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The 3D index of an ideal triangulation and angle structures

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With an appendix by Sander Zwegers

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Abstract The 3D index of Dimofte–Gaiotto–Gukov is a partially defined function on the set of ideal triangulations of 3-manifolds with r tori boundary components. For a fixed $2r$ tuple of integers, the index takes values in the set of q-series with integer coefficients. Our goal is to give an axiomatic definition of the tetrahedron index and a proof that the domain of the 3D index consists precisely of the set of ideal triangulations that support an index structure. The latter is a generalization of a strict angle structure. We also prove that the 3D index is invariant under 3–2 moves, but not in general under 2–3 moves.

Keywords 3D Tetrahedron index · Quantum dilogarithm · Neumann–Zagier equations · Hyperbolic geometry · Ideal triangulations · Angle structures

Mathematics Subject Classification Primary 57N10 · Secondary 57M25

1 Introduction

In a series of papers [6,7], Dimofte–Gaiotto–Gukov studied topological gauge theories using as input an ideal triangulation $\mathcal{T}$ of a 3-manifold M. These gauge theories play an important role in

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- Chern–Simons perturbation theory (that fits well with the earlier work on quantum Riemann surfaces [4] and the later work on the perturbative invariants [5]) and
- categorification and Khovanov Homology, which fits with the earlier work [28].

Although the gauge theory depends on the ideal triangulation $\mathcal{T}$, and the 3D index in general may not converge, physics predicts that the gauge theory ought to be a topological invariant of the underlying 3-manifold M. When ∂M consists of r tori, the low energy description of these gauge theories gives rise to a *partially* defined function

$$\begin{aligned} I : \{\text{ideal triangulations}\} &\longrightarrow \mathbb{Z}((q^{1/2}))^{\mathbb{Z}^r \times \mathbb{Z}^r}, \\ \mathcal{T} &\mapsto I_{\mathcal{T}}(m_1, \dots, m_r, e_1, \dots, e_r) \in \mathbb{Z}((q^{1/2})) \end{aligned} \tag{1.1}$$

for integers m_i and e_i, which is invariant under some *partial* 2–3 moves. The building block of the 3D index $I_{\mathcal{T}}$ is the *tetrahedron index* $I_\Delta(m, e)(q) \in \mathbb{Z}[[q^{1/2}]]$ defined by[1]

$$I_\Delta(m, e) = \sum_{n=(-e)_+}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+1)-\left(n+\frac{1}{2}e\right)m}}{(q)_n (q)_{n+e}}, \tag{1.2}$$

where

$$e_+ = \max\{0, e\}$$

and $(q)_n = \prod_{i=1}^n (1 - q^i)$. If we wish, we can sum in the above equation over the integers, with the understanding that $1/(q)_n = 0$ for $n < 0$.

Roughly, the 3D index $I_{\mathcal{T}}$ of an ideal triangulation $\mathcal{T}$ is a sum over tuples of integers of a finite product of tetrahedron indices evaluated at some linear forms in the summation variables. Convergence of such sums is not obvious and thus not always expected on physics grounds. For instance, the following sum

$$\sum_{e \in \mathbb{Z}} I_\Delta(0, e) q^{ve}$$

converges in $\mathbb{Z}((q^{1/2}))$ if and only if $v > 0$. This follows easily from the fact that the degree $\delta(e)$ of the summand is given by

$$\delta(e) = ve + \begin{cases} 0 & \text{if } e \geq 0, \\ \frac{e^2}{2} - \frac{e}{2} & \text{if } e \leq 0. \end{cases}$$

Our goal is to

(a) prove that the 3D index $I_{\mathcal{T}}$ exists if and only if $\mathcal{T}$ admits an index structure (a generalization of a strict angle structure)—see Theorem 2.12;

[1] The variables (m, e) are named after the magnetic and electric charges of [6].

(b) give a complete axiomatic definition of the tetrahedron index I_Δ focusing on the combinatorial and q-holonomic aspects—see Sect. 3; and
(c) show that the 3D index is invariant under $3 \to 2$ moves, but not in general under $2 \to 3$ moves, and give a necessary and sufficient criterion for invariance under $2 \leftrightarrow 3$ moves—see Sect. 6.

2 Index structures, angle structures, and the 3D index

2.1 Index structures

Consider two $r \times s$ matrices $\mathbf{A}$ and $\mathbf{B}$ with integer entries and a column vector $\nu \in \mathbb{Z}^r$, and let $\mathbf{M} = (\mathbf{A}|\mathbf{B}|\nu)$.

Definition 2.1 (a) We say that $\mathbf{M}$ supports an *index structure* if the rank of $(\mathbf{A}|\mathbf{B})$ is r and for every $Q : \{1, \dots, s\} \to \{1, 2, 3\}$ there exists $(\alpha, \beta, \gamma) \in \mathbb{Q}^{3s}$ that satisfies

$$\mathbf{A}\alpha + \mathbf{B}\gamma = \nu, \qquad \alpha + \beta + \gamma = (1, \dots, 1)^T, \tag{2.1}$$

and $Q(\alpha, \beta, \gamma) > 0$. The latter means that for every $i = 1, \dots, s$ the following inequalities are satisfied:

$$\begin{cases} \alpha_i > 0 & \text{if} \quad Q(i) = 1, \\ \beta_i > 0 & \text{if} \quad Q(i) = 2, \\ \gamma_i > 0 & \text{if} \quad Q(i) = 3. \end{cases} \tag{2.2}$$

(b) We say that $\mathbf{M}$ supports a *strict index structure* if the rank of $(\mathbf{A}|\mathbf{B})$ is r and there exists $(\alpha, \beta, \gamma) \in \mathbb{Q}_+^{3s}$ that satisfies (2.1), where $\mathbb{Q}^+$ is the set of positive rational numbers.

It is easy to see that if $\mathbf{M}$ supports a strict index structure, then it supports an index structure, but not conversely. As we will see in Sect. 2.2, ideal triangulations $\mathcal{T}$ give rise to matrices $\mathbf{M}$, and a strict index structure on $\mathbf{M}$ is a strict angle structure on $\mathcal{T}$. On the other hand, index structures are new and motivated by Theorem 2.4.

The next definition discusses two actions on $\mathbf{M}$: an action of $\mathrm{GL}(r, \mathbb{Z})$ on the left which allows for row operations on $\mathbf{M}$ and a cyclic action of order three at the pair of the ith columns of $(\mathbf{A}|\mathbf{B})$.

Definition 2.2 (a) There is a left action of $\mathrm{GL}(r, \mathbb{Z})$ on $\mathbf{M}$, defined by

$$P \in \mathrm{GL}(r, \mathbb{Z}), \qquad \mathbf{M} = (\mathbf{A}|\mathbf{B}|\nu), \qquad P\mathbf{M} = (P\mathbf{A}|P\mathbf{B}|P\nu).$$

An index structure on $\mathbf{M}$ is also an index structure on $P\mathbf{M}$. (b) There is a left action of $(\mathbb{Z}/3)^s$ on $\mathbf{M}$ acting on the ith columns $(a_i|b_i)$ of $(\mathbf{A}|\mathbf{B})$ (and fixing all other columns) given by

$$(a_i|b_i|\nu) \overset{S}{\mapsto} (-b_i|a_i - b_i|\nu - b_i)\,, \tag{2.3}$$

where

$$S(a|b|v) = (-b|a-b|v-b) \tag{2.4}$$

satisfies $S^3 = \mathrm{Id}$. We extend S to act on an index structure (α, β, γ) of $\mathbf{M}$ by

$$(\alpha_i, \beta_i, \gamma_i) \overset{S}{\mapsto} (\beta_i, \gamma_i, \alpha_i) \tag{2.5}$$

and fixing all other coordinates of (α, β, γ). It is easy to see that if (α, β, γ) is an index structure on $\mathbf{M}$ and $S \in (\mathbb{Z}/3)^s$, then $S(\alpha, \beta, \gamma)$ is an index structure of $S\mathbf{M}$.

Definition 2.3 Given $\mathbf{M}$, and $m = (m_1, \dots, m_s), e = (e_1, \dots, e_s) \in \mathbb{Z}^s$, consider the sum

$$I_{\mathbf{M}}(m, e)(q) = \sum_{k \in \mathbb{Z}^r} q^{\frac{1}{2} v \cdot k} \prod_{i=1}^{s} I_\Delta(m_i - b_i \cdot k, e_i + a_i \cdot k). \tag{2.6}$$

Theorem 2.4 *$I_{\mathbf{M}}(m, e)(q) \in \mathbb{Z}((q^{1/2}))$ is convergent for all $m, e \in \mathbb{Z}^s$ if and only if $\mathbf{M}$ supports an index structure. In that case, $I_{\mathbf{M}}$ is q-holonomic in the variables (m, e).*

Remark 2.5 q-holonomicity in Theorem 2.4 follows immediately from [27]. Convergence is the main difficulty.

Remark 2.6 By definition, $I_{\mathbf{M}}$ is a generalized Nahm sum in the sense of [10], where the summation is over a lattice.

Corollary 2.7 *Applying Theorem 2.4 to the case $r = 1$, $s = 3$, $\mathbf{M} = (\mathbf{A}|\mathbf{B}|v) = (1\,1\,1|0\,0\,0|2)$, and the strict index structure $2 = \frac{2}{3} \cdot 1 + \frac{2}{3} \cdot 1 + \frac{2}{3} \cdot 1$, it follows that the right-hand side of the pentagon identity (3.6) is convergent in $\mathbb{Z}((q^{1/2}))$.*

The next remark discusses the invariance of the index under the actions of Definition 2.2.

Remark 2.8 Fix $\mathbf{M}$ that supports an index structure. Then, for $P \in \mathrm{GL}(r, \mathbb{Z})$ and $S \in (\mathbb{Z}/3)^s$, it follows that $P\mathbf{M}$ and $S\mathbf{M}$ also support an index structure. In that case, Theorem 2.4 implies that $I_{\mathbf{M}}$, $I_{P\mathbf{M}}$, and $I_{S\mathbf{M}}$ are all convergent. We claim that

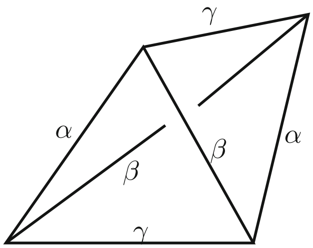
$$I_{P\mathbf{M}} = I_{\mathbf{M}}, \qquad I_{S\mathbf{M}} = I_{\mathbf{M}}.$$

The first equality follows by changing variables $k \mapsto Pk$ in the definition of $I_{\mathbf{M}}$ given by (2.6). The second equality follows from the fact that the tetrahedron index I_Δ satisfies Eq. (3.2); this is shown in part (a) of Theorem 3.7.

The next corollary follows easily from Theorem 2.4 and the definition of an index structure on $(\mathbf{A}|0|v_0)$.

Corollary 2.9 *Fix an $r \times s$ matrix $\mathbf{A}$ with integer entries and columns v_i for $i = 1, \dots, s$, and let $v_0 \in \mathbb{Z}^r$ and $\mathbf{M} = (\mathbf{A}|0|v_0)$. The following are equivalent:*

Fig. 1 Angles of a tetrahedron



(a) $I_{\mathbf{M}}(q)$ *converges.*
(b) $\mathrm{rk}(\mathbf{A}) = r$ *and there exists* $\alpha_i > 0$ *for* $i = 1, \dots, s$ *such that* $v_0 = \sum_{i=1}^{s} \alpha_i v_i$.

Question 2.10 Compare the q-series $I_{(\mathbf{A}|0|v)}$ with the vector partition functions of Sturmfels [24] and Brion–Vergne [1], and the q-hypergeometric systems of equations of [23].

2.2 Angle structures

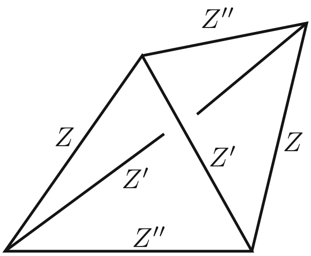
In this section, we define the 3D index of an ideal triangulation. A *generalized angle structure* on a combinatorial ideal tetrahedron Δ is an assignment of real numbers (called *angles*) at each edge of Δ such that the sum of the three angles around each vertex is 1.[2] It is easy to see that opposite edges are assigned the same angle, and thus a generalized angle structure is determined by a triple $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ that satisfies $\alpha + \beta + \gamma = 1$; see Fig. 1.

A generalized angle structure is *strict* if $\alpha, \beta, \gamma > 0$. Let $\mathcal{T}$ denote an ideal triangulation of an oriented 3-manifold M with torus boundary. A *generalized angle structure* on $\mathcal{T}$ is the assignment of angles at each tetrahedron of $\mathcal{T}$ such that the sum of angles around every edge of $\mathcal{T}$ is 2. A generalized angle structure on $\mathcal{T}$ is *strict* if its restriction to each tetrahedron is strict. For a detailed discussion of angle structures and their duality with normal surfaces, see [11,16,26]. Generalized angle structures are linearizations of the gluing equations, which may be used to construct complete hyperbolic structures, and intimately connected with the theory of normal surfaces on M [12].

The existence of a strict angle structure imposes restrictions on the topology of M: it implies that M is irreducible and atoroidal, and each boundary component of M is a torus; see for example [16]. On the other hand, if M is a hyperbolic link complement, then there exist triangulations which admit a strict angle structure [11]. In fact, such triangulations can be constructed by a suitable refinement of the Epstein–Penner ideal cell decomposition of M. Note that not all such triangulations are geometric [11].

[2] The sum of the 3 angles around each vertex is traditionally π.

Fig. 2 Shapes of a tetrahedron



2.3 The Neumann–Zagier matrices

Fix an oriented ideal triangulation $\mathcal{T}$ with N tetrahedra of a 3-manifold M with tori boundary components. Assign variables Z_i, Z_i', Z_i'' at the opposite edges of each tetrahedron Δ_i respecting its orientation as in Fig. 2.

Then we can read off matrices $N \times N$ matrices $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$, and $\bar{\mathbf{C}}$ whose rows are indexed by the N edges of $\mathcal{T}$ and whose columns are indexed by the Z_i, Z_i', Z_i'' variables. These are the so-called *Neumann–Zagier matrices* that encode the exponents of the *gluing equations* of $\mathcal{T}$, originally introduced by Thurston [20,25]. In terms of these matrices, a generalized angle structure is a triple of vectors α, β, $\gamma \in \mathbb{R}^N$ that satisfy the equations

$$\bar{\mathbf{A}}\alpha + \bar{\mathbf{B}}\beta + \bar{\mathbf{C}}\gamma = (2, \dots, 2)^T, \qquad \alpha + \beta + \gamma = (1, \dots, 1)^T . \tag{2.7}$$

A *quad* Q for $\mathcal{T}$ is a choice of pair of opposite edges at each tetrahedron Δ_i for $i = 1, \dots, N$. Q can be used to eliminate one of the three variables α_i, β_i, γ_i at each tetrahedron using the relation $\alpha_i + \beta_i + \gamma_i = 1$. Doing so, Eq. (2.7) takes the form

$$\mathbf{A}\alpha + \mathbf{B}\gamma = \nu .$$

The matrices $(\mathbf{A}|\mathbf{B})$ have some key *symplectic properties*, discovered by Neumann and Zagier when M is a hyperbolic 3-manifold (and $\mathcal{T}$ is well adapted to the hyperbolic structure) [20], and later generalized to the case of arbitrary 3-manifolds in [19]. Neumann and Zagier show that the rank of $(\mathbf{A}|\mathbf{B})$ is $N - r$, where r is the number of boundary components of M; all assumed tori. If we choose $N - r$ linearly independent rows of $(\mathbf{A}|\mathbf{B})$, then we obtain matrices $(\mathbf{A}'|\mathbf{B}')$ and a vector ν', which combine to form $\mathbf{M} = (\mathbf{A}'|\mathbf{B}'|\nu')$. In addition, the exponents of meridian and longitude loops (the latter, divided by 2) at each boundary torus give additional matrices (a^T, b^T) and (c^T, d^T) of size $r \times 2N$.

Definition 2.11 The 3D index of $\mathcal{T}$ is defined by

$$I_{\mathcal{T}}(m, e)(q) = I_{\mathbf{M}}(dm - be, -cm + ae)(q). \tag{2.8}$$

Implicit in the above definition are a choice of quad Q and a choice of rows to remove. However, the index is independent of these choices; see Remark 2.8. Keep in mind the action of $(\mathbb{Z}/3)^N$ given by acting on the ith columns $\bar{a}_i$, $\bar{b}_i$ and $\bar{c}_i$ of $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$, and $\bar{\mathbf{C}}$ by

$$S(\bar{a}_i|\bar{b}_i|\bar{c}_i) = (\bar{b}_i|\bar{c}_i|\bar{a}_i)$$

(and fixing all other columns) and on the ith coordinates of an angle structure by

$$S(\alpha_i, \beta_i, \gamma_i) = (\beta_i, \gamma_i, \alpha_i)$$

(and fixing all other coordinates) and on the ith columns a_i and b_i of $\mathbf{A}$ and $\mathbf{B}$ by

$$S(a_i|b_i|\nu) = (-b_i|a_i - b_i|\nu - b_i)$$

(and fixing all other columns). Since the rank of $(\mathbf{A}|\mathbf{B})$ is $N-r$ and $\mathbf{A}$, $\mathbf{B}$ are $(N-r)\times N$ matrices, it follows that $\mathbf{M}$ admits a strict structure if and only if $\mathcal{T}$ admits a strict angle structure. In addition, $\mathcal{T}$ admits an index structure if for every choice of quad Q there exists a solution (α, β, γ) of Eq. (2.7) that satisfies the inequalities (2.2). Theorem 2.4 implies the following.

Theorem 2.12 *The index $I_{\mathcal{T}} : \mathbb{Z}^r \times \mathbb{Z}^r \longrightarrow \mathbb{Z}((q^{1/2}))$ is well defined if and only if $\mathcal{T}$ admits an index structure. In particular, $I_{\mathcal{T}}$ exists if $\mathcal{T}$ admits a strict angle structure.*

See Sect. 6.3 for an example of an ideal triangulation $\mathcal{T}$ of the census manifold `m136` [3] which admits a semi-strict angle structure (i.e., angles are nonnegative real numbers), does not admit a strict angle structure, and which has a solution of the gluing equations that recover the complete hyperbolic structure. A case-by-case analysis shows that this example admits an index structure, and thus the index $I_{\mathcal{T}}$ exists. This example appears in [11, Example 7.7]. We thank H. Segerman for a detailed analysis of this example.

2.4 On the topological invariance of the index

Physics predicts that when defined, the 3D index $I_{\mathcal{T}}$ depends only on the underlying 3-manifold M. Recall that Hodgson et al. [11] prove that every hyperbolic 3-manifold M that satisfies

$$H_1(M, \mathbb{Z}/2) \to H_1(M, \partial M, \mathbb{Z}/2) \qquad \text{is the zero map} \tag{2.9}$$

(e.g., a hyperbolic link complement) admits an ideal triangulation with a strict angle structure, and conversely if M has an ideal triangulation with a strict angle structure, then M is irreducible and atoroidal, and each boundary component of M is a torus [16].

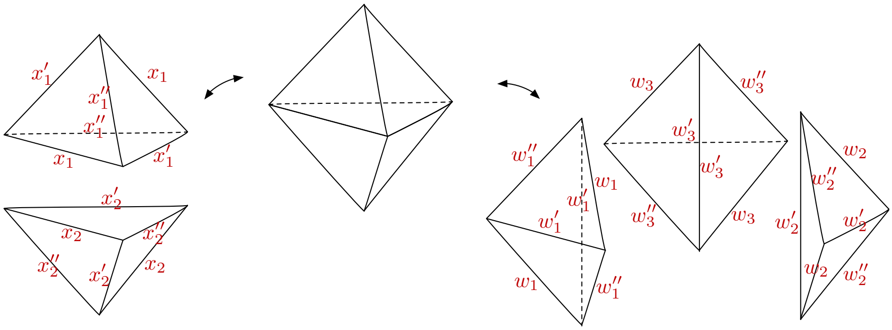


Fig. 3 A 2–3 move: a bipyramid splits into N tetrahedra for $\mathcal{T}$ and $N+1$ tetrahedra for $\widetilde{\mathcal{T}}$

A simple way to construct a topological invariant using the index would be a map

$$M \mapsto \{I_{\mathcal{T}} \mid \mathcal{T} \in \mathcal{S}_M\},$$

where M is a cusped hyperbolic 3-manifold with at least one cusp and $\mathcal{S}_M$ is the set of ideal triangulations of M that support an index structure. The latter is a nonempty (generally infinite) set by [11], assuming that M satisfies (2.9). If we want a finite set, we can use the subset $\mathcal{S}_M^{\mathrm{EP}}$ of ideal triangulations $\mathcal{T}$ of M which are a refinement of the Epstein–Penner cell decomposition of M. Again, [11] implies that $\mathcal{S}_M^{\mathrm{EP}}$ is nonempty assuming (2.9). But really, we would prefer a single 3D index for a cusped manifold M, rather than a finite collection of 3D indices.

It is known that every two combinatorial ideal triangulations of a 3-manifold are related by a sequence of 2–3 *moves* [17,18,22]. Thus, topological invariance of the 3D index follows from invariance under 2–3 moves.

Consider two ideal triangulations $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ with N and $N+1$ tetrahedra related by a 2–3 *move* as shown in Fig. 3.

Proposition 2.13 *If $\widetilde{\mathcal{T}}$ admits a strict angle structure, so does $\mathcal{T}$ and $I_{\widetilde{\mathcal{T}}} = I_{\mathcal{T}}$.*

For the next proposition, a special index structure on $\mathcal{T}$ is given in Definition 6.2.

Proposition 2.14 *If $\mathcal{T}$ admits a special strict angle structure, then $\widetilde{\mathcal{T}}$ admits a strict angle structure and $I_{\widetilde{\mathcal{T}}} = I_{\mathcal{T}}$.*

Remark 2.15 The asymmetry in Propositions 2.13 and 2.13 is curious, but also necessary. The origin of this asymmetry is the fact that 3–2 moves always preserve strict angle structures but 2–3 moves sometimes do not. If 2–3 moves always preserved strict angle structures, then all ideal triangulations of a fixed manifold would admit strict angle structures as long as one of them does. On the other hand, an ideal triangulation that contains an edge which belongs to exactly one (or two) ideal tetrahedron(a) does not admit a strict angle structure since the angle equations around that edge should add to 2. Such triangulations are easy to construct, even for hyperbolic 3-manifolds (e.g., the 4_1 knot).

3 Axioms for the tetrahedron index

In this section, we discuss an axiomatic approach to the tetrahedron index. Let $\mathbb{Z}((q^{1/2}))$ (resp., $\mathbb{Z}[[q^{1/2}]]$) denote the ring of series of the form

$$f(q) = \sum_{n \in \frac{1}{2}\mathbb{Z}} a_n q^n,$$

where there exists $n_0 = n_0(f)$ such that $a_n = 0$ for all $n < n_0$ (resp., $n < 0$). For $f(q) \in \mathbb{Z}((q^{1/2}))$, its *degree* $\delta(f(q))$ is the largest half-integer (or infinity) such that $f(q) \in q^{d(f)}\mathbb{Z}[[q^{1/2}]]$. We will say that $f(q) \in \mathbb{Z}((q^{1/2}))$ is *q-positive* if $\delta(f(q)) \geq 0$.

Definition 3.1 A *tetrahedron index* is a function $f : \mathbb{Z}^2 \longrightarrow \mathbb{Z}((q^{1/2}))$ that satisfies the equations

$$q^{\frac{e}{2}} f(m+1, e) + q^{-\frac{m}{2}} f(m, e+1) - f(m, e) = 0, \tag{3.1a}$$
$$q^{\frac{e}{2}} f(m-1, e) + q^{-\frac{m}{2}} f(m, e-1) - f(m, e) = 0 \tag{3.1b}$$

for all integers m, e, together with the parity condition $f(m, e) \in q^{\frac{em}{2}}\mathbb{Z}((q))$ for all m and e. Let V denote the set of all tetrahedron indices and V_+ denote the set of all q-positive tetrahedron indices.

Theorem 3.2 (a) *V is a free q-holonomic $\mathbb{Z}((q))$-module of rank* 2.
(b) *V_+ is a free q-holonomic $\mathbb{Z}[[q]]$-module of rank* 1.
(c) *If $f \in V$, then it satisfies the equation*

$$\begin{aligned} f(m, e)(q) &= \left(-q^{\frac{1}{2}}\right)^{-e} f(e, -e-m)(q) \\ &= \left(-q^{\frac{1}{2}}\right)^{m} f(-e-m, m)(q) \end{aligned} \tag{3.2}$$

for all integers m and e.
(d) *If $f \in V$, then it satisfies the equations*

$$f(m, e+1) + \left(q^{e+\frac{m}{2}} - q^{-\frac{m}{2}} - q^{\frac{m}{2}}\right) f(m, e) + f(m, e-1) = 0, \tag{3.3a}$$
$$f(m+1, e) + \left(q^{-\frac{e}{2}-m} - q^{-\frac{e}{2}} - q^{\frac{e}{2}}\right) f(m, e) + f(m-1, e) = 0 \tag{3.3b}$$

for all integers m, e.
(e) *If $f \in V$, then it satisfies the equation*

$$f(m,\ e) = f(-e,\ -m) \tag{3.4}$$

for all integers m, e.

Question 3.3 What is a basis for V?

Remark 3.4 The proof of part (a) of Theorem 3.2 implies that if $f(m, e)$ is a tetrahedron index, then $f(m, e)$ is a unique $\mathbb{Z}[q^{\pm 1/2}]$-linear combination of A and B

where $(f(0,0), f(0,1)) = (A, B)$. For example, if $C = (f(m,e))_{-2\leq m,e\leq 2}$, then $C = M_A A + M_B B$, where

$$M_A = \begin{pmatrix} 1-\frac{1}{q^3}+\frac{1}{q^2}+\frac{1}{q}-q^2 & \frac{1}{q}-q & -1 & -\frac{1}{q} & -\frac{1}{q^2}+\frac{1}{q} \\ 1-\frac{1}{q^2}+\frac{1}{q} & \frac{1}{\sqrt{q}} & 0 & -\frac{1}{\sqrt{q}} & -\frac{1}{q} \\ 1-\frac{1}{q} & 1 & 1 & 0 & -1 \\ -1 & 0 & 1 & \frac{1}{\sqrt{q}} & \frac{1}{q}-q \\ -q & -1 & 1-\frac{1}{q} & 1-\frac{1}{q^2}+\frac{1}{q} & 1-\frac{1}{q^3}+\frac{1}{q^2}+\frac{1}{q}-q^2 \end{pmatrix},$$

$$M_B = \begin{pmatrix} \frac{1}{q^3}-\frac{2}{q^2}-\frac{1}{q}+q+2q^2-q^3 & 1-\frac{1}{q}+2q-q^2 & 2-q & -1+\frac{1}{q} & \frac{1}{q^2}-\frac{2}{q} \\ -1+\frac{1}{q^2}-\frac{2}{q}+q & -\frac{1}{\sqrt{q}}+\sqrt{q} & 1 & \frac{1}{\sqrt{q}} & -1+\frac{1}{q} \\ -2+\frac{1}{q} & -1 & 0 & 1 & 2-q \\ 1-q & -\sqrt{q} & -1 & -\frac{1}{\sqrt{q}}+\sqrt{q} & 1-\frac{1}{q}+2q-q^2 \\ 2q-q^2 & 1-q & -2+\frac{1}{q} & -1+\frac{1}{q^2}-\frac{2}{q}+q & \frac{1}{q^3}-\frac{2}{q^2}-\frac{1}{q}+q+2q^2-q^3 \end{pmatrix}.$$

Remark 3.5 The proof of part (b) of Theorem 3.2 implies that if $f(m,e)$ is a tetrahedron index, then $f(m,e)$ is uniquely determined by $f(0,0) = \sum_{n=0}^{\infty} a_n q^n$. In particular, if $f(0,1) = \sum_{n=0}^{\infty} b_n q^n$, then b_n are $\mathbb{Z}$-linear combinations of a_k for $k \leq n$. For example, we have

$$\begin{aligned}
b_0 &= a_0 \\
b_1 &= a_0 + a_1 \\
b_2 &= 2a_0 + a_1 + a_2 \\
b_3 &= 4a_0 + 2a_1 + a_2 + a_3 \\
b_4 &= 9a_0 + 4a_1 + 2a_2 + a_3 + a_4 \\
b_5 &= 20a_0 + 9a_1 + 4a_2 + 2a_3 + a_4 + a_5 \\
b_6 &= 46a_0 + 20a_1 + 9a_2 + 4a_3 + 2a_4 + a_5 + a_6 \\
b_7 &= 105a_0 + 46a_1 + 20a_2 + 9a_3 + 4a_4 + 2a_5 + a_6 + a_7 \\
b_8 &= 242a_0 + 105a_1 + 46a_2 + 20a_3 + 9a_4 + 4a_5 + 2a_6 + a_7 + a_8 \\
b_9 &= 557a_0 + 242a_1 + 105a_2 + 46a_3 + 20a_4 + 9a_5 + 4a_6 + 2a_7 + a_8 + a_9 \\
b_{10} &= 1285a_0 + 557a_1 + 242a_2 + 105a_3 + 46a_4 + 20a_5 + 9a_6 + 4a_7 + 2a_8 + a_9 + a_{10} \\
b_{11} &= 2964a_0 + 1285a_1 + 557a_2 + 242a_3 + 105a_4 + 46a_5 + 20a_6 + 9a_7 + 4a_8 + 2a_9 + a_{10} + a_{11} \\
b_{12} &= 6842a_0 + 2964a_1 + 1285a_2 + 557a_3 + 242a_4 + 105a_5 + 46a_6 + 20a_7 + 9a_8 + 4a_9 + 2a_{10} \\
&\quad + a_{11} + a_{12}.
\end{aligned}$$

In fact, it appears that b_n is a $\mathbb{N}$-linear combination of a_k for $k \leq n$, although we do not know how to show this, nor do we know of a geometric significance of this experimental fact.

The next lemma computes the degree of the tetrahedron index.

Lemma 3.6 *The degree $\delta(m,e)$ of $I_\Delta(m,e)(q)$ is given by*

$$\begin{aligned}
\delta(m,e) = \frac{1}{2}\big(&m_+(m+e)_+ + (-m)_+ e_+ + (-e)_+(-e-m)_+ \\
&+ \max\{0, m, -e\}\big).
\end{aligned} \tag{3.5}$$

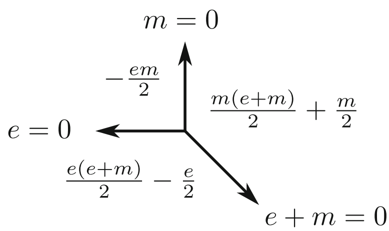


Fig. 4 The degree of the tetrahedron index

It follows that $\delta(m, e)$ *is a piece-wise quadratic polynomial as given in Fig. 4.*

The next theorem gives an axiomatic characterization of the tetrahedron index I_Δ.

Theorem 3.7 I_Δ *is uniquely characterized by the following equations:*

(a) $I_\Delta \in V_+$, $I_\Delta(0, 0)(0) \neq 0$
(b) I_Δ *satisfies the pentagon identity*

$$\begin{aligned} & I_\Delta(m_1 - e_2, e_1) I_\Delta(m_2 - e_1, e_2) \\ & \quad = \sum_{e_3 \in \mathbb{Z}} q^{e_3} I_\Delta(m_1, e_1 + e_3) I_\Delta(m_2, e_2 + e_3) I_\Delta(m_1 + m_2, e_3) \end{aligned} \tag{3.6}$$

for all integers m_1, m_2, e_1, e_2.

Remark 3.8 The unique part of Theorem 3.7 uses only the facts that $I_\Delta \in V$, $\delta(I_\Delta(0, e)) \geq 0$ for all e and I_Δ satisfy the special pentagon

$$I_\Delta(0, 0)^2 = \sum_{e \in \mathbb{Z}} I_\Delta(0, e)^3 q^e .$$

4 Properties of a tetrahedron index

4.1 Part (d) of Theorem 3.2

Consider a function $f(m, e)$ of two discrete integer variables e, m which satisfies Eqs. (3.1a) and (3.1b). An application of the `HolonomicFunctions.m` computer algebra package [15] implies that $f(m, e)$ also satisfies Eqs. (3.3a) and (3.3b).

4.2 The rank of V: part (a) of Theorem 3.2

An application of the `HolonomicFunctions.m` computer algebra package [15] implies that the linear q-difference operators corresponding to the recursions of Eqs. (3.1a) and (3.1b) is a Gröbner basis and the corresponding module has rank 2. Said differently, $f(m, e)$ is a unique $\mathbb{Z}[q^{\pm 1/2}]$-linear combination of A and B where $A = f(0, 0)$ and $B = f(0, 1)$.

4.3 The rank of V_+: part (b) of Theorem 3.2

Consider a function $f(m, e)$ of two discrete integer variables e, m which satisfies Eqs. (3.1a) and (3.1b). Section 4.1 implies that $f(0, e)$ satisfies the 3-term recursion

$$f(0,e) - \left(2 - q^{e-1}\right) f(0, e-1) + f(0, e-2) = 0 \tag{4.1}$$

for all integers e. It follows that for every integer e, $f(0, e)$ is a $\mathbb{Z}[q^{\pm 1}]$-linear combination of A and B where $f(0, 0) = A$ and $f(0, 1) = B$. An induction on $e < 0$ using the recursion relation (4.1) shows that for all $e < 0$ we have

$$f(0,e) = q^{-\frac{e^2}{2}-\frac{e}{2}} \left(p_1(e)A + p_2(e)B\right),$$

where $p_1(e), p_2(e) \in \mathbb{Z}[q]$ are polynomials of maximum q-degree $e^2/2 + e/2$ and constant term $(-1)^{e-1}$ and $(-1)^e$, respectively. For example, we have

$$\begin{aligned}
f(0,-1) &= A - B,\\
qf(0,-2) &= A(-1+q) + B(1-2q),\\
q^3 f(0,-3) &= A\left(1 - q - 2q^2 + q^3\right) + B\left(-1 + 2q + 2q^2 - -3q^3\right),\\
q^6 f(0,-4) &= A\left(-1 + q + 2q^2 + q^3 - -2q^4 - 3q^5 + q^6\right)\\
&\quad + B\left(1 - 2q - 2q^2 + q^3 + 4q^4 + 3q^5 - 4q^6\right),\\
q^{10} f(0,-5) &= A\left(1 - q - 2q^2 - q^3 + 5q^5 + 3q^6 + q^7 - 3q^8 - 4q^9 + q^{10}\right)\\
&\quad + B\left(-1 + 2q + 2q^2 - q^3 - -2q^4 - 7q^5 + 3q^7 + 6q^8 + 4q^9 - 5q^{10}\right).
\end{aligned}$$

Let us write

$$A = \sum_{n=0}^{\infty} a_n q^n, \qquad B = \sum_{n=0}^{\infty} b_n q^n.$$

If we assume that $f(0, e) \in \mathbb{Z}[[q]]$, this imposes a system of linear equations on the coefficients a_n and b_n of A and B. In fact, for fixed $e < 0$, the system of equations $\mathrm{coeff}(f(0, e), q^j) = 0$ for $j = -e^2/2 - e/2, \ldots, -2, -1$ is a triangular system of linear equations with unknowns b_j for $j = 0, 1, \ldots, e^2/2 - e/2 - 1$ where all diagonal entries of the coefficient matrix are 1. For example, we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ -2 & 1 & 0 & 0 & 0 & 0\\ -2 & -2 & 1 & 0 & 0 & 0\\ 1 & -2 & -2 & 1 & 0 & 0\\ 4 & 1 & -2 & -2 & 1 & 0\\ 3 & 4 & 1 & -2 & -2 & 1 \end{pmatrix} \begin{pmatrix} b_0\\ b_1\\ b_2\\ b_3\\ b_4\\ b_5 \end{pmatrix} = \begin{pmatrix} -a_0\\ a_0 - a_1\\ 2a_0 + a_1 - a_2\\ a_0 + 2a_1 + a_2 - a_3\\ -2a_0 + a_1 + 2a_2 + a_3 - a_4\\ -3a_0 - 2a_1 + a_2 + 2a_3 + a_4 - a_5 \end{pmatrix}.$$

It follows that b_n is a $\mathbb{Z}$-linear combination of a_k for $k \leq n$. This proves that the rank of the $\mathbb{Z}[[q]]$-module V_+ is at most 1. Since $I_\Delta \in V_+$ (as follows from the proof of Theorem 3.7), it follows that the rank of the $\mathbb{Z}[[q]]$-module V_+ is exactly 1. This proves part (b) of Theorem 3.2. □

Corollary 4.1 *The above proof implies that $f \in V_+$ is uniquely determined by its initial condition $f(0,0) \in \mathbb{Z}[[q]]$. It follows that if $f, g \in V_+$, then*

$$g(0,0)f(m,e) = f(0,0)g(m,e) \tag{4.2}$$

for all integers m and e.

4.4 Proof of triality: part (c) of Theorem 3.2

In this section, we prove part (c) of Theorem 3.2. Equation (3.2) concerns the following $\mathbb{Z}/3$-action on V.

Definition 4.2 Consider the action $f \mapsto Sf$ on a function $f : \mathbb{Z}^2 \longrightarrow \mathbb{Z}((q^{1/2}))$ given by

$$Sf(m,e) = \left(-q^{\frac{1}{2}}\right)^{-e} f(e, -e-m)\,. \tag{4.3}$$

Proposition 4.3 (a) *We have $S^3 = Id$.*
(b) *If $f \in V$, then $Sf = f$, and of course, also $S^2 f = f$.*

Part (c) of Theorem 3.2 follows from part (b) of the above proposition.

Proof (of Proposition 4.3) Part (a) is elementary. For part (b), assume that f satisfies Eq. (3.1a) for all (m,e). Replace (m,e) by $(e, -1-e-m)$ in (3.1a) and we obtain that

$$\begin{aligned} -f(e,-1-e-m) + q^{-\frac{e}{2}} f(e,-e-m) \\ +q^{-\frac{1}{2}-\frac{e}{2}-\frac{m}{2}} f(1+e,-1-e-m) = 0\,. \end{aligned} \tag{4.4}$$

Now, replace f by Sf on the left-hand side of Eq. (3.1a), and the result is given by

$$\begin{aligned} (-1)^{e+1}\big(-f(e,-1-e-m) + q^{-\frac{e}{2}} f(e,-e-m) \\ +q^{-\frac{1}{2}-\frac{e}{2}-\frac{m}{2}} f(1+e,-1-e-m)\big). \end{aligned}$$

The above vanishes from Eq. (4.4).

Likewise, assume that f satisfies Eq. (3.1b) for all (m,e). Replace (m,e) by $(e, 1-e-m)$ in (3.1b) and we obtain that

$$q^{\frac{1}{2}-\frac{e}{2}-\frac{m}{2}} f(-1+e, 1-e-m) + q^{-\frac{e}{2}} f(e,-e-m) - f(e,1-e-m) = 0. \tag{4.5}$$

Now, replace f by Sf on the left-hand side of Eq. (3.1b), and the result is given by

$$(-1)^{e+1}\Big(q^{\frac{1}{2}-\frac{e}{2}-\frac{m}{2}}f(-1+e,1-e-m)+q^{-\frac{e}{2}}f(e,-e-m)-f(e,1-e-m)\Big).$$

It follows that if $f\in V$, then the above vanishes from Eq. (4.5). In other words, if $f\in V$, then $Sf\in V$. To conclude that $f=Sf$, it suffices to show (by part (a) of Theorem 3.2) that $f(0,0)=(Sf)(0,0)$. If $f(0,0)=A$ and $f(0,1)=B$, using Remark 3.4, we have

$$\begin{aligned}(Sf)(0,0)&=f(0,0)=A,\\(Sf)(0,1)&=f(0,1)+q^{-\frac{1}{2}}f(1,-1)=B+q^{-\frac{1}{2}}\big(-Bq^{\frac{1}{2}}\big)=0.\end{aligned}$$

This concludes the proof of Proposition 4.3. □

4.5 I_Δ is a tetrahedron index

Observe that by its definition

$$I_\Delta(m,e)=\sum_{e\in\mathbb{Z}}S(m,e,n)$$

is given by a one-dimensional sum of a *proper q-hypergeometric term* [21,27]

$$S(m,e,n)=(-1)^n\frac{q^{\frac{1}{2}n(n+1)-\left(n+\frac{1}{2}e\right)m}}{(q)_n(q)_{n+e}}.$$

It follows from [27] that $I_\Delta(m,e)$ is q-holonomic in both variables m and e. Moreover, recursion relations for $I_\Delta(m,e)$ can be found by the creative telescoping method of [27]. For instance, S satisfies the recursion

$$q^{\frac{e}{2}}S(m-1,e,n)+q^{-\frac{m}{2}}S(m,e-1,n)-S(m,e,n)=0 \tag{4.6}$$

which implies that I_Δ satisfies Eq. (3.1b). To prove Eq. (4.6), divide it by $S(m,e,n)$ and use the fact that

$$q^{\frac{e}{2}}\frac{S(m-1,e,n)}{S(m,e,n)}=q^{e+n},\qquad q^{-\frac{m}{2}}\frac{S(m,e-1,n)}{S(m,e,n)}=1-q^{e+n}.$$

The proof of Eq. (3.1a) is similar. For an alternative proof, using the quantum dilogarithm, see Sect. 1.

4.6 The degree of I_Δ

Proof (of Lemma 3.6) Consider the fan F of $\mathbb{R}^2$ with rays $(1,0)$, $(0,1)$, and $(1,-1)$. Observe that the linear transformation $(m,e) \mapsto (e,-e-m)$ (which appears in Definition 4.2) rotates the three cones of the fan F and preserves the piece-wise quadratic polynomial that appears in Lemma 3.6. Since $I_\Delta \in V$ (by Sect. 4.5) and V is pointwise invariant under S (by Proposition 4.3), it suffices to compute $\delta(m,e)$ when (m,e) lies in the cone $m \leq 0$, $e \geq 0$. In that case, Eq. (1.2) gives

$$I_\Delta(m,e) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+1)-\left(n+\frac{1}{2}e\right)m}}{(q)_n (q)_{n+e}}.$$

If $\delta(m,e,n)$ denotes the degree of the summand, using $m \leq 0$, $n \geq 0$ we get

$$\delta(m,e,n) = \frac{1}{2}\left(n(n+1)\right) - \left(n + \frac{1}{2}e\right) m \geq -\frac{em}{2},$$

with equality achieved uniquely at $n = 0$. It follows that the degree of $I(m,e)$ in this cone is given by $-em/2$. □

4.7 Proof of Theorem 3.7

First we show that I_Δ satisfies the required equations:

(a) $I_\Delta \in V$ from Sect. 4.5. Lemma 3.6 and Eq. (3.5) manifestly imply that $\delta(I_\Delta(m,e)) \geq 0$ for all integers m and e. Thus, $I_\Delta \in V_+$. Moreover, $I_\Delta(0,0) = 1 + O(q)$.

(b) I_Δ satisfies the pentagon identity from Sect. 1.

It remains to show the unique part in Theorem 3.7. Suppose $f \in V_+$ satisfies the pentagon and $f(0,0)(0) \neq 0$. Corollary 4.1 implies that $f(m,e)(q) = C(q) I_\Delta(e,m)(q)$ for some $C(q) \in \mathbb{Q}((q))$. Consider the special pentagon for f and I_Δ:

$$f(0,0)^2 = \sum_{e\in\mathbb{Z}} f(0,e)^3 q^e, \qquad I_\Delta(0,0)^2 = \sum_{e\in\mathbb{Z}} I_\Delta(0,e)^3 q^e.$$

It follows that $C(q)^2 = C(q)^3$, and since $C(q) \neq 0$, we get $C(q) = 1$. This concludes the uniqueness part of Theorem 3.7. □

5 Convergence of the 3D index

5.1 Proof of Theorem 2.4

In this section, we prove Theorem 2.4. We begin by a well-known lemma due to Farkas [30].

Lemma 5.1 *Fix finite collections $\mathcal{A} = \{a_1, \dots, a_r\}$ and $\mathcal{B} = \{b_1, \dots, b_s\}$ of vectors in $\mathbb{R}^N$. The following are equivalent:*

(a) *there does not exist $v \neq 0$ such that $a_i \cdot v \geq 0$ for $i = 1, \dots, r$ and $b_j \cdot v = 0$ for $j = 1, \dots, s$.*
(b) *$\mathcal{A} \cup \mathcal{B}$ spans $\mathbb{R}^N$ and there exist $\alpha_i > 0$ for $i = 1, \dots, r$ and $\gamma_j \in \mathbb{R}$ for $j = 1, \dots, s$ such that $0 = \sum_i \alpha_i a_i + \sum_j \gamma_j b_j$.*

Proof (a) is equivalent to

(c) there does not exist $v \neq 0$ such that $a_i \cdot v \geq 0$ for $i = 1, \dots, r$, $b_j \cdot v \geq 0$ for $j = 1, \dots, s$, and $(-b_j) \cdot v \geq 0$ for $j = 1, \dots, s$.

(c) implies (b). Let C denote the cone spanned by $\mathcal{A} \cup \mathcal{B} \cup -\mathcal{B}$. (c) states that C is not contained in any half-space through the origin. By Farkas' lemma [30], it follows that $C = \mathbb{R}^N$. Thus, $\mathcal{A} \cup \mathcal{B} \cup -\mathcal{B}$ spans $\mathbb{R}^N$ and $-\sum_i a_i \in C$. (b) follows.

(b) implies (c): consider v such that $a_i \cdot v \geq 0$ and $b_j \cdot v = 0$ for all i, j. We know there exist $\alpha_i > 0$ and γ_j real such that $0 = \sum_i \alpha_i a_i + \sum_j \gamma_j b_j$. Taking inner product with v, it follows that $0 = \sum_i \alpha_i a_i \cdot v$. Since $\alpha_i > 0$ and $a_i \cdot v \geq 0$ for all i, it follows that $a_i \cdot v = 0$ for all i. Thus, v is perpendicular to $\mathcal{A} \cup \mathcal{B}$ which is assumed to span $\mathbb{R}^N$. Thus $v = 0$ and (c) follows. □

The next lemma concerns super-linear polynomial functions on a cone.

Lemma 5.2 *Suppose C is a closed cone in $\mathbb{R}^r$ and $p : C \longrightarrow \mathbb{R}$ is a polynomial that satisfies $p(nx) \geq c_x n$ for $n > 0$, $x \in C \setminus \{0\}$, and $c_x > 0$. Then, there exist $c > 0$ and $c' > 0$ such that $p(x) \geq c|x|$ for all $x \in C$ with $|x| \geq c'$.*

Proof Let $S = \{x \in \mathbb{R}^r \mid |x| = 1\}$ denote the unit sphere and let $p = \sum_{k=0}^d p_k(x)$ denote the decomposition of p into homogeneous polynomials p_k of degree k. Since $p(nx) = \sum_k n^k p_k(x)$, it follows that for every $x \in S \cap C$ there exists i such that $p_j(x) = 0$ for $j > i$ and $p_i(x) > 0$. In particular, $p_d : S \cap C \longrightarrow [0, \infty)$.

Case 1: $p_d(S \cap C) \subset (0, \infty)$. By compactness, $p_d(x) \geq c_0 > 0$ for $x \in S \cap C$ and $|p_k(x)| \leq c_k$ for $x \in S \cap C$ and $k = 1, \dots, d-1$. Thus $p(x) \geq c_0 |x|^d - \sum_{k=0}^{d-1} |x|^k c_k \geq c|x|$ for some $c > 0$.

Case 2: There exists $x \in S \cap C$ such that $p_d(x) = 0$ and $p_{d-1}(x) > 0$. Argue as above using the complement of a neighborhood of x where p_1 is strictly positive, and conclude the proof by induction on the depth of a point. □

Consider the restriction

$$I^{\rho}_{\mathbf{M}}(m, e)(q) = \sum_{n \in \mathbb{N}} q^{\frac{n}{2} v \cdot k_0} \prod_{i=1}^{r} I_{\Delta}(m_i - n b_i \cdot k_0, e_i + n a_i \cdot k_0) \tag{5.1}$$

of the sum that defines $I_{\mathbf{M}}$ on a ray $\rho = \mathbb{N} k_0$ for $k_0 \in \mathbb{Z}^r$, $k_0 \neq 0$. Consider the union R of the 3 rays in $\mathbb{R}^2$ as shown in Fig. 5.

If $\rho = \mathbb{N} k_0$ is a fixed ray, let $x = \mathbf{B}^T k_0 = (x_1, \dots, x_s)$ and $y = \mathbf{A}^T k_0 = (y_1, \dots, y_s)$.

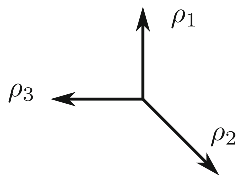


Fig. 5 The degree of the tetrahedron index

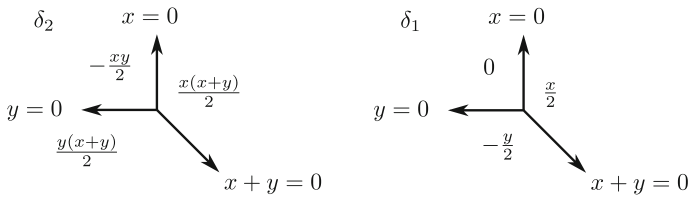


Fig. 6 Piece-wise quadratic and linear functions δ_2 and δ_1

Lemma 5.3 (a) *If $(-x_i, y_i) \notin R$ for some $i = 1, \dots, s$, then $I^{\rho}_{\mathbf{M}}(m, e)$ converges for all m, e.*

(b) *If $(-x_i, y_i) \in R$ for all $i = 1, \dots, s$. Then, there exists $Q \in \{1, \dots, s\} \to \{1, 2, 3\}$ such that $(-x_i, y_i) \in \rho_{Q(i)}$ for all $i = 1, \dots, s$. Then $I^{\rho}_{\mathbf{M}}$ does not converge if and only if all of the following inequalities hold:*

$$b_i \cdot k_0 = 0, \qquad a_i \cdot k_0 \geq 0, \qquad (-v) \cdot k_0 \leq 0 \quad \text{if} \quad Q(i) = 1, \tag{5.2a}$$

$$(a_i - b_i) \cdot k_0 = 0, \qquad (-b_i) \cdot k_0 \geq 0, \quad (-v + b_i) \cdot k_0 \leq 0 \quad \text{if} \quad Q(i) = 2, \tag{5.2b}$$

$$(-a_i) \cdot k_0 = 0, \quad (-a_i + b_i) \cdot k_0 \geq 0, \quad (-v + a_i) \cdot k_0 \leq 0 \quad \text{if} \quad Q(i) = 3. \tag{5.2c}$$

Proof (a) Without loss of generality, let us assume $m = e = 0$. In that case, the degree of the summand in Eq. (5.1) is given by

$$n^2 \sum_{i=1}^{s} \delta_2(-x_i, y_i) + n \sum_{i=1}^{s} \delta_1(-x_i, y_i) + \frac{n}{2} v \cdot k_0,$$

where δ_1 and δ_2 are piece-wise quadratic and linear functions as given in Fig. 6.

If $(-x_i, y_i) \notin R$ for some $i = 1, \dots, s$, it follows that the degree of the summand is a quadratic function of n with nonvanishing leading term, and thus $I^{\rho}_{\mathbf{M}}$ converges.

(b) The above computation shows that $I^{\rho}_{\mathbf{M}}(0, 0)$ diverges if and only if $\delta_2(-x_i, y_i) = 0$ for all $i = 1, \dots, s$ and in addition the coefficient of n is less than or equal to zero. The first condition is equivalent to $(-x_i, y_i) \in R$ for all i, and together with the second one, they are equivalent to the inequalities (5.2). □

Proof (of Theorem 2.4) Lemma 5.2 implies that $I_{\mathbf{M}}$ converges if and only if $I^{\rho}_{\mathbf{M}}$ converges for all rays ρ. This is true since the degree of the summand of $I_{\mathbf{M}}$ is a piece-wise quadratic polynomial. Lemma 5.3 gives necessary and sufficient conditions for the convergence of $I^{\rho}_{\mathbf{M}}$. It remains to match these conditions with the definition of an index structure on $\mathbf{M}$ using Lemma 5.1.

The above discussion implies that $I_{\mathbf{M}}$ is convergent if and only if for every $Q\colon \{1,\dots,s\} \to \{1,2,3\}$, there does not exist $k_0 \neq 0$ such that Eq. (5.2) holds. Assume for simplicity that $s=1$.

Case 1: If $Q(1)=1$, Inequality (5.2) and Lemma 5.1 imply that there exist $\alpha_1 > 0$ and γ_1 real such that $v = \alpha_1 a_1 + \gamma_1 b_1$. Define $\beta_1 = 1-\alpha_1-\gamma_1$.

Case 2: If $Q(1)=2$, Inequality (5.2) and Lemma 5.1 imply that there exist $\alpha'_1 > 0$ and γ'_1 real such that $v - b_1 = \alpha'_1(-b_1) + \gamma'_1(a_1-b_1)$. Letting $(\alpha_1,\beta_1,\gamma_1) = (\gamma'_1, \alpha'_1, -\gamma'_1-\alpha'_1+1)$, it follows that

$$v = \alpha_1 a_1 + \gamma_1 b_1, \qquad \beta_1 > 0.$$

Case 3: If $Q(1)=3$, Inequality (5.2) and Lemma 5.1 imply that there exist $\alpha'_1 > 0$ and γ'_1 real such that $v - a_1 = \alpha'_1(-a_1+b_1) + \gamma'_1(-a_1)$. Letting $(\alpha_1,\beta_1,\gamma_1) = (1-\alpha'_1-\gamma'_1, \gamma'_1, \alpha'_1)$, it follows that

$$v = \alpha_1 a_1 + \gamma_1 b_1, \qquad \gamma_1 > 0.$$

It follows that $\mathbf{M}$ admits an index structure.

The general case of s follows as above. Indeed for each $Q : \{1,\dots,s\} \to \{1,2,3\}$, assume $(-x_i, y_i) \in \rho_{Q(i)}$ for $i = 1,\dots,s$. Then $I^{\rho}_{\mathbf{M}}$ convergences if and only if there exists (α,β,γ) that satisfies Eq. (2.1) and inequalities (2.2). This completes the convergence proof of Theorem 2.4. q-holonomicity follows from the main theorem of Wilf-Zeilberger [27], using the fact that $I_{\mathbf{M}}(m,e)$ is a $2r$-dimensional sum of a proper q-hypergeometric summand. □

5.2 An independent proof of convergence for strict index structures

Theorem 2.4 implies that $I_{\mathbf{M}}$ converges when $\mathbf{M}$ admits a strict index structure. In this section, we give an independent proof of this fact without using the restriction of the summand of the index to a ray.

Proposition 5.4 *If* $\mathbf{M}$ *supports a strict index structure, then* $I_{\mathbf{M}}(m,e)(q) \in \mathbb{Z}((q^{1/2}))$ *is convergent for all* $m, e \in \mathbb{Z}^s$.

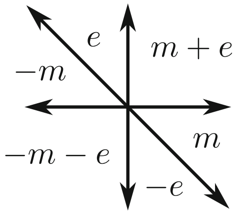
The proof of proposition 5.4 requires some lemmas.

Lemma 5.5 *Fix positive real numbers* $\alpha, \beta > 0$ *with* $\alpha+\beta < 1/2$ *and let* $\gamma = \min\{\alpha, \beta, 1/2-\alpha-\beta\}$. *Then for all integers* m, e *we have*

$$\delta(I_{\Delta}(m,e) q^{-\beta m + \alpha e}) \geq \gamma \max\{|m|, |e|, |m+e|\}\,.$$

Proof (of Lemma 5.5) Let $L_+(m,e) = \max\{|m|,|e|,|m+e|\}$. $L_+(m,e)$ is a piecewise linear function as given in Fig. 7.

Fig. 7 A piece-wise linear function L_+



With the notation of Lemma 3.6, we need to show that

$$\delta(m, e) - \beta m + \alpha e \geq \gamma L_+(m, e). \tag{5.3}$$

First, consider the three rays of $\delta(m, e)$:

Ray	Left-hand side of (5.3)	Right-hand side of (5.3)
$m = 0, e \geq 0$	αe	γe
$e = 0, m \leq 0$	$-\beta m$	$-\gamma m$
$m = -e \geq 0$	$m(1/2 - \alpha - \beta)$	γm

This proves inequality (5.3) in the three rays and shows that the choice of γ is optimal. Now, in the interior of each of the 6 cones of linearity of L_+, $\delta(m, e) - \beta m + \alpha e$ is given by a quadratic polynomial of m, e. The degree 2 (resp. 1) part of this polynomial is always greater than or equal to $1/2 L_+(m, e)$ (resp. $(1/2 - \gamma)L_+(m, e)$) by a case computation. For example, in the cone $m \geq 0, e \leq 0, e + m \geq 0$ with rays $\mathbb{R}_+(1, 0)$ and $\mathbb{R}_+(1, -1)$, we have $\delta(m, e) = m(m + e)/2 + m/2$ and $L_+(m, e) = m$ and

$$\begin{aligned}
\delta(m, e) - \beta m + \alpha e &= \frac{m(e + m)}{2} + \frac{m}{2} - \beta m + \alpha e \\
&\geq \frac{m}{2} + \frac{m}{2} - \beta m + \alpha e \\
&= (1 - \beta - \alpha)m + \alpha(m + e) \\
&\geq (1 - \beta - \alpha)m \geq (1 - \beta - \alpha)L_+(m, e).
\end{aligned}$$

The other cases are similar. □

The next lemma is well known [30].

Lemma 5.6 *Consider the convex polytope P in $\mathbb{R}^r$ defined by*

$$P = \{x \in \mathbb{R}^r \,|v_i \cdot x \leq c_i \quad i = 1, \ldots s\},$$

where $v_i \in \mathbb{R}^r$ and $c_i \in \mathbb{R}$ for $i = 1, \ldots, s$. Then P is compact if and only if the linear span of the set $\{v_i | i = 1, \ldots, s\}$ is $\mathbb{R}^r$ and 0 is a $\mathbb{R}_{\geq 0}$-linear combination of elements of $\{v_i | i = 1, \ldots, s\}$.

Proof (of Proposition 5.4) Let a_i and b_i for $i = 1, \dots, s$ denote the columns of $(\mathbf{A}|\mathbf{B})$. If $\mathbf{M}$ admits a strict index structure, then there exist $\alpha_i, \gamma_i > 0$ that satisfy $a_i + \gamma_i < 1$ for all i such that

$$\sum_{i=1}^{s} \alpha_i a_i + \gamma_i \beta_i = \nu \,.$$

It follows that

$$I_{\mathbf{M}}(m, e)(q) = \sum_{k \in \mathbb{Z}^r} \prod_{i=1}^{s} I(m_i - b_i \cdot k, e_i + a_i \cdot k)(q) q^{\frac{\beta_i}{2} b_i \cdot k + \frac{\alpha_i}{2} a_i \cdot k} \,.$$

Applying Lemma 5.5, it follows that for every $k \in \mathbb{Z}^d$, the degree of the summand is bounded below by

$$\sum_{i=1}^{s} (\beta_i m_i - \alpha_i e_i) + \gamma' \sum_{i=1}^{s} (|-m_i + b_i \cdot k| + |e_i + a_i \cdot k|) \,.$$

Now, Lemma 5.6 and admissibility imply that for fixed N_0, there are finitely many $k \in \mathbb{Z}^d$ such that the above degree is less than N_0. Proposition 5.4 follows. □

6 Invariance of the 3D index under 2 ↔ 3 moves and 2 ↔ 0 moves

6.1 Invariance under the 3 → 2 move

Consider two ideal triangulations $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ with N and $N+1$ tetrahedra, respectively, related by a 2–3 move as shown in Fig. 8.

The above figure matches the conventions of [5, Sec. 3.6]. For a variable, matrix, or vector f associated to $\mathcal{T}$, we will denote by $\widetilde{f}$ the corresponding variable, matrix,

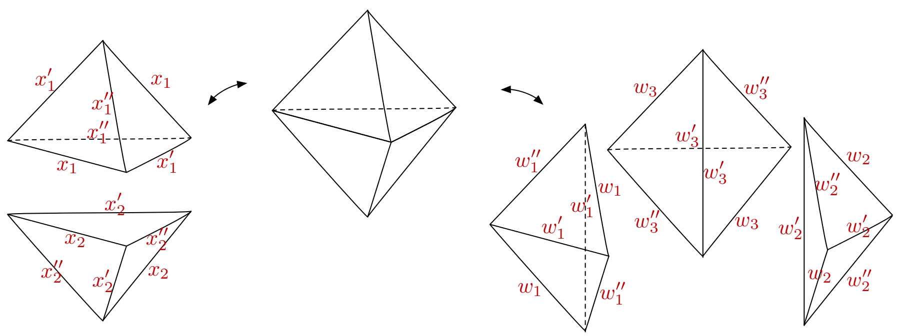


Fig. 8 A 2–3 move

or vector associated to $\widetilde{\mathcal{T}}$. Let us use variables (Z, Z', Z'') and $(\widetilde{Z}, \widetilde{Z}', \widetilde{Z}'')$ to denote the angles of $\mathcal{T}$ and $\widetilde{\mathcal{T}}$, respectively, where

$$Z := (X_1, X_2, Z_3, ..., Z_N), \qquad \tilde{Z} := (W_1, W_2, W_3, Z_3, ..., Z_N). \tag{6.1}$$

We fix a quad type assigning these variables to $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ as in Fig. 3. When calculating the Neumann–Zagier matrices, we will assume that we keep the edge equation which comes from the internal edge of the 2–3 bipyramid.

There are nine linear relations among the shapes of the tetrahedra involved in the move—three come from adding dihedral angles on the equatorial edges of the bipyramid:

$$W_1' = X_1 + X_2, \qquad W_2' = X_1' + X_2'', \qquad W_3' = X_1'' + X_2', \tag{6.2}$$

and six from the longitudinal edges:

$$\begin{aligned} X_1 &= W_2 + W_3'', \qquad & X_1' &= W_3 + W_1'', \qquad & X_1'' &= W_1 + W_2'', \\ X_2 &= W_2'' + W_3, \qquad & X_2' &= W_1'' + W_2, \qquad & X_2'' &= W_3'' + W_1. \end{aligned} \tag{6.3}$$

Moreover, due to the central edge of the bipyramid, there is an extra gluing constraint in $\widetilde{\mathcal{T}}$:

$$W_1' + W_2' + W_3' = 2\pi i. \tag{6.4}$$

Let $\mathrm{GA}(\mathcal{T})$ and $\mathrm{A}(\mathcal{T})$ denote, respectively, the sets of generalized and strict angle structures of $\mathcal{T}$.

Lemma 6.1 *Consider the map*

$$\mu_{3\to 2}\colon \mathrm{GA}(\widetilde{\mathcal{T}}) \to \mathrm{GA}(\mathcal{T}), \qquad \mu_{3\to 2}(\widetilde{Z}, \widetilde{Z}', \widetilde{Z}'') = (Z, Z', Z'') \tag{6.5}$$

defined by Eq. (6.3). *It induces a map*

$$\mu_{3\to 2}\colon \mathrm{A}(\widetilde{\mathcal{T}}) \to \mathrm{A}(\mathcal{T}).$$

Proof To check that $\mu_{3\to 2}$ is well defined, we need to show that $X_i + X_i' + X_i'' = 1$ is satisfied for $i = 1, 2$, assuming that Eq. (6.4) holds and $W_i + W_i' + W_i'' = 1$ for $i = 1, 2, 3$. This is easy to check. If $(\widetilde{Z}, \widetilde{Z}', \widetilde{Z}'') \in \mathbb{R}_+^{3(N+1)}$ (where $\mathbb{R}_+$ is the set of positive real numbers), it is evident from the definition that $(Z, Z', Z'') \in \mathbb{R}_+^{3N}$. In other words, $\mu_{3\to 2}$ sends strict angle structures on $\widetilde{\mathcal{T}}$ to those on $\mathcal{T}$. □

This proves the first part of Proposition 2.13. To prove the remaining part, we study how the gluing equation matrices of $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ are related. Let $(\bar{\mathbf{A}}|\bar{\mathbf{B}}|\bar{\mathbf{C}})$ denote the matrix of exponents of the gluing equations of $\mathcal{T}$. We will use column notation and write

$$\bar{\mathbf{A}} = (\bar{a}_1,\ \bar{a}_2,\ \bar{a}_i), \quad \bar{\mathbf{B}} = (\bar{b}_1,\ \bar{b}_2,\ \bar{b}_i), \quad \bar{\mathbf{C}} = (\bar{c}_1,\ \bar{c}_2,\ \bar{c}_i),$$

where $\bar{a}_i$ signifies $(\bar{a}_3, \bar{a}_4, ..., \bar{a}_N)$ and similarly for $\bar{b}_i$ and $\bar{c}_i$. Eliminating the Z' variables, we obtain

$$\mathbf{A} = \bar{\mathbf{A}} - \bar{\mathbf{B}}, \qquad \mathbf{B} = \bar{\mathbf{C}} - \bar{\mathbf{B}}\,.$$

In other words,

$$\begin{aligned}(a_1, a_2, a_i) &= (\bar{a}_1 - \bar{b}_1, \bar{a}_2 - \bar{b}_2, \bar{a}_i - \bar{b}_i),\\ (b_1, b_2, b_i) &= (\bar{c}_1 - \bar{b}_1, \bar{c}_2 - \bar{b}_2, \bar{c}_i - \bar{b}_i)\,.\end{aligned} \tag{6.6}$$

To compute the corresponding matrices of $\widetilde{\mathcal{T}}$, use

$$\begin{aligned}2 &= \bar{a}_1 X_1 + \bar{a}_2 X_2 + \bar{a}_i Z_i + \bar{b}_1 X_1' + \bar{b}_2 X_2' + \bar{b}_i Z_i' + \bar{c}_1 X_1'' + \bar{c}_2 X_2'' + \bar{c}_i Z_i''\\ &= \bar{a}_1(W_2 + W_3'') + \bar{a}_2(W_2'' + W_3) + \bar{a}_i Z_i\\ &\quad + \bar{b}_1(W_3 + W_1'') + \bar{b}_2(W_1'' + W_2) + \bar{b}_i Z_i'\\ &\quad + \bar{c}_1(W_1 + W_2'') + \bar{c}_2(W_3'' + W_1) + \bar{c}_i Z_i''\,.\end{aligned}$$

Collecting the coefficients of $\tilde{Z}, \tilde{Z}', \tilde{Z}''$, it follows that the matrix of exponents of the gluing equations of $\widetilde{\mathcal{T}}$ is given by

$$\tilde{\bar{\mathbf{A}}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \bar{c}_1 + \bar{c}_2 & \bar{a}_1 + \bar{b}_2 & \bar{a}_2 + \bar{b}_1 & \bar{a}_i \end{pmatrix}, \qquad \tilde{\bar{\mathbf{B}}} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & \bar{b}_i \end{pmatrix},$$
$$\tilde{\bar{\mathbf{C}}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \bar{b}_1 + \bar{b}_2 & \bar{a}_2 + \bar{c}_1 & \bar{a}_1 + \bar{c}_2 & \bar{c}_i \end{pmatrix}.$$

Using a row operation via $P = \begin{pmatrix} 1 & 0 \\ \bar{b}_1 + \bar{b}_2 & \mathbf{I} \end{pmatrix}$, it follows that

$$P\tilde{\bar{\mathbf{A}}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \bar{c}_1 + \bar{c}_2 & \bar{a}_1 + \bar{b}_2 & \bar{a}_2 + \bar{b}_1 & \bar{a}_i \end{pmatrix}, \qquad P\tilde{\bar{\mathbf{B}}} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ \bar{b}_1 + \bar{b}_2 & \bar{b}_1 + \bar{b}_2 & \bar{b}_1 + \bar{b}_2 & \bar{b}_i \end{pmatrix},$$
$$P\tilde{\bar{\mathbf{C}}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \bar{b}_1 + \bar{b}_2 & \bar{a}_2 + \bar{c}_1 & \bar{a}_1 + \bar{c}_2 & \bar{c}_i \end{pmatrix}.$$

Since $\widetilde{\mathbf{A}} = \tilde{\bar{\mathbf{A}}} - \tilde{\bar{\mathbf{B}}}$ and $\widetilde{\mathbf{B}} = \tilde{\bar{\mathbf{C}}} - \tilde{\bar{\mathbf{B}}}$, the above inequalities combined with Eq. (6.6) imply that

$$P\widetilde{\mathbf{A}} = \begin{pmatrix} -1 & -1 & -1 & 0 \\ b_1 + b_2 & a_1 & a_2 & a_i \end{pmatrix}, \qquad P\widetilde{\mathbf{B}} = \begin{pmatrix} -1 & -1 & -1 & 0 \\ 0 & a_2 + b_1 & a_1 + b_2 & b_i \end{pmatrix}. \tag{6.7}$$

Since the 3D index is invariant under row operations (see Remark 2.8), Eq. (6.7) and the pentagon identity (3.6) conclude that $I_{\widetilde{\mathcal{T}}} = I_{\mathcal{T}}$. □

6.2 Invariance under the $2 \to 3$ move

In this section, we will define what is a special angle structure on $\mathcal{T}$ and show the partial invariance of the 3D index under a $2 \to 3$ move. We will use the same notation as in Sect. 6.1. To define a map $(Z, Z', Z'') \mapsto (\widetilde{Z}, \widetilde{Z}', \widetilde{Z}'')$, we need to solve for W_i, W_i', W_i'' for $i = 1, 2, 3$ in terms of X_i, X_i', X_i'' for $i = 1, 2$ using Eqs. (6.2) and (6.3). The answer involves one free variable (say, W_1) and it is given by

$$(W_1, W_2, W_3) = (W_1, W_1 + X_1 + X_2 + X_2'' - 1, W_1 + X_1 + X_2 + X_1' - 1), \tag{6.8a}$$

$$(W_1', W_2', W_3') = (X_1 + X_2, X_1' - X_2 - X_2'' + 1, -X_1 - X_1' + X_2' + 1), \tag{6.8b}$$

$$(W_1'', W_2'', W_3'') = (-W_1 + 1 - X_1 - X_2, -W_1 - X_1 - X_1' + 1, -W_1 - X_2 - X_2' + 1). \tag{6.8c}$$

If (Z, Z', Z'') is a strict angle structure on $\mathcal{T}$, then $(\widetilde{Z}, \widetilde{Z}', \widetilde{Z}'')$ is a strict angle structure if and only if Eq. (6.8) has a strictly positive solution. It is easy to see that this is equivalent to the following condition:

$$X_1 + X_2 < 1, \qquad X_1'' + X_2' < 1, \qquad X_1' + X_2'' < 1\,. \tag{6.9}$$

These conditions are precisely equivalent to the conditions $W_1', W_2', W_3' < 1$, as follows in Eq. (6.2). In other words, a special strict angle structure is an angle structure such that all angles of the bipyramid are less than 1.

Definition 6.2 We will say that (Z, Z', Z'') is a *special strict angle structure* on $\mathcal{T}$ if the inequality (6.9) is satisfied.

Let $\mathrm{A}^{\mathrm{sp}}(\mathcal{T})$ denote the set of special strict angle structures on $\mathcal{T}$. Then, we have a map (more precisely, a section of $\mu_{3\to 2}$)

$$\mu_{2\to 3} \colon \mathrm{A}^{\mathrm{sp}}(\mathcal{T}) \to A(\widetilde{\mathcal{T}}), \qquad \mu_{2\to 3}(Z, Z', Z'') = (\widetilde{Z}, \widetilde{Z}', \widetilde{Z}'')\,.$$

The conclusion is that if $\mathcal{T}$ admits a special strict angle structure, then so does $\widetilde{\mathcal{T}}$. In that case, $I_{\mathcal{T}}$ and $I_{\widetilde{\mathcal{T}}}$ both exist. An application of the pentagon identity as in Sect. 6.1 implies that $I_{\mathcal{T}} = I_{\widetilde{\mathcal{T}}}$. □

6.3 An ideal triangulation of `m136`

Let $\mathcal{T}$ denote the ideal triangulation [11, Ex. 7.7] of the 1-cusped census manifold `m136` using 7 tetrahedra. Its gluing equation matrices around the edges are given by

$$\bar{\mathbf{A}} = \begin{pmatrix} 1&0&0&0&1&1&1\\ 0&0&0&1&0&1&0\\ 0&0&1&0&0&0&0\\ 0&1&0&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&1&1&1&0&0&1\\ 0&0&0&0&1&0&0 \end{pmatrix}, \quad \bar{\mathbf{B}} = \begin{pmatrix} 0&0&0&1&0&0&1\\ 1&0&0&0&1&0&0\\ 0&1&1&1&1&1&0\\ 1&0&0&0&0&0&1\\ 0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&1&0&0&1&0 \end{pmatrix}, \quad \bar{\mathbf{C}} = \begin{pmatrix} 1&0&0&1&0&1&1\\ 0&0&0&0&1&1&0\\ 1&2&0&1&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&1&0&1&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix}.$$

A generalized angle structure is a solution to Eq. (2.7). In our example, the set of generalized angle structures $\mathrm{GA}(\mathcal{T})$ is an affine 8-dimensional subspace of $\mathbb{R}^{21}$ and the intersection $\mathrm{SA}(\mathcal{T}) = \mathrm{GA}(\mathcal{T}) \cap [0, \infty)^{21}$ is the polytope of semi-angle structures. `Regina` [2] gives that $\mathrm{SA}(\mathcal{T})$ is the convex hull of the following set of 11 points $(\alpha_1, \beta_1, \gamma_1, \ldots, \alpha_7, \beta_7, \gamma_7)$ in $\mathbb{R}^{21}$:

$$\left(\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc|ccc}
0&0&1&1&0&0&0&0&1&1&0&0&0&0&1&0&1&0&0&0&1\\
0&1&0&0&1&0&0&0&1&1&0&0&1&0&0&0&1&0&1&0&0\\
1&0&0&1&0&0&0&1&0&1&0&0&0&0&1&0&1&0&0&1&0\\
1&0&0&1&0&0&0&0&1&1&0&0&0&1&0&0&1&0&0&0&1\\
1/2&1/2&0&1&0&0&0&1/2&1/2&1/2&1/2&0&0&0&1&0&1&0&1/2&0&1/2\\
1/2&1/2&0&1&0&0&0&1/2&1/2&1/2&0&1/2&0&0&1&0&1&0&1/2&0&1/2\\
1/2&1/2&0&1&0&0&1/2&0&1/2&1/2&1/2&0&0&0&1&0&1&0&0&0&1\\
1/2&1/2&0&1&0&0&1/2&0&1/2&1/2&0&1/2&0&0&1&0&1&0&0&0&1\\
1/2&1/2&0&1/2&0&1/2&0&0&1&1&0&0&1/2&0&1/2&0&1&0&1/2&0&1/2\\
1/2&1/2&0&1/2&1/2&0&1/2&0&1/2&1&0&0&1/2&0&1/2&0&1&0&0&1/2&1/2\\
2/3&1/3&0&2/3&0&1/3&1/3&0&2/3&1&0&0&1/3&0&2/3&0&1&0&0&1/3&2/3
\end{array}\right).$$

A computation shows that if $(\alpha, \beta, \gamma) \in \mathrm{SA}(\mathcal{T})$, then $(a_6, b_6, c_6) = (t, 1, -t)$ for some $t \in \mathbb{R}$ which explains why $\mathcal{T}$ has no strict angle structure. On the other hand, Hodgson et al. [11, Example 7.7] mention that $\mathcal{T}$ has a solution

$$(z_1, \ldots, z_6) = \left(2i, -1 + 2i, \frac{3}{5} + \frac{1}{5}i, -1, \frac{1}{5} + \frac{2}{5}i, 2, \frac{1}{2} + \frac{1}{2}i\right)$$

of the gluing equations which recover the complete hyperbolic structure on `m136`.

6.4 An ideal triangulation of `m064`

There is an explicit triangulation of `m064` that uses 7 ideal tetrahedra, communicated to us by Henry Segerman. Its gluing equation matrices are given by

$$\bar{\mathbf{A}} = \begin{pmatrix} 2\,0\,1\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,1\,0\,0\,0 \\ 0\,2\,1\,1\,1\,1\,0 \\ 0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,1 \\ 0\,0\,0\,0\,1\,1\,1 \end{pmatrix}, \quad \bar{\mathbf{B}} = \begin{pmatrix} 0\,2\,0\,0\,0\,0\,1 \\ 2\,0\,0\,1\,0\,0\,0 \\ 0\,0\,1\,0\,0\,0\,1 \\ 0\,0\,0\,0\,1\,1\,0 \\ 0\,0\,0\,0\,1\,1\,0 \\ 0\,0\,1\,0\,0\,0\,0 \\ 0\,0\,0\,1\,0\,0\,0 \end{pmatrix}, \quad \bar{\mathbf{C}} = \begin{pmatrix} 1\,0\,0\,1\,1\,1\,0 \\ 0\,0\,1\,0\,0\,0\,0 \\ 1\,0\,0\,0\,0\,0\,0 \\ 0\,1\,0\,0\,0\,0\,1 \\ 0\,1\,0\,0\,1\,1\,0 \\ 0\,0\,1\,1\,0\,0\,1 \\ 0\,0\,0\,0\,0\,0\,0 \end{pmatrix}$$

This triangulation has no semi-angle structure, and its gluing equations have the following numerical shape solution:

$$(1.60 + 0.34i, 0.74 + 0.40i, 0.86 - 0.33i, 1.68 + 0.39i, 0.51 + 0.54i, 0.51 + 0.54i, -0.61 + 1.25i)$$

which gives rise to the discrete faithful representation of `m064`. An explicit computation shows that this triangulation admits an index structure.

6.5 An ideal triangulation with no index structure

Consider an ideal triangulation $\mathcal{T}$ which contains an edge e and a tetrahedron Δ_1 that goes around e five times with shapes Z, Z', Z', Z'', and Z''. Suppose that no other tetrahedron touches e. Then the equation for a generalized angle structure around e reads

$$\alpha + 2\beta + 2\gamma = 2, \qquad \alpha + \beta + \gamma = 1.$$

This forces $\alpha = 0$, so no generalized angle structure has $\alpha > 0$. Note that the corresponding gluing equations around the edge e read

$$z(z')^2(z'')^2 = 1, \qquad zz'z'' = -1, \qquad z' = (1-z)^{-1}$$

which forces $z = 1$. Thus the gluing equations have no nondegenerate solution, i.e., no solution with shapes in $\mathbb{C} \setminus \{0, 1\}$.

More complicated examples can be arranged using special configurations of two or more edges and tetrahedra. In all examples that we could generate with no index structure, the triangulation is degenerate.

Of course, the argument of a shape solution to the gluing equations is a generalized angle structure. The latter, however, need not be an index structure if some of the shapes are real, or have negative imaginary part; see for instance the triangulation of `m064` in Sect. 6.4.

6.6 Invariance under the 2 $\leftrightarrow$ 0 move

The next lemma implies the invariance of the index of an ideal triangulation under a 2 $\leftrightarrow$ 0 move. Such a move is also known as a pillowcase move, described in detail in [[9]Sec.6].

Lemma 6.3 *For integers m, e, c, we have*

$$\sum_e I_\Delta(m,e)I_\Delta(m,e+c)q^e = \delta_{c,0}\,.$$

Proof Equations (6.14) and (6.15) imply that

$$\sum_e I_\Delta(m,e)x^e = \frac{(q^{-\frac{m}{2}+1}x^{-1})_\infty}{(q^{-\frac{m}{2}}x)_\infty}\,.$$

Since

$$\frac{(q^{-\frac{m}{2}+1}x^{-1})_\infty}{(q^{-\frac{m}{2}}x)_\infty}\cdot\frac{(q^{-\frac{m}{2}+1}(qx^{-1})^{-1})_\infty}{(q^{-\frac{m}{2}}(qx^{-1}))_\infty} = 1\,,$$

it follows that

$$\sum_{e,e'} I_\Delta(m,e)x^e I_\Delta(m,e')q^{e'}x^{-e'} = 1\,.$$

Therefore,

$$\sum_{e,e':e-e'=c} I_\Delta(m,e)I_\Delta(m,e')q^{e'} = \delta_{c,0}\,.$$

This implies that

$$\sum_{e'} I_\Delta(m,e'+c)I_\Delta(m,e')q^{e'} = \delta_{c,0}\,.$$

The result follows. □

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Appendix A: I_Δ satisfies the pentagon identity

There are several proofs of the key pentagon identity of the tetrahedron index I_Δ. The proofs may use an integral representation of the quantum dilogarithm, or q-holonomic recursion relations, or algebraic identities of generating series of q-series of Nahm type [10].

A generating series proof of the pentagon identity

In this section, we will prove that I_Δ satisfies the pentagon identity using generating series. We will abbreviate the Pochhammer symbol

$$(x;q)_\infty = \prod_{n=0}^{\infty}(1-xq^n)$$

by $(x)_\infty = (x;q)_\infty$. The proof

- starts from an associativity identity

$$\frac{(z_1z_2)_\infty}{(z_1)_\infty(z_2)_\infty}\cdot\frac{(x_1z_1^{-1}q)_\infty(x_2z_2^{-1}q)_\infty}{(x_1x_2z_1^{-1}z_2^{-1}q)_\infty} = \frac{(x_1z_1^{-1}q)_\infty}{(z_1)_\infty}\cdot\frac{(x_2z_2^{-1}q)_\infty}{(z_2)_\infty}\cdot\frac{(z_1z_2)_\infty}{(x_1x_2z_1^{-1}z_2^{-1}q)_\infty}$$

  that uses four additional variables $\{x_1, x_2, z_1, z_2\}$ in addition to the other four variables $\{m_1, m_2, e_1, e_2\}$,
- extracts coefficients with respect to (z_1, z_2), and
- specializes $(x_1, x_2) = (q^{-m_1}, q^{-m_2})$. This last part is not algebraic and required to show convergence. The latter follows from Corollary 2.7.

Let us now give the details. Consider

$$F_e(x) = \sum_n (-1)^n \frac{q^{\frac{1}{2}n(n+1)}x^n}{(q)_n(q)_{n+e}} \in \mathbb{Z}[[x,q]]\,. \tag{6.10}$$

Here and below, summation is over the set of integers, with the understanding that $1/(q)_n = 0$ for $n < 0$.

We will show that

$$q^{e_1e_2}F_{e_1}(q^{e_2}x_1)F_{e_2}(q^{e_1}x_2) = \sum_{e_3}(x_1x_2q)^{e_3}F_{e_1+e_3}(x_1)F_{e_2+e_3}(x_2)F_{e_3}(x_1x_2) \tag{6.11}$$

in the ring $\mathbb{Z}((x_1, x_2, q))$. Since

$$F_e(q^{-m}) = q^{\frac{em}{2}}I_\Delta(m,e)\,,$$

the substitution $(x_1, x_2) = (q^{-m_1}, q^{-m_2})$ (which converges by Corollary 2.7) implies the pentagon identity of Eq. 3.6.

Lemma 6.4 *For $|q| < 1$, we have*

$$\frac{1}{(x)_\infty} = \sum_n \frac{x^n}{(q)_n}, \qquad |x| < 1,$$

$$(xq)_\infty = \sum_n (-1)^n \frac{q^{\frac{1}{2}n(n+1)} x^n}{(q)_n},$$

$$\frac{(xy)_\infty}{(x)_\infty} = \sum_n \frac{(y)_n x^n}{(q)_n}, \qquad |x| < 1,$$

$$\frac{(xy)_\infty}{(x)_\infty (y)_\infty} = \sum_{r,s} \frac{q^{rs} x^r y^s}{(q)_r (q)_s}, \qquad |x| < 1, \ |y| < 1,$$

$$\frac{(xq)_\infty (yq)_\infty}{(xyq)_\infty} = \sum_{r,s} (-1)^{r+s} \frac{q^{\frac{1}{2}(r-s)^2 + \frac{1}{2}(r+s)} x^r y^s}{(q)_r (q)_s}, \qquad |xyq| < 1.$$

Proof The first three identities are well known and appear in [29, Prop. 2]. The last two follow from the first three:

$$\begin{aligned}
\sum_{r,s} \frac{q^{rs} x^r y^s}{(q)_r (q)_s} &= \sum_r \frac{x^r}{(q)_r} \sum_s \frac{(q^r y)^s}{(q)_s} = \sum_r \frac{x^r}{(q)_r} \frac{1}{(q^r y)_\infty} \\
&= \frac{1}{(y)_\infty} \sum_r \frac{(y)_r x^r}{(q)_r} = \frac{(xy)_\infty}{(x)_\infty (y)_\infty},
\end{aligned}$$

$$\begin{aligned}
\sum_{r,s} (-1)^{r+s} \frac{q^{\frac{1}{2}(r-s)^2 + \frac{1}{2}(r+s)} x^r y^s}{(q)_r (q)_s} &= \sum_r \frac{(-1)^r q^{\frac{1}{2}r^2 + \frac{1}{2}r} x^r}{(q)_r} \sum_s \frac{(-1)^r q^{\frac{1}{2}s^2 + \frac{1}{2}s} (q^{-r} y)^s}{(q)_s} \\
&= \sum_r \frac{(-1)^r q^{\frac{1}{2}r^2 + \frac{1}{2}r} x^r}{(q)_r} (q^{1-r} y)_\infty \\
&= (yq)_\infty \sum_r \frac{(y^{-1})_r (xyq)^r}{(q)_r} = \frac{(xq)_\infty (yq)_\infty}{(xyq)_\infty}.
\end{aligned}$$

□

Remark 6.5 The identities of Lemma 6.4 also hold in the ring $\mathbb{Z}((x, y, q))$.

Observe that $F_e(x)$ is an analytic function of (x, q) when $|q| < 1$ and $x \in \mathbb{C}$. With $|q| < 1$ and $|y| < 1$, Lemma 6.4 gives

$$\begin{aligned}
\sum_e F_e(x) y^e &= \sum_n \frac{(-1)^n q^{\frac{1}{2}n^2 + \frac{1}{2}n} x^n}{(q)_n} \sum_e \frac{y^e}{(q)_{n+e}} \\
&= \frac{1}{(y)_\infty} \sum_n \frac{(-1)^n q^{\frac{1}{2}n^2 + \frac{1}{2}n} (xy^{-1})^n}{(q)_n} = \frac{(xy^{-1} q)_\infty}{(y)_\infty}.
\end{aligned}$$

Thus, the generating function of the left-hand side of Eq. (6.11) is

$$\begin{aligned}
&\sum_{e_1,e_2} q^{e_1e_2} F_{e_1}(q^{e_2}x_1)F_{e_2}(q^{e_1}x_2)z_1^{e_1}z_2^{e_2} \\
&\quad = \sum_{n_1,n_2} \frac{(-1)^{n_1+n_2} q^{\frac{1}{2}n_1^2+\frac{1}{2}n_2^2+\frac{1}{2}n_1+\frac{1}{2}n_2} x_1^{n_1}x_2^{n_2}}{(q)_{n_1}(q)_{n_2}} \sum_{e_1,e_2} \frac{q^{e_1e_2+n_2e_1+n_1e_2} z_1^{e_1}z_2^{e_2}}{(q)_{n_1+e_1}(q)_{n_2+e_2}} \\
&\quad = \frac{(z_1z_2)_\infty}{(z_1)_\infty(z_2)_\infty} \sum_{n_1,n_2} \frac{(-1)^{n_1+n_2} q^{\frac{1}{2}(n_1-n_2)^2+\frac{1}{2}n_1+\frac{1}{2}n_2} (x_1z_1^{-1})^{n_1}(x_2z_2^{-1})^{n_2}}{(q)_{n_1}(q)_{n_2}} \\
&\quad = \frac{(z_1z_2)_\infty}{(z_1)_\infty(z_2)_\infty} \cdot \frac{(x_1z_1^{-1}q)_\infty(x_2z_2^{-1}q)_\infty}{(x_1x_2z_1^{-1}z_2^{-1}q)_\infty} .
\end{aligned}$$

Likewise, the generating function of the right-hand side of Eq. (6.11) is the same

$$\begin{aligned}
&\sum_{e_1,e_2} \left(\sum_{e_3} (x_1x_2q)^{e_3} F_{e_1+e_3}(x_1)F_{e_2+e_3}(x_2)F_{e_3}(x_1x_2) \right) z_1^{e_1}z_2^{e_2} \\
&\quad = \left(\sum_{e_1} F_{e_1}(x_1)z_1^{e_1} \right)\left(\sum_{e_2} F_{e_2}(x_2)z_2^{e_2} \right)\left(\sum_{e_3} F_{e_3}(x_1x_2)(x_1x_2z_1^{-1}z_2^{-1}q)^{e_3} \right) \\
&\quad = \frac{(x_1z_1^{-1}q)_\infty}{(z_1)_\infty} \cdot \frac{(x_2z_2^{-1}q)_\infty}{(z_2)_\infty} \cdot \frac{(z_1z_2)_\infty}{(x_1x_2z_1^{-1}z_2^{-1}q)_\infty} .
\end{aligned}$$

The above identities for each side of Eq. (6.11) hold when $|q| < 1$, $|z_1| < 1$, $|z_2| < 1$, and $|x_1x_2z_1^{-1}z_2^{-1}q| < 1$. Remark 6.5 implies that they also hold in the ring $\mathbb{Z}((x_1, x_2, z_1, z_2, q))$. Extracting the coefficient of $z_1^{e_1}z_2^{e_2}$ from the above concludes the proof of Eq. (6.11). □

A second proof of the pentagon identity

In this section, we give a second proof of the pentagon identity using

$$\begin{aligned}
\frac{1}{(q)_m(q)_n} &= \sum_{\substack{r,s,t \\ r+s=m \\ s+t=n}} \frac{q^{rt}}{(q)_r(q)_s(q)_t} , \\
\frac{q^{mn}}{(q)_m(q)_n} &= \sum_{\substack{r,s,t \\ r+s=m \\ s+t=n}} \frac{(-1)^s q^{\frac{1}{2}s^2-\frac{1}{2}s}}{(q)_r(q)_s(q)_t} = \sum_s \frac{(-1)^s q^{\frac{1}{2}s^2-\frac{1}{2}s}}{(q)_{m-s}(q)_{n-s}(q)_s} . \qquad (6.12)
\end{aligned}$$

The first identity is well known [29, Eqn. (13)], and the second follows from the first by replacing q with q^{-1} and multiplying both sides by $(-1)^{m+n}q^{-\frac{1}{2}(m-n)^2-\frac{1}{2}(m+n)}$.

Using these equations, we will show here that

$$q^{e_1e_2}\frac{(-1)^{n_1+n_2}q^{\frac{1}{2}n_1^2+\frac{1}{2}n_2^2+\frac{1}{2}n_1+\frac{1}{2}n_2+e_2n_1+e_1n_2}}{(q)_{n_1}(q)_{n_2}(q)_{n_1+e_1}(q)_{n_2+e_2}}$$
$$=\sum_{\substack{r_1,r_2,r_3,e_3\\ r_1+r_3+e_3=n_1\\ r_2+r_3+e_3=n_2}}\frac{(-1)^{r_1+r_2+r_3}q^{\frac{1}{2}r_1^2+\frac{1}{2}r_2^2+\frac{1}{2}r_3^2+\frac{1}{2}r_1+\frac{1}{2}r_2+\frac{1}{2}r_3+e_3}}{(q)_{r_1}(q)_{r_1+e_1+e_3}(q)_{r_2}(q)_{r_2+e_2+e_3}(q)_{r_3}(q)_{r_3+e_3}}. \tag{6.13}$$

The sum on the right actually only has a finite number of nonzero terms, so there is no issue with convergence. If we multiply both sides by $x_1^{n_1}x_2^{n_2}$ and sum over all n_1 and n_2, then we again find

$$q^{e_1e_2}F_{e_1}(q^{e_2}x_1)F_{e_2}(q^{e_1}x_2)=\sum_{e_3}(x_1x_2q)^{e_3}F_{e_1+e_3}(x_1)F_{e_2+e_3}(x_2)F_{e_3}(x_1x_2)\,.$$

To prove (6.13), we use Eq. (6.12) which gives

$$\frac{q^{(n_1+e_1)(n_2+e_2)}}{(q)_{n_1}(q)_{n_2}(q)_{n_1+e_1}(q)_{n_2+e_2}}$$
$$=\sum_{\substack{r_1,r_2,r_3,e_3\\ r_1+e_3=n_1\\ r_2+e_3=n_2}}\frac{(-1)^{r_3}q^{\frac{1}{2}r_3^2-\frac{1}{2}r_3+r_1r_2}}{(q)_{r_1}(q)_{r_2}(q)_{n_1+e_1-r_3}(q)_{n_2+e_2-r_3}(q)_{r_3}(q)_{e_3}}\,.$$

Replacing e_3 by e_3+r_3 in this sum, we get

$$\frac{q^{(n_1+e_1)(n_2+e_2)}}{(q)_{n_1}(q)_{n_2}(q)_{n_1+e_1}(q)_{n_2+e_2}}$$
$$=\sum_{\substack{r_1,r_2,r_3,e_3\\ r_1+r_3+e_3=n_1\\ r_2+r_3+e_3=n_2}}\frac{(-1)^{r_3}q^{\frac{1}{2}r_3^2-\frac{1}{2}r_3+r_1r_2}}{(q)_{r_1}(q)_{r_2}(q)_{n_1+e_1-r_3}(q)_{n_2+e_2-r_3}(q)_{r_3}(q)_{r_3+e_3}}$$
$$=\sum_{\substack{r_1,r_2,r_3,e_3\\ r_1+r_3+e_3=n_1\\ r_2+r_3+e_3=n_2}}\frac{(-1)^{r_3}q^{\frac{1}{2}r_3^2-\frac{1}{2}r_3+r_1r_2}}{(q)_{r_1}(q)_{r_2}(q)_{r_1+e_1+e_3}(q)_{r_2+e_2+e_3}(q)_{r_3}(q)_{r_3+e_3}}\,.$$

Now multiplying both sides by $(-1)^{n_1+n_2}q^{\frac{1}{2}(n_1-n_2)^2+\frac{1}{2}n_1+\frac{1}{2}n_2}$ gives Eq. (6.13).

Appendix B: The tetrahedron index and the quantum dilogarithm

Gukov–Gaiotto–Dimofte came up with the beautiful formula (1.2) for the tetrahedron index from a Fourier transform of the quantum dilogarithm. For completeness, we include this relation here, taken from [6]. The quantum dilogarithm of Faddeev and Kashaev is a fundamental building block of quantum topology [8,13,14]. The q-series version of this analytic function is given by

$$L(m,x,q)=\frac{(q^{-\frac{m}{2}+1}x^{-1})_\infty}{(q^{-\frac{m}{2}}x)_\infty}\in\mathbb{Z}((x))[[q^{1/2}]]. \tag{6.14}$$

We claim that

$$\sum_e I(m,e)(q)x^e=L(m,x,q). \tag{6.15}$$

To prove this, use the definition of $I(m,e)$, shift e to $e-n$, and use the first two identities of Lemma 6.4. We get

$$\begin{aligned}\sum_e I(m,e)(q)x^e &= \sum_{n,e}(-1)^n\frac{q^{\frac{1}{2}n(n+1)-\left(n+\frac{1}{2}e\right)m}x^e}{(q)_n(q)_{n+e}}\\ &= \sum_{n,e}(-1)^n\frac{q^{\frac{1}{2}n(n+1)}\left(q^{-\frac{m}{2}}x^{-1}\right)^n\left(q^{-\frac{m}{2}}x\right)^e}{(q)_n(q)_e}\\ &= \frac{(q^{-\frac{m}{2}+1}x^{-1})_\infty}{(q^{-\frac{m}{2}}x)_\infty}.\end{aligned}$$

Each of the recursion relations (3.1a), (3.1b), (3.3a), and (3.3b) is equivalent to the corresponding relations (6.16a–6.16d) for the generating series $L(m,x,q)$:

$$(-1+q^{-\frac{m}{2}}x^{-1})L(m,x,q)+L(m+1,q^{\frac{m}{2}}x,q)=0, \tag{6.16a}$$

$$(1-q^{-\frac{m}{2}}x^{-1})L(m,x,q)+L(m-1,q^{\frac{m}{2}}x,q)=0, \tag{6.16b}$$

$$(1+x^2-(q^{\frac{m}{2}}+q^{-\frac{m}{2}}))L(m,x,q)+xq^{\frac{m}{2}}L(m,qx,q)=0, \tag{6.16c}$$

$$\begin{aligned}&L(m-2,x,q)-(L(m-1,q^{\frac{1}{2}}x,q)+L(m-1,q^{-\frac{1}{2}}x,q)\\ &\quad -q^{1-m}L(m-1,q^{-\frac{1}{2}}x,q))+L(m,x,q)=0\end{aligned} \tag{6.16d}$$

Equations (6.16a–6.16d) are easy to verify using the fact that $L(m,q,x)$ is a proper hypergeometric function of (m,q). This gives an alternative proof of part (a) of Theorem 3.7.

Observe finally that the recursions (3.1a) and (3.1b) have a solution space of rank 2. On the other hand, the recursions (6.16a) and (6.16b) have a solution space of rank 1.

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