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On finite type 3-manifold invariants IV: comparison of definitions

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Abstract

The present paper is a continuation of [**Ga**], [**GL1**] and [**GO**]. Using a key lemma we compare two currently existing definitions of finite type invariants of oriented integral homology spheres and show that type $3m$ invariants in the sense of Ohtsuki [**Oh**] are included in type m invariants in the sense of the first author [**Ga**]. This partially answers question 1 of [**Ga**]. We show that type $3m$ invariants of integral homology spheres in the sense of Ohtsuki map to type $2m$ invariants of knots in S^3, thus answering question 2 from [**Ga**].

1. *Introduction*

Definitions. We begin by recalling some definitions from [**Ga**], [**GO**] and [**GL1**] and establishing some notation that will be followed in the present paper.

All 3-manifolds considered are *oriented integral homology spheres*. A link L in an integral homology sphere is called *algebraically split* (denoted AS) if the linking numbers between its components vanish. A link L is called *boundary* if each component bounds a Seifert surface, and the Seifert surfaces are disjoint from each other. A (integral) *framing* f for a link L in an integral homology sphere M is a sequence of integers indicating the linking numbers of longitudes of L with the corresponding components. This requires a choice of orientation, but if one gives the longitudes the parallel orientation then the framing number is independent of the choice of orientation. A link is called *unit-framed* if the framing on each component is ± 1. A framed link (L,f) is called *AS-admissible* (resp. *B-admissible*) if it is AS (resp. boundary) and unit-framed. Let $\mathscr{M}$ denote the $\mathbb{Q}$-vector space generated by the diffeomorphism classes of oriented integral homology 3-spheres. Let

$$[M,L,f] = \sum_{L' \subseteq L} (-1)^{|L'|} M_{L',f'}, \tag{1}$$

where f denotes a framing of L, f' the restriction of f to L' and $M_{L,f}$ Dehn surgery on the framed (unoriented) link L in M. $|L|$ denotes the number of components. Let $\mathscr{F}^{as}_n \mathscr{M}$ (resp. $\mathscr{F}^{b}_n \mathscr{M}$) be the subspace of $\mathscr{M}$ spanned by all $[M,L,f]$ for AS-

admissible (resp. B-admissible) links L of n components in integral homology spheres M. Obviously $\mathscr{F}^{as}_{*}\mathscr{M}, \mathscr{F}^{b}_{*}\mathscr{M}$ are decreasing filtrations on $\mathscr{M}$.

We call a map $v: \mathscr{M} \to \mathbb{Q}$ a *type* m invariant of integral homology spheres if $v(\mathscr{F}^{as}_{m+1}\mathscr{M}) = 0$, see [**Oh**]. Similarly, we call v a *b-type* m invariant of integral homology spheres if $v(\mathscr{F}^{b}_{m+1}\mathscr{M}) = 0$, see [**Ga**]. We denote the space of type m invariants of integral homology spheres by $\mathscr{F}_m\mathcal{O}$.

Statement of the results

Theorem 1. *For every non-negative integer n we have*

$$\mathscr{F}^{b}_{n}\mathscr{M} \subseteq \mathscr{F}^{as}_{3n}\mathscr{M}. \tag{2}$$

Corollary 1·1. *Type* $3m$ *invariants of integral homology spheres are included in b-type m invariants of integral homology spheres.*

Corollary 1·2. *Let* λ *be a type* $3m$ *invariant of integral homology spheres and K a knot in an integral homology sphere M. Then the map* $n \to \lambda(M_{K,1/n})$ *(defined for every non-zero integer n) is a polynomial in n of degree m.*

Theorem 2. *We have the following equality of filtrations:*

$$\mathscr{F}^{as}_{3n}\mathscr{M} = \bigcap_{k \geqslant 0} (\mathscr{F}^{b}_{n}\mathscr{M} + \mathscr{F}^{as}_{k}\mathscr{M}). \tag{3}$$

Assuming that for every $n \geqslant 0$ *there is a* $k \geqslant 0$ *such that* $\mathscr{F}^{as}_{k}\mathscr{M} \subseteq \mathscr{F}^{b}_{n}\mathscr{M}$, *we obtain* $\mathscr{F}^{as}_{3n}\mathscr{M} = \mathscr{F}^{b}_{n}\mathscr{M}$.

Conjecture 1. *For every* $n \geqslant 0$ *there is a* $k \geqslant 0$ *such that* $\mathscr{F}^{as}_{k}\mathscr{M} \subseteq \mathscr{F}^{b}_{n}\mathscr{M}$.

Recall from [**Ga**] that there is a well-defined map $\Phi: \mathscr{F}_n\mathcal{O} \to \mathscr{F}_{n-1}\mathscr{V}$, where $\mathscr{F}_{n-1}\mathscr{V}$ denotes the space of type $n-1$ invariants of knots in S^3. In [**Ga**] asked the question whether Φ descends to a map $\mathscr{F}_{3n}\mathcal{O} \to \mathscr{F}_{2n}\mathscr{V}$. In [**GrLi**] it was shown that Φ descends to a map $\mathscr{F}_n\mathcal{O} \to \mathscr{F}_{n-2}\mathscr{V}$ if $n \geqslant 4$. In [**GL1**] we showed that Φ descends to a map $\mathscr{F}_{5n+1}\mathcal{O} \to \mathscr{F}_{4n}\mathscr{V}$. Recently N. Habegger [**Ha**] gave a positive answer to the question. We will give a different proof along the lines of our argument in [**GL1**]. We shall first show:

Theorem 3. *If L is an AS-admissible link containing a* $2m+1$*-component trivial sublink, then* $[S^3, L, f] \in \mathscr{F}^{as}_{3m}\mathscr{M}$.

As in [**GL1**], Theorem 3 implies:

Theorem 4 [**Ha**]. *The map* Φ *above factors through a map:*

$$\mathscr{F}_{3m}\mathcal{O} \to \mathscr{F}_{2m}\mathscr{V}. \tag{4}$$

Question 1. Can every integral homology sphere be obtained by Dehn surgery on a unit-framed *boundary* link in S^3?

Remark 1·3. It was recently shown by [**Au**] and [**GoLu**] that there are integral homology spheres that cannot be obtained by surgery on a knot. A positive answer to Question 1 will be given in forthcoming work of the authors [**GL3**].

In Section 2 we prove a key Lemma 2·1. In Section 3·1 we give a proof of Theorem 1 and Corollaries 1·1 and 1·2. In Section 3·2 we give a proof of Theorem 2. In Section 4 we give a proof of Theorems 3 and 4.

2. *A key lemma*

This section is devoted to the proof of the following lemma, which is the key to the proof of Theorems 1 and 3. Note that all links considered in the rest of the paper are unit-framed.

Lemma 2.1. *Let L be an AS-admissible link containing a sublink with two components k_1, k_2 which bound discs D_1, D_2 in U so that $D_1 \cap D_2$ is a single arc α in the interior of D_1 (a ribbon intersection). Suppose U is a ball containing $D_1 \cap D_2$ whose intersection with L is as pictured in Fig.* 1. *Let L_α be the link obtained from K by replacing k_1 with k_1', a small circle in D_1 about α. See Fig.* 2. *Then*

$$[S^3, L, f] = [S^3, L_\alpha, f] + \text{a linear combination of } [S^3, L(\nu), f(\nu)], \tag{5}$$

where each link $L(\nu)$ contains L as a proper sublink such that $L(\nu) - L \subseteq U$ (we will say such links are subordinate to L).

Proof. Let L_{twist} be the link obtained from L by replacing $L \cap U$ with Fig. 3. We first show:

Claim 2·2.

$$[S^3, L_{twist}, f] = -[S^3, L, f] + 2[S^3, L_\alpha, f] + \text{a linear combination of } [S^3, L(\nu), f(\nu)], \tag{6}$$

where the $\{L(\nu)\}$ are subordinate to L.

Now apply theorem 5 from [**GL1**] to the disc D_1, where we use three bands. The last two are the ones seen penetrating D_1 in Fig. 4 and the first one contains all the other strands of $L_{untwist}$ penetrating D_1. Then theorem 5 implies that $[S^3, L_{untwist}, f]$ is a sum of six terms in which D_1 is replaced by smaller subdiscs and others in which D_1 is replaced by more than one subdisc. These last terms are all subordinate to L. Three of the first six terms are just $[S^3, L, f]$, $[S^3, L_{twist}, f]$ and $[S^3, L_{notwist}, f]$, where $L_{notwist}$ is obtained from $L_{untwist}$ by replacing D_1 with a subdisc which only encloses the two penetrations of the third band β. But $[S^3, L_{untwist}, f] = [S^3, L_{notwist}, f] = 0$ because we can obviously isotop β to miss D_1 and then k_2 bounds a disc in the complement of the rest of the link. Two of the remaining three terms are $-[S^3, L_\alpha, f]$ and the last term is given by a link obtained from L by replacing D_1 with a subdisc disjoint from β but intersected by all the other strands of L which intersect D_1. As above this term vanishes, since k_2 bounds a disc in the complement of the rest of the link. ▌

Claim 2·3. $[S^3, L_{twist}, f] = [S^3, L, f]$ + *a linear combination of* $[S^3, L(\nu), f(\nu)]$, *where the* $\{L(\nu)\}$ *are all subordinate to* L.

Proof of claim 2·3. After an isotopy, $L_{twist} \cap U$ appears as in Fig. 5. Three crossing changes, from Fig. 5, will convert this into $L \cap U$. These crossing changes are effected by surgeries along three circles. In Fig. 6 we see $L \cap U$ with the three circles added. Thus we conclude that

$$[S^3, L_{twist}, f] = [S^3, L, f] + \text{a linear combination of } [S^3, L(\nu), f(\nu)], \tag{7}$$

where the $L(\nu)$ consist of L together with one or more of the extra circles in Fig. 6. ▌

Obviously Lemma 2·1 follows from Claims 2·2 and 2·3.

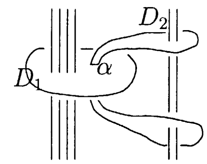

Fig. 1. Shown here is the intersection of L with U. Note that $k_i = \partial D_i$ for $i = 1, 2$ and that the discs D_1 and D_2 intersect in a ribbon arc α.

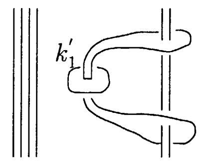

Fig. 2. Shown here is the intersection of U with the link L_α obtained by changing k_1 to k_1'. Note that $L_\alpha - L \subseteq U$.

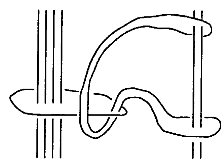

Fig. 3. Shown here is the intersection of U with the link L_{twist}. Note that $L_{twist} - L \subseteq U$.

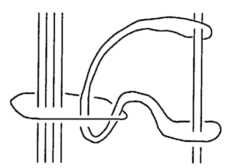

Fig. 4. Shown here is the intersection of U with the link $L_{untwist}$. Note that $L_{untwist} - L \subseteq U$.

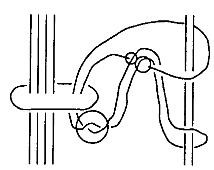

Fig. 5. Shown here is the result of an isotopy fixing the boundary of the intersection of U with the link L_{twist}. Note that $L_{twist} - L \subseteq U$. Also circled are 3 crossings to be changed.

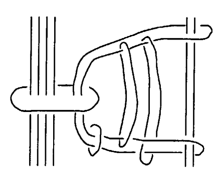

Fig. 6. Yet another intersection of U with a link.

3. *Proof of Theorems* 1 *and* 2

This section is devoted to the proof of Theorems 1 and 2.

3·1. *Proof of Theorem* 1. We divide the proof of Theorem 1 into 6 steps. We begin

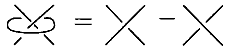

Fig. 7. A relation in $\mathcal{M}$. Here an unknot circles the same component of a link, with linking number zero.

Fig. 8. A surface of genus 2 (and 4 bands) whose boundary is an unknot.

with some definitions. A pair of links (L, L_b) is called *n-boundary* if L_b is a sublink of $L \subseteq S^3$ and L_b is a boundary n component link in the complement of $L - L_b$. The *goodness* $k(L, L_b)$ of an n-boundary pair is the number of components of $L - L_b$. The *genus* $g(L, L_b)$ of an n-boundary pair (L, L_b) is the minimal total genus of disjoint Seifert surfaces of L_b in the complement of $L - L_b$.

Step 1. $\mathcal{F}_n^b \mathcal{M}$ is generated by elements of the form $[S^3, L, f]$ for all n-boundary pairs (L, L_b).

Proof. Let $\widetilde{\mathcal{F}_n^b \mathcal{M}}$ denote the subspace spanned by all $[S^3, L, f]$ for all n-boundary pairs (L, L_b). We first show $\mathcal{F}_n^b \mathcal{M} \subseteq \widetilde{\mathcal{F}_n^b \mathcal{M}}$. Let L be an n-component boundary link in an integral homology sphere M. Write $M = S^3_{L', \delta}$ for an algebraically split unit-framed link L' in S^3. Since L is an n-component boundary link in M, we can assume that L bounds Seifert surfaces Σ such that $\Sigma \cap L'$ is empty (here we mean by L' the corresponding tubes of M). Thus $L \cup L'$ becomes a link in S^3 and $(L \cup L', L)$ is an n-boundary pair. We now proceed by upward induction on the number of components $|L'|$ of L'. If L' is empty, we are done by definition. Otherwise, using (1) we get

$$[S^3, L \cup L', f \cup \delta] = \pm [M, L, f] + \sum_{L'' \subsetneq L'} \pm [S^3_{L'', \delta''}, L, f]. \tag{8}$$

By induction, all the terms in the summation on the right-hand belong to $\widetilde{\mathcal{F}_n^b \mathcal{M}}$ and so we conclude that $[M, L, f]$ does also.

The fact that $\widetilde{\mathcal{F}_n^b \mathcal{M}} \subseteq \mathcal{F}_n^b \mathcal{M}$ is an immediate consequence of (8). ▌

Let (L, L_b) be an n-boundary pair. We want to show that $[S^3, L, f] \in \mathcal{F}_{3n}^{as} \mathcal{M}$. We proceed by primary downward induction on the goodness $k(L, L_b)$, and secondary upward induction on the genus $g(L, L_b)$. If $k(L, L_b) \geqslant 2n$ we are done by definition. If $g(L, L_b) = 0$ we are also done, since $[S^3, L, f] = 0$.

Step 2. We may assume that the components of $L - L_b$ are all unknotted.

Proof. This can be achieved by crossing changes in $L - L_b$ and, since this is the result of *a* ± 1-surgery along a small circle C enclosing the crossing, the change to $[S^3, L, f]$ is given by an element $[S^3, L \cup C, f \cup \pm 1]$; see Fig. 7. Since $(L \cup C, L_b)$ remains an n-boundary pair, whose goodness is one more than the goodness of (L, L_b), it follows by the primary inductive hypothesis that $[S^3, L \cup C, f \cup \pm 1] \in \mathcal{F}_{3n}^{as} \mathcal{M}$. ▌

Step 3. Suppose that $L_b = \partial \Sigma_b$, where Σ_b is a union of Seifert surfaces in the complement of $L - L_b$. We may assume that Σ_b is embedded in a standard, almost planar (except for the necessary band crossings) way; see Fig. 8.

Proof. This can be achieved by band crossing changes, which are the result of a ± 1-surgery along a circle enclosing the band crossing. This surgery will introduce

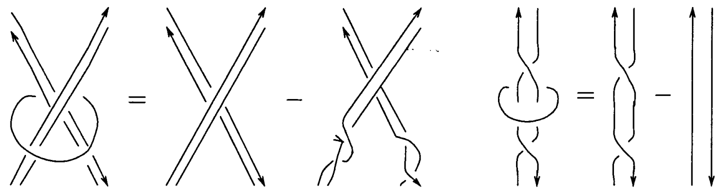

Fig. 9. A few more identities in $\mathscr{M}$.

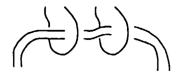

Fig. 10. A band of a surface penetrating two pieces of discs.

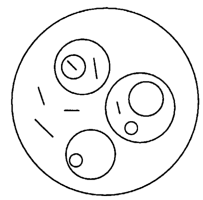

Fig. 11. An intersection of a disc with the surface Σ_b.

some extra twists into the bands, but further surgery along circles enclosing these twists will remove them; see Fig. 9. As in step 2, the changes to $[S^3, L, f]$ are linear combinations of $[S^3, L', f']$ for n-boundary pairs (L', L_b) with strictly higher goodness than that of (L, L_b). By appealing to the primary inductive hypothesis, the changes to $[S^3, L, f]$ lie in $\mathscr{F}^{as}_{3n}\mathscr{M}$. ▮

Let $\{K_i\}$ denote the components of $L - L_b$. Since by step 2 they are unknotted, we may choose embedded discs D_i so that $K_i = \partial D_i$. Furthermore, since Σ_b is just a thickening of a wedge of circles, we may choose the D_i so that their intersections with Σ_b consist of a number of transverse penetrations of the interiors of the D_i by the bands of Σ_b; see Fig. 10. We will be interested in counting the number of 'band penetrations'.

Step 4. We may assume that every band of Σ_b penetrates at least one D_i.

Proof. Suppose that a band β from one of the components Σ of Σ_b penetrates no D_i. We will show how to replace Σ by a surface of lower genus and then appeal to the secondary inductive hypothesis. This will also involve a number of changes to L, but using the primary inductive hypothesis each of these changes will only be by an element of $\mathscr{F}^{as}_{3n}\mathscr{M}$.

Let C be the circle in Σ which goes once around the band β. C bounds an obvious disc in the plane containing Σ. We push this disc slightly off the plane (except on C) to obtain a disc D such that $D \cap \Sigma = \varnothing$ and $\partial D = C$. Now, since $C \cap \cup_i D_i = \varnothing$, $D \cap \cup_i D_i$ consists of circles and interior arcs; see Fig. 11. If $D \cap \cup_i D_i = \varnothing$, then we can perform a surgery on σ along D to obtain the desired surface of lower genus. Thus we only have to see how to remove these intersections. We claim that we can first remove the arc intersections and then (by using an innermost circle argument) remove the circle intersections. In fact we only have to remove the arcs since it is only necessary that $D \cap (L - L_b) = \varnothing$. Suppose α is an arc and a component of $D_j \cap D$. We

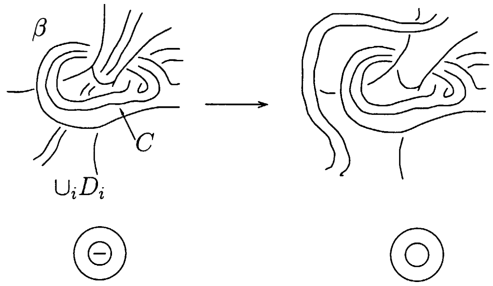

Fig. 12. In the left side of this picture is shown a band β that does not penetrate any of the discs D_i its associated circle C. The disc which the circle C bounds intersects the union $\cup_i D_i$ as shown in the lower part of the left-hand side. After a band change move, shown on the right-hand side, we can arrange so that the new disc of the new circle has one less band intersection with the union of the discs $\cup_i D_i$.

can perform an isotopy of D_j to move α adjacent to the boundary C of D. This may require α to cross some circle components of $D \cap \cup D_i$ which means that D_j may cut through some D_i during the isotopy. If $i = j$ the result will be that D_j is now only immersed, but this will not be important. We have only a regular homotopy of D_j but still an isotopy of K_j. Now a neighbourhood of α in D_j is a band which is adjacent to the band β. If we change this band crossing, the result will be to eliminate α. As above, this crossing change can be produced by a ± 1-surgery on a small circle enclosing the two bands and so the change in $[S^3, L, f]$ is, by primary induction, an element of $\mathscr{F}^{as}_{3n}\mathscr{M}$. Note that we have not changed Σ. See Fig. 12. ▮

Step 5. We may assume that each disc D_i has at most two band penetrations.

Proof. We want to apply theorem 5 of [**GL1**], to every disc D_i. The bands in theorem 5 are all but one the bands of Σ_b penetrating D_i and the remaining band consists of all the strands of $L - L_b$ penetrating D_i. Thus $[S^3, L, f]$ is a sum of elements in which D_i is replaced by one or more discs inside D_i containing no more than two bands of Σ_b. Thus $L - L_b$ is changed, but not Σ_b and so the change in $[S^3, L, f]$ comes from n-boundary pairs of higher goodness than (L, L_b), and thus lie in $\mathscr{F}^{as}_{3n}\mathscr{M}$, by the primary inductive hypothesis. ▮

Let $\{\Sigma_j | 1 \leqslant j \leqslant n\}$ denote the connected components of Σ_b.

Step 6. We may assume that if Σ_j has genus one and a band of Σ_j penetrates only one disc D_i then D_i is penetrated by no other band of Σ_b.

Proof. Suppose one of the bands β of Σ_j penetrates D_i once. Let L_β be defined from L by replacing k_i with a small meridian circle about β. Then Step 6 will be confirmed by:

Claim 3·1. *We have:*

$$[S^3, L, f] = [S^3, L_\beta, f] \bmod \mathscr{F}^{as}_{3n}\mathscr{M}. \tag{9}$$

Proof of Claim 3·1. This claim follows from Lemma 2·1 as follows. We can draw $L \cap U$, where U is a ball containing Σ_j, as in Fig. 13. If we expunge Σ_j from the picture we have exactly the situation in Figs. 1 and 2 of Lemma 2·1. If we put Σ_j back into any of the $L(\nu)$ of Lemma 2·1, we see that it may intersect the additional components

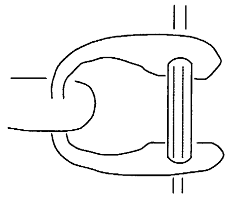

Fig. 13. An intersection of L with U. Shown also is the genus 1 surface Σ_j.

of $L(\nu)$. However, we can add tubes to Σ_j to eliminate these intersections and, since none of other Σ_i intersect U, we may conclude from Lemma 2·1 that

$$[S^3, L, f] = [S^3, L_\beta, f] + \text{a linear combination of } [S^3, L(\nu), f(\nu)], \tag{10}$$

where each of the pairs $(L(\nu), L_b)$ are n-boundary with strictly higher goodness than that of (L, L_b). By the primary inductive hypothesis, we conclude the proof of Claim 3·1. ∎

We can now complete the proof of Theorem 1. We define:

r_2 = number of D_i penetrated by two bands,

r_1 = number of D_i penetrated by one band,

m_2 = number of $\sum_j$ of genus > 1,

m_1 = number of $\sum_j$ of genus $= 1$,

p = number of bands of surfaces of genus one penetrating only one disc.

Obviously $k \geqslant r_1 + r_2$ and $n = m_1 + m_2$. By Step 5, the total number of band penetrations is $2r_2 + r_1$. From Step 4 we thus conclude $2r_2 + r_1 \geqslant 4m_1 + 4m_2 - p$ and, by Step 6, we have $r_1 \geqslant p$. Adding these two equations together gives $2r_1 + 2r_2 \geqslant 4m_1 + 4m_2$ and so $k \geqslant r_1 + r_2 \geqslant 2m_1 + 2m_2 = 2n$. This concludes the proof of Theorem 1.

Proof of Corollary 1·1. If $v: \mathscr{M} \to \mathbb{Q}$ is of type $3m$, then $v(\mathscr{F}^{as}_{3m+1}\mathscr{M}) = 0$, and therefore, by Theorem 1, $v(\mathscr{F}^{b}_{m+1}\mathscr{M}) = 0$.

Proof of Corollary 1·2. It follows by Exercise 4·2 of [**Ga**], using the remark that the j-fold parallel (with zero framing)[1] of a knot K in a integral homology sphere M is a boundary link of j components.

3·2. *Proof of Theorem* 2. The proof will use the AS and IHX relations on $\mathscr{M}$ proved in [**GO**]. We recall the notation and terminology from [**GO**]. A Chinese manifold character is a trivalent graph with vertex orientation. The degree of a Chinese manifold character is the number of edges of it. Let $\mathscr{C}\mathscr{M}$ denote the vector space on the set of Chinese manifold characters and let $\mathscr{B}\mathscr{M}$ be the quotient space $\mathscr{C}\mathscr{M}/\{AS, IHX\}$, where we quotient by the AS and the IHX relations of [**GO**]. Note that $\mathscr{B}\mathscr{M}$ is a graded (and therefore a filtered) space. In general, for a filtered space $\mathscr{F}_*$ space, we denote the associated graded space by $\mathscr{G}_*$ space. Examples of filtered spaces that we will consider here are $\mathscr{B}\mathscr{M}$ and $\mathscr{F}^{as}_*\mathscr{M}$. With this notation and terminology, we recall the following theorem from [**GO**]:

THEOREM 5 [**GO**]. *There is an onto map* $O^*_m: \mathscr{G}_m\mathscr{B}\mathscr{M} \to \mathscr{G}^{as}_m\mathscr{M}$.

[1] We thank the referee for suggesting to us the term 'parallel'.

We need:

LEMMA 3·2. *Let* $\Gamma \in \mathscr{G}_{3m}\mathscr{B}\mathscr{M}$ *be a Chinese manifold character of* $3m$ *edges. Then* $O^*_{3m}(\Gamma) \in \mathscr{G}^{as}_{3m}\mathscr{M}$ *actually lies in* $\mathscr{F}^{\,b}_{m}\mathscr{M}$.

Proof. Choose a circuit in Γ (that is a sequence of edges, the beginning of which is the end of the previous, such that the end of the last is the beginning of the first, and such that the edges in the sequence are distinct). Colour the edges of the circuit red. Thinking of the red coloured edges of Γ as the external circle, and using repeatedly the *IHX* relation of [**GO**], we can write $O^*_m(\Gamma)$ as a linear combination of values (under O^*_m) of chord diagrams based on the red circle. By counting degrees, we see that each of the above-mentioned chord diagrams have m chords. Now using Lemma 3·4 from [**Ga**] we see that the pair (O^*_m (chord diagram), O^*_m (m-chords)) is an m-boundary pair, from which our conclusion follows.

Proof of Theorem 2. We can now finish the proof of Theorem 2 as follows. Theorem 5 and Lemma 3·2 show that $\mathscr{F}^{as}_{3n}\mathscr{M} = \mathscr{F}^{\,b}_{n}\mathscr{M} + \mathscr{F}^{as}_{3n+1}\mathscr{M}$. Iterating this equation we obtain (3), as required.

4. *Proof of Theorem* 3

Let (L,f) be an *AS*-admissible link containing a trivial sublink $L_{trivial}$ of $2m+1$ components. We will use downward induction on the number r of components of $L - L_{trivial}$ to show first that $[S^3, L, f] \in \mathscr{F}^{as}_{3m+2}\mathscr{M}$. Obviously if $r \geqslant m+1$ we are done. We refer the reader to the proof of theorem 7 in [**GL1**] for the first part of the argument. Let us denote the components of $L_{trivial}$ by $\{L_i\}^{2m}_{i=1}$. Then $L_i = \partial D_i$, where the $\{D_i\}$ are disjoint discs. We showed in [**GL1**] that we may assume that $L - L_{trivial}$ consists of components $\{l_k\}$ such that each l_k is either of the form σ_{ij}, $i \neq j$ (where σ_{ij} is pictured in Fig. 15 of [**GL1**]) or a band sum of two σ_{ij}. We will refer to l_k as *simple* in the former and *composite* in the latter case. Note that σ_{ij} intersects the discs D_i and D_j, but no others. We will say that L_i is k-special if the only component l_s of $L_{trivial}$ intersecting D_i is l_k, and if $l_k = \sigma_{ij} \# \sigma_{rs}$ then $i \neq r, s$.

We now use Lemma 2·1 to make an important observation.

Claim 4·1. *We may assume that if* L_i *is* k*-special then* l_k *is simple.*

Proof. Suppose that l_k is composite. Then there is a ball U which intersects L as in Fig. 13. But we can redraw this so that it looks like Fig. 1 of Lemma 2·1, with the two component distinguished sublink (l_k, L_i) of L (substituted for (k_1, k_2) in Lemma 2·1). For the subordinate links $L(\nu)$ of Lemma 2·1, we see that $[S^3, L(\nu), f(\nu)] \in \mathscr{F}^{O}_{3m}\mathscr{M}$ by induction. Thus, using Lemma 2·1, we can assume that each l_k is simple. ∎

We now complete the proof of Theorem 3 by a counting argument. We define:

$$\begin{aligned} a &= \text{number of simple } l_k,\\ b &= \text{number of composite } l_k,\\ c &= \text{number of } k_i \text{ which are } k\text{-special for some } k,\\ 2m &= |L_{trivial}|,\\ d &= 2m+1-c,\\ r &= |L - L_{trivial}|. \end{aligned}$$

Obviously $r = a+b$. As pointed out in [**GL1**], we may as well assume that every D_i is intersected by at least one l_k (or else $[S^3, L, f] = 0$). Counting intersections of the $\{l_k\}$ with the $\{D_i\}$, we have $2a+4b \geqslant c+2d$. From Claim 4·1 we obtain the inequality $2a \geqslant c$. Adding these last two inequalities we get $4a+4b \geqslant 2c+2d$ or $2r \geqslant 2m+1$ or else $r \geqslant m+1$.

This concludes the proof that $[S^3, L, f] \in \mathscr{F}^{as}_{3m+2}\mathscr{M}$. Using corollary 3·5 of [**GL1**] (see also corollary 1·6 of [**GO**]) we deduce that $\mathscr{F}^{as}_{3m+2}\mathscr{M} = \mathscr{F}^{as}_{3m}\mathscr{M}$, which concludes the proof of Theorem 3.

Proof of Theorem 4. This follows *verbatim* as in proposition 3·9 of [**GL1**], using Theorem 3 of the present paper.

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