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DIFFERENCE AND DIFFERENTIAL EQUATIONS FOR THE COLORED JONES FUNCTION

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ABSTRACT

The colored Jones function of a knot is a sequence of Laurent polynomials. It was shown by Le and the author that such sequences are q-holonomic, that is, they satisfy linear q-difference equations with coefficients Laurent polynomials in q and q^n. We show from first principles that q-holonomic sequences give rise to modules over a q-Weyl ring. Frohman–Gelca–LoFaro have identified the latter ring with the ring of even functions of the quantum torus, and with the Kauffman bracket skein module of the torus. Via this identification, we study relations among the orthogonal, peripheral and recursion ideal of the colored Jones function, introduced by the above mentioned authors. In the second part of the paper, we convert the linear q-difference equations of the colored Jones function in terms of a hierarchy of linear ordinary differential equations for its loop expansion. This conversion is a version of the WKB method, and may shed some information on the problem of asymptotics of the colored Jones function of a knot.

Keywords: Holonomic function; colored Jones function; recursion ideal; peripheral ideal; orthogonal ideal; Kauffman bracket skein module; loop expansion; hierarchy of ODE; WKB.

Mathematics Classification 2000: Primary 57N10; Secondary 57M25

1. Introduction

1.1. *The colored Jones function and its loop expansion*

The *colored Jones function* of a knot K in 3-space is a sequence

$$J_K : \mathbb{Z} \longrightarrow \mathbb{Z}[q^{\pm/2}]$$

of Laurent polynomials that encodes the *Jones polynomial* of a knot and its parallels [20, 31]. Technically, $J_{K,n}(q)$ is the quantum group invariant using the n-dimensional representation of $\mathfrak{sl}_2$ for $n \geq 0$, normalized by $J_{\text{unknot},n}(q) = [n]$ (where $[n] = (q^{n/2} - q^{-n/2})/(q^{1/2} - q^{-1/2})$), and extended to integer indices by $J_{K,n} = -J_{K,-n}$.

In the spring of 2005, Le and the author proved that the colored Jones function is *q-holonomic* [11]. In other words, it satisfies a *linear q-difference equation* with coefficients Laurent polynomials in q and q^n.

In [28], Rozansky introduced a *loop expansion* of the colored Jones function. Namely, he associated to a knot K an invariant

$$J_K^{\mathrm{rat}}(q,u) = \sum_{k=0}^{\infty} Q_{K,k}(u)(q-1)^k \in \mathbb{Q}'(u)[[q-1]],$$

where $\mathbb{Q}'(u)[[q-1]]$ is the ring of power series in $q-1$ with coefficients in the ring $\mathbb{Q}'(u)$ of rational functions in u which do not have a pole at $u=1$, and

$$Q_{K,k}(u) = \frac{P_{K,k}(u)}{\Delta_K(u)^{2k+1}},$$

where $P_{K,k}(u) \in \mathbb{Q}[u^{\pm}]$ and $\Delta_K(t)$ is the *Alexander polynomial* normalized by $\Delta_K(t) = \Delta_K(t^{-1})$, $\Delta_K(1) = 1$ and $\Delta_{\mathrm{unknot}}(t) = 1$.

The relation between J_K^{rat} and J_K is the following equality, valid in the power series ring $\mathbb{Q}[[q-1]]$

$$[n] J_K^{\mathrm{rat}}(q,q^n) = J_{K,n}(q) \in \mathbb{Q}[[q-1]], \tag{1.1}$$

for all $n > 0$. Notice that $\Delta_K(1) = 1$, thus $1/\Delta_K(u) \in \mathbb{Q}'(u)$ and $1/\Delta_K(q^n)$ can always be expanded in power series of $q-1$. Notice moreover that J_K^{rat} determines J_K and vice-versa, via the above equation.

In this paper, we convert linear q-difference equations for the colored Jones function J_K into a *hierarchy of linear differential equations* for the loop expansion J_K^{rat}.

Moreover, we study holonomicity of the colored Jones function from the point of view of quantum field theory, and compare it with the skein theory approach of the colored Jones function initiated by Frohman and Gelca.

The paper was written in the spring of 2003, following stimulating conversations with A. Sikora, who kindly explained to the author the work of Gelca–Frohman–LoFaro and others on the skein theory approach to the colored Jones function. The author wishes to thank A. Sikora for enlightening conversations. The paper remained as a preprint for over two years. The current version is substantially revised to take into account the recent developments of the last two years.

1.2. *Holonomic functions and the q-Weyl ring $\mathcal{C}$*

A holonomic function $f(x)$ in one continuous variable x is one that satisfies a nontrivial linear differential equation with polynomial coefficients.

In this section, we will see that the notion of holonomicity for a sequence of Laurent polynomials *naturally leads* to a q-Weyl ring $\mathcal{C}$ defined below.

Holonomicity was introduced by Bernstein [1, 2] in relation to algebraic geometry, D-modules and differential Galois theory. In a stroke of brilliance, Zeilberger noticed that holonomicity can be applied to verify, in a systematic way, combinatorial identities among special functions [34]. This was later implemented on a computer [27, 33].

A key idea is to study the recursion relations that a function satisfies, rather than the function itself. This idea leads in a natural way to noncommutative algebras of operators that act on a function, together with left ideals of annihilating operators.

To explain this idea concretely, consider the operators x and ∂ which act on a smooth function f defined on $\mathbb{R}$ (or a distribution, or whatever else can be differentiated) by

$$(xf)(x) = xf(x), \quad (\partial f)(x) = \frac{\partial}{\partial x} f(x).$$

Leibnitz's rule $\partial(xf) = x\partial(f) + f$ written in operator form states that $\partial x = x\partial + 1$. The operators x and ∂ generate the *Weyl algebra* which is a free noncommutative algebra on x and ∂ modulo the two sided ideal $\partial x - x\partial - 1$:

$$\mathcal{A} = \frac{\mathbb{C}\langle x, \partial\rangle}{(\partial x - x\partial - 1)}.$$

The Weyl algebra is nothing but the *algebra of differential operators in one variable with polynomial coefficients*. Given a function f of one variable, let us define the *recursion ideal* $\mathcal{I}_f$ by

$$\mathcal{I}_f = \{P \in \mathcal{A} |\, Pf = 0\}.$$

It is easy to see that $\mathcal{I}_f$ is a left ideal of $\mathcal{A}$. Following Zeilberger and Bernstein, we say that f is *holonomic* if and only if $\mathcal{I}_f \neq 0$. In other words, a holonomic function is one that satisfy a linear differential equation with polynomial coefficients.

A key property of the Weyl algebra $\mathcal{A}$ (shared by its cousins, $\mathcal{B}$ and $\mathcal{C}$ defined below) is that it is *Noetherian*, which implies that every left ideal is *finitely generated*. In particular, a holonomic function is uniquely determined by a finitely list, namely the generators of its recursion ideal and a finite set of initial conditions.

The set of holonomic functions is closed under summation and product. Moreover, holonomicity can be extended to functions of several variables. For an excellent exposition of these results, see [3].

Zeilberger expanded the definition of holonomic functions of a continuous variable to *discrete functions* f (that is, functions with domain $\mathbb{Z}$; otherwise known as bi-infinite sequences) by replacing differential operators by *shift* operators. More precisely, consider the operators N and E which act on a discrete function (f_n) by

$$(Nf)_n = nf_n, \quad (Ef)_n = f_{n+1}.$$

It is easy to see that $EN = NE + E$. The *discrete Weyl algebra* $\mathcal{B}$ is a noncommutative algebra with presentation

$$\mathcal{B} = \frac{\mathbb{Q}\langle N^{\pm}, E^{\pm}\rangle}{(EN - NE - E)}.$$

The field coefficients $\mathbb{Q}$ are not so important, and neither is the fact that we allow positive as well as negative powers of N and E. Given a discrete function f, one can define the *recursion ideal* $\mathcal{I}_f$ in $\mathcal{B}$ as before. We will call a discrete function f *holonomic* if and only if the ideal $\mathcal{I}_f \neq 0$.

In our paper, we will consider a q-variant of the Weyl algebra. Let

$$\mathcal{R} = \mathbb{Z}[q^{\pm/2}]. \tag{1.2}$$

Consider the operators E and Q which act on a discrete function $f : \mathbb{Z} \longrightarrow \mathcal{R}$ by:

$$(Qf)_n(q) = q^n f_n(q), \quad (Ef)_n(q) = f_{n+1}(q). \tag{1.3}$$

It is easy to see that $EQ = qQE$. We define the *q-Weyl ring* $\mathcal{C}$ to be a noncommutative ring with presentation

$$\mathcal{C} = \frac{\mathcal{R}\langle Q, E\rangle}{(EQ - qQE)}. \tag{1.4}$$

Given a discrete function $f : \mathbb{Z} \longrightarrow \mathcal{R}$, one can define the left ideal $\mathcal{I}_f$ in $\mathcal{C}$ as before, and call a discrete function f *q-holonomic* if and only if the ideal $\mathcal{I}_f \neq 0$. Concretely, a discrete function $f : \mathbb{Z} \longrightarrow \mathcal{R}$ is q-holonomic if and only if there exists a nonzero element $\sum_{a,b} c_{a,b} E^a Q^b \in \mathcal{C}$ such that

$$\sum_{a,b} c_{a,b} q^{(n+a)b} f_{n+a}(q) = 0. \tag{1.5}$$

The sequence $J : \mathbb{Z} \longrightarrow \mathcal{R}$ of Laurent polynomials that we have in mind is the celebrated *colored Jones function.*

Theorem 1.1 [11]. *The colored Jones function of every knot in S^3 is q-holonomic.*

1.3. *The q-torus and the Kauffman bracket skein module of the torus*

In the above section, we introduced the q-Weyl ring $\mathcal{C}$ to define the notion of holonomicity of a sequence of Laurent polynomials.

The q-Weyl ring is not new to quantum topology. It has already appeared in the theory of *quantum groups*, (see [22, Chap. IV], [24]), under the name: the algebra of functions of the *quantum torus.*

It has also appeared in the skein theory approach of the colored Jones polynomial, via the Kauffman bracket. Let us review this important discovery of Frohman, Gelca and Lofaro [7]. As a side bonus, we can associate two further natural knot invariants: the quantum peripheral and orthogonal ideals.

1.4. *The quantum peripheral and orthogonal ideals of a knot*

The recursion relations (1.5) for the colored Jones function are motivated by the work of Frohman, Gelca, Przytycki, Sikora and others on the Kauffman bracket skein module, and its relation to the colored Jones function. Let us recall in brief these beautiful ideas.

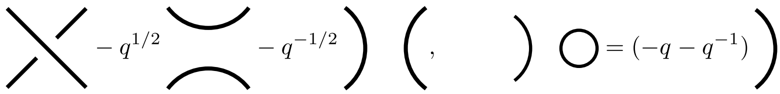

Fig. 1. The relations of the Kauffman bracket skein module.

For a manifold N, of any dimension, possibly with nonempty boundary, let $\mathcal{S}_q(N)$ denote the *Kauffman bracket skein module*, which is an $\mathcal{R}$-module (where $\mathcal{R}$ is defined in (1.2)) generated by the isotopy classes of *framed unoriented* links in N (including the empty one), modulo the relations of Fig. 1.

Remark 1.2. Our notation differs slightly from Gelca's *et al.* [7, 16, 17]. Gelca's t^2 equals to q and further, Gelca's $(-1)^n J_n$ is our J_{n+1} used below.

Let us recall some elementary facts of skein theory, reminiscent of TQFT:

Fact 1. If $N = N' \times I$, where N' is a closed manifold, then $\mathcal{S}_q(N)$ is an algebra.
Fact 2. If N is a manifold with boundary ∂N, then $\mathcal{S}_q(N)$ is a module over the algebra $\mathcal{S}_q(\partial N \times I)$.
Fact 3. If $N = N_1 \cup_Y N_2$ is the union of N_1 and N_2 along their common boundary Y, then there is a map:

$$\langle \, , \, \rangle : \mathcal{S}_q(N_1) \otimes_{\mathcal{S}_q(\partial Y \times I)} \mathcal{S}_q(N_1) \longrightarrow \mathcal{S}_q(N)$$

We will apply the previous discussion in the following situation. Let K denote a knot in a homology sphere N, and let M denote the complement of a thickening of K. Then, using the abbreviation $\mathcal{S}_q(\mathbb{T}) := \mathcal{S}_q(\mathbb{T}^2 \times I)$, Gelca and Frohman introduced in [6, 7] the quantum peripheral and orthogonal ideals of K (the latter was called *formal* in [7, Sec. 5]):

Definition 1.3. (a) We define the *quantum peripheral ideal* $\mathcal{P}(K)$ of K to be the *annihilator* of the action of $\mathcal{S}_q(\mathbb{T})$ on $\mathcal{S}_q(M)$. In other words,

$$\mathcal{P}(K) = \{P \in \mathcal{S}_q(\mathbb{T}) | P.\emptyset = 0\}.$$

(b) We define the *quantum orthogonal ideal* of a knot K to be

$$\mathcal{O}(K) = \{v \in \mathcal{S}_q(\mathbb{T}) | \, \langle \mathcal{S}_q(S^1 \times D^2)v, \emptyset \rangle = 0\}.$$

Note that $\mathcal{O}(K)$ and $\mathcal{P}(K)$ are left ideals in $\mathcal{S}_q(\mathbb{T})$ and that $\mathcal{P}(K) \subset \mathcal{O}(K)$. Unfortunately, the quantum peripheral and orthogonal ideals of a knot do not seem to be algorithmically computable objects.

To understand the quantum peripheral and orthogonal ideals requires a better description of the ring $\mathcal{S}_q(\mathbb{T})$, and its module $\mathcal{S}_q(S^1 \times D^2)$.

The skein module $\mathcal{S}_q(F \times I)$ is well-studied for a closed surface F. It is a free $\mathcal{R}$-module on the set of free homotopy classes of finite (possibly empty) collections of disjoint unoriented curves in F without contractible components. In particular, for a 2-torus $\mathbb{T}$, $\mathcal{S}_q(\mathbb{T})$ is the quotient of the free $\mathcal{R}$-module on the set $\{(a,b)\,|a,b\in\mathbb{Z}\}$ modulo the relations $(a,b)=(-a,-b)$. The multiplicative structure of $\mathcal{S}_q(\mathbb{T})$ is well-known, and related to the *even trigonometric polynomials* of the *quantum torus* [6]. Let us recall this description due to Gelca and Frohman. Consider the ring involution given by

$$\tau : \mathcal{C} \longrightarrow \mathcal{C} \quad E^aQ^b \longrightarrow E^{-a}Q^{-b} \tag{1.6}$$

and let $\mathcal{C}^{\mathbb{Z}_2}$ denote the invariant subring of $\mathcal{C}$. Frohman–Gelca [6] prove that

Fact 4. The map

$$\Phi : \mathcal{S}_q(\mathbb{T}) \longrightarrow \mathcal{C}^{\mathbb{Z}_2} \tag{1.7}$$

given by

$$(a,b) \longrightarrow (-1)^{a+b}q^{-ab/2}(E^aQ^b + E^{-a}Q^{-b}),$$

when $(a,b)\neq(0,0)$ and $\Phi(0,0)=1$, is an isomorphism of rings.

Thus, to a knot one can associate three ideals: the recursion ideal in $\mathcal{C}$ and the quantum peripheral and the orthogonal ideal in $\mathcal{S}_q(\mathbb{T})$. The next theorem explains the relation between the recursion and quantum orthogonal ideals.

Theorem 1.4. (a) *We have*:

$$\Phi(\mathcal{O}) = \mathcal{C}^{\mathbb{Z}_2} \cap \mathcal{I}.$$

In particular, the colored Jones function of a knot determines its quantum orthogonal ideal.

(b) *$\mathcal{I}$ is invariant under the ring involution τ.*

The proof of the above theorem proves the following corollary which compares the *orthogonality relations* of Gelca [16, Sec. 3] with the recursion relations given here:

Corollary 1.5. *Fix an element x of the quantum orthogonal ideal of a knot. The orthogonality relation for the colored Jones function is $(E-E^{-1})xJ=0$. On the other hand, Theorem* 1.4 *implies that $xJ=0$. It follows that a $(d+2)$-term recursion relation for the colored Jones function given by Gelca is implied by a d-term recursion relation.*

Remark 1.6. Frohman and Gelca claimed that the quantum peripheral ideal is nonzero [7, Proposition 8] and [16, Proof of Corollary 1], by specializing at $q=1$ and using the fact that the classical peripheral ideal is nonzero. At the time of [7], the nontriviality of the classical peripheral ideal was known for hyperbolic knots. Later on, the nontriviality was shown by Dunfield and the author for all knots in S^3 [5].

Combined with our Theorem 1.4, the surjection of the specialization map (that sets $q=1$) would prove Theorem 1.1. Unfortunately, there is an error in the argument of Frohman and Gelca. Instead, the surjection of the specialization map became known as the AJ Conjecture, formulated by the author in [13]. For a state-of-the art knowledge on the AJ Conjecture, see [18] and especially [23]. The latter gives a friendly discussion of the classical and quantum peripheral and orthogonal ideals of a knot.

1.5. *Converting difference into differential equations*

We now have all the ingredients to translate difference equations for $\{J_{K,n}\}$ to differential equations for $\{Q_{K,k}\}$.

Theorem 1.7. (a) *Theorem* 1.1 *implies a hierarchy of ODEs for* $\{Q_{K,k}\}$. *More precisely, for every knot there exist a lower diagonal matrix of infinite size*

$$D = \begin{pmatrix} D_0 & 0 & 0 & \cdots \\ D_1 & D_0 & 0 & \ddots \\ D_2 & D_1 & D_0 & \ddots \\ \cdots & \cdots & \cdots & \end{pmatrix} \tag{1.8}$$

such that $D_i \in A_1, 0 \neq D_0$ *and* $DQ_K = 0$ *where* $Q_K = (Q_{K,1}, Q_{K,2}, \dots)^T$.

(b) *The above hierarchy uniquely determines the sequence* $\{Q_{K,k}\}$ *up to a finite number of initial conditions* $\{\frac{d^j}{du^j}|_{u=j} Q_{K,k}(u)\}$ *for* $0 \leq j \leq \deg(D_0)$.

Notice that the hierarchy (1.8) depends on a linear q-difference equation for $\{J_{K,n}\}$. The degree of D_0 can be computed from a linear q-difference equation for $\{J_{K,n}\}$, see (3.8) below.

1.6. *Regular knots*

In this section, we introduce the notion of a regular knot, and explain the importance of this class of knots. The smallest degree of a nontrivial differential operator is 1.

Definition 1.8. A knot K is *regular* if J_K satisfies a q-difference equation so that $\deg(D_0) = 1$.

The explicit formulas of [11] imply that the knots 3_1 and 4_1 are regular. In a forthcoming paper [15], we will prove that many *twist knots* are regular.

Among other reasons, regularity is important because of the following.

Corollary 1.9. *If a knot* K *is regular, then* $J_K^{\mathrm{rat}}(q,u)$ (*and thus, also* J_K) *is uniquely determined by the hierarchy* (1.8) *and the initial condition*

$$J_K^{\mathrm{rat}}(q,1) = \sum_{k=0}^{\infty} Q_{K,k}(1)(q-1)^k \in \mathbb{Q}[[q-1]] \tag{1.9}$$

The power series invariant $J_K^{\mathrm{rat}}(q,1)$ is a disguized form of the *Kashaev invariant* of a knot, which plays a prominent role in the Volume Conjecture, due to Kashaev, and H. and J. Murakami; see [21, 25]. Recall that the *Volume Conjecture* states that for a hyperbolic knot K, we have

$$\lim_{n\to\infty}\frac{1}{n}\log|J_{K,n}(e^{2\pi i/n})| = \frac{1}{2\pi}\mathrm{vol}(K),$$

where $\mathrm{vol}(K)$ is the volume of K [30]. In [19], Le and Vu reformulated the sequence $(J_{K,n}(e^{2\pi i/n}))$ in terms of the evaluation of a single function

$$\kappa_K(q)\in\hat{\Lambda} := \mathrm{invlim}_j\mathbb{Z}[q^{\pm}]/((1-q)(1-q^2)\cdots(1-q^j)).$$

More precisely, Le constructed an element $\kappa_K(q)$ in the above ring so that for all $n\geq 1$ we have:

$$\kappa_K(e^{2\pi i/n}) = J_{K,n}(e^{2\pi i/n}).$$

There is a Taylor series map:

$$\mathrm{Taylor}:\hat{\Lambda}\longrightarrow\mathbb{Z}[[q-1]].$$

Then, we have:

$$J_K^{\mathrm{rat}}(q,1) = \mathrm{Taylor}(\kappa_K(q)). \tag{1.10}$$

The reader may deduce a proof of the above equation from [11, Sec. 3]. Thus, for regular knots, the Kashaev invariant together with the ODE hierarchy (1.8) *uniquely determines* the colored Jones function $\{J_K\}$, and its growth-rate in the Generalized Volume Conjecture.

1.7. *Questions*

The above hierarchy is reminiscent of *matrix models* discussed in physics. See, for example [4] and Question 1.10 below.

Let us mention that the conversion of difference into differential equations can actually be interpreted as an application of the *WKB method* for the linear q-difference equation. This remark, and its implications to asymptotics of the colored Jones function is explained in a later publication [14].

Let us end with some questions.

Question 1.10. Is there a *physical meaning* to the recursion relations of the colored Jones function, and in particular of the hierarchy of ODE which is satisfied by its loop expansion? Differential equations often hint at a hidden *matrix model*, or an *M-theory* explanation.

Question 1.11. The hierarchy of ODEs that appear in Theorem 1.7 also appears, under the name of *semi-pfaffian chain*, in complexity questions of real algebraic geometry. We thank Basu for pointing this out to us. For a reference, see [8]. Is this a coincidence?

Question 1.12. In [9], Kricker and the author constructed a rational form Z^{rat} of the Kontsevich integral of a knot. As was explained in [10], this rational form becomes the loop expansion of the colored Jones function, on the level of Lie algebras. In [11] it is shown that the $\mathfrak{g}$-colored Jones function for any simple Lie algebra $\mathfrak{g}$ is q-holonomic. Holonomicity gives rise to a ring $\mathcal{C}_{q,\mathfrak{g}}$ with an action of the Weyl group W of $\mathfrak{g}$. The ring $\mathcal{C}_{q,\mathfrak{g}}$ specializes to the coordinate ring of the $\mathfrak{g}$-character variety of the torus, introduced by Przytycki–Sikora, at least in case $\mathfrak{g} = \mathfrak{sl}_n$; see, for example [26, 29].

Is there a Kauffman bracket skein theory $\mathcal{S}_{q,\mathfrak{g}}$ that depends on $\mathfrak{g}$, in such a way that we have a ring isomorphism:

$$\Phi_{\mathfrak{g}} : \mathcal{S}_{q,\mathfrak{g}}(\mathbb{T}) \longrightarrow \mathcal{C}_{\mathfrak{g}}^{W}?$$

If so, one could define the $\mathfrak{g}$-quantum peripheral and orthogonal ideals of a knot, and ask whether the analogue of Theorem 1.4 holds:

$$\Phi_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}}) = \mathcal{C}_{\mathfrak{g}}^{W} \cap \mathcal{I}_{\mathfrak{g}}?$$

In addition, one may ask for an analogue of Theorem 1.7 for simple Lie algebras $\mathfrak{g}$. Notice, however, that this analogue is going to be a hierarchy of PDEs for functions of as many variables as the rank of the Lie algebra.

Question 1.13. Theorem 1.4 proves that $\mathcal{I}$ is an ideal in $\mathcal{C}^{\mathbb{Z}_2}$ invariant under the ring involution τ. Is it true that $\mathcal{I}$ is generated by its $\mathbb{Z}_2$-invariant part? In other words, is it true that $\mathcal{I} = \mathcal{C}(\mathcal{C}^{\mathbb{Z}_2} \cap \mathcal{I})$?

2. Proof of Theorem 1.4

Let us begin by discussing the recursion relation for the colored Jones function which is obtained by a nonzero element of the quantum orthogonal ideal of a knot. This uses work of Gelca, which we will quote here. For proofs, we refer the reader to [16].

The problem is to understand the right action of the ring $\mathcal{S}_q(\mathbb{T})$ on the skein module $\mathcal{S}_q(S^1 \times D^2)$.

To begin with, the skein module $\mathcal{S}_q(S^1 \times D^2)$ can be identified with the polynomial ring $\mathcal{R}[\alpha]$, where α is a longitudonal curve in the solid torus, and $\mathcal{R} = \mathbb{Z}[q^{\pm 1/2}]$. Rather than using the $\mathcal{R}$-basis for $\mathcal{S}_q(S^1 \times D^2)$ given by $\{\alpha^n\}_n$, Gelca uses the basis given by $\{T_n(\alpha)\}_n$, where $\{T_n\}$ is a sequence of Chebytchev-like polynomials defined by $T_0(x) = 2$, $T_1(x) = x$ and $T_{n+1}(x) = xT_n(x) - T_{n-1}(x)$.

Recall that the ring $\mathcal{S}_q(\mathbb{T})$ is generated by symbols (a, b) for integers a and b and relations $(a, b) = (-a, -b)$. Gelca [16, Lemma 1] describes the right action of $\mathcal{S}_q(\mathbb{T})$ on $\mathcal{S}_q(S^1 \times D^2)$ as follows:

$$\begin{aligned} T_n(\alpha) \cdot (a, b) = q^{ab/2}(-1)^b(q^{nb}[q^b S_{n+a}(\alpha) - q^{-b} S_{n+a-2}(\alpha)] \\ + q^{-nb}[-q^b S_{n-a-2}(\alpha) + q^{-b} S_{n-a}(\alpha)]), \end{aligned} \tag{2.1}$$

where $\{S_n\}$ is a sequence of Chebytchev polynomials defined by $S_0(x) = 1$, $S_1(x) = x$ and $S_{n+1}(x) = xS_n(x) - S_{n-1}(x)$.

Consider an element $x = \sum_{a,b} c_{a,b}(a,b)$ of the quantum orthogonal ideal of a knot and recall the pairing $\langle\, , \rangle$ from Fact 3. Since the (shifted) nth colored Jones polynomial of a knot is given by $J_n = (-1)^{n-1}\langle S_{n-1}(\alpha), \emptyset\rangle$, Eq. (2.1) implies the recursion relation

$$0 = \sum_{a,b} c_{a,b} q^{ab/2}(-1)^{a+b}(q^{nb}[q^b J_{n+a+1}(K) - q^{-b} J_{n+a-1}(K)] \\ + q^{-nb}[-q^b J_{n-a-1}(K) + q^{-b} J_{n-a+1}(K)]) \tag{2.2}$$

for the colored Jones function corresponding to an element $\sum_{a,b} c_{a,b}(a,b)$ in the quantum orthogonal ideal.

Let us write the above recursion relation in operator form. Recall that the operators E and Q act on the discrete function J by $(EJ)(n) = J(n+1)$ and $QJ(n) = q^n J(n)$, and satisfy the commutation relation $EQ = qQE$. Then, Eq. (2.2) becomes:

$$0 = \sum_{a,b} c_{a,b} q^{ab/2}(-1)^{a+b}(q^b Q^b E^{a+1} - q^{-b} Q^b E^{a-1} - q^b Q^{-b} E^{-a-1} \\ + q^{-b} Q^{-b} E^{-a+1})J.$$

Using the commutation relation $E^k Q^l = q^{kl} Q^l E^k$ for integers k, l and moving the E's on the left and the Q's on the right, we obtain that

$$0 = (E - E^{-1}) \sum_{a,b} c_{a,b} q^{-ab/2}(E^a Q^b + E^{-a} Q^{-b})J.$$

Recall the isomorphism Φ of Eq. (1.7). Using this isomorphism, our discussion so far implies that $x \in \mathcal{C}^{\mathbb{Z}_2}$ is an element of the quantum orthogonal ideal of a knot if and only if $(E - E^{-1})x$ lies in the recursion ideal. It remains to show that for every $x \in \mathcal{C}^{\mathbb{Z}_2}$, $(E - E^{-1})x \in \mathcal{I}$ if and only if $x \in \mathcal{I}$. One direction is obvious since $\mathcal{I}$ is a left ideal. For the opposite direction, consider $x \in \mathcal{C}^{\mathbb{Z}_2}$, and let $y = (E - E^{-1})x$ and $f = xJ$. Assume that $y \in \mathcal{I}$. We need to show that $x \in \mathcal{I}$; in other words that $f = 0$.

We have $(E^2 - I)f = E(E - E^{-1})f = 0$, which implies that

$$f(n+2) = f(n) \tag{2.3}$$

for all $n \in \mathbb{Z}$.

Recall the symmetry relation $J_n + J_{-n} = 0$ for the colored Jones function. In order to write it in operator form, consider the operator S that acts on a discrete function f by $(Sf)(n) = f(-n)$. Then, $(S + I)J = 0$.

It is easy to see that

$$SE = E^{-1}S \quad SQ = Q^{-1}S. \tag{2.4}$$

Since $\mathcal{C}^{\mathbb{Z}_2}$ is generated by $E^aQ^b + E^{-a}Q^{-b}$, it follows that S commutes with every element of $\mathcal{C}^{\mathbb{Z}_2}$; in particular $Sx = xS$, and thus $(S+I)f = (S+I)xJ = x(S+I)J = 0$. In other words,

$$f(n) + f(-n) = 0 \tag{2.5}$$

for all n. Equations (2.3) and (2.5) imply that $f(2n) = f(0)$, $f(2n+1) = f(1)$, $f(0) = f(1) = 0$. Thus, $f = 0$. This completes part (a) of Theorem 1.4.

For part (b), consider $x \in \mathcal{I}$ and recall the involution τ of (1.6). Then, we have $x = x_+ + x_-$ where $x_\pm = 1/2(x \pm \tau(x)) \in \mathcal{C}_\pm$, where $\mathcal{C}_\pm$ is generated by $E^aQ^b \pm E^{-a}Q^{-b}$. (2.4) implies that $Sx_+ = x_+S$ and $Sx_- = x_-S$. Now, we have

$$0 = xJ = x(-J) = xSJ = (x_+ + x_-)SJ = S(x_+ - x_-)J.$$

Since $S^2 = I$, it follows that $(x_+ - x_-)J = 0$. This, together with $0 = (x_+ + x_-)J$, implies that $x_\pm J = 0$. In other words, $x_\pm \in \mathcal{I}$. Since $\tau(x) = x_+ - x_-$, it follows that $\mathcal{I}$ is invariant under τ.

The proof of Theorem 1.4 also proves Corollary 1.5.

Example 2.1. In [17], Gelca computes that the following element

$$(1, -2k-3) - q^{-4}(1, -2k+1) + q^{(2k-5)/2}(0, 2k+3) - q^{(2k-1)/2}(0, 2k-1)$$

lies in the quantum peripheral (and thus quantum orthogonal) ideal of the left handed $(2, 2k+1)$ torus knot. Using the isomorphism Φ and Theorem 1.4 and a simple calculation, it follows that the following element

$$\begin{aligned} &-q^2(EQ^{-2k-3} + E^{-1}Q^{2k+3}) + q^{-4}(EQ^{-2k+1} + E^{-1}Q^{2k-1}) \\ &+ q^{-2}(Q^{2k+3} + Q^{-2k-3}) - (Q^{2k-1} + Q^{-2k+1}) \end{aligned}$$

lies in the recursion ideal of the left handed $(2, 2k+1)$ torus knot. This element gives rise to a 3-term recursion relation for the colored Jones function.

3. Proof of Theorem 1.7

3.1. *Three q-difference rings*

In this section, we consider some auxiliary rings and their associated $\mathcal{C}$-module structure.

The colored Jones function is a sequence of Laurent polynomials, in other words an element of the ring $\mathbb{Z}[q^{\pm/2}]^{\mathbb{Z}}$. The ring $\mathbb{Z}[q^{\pm/2}]^{\mathbb{Z}}$ is a $\mathcal{C}$-module via the action (1.3). In other words, for $f \in \mathbb{Z}[q^{\pm/2}]^{\mathbb{Z}}$, we define:

$$(Qf)_n(q) = q^n f_n(q), \quad (Ef)_n(q) = f_{n+1}(q).$$

Consider the ring $\mathbb{Q}'(u)[[q-1]]$ from Sec. 1.1. It is also a $\mathcal{C}$-module, where for $f(q,u) \in \mathbb{Q}'(u)[[q-1]]$, we define:

$$(Qf)(q,u) = uf(q,u), \quad (Ef)(q,u) = f(q,qu).$$

Consider in addition the ring $\mathbb{Q}[[q-1]]^{\mathbb{Z}}$. It is a $\mathcal{C}$-module, where for $(f_n(q)) \in \mathbb{Q}[[q-1]]$, we define:

$$(Qf)_n(q) = q^n f_n(q), \quad (Ef)_n(q) = f_{n+1}(q).$$

In the language of *differential Galois theory*, $\mathbb{Z}[q^{\pm/2}]^{\mathbb{Z}}$, $\mathbb{Q}'(u)[[q-1]]$ and $\mathbb{Q}[[q-1]]^{\mathbb{Z}}$ are *q-difference rings*, see, for example [32]. There is a ring homomorphism:

$$\mathrm{Ev} : \mathbb{Q}'(u)[[q-1]] \longrightarrow \mathbb{Q}[[q-1]]^{\mathbb{Z}}$$

given by

$$f(q,u) \mapsto (f(q,q^n)).$$

which respects the $\mathcal{C}$-module structure. In other words, for $f(q,u) \in \mathbb{Q}'(u)[[q-1]]$, we have:

$$\mathrm{Ev}(Qf) = Q\mathrm{Ev}(f), \quad \mathrm{Ev}(Ef) = E\mathrm{Ev}(f).$$

It is not hard to see that Ev map is injective. Equation (1.1) then states that for every knot K we have:

$$\mathrm{Ev}(J_K^{\mathrm{rat}}) = J_K/J_{\mathrm{unknot}}.$$

Notice further that since J_K is holonomic and $J_{\mathrm{unknot},n}(q) = [n]$ is *closed-form* (that is, $[n+1] \in \mathbb{Q}(q^{n/2}, q^{1/2})$), it follows that J_K/J_{unknot} is holonomic, too. In the proof of Theorem 1.7 below, J and J^{rat} will stand for J_K/J_{unknot} and J_K^{rat} respectively.

3.2. *Proof of Theorem 1.7*

Consider a function

$$J^{\mathrm{rat}}(q,u) = \sum_{k=0}^{\infty} Q_k(u)(q-1)^k \in \mathbb{Q}'(u)[[q-1]] \tag{3.1}$$

such that $J := \mathrm{Ev}(J^{\mathrm{rat}}) \in \mathbb{Q}[[q-1]]^{\mathbb{Z}}$ is holonomic. Thus, $XJ = 0$ where $X = \sum_{a,b} c_{a,b}(q) E^a Q^b \in \mathcal{C}$, and the sum is finite. In other words, we have:

$$0 = \sum_{a,b} c_{a,b}(q) q^{(n+a)b} J_{n+a}(q).$$

Since Ψ is a map of $\mathcal{C}$-modules, it follows that

$$0 = \mathrm{Ev}\left(\sum_{a,b} \widetilde{c}_{a,b}(q) Q^b E^a J^{\mathrm{rat}} \right),$$

where

$$\widetilde{c}_{a,b}(q) = c_{a,b}(q) q^{ab}.$$

Since Ψ is an injection, it follows that

$$0 = \sum_{a,b} \widetilde{c}_{a,b}(q) Q^b E^a J^{\mathrm{rat}}(q,u).$$

Using Eq. (3.1) and interchanging the order of summation, we obtain that

$$\begin{aligned} 0 &= \sum_{a,b} \widetilde{c}_{a,b}(q) u^b J^{\mathrm{rat}}(q, q^a u) \\ &= \sum_{k=0}^{\infty} (q-1)^k \mathcal{X} Q_k, \end{aligned} \tag{3.2}$$

where

$$\mathcal{X} : \mathbb{Q}'(u) \longrightarrow \mathbb{Q}'(u)[[q-1]]$$

is the operator defined by

$$\begin{aligned} f \mapsto \mathcal{X} f &= \sum_{a,b} c_{a,b}(q) u^b q^{ab} f(uq^a) \\ &= \sum_{a,b} \widetilde{c}_{a,b}(q) s^b f(uq^a) \end{aligned}$$

Let us define

$$P(\lambda, u, q) = \sum_{a,b} \widetilde{c}_{a,b}(q) u^b \lambda^a \in \mathbb{Z}[\lambda^{\pm}, u^{\pm}, q^{\pm}].$$

In the language of q-difference equations, $P(\lambda, u, q)$ is often called the *characteristic polynomial* of X. Let us denote by $\langle f \rangle_m$ the coefficient of $(q-1)^m$ in a power series f. Applying $\langle \cdot \rangle_m$ to (3.2), it follows that for all $m \geq 0$ we have

$$0 = \langle \mathcal{X} Q_0 \rangle_m + \langle \mathcal{X} Q_1 \rangle_{m-1} + \cdots + \langle \mathcal{X} Q_m \rangle_0 \tag{3.3}$$

The chain rule implies that

$$\langle \mathcal{X} f \rangle_m = \mathcal{D}_m f \tag{3.4}$$

for some differential operator $\mathcal{D}_m$ with polynomial coefficients of degree at most m. For example, we have:

$$\begin{aligned} (\mathcal{D}_0 f)(u) &= P(1, u, 1) f(u), \\ (\mathcal{D}_1 f)(u) &= P_q(1, u, 1) f(u) + P_\lambda(1, u, 1) u f'(u), \\ (\mathcal{D}_2 f)(u) &= P_{qq}(1, u, 1) f(u) + P_{\lambda q}(1, u, 1) u f'(u) + P_{\lambda\lambda}(1, u, 1) u^2 f''(u) \\ &\quad + (P_{\lambda\lambda}(1, u, 1) - P_\lambda(1, u, 1)) u f'(u), \end{aligned}$$

where the subscripts $._q$ and $._\lambda$ denote $\partial/\partial q$ and $\lambda \partial/\partial \lambda$ respectively, and the superscript denotes derivative with respect to u.

Thus, we obtain that for all $m \geq 0$

$$0 = \mathcal{D}_m Q_0 + \mathcal{D}_{m-1} Q_1 + \cdots + \mathcal{D}_0 Q_m. \tag{3.5}$$

We will show shortly that $\mathcal{D}_m \neq 0$ for some m. Assuming this, let $l = \min\{m|\mathcal{D}_m \neq 0\}$ and define $D_m = \mathcal{D}_{l+m}$. Equation (3.5) for $l+m$ implies that

$$0 = D_{m+l}Q_0 + D_{m+l-1}Q_1 + \cdots + D_0 Q_{m+l}.$$

In other words,

$$\begin{pmatrix} D_0 & 0 & 0 & \dots \\ D_1 & D_0 & 0 & \ddots \\ D_2 & D_1 & D_0 & \ddots \\ \dots & \dots & \dots & \ddots \end{pmatrix} \begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

as needed.

It remains to prove that $\mathcal{D}_m \neq 0$ for some m. The definition of $\mathcal{D}_m$ implies easily that $\mathcal{D}_k = 0$ for $k \leq m$ if and only if for all multiindices $I = (i_1, \dots, i_k)$ with $i_j \in \{q, \lambda\}$ and $k \leq m$, we have:

$$P_I(1, u, 1) = 0. \tag{3.6}$$

Since $P(\lambda, u, q)$ is a Laurent polynomial, if $\mathcal{D}_m = 0$ for all m, then $P(\lambda, u, q) = 0$.

Additively, there is a $\mathbb{Z}$-linear isomorphism $\mathcal{C} \leftrightarrow \mathbb{Z}[\lambda^{\pm}, u^{\pm 1}, q^{\pm 1}]$, given by $Q^b E^a \mapsto u^b v^a$, and sends $X \leftrightarrow P(\lambda, u, q)$. Thus, $X = 0$, a contradiction to our hypothesis. This concludes part (a) of Theorem 1.7. Part (b) follows from Lemma 3.1 below.

Let us finally compute l and the degree d of $D_0\mathcal{D}_l$ from X. Notice that d equals to the number of initial conditions needed to determine the sequence $\{Q_k\}$ from the ODE hierarchy (1.8).

For natural numbers n, m with $n \geq m$, let us denote by $I(n, m) = (\lambda, \dots, \lambda, q, \dots, q)$ the multiindex of length $|I(n, m)| = n$ where λ appears m times and q appears $n - m$ times.

Equation (3.6) implies that

$$l = \min\{n \,|P_{I(n,m)}(1, u, 1) \neq 0 \text{ for some } m\} \tag{3.7}$$

and

$$d = \min\{m \,|P_{I(l,m)}(1, u, 1) \neq 0\}. \tag{3.8}$$

Lemma 3.1. *If $a_i, c \in \mathbb{C}[u^{\pm 1}], a_n \neq 0$, the ODE*

$$a_n f^{(n)} + a_{n-1} f^{(n-1)} + \cdots + a_0 f = c$$

has at most one solution which is a rational function with fixed initial condition for $f^{(k)}(x_0)$ for $k = 0, \dots, n-1$, where $a_n(x_0) \neq 0$.

Proof. Consider the set of real numbers u such that $a_0(u) \neq 0$. It is a finite set of open intervals. Uniqueness of the solution (modulo initial conditions) is well-known. Since f is a rational function, it is uniquely determined by its restriction on an open interval. The result follows. □

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