Chapter 14

Resurgence of Faddeev's quantum dilogarithm

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Abstract The quantum dilogarithm function of Faddeev is a special function that plays a key role as the building block of quantum invariants of knots and 3-manifolds, of quantum Teichmüller theory and of complex Chern–Simons theory. Motivated by conjectures on resurgence and the recent interest in wall-crossing phenomena, we prove that the Borel summation of a formal power series solution of a linear difference equation produces Faddeev's quantum dilogarithm. Along the way, we give an explicit formula for the Borel transform, a meromorphic function in the Borel plane, locate its poles and residues and describe the Stokes phenomenon of its Laplace transforms along the Stokes rays.

1 Introduction

A well-known problem in quantum topology is the Volume Conjecture which asserts that the Kashaev invariant of a hyperbolic knot grows exponentially at a rate proportional to the volume of the knot [19–21]. There are several strengthenings of this conjecture that involve the analytic properties of the asymptotics of the Kashaev invariant to all orders (see, e.g., [11, 16] and references therein). Such factorially divergent formal power series have been conjectured to lead to resurgent functions [13], and this in turn leads to astonishing numerically testable conjectures [14, 16, 18]. The Kashaev invariant of a knot is a finite state-sum whose building block is the quantum n factorial $(q;q)_n = \prod_{j=1}^n (1-q^j)$, evaluated at complex roots of unity. The latter is intimately related to another special function, the Faddeev quantum dilogarithm [12], evaluated at rational points. Although the conjectured resurgence properties of quantum knot invariants are largely unproven, in an unfinished manuscript from 2006 we studied the resurgence properties of their building block, namely the Faddeev quantum dilogarithm. This special function plays a key role in quantum Teichmüller theory [5, 22] and complex Chern–Simons theory [6, 9, 10]. Since there is renewed interest in this subject with applications to resurgence and wall-crossing phenomena (see for instance [25]), we decided to update our manuscript and make it widely available.

To begin the story, in the quantization of Teichmüller theory one considers the difference equation

$$f_\tau(z - i\pi\tau) = (1 + e^z) f_\tau(z + i\pi\tau) \tag{1.1}$$

whose motivation is explained in detail in [22, Prop. 8] and also in [5, Prop. 1, Eqn. (9)] (after some minor change in notation). The above difference equation appears, among other places, in quantum integrable systems (see Ruijsenaars [31, Eqn. (1.17)]) and in holomorphic dynamics (see Marmi–Sauzin [28]).

It it easy to see (see Lemma 2.1 below) that if $f_\tau(z)$ satisfies equation (1.1) and the limiting value $\lim_{z\to-\infty} f_\tau(z) = 1$, then for a fixed z, $f_\tau(z)$ admits an asymptotic expansion of the form

$$\log f_\tau(z) \sim \frac{1}{2\pi i \tau}\mathrm{Li}_2(-e^z) + \hat{\phi}_\tau(z), \quad (\tau \to 0) \tag{1.2}$$

where

$$\hat{\phi}_\tau(z) = \sum_{n=1}^{\infty}(2\pi i)^{2n-1}\frac{B_{2n}(1/2)}{(2n)!}\partial_z^{2n}\mathrm{Li}_2(-e^z)\tau^{2n-1}, \tag{1.3}$$

$\mathrm{Li}_2(z) = \sum_{k\geq 1} z^k/k^2$ is Euler's dilogarithm function and the differentiation operator ∂_z^{2n} (defined by $\partial_z g(z) = g'(z)$) acts on $\mathrm{Li}_2(-e^z)$.

The goal of this paper is to identify the Borel summation of the factorially divergent series $\hat{\phi}_\tau(z)$ with Faddeev's quantum dilogarithm function

$$f_\tau(z) = \Phi_{\mathsf{b}}(z/(2\pi \mathsf{b})) \tag{1.4}$$

(see Corollary 1.5 below) when $\tau = \mathsf{b}^2 > 0$. Along the way, we give an explicit formula for the Borel transform $G(\xi, z)$ of the power series $\hat{\phi}_\tau(z)$ (see Theorem 1.1 below).

It turns out that $G(\xi, z)$ is a meromorphic function of ξ with poles that lie discretely in a countable union $L(z)$ of lines through the origin given in equation (1.11) below. The arrangement $L(z)$ depends on z and accumulates to the imaginary axis. Such an arrangement is reminiscent of the parametric resurgence of non-linear equations (see, e.g., [30]), the exact and perturbative invariants of Chern–Simons theory (predicted for instance in [13] and the figure below Definition 2.3 of ibid), and the wall-crossing formulas of Kontsevich–Soibelman (see [26] and also [25]).

Theorem 1.3 identifies the Laplace transform of $G(\cdot, z)$ with the logarithm of Faddeev's quantum dilogarithm function $f_\tau(z)$ given in equation (1.4). We also consider the Laplace transform of the function $G(\cdot, z)$ along any ray in the complement of $L(z)$ and describe the Stokes phenomenon, i.e., the change of the Laplace transform as one crosses a Stokes line $\mathbb{R}\xi_m(z)$. One may think of this as an instance of a wall-crossing formula, in the spirit of Kontsevich–Soibelman.

Another noteworthy phenomenon is the Laplace transform of $G(\cdot, z)$ along the vertical rays $\pm i\mathbb{R}_+$, which no longer lie in an open cone in the complement of $L(z)$. This is the case considered by Marmi–Sauzin [28] who prove (with a careful analysis) that the Laplace transform f_τ^- (resp. f_τ^+) is defined in the upper half-plane $\mathrm{Im}(\tau) > 0$ (resp. lower half-plane $\mathrm{Im}(\tau) < 0$) thus leading to two distinguished solutions $f_\tau^\pm(z)$ of equation (1.1).

1.1 Our results.

Recall the quantum dilogarithm function of Faddeev [12]

$$\Phi_{\mathsf{b}}(z) = \exp\left(\int_{\mathbb{R}+\mathrm{i}\epsilon} \frac{e^{-2\mathrm{i}xz}}{4\sinh(x\mathsf{b})\sinh(x\mathsf{b}^{-1})}\frac{\mathrm{d}x}{x}\right), \tag{1.5}$$

a function with remarkable analytic properties that satisfies a pentagon identity and an inversion relation summarized in Section 3 below. $\Phi_{\mathsf{b}}(z)$ is a meromorphic, quasi-periodic function of z that satisfies

$$\Phi_{\mathsf{b}}(z - i\mathsf{b}/2) = (1 + e^{2\pi \mathsf{b} z})\, \Phi_{\mathsf{b}}(z + i\mathsf{b}/2). \tag{1.6}$$

The function $\Phi_{\mathsf{b}}(z)$ is used as the building block of topological invariants of 3-manifolds via quantum Teichmüller theory [3–5, 24].

Consider the Borel transform

$$\mathcal{B}\colon \tau\mathbb{C}[[\tau]] \to \mathbb{C}[[\xi]], \quad \mathcal{B}(\tau^{n+1}) = \frac{\xi^n}{n!}. \tag{1.7}$$

Let

$$G(\xi, z) = \mathcal{B}(\hat{\phi}_\tau(z)) \tag{1.8}$$

denote the Borel transform of $\hat{\phi}_\tau(z)$. Our first result describes a global formula for $G(\xi, z)$ in the complex Borel ξ-plane.

Theorem 1.1. *When $z \in \mathbb{C}$ with $|\mathrm{Im}(z)| < \pi$ and $\xi \in \mathbb{C}$ with $|\xi| < \pi - |\mathrm{Im}(z)|$, we have*

$$G(\xi, z) = \frac{1}{2\pi i} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \left(\frac{1}{1 + e^{\frac{\xi}{n} - z}} + \frac{1}{1 + e^{-\frac{\xi}{n} - z}} \right), \tag{1.9}$$

where the right-hand side is expanded as a formal power series in ξ around zero with radius of convergence $\pi - |\mathrm{Im}(z)|$.

It follows that $G(\xi, z)$ is a meromorphic function of ξ with simple poles at $\xi = n\xi_m(z)$ (shown in Figure 1) with residue C_n, where

$$\xi_m(z) = z + (2m+1)\pi i, \quad n \in \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}, \quad m \in \mathbb{Z}, \quad C_n = \frac{(-1)^n}{2\pi i n}. \tag{1.10}$$

Note that the singularities of $G(\cdot, z)$ form a lattice in a countable union $L(z)$ of lines through the origin

$$L(z) = \bigcup_{m \in \mathbb{Z}} \mathbb{R}\xi_m(z). \tag{1.11}$$

The formula of the above theorem is similar to the one of Marmi–Sauzin [28, Thm. 4.3]. It is also similar to the Euler–MacLaurin summation formula given in Costin–Garoufalidis [8, Thm. 2]. This is not an accident. When $\mathrm{Im}(\tau) > 0$, the quantum dilogarithm has an infinite product expansion (see equation (3.1) below) whose logarithm can be written as an infinite sum which one can analyze with the Euler–MacLaurin summation method. However, such a manipulation is meaningless since the product formula (3.1), although convergent when $\mathrm{Im}(\tau) > 0$, diverges when $\tau > 0$.

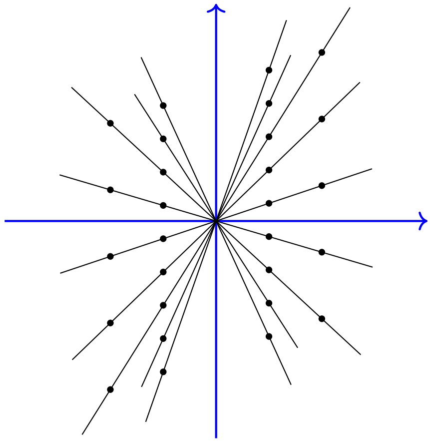


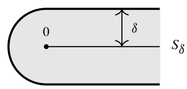
Figure 1. The poles of $G(\xi, z)$ in the ξ-plane are points $n\xi_m(z)$ lie in an arrangement of lines passing through the origin.

Recall the Laplace transform

$$(\mathcal{L}f)(\tau) = \int_0^\infty e^{-\xi/\tau} f(\xi)\, \mathrm{d}\xi \tag{1.12}$$

of a function $f \in L^1(\mathbb{R})$. Our next theorem concerns the analytic properties of the meromorphic function $G(\xi, z)$ and its Laplace transform $(\mathcal{L}G)(\tau, z)$. For a positive real number δ, define

$$S_\delta = \big\{\xi \in \mathbb{C} \mid \mathrm{dist}(\xi, [0, \infty)) < \delta\big\}, \tag{1.13}$$



Theorem 1.2. (a) *When $z \in \mathbb{C}$ with $0 < \delta < \pi - |\mathrm{Im}(z)|$ and $\xi \in S_\delta$ we have*

$$|G(\xi, z)| \leq \max\left\{2, \frac{\pi^2}{3\sin(|\mathrm{Im}(z)| + \delta)}\right\}. \tag{1.14}$$

(b) *Fix $0 < \delta < \pi$. Then we have*

$$\begin{aligned} &\left|(\mathcal{L}G)(\tau, z) - \sum_{n=1}^{N} (2\pi i)^{2n-1} \frac{B_{2n}(1/2)}{(2n)!} \partial_z^{2n} \mathrm{Li}_2(-e^z)\tau^{2n-1}\right| \\ &\leq LM^{2N}(2N)!\mathrm{Re}(\tau)|\tau|^{2N} \end{aligned} \tag{1.15}$$

for all $\tau \in \mathbb{C}$ with $\mathrm{Re}(\tau) > 0$ and all $z \in \mathbb{C}$ with $|\mathrm{Im}(z)| < \pi - \delta$ and all natural numbers N, where $M = M_\delta = 2/\delta$ and $L = L_{z,\delta} = \max\{2, \frac{\pi^2}{3\sin(|\mathrm{Im}(z)|+\delta)}\}$.

Our next result identifies the Borel transform of $G(\xi, z)$ with the quantum dilogarithm function.

Theorem 1.3. *When $\tau > 0$ and $z \in \mathbb{C}$ with $|\mathrm{Im}(z)| < \pi$, we have*

$$\log \Phi_{\mathsf{b}}\Big(\frac{z}{2\pi \mathsf{b}}\Big) = \frac{1}{2\pi i \tau}\mathrm{Li}_2(-e^z) + (\mathcal{L}G)(\tau, z) \tag{1.16}$$

where $\mathsf{b}^2 = \tau$ and $G(\xi, z)$ is as in (1.9).

This theorem follows from the explicit formula for $G(\xi, z)$ in Theorem 1.1 which agrees with the integral formula of Woronowicz for $\Phi_{\mathsf{b}}(z)$. Theorems 1.2 and 1.3 give the following.

Corollary 1.4. *With the assumptions of part* (b) *of Theorem 1.2, we have*

$$\begin{aligned}&\left|\log \Phi_{\mathsf{b}}\Big(\frac{z}{2\pi \mathsf{b}}\Big) - \sum_{n=0}^{N} (2\pi i)^{2n-1} \frac{B_{2n}(1/2)}{(2n)!} \partial_z^{2n} \mathrm{Li}_2(-e^z)\tau^{2n-1}\right| \\ &\le LM^{2N}(2N)!\mathrm{Re}(\tau)|\tau|^{2N}.\end{aligned} \tag{1.17}$$

The special case of (1.17) with $N = 0$ is equivalent to Lemma 7.13 of [7], which itself is an improvement of an earlier Lemma 3 of Andersen–Hansen [2]. An alternative proof of the above inequality (1.17) was given by Andersen [1].

The process of replacing a factorially divergent series with its Borel transform, followed by the Laplace transform is known as Borel summation. Theorems 1.1 and 1.3 imply the following.

Corollary 1.5. *When $\tau > 0$ and $z \in \mathbb{C}$ with $|\mathrm{Im}(z)| < \pi$, the Borel summation of the series* (1.3) *reproduces the logarithm of Faddeev's quantum dilogarithm function, namely,* $\log \Phi_{\mathsf{b}}(z/(2\pi \mathsf{b}))$.

The above corollary has some surprising consequences. A priori, a solution of (1.1) is well-defined up to multiplication with $2\pi i \tau$-periodic functions, and Borel summation chooses exactly the one that agrees with the quantum dilogarithm function. What's more, the quantum dilogarithm function satisfies the symmetry of equation (3.3), and hence satisfies a second difference equation (obtained by replacing b by b^{-1} in (1.6)). This, together with Corollary 1.5, implies the following.

Corollary 1.6. *The Borel summation of $\hat{\phi}_\tau(z)$ satisfies the additional functional equation*

$$f_\tau(z - i\pi) = \big(1 + e^{\frac{z}{\tau}}\big) f_\tau(z + i\pi). \tag{1.18}$$

The two functional equations (1.1) and (1.18) determine f_τ up to multiplication by a doubly periodic function, and when $\tau > 0$ and irrational, such functions are constant; see for example [12, p. 251].

Our last topic concerns the Stokes phenomenon of the Laplace transform of $G(\cdot, z)$. Let $\rho_\theta = [0, \infty)e^{i\theta}$ denote the ray in the complex plane and let

$$(\mathcal{L}^\theta f)(\tau) = \int_{\rho_\theta} e^{-\xi/\tau} f(\xi)\, \mathrm{d}\xi \tag{1.19}$$

denote the Laplace transform of a function f, integrable along ρ_θ. Recall that the singularities of the meromorphic function $G(\cdot, z)$ are in an arrangement of lines $L(z)$ through the origin whose complement $\mathbb{C} \setminus L(z) = \bigcup_{m\in\mathbb{Z}} C_m(z)$ is a union of open cones

$$C_m(z) = \big\{\xi \in \mathbb{C}^* \mid \arg(\xi_{m-1}(z)) < \arg(\xi) < \arg(\xi_m(z))\big\}. \tag{1.20}$$

It follows that when $\theta \in C_m(z)$, the Laplace transform $(\mathcal{L}^\theta G)(\tau, z)$ is independent of θ and defines a holomorphic function of τ for $\arg(\xi_{m-1}(z)) - \pi/2 < \arg(\tau) < \arg(\xi_m(z)) + \pi/2$. When $\theta = 0 \in C_0(z)$, Theorem 1.3 implies that $(\mathcal{L}^\theta G)(\tau, z)$ is, up to a dilogarithm term, equal to $f_\tau(z)$ and the latter is equal to

$$\log f_\tau(z) = \log\big(-q^{\frac{1}{2}}e^z; q\big)_\infty - \log\big(-\tilde{q}^{\frac{1}{2}}e^{z/\tau}; \tilde{q}\big)_\infty \tag{1.21}$$

when $\mathrm{Im}(\tau) > 0$ and $\mathrm{Re}(z) < 0$, as follows from equation (3.1). On the other hand, by crossing the walls of $L(z)$, we get

$$\big(\mathcal{L}^{\pi/2}G\big)(\tau, z) - \big(\mathcal{L}^0 G\big)(\tau, z) = \sum_{m=0}^{\infty} f_m(\tau, z) - f_{m+1}(\tau, z) \tag{1.22}$$

where $f_m(\tau, z) = (\mathcal{L}^{\theta_m(z)}G)(\tau, z)$ is the Laplace transform of $G(\cdot, z)$ along a ray $\theta_m(z) \in C_m(z)$. When $\mathrm{Re}(z) < 0$ and $|\mathrm{Im}(z)| < \pi$ and $\mathrm{Im}(\tau) > 0$, the difference $f_m(\tau, z) - f_{m+1}(\tau, z)$ is obtained by adding the poles $-n\xi_{-m-1}(z)$ with $n > 0$ of $G(\cdot, z)$ at the corresponding ray. Using the residue of $G(\cdot, z)$ at these points given in Theorem 1.1 and adding up, it follows that

$$\begin{aligned} f_m(\tau, z) - f_{m+1}(\tau, z) &= \sum_{n=1}^{\infty} 2\pi i C_{-n} e^{\xi_{-m-1}(z)n/\tau} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{\xi_{-m-1}(z)n/\tau} \\ &= \log\big(1 + e^{\xi_{-m-1}(z)/\tau}\big) = \log\big(1 + e^{z/\tau}\tilde{q}^{m+\frac{1}{2}}\big). \end{aligned} \tag{1.23}$$

This, combined with equations (1.21) and (1.22), implies that

$$\big(\mathcal{L}^{\pi/2}G\big)(\tau, z) = \log\big(-q^{\frac{1}{2}}e^z; q\big)_\infty \tag{1.24}$$

in agreement with the result of Marmi–Sauzin proven in [28, Sec. 1.1 and Thm. 4.3].

2 Proofs

2.1 The formal power series solution of the difference equation. The following lemma is well-known and standard (see, e.g., [5, Sec. 13.4, Prop. 6]), but we include its proof for completeness.

Lemma 2.1. *If $f_\tau(z)$ satisfies equation (1.1) and $\lim_{z\to-\infty} f_\tau(z) = 1$, then for fixed z, $f_\tau(z)$ admits an asymptotic expansion of the form (1.2) with $\hat{\phi}_\tau(z)$ given in (1.3).*

Proof. Letting $\phi_\tau(z) = \log f_\tau(z)$, it follows that

$$\phi_\tau(z+\pi i\tau) - \phi_\tau(z-\pi i\tau) = -\log(1+e^z).$$

Taylor's theorem combined with $-\log(1+e^z) = \partial_z \mathrm{Li}_2(-e^z)$ implies that

$$2\sinh(\pi i\tau\partial_z)\phi_\tau(z) = \partial_z \mathrm{Li}_2(-e^z)$$

hence, that

$$2\pi i\phi_\tau(z) = \frac{\pi i\partial_z}{\sinh(\pi i\tau\partial_z)}\mathrm{Li}_2(-e^z).$$

The expansion

$$\frac{z}{\sinh(z)} = \sum_{n=0}^{\infty} B_{2n}(1/2)\frac{(2z)^{2n}}{(2n)!}$$

concludes the proof of the lemma. ∎

2.2 The Borel transform. Consider the formal power series

$$\phi_f(\tau,z) = \sum_{n=1}^{\infty} \frac{B_{2n}(1/2)}{(2n)!} f^{(2n)}(z)(2\pi i)^{2n-1}\tau^{2n-1} \tag{2.1}$$

for a function f analytic on z with $|\mathrm{Im}(z)| < \pi$, and let $G_f(\xi,z) = \mathcal{B}(\phi_f(\cdot,z))$ denote the corresponding Borel transform.

Proposition 2.2. *We have*

$$G_f(\xi,z) = \frac{i}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(f''\left(z+\frac{\xi}{n}\right) + f''\left(z-\frac{\xi}{n}\right)\right). \tag{2.2}$$

Proof. The proof is rather standard. It uses the Hadamard product $\circledast$ of power series (which was also used in [8]) whose definition we recall

$$\left(\sum_{n=0}^{\infty} b_n\xi^n\right) \circledast \left(\sum_{n=0}^{\infty} c_n\xi^n\right) = \sum_{n=0}^{\infty} b_n c_n \xi^n. \tag{2.3}$$

We have

$$\begin{aligned} G_f(\xi,z) &= \sum_{n=1}^{\infty} \frac{B_{2n}(1/2)}{(2n)!} \frac{f^{(2n)}(z)}{(2n-2)!} (2\pi i)^{2n-1} \xi^{2n-2} \\ &= \left(\sum_{n=1}^{\infty} \frac{B_{2n}(1/2)}{(2n)!} \xi^{2n-2} \right) \circledast \left(\sum_{n=1}^{\infty} \frac{f^{(2n)}(z)}{(2n-2)!} (2\pi i)^{2n-1} \xi^{2n-2} \right) \\ &= f_1(\xi) \circledast f_2(\xi,z) \end{aligned}$$

where

$$f_1(\xi) = \sum_{n=1}^{\infty} \frac{B_{2n}(1/2)}{(2n)!} \xi^{2n-2}, \quad f_2(\xi,z) = 2\pi i \sum_{n=1}^{\infty} \frac{f^{(2n)}(z)}{(2n-2)!} (2\pi i \xi)^{2n-2}. \tag{2.4}$$

Now, since $B_m(1/2) = 0$ for every odd m, we have

$$\begin{aligned} f_1(\xi) &= \sum_{n=1}^{\infty} \frac{B_{2n}(1/2)}{(2n)!} \xi^{2n-2} = \frac{1}{\xi^2} \sum_{n=1}^{\infty} \frac{B_{2n}(1/2)}{(2n)!} \xi^{2n} = \frac{1}{\xi^2} \sum_{n=1}^{\infty} \frac{B_n(1/2)}{n!} \xi^n \\ &= \frac{1}{\xi^2} \Big(\frac{e^{\xi/2}\xi}{e^{\xi}-1} - 1 \Big) = \frac{1}{\xi(e^{\xi/2} - e^{-\xi/2})} - \frac{1}{\xi^2} \end{aligned}$$

and Taylor's theorem gives

$$\begin{aligned} f_2(\xi,z) &= 2\pi i \sum_{n=1}^{\infty} \frac{f^{(2n)}(z)}{(2n-2)!} (2\pi i \xi)^{2n-2} \\ &= (2\pi i)^{-1} \partial_\xi^2 \sum_{n=1}^{\infty} \frac{f^{(2n)}(z)}{(2n)!} (2\pi i \xi)^{2n} \\ &= (2\pi i)^{-1} \partial_\xi^2 \Big(\frac{1}{2} (f(z + 2\pi i \xi) + f(z - 2\pi i \xi)) - f(z) \Big) \\ &= \pi i \big(f''(z + 2\pi i \xi) + f''(z - 2\pi i \xi) \big). \end{aligned}$$

Now, using Cauchy's theorem, it follows that

$$G_f(\xi,z) = \frac{1}{2\pi i} \int_\gamma f_1(s) f_2\Big(\frac{\xi}{s}, z\Big) \frac{ds}{s}$$

where γ is a small circle around 0. The function $f_1(s)$ has simple poles at $2\pi i m$ for $m \in \mathbb{Z} \setminus \{0\}$ with residue $(-1)^m/(2\pi i m)$. Now, deform the integration contour to circles of increasing radii and collect the residues. Since $f_1(s) = O(1/s)$ and $f_2(\xi/s) = O(1/s^2)$, it follows that the integrand is $O(1/s^3)$, thus the contribution from infinity is zero. The residue of the integrand for $m \in \mathbb{Z} \setminus \{0\}$ is given by

$$\begin{aligned}\operatorname{Res}\Big(f_1(s) f_2\Big(\frac{\xi}{s}, z\Big)\frac{1}{s}, s = 2\pi i m\Big) &= \frac{1}{2\pi i m} f_2\Big(\frac{\xi}{2\pi i m}, z\Big)\operatorname{Res}(f_1(s), s = 2\pi i m)\\ &= \frac{(-1)^m}{4\pi i m^2}\Big(f''\Big(z + \frac{\xi}{m}\Big) + f''\Big(z - \frac{\xi}{m}\Big)\Big).\end{aligned}$$

Thus, collecting the residues, it follows that

$$\begin{aligned} G_f(\xi, z) &= -\sum_{m \in \mathbb{Z}\setminus 0} \frac{(-1)^m}{4\pi i m^2}\Big(f''\Big(z + \frac{\xi}{m}\Big) + f''\Big(z - \frac{\xi}{m}\Big)\Big)\\ &= \frac{i}{2\pi}\sum_{m=1}^{\infty} \frac{(-1)^m}{m^2}\Big(f''\Big(z + \frac{\xi}{m}\Big) + f''\Big(z - \frac{\xi}{m}\Big)\Big).\end{aligned}$$

For fixed z and ξ, the above sum is dominated by $\sum_{m=1}^{\infty} 1/m^2$ and thus the convergence is uniform on compact sets. This concludes the proof of Proposition 2.2. ∎

Proof of Theorem 1.1. Apply Proposition 2.2 to the function $f(z) = \mathrm{Li}_2(-e^z)$ which satisfies

$$f''(z) = -\frac{1}{1 + e^{-z}}.$$ ∎

2.3 Bounds. In this section we give a proof of Theorem 1.2.

We begin with the following lemma.

Lemma 2.3. *When* $z \in \mathbb{C}$ *with* $|\mathrm{Im}(z)| < \pi$ *we have*

$$\frac{1}{|1 + e^z|} \le \begin{cases} 1 & \textit{if } \cos(\mathrm{Im}(z)) \ge 0, \\ \frac{1}{|\sin(\mathrm{Im} z)|} & \textit{if } \cos(\mathrm{Im}(z)) \le 0. \end{cases} \tag{2.5}$$

Proof. With $z = t + ia$, we have

$$|1 + e^z|^2 = e^{2t} + 2e^t \cos a + 1$$

and the right-hand side, as a function of $t \in \mathbb{R}$, has critical points in $t_0 \in \mathbb{R}$ such that $e^{t_0} + \cos a = 0$. When $\cos a > 0$, it follows that $\inf_{t \in \mathbb{R}} |1 + e^z|^2 = 1$, and when $\cos a \le 0$, it follows that t_0 is a global minimum and $e^{2t_0} + 2e^{t_0}\cos a + 1 = \sin^2 a$. The result follows. ∎

Proof of Theorem 1.2. The first part follows from equation (1.9), Lemma 2.3 (applied to $\xi/n - z$ for $n \in \mathbb{Z}^*$) and the fact that $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$. In particular, it implies that the Laplace transform $(\mathcal{L}G)(\tau, z)$ is well-defined and even extends to $\tau \in \mathbb{C}$ with $\mathrm{Re}(\tau) > 0$.

The second part follows from what is known in the literature as Watson's lemma [32], a modern proof of which may be found for instance in Miller [29, p. 53, Prop. 2.1]. It is also known as the fine Borel–Laplace transform in Mitschi–Sauzin [30, Sec. 5.7, Thm. 5.20] whose proof follows Malgrange [27]. Our proof of the second part follows directly from Mitschi–Sauzin [30, Sec. 5.7, Thm. 5.20], together with the upper bound from the first part (which implies that $c_0 = 0$ and $c_1 = \epsilon$ in the notation of Theorem 5.20 of [30]). ∎

2.4 The Laplace transform. Woronowicz [33], while studying the quantum exponential function via functional analysis, introduced the function

$$W_\theta(z) = \int_{\mathbb{R}} \frac{\log(1+e^{\theta\xi})}{1+e^{\xi-z}}\, \mathrm{d}\xi \tag{2.6}$$

defined for $\theta > 0$ and $|\mathrm{Im}(z)| < \pi$, and proved that (after some elementary change of variables) it satisfies the functional equation (1.1); see [33, Eqn. (B.3)]. Thus, one can relate Woronowicz's function with the quantum dilogarithm as is done in [23, Eqns. (1), (2)] without proof. The next proposition provides a formal proof of this fact.

Proposition 2.4. *When $\tau > 0$ and $z \in \mathbb{C}$ with $|\mathrm{Im}(z)| < \pi$, we have*

$$W_{\frac{1}{\tau}}(z) = -2\pi i \log \Phi_{\mathsf{b}}\Big(\frac{z}{2\pi \mathsf{b}}\Big), \quad \mathsf{b}^2 = \tau. \tag{2.7}$$

Proof. The proof uses a mixture of real and complex analysis. Let ϵ be a positive real number such that $0 < \epsilon < \min(1,\theta)$. We remark that $W_\theta(z)$ can be interpreted as a value of the scalar product in the complex Hilbert space $L^2(\mathbb{R})$ of square integrable functions on the real line with respect to the Lebesgue measure

$$W_\theta(z) = \langle f|g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}\, \mathrm{d}x \tag{2.8}$$

where $f, g \in L^2(\mathbb{R})$ are defined by

$$f(x) = e^{-\epsilon x}\log(1+e^{\theta x}), \quad g(x) = \frac{e^{\epsilon x}}{1+e^{x-\bar{z}}}. \tag{2.9}$$

As the Fourier transformation

$$(F\psi)(x) = \int_{\mathbb{R}} \psi(y) e^{2\pi i x y}\, \mathrm{d}y \tag{2.10}$$

is a unitary operator in $L^2(\mathbb{R})$, we have the equality

$$\langle f|g\rangle = \langle Ff|Fg\rangle. \tag{2.11}$$

By using Lemma 2.5 below, we can calculate explicitly the elements $Ff, Fg \in L^2(\mathbb{R})$. Indeed, denoting $\zeta = 2\pi x + i\epsilon$, we have

$$(Ff)(x) = \int_{\mathbb{R}} e^{i\zeta y} \log(1+e^{\theta y})\,\mathrm{d}y = \frac{i\theta}{\zeta}\int_{\mathbb{R}} \frac{e^{i\zeta y}}{1+e^{-\theta y}}\,\mathrm{d}y \tag{2.12}$$

$$= \frac{2\pi i}{\zeta}\int_{\mathbb{R}} \frac{e^{2\pi i\zeta y/\theta}}{1+e^{-2\pi y}}\,\mathrm{d}y = \frac{\pi i}{\zeta}\int_{\mathbb{R}} \frac{e^{\left(\frac{\zeta}{\theta}-\frac{i}{2}\right)2\pi i y}}{\cosh(\pi y)}\,\mathrm{d}y \tag{2.13}$$

$$= \frac{\pi i}{\zeta\cosh\left(\frac{\pi\zeta}{\theta}-\frac{\pi i}{2}\right)} = \frac{\pi i}{\zeta\cos\left(\frac{\pi}{2}-\frac{\pi\zeta}{i\theta}\right)} = \frac{\pi i}{\zeta\sin\left(\frac{\pi\zeta}{i\theta}\right)} = -\frac{\pi}{\zeta\sinh\left(\frac{\pi\zeta}{\theta}\right)} \tag{2.14}$$

where in the second equality we integrated by parts, and

$$\overline{(Fg)(x)} = \int_{\mathbb{R}} \frac{e^{-i\zeta y}}{1+e^{y-z}}\,\mathrm{d}y = \int_{\mathbb{R}-z} \frac{e^{-i\zeta(y+z)}}{1+e^{y}}\,\mathrm{d}y \tag{2.15}$$

$$= 2\pi e^{-i\zeta z}\int_{\mathbb{R}-\frac{z}{2\pi}} \frac{e^{-2\pi i\zeta y}}{1+e^{2\pi y}}\,\mathrm{d}y = \pi e^{-i\zeta z}\int_{\mathbb{R}-\frac{z}{2\pi}} \frac{e^{\left(\frac{i}{2}-\zeta\right)2\pi i y}}{\cosh(\pi y)}\,\mathrm{d}y \tag{2.16}$$

$$= \frac{\pi e^{-i\zeta z}}{\cosh\left(\frac{\pi i}{2}-\pi\zeta\right)} = \frac{\pi e^{-i\zeta z}}{\cos\left(\frac{\pi}{2}+\pi i\zeta\right)} = \frac{\pi e^{-i\zeta z}}{\sin(-\pi i\zeta)} = \frac{\pi i e^{-i\zeta z}}{\sinh(\pi\zeta)} \tag{2.17}$$

where in the fifth equality we used the condition $|\mathrm{Im}(z)| < \pi$. Thus, we obtain

$$W_\theta(z) = \langle Ff|Fg\rangle = \int_{\mathbb{R}} (Ff)(x)\overline{(Fg)(x)}\,\mathrm{d}x = \int_{\mathbb{R}+i\epsilon} \frac{-\pi i e^{-i\zeta z}}{2\sinh\left(\frac{\pi\zeta}{\theta}\right)\sinh(\pi\zeta)}\frac{\mathrm{d}\zeta}{\zeta} \tag{2.18}$$

which implies that

$$\frac{i}{2\pi}W_{\frac{1}{\tau}}(z) = \int_{\mathbb{R}+i\pi\mathsf{b}\epsilon} \frac{e^{-i\frac{\zeta z}{\pi\mathsf{b}}}}{4\sinh(\mathsf{b}\zeta)\sinh(\zeta/\mathsf{b})}\frac{\mathrm{d}\zeta}{\zeta} = \log\Phi_{\mathsf{b}}\left(\frac{z}{2\pi\mathsf{b}}\right). \tag{2.19}$$

■

The next lemma is well-known, see, e.g., Godement [17, VII and 3.15]. We will also give a proof using [15, Lem. 2.1].

Lemma 2.5. *When $w, \sigma \in \mathbb{C}_{|\mathrm{Im}|<\frac{1}{2}} = \{u \in \mathbb{C} \mid |\mathrm{Im}(u)| < \frac{1}{2}\}$, we have*

$$\int_{\mathbb{R}+\sigma} \frac{e^{2\pi i w z}}{\cosh(\pi z)}\,\mathrm{d}z = \frac{1}{\cosh(\pi w)}. \tag{2.20}$$

Proof. By using [15, Lem. 2.1] with $f(z) = \frac{e^{2\pi i w z}}{\cosh(\pi z)}$ and $a = i$, we have

$$\frac{f(z+a)}{f(z)} = \frac{\cosh(\pi z)e^{2\pi i w(z+i)}}{\cosh(\pi z+\pi i)e^{2\pi i w z}} = -e^{-2\pi w} \tag{2.21}$$

so that

$$\int_{\mathbb{R}+\sigma} \frac{e^{2\pi i w z}}{\cosh(\pi z)}\,\mathrm{d}z = \left(\int_{\mathbb{R}+\sigma} - \int_{\mathbb{R}+\sigma+i}\right) \frac{e^{2\pi i w z}}{(1+e^{-2\pi w})\cosh(\pi z)}\,\mathrm{d}z \tag{2.22}$$

$$= 2\pi i \operatorname{Res}_{z=\frac{i}{2}}\left(\frac{e^{2\pi i w z}}{(1+e^{-2\pi w})\cosh(\pi z)}\right) \tag{2.23}$$

$$= \frac{\pi i}{\cosh(\pi w)} \operatorname{Res}_{z=0}\left(\frac{1}{\sin(\pi i z)}\right) = \frac{1}{\cosh(\pi w)} \tag{2.24}$$

where, in the second equality, the application of the residue theorem is justified by the limits

$$\lim_{x\to\pm\infty} |f(x+\sigma+it)| \le \left|e^{-2\pi w t}\right| \lim_{x\to\pm\infty}\left(\frac{e^{2\pi|\mathrm{Im}(w)||x+\mathrm{Re}(\sigma)|}}{\sinh|\pi(x+\mathrm{Re}(\sigma))|}\right) = 0. \tag{2.25}$$

∎

The next proposition identifies Woronowicz's formula for the quantum dilogarithm with the Laplace transform of the function $G(\xi, z)$.

Proposition 2.6. *When $\tau > 0$ and $z \in \mathbb{C}$ with $|\mathrm{Im}(z)| < \pi$, we have*

$$W_{\frac{1}{\tau}}(z) = -\frac{1}{\tau}\mathrm{Li}_2(-e^z) - 2\pi i (\mathcal{L}G)(\tau, z). \tag{2.26}$$

Proof. We have

$$\begin{aligned} W_{\frac{1}{\tau}}(z) &= \int_{-\infty}^{\infty} \frac{\log(1+e^{\xi/\tau})}{e^{\xi-z}+1}\,\mathrm{d}\xi = \int_{-\infty}^{0} \frac{\log(1+e^{\xi/\tau})}{1+e^{\xi-z}}\,\mathrm{d}\xi + \int_{0}^{\infty} \frac{\log(1+e^{\xi/\tau})}{1+e^{\xi-z}}\,\mathrm{d}\xi \\ &= \int_0^\infty \left(\frac{\log(1+e^{\xi/\tau})}{1+e^{\xi-z}} + \frac{\log(1+e^{-\xi/\tau})}{1+e^{-\xi-z}}\right)\mathrm{d}\xi \\ &= \frac{1}{\tau}\int_0^\infty \frac{\xi}{1+e^{\xi-z}}\,\mathrm{d}\xi + \int_0^\infty \log(1+e^{-\xi/\tau})\left(\frac{1}{1+e^{\xi-z}} + \frac{1}{1+e^{-\xi-z}}\right)\mathrm{d}\xi \\ &= \frac{1}{\tau}\int_0^\infty \frac{\xi}{1+e^{\xi-z}}\,\mathrm{d}\xi - \int_0^\infty \sum_{n=1}^{\infty}\frac{(-1)^n}{n} e^{-n\xi/\tau}\left(\frac{1}{1+e^{\xi-z}} + \frac{1}{1+e^{-\xi-z}}\right)\mathrm{d}\xi \end{aligned}$$

where the last equality follows from expanding the logarithm. Rescaling $\xi \to \xi/n$ and using the identity

$$\int_0^\infty \frac{\xi}{1+e^{\xi-z}}\,\mathrm{d}\xi = -\mathrm{Li}_2(-e^z) \tag{2.27}$$

(which can be verified for instance by integrating by parts) it follows that

$$\begin{aligned} W_{\frac{1}{\tau}}(z) + \frac{1}{\tau}\mathrm{Li}_2(-e^z) &= -\int_0^\infty e^{-\xi/\tau} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\left(\frac{1}{1+e^{\frac{\xi}{n}-z}} + \frac{1}{1+e^{-\frac{\xi}{n}-z}}\right)\mathrm{d}\xi \\ &= -2\pi i \int_0^\infty e^{-\xi/\tau} G(\xi, z)\,\mathrm{d}\xi = -2\pi i (\mathcal{L}G)(\tau, z) \end{aligned}$$

where we used equation (1.9). ∎

We are now ready to give a proof of Theorem 1.3.

Proof of Theorem 1.3. Propositions 2.4 and equation (2.26) imply that

$$-2\pi i \log \Phi_{\mathsf{b}}\Big(\frac{z}{2\pi \mathsf{b}}\Big) = W_{\frac{1}{\tau}}(z) = -\frac{1}{\tau}\mathrm{Li}_2(-e^z) - 2\pi i(\mathcal{L}G)(\tau, z) \tag{2.28}$$

which is equivalent to (1.16). ∎

3 Useful properties of the dilogarithm function

In this section we collect some useful properties of the quantum dilogarithm function

$$\Phi_{\mathsf{b}}(z) = \frac{(e^{2\pi b(z+c_{\mathsf{b}})}; q)_\infty}{(e^{2\pi b^{-1}(z-c_{\mathsf{b}})}; \tilde{q})_\infty} \tag{3.1}$$

where

$$q = e^{2\pi i \mathsf{b}^2}, \quad \tilde{q} = e^{-2\pi i \mathsf{b}^{-2}}, \quad c_b = \frac{i}{2}(\mathsf{b} + \mathsf{b}^{-1}), \quad \mathrm{Im}(\mathsf{b}^2) > 0. \tag{3.2}$$

The above function can also be defined in the lower half-plane $\mathrm{Im}(\mathsf{b}^2) < 0$ using the symmetry

$$\Phi_{\mathsf{b}}(z) = \Phi_{\mathsf{b}^{-1}}(z), \tag{3.3}$$

and remarkably the function of $\mathsf{b}^2 \in \mathbb{C} \setminus \mathbb{R}$ admits an extension to $\mathsf{b}^2 \in \mathbb{C}' = \mathbb{C} \setminus (-\infty, 0]$. The integral representation (1.5) implies the additional symmetry

$$\Phi_{\mathsf{b}}(z) = \Phi_{-\mathsf{b}}(z). \tag{3.4}$$

$\Phi_{\mathsf{b}}(z)$ is a meromorphic function of z with

$$\text{poles:} \quad c_{\mathsf{b}} + i\mathbb{N}\mathsf{b} + i\mathbb{N}\mathsf{b}^{-1}, \quad \text{zeros:} \quad -c_{\mathsf{b}} - i\mathbb{N}\mathsf{b} - i\mathbb{N}\mathsf{b}^{-1}.$$

It satisfies the inversion relation

$$\Phi_{\mathsf{b}}(z)\,\Phi_{\mathsf{b}}(-z) = e^{\pi i z^2}\,\Phi_{\mathsf{b}}(0)^2, \quad \Phi_{\mathsf{b}}(0) = q^{\frac{1}{24}}\tilde{q}^{-\frac{1}{24}}.$$

It is a quasi-periodic function satisfying

$$\Phi_{\mathsf{b}}(z - i\mathsf{b}/2) = (1 + e^{2\pi \mathsf{b} z})\,\Phi_{\mathsf{b}}(z + i\mathsf{b}/2) \tag{3.5}$$

which, due to the symmetry (3.3), implies a second functional equation

$$\Phi_{\mathsf{b}}(z - i\mathsf{b}^{-1}/2) = (1 + e^{2\pi \mathsf{b}^{-1} z})\,\Phi_{\mathsf{b}}(z + i\mathsf{b}^{-1}/2) \tag{3.6}$$

obtained from (3.5) by replacing b by b^{-1}. Note that equation (3.5) (resp. (3.6)) implies that the function $\Phi_{\mathsf{b}}(z/(2\pi\mathsf{b}))$ satisfies equation (1.1) (resp. (1.18)) with $\tau = \mathsf{b}^2$.

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