

# Asymptotics of quantum spin networks at a fixed root of unity

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**Abstract** A classical spin network consists of a ribbon graph (i.e., an abstract graph with a cyclic ordering of the vertices around each edge) and an admissible coloring of its edges by natural numbers. The standard evaluation of a spin network is an integer number. In a previous paper, we proved an existence theorem for the asymptotics of the standard evaluation of an arbitrary classical spin network when the coloring of its edges are scaled by a large natural number. In the present paper, we extend the results to the case of an evaluation of quantum spin networks of arbitrary valency at a fixed root of unity. As in the classical case, our proofs use the theory of  $G$ -functions of André, together with some new results concerning holonomic and  $q$ -holonomic sequences of Wilf-Zeilberger.

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## Contents

1	Introduction	988
1.1	History	988
1.2	Classical spin networks	989
1.3	Quantum spin networks	989
1.4	Sequences of Nilsson type and $G$ -functions	990
1.5	Statement of our results	991
2	Evaluation of quantum spin networks	993
2.1	Local projectors	993
2.2	The standard evaluation of a quantum spin network	993
2.3	Evaluation of spin networks using the shadow formula	996
2.4	Integrality	999
3	Proof of Theorems 1.3, 1.4, 1.5 and 1.6	1001
3.1	Behavior of $q$ -holonomic sequences under differentiation	1001
3.2	Evaluations of $q$ -holonomic sequences at roots of unity are holonomic	1001
3.3	Proof of Theorem 1.6	1004
3.4	Proof of Theorem 1.3	1006
4	Examples	1006
4.1	Evaluations of the $q$ -multinomial coefficients at a root of unity	1006
4.2	The $q$ -difference equation of the regular $6j$ -symbol	1007
4.3	The regular $6j$ -symbol at roots of unity	1009
5	Open problems	1010

## 1 Introduction

### 1.1 History

Spin networks originally arose from calculations of angular momentum in quantum mechanics [35]. They were formalized in the sixties by Penrose [25, 26] in an attempt to quantize gravity combinatorially. Similar ideas were developed by Ponzano and Regge who related the semi-classical expansion of 3D gravity to the Regge action [5, 6, 24, 37]. In recent years, spin networks have played an important role in the development of loop quantum gravity; see [4, 10, 27, 31] and references therein. In the eighties, quantum spin networks were used by Reshetikhin, Turaev and Viro as building blocks of combinatorially defined invariants of knotted three-dimensional objects; see [32, 33]. Those topological invariants that are collectively called TQFT generalize the famous Jones [17] polynomial of a knot.

A key problem in classical and quantum spin networks is their asymptotic behavior for large spins. In the case of the  $6j$ -symbols Ponzano-Regge [24] conjectured the leading term of an explicit asymptotic expansion and gave ample numerical evidence. The Ponzano-Regge conjecture was proven by Roberts [30] using methods of geometric quantization. The existence of a general asymptotic expansion for all classical spin networks (to all orders in perturbation theory, in a constructive way) was recently obtained by the authors in [16]. The method of [16] was to convert the problem of asymptotic expansions to questions in Algebraic Geometry and Number Theory, and use the highly developed theory of  $G$ -functions. An important part in this conversion was the use of holonomic functions, developed by Zeilberger.

## 1.2 Classical spin networks

A *classical spin network* consists of a ribbon graph  $\Gamma$  (i.e., an abstract graph with a cyclic ordering of the vertices around each edge) and an admissible coloring  $\gamma$  of its edges by natural numbers. The standard evaluation  $\langle \Gamma, \gamma \rangle$  of a spin network is an integer number. In a previous paper [16], we proved an existence theorem for the asymptotics of the sequence  $\langle \Gamma, n\gamma \rangle$  of an arbitrary trivalent classical spin network when the coloring of its edges is scaled by a large natural number  $n$ . In addition, we presented several ways (algebraic-geometric, combinatorial, and numerical) for computing the asymptotics of  $\langle \Gamma, n\gamma \rangle$  for large  $n$ .

The goal of the present paper is to extend the results of [16] to the case of quantum spin networks at a fixed root of unity. As in the case of classical spin networks, our proofs use the theory of  $G$ -functions of André. There are two new ingredients which allow us to use the theory of  $G$ -functions. They come from the theory of holonomic and  $q$ -holonomic functions of Wilf-Zeilberger:

- (a) Theorem 1.4 which states that the derivative of a  $q$ -holonomic sequence with respect to  $q$  is  $q$ -holonomic.
- (b) Theorem 1.5 which states that the evaluation of a  $q$ -holonomic sequence at a fixed root of unity is holonomic.

## 1.3 Quantum spin networks

Quantum spin networks differ from their classical versions in two ways:

- (a) the underlying graphs  $\Gamma$  are knotted (i.e., embedded in  $S^3$ ) and not abstract,
- (b) their evaluations are polynomials of  $q$ , and not simply integer numbers.

Recall that a *knotted ribbon graph* is an embedded framed graph in  $S^3$ , of arbitrary valency, together with a cyclic ordering of the edges around each vertex. We will restrict ourselves to integer framing so that thickening the graph gives rise to an oriented surface. A *quantum spin network* consists of a knotted ribbon graph  $\Gamma$  together with a pair of functions  $\gamma = (\gamma_E, \gamma_V)$ . Here  $\gamma_E : \text{Edges}(\Gamma) \rightarrow \mathbb{N}$  is an admissible coloring  $\gamma$  of its edges, and  $\gamma_V : \text{Vert}(\Gamma) \rightarrow \{\text{projectors}\}$  a choice of a local projector at each vertex of  $\Gamma$ . Local projectors are explained in detail in Sect. 2, and can be ignored in case  $\Gamma$  is a trivalent graph.

The *standard evaluation*  $\langle \Gamma, \gamma \rangle(q)$  of a quantum spin network is a rational function of  $q^{1/4}$ ; see [32]. Recently, Costantino [8] obtained an integrality result for the standard evaluation of a quantum spin network, namely:

$$\langle \Gamma, \gamma \rangle(q) \in q^{\frac{n}{4}} \mathbb{Z}[q^{\pm \frac{1}{2}}]. \quad (1)$$

For some  $n$  depending on the network. Our first result deals with fixing the fractional power of  $q$ .

**Proposition 1.1** *For every quantum spin network  $(\Gamma, \gamma)$  there exist a  $\mathbb{Z}/4\mathbb{Z}$ -valued quadratic form  $Q(\gamma)$  such that*

$$\langle \Gamma, \gamma \rangle(q) = q^{\frac{1}{4}Q(\gamma)} \langle\langle \Gamma, \gamma \rangle\rangle(q), \quad \langle\langle \Gamma, \gamma \rangle\rangle(q) \in \mathbb{Z}[q^{\pm 1}]. \tag{2}$$

The above proposition allows us to define a modified evaluation  $\langle\langle \Gamma, \gamma \rangle\rangle(\zeta) \in \mathbb{C}$  that can be evaluated a *fixed root of unity*  $\zeta$ . To make  $\langle\langle \Gamma, \gamma \rangle\rangle$  well defined we use the convention that  $0 \leq Q(\gamma) \leq 3$ . When  $\zeta = 1$ , it follows from the definitions that  $\langle\langle \Gamma, \gamma \rangle\rangle(1)$  equals the classical evaluation and is hence independent of the embedding of  $\Gamma$  in 3-space. The goal of this paper is to extend the results of [16] to the case of the standard evaluation of a quantum spin network at a *fixed* root of unity  $\zeta$ .

### 1.4 Sequences of Nilsson type and $G$ -functions

To state our results, we need to recall what is a  $G$ -function and what is a sequence of *Nilsson type*. Sequences of Nilsson type are discussed in detail in [14]. We recall some definitions here for the convenience of the reader.

**Definition 1.2** We say that a sequence  $(a_n)$  is of *Nilsson type* if it has an *asymptotic expansion* of the form

$$a_n \sim \sum_{\lambda, \alpha, \beta} \lambda^n n^\alpha (\log n)^\beta S_{\lambda, \alpha, \beta} h_{\lambda, \alpha, \beta} \left( \frac{1}{n} \right) \tag{3}$$

where

- (a) the summation is over a finite set of triples  $(\lambda, \alpha, \beta)$ ,
- (b) the *growth rates*  $\lambda$  are algebraic numbers of equal absolute value,
- (c) the *exponents*  $\alpha$  are rational and the *nilpotent exponents*  $\beta$  are natural numbers,
- (d) the *Stokes constants*  $S_{\lambda, \alpha, \beta}$  are complex numbers,
- (e) the formal power series  $h_{\lambda, \alpha, \beta}(x) \in K[[x]]$  are Gevrey-1 (i.e., the coefficient of  $x^n$  is bounded by  $C^n n!$  for some  $C > 0$ ),
- (f)  $K$  is a *number field* generated by the coefficients of  $h_{\lambda, \alpha, \beta}(x)$  for all  $\lambda, \alpha, \beta$ .

**Definition 1.3** We say that series  $G(z) = \sum_{n=0}^\infty a_n z^n$  is a  $G$ -function if

- (a) the coefficients  $a_n$  are algebraic numbers,
- (b) there exists a constant  $C > 0$  so that for every  $n \in \mathbb{N}$  the absolute value of every Galois conjugate of  $a_n$  is less than or equal to  $C^n$ ,
- (c) the common denominator of  $a_0, \dots, a_n$  is less than or equal to  $C^n$ , (where the common denominator  $d$  of  $a_0, \dots, a_n$  is the least natural number such that  $da_i$  is an algebraic integer for  $i = 1, \dots, n$ ),
- (d)  $G(z)$  is holonomic, i.e., it satisfies a linear differential equation with coefficients polynomials in  $z$ .

The following connection between sequences of Nilsson type and  $G$ -functions was observed in [13, Prop.2.5].

**Theorem 1.1** [13, Prop.2.5] *If  $G(z) = \sum_{n=0}^\infty a_n z^n$  is a  $G$ -function, then  $(a_n)$  is a sequence of Nilsson type.*

The above existence theorem has a valuable, effective corollary, discussed in detail in [13].

**Corollary 1.4** *If  $G(z) = \sum_{n=0}^{\infty} a_n z^n$  is a  $G$ -function, and suppose that we are given either (a) a linear differential equation for  $G(z)$  with coefficients in  $L[x]$  for some number field  $L$ , or (b) a linear recursion for  $(a_n)$  with coefficients in  $L[n]$ , then one can effectively compute  $\lambda, \alpha, \beta$  and  $h_{\lambda, \alpha, \beta}(x) \in K[[x]]$  for a number field  $K$  such that the asymptotic expansion (3) holds, for some unknown Stokes constants  $S_{\lambda, \alpha, \beta}$ .*

In other words, a linear recursion for  $(a_n)$  or a linear differential equation for  $G(z)$  allows us to compute the asymptotic expansion (3) to all orders in  $1/n$ , up to a finite set of unknown Stokes constants. For explicit illustrations of the above corollary, see [12, 38] and also [16, Sec.10] where the authors discuss in detail effective asymptotic expansions of evaluations of classical spin networks.

### 1.5 Statement of our results

Let  $f^{(r)}(q) = d^r/dq^r f(q)$  denote the  $r$ th derivative of a Laurent polynomial  $f(q) \in \mathbb{Z}[q^{\pm 1}]$ .

**Theorem 1.2** *For every quantum spin network  $(\Gamma, \gamma)$ , every complex root of unity  $\zeta$  and every natural number  $r \geq 0$ , the sequence  $\langle\langle \Gamma, n\gamma \rangle\rangle^{(r)}(\zeta)$  is of Nilsson type.*

Theorem 1.2 follows from Theorem 1.1 and the following theorem.

**Theorem 1.3** *For every quantum spin network  $(\Gamma, \gamma)$ , every complex root of unity  $\zeta$  and every natural number  $r$ , the generating function*

$$F_{\Gamma, \gamma, \zeta, r}(z) = \sum_{n=0}^{\infty} \langle\langle \Gamma, n\gamma \rangle\rangle^{(r)}(\zeta) z^n \quad (4)$$

*is a  $G$ -function.*

Theorem 1.3 follows from Theorems 1.4, 1.5 and 1.6 below, which involve structural properties of the classes of holonomic and  $q$ -holonomic sequences and may be of independent interest. To state them, recall that a sequence  $(f_n)$  of complex numbers is *holonomic* if it satisfies a linear recursion of the form

$$c_d(n) f_{n+d} + \cdots + c_0(n) f_n = 0 \quad (5)$$

for all  $n$  where  $c_j(n) \in K[n]$  are polynomials in  $n$  with coefficients in a number field  $K$  for  $j = 0, \dots, d$  with  $c_d \neq 0$ . Likewise, a sequence  $(f_n(q))$  of rational functions of  $q$  is  *$q$ -holonomic* if it satisfies a linear recursion of the form

$$c_d(q^n, q) f_{n+d}(q) + \cdots + c_0(q^n, q) f_n(q) = 0 \quad (6)$$

for all  $n$  where  $c_j(u, v) \in K[u, v]$  are polynomials in two variables  $u$  and  $v$  for  $j = 0, \dots, d$  with  $c_d \neq 0$ . Holonomic and  $q$ -holonomic sequences were studied in detail by Zeilberger; see [39, 41].

**Theorem 1.4** *The derivative with respect to  $q$  of a  $q$ -holonomic sequence is  $q$ -holonomic.*

**Theorem 1.5** *For every  $q$ -holonomic sequence  $f_n(q) \in \mathbb{Z}[q^{\pm 1}]$  and every complex root of unity  $\zeta$ , the evaluation  $f_n(\zeta)$  is a holonomic sequence.*

Let us make some remarks.

*Remark 1.5* Theorem 1.5 fails when  $\zeta$  is not a complex root of unity. For example,  $f_n(q) = q^{n^2}$  is  $q$ -holonomic and satisfies the recursion  $f_{n+1}(q) - q^{2n+1} f_n(q) = 0$ . On the other hand,  $f_n(\omega)$  is holonomic only when  $\omega$  is a complex root of unity. Theorem 1.5 is another manifestation of the importance of roots of unity in Quantum Topology. See also Sect. 3.

*Remark 1.6* Theorems 1.4 and 1.5 presumably hold for multi-variable  $q$ -holonomic sequences  $f_{n_1, \dots, n_r}(q) \in \mathbb{Z}[q^{\pm 1}]$ . Multi-variable holonomic and  $q$ -holonomic sequences were introduced and studied in [39, 41]. In the present paper, we will not use them. However, the reader should keep in mind that for every quantum spin network  $(\Gamma, \gamma)$  the multi-variable sequence  $\gamma \mapsto \langle \Gamma, \gamma \rangle(q)$  is  $q$ -holonomic. This follows from Sect. 2.3 where it is shown that  $\langle \Gamma, \gamma \rangle(q)$  is a multi-sum of a  $q$ -proper hypergeometric term. By the fundamental theorem of WZ-theory (see [39]),  $\langle \Gamma, \gamma \rangle(q)$  is  $q$ -holonomic in all  $\gamma$ -variables.

*Remark 1.7* The  $q$ -holonomic sequence  $f_n(\zeta)$  in Theorem 1.5 need not be exponentially bounded. For example,

$$f_n(q) = \prod_{k=1}^n \frac{1 - q^k}{1 - q} \in \mathbb{Z}[q]$$

satisfies the linear recursion

$$(q - 1)f_{n+1}(q) - (q^{n+1} - 1)f_n(q) = 0.$$

Thus,  $f_n(q)$  is  $q$ -holonomic. Its evaluation at  $\zeta = 1$  is given by  $f_n(1) = n!$  which is holonomic, since it satisfies the linear recursion

$$f_{n+1}(1) - (n + 1)f_n(1) = 0$$

However,  $n!$  is not exponentially bounded.

Despite the above remark, the evaluation of a quantum spin network at a fixed root of unity is an exponentially bounded sequence.

**Theorem 1.6** *For every quantum spin network  $(\Gamma, \gamma)$ , every complex root of unity  $\zeta$  and every natural number  $r$ , there exists  $C > 0$  (which depends on  $\Gamma, \gamma$  and  $r$ ) such that*

$$|\langle \langle \Gamma, n\gamma \rangle \rangle^{(r)}(\zeta)| \leq C^n \tag{7}$$

for all  $n \in \mathbb{N}$ .

## 2 Evaluation of quantum spin networks

### 2.1 Local projectors

We start by defining classical spin networks of arbitrary valency using the notion of a local projector. A *classical spin network* will be a pair  $(\Gamma, \gamma)$  of an abstract ribbon graph  $\Gamma$  (of arbitrary valency), together with a pair of functions  $\gamma = (\gamma_E, \gamma_V)$  where  $\gamma_E : \text{Edges}(\Gamma) \rightarrow \mathbb{N}$  is an admissible coloring  $\gamma$  of its edges, and  $\gamma_V : \text{Vert}(\Gamma) \rightarrow \{\text{projectors}\}$  a choice of a local projector at each vertex of  $\Gamma$ . A local projector is defined as follows. Given an admissible coloring  $(c_1, \dots, c_d)$  of the  $d$  edges around a vertex  $v$  of a ribbon graph, place a collection of  $c_1 + \dots + c_d$  points on a disk. A *local projector* is a planar way to connect these points with *disjoint arcs* on the disk and *no U-turns*, as in the example shown in Fig. 1.

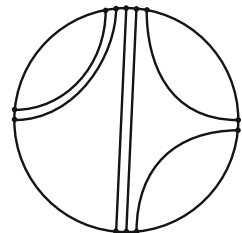
Note that if  $p$  is a local projector and  $n$  is a natural number, then there is a canonical choice of a local projector  $np$  by cabling each arc of  $p$  into  $n$  arcs. In [21, p.118], Kuperberg uses the term *clasped web space*  $W(c_1, \dots, c_d)$  which has a basis the local projectors defined above. Local projectors are a pictorial way to encode dual canonical bases for  $\mathfrak{sl}_2$ , as was shown in [11]. However, clasped web spaces and dual canonical bases do *not* coincide for  $\mathfrak{sl}_3$ ; see [19]. Finally note that there are alternative methods of defining multi-valent vertices, for example [3,40]. However our definition using projectors works without change for Lie algebras of arbitrary rank.

### 2.2 The standard evaluation of a quantum spin network

Recall from the introduction that a quantum spin network is a pair  $(\Gamma, \gamma)$  where  $\Gamma$  is a framed ribbon graph embedded in  $S^3$  and  $\gamma = (\gamma_E, \gamma_V)$  as in the classical case. We allow  $\Gamma$  to have multiple edges and loops and it may be disconnected. Knots and links are quantum spin networks with zero vertices. However we restrict ourselves to integral framing of  $\Gamma$ . In this section we define the evaluation of quantum spin networks in terms of the Kauffman bracket. Recall that the *quantum integer*  $[n]$  and the balanced *quantum factorial*  $[n]!$  is defined by

$$[n] = \frac{A^{2n} - A^{-2n}}{A^2 - A^{-2}}, \quad [n]! = \prod_{k=1}^n [k] \quad (8)$$

**Fig. 1** An example of a projector for a degree 4 vertex with edge coloring  $(2, 5, 2, 3)$



where

$$A^4 = q. \tag{9}$$

The *Kauffman bracket* evaluation is determined by the following rules:

$$\begin{aligned} \begin{array}{c} \diagup \\ \diagdown \end{array} &= A \begin{array}{c} \frown \\ \smile \end{array} + A^{-1} \begin{array}{c} \smile \\ \frown \end{array} \quad \left( \right. \\ \bigcirc \cup D &= -[2] \cdot D \end{aligned}$$

To emphasize the similarity to the evaluation of classical spin networks we use the explicit expression for the Jones-Wenzl idempotent from [20, Chpt.3]. This is defined as the following formal sum of braids:

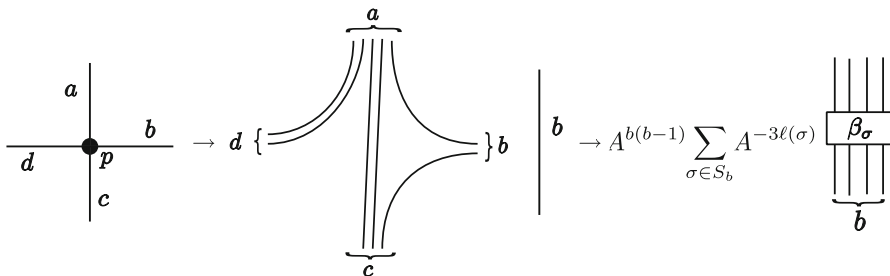
$$A^{b(b-1)} \sum_{\sigma \in S_b} A^{-3\ell(\sigma)} \beta_\sigma$$

Here  $b$  is the label of the edge, and for any permutation  $\sigma$  we denote by  $\beta_\sigma$  the unique negative (with respect to orientation downwards) permutation braid corresponding to  $\sigma$ . By  $\ell(\sigma)$  we mean the minimal length of  $\sigma$  written as a product of transpositions. Note that we leave out the quantum factorial  $[b]!$  that is used in [20].

**Definition 2.1** (a) We say a quantum spin network is *admissible* if for every vertex  $v$  the projector  $\gamma_V(v)$  matches the labels of the edges given by  $\gamma_E$ .

(b) The evaluation  $\langle \Gamma, \gamma \rangle^P$  of a quantum spin network  $(\Gamma, \gamma)$  is defined to be zero if it is not admissible. An admissible quantum spin network is evaluated by the following algorithm.

- Use the ribbon structure to thicken the vertices into disks and the edges into untwisted bands.
- Replace each vertex  $v$  by the pattern of the projector  $\gamma_V(v)$  and replace each edge by the linear combination of braids as shown in Fig. 2.
- Finally the resulting linear combination of links is evaluated calculating the Kauffman bracket.



**Fig. 2** The rules for evaluating a quantum spin network. Replace vertices and edges to get a linear combination of links that is evaluated using the Kauffman bracket



In the case  $A = -1$  this definition agrees with the Penrose evaluation defined in [16]. For general  $A$  the definition agrees with that given in [7, 20, 22] except for the missing quantum factorial  $[b]!$  in the denominator, one for every edge. The reason for leaving out this factor is that there is a better way to normalize (see Definition 2.2 below) which results in Laurent polynomials in  $q^{1/4}$  (as opposed to rational functions in  $q^{1/4}$ ) while the value of  $\langle \Gamma, n\gamma \rangle(q)$  at a fixed root of unity  $q$  grows at most exponentially with respect to  $n$ , see Theorem 1.6.

**Definition 2.2** The *standard evaluation* of a quantum spin network is defined by

$$\langle \Gamma, \gamma \rangle = \frac{1}{[\mathcal{I}]!} \langle \Gamma, \gamma \rangle^P.$$

Here  $[\mathcal{I}]! = \prod_v [\gamma_V(v)]!$ , where for a projector  $[p]!$  means grouping all strands connecting the same two edges and forming the product of the quantum factorials of these numbers.

Note that with this normalization subdividing an edge colored  $a$  into two edges colored  $a$  connected by a vertex whose third edge is colored 0 does not change the value of the evaluation. This is the way in which the value of the unknot should be interpreted in order to obtain the correct value  $(-1)^a [a + 1]$  instead of its quantum factorial.

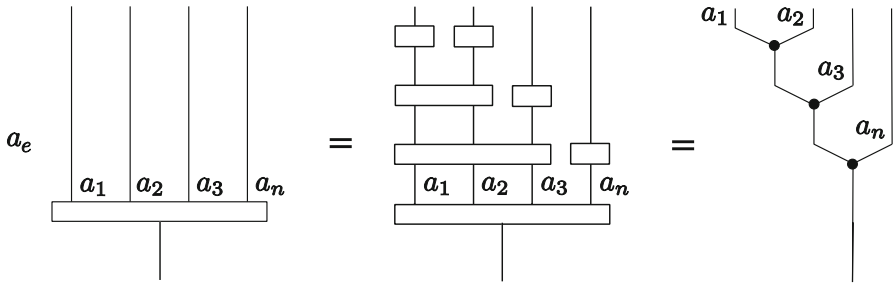
We end this section by proving that any quantum spin network evaluation actually reduces to a trivalent quantum spin network evaluation. Hence all results previously known in the trivalent case extend to the general case.

**Lemma 2.3** *Let  $(\Gamma, \gamma)$  be a quantum spin network. There exists a trivalent quantum spin network  $(\Gamma', \gamma')$  such that for all  $n \in \mathbb{N}$*

$$\langle \Gamma, n\gamma \rangle = \langle \Gamma', n\gamma' \rangle.$$

*Proof* It is convenient to take a slightly different view of the evaluation algorithm defined in 2.1. Instead of expanding out the linear combination at every edge, we view the Jones-Wenzl idempotent as box with arcs coming out. After expanding the vertices the whole network becomes a number of boxes connected by arcs. The idea is to add extraneous boxes and to reinterpret the result as the evaluation of a trivalent quantum spin network. The key property that makes this work is the fundamental fact that once one includes the factor  $\frac{1}{[b]!}$  in a box with  $b$  strands, it becomes an idempotent [22]. Since we're not using this normalization we get that adding an extra box to  $b$  parallel adjacent arcs coming out of a box multiplies the evaluation by  $[b]!$ . This factor will cancel with the normalization factor due to the trivalent vertices that will be created in the process.

Let us first describe how to modify a single half-edge pointing into a multivalent vertex  $v$  with projector  $p$ . The arcs coming into  $v$  from half edge  $e$  are grouped into  $a_j$  parallel arcs according to the projector  $p$ . To the box that already marks the beginning of  $e$  we now add an extra stack of boxes as shown in the middle of Fig. 3. This will



**Fig. 3** Turning an edge into a trivalent network

multiply the Penrose evaluation of  $\Gamma$  by a factor  $a_e = ([a_2]! \dots [a_n]!)([a_1 + a_2 \dots + a_{n-1}]! \dots [a_1 + a_2]![a_1]!)$

Doing this for every half edge gives just enough boxes to split into a trivalent quantum spin network  $\Gamma'$  as shown in the above figure. However the smallest boxes at the tip of the tree meet another equal sized box on the other side which are not there in the evaluation of  $\Gamma'$ . Therefore we have to remove those boxes again, which reduces the Penrose evaluation by exactly a factor  $[Z]! = [Z(\Gamma)]!$ . Therefore,

$$\langle \Gamma, \gamma \rangle^P \prod_e a_e = [Z]! \langle \Gamma', \gamma' \rangle^P$$

And hence passing to the standard normalization we find:

$$\langle \Gamma, \gamma \rangle = \frac{\langle \Gamma, \gamma \rangle^P}{[Z]!} = \frac{\langle \Gamma', \gamma' \rangle^P}{\prod_e a_e} = \langle \Gamma', \gamma' \rangle$$

concluding the proof. □

### 2.3 Evaluation of spin networks using the shadow formula

In this subsection we describe a way of evaluating spin networks in terms of the shadow formula. We restrict ourselves to trivalent  $\Gamma$ . Since  $\Gamma$  is supposed to be oriented, we can choose a blackboard framed diagram  $D$  of  $\Gamma$ .

The shadow formula expresses the evaluation in terms of a multi-dimensional sum of  $1j$ ,  $3j$  and  $6j$ -symbols. The latter are the evaluations of three basic spin networks, shown in Fig. 4.

Let

$$\left[ \begin{matrix} a \\ a_1, a_2, \dots, a_r \end{matrix} \right] = \frac{[a]!}{[a_1]! \dots [a_r]!} \tag{10}$$

denote the quantum multinomial coefficient when  $a_1 + \dots + a_r = a$ . The value of the  $1j$ ,  $3j$  and  $6j$ -symbols is given by the following lemma of [20] (see also, [22,36]), using our normalization.

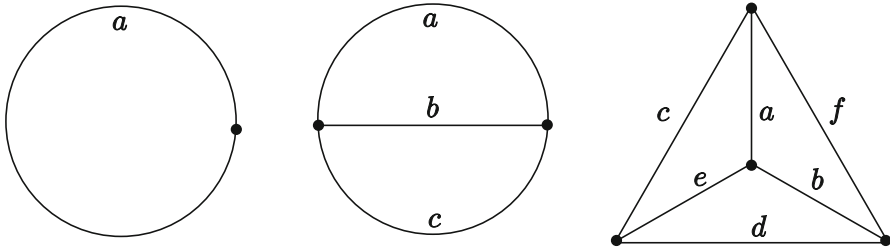


Fig. 4 Three basic spin networks

Lemma 2.4 (a) We have

$$\langle \bigcirc, a \rangle = (-1)^a [a + 1]$$

(b) Let  $(\Theta, \gamma)$  denote the  $\Theta$  spin network admissibly colored by  $\gamma = (a, b, c)$  as in Fig. 4. Then we have

$$\langle \Theta, \gamma \rangle = (-1)^{\frac{a+b+c}{2}} \left[ \frac{a+b+c}{2} + 1 \right] \left[ \frac{a+b+c}{2}, \frac{-a+b+c}{2}, \frac{a-b+c}{2}, \frac{a+b-c}{2} \right] \tag{11}$$

(c) Let  $\triangle$  denote a tetrahedron labeled and oriented as in Fig. 4 and admissibly colored by  $\gamma = (a, b, c, d, e, f)$ . Then we have

$$\langle \triangle, \gamma \rangle = \sum_{k=\max T_i}^{\min S_j} (-1)^k [k + 1] \left[ S_1 - k, S_2 - k, S_3 - k, k - T_1, k - T_2, k - T_3, k - T_4 \right] \tag{12}$$

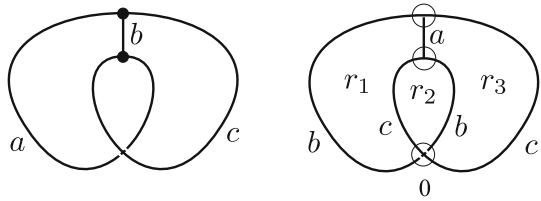
where  $S_i$  are the half sums of the labels in the three quadrangular curves in the tetrahedron and  $T_j$  are the half sums of the three edges emanating from a given vertex. In other words, the  $S_i$  and  $T_j$  are given by

$$S_1 = \frac{1}{2}(a + d + b + c) \quad S_2 = \frac{1}{2}(a + d + e + f) \quad S_3 = \frac{1}{2}(b + c + e + f) \tag{13}$$

$$T_1 = \frac{1}{2}(a + b + e) \quad T_2 = \frac{1}{2}(a + c + f) \quad T_3 = \frac{1}{2}(c + d + e) \\ T_4 = \frac{1}{2}(b + d + f). \tag{14}$$

In addition to the three basic spin networks above, we choose to consider a variant of the tetrahedron that represents a crossing. The crossing can be reduced

**Fig. 5** An example spin network  $\Gamma, \gamma$  (left) and its diagram with encircled crossings and vertices and a coloring of the regions



to a  $6j$ -symbol using the half twist formula, [22].

$$\begin{matrix} r_1 & & r_2 \\ \circlearrowleft & & \circlearrowright \\ r_4 & & r_3 \end{matrix} = (-1)^{\frac{-r_1 - r_3 + r_2 + r_4}{2}} q^{\frac{-r_1(r_1+2) - r_3(r_3+2) + r_2(r_2+2) + r_4(r_4+2)}{8}} \begin{matrix} r_1 & & r_2 \\ \circlearrowleft & & \circlearrowright \\ r_4 & & r_3 \end{matrix}$$

We are now ready to explain how to evaluate a spin network using the shadow formula. A blackboard framed diagram of  $\Gamma$  gives rise to a planar graph  $D$ , whose edges are colored by  $\gamma$  and whose vertices are either vertices of  $\Gamma$  or crossings. Let  $V, E, F, C$  denote the sets of trivalent vertices, edges, faces and crossings (4-valent vertices) of  $D$ . We will express the evaluation of the quantum spin network as a sum over all admissible colorings of  $F$  by natural numbers. Here admissible means that for any two faces colored  $r_1, e, r_2$  and separated by edge  $e$ , the numbers  $(r_1, r_2, \gamma(e))$  satisfy the triangle inequalities. Moreover the outer face should get color 0.

$$\langle \Gamma, \gamma \rangle = \sum_r \prod_{f \in F} \circ \prod_{e \in E} \Theta^{-1} \prod_{v \in V} \triangle \prod_{c \in C} \otimes \tag{15}$$

The sum is over admissible colorings  $r$  of the faces  $F$  and the labels of the symbols in the formula are found from the coloring and  $\gamma$  as follows. The  $1j$  symbol gets the color of the face it corresponds to. The theta gets the color of the edge it corresponds to plus the two colors of the faces it bounds. For every vertex the corresponding  $6j$  symbol is colored by the three adjacent faces and the three adjacent edge labels. Finally every crossing is counted by a skew  $6j$ -symbol obtained by encircling the crossing and coloring according to the faces one crosses. For a more elaborate discussion see [8] (Fig. 5).

Consider the following simple example of a spin network evaluation using the shadow formula.

Note that one can find the contributing  $6j$ -symbols from the diagram by encircling the vertices and crossings. This is useful in keeping track of the labels. Similarly one could also encircle the edges and faces to obtain the  $3j$  and  $1j$ -symbols.

$$\langle \Gamma, a, b, c \rangle = \sum_{r_1, r_2, r_3} \frac{\Theta(0, b, r_1)\Theta(0, c, r_3)\Theta(r_1, c, r_2)\Theta(r_2, b, r_3)\Theta(r_1, a, r_3)}{\Theta(0, b, r_1)\Theta(0, c, r_3)\Theta(r_1, c, r_2)\Theta(r_2, b, r_3)\Theta(r_1, a, r_3)}$$

Since the sum is over admissible colors only, we see that in this special case there is quite some simplification. The fact that the outside color is 0 forces  $r_1 = b, r_2 = c$ . Actually one can check that after substituting the formulas for the  $6j, 3j$  and  $1j$ -symbols one gets that the answer is 0 unless  $a = 0$  and  $b = c$  in which case we get a twisted unknot.

### 2.4 Integrality

In [8] it was shown that  $\langle \Gamma, \gamma \rangle \in A^f \mathbb{Z}[A^{\pm 2}]$  for some  $f$  depending on  $(\Gamma, \gamma)$ . In this subsection we use the shadow formula to extend this result to prove that actually  $\langle \Gamma, \gamma \rangle \in A^{Q(\gamma)} \mathbb{Z}[A^{\pm 4}]$ . Here  $Q(\gamma)$  is a quadratic form depending on  $\Gamma$  in the labels  $\gamma$ , that takes values in  $\mathbb{Z}/4\mathbb{Z}$ . The proof below will indicate how to obtain an expression for  $Q(\gamma)$  from an admissible coloring of the faces of a diagram of  $\Gamma$ . The form itself is independent of the choice of diagram, and coloring. By identifying  $\mathbb{Z}/4\mathbb{Z}$  with  $\{0, 1, 2, 3\}$  we have defined  $\langle \Gamma, \gamma \rangle = A^{-Q(\gamma)} \langle \Gamma, \gamma \rangle$  precisely.

*Proof* (of Proposition 1.1). Note that by Lemma 2.3 we can restrict ourselves to trivalent  $\Gamma$ . Pick any blackboard framed diagram  $D$  of  $(\Gamma, \gamma)$  and consider applying the shadow formula (15). Since by Costantino’s result [8] we already know the evaluation is a polynomial in  $A$  we can look at the degree of the terms in the sum.

Now define  $\phi : \mathbb{Z}[A^{\pm 1}] \rightarrow \mathbb{Z}/4\mathbb{Z}$  to be a function such that  $P(a) \in A^{\phi(P)} \mathbb{Z}[A^{\pm 4}]$ . As a first step we calculate  $\phi$  for the building blocks of the shadow formula. By dividing the leading powers of the denominator in the formulas for the  $6j$ -symbol we see that:

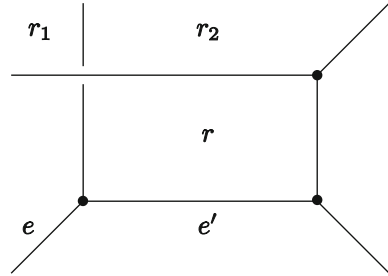
$$\begin{aligned} \phi([k]) &= 2k - 2 \\ \phi(\Theta, a, b, c) &= a^2 + b^2 + c^2 \\ \phi(\triangle, \gamma) &= \sum_i \gamma_i^2 + \sum_{i < j} \gamma_i \gamma_j \\ \phi\left(\begin{array}{c|c} r_1 & r_2 \\ \hline r_4 & r_3 \end{array}\right) &= \frac{-r_1^2 - r_3^2 + r_2^2 + r_4^2}{2} - r_1 - r_3 + r_2 + r_4 + \phi(\triangle) \end{aligned}$$

It suffices to show that every term in the shadow formula has the same value of  $\phi$  (modulo 4). In order to check this we consider the effect on  $\phi$  of a term when we increase one of the variables  $r$  by two. This is sufficient since a simple argument shows that any state can be reached from any other state by repeatedly increasing or decreasing one of the variables by two.

So let  $\phi(r)$  be the value of  $\phi$  on a particular summand in which we ignore the terms not containing  $r$ . With respect to the region labeled  $r$  we make a distinction between the edges and regions directly adjacent to it (notation:  $e|r$  or  $r_i|r$ ) and those edges and regions transverse to the region  $r$  (notation:  $e \perp r$ ). See also Fig. 6.

Looking at the above formulae for  $\phi$  and counting the crossings next to  $r$  with sign  $\sigma$ , we get the following expression for  $\phi(r)$ . Here we abused the notation to make  $r_i$

**Fig. 6** The region labeled  $r$ . The region  $r_1$  the edge  $e$  are transverse to  $r$ . The region  $r_2$  and the edge  $e'$  are adjacent to  $r$



denote both the face and its value and  $\sigma(r_i)$  is the sign of the corresponding crossing between  $r$  and  $r_i$ .

$$\begin{aligned} \phi(r) = r^2 & \left( \# \text{adjacent edges} + \# \text{adjacent vertices} + \frac{1}{2} \# \text{crossings} + \# \text{unsigned crossings} \right) \\ & + r \left( 2 + 2 \sum_{e|r} \gamma(e) + 2 \sum_{r_i|r} r_i + \sum_{e \perp r} \gamma(e) + \sum_{r_i \perp r} r_i + \sigma(r_i) \right) \end{aligned}$$

We will show that  $\Delta = \phi(r + 2) - \phi(r)$  is divisible by 4. Since modulo 4 we have

$$\Delta \equiv 2((r + 1)\# \text{crossings} + \sum_{e \perp r} \gamma(e) + \sum_{r_i \perp r} r_i + \sigma(r_i)) \pmod{4}$$

we need to check that the term in brackets is even. To do so we use admissibility and write congruences modulo 2. By admissibility of the labels at the trivalent vertices and admissibility of the thetas coming from the edges in the shadow formula respectively we have:

$$\gamma(e) \equiv \gamma(e') + \gamma(e'') \quad r_i \equiv \gamma(e_1) + \gamma(e_2) + r \pmod{2}$$

Here  $e, e', e''$  are the edges at a vertex and  $e_1, e_2, r, r_i$  are the edges and two opposite regions at a crossing. Summing all these identities we see that every edge adjacent to  $r$  appears exactly twice, which shows that

$$\sum_{e \perp r} \gamma(e) + \sum_{r_i \perp r} r_i \equiv r \# \text{crossings} \pmod{2}$$

Finally  $\sum_{r_i \perp r} \sigma(r_i) \equiv \# \text{crossings}$ , therefore  $\Delta$  is divisible by 4 and the proof is complete. □

### 3 Proof of Theorems 1.3, 1.4, 1.5 and 1.6

#### 3.1 Behavior of $q$ -holonomic sequences under differentiation

In this section we prove Theorem 1.4.

*Proof* Consider a  $q$ -holonomic sequence  $f_n(q) \in \mathbb{Q}(q)$  that satisfies a linear recursion relation

$$\sum_{j=0}^d a_j(q^n, q) f_{n+j}(q) = 0$$

where  $a_j(u, v) \in \mathbb{Q}[u, v]$  for  $j = 0, \dots, d$ . In the proof, the choice of coefficients  $\mathbb{Q}[u, v]$  versus  $K[u, v]$  for a number field  $K$  does not play an important role, and therefore we assume  $K = \mathbb{Q}$ . Differentiate with respect to  $q$ . We obtain that

$$\sum_{j=0}^d a_j(q^n, q) f'_{n+j}(q) + nq^{n-1} \sum_{j=0}^d a_{j,u}(q^n, q) f_{n+j}(q) + \sum_{j=0}^d a_{j,v}(q^n, q) f_{n+j}(q) = 0 \tag{16}$$

where  $a_{j,u} = \partial a_j / \partial u$ ,  $a_{j,v} = \partial a_j / \partial v$  and  $f'(q) = df(q)/dq$ . Recall now that the product and the  $\mathbb{Q}[q]$ -linear combination of two  $q$ -holonomic sequences is  $q$ -holonomic; see [39,41]. Lemma 3.1 implies that for every  $j$ , the sequences  $(nq^{n-1} a_{j,u}(q^n, q) f_{n+j}(q))$  and  $(a_{j,v}(q^n, q) f_{n+j}(q))$  are  $q$ -holonomic. It follows that the second and third sum in Eq. (16) is  $q$ -holonomic, i.e., it satisfies a linear recursion relation with coefficients in  $\mathbb{Q}[q^n, q]$ . Substituting in this linear recursion the first sum of Eq. (16), it follows that the sequence  $f'_n(q)$  is  $q$ -holonomic.  $\square$

**Lemma 3.1** *For every fixed integer  $c$ , the sequence  $(nq^{cn})$  is  $q$ -holonomic.*

*Proof* It is easy to verify that the sequence  $f_n(q) = nq^{cn}$  satisfies linear recursion relation

$$f_{n+2}(q) - 2q^c f_{n+1}(q) + q^{2c} f_n(q) = 0$$

$\square$

#### 3.2 Evaluations of $q$ -holonomic sequences at roots of unity are holonomic

Consider a  $q$ -holonomic sequence  $f_n(q) \in \mathbb{Z}[q^{\pm 1}]$  that satisfies a linear recursion relation of the form

$$c_d(q^n, q) f_{n+d}(q) + \dots + c_0(q^n, q) f_n(q) = 0 \tag{17}$$

for all  $n$  where  $c_j(u, v) \in \mathbb{Q}[u, v]$  are polynomials in two variables for  $j = 0, \dots, d$ . The field  $\mathbb{Q}$  of coefficients does not matter here and can be replaced by  $\mathbb{C}$  without any change. Fix a complex root of unity  $\zeta$ . The idea of the proof consists of a lucky case, and a general case which reduces to a lucky case after sufficient differentiation. The differentiation trick appears in an efficient algorithm to compute the Kashaev invariant of some knots in [15], and was also inspired by conversations of the first author with D. Zagier. Let us first discuss the proof of Theorem 1.5 when  $\zeta = 1$ . In the lucky case, the set

$$S = \{j \mid 0 \leq j \leq d, c_j(1, 1) \neq 0\}$$

is non-empty. Expand (17) as a power series in  $q - 1$ . The vanishing of the constant term implies that  $f_n(1)$  satisfies a non-trivial linear recursion with constant coefficients

$$\sum_{j \in S} c_j(1, 1) f_{n+j}(1) = 0$$

Thus, the sequence  $(f_n(1))$  is holonomic. In the general case, there exists a unique natural number  $s \geq 0$  such that  $c_j(q^n, q) = \gamma_j(q - 1)^s + O((q - 1)^{s+1})$  and some  $\gamma_j$  is nonzero. In other words,  $c_j(q^n, q)$  vanish to order  $s - 1$  at  $q = 1$  for all  $i$ , and some  $c_j(q^n, q)$  does not vanish to order  $s$  at  $q = 1$ . In that case, observe that

$$\sum_{j \in S'} c_j^{(s)}(1, 1) f_{n+j}(1) = 0$$

where

$$S' = \{j \mid 0 \leq j \leq d, \gamma_j \neq 0\} \neq \emptyset$$

and  $c_j^{(s)}(1, 1) \in \mathbb{Q}[n]$ . This concludes the proof of Theorem 1.5 when  $\zeta = 1$ .

Suppose now that  $\zeta$  is a complex root of unity of order  $N$ . The problem is that  $c_j^{(s)}(\zeta^n, \zeta)$  is no longer a polynomial of  $n$  even when  $s = 0$ . To overcome this obstacle, we consider  $n$  to be in a fixed arithmetic progression modulo  $N$ . In other words, fix  $i$  with  $0 \leq i \leq N - 1$  and replace  $n$  by  $Nn + i$  in (17). Then,  $c_j(q^{Nn+i}, q)$  can be expanded in powers of  $q - \zeta$  with coefficients in  $\mathbb{Q}(\zeta)[n]$  for all  $i$  and  $j$ . In other words, for all  $i, j$  we have  $c_{ij}(q) := c_j(q^{Nn+i}, q) \in \mathbb{Q}(\zeta)[n][[q - \zeta]]$ . Let us consider the column vector  $x_n = (f_{Nn}(\zeta), \dots, f_{Nn+N-1}(\zeta)) \in \mathbb{C}^N$ . If  $c_{i,j}(\zeta) \neq 0$ , then  $f_{Nn+i+j}(\zeta)$  is a  $\mathbb{Q}(\zeta)[n]$ -linear combination of  $f_{Nn+i+j'}(\zeta)$  for  $j' \neq j$ . Of course,  $f_{Nn+i+j}(\zeta)$  is the  $k$ th coordinate of  $x_n$  where  $k \equiv i + j \pmod N$ .

Now we consider two cases. In the lucky Case 1, the following set

$$S = \{(i, j) \in [0, N - 1] \times [0, d] \mid c_{i,j}(\zeta) \neq 0\}$$

is non-empty.  $S$  defines a graph  $G(S)$  as follows. It has vertex set  $\{0, 1, \dots, N - 1\}$ , and an edge  $e$  between vertices  $k$  and  $k'$  iff there exist  $(i, j), (i', j') \in S$



such that  $k \equiv i + j \pmod N$  and  $k' \equiv i + j' \pmod N$ . We consider two subcases. In the very lucky Case 1.1,  $G(S)$  is connected. The above discussion implies that the vector  $x_n$  and finitely many of its translates, put together in a column vector  $y_n$  satisfy a first order linear recursion of the form

$$y_{n+1} = A(n)y_n$$

for a square matrix  $A(n)$  with coefficients in the field  $\mathbb{Q}(\zeta)(n)$ . Lemma 3.2 below implies that every coordinate of  $y_n$  is holonomic. It follows that  $(f_{Nn+i}(\zeta))$  is holonomic (with respect to  $n$ ) for every fixed  $i \in [0, N - 1]$ . Since  $(a_n)$  is holonomic if and only if  $(a_{Nn+i})$  is holonomic for all  $i \in [0, N - 1]$  (see [29]), it follows that  $f_n(\zeta)$  is holonomic in the very lucky Case 1.1.

In the not-so-lucky Case 1.2, the set  $S$  is non-empty but the graph  $G(S)$  is disconnected. In that case, differentiate (17) once and consider the pair of column vectors  $x_n^r = (f_{Nn}^{(r)}(\zeta), \dots, f_{Nn+N-1}^{(r)}(\zeta)) \in \mathbb{C}^N$  for  $r = 0, 1$ . Consider also the set

$$S^r = \{(i, j) \in [0, N - 1] \times [0, d] \mid c_{i,j}^{(r)}(\zeta) \neq 0\}$$

for  $r = 0, 1$ .  $S^0 \cup S^1$  gives rise to a graph  $G(S^0 \cup S^1)$  with vertices pairs  $(r, k)$  for  $r = 0, 1$  and  $k = 0, \dots, N - 1$ . There is an edge between  $(r, k)$  and  $(r', k')$  if there exist  $(i, j) \in S^r, (i, j') \in S^{r'}$  such that  $k \equiv i + j \pmod N$  and  $k' \equiv i + j' \pmod N$ . Note that by assumption,  $S^0 = S$  is non-empty, and there is a natural inclusion of  $G(S)$  into  $G(S^0 \cup S^1)$ . In the not-so-lucky Case 1.2, the graph  $G(S^0 \cup S^1)$  is disconnected. The argument of Case 1.1 shows that  $f_n(\zeta)$  is holonomic. If  $G(S^0 \cup S^1)$  is disconnected, differentiate (17) once more. It is easy to see that after a finite number  $t$  of differentiations, the graph  $G(S^0 \cup \dots \cup S^t)$  will be connected and consequently  $f_n(\zeta)$  is holonomic. Differentiating (17)  $t$  times and using induction implies that  $f_n(\zeta)$  is holonomic.

This concludes the proof of Theorem 1.5 when  $S$  is non-empty. In the unlucky Case 2 where all functions  $c_{i,j}(q)$  vanish to first order at  $q = \zeta$ , there is a natural number  $s$  such that  $c_{i,j}^{(s')}(\zeta) = 0$  for all  $s' < s$  and all  $i, j$  and in addition  $c_{i,j}^{(s)}(\zeta) \neq 0$  for some  $i, j$ . In that case, replace  $S$  by the set

$$S' = \{(i, j) \mid 0 \leq j \leq d, 0 \leq i < N - 1, c_{i,j}^{(s)}(\zeta) \neq 0\}$$

and repeat the above proof.

**Lemma 3.2** Consider a sequence of vectors  $x_n \in \mathbb{C}^l$  that satisfy a first order linear recursion of the form

$$x_{n+1} = A(n)x_n$$

where  $A(n)$  is an  $l \times l$  matrix with coefficients in  $\mathbb{C}[n]$ . Then every coordinate of  $x_n$  is a holonomic sequence.

*Proof* Consider the operators  $\nu$  and  $N$  which act on a sequence  $(a_n)$  by

$$(\nu a)_n = na_n, \quad (Na)_n = a_{n+1}.$$

These operators satisfy the commutation relation  $N\nu = \nu N + N$  and generate an Ore ring  $R = \mathbb{C}[\nu](N)$ . The matrix  $A(n)$  defines an  $R$ -module  $M$  as in [34]. The lemma follows from the *cyclic vector theorem* applied to the module  $M$ ; see [34, Prop.2.9] and also [18]. The proof is constructive and can be implemented for example in [1]. □

This finishes the proof of Theorem 1.5. □

Note that Theorem 1.5 fails when  $q$  is not a root of unity. For example, consider the  $q$ -holonomic sequence  $a_n(q) = q^{n^2}$  and fix  $q = \omega$ , a complex number which is not a root of unity. Suppose that the sequence  $b_n = \omega^{n^2}$  is holonomic. So for all natural numbers  $n$  we have

$$\sum_{k=0}^d c_k(n)b_{n+k} = 0$$

where  $c_j(n) \in \mathbb{Q}[n]$ . Dividing by  $b_n$  we get

$$\sum_{k=0}^d c_k(n)\omega^{2nk+k^2} = 0.$$

Collecting the coefficients of a fixed power of  $n$ , we find that there are  $C_1 \dots C_D \in \mathbb{C}$  such that

$$\sum_{k=0}^D C_k \omega^{nk} = 0$$

In other words, for all  $n$ ,  $\omega^n$  is a root of the polynomial  $\sum_{k=0}^D C_k x^k$ . It follows that  $\omega^m = 1$  for some  $m$ .

### 3.3 Proof of Theorem 1.6

In this section we prove Theorem 1.6. Recall from Remark 1.7 that this theorem is special to the quantum spin network evaluations and not valid for general  $q$ -holonomic sequences.

The main ideas that go into the proof of Theorem 1.6 are the following:

- (a) its validity for the local building block of quantum spin networks (namely the quantum  $6j$ -symbol),
- (b) The shadow formula (15) for the evaluation of an arbitrary quantum spin network where the summand is roughly a product of quantum  $6j$ -symbols divided by quantum  $3j$ -symbols,

- (c) a lemma for the growth rate of the coefficients of inverse quantum factorials, in the spirit of [15, Thm.15].

Let us begin with the first ingredient.

**Lemma 3.3** *Theorem 1.6 holds for the quantum  $6j$ -symbols.*

*Proof* Equation (12) shows that  $f_n(q) = \langle \triangle, n\gamma \rangle$  is a Laurent polynomial in  $q^{1/2}$  which is written as a one-dimensional sum. Let  $\|h(q)\|_1$  denote the sum of the absolute values of a Laurent polynomial in  $q^{1/4}$ . It is well known that the quantum binomial coefficient (10) is a Laurent polynomial in  $q^{1/2}$  with nonnegative integer coefficients. Moreover, its evaluation at  $q = 1$  is the usual multinomial coefficient. The sum of the latter is at most  $r^a$ . It follows that

$$\left\| \begin{bmatrix} a \\ a_1, a_2, \dots, a_r \end{bmatrix} \right\| = \frac{a!}{a_1! \dots a_r!} \leq r^a.$$

Equation (12) and the above inequality conclude the proof of the lemma. □

Now consider an arbitrary quantum spin network  $(\Gamma, \gamma)$  and its evaluation  $f_N(q) = \langle \Gamma, N\gamma \rangle(q)$ . Viewing  $1j$ -symbols as special cases of  $6j$ -symbols and crossings as a power of  $q$  times a  $6j$ -symbol, the shadow state-sum formula (15) expresses  $f_N(q)$  by

$$f_N(q) = \sum_{k \in NP} \frac{M(k)}{D(k)} \tag{18}$$

where the summation is over the set of lattice points of  $NP$  (for some rational convex polytope  $P$ ) and  $M(k)$  is a product a power of  $q$  and of quantum  $6j$ -symbols at linear forms of  $k$ , and  $D(k)$  is a product of quantum factorials at linear forms of  $k$ . The fact that  $D$  is a product of quantum factorials only follows from the formula for the  $3j$ -symbol, see Eq. (11). We know that  $f_N(q) \in \mathbb{Z}[q^{\pm 1/4}]$  by Proposition 1.1. Theorem 1.5 implies that the exponents of  $q$  in  $\langle \Gamma, N\gamma \rangle(q)$  are bounded above and below by quadratic functions of  $N$ .

Let  $(q)_n = \prod_{j=1}^n (1 - q^j)$  for  $n \in \mathbb{N} \cup \{\infty\}$ . It is well-known that

$$\frac{1}{(q)_\infty} = \sum_{k=0}^{\infty} p_k q^k$$

where  $p_k \in \mathbb{N}$  is the number of partitions of  $k$ ; see [2]. Moreover, for every  $k$  and  $n$ , the coefficient of  $q^k$  in  $1/(q)_n$  is a natural number which is at most  $p_k$ . In addition, we have

$$p_k \leq e^{Ck^{1/2} + o(1)}$$

where  $C = \pi\sqrt{2/3}$ ; see [2]. It follows that the coefficient of  $q^n$  in  $1/D(k)$  is exponentially bounded by  $n^2$ , and since  $n = O(N^2)$ , the coefficient is exponentially bounded

by  $N$ . Since the number of  $k$ -summation terms in the state-sum (18) is bounded by a polynomial function of  $N$ , this completes the proof of Theorem 1.6.  $\square$

*Remark 3.4* The above proof is similar in spirit with the proof of [15, Thm.15].

### 3.4 Proof of Theorem 1.3

Fix a quantum spin network  $(\Gamma, \gamma)$ . It follows from the shadow state sum in Sect. 2.3 and Lemma 2.3 that the standard evaluation  $\langle \Gamma, n\gamma \rangle(q)$  is a sum of a balanced  $q$ -hypergeometric term. Hence, by the fundamental theorem of WZ theory (see [39]), it follows that  $\langle \Gamma, n\gamma \rangle(q)$  is  $q$ -holonomic. Since  $\langle\langle \Gamma, n\gamma \rangle\rangle(q) = q^{Q(n\gamma)/4} \langle \Gamma, n\gamma \rangle(q)$  and  $Q(n\gamma)$  is periodic,  $\langle\langle \Gamma, n\gamma \rangle\rangle(q)$  is  $q$ -holonomic as well.

Fix a complex root of unity  $\zeta$ , and a natural number  $r \geq 0$ , and let  $a_n = \langle\langle \Gamma, n\gamma \rangle\rangle^{(r)}(\zeta)$ . We will apply Theorem 1.5 to check conditions (a)–(d) of Definition 1.3.

- (a)  $a_n \in \mathbb{Z}[\zeta]$  as follows from Proposition 1.1.
- (b) Theorem 1.6 shows that the coefficients  $\langle\langle \Gamma, n\gamma \rangle\rangle^{(r)}(\zeta)$  of the generating function are exponentially bounded. The same goes for their Galois conjugates since the action of the Galois group can only replace the root of unity  $\zeta$  by another root of unity, so Theorem 1.6 still applies.
- (c) Since  $a_n \in \mathbb{Z}[\zeta]$  are algebraic integers, their common denominator is 1.
- (d) By Theorems 1.4 and 1.5 the coefficients of the power series are holonomic, hence the series itself is holonomic as well.

This concludes the proof of Theorem 1.3.  $\square$

## 4 Examples

In this section we consider various simple examples to illustrate the theorems. For the sake of simplicity we suppress the function  $A^{-Q(\gamma)}$  so we choose to work with  $\langle \Gamma, \gamma \rangle$ , rather than the more correct  $\langle\langle \Gamma, \gamma \rangle\rangle$ .

### 4.1 Evaluations of the $q$ -multinomial coefficients at a root of unity

**Lemma 4.1** *Let  $q = e^{\frac{2\pi i}{N}}$  and set  $a_i = A_i N + \alpha_i$ . We have*

$$\begin{bmatrix} a_1 + \dots + a_n \\ a_1, a_2, \dots, a_n \end{bmatrix} (q) = q^{\frac{1}{2} \sum_{j < k} \alpha_j \alpha_k - a_j a_k} \binom{A_1 + \dots + A_n}{A_1, A_2, \dots, A_n} \begin{bmatrix} \alpha_1 + \dots + \alpha_n \\ \alpha_1, \alpha_2, \dots, \alpha_n \end{bmatrix} (q) \tag{19}$$

This was proven for Gaussian (asymmetric) binomial coefficients in [9]. The extension to the multinomial case is straightforward and the power of  $q$  appears when one converts to symmetric multinomial coefficients. Note that the power of  $q$  is actually a sign since  $\alpha_j \alpha_k - a_j a_k = -N(A_j A_k N + \alpha_j A_k + A_j \alpha_k)$ .

Now consider the growth rates of the theta evaluations. Fix and admissible triple  $a, b, c \in \mathbb{N}$ , and set  $A = \frac{-a+b+c}{2}$ ,  $B = \frac{a-b+c}{2}$ ,  $C = \frac{a+b-c}{2}$  then we have

$$\langle \Theta, an, bn, cn \rangle = \langle \bigcirc, n(A + B + C) \rangle \begin{bmatrix} n(A + B + C) \\ nA, nB, nC \end{bmatrix}$$

At  $q = 1$  we find using *Stirling's formula* (see [23])

$$\Lambda_1 = \left\{ (-1)^{A+B+C} \frac{(A + B + C)^{A+B+C}}{A^A B^B C^C} \right\}$$

At the  $N$ th root of unity we find

$$\langle \Theta, an, bn, cn \rangle (e^{\frac{2\pi i}{N}})^n = P(n) \begin{pmatrix} \lfloor \frac{nA}{N} \rfloor + \lfloor \frac{nB}{N} \rfloor + \lfloor \frac{nC}{N} \rfloor \\ \lfloor \frac{nA}{N} \rfloor, \lfloor \frac{nB}{N} \rfloor, \lfloor \frac{nC}{N} \rfloor \end{pmatrix}$$

where  $P(n)$  is a periodic function in  $n$  of period  $2N$ . Another application of Stirling shows that  $\Lambda_N \subset \Omega_{2N} |\Lambda_1|^{\frac{1}{N}}$ , where  $\Omega_N$  denotes the set of all  $N$ th roots of unity.

### 4.2 The $q$ -difference equation of the regular $6j$ -symbol

We compute a second order recursion relation for the regular quantum  $6j$ -symbol

$$a_n(q) = \langle \triangle, n \gamma \rangle(q), \quad \gamma = (2, 2, 2, 2, 2, 2).$$

using the  $q$ -WZ method, implemented in *Mathematica* by [28] and used as in [16]. The recursion has the form

$$\sum_{k=0}^2 c_k(q, q^n) a_{n-k}(q) = 0 \tag{20}$$

where

$$\begin{aligned} c_0(q, q^n) &= q^{2+12n} (q - q^n)^3 (-1 + q^n)^5 (q + q^n) \left( 4q^{10} + 4q^{10n} + 6q^{9+n} + 4q^{8+2n} - q^{6+3n} \right. \\ &\quad + 5q^{7+3n} - q^{8+3n} - 6q^{5+4n} - 2q^{6+4n} - 6q^{7+4n} - 4q^{4+5n} + 2q^{5+5n} - 4q^{6+5n} \\ &\quad \left. - 6q^{3+6n} - 2q^{4+6n} - 6q^{5+6n} - q^{2+7n} + 5q^{3+7n} - q^{4+7n} + 4q^{2+8n} + 6q^{1+9n} \right) \\ c_1(q, q^n) &= -q^{-1+6n} (q - q^n)^3 (q - q^{2n}) \left( -4q^{14} - 4q^{15} + q^{18n} + 7q^{19n} + 10q^{20n} + 10q^{21n} \right. \\ &\quad + 7q^{22n} + q^{23n} - 10q^{13+n} - 8q^{14+n} - 10q^{15+n} - 10q^{12+2n} + 6q^{13+2n} + 6q^{14+2n} \\ &\quad - 10q^{15+2n} + q^{10+3n} - 9q^{11+3n} + q^{12+3n} + 28q^{13+3n} + q^{14+3n} - 9q^{15+3n} + q^{16+3n} \\ &\quad + 7q^{9+4n} + 4q^{10+4n} + 14q^{11+4n} + 13q^{12+4n} + 13q^{13+4n} + 14q^{14+4n} + 4q^{15+4n} \\ &\quad + 7q^{16+4n} + 10q^{8+5n} + 2q^{9+5n} + 22q^{10+5n} + 36q^{11+5n} + 36q^{13+5n} + 22q^{14+5n} \\ &\quad \left. + 2q^{15+5n} + 10q^{16+5n} + 10q^{7+6n} - 3q^{8+6n} - 16q^{9+6n} + 4q^{10+6n} - 17q^{11+6n} \right) \end{aligned}$$

$$\begin{aligned}
 & -17q^{12+6n} + 4q^{13+6n} - 16q^{14+6n} - 3q^{15+6n} + 10q^{16+6n} + 7q^{6+7n} - 3q^{7+7n} \\
 & + q^{8+7n} + 6q^{9+7n} - 3q^{10+7n} + q^{11+7n} - 3q^{12+7n} + 6q^{13+7n} + q^{14+7n} - 3q^{15+7n} \\
 & + 7q^{16+7n} + q^{5+8n} - 18q^{6+8n} - 31q^{7+8n} - 7q^{8+8n} - 47q^{9+8n} - 37q^{10+8n} \\
 & - 37q^{11+8n} - 47q^{12+8n} - 7q^{13+8n} - 31q^{14+8n} - 18q^{15+8n} + q^{16+8n} - 8q^{5+9n} \\
 & - 18q^{6+9n} + 4q^{7+9n} - 18q^{8+9n} - 12q^{9+9n} - 8q^{10+9n} - 12q^{11+9n} - 18q^{12+9n} \\
 & + 4q^{13+9n} - 18q^{14+9n} - 8q^{15+9n} - 9q^{4+10n} - 8q^{5+10n} + 9q^{6+10n} - 7q^{7+10n} \\
 & + 13q^{8+10n} + 13q^{9+10n} + 13q^{10+10n} + 13q^{11+10n} - 7q^{12+10n} + 9q^{13+10n} - 8q^{14+10n} \\
 & - 9q^{15+10n} - 4q^{3+11n} - 2q^{4+11n} - q^{5+11n} - 13q^{6+11n} - 24q^{7+11n} + 2q^{8+11n} \\
 & - 43q^{9+11n} + 2q^{10+11n} - 24q^{11+11n} - 13q^{12+11n} - q^{13+11n} - 2q^{14+11n} - 4q^{15+11n} \\
 & + 11q^{3+12n} + 31q^{4+12n} + 21q^{5+12n} + 44q^{6+12n} + 53q^{7+12n} + 56q^{8+12n} + 56q^{9+12n} \\
 & + 53q^{10+12n} + 44q^{11+12n} + 21q^{12+12n} + 31q^{13+12n} + 11q^{14+12n} - 4q^{2+13n} + 18q^{3+13n} \\
 & - 12q^{4+13n} - 14q^{5+13n} - 26q^{6+13n} - 40q^{7+13n} - 36q^{8+13n} - 40q^{9+13n} - 26q^{10+13n} \\
 & - 14q^{11+13n} - 12q^{12+13n} + 18q^{13+13n} - 4q^{14+13n} + 11q^{2+14n} + 31q^{3+14n} + 21q^{4+14n} \\
 & + 44q^{5+14n} + 53q^{6+14n} + 56q^{7+14n} + 56q^{8+14n} + 53q^{9+14n} + 44q^{10+14n} + 21q^{11+14n} \\
 & + 31q^{12+14n} + 11q^{13+14n} - 4q^{1+15n} - 2q^{2+15n} - q^{3+15n} - 13q^{4+15n} - 24q^{5+15n} \\
 & + 2q^{6+15n} - 43q^{7+15n} + 2q^{8+15n} - 24q^{9+15n} - 13q^{10+15n} - q^{11+15n} - 2q^{12+15n} \\
 & - 4q^{13+15n} - 9q^{1+16n} - 8q^{2+16n} + 9q^{3+16n} - 7q^{4+16n} + 13q^{5+16n} + 13q^{6+16n} \\
 & + 13q^{7+16n} + 13q^{8+16n} - 7q^{9+16n} + 9q^{10+16n} - 8q^{11+16n} - 9q^{12+16n} - 8q^{1+17n} \\
 & - 18q^{2+17n} + 4q^{3+17n} - 18q^{4+17n} - 12q^{5+17n} - 8q^{6+17n} - 12q^{7+17n} - 18q^{8+17n} \\
 & + 4q^{9+17n} - 18q^{10+17n} - 8q^{11+17n} - 18q^{1+18n} - 31q^{2+18n} - 7q^{3+18n} - 47q^{4+18n} \\
 & - 37q^{5+18n} - 37q^{6+18n} - 47q^{7+18n} - 7q^{8+18n} - 31q^{9+18n} - 18q^{10+18n} + q^{11+18n} \\
 & - 3q^{1+19n} + q^{2+19n} + 6q^{3+19n} - 3q^{4+19n} + q^{5+19n} - 3q^{6+19n} + 6q^{7+19n} + q^{8+19n} \\
 & - 3q^{9+19n} + 7q^{10+19n} - 3q^{1+20n} - 16q^{2+20n} + 4q^{3+20n} - 17q^{4+20n} - 17q^{5+20n} \\
 & + 4q^{6+20n} - 16q^{7+20n} - 3q^{8+20n} + 10q^{9+20n} + 2q^{1+21n} + 22q^{2+21n} + 36q^{3+21n} \\
 & + 36q^{5+21n} + 22q^{6+21n} + 2q^{7+21n} + 10q^{8+21n} + 4q^{1+22n} + 14q^{2+22n} + 13q^{3+22n} \\
 & + 13q^{4+22n} + 14q^{5+22n} + 4q^{6+22n} + 7q^{7+22n} - 9q^{1+23n} + q^{2+23n} + 28q^{3+23n} + q^{4+23n} \\
 & - 9q^{5+23n} + q^{6+23n} - 10q^{1+24n} + 6q^{2+24n} + 6q^{3+24n} - 10q^{4+24n} - 10q^{1+25n} \\
 & - 8q^{2+25n} - 10q^{3+25n} - 4q^{1+26n} - 4q^{2+26n} )
 \end{aligned}$$

$$\begin{aligned}
 c_2(q, q^n) = & \frac{1}{q^8} (1 + q^n) (q^2 - q^{3n})^4 (q^4 - q^{3n})^4 (q^2 + q^{2n} + q^{1+n})^4 \left( -4q + q^{3n} + 6q^{4n} + 4q^{5n} \right. \\
 & + 6q^{6n} + q^{7n} - 6q^{1+n} - 4q^{1+2n} - 5q^{1+3n} + q^{2+3n} + 2q^{1+4n} + 6q^{2+4n} - 2q^{1+5n} \\
 & \left. + 4q^{2+5n} + 2q^{1+6n} + 6q^{2+6n} - 5q^{1+7n} + q^{2+7n} - 4q^{1+8n} - 6q^{1+9n} - 4q^{1+10n} \right).
 \end{aligned}$$

The above recursion can be used to deduce a recursion relation for the sequence  $a_n(\zeta)$  at every fixed root of unity  $\zeta$ , following and illustrating the proof of Theorem 1.5. For example, to obtain the recursion of the sequence  $a_n(1)$ , use:

$$\begin{aligned}
 c_0(q) &= -4((-1 + n)^3 n^5 (106 - 230n + 115n^2))(q - 1)^{10} + O((q - 1)^{11}) \\
 c_1(q) &= 4(-1 + n)^3 (-1 + 2n)(-864 + 4470n + 2114n^2 - 51003n^3 + 120089n^4 \\
 & \quad - 113505n^5 + 37835n^6)(q - 1)^{10} + O((q - 1)^{11}) \\
 c_2(q) &= -324((8 - 18n + 9n^2)^4 (-9 + 115n^2))(q - 1)^{10} + O((q - 1)^{11}).
 \end{aligned}$$

An alternative method to obtain the recursion relation of the sequence  $a_n(\zeta)$ , using the WZ method, is described in the next section.

### 4.3 The regular 6j-symbol at roots of unity

The regular 6j-symbol corresponds to the labeling  $\gamma = (2, 2, 2, 2, 2, 2)$  and set

$$a_n = \langle \triangle, n \gamma \rangle(1), \quad b_n = \langle \triangle, n \gamma \rangle(-1).$$

Equation (12) gives

$$a_n = \sum_{k=3n}^{4n} (-1)^k \frac{(k + 1)!}{(k - 3n)!^4 (4n - k)!^3}.$$

Using the WZ method (implemented in Mathematica by [28] and using it as in [16]) it follows that  $(a_n)$  satisfies the recursion relation

$$\begin{aligned} & -81(2 + 3n)^4(4 + 3n)^4(451 + 460n + 115n^2)a[n] + \\ & (1 + n)^3(3 + 2n)(319212 + 1427658n + 2578232n^2 + 2423109n^3 + 1255139n^4 + 340515n^5 + 37835n^6)a[1 + n] - \\ & (1 + n)^3(2 + n)^5(106 + 230n + 115n^2)a[2 + n] = 0 \end{aligned}$$

This linear recursion has two complex conjugate formal power series solutions  $a_{\pm, n}$  where

$$\begin{aligned} a_{+, n} = & \frac{1}{n^{3/2}}(329 + 460i\sqrt{2})^n \left( 1 + \frac{-304 - 31i\sqrt{2}}{576n} + \frac{25879 + 18352i\sqrt{2}}{331776n^2} \right. \\ & + \frac{71176912 + 6323071i\sqrt{2}}{573308928n^3} + \frac{5(-2742864803 - 10265264480i\sqrt{2})}{660451885056n^4} \\ & + \frac{-82823449457840 - 12750219708659i\sqrt{2}}{380420285792256n^5} \\ & \left. + \frac{-61273664686901989 + 213495822835779152i\sqrt{2}}{657366253849018368n^6} + O\left(\frac{1}{n^7}\right) \right) \end{aligned}$$

When  $q = -1$ , Eq. (12) and Lemma 4.1 imply that  $b_n = 0$  for odd  $n$ , and

$$b_{2n} = \sum_{k=3n}^{4n} (-1)^k \frac{k!}{(k - 3n)!^4 (4n - k)!^3}.$$

It follows that the sequence  $(b_{2n})$  satisfies the following recursion relation

$$\begin{aligned} & 81(1 + 3n)^4(2 + 3n)^4(3822 + 10779n + 11352n^2 + 5290n^3 + 920n^4)a[n] - \\ & 4(1 + n)^3(41796 + 522342n + 2792052n^2 + 8292059n^3 + 15002105n^4 + 17158765n^5 + 12460766n^6 + 5571784n^7 + 1399895n^8 + 151340n^9) \\ & a[1 + n] + (1 + n)^3(2 + n)^3(25 + 265n + 1002n^2 + 1610n^3 + 920n^4)a[2 + n] = 0 \end{aligned}$$

This linear recursion has two complex conjugate formal power series solutions  $b_{\pm,2n}$  where

$$\begin{aligned}
 b_{+,2n} = & \frac{1}{n^{5/2}} (329 + 460i\sqrt{2})^n \left( 1 + \frac{-472 - 19i\sqrt{2}}{576n} + \frac{105199 + 20632i\sqrt{2}}{331776n^2} \right. \\
 & + \frac{22386472 - 9304565i\sqrt{2}}{573308928n^3} + \frac{-30711007135 - 48640734448i\sqrt{2}}{660451885056n^4} \\
 & + \frac{-80744339543960 + 2822061829369i\sqrt{2}}{380420285792256n^5} \\
 & \left. + \frac{17678244315725891 + 219394408835134568i\sqrt{2}}{657366253849018368n^6} + O\left(\frac{1}{n^7}\right) \right)
 \end{aligned}$$

## 5 Open problems

Fix a cubic ribbon graph  $\Gamma$  with edge set  $E(\Gamma)$ , a complex root of unity  $\zeta$ , and consider the *generating series*

$$S_{\Gamma,\zeta} = \sum_{\gamma} \langle\langle \Gamma, \gamma \rangle\rangle(\zeta) \prod_{e \in E(\Gamma)} z_e^{\gamma_e} \in \mathbb{Q}(\zeta)[[z_e, e \in E(\Gamma)]]$$

In [16, Thm.5], we proved that when  $\zeta = 1$ ,  $S_{\Gamma,1}$  is a rational function, i.e., belongs to the field  $\mathbb{Q}(z_e, e \in E(\Gamma))$ . It was explained in [16, Thm.6] that the generating series  $F_{\Gamma,\gamma,\zeta}$  (4) is a diagonal of  $S_{\Gamma,\zeta}$ , and consequently the rationality of  $S_{\Gamma,\zeta}$  implies that  $F_{\Gamma,\gamma,\zeta}$  is a  $G$ -function. The rationality of  $S_{\Gamma,\zeta}$  follows from the so-called *chromatic evaluation* of classical spin networks. For details, see [16] and also [36]. The chromatic evaluation seems to break down in the case of complex roots of unity.

**Question 5.1** Is it true that for all cubic ribbon graphs  $\Gamma$  and all complex roots of unity,  $S_{\Gamma,\zeta}$  is a rational function, i.e., belongs to the field  $\mathbb{Q}(\zeta)(z_e, e \in E(\Gamma))$ ?

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