K_2 \rangle_N|}{N}.$$

Since  $\langle K_2 \rangle_N = \tau_{K_2,N}(e^{2\pi i/N})$ , the above expression gives the numerical value

$$a_{100} = 3.22309\dots,$$

which is a rather poor approximation of the volume  $21(K_2) = 2.8281220883307827\dots$  of  $K_2$ . On the other hand, for  $N = 990, \dots, 1000$  we have the values shown in Table 1.

A plot of the above data is shown in Figure 3. It numerically fits the curve

$$2.82813 + 9.41764 \frac{\log(n)}{n} - 3.89193 \frac{1}{n},$$

which is a four-digit approximation to the volume. In [Garoufalidis and Zagier 12], a more precise approximation to the volume and its correction is given.

### 3. THE COMPUTATION

We use the computer to guess the recurrences for the colored Jones polynomials  $J_{p,n}(q)$  [Kauers 09a, Koutschan 09, Koutschan 10, Garoufalidis and Koutschan 11]. By *guessing*, we mean the method of making an ansatz with undetermined coefficients. The first values of the sequence  $J_{p,n}(q)$  can be computed explicitly using (1–1) (see Table 2 for an example). These values are then plugged into an ansatz with undetermined coefficients in order to produce an overdetermined

linear system of equations. The more equations are used (note that at least as many equations as unknowns are needed to obtain a reliable result), the higher is the certainty that the result is the recurrence for the sequence. Alternatively, we can verify that the recurrence is satisfied for some values of the sequence that were not used for guessing, gaining further confidence in the result.

For guessing  $q$ -difference equations, there are two different choices for the ansatz and the nature of its coefficients.

The first ansatz (we consider it the more classical one) is of the form

$$\sum_{(\alpha,\beta) \in S} c_{\alpha,\beta} M^\alpha L^\beta, \tag{3-1}$$

where the unknown coefficients  $c_{\alpha,\beta}$  have to be determined in  $\mathbb{Q}(q)$ . We will refer to the finite set  $S \subseteq \mathbb{N}^2$  as the *structure set* of the ansatz. It is easy to see that with ansatz (3–1), at least the first  $(o + 1)(d + 2) - 1$  (respectively  $(o + 2)(d + 2) - 2$ ) values of a sequence are needed in order to guess a homogeneous (respectively inhomogeneous) recurrence of order  $o$  and coefficient degree  $d$ , i.e., when  $0 \leq \alpha \leq d$  and  $0 \leq \beta \leq o$ . Table 3 illustrates how fast the entries of the colored Jones polynomials grow, and it becomes obvious that we cannot go very far with this ansatz.

The second ansatz is of the form

$$\sum_{(\alpha,\beta,\gamma) \in S} c_{\alpha,\beta,\gamma} q^\gamma M^\alpha L^\beta, \tag{3-2}$$

$N$	990	991	992	993	994	995	996	997	998	999	1000
$a_N$	2.88981	2.88976	2.88971	2.88965	2.8896	2.88955	2.8895	2.88944	2.88939	2.88934	2.88929

TABLE 1. Numerical values of the normalized Kashaev invariant  $a_N$  of  $K_2$  for  $N = 990, \dots, 1000$ .

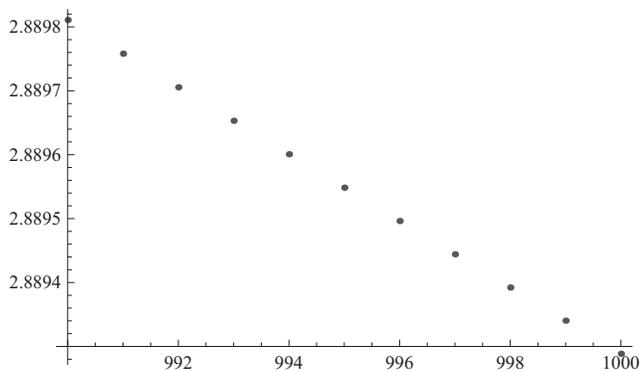


FIGURE 3. A plot of Table 1.

$p$	$d(J_{p,10}(q))$	$d(J_{p,20}(q))$	$d(J_{p,30}(q))$
-5	453	1919	4400
-4	363	1546	3549
-3	282	1197	2735
-2	225	950	2175
-1	225	950	2175
0	265	1130	2595
1	330	1410	3240
2	406	1736	3991
3	491	2098	4821
4	579	2469	5671
5	667	2843	6529

TABLE 3. Size of the colored Jones polynomial at  $n = 10, 20, 30$  for the pretzel knot family, where  $d(p) = d_1 + d_2$  for a Laurent polynomial  $p = \sum_{i=-d_1}^{d_2} c_i q^i$  with  $c_{-d_1} \neq 0$  and  $c_{d_2} \neq 0$ .

where the unknowns  $c_{\alpha,\beta,\gamma}$  are elements of  $\mathbb{Q}$  and  $S \subseteq \mathbb{N}^3$  is again a finite structure set. This alternative is particularly promising when the  $q$ -degrees of the sequence grow very fast, as is the case with the colored Jones polynomials. On the one hand, we have many more unknowns than in (3-1), but already the first 30 values of  $J_{p,n}(q)$  suffice to generate thousands of equations. However, this method has its limits, too, as can be seen from Table 4. Guessing the last entry ( $p = 5$ ) would require solving a linear system over  $\mathbb{Q}$  with  $17 \cdot 289 \cdot 2175$  (about 10 million) unknowns.

The small instances in our family of problems can be done with either technique and without further previous knowledge. But for finding the larger recurrences presented in this paper, some optimizations are necessary. What helped considerably in reducing the size of the computations is the fact that certain properties of the structure sets in (3-1) or (3-2) can be deduced a priori. In particular, the AJ conjecture of [Garoufalidis 04] and the Newton polygon of the  $A$ -polynomial of the pretzel knots  $K_p$  from [Mattman 02] allow us to guess the exponents  $(\alpha, \beta)$  that appear in (3-2), up to an overall translation in the  $M$ -direction. The only missing link to obtaining the structure set is how far we have to translate the Newton polygon in the  $M$ -direction (this has to

be found out by trial and error); see Figure 4. Obviously, this translated Newton polygon has many fewer lattice points than the rectangular box, and this helps considerably in the computations.

The second trick that allows us to compute the large recurrences listed in Table 4 is to use ansatz (3-1) in connection with modular computations. This means that we compute the sequence  $J_{p,n}(q)$  only for specific integers  $q$  and modulo some prime number  $m$ . Then the monstrous Laurent polynomials (see Tables 2 and 3) shrink to a natural number between 0 and  $m - 1$ . Thus we can compute the first few hundred values of the colored Jones sequence “easily” (i.e., in a few hours). The guessing then is done efficiently, since only a single null-space computation modulo  $m$  of a moderately sized matrix (less than 1000 columns) is required. This procedure has to be performed for many specific values of  $q$  for us to be able to interpolate and reconstruct the polynomial coefficients  $c_{\alpha,\beta}$ . In general, we also would have to use several prime numbers in order to recover the integer coefficients via Chinese remaindering and rational reconstruction. However, the integer coefficients in the present problems are so small (see Table 4) that

$n$	$J_{0,n}(q)$
1	1
2	$-q^8 + q^5 + q^3$
3	$q^{23} - q^{22} + q^{20} - q^{19} - q^{16} - q^{13} + q^{12} + q^9 + q^6$
4	$-q^{43} + q^{41} + q^{40} - q^{39} + q^{37} - q^{35} + q^{33} - q^{31} + q^{29} - q^{27} - q^{26} + q^{25} - q^{23} - q^{22} + q^{21} - q^{19} + q^{17} + q^{13} + q^9$
5	$q^{70} - q^{69} + q^{65} - 2q^{64} + q^{60} - q^{59} + q^{57} + q^{55} - q^{54} + q^{52} - q^{49} + q^{47} - q^{44} + q^{42} - q^{39} + q^{37} - q^{35} - q^{34} + q^{32} - q^{30} - q^{29} + q^{27} - q^{25} + q^{22} + q^{17} + q^{12}$

TABLE 2. The first elements of the colored Jones polynomial of the  $(-2, 3, 3)$  pretzel knot  $K_0$ .

$p$	-5	-4	-3	-2	-1	0	1	2	3	4	5
ByteCount	$5.7 \times 10^7$	$1.1 \times 10^7$	$1.1 \times 10^6$	32032	1192	1616	1616	47016	$2.3 \times 10^6$	$1.9 \times 10^7$	$8.6 \times 10^7$
$L$ -degree	12	9	6	3	1	2	2	6	9	12	15
$M$ -degree	125	66	27	12	6	13	16	58	114	191	288
$q$ -degree	946	392	85	19	3	13	16	233	514	1151	2174
largest cf.	$3.0 \times 10^8$	12345	33	4	1	2	2	6	118	386444	$2.2 \times 10^{11}$
translation	68	39	18	5	1	1	1	3	15	36	65

**TABLE 4.** Some data concerning the recurrence relations for the colored Jones polynomial of the pretzel knots  $K_p$ ,  $p = -5, \dots, 5$ : the size of the recurrence (in bytes, using the Mathematica command `ByteCount`), the order (or  $L$ -degree), the coefficient degree (or  $M$ -degree), the degree of  $q$  in the coefficients, the largest integer coefficient, and how far the Newton polygon had to be translated in order to find this recurrence.

a single prime suffices to recover the coefficients in  $\mathbb{Z}$ . Note that this strategy is perfectly suited for parallel computations.

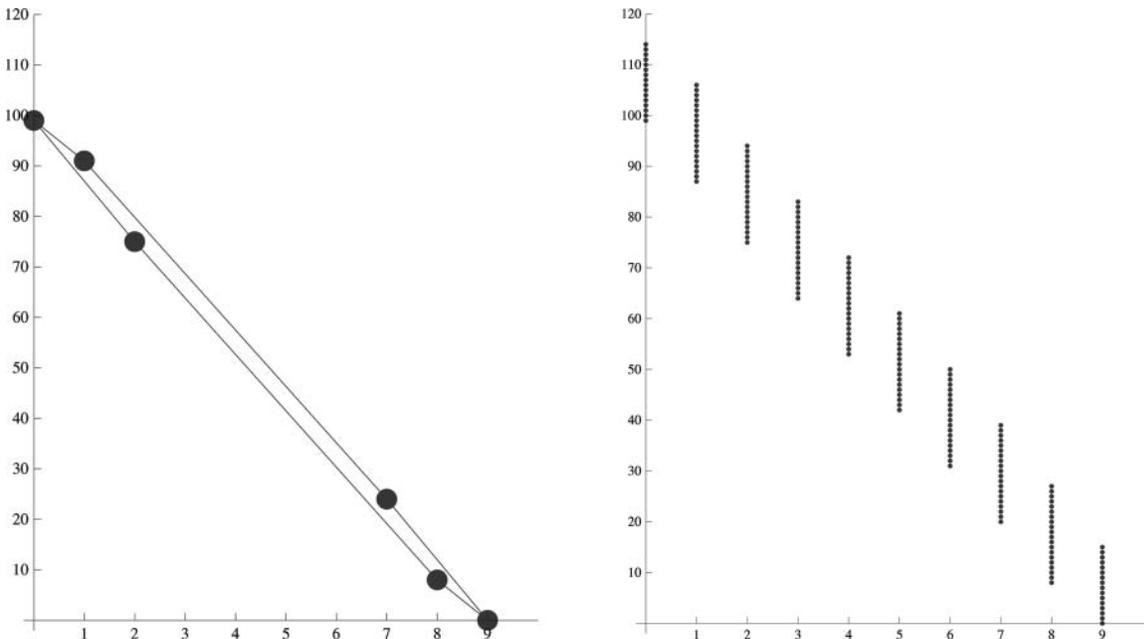
We have seen that the number of sequence entries that we need to compute is determined by the number of unknowns in the ansatz, and the number of interpolation points (specific values for  $q$ ) by the  $q$ -degree of the coefficients in the recurrence. We mention some further tricks for reducing these two parameters. The  $q$ -degree can be decreased by considering a slight variation of the original sequence  $f(n)$ , namely  $g(n) := f(n + s)$  for some  $s \in \mathbb{Z}$ . Even more, a similar trick can be used to halve the number of unknowns, by exploiting the fact there is an  $s = t/2$  for  $t \in \mathbb{Z}$  such that the substitution  $n \rightarrow n + s$  in the coefficients of the recurrence reveals the following

symmetry:

$$c_{\alpha,\beta} = \begin{cases} c_{m-\alpha,l-\beta} & \text{for } t \text{ even,} \\ -c_{m-\alpha,l-\beta} & \text{for } t \text{ odd,} \end{cases}$$

where  $m$  is the  $M$ -degree and  $l$  is the  $L$ -degree of the recurrence. The above symmetry is equivalent to the palindromic property of the sought recurrence.

We want to remark that most of the computation time was used to compute the data, which then later served as input for the guessing. In the most difficult example presented here, the recurrence for  $K_5$ , it took 1607 CPU days to produce the first 744 entries of  $J_{5,n}(q)$  for 700 different values of  $q$ . Since we ran this computation on a cluster with several hundred processors, it finished within a few days.



**FIGURE 4.** The Newton polygon for the  $(-3, 2, 9)$  pretzel knot  $K_3$  (left) and the structure set of the recurrence for its colored Jones polynomial (right).

#### 4. THE RECURRENCE FOR $K_{-2}$ AND $K_2$

The operators  $A_{\pm 2}(M, L, q)$  and  $b_{\pm 2}(M, q)$  are given as follows:<sup>1</sup>

$$A_{-2}(M, L, q) = -(q^2 M - 1)(q^2 M + 1)(q^4 M - 1)(q^3 M^2 - 1) L^3 - q(q^3 M - 1)^2(q^3 M + 1)(q^3 M^2 - 1)(q^{14} M^5 - q^{11} M^4 - (q^{10} - q^9 - q^8 + q^7)M^3 + (q^7 + q^4)M^2 + 2q^3 M - 1)L^2 + q^7 M^2(q^2 M - 1)^2(q^2 M + 1)(q^7 M^2 - 1)(q^{11} M^5 - 2q^9 M^4 - (q^8 + q^5)M^3 + (q^6 - q^5 - q^4 + q^3)M^2 + q^2 M - 1)L - q^{16} M^7(q M - 1)(q^3 M - 1)(q^3 M + 1)(q^7 M^2 - 1),$$

$$b_{-2}(M, q) = q^6 M^2(q^2 M + 1)(q^3 M + 1)(q^3 M^2 - 1)(q^5 M^2 - 1)(q^7 M^2 - 1),$$

$$A_2(M, L, q) = q^{59}(q^2 M - 1)(q^3 M - 1)(q^7 M - 1) L^6 - q^{112} M^8(q^2 M - 1)(q^3 M - 1)(q^6 M - 1)^3 L^5 - q^{167} M^{18}(q^2 M - 1)(q^5 M - 1)^2(q^2 M + q + 1)L^4 + (q - 1)(q + 1)q^{207} M^{27}(q^2 M - 1)(q^4 M - 1)^2(q^6 M - 1)L^3 + q^{240} M^{36}(q^3 M - 1)^2(q^4 M + q^3 M + 1)(q^6 M - 1)L^2 + q^{263} M^{45}(q^2 M - 1)^3(q^5 M - 1)(q^6 M - 1) L - q^{279} M^{55}(q M - 1)(q^5 M - 1)(q^6 M - 1),$$

$$b_2(M, q) = q^{89} M^5(q^{192} M^{48} - q^{186}(q + 1)M^{47} + q^{181} M^{46} - q^{187} M^{45} + q^{181}(q + 1)M^{44} - q^{176}(q^7 + 1)M^{43} + q^{177}(q^4 + q + 1)M^{42} - q^{172}(q^7 + q^4 + q^3 + 1)M^{41} + q^{170}(q^4 + q^3 + 1)M^{40} + q^{168}(q^6 - 1)M^{39} - q^{168}(-q^3 + q + 1)M^{38} - q^{163}(q^6 + q^3 + q^2 - 1)M^{37} + q^{160}(q^4 + q^3 + 1)M^{36} - q^{158}(q^5 + 1)M^{35} + q^{157}(q^4 - q^3 + q + 1)M^{34} + q^{152}(q^6 - q^4 + q^2 - 1)M^{33} - q^{149}(q^3 - q + 1)M^{32} - q^{148}(q^3 + 1)M^{31} + q^{142}(q^3 + 1)M^{30} + q^{142}(q^2 - 1)M^{29} + q^{136}(q^4 - q^2 + 1)M^{28} + q^{134}(q^2 - 1)M^{27} - q^{134} M^{26} - q^{124} M^{25} + 2q^{122} M^{24} - q^{116} M^{23} - q^{118} M^{22} + q^{110}(q^2 - 1)M^{21} + q^{104}(q^4 - q^2 + 1)M^{20} + q^{102}(q^2 - 1)M^{19} + q^{94}(q^3 + 1)M^{18} - q^{92}(q^3 + 1)M^{17} - q^{85}(q^3 - q + 1)M^{16} + q^{80}(q^6 - q^4 + q^2 - 1)M^{15} + q^{77}(q^4 - q^3 + q + 1)M^{14} - q^{70}(q^5 + 1)M^{13} + q^{64}(q^4 + q^3 + 1)M^{12} - q^{59}(q^6 + q^3 + q^2 - 1)M^{11} - q^{56}(-q^3 + q + 1)M^{10} + q^{48}(q^6 - 1)M^9 + q^{42}(q^4 + q^3 + 1)M^8 - q^{36}(q^7 + q^4 + q^3 + 1)M^7 + q^{33}(q^4 + q + 1)M^6 - q^{24}(q^7 + 1)M^5 + q^{21}(q + 1)M^4 - q^{19} M^3 + q^5 M^2 - q^2(q + 1)M + 1).$$

#### 5. THE $M$ -FACTORS

The  $M$ -factors  $\epsilon_p(M) \in \mathbb{Q}[M]$  are given as follows for  $p = -5, \dots, 5$ :

$$\epsilon_{-5}(M) = -(-1 + M)^9(1 + M)^5(1 + M + M^2)(1 - 2M + 5M^2 + 6M^3 - 14M^4 + 20M^5 + 17M^6 - 48M^7 + 43M^8 + 40M^9 - 67M^{10} + 40M^{11} + 43M^{12} - 48M^{13} + 17M^{14} + 20M^{15} - 14M^{16} + 6M^{17} + 5M^{18} - 2M^{19} + M^{20})(12 - 32M + 34M^2 + 18M^3 - 171M^4 + 462M^5 - 680M^6 + 240M^7 + 1054M^8 - 2126M^9 + 1332M^{10} + 1080M^{11} - 3016M^{12} + 2558M^{13} - 227M^{14} - 2256M^{15} + 3187M^{16} - 2256M^{17} - 227M^{18} + 2558M^{19} - 3016M^{20} + 1080M^{21} +$$

$$1332M^{22} - 2126M^{23} + 1054M^{24} + 240M^{25} - 680M^{26} + 462M^{27} - 171M^{28} + 18M^{29} + 34M^{30} - 32M^{31} + 12M^{32}),$$

$$\epsilon_{-4}(M) = (-1 + M)^9(1 + M)^6(1 + M^4)^2(4 - 2M + 7M^2 + 10M^3 - 14M^4 + 34M^5 - 7M^6 + 6M^7 + 24M^8 + 6M^9 - 7M^{10} + 34M^{11} - 14M^{12} + 10M^{13} + 7M^{14} - 2M^{15} + 4M^{16}),$$

$$\epsilon_{-3}(M) = 2(-1 + M)^5(1 + M)^5(1 - M + M^2)^3(1 + M + M^2),$$

$$\epsilon_{-2}(M) = -(-1 + M)^3(1 + M)^2,$$

$$\epsilon_{-1}(M) = -1 + M,$$

$$\epsilon_0(M) = -1 + M,$$

$$\epsilon_1(M) = 1 - M,$$

$$\epsilon_2(M) = (-1 + M)^3,$$

$$\epsilon_3(M) = 2(-1 + M)^5(1 + M)^2(1 + M + M^2)^2(1 - 6M + 13M^2 - 6M^3 + M^4),$$

$$\epsilon_4(M) = -(-1 + M)^7(1 + M)^3(1 + M^2)(1 - 4M + 4M^2 + 4M^3 - 8M^4 + 4M^5 + 4M^6 - 4M^7 + M^8)(4 - 14M + 33M^2 + 12M^3 - 330M^4 + 328M^5 - 9M^6 + 226M^7 + 650M^8 + 226M^9 - 9M^{10} + 328M^{11} - 330M^{12} + 12M^{13} + 33M^{14} - 14M^{15} + 4M^{16}),$$

$$\epsilon_5(M) = (-1 + M)^9(1 + M)^2(1 + M + M^2)(1 - 2M + 4M^2 - 2M^3 - M^4 - 109M^6 - 78M^7 + 406M^8 + 162M^9 - 417M^{10} + 162M^{11} + 406M^{12} - 78M^{13} - 109M^{14} - M^{16} - 2M^{17} + 4M^{18} - 2M^{19} + M^{20})(12 - 88M + 318M^2 - 698M^3 + 381M^4 + 4924M^5 - 20623M^6 + 30440M^7 + 4694M^8 - 61074M^9 + 47268M^{10} + 15096M^{11} - 11350M^{12} - 42702M^{13} + 37078M^{14} + 55502M^{15} - 112131M^{16} + 55502M^{17} + 37078M^{18} - 42702M^{19} - 11350M^{20} + 15096M^{21} + 47268M^{22} - 61074M^{23} + 4694M^{24} + 30440M^{25} - 20623M^{26} + 4924M^{27} + 381M^{28} - 698M^{29} + 318M^{30} - 88M^{31} + 12M^{32}).$$

An explicit calculation shows that  $\epsilon_p(M)$  are palindromic polynomials, i.e.,  $\epsilon_p(M)/e_p(1/M) = -M^{\delta_p}$ , where  $\delta_p$  is given by

$$\{68, 39, 18, 5, 1, 1, 1, 3, 15, 36, 65\}$$

for  $p = -5, \dots, 5$ . One factor of  $\epsilon_p(M)$  is easy to spot: it is the Alexander polynomial  $\Delta_p(M)$ , in accordance with the loop expansion; see (1–4). The other factors of  $\epsilon_p(M)$  are more mysterious, with no clear geometric definition. Note, finally, that the values of  $\delta_p$  given above agree with the translation factor of the Newton polygon of  $K_p$  given in the last row of Table 4. This agreement is consistent with the fact that our computed recurrence relations  $(A_p(M, L, q), b_p(M, q))$  are palindromic for  $p = -5, \dots, 5$ .

#### ACKNOWLEDGMENTS

The paper came into maturity following a request by D. Zagier for an explicit formula for the recurrence of the colored Jones polynomial of a knot, during visits of the first author at the Max-Planck-Institut für Mathematik in 2008–2010. The

<sup>1</sup>The values of  $(A_p(M, L, q), b_p(M, q))$  for  $p = -5, \dots, 5$  are available from <http://www.risc.jku.at/people/ckoutsch/pretzel/> or <http://www.math.gatech.edu/~stavros/publications/pretzel.data/>.

authors met during the conference in honor of Doron Zeilberger's 60th birthday at Rutgers University in 2010, and we wish to thank the organizers of the Zeilberger Fest, and especially D. Zagier and D. Zeilberger, for their interest and for many stimulating conversations.

C. K. was supported by grants DMS-0070567 of the National Science Foundation of the United States and FWF P20162-N18 of the Austrian Science Fund. S. G. was supported in part by grant DMS-0805078 of the National Science Foundation of the United States.

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