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ГЕОМЕТРИЯ И ТОПОЛОГИЯ

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КРИТЕРИЙ КРИСТАЛЛА И ЛОКАЛЬНО АНТИПОДАЛЬНЫЕ МНОЖЕСТВА ДЕЛОНЕ*

Н. П. Долбилин

Доказывается, что в дискретном множестве точек повторяемость локальных конфигураций при определенных условиях имплицирует так называемый «глобальный порядок», который включает в себя наличие у множества кристаллографической группы симметрий. Доказывается также, что множество Делоне, в котором все $2R$-кластеры антиподальны, то есть центрально-симметричны, само является центрально-симметричным в целом относительно каждой своей точки. Более того, если кроме этого кластеры идентичны, то множество является правильным, то есть таким, что его группа симметрий действует транзитивно.

Статья написана по материалам лекции, прочитанной на международной конференции «Квантовая топология» (5–17 июля 2014 г.), организованной лабораторией квантовой топологии Челябинского государственного университета.

Ключевые слова: *множество Делоне, кластер, правильная система, кристаллографическая группа.*

Введение

В работе продолжается начатое в [1] исследование локальных условий, при которых данное множество является правильной системой. Это направление было мотивировано попыткой ответить на вопрос, почему в процессе кристаллизации из аморфного состояния, в котором находятся атомы раствора или расплава, рождается кристаллическая структура, обладающая пространственной группой симметрий.

Физики, кристаллографы (Л. Полинг, Р. Фейнман и др.) считают, что глобальный порядок атомной структуры кристалла вытекает из повторяемости локальных конфигураций, которые возникают в окрестности атомов одного сорта. В частности, Р. Фейнман пишет:

«Если атомы в веществе движутся не слишком активно, они сцепляются и располагаются в конфигурации с минимально возможной энергией. Если атомы где-то разместились так, что их расположения отвечают самой низкой энергии, то в другом месте атомы создадут такое же расположение. Поэтому в твердом веществе расположение атомов повторяется.

Иными словами, условия в кристалле таковы, что каждый атом окружен определенно расположенными другими атомами, и если посмотреть на атом такого же сорта в другом месте, то обнаружится, что окружение его и в новом месте точно такое же. Если вы выберете атом еще дальше, то еще раз найдете точно такие же условия. Порядок повторяется снова и снова, и конечно, во всех трех измерениях…».**

Однако никаких строгих рассуждений в пользу этой концепции не приводилось. В 1974 г. Б. Н. Делоне (совместно с Р. В. Галиулиным) инициировал задачу поиска локальных условий, выполнение которых гарантирует так называемую правильность структуры (правильные

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** Р. Фейнман, Р. Лейтон, М. Сэндс. Фейнмановские лекции по физике. 1965. Вып. 7. С. 5.

системы точек есть важный случай кристаллической структуры). Кристаллограф Н. В. Белов также выдвигал, хотя и не вполне отчетливо, нечто подобное в задаче «про 501-й элемент».

Связь между локальной идентичностью структуры и ее глобальной правильностью представлялась совершенно очевидной, и поиск точной формулировки и строгого доказательства, казалось, имел лишь отвлеченный интерес. Однако приблизительно в то же время (1977 г.) Р. Пенроуз представил знаменитые ныне мозаики, в которых, с одной стороны, локальные конфигурации повторяются, подобно тому, как это происходит в кристалле. С другой стороны, в мозаиках Пенроуза отсутствует периодичность и присутствуют повторяющиеся сколь угодно большие фрагменты с пятиугольной симметрией, что не возможно в кристаллических структурах.

В 1982 г. физик Д. Шехтман получил в лабораторных условиях быстроохлажденный сплав алюминия и марганца с трехмерной квазикристаллической структурой, обладающей симметриями 5-го порядка (Нобелевская премия 2011 г.).

Открытия Пенроуза и Шехтмана указывают на то, что связь между ближним и дальним порядками в структурах не является столь очевидной. Задача здесь состоит в том, чтобы найти правильные формулировки и доказать их.

В следующем параграфе мы введем необходимые понятия и сформулируем основные результаты, полученные по локальной теории кристаллов ранее, а также три результата, доказательство которых будет дано в заключительных трех параграфах этой работы.

1. Основные определения и результаты

Множество $X \subset \mathbb{R}^d$ называется *множеством Делоне* с параметрами r и R (или (r,R)-*системой*, см. [2; 3]), где r, $R > 0$, если для него выполняются два условия:

(1) открытый d-шар $B_y^o(r)$ радиуса r с центром в *произвольной* точке $y \in \mathbb{R}^d$ содержит не более одной точки из X:

$$\#(B_y^o(r) \cap X) \le 1; \qquad (r)$$

(2) любой замкнутый d-шар $B_y(R)$ радиуса R содержит хотя бы одну точку из X:

$$\#(B_y(R) \cap X) \ge 1. \qquad (R)$$

Заметим, что в силу условия (r) расстояние между любыми двумя точками не меньше r.

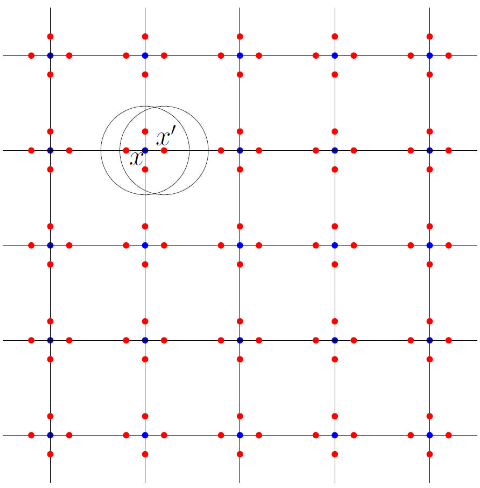

Рис. 1

Пусть $x \in X$, обозначим $C_x(\rho) := X \cap B_x(\rho)$ и будем говорить, что подмножество $C_x(\rho)$ есть ρ-*кластер* точки x. В принципе, под ρ-кластером $C_x(\rho)$ понимается пара (центр, множество точек): $(x, C_x(\rho))$. Информация о ней содержится в самом обозначении $C_x(\rho)$. Подчеркнем, что мы различаем ρ-кластеры $C_x(\rho)$ и $C_{x'}(\rho)$ разных точек x, x', даже если множества точек, входящих в эти кластеры, совпадают (рис. 1).

Два ρ-кластера $C_x(\rho)$ и $C_{x'}(\rho)$ назовем *эквивалентными*, если существует движение g такое, что $g: x \mapsto x'$ и $g: C_x(\rho) \to C_{x'}(\rho)$.

Подчеркнем, что требование эквивалентности кластеров несколько сильнее, чем требование только конгруэнтности множеств точек, входящих в них. Действительно, множества точек в кластерах, рассмотренных в примере (рис. 1), совпадают и, следовательно, конгруэнтны. Хотя сами ρ-кластеры $C_x(\rho)$ и $C_{x'}(\rho)$ не эквивалентны, так как нет изометрии пространства, одновременно совмещающего центры x и x', а также их кластеры $C_x(\rho)$ и $C_{x'}(\rho)$.

Если для данного множества Делоне X при каждом $\rho > 0$ число классов эквивалентных ρ-кластеров конечно, то говорят, что множество X *конечного типа*. Пусть X — множество Делоне конечного типа, обозначим число классов ρ-кластеров через $N(\rho)$.

Нетрудно показать, что конечность числа классов $2R$-кластеров $N(2R) < \infty$ гарантирует конечность числа классов ρ-кластеров для любого фиксированного $\rho > 0$, то есть

$$N(2R) < \infty \Rightarrow N(\rho) < \infty \forall \rho > 2R.$$

Основной аргумент здесь следующий. Из условия $N(2R) < \infty$ вытекает, что в разбиении Делоне для множества X [3] имеется лишь конечное число попарно неконгруэнтных многогранников Делоне. Заметим, что вершины разбиения Делоне суть множества X. Далее, заметим, что два выпуклых многогранника P и Q могут склеиваться по $(d-1)$-мерной грани лишь в конечное число неконгруэнтных между собой пар.

Возьмем точку $x \in X$ и кластер $C_x(2R)$. Точки, входящие в кластер, однозначно определяют все многогранники Делоне разбиения относительно X, сходящиеся в точке x. Так как $N(2R) < \infty$, то число попарно неконгруэнтных многогранников Делоне, встречающихся в разбиении Делоне для данного X, конечно. В силу конечности числа неконгруэнтных многогранников Делоне и упомянутой ранее конечности числа склеек по гиперграням при данном кластере $C_x(2R)$ допускается лишь конечное число различных заполнений шара радиуса ρ многогранниками Делоне, смежными по целым граням. Отсюда следует, что каждый $2R$-кластер $C_x(2R)$ допускает лишь конечное число различных расширений до ρ-кластера $C_x(\rho)$. А так как попарно не конгруэнтных $2R$-кластеров, по предположению, конечное число, то и ρ-расширений также конечное число для любого $\rho > 0$.

Итак, мы будем рассматривать лишь множества Делоне конечного типа. Заметим, что число $N(\rho)$ классов ρ-кластеров в таком множестве Делоне есть положительная, целочисленная, неубывающая, кусочно-постоянная, непрерывная справа функция.

Важными примерами множеств Делоне конечного типа являются понятия правильной системы и кристалла.

Правильная система — это множество Делоне, группа симметрий которого действует транзитивно, то есть для любой пары точек x и x' из X найдется движение g пространства $\mathbb{R}^d$ такое, что $g : x \mapsto x'$ и $g : X \to X$.

Множество $X \subset \mathbb{R}^d$ является правильной системой тогда и только тогда, когда оно является орбитой точки $x \in \mathbb{R}^d$ относительно некоторой кристаллографической группы G, действующей в $\mathbb{R}^d$.

Напомним, что подгруппа $G \subset \mathrm{Iso}(d)$, где $\mathrm{Iso}(d)$ есть группа всех изометрий пространства $\mathbb{R}^d$, называется *кристаллографической группой*:

(1) если G действует разрывно в каждой точке $x \in \mathbb{R}^d$, то есть если орбита $G \cdot x$ дискретна;

(2) имеется компактная фудаментальная область.

Кристаллом называется множество Делоне, которое является орбитой **конечного** множества X_0 относительно некоторой кристаллографической группы G: $G \cdot X_0$.

Таким образом, правильная система является важным случаем кристалла. В терминах перечисляющей функции $N(\rho)$ эти множества выделяются следующим образом. Множество Делоне конечного типа является правильной системой тогда и только тогда, когда $N(\rho) \equiv 1$ на R_+. Множество Делоне является кристаллом тогда и только тогда, когда перечисляющая функция ограничена: $N(\rho) \le m < \infty$, где $m \le \#(X_0)$. Если $m = 1$, то кристалл является правильной системой.

Приведенное определение правильной системы и кристалла восходит к Е. С. Федорову [4]. До него кристалл рассматривался как совокупность конгруэнтных и параллельных друг другу решеток. Федоровское определение кристалла как объединения правильных систем не отрицает, как могло бы показаться на первый взгляд, исходную решеточную концепцию кристалла. Федоров был уверен, что всякая кристаллографическая группа содержит подгруппу трансляций. Более того, он представил рассуждение, которое считал доказательством. В действительности его рассуждение содержало принципиальный пробел. Тем не менее утверждение о существовании подгруппы трансляций верно (см. ниже). Для $d = 2$ его доказательство несложно. Для $d = 3$ теорема была доказана А. Шенфлисом [5]. Задача доказать теорему Шенфлиса для любого $d > 3$ содержалась в 18-й проблеме Гильберта [6] (вопрос о конечности числа (неизоморфных) кристаллографических групп для каждой данной размерности d).

Теорема 1. [Шенфлис: $d = 3$ [5], Бибербах: $\forall d \geq 4$ [7]]. *Кристаллографическая группа $G \subset \mathrm{Iso}(\mathbb{R}^d)$ содержит подгруппу T параллельных переносов пространства конечного индекса h: $G = T \cup Tg_2 \cup \ldots \cup Tg_h$, где индекс h ограничен константой, зависящей от d: $h \leq H(d)$.*

В силу этой теоремы кристалл $G \cdot X_0$ распадается в конечное число ($\leq mh$) конгруэнтных и параллельно расположенных решеток ранга d:

$$G \cdot X_0 = \cup_i^m (T \cdot x_i \cup T \cdot g_2(x_i) \cup \ldots \cup T \cdot g_h(x_i)), x_i \in X_0.$$

Для дальнейшего рассмотрения введем *группу* ρ-кластера $C_x(\rho)$ как подгруппу $S_x(\rho)$ группы $\mathrm{Iso}(d)$, состоящую из тех изометрий s, для которых

$$s\colon x \mapsto x,\ s : C_x(\rho) \mapsto C_x(\rho).$$

Через $M_x(\rho)$ обозначим порядок группы $S_x(\rho)$. Понятно, что функция $M_x(\rho) \geq 1$ — целочисленная функция, определенная на $[0, \infty)$. Она непрерывна слева, кусочно-постоянна, не возрастает. Последнее связано с тем, что при увеличении радиуса ρ в кластер $C_x(\rho)$ вовлекаются новые точки. Поэтому группа $S_x(\rho')$ большего кластера $C_x(\rho')$, $\rho' > \rho$, либо совпадает с $S_x(\rho)$, либо является ее собственной подгруппой.

Пусть X — множество конечного типа. Тогда множество X разбивается в конечное число $N(\rho)$ непересекающихся подмножеств $Y_1, Y_2, \ldots, Y_{N(\rho)}$ таких, что точки x и x' из одного Y_i имеют эквивалентные ρ-кластеры $C_x(\rho)$ и $C_{x'}(\rho)$, а у точек из разных подмножеств Y_i и Y_j ρ-кластеры не эквивалентны. Подчеркнем, что группы эквивалентных ρ-кластеров сопряжены в $\mathrm{Iso}(d)$ и, следовательно, имеют одинаковый порядок $M_i(\rho)$, где i — индекс подмножества Y_i точек с данным классом ρ-кластеров. В дальнейшем у нас Y_i будет обозначать множество точек, ρ-кластеры которых принадлежат классу, помеченному индексом i.

Теперь все готово к тому, чтобы сформулировать некоторые результаты локальной теории кристалла. Результаты были получены в основном в работах М. И. Штогрина и Н. П. Долбилина. Первый строгий результат — критерий правильной системы — был получен в работе [1].

Теорема 2. [Критерий правильной системы]. *Множество Делоне $X \subset \mathbb{R}^d$ с параметрами r, R является правильной системой тогда и только тогда, когда для некоторого $\rho_0 > 0$ выполняются два условия:*

(I) $N(\rho_0 + 2R) = 1$;

(II) $M(\rho_0) = M(\rho_0 + 2R)$.

Условие (I) означает, что $(\rho_0 + 2R)$-кластеры $C_x(\rho_0 + 2R)$ для всех $x \in X$ эквивалентны. Поэтому такие кластеры имеют сопряженные группы симметрий. Условие (II) означает, что при этом для каждого x группы ρ_0- и $\rho_0 + 2R$-кластеров, соответственно, совпадают.

Из критерия правильной системы можно вывести следующее.

Теорема 3. *Для любых d, r, R существует такое $\overline{\rho} = \overline{\rho}(d, r, R)$, что для любого множества Делоне $X \subset \mathbb{R}^d$ с параметрами r и R имеем: если $N(\overline{\rho}) = 1$, то X — правильная система.*

Требование эквивалентности кластеров очень большого радиуса $\overline{\rho}$ объясняется тем, что $2R$-кластеры могут иметь очень богатые группы симметрий, а гарантировать стабилизацию в последовательности групп кластеров на $2R$-шаге мы можем лишь в тот момент, когда последовательность групп «падает» до тривиальной.

Напротив, если же группа $S_x(2R)$ $2R$-кластера тривиальна, то из локального критерия вытекает следующее предложение.

Следствие 1. *Пусть для множества Делоне* $X \subset \mathbb{R}^d$ *имеем* $N(4R)=1$ *и* $M(2R)=1$ *(то есть группа* $2R$*-кластера тривиальна). Тогда* $N(\rho)\equiv 1$ *при каждом* $\rho > 2R$*, то есть* X *— правильная система.*

Отметим, что в силу следующей теоремы требование $N(4R)=1$ нельзя ослабить.

Теорема 4. [О $4R-\varepsilon$]. *Для любого* $\varepsilon > 0$ *существует множество Делоне* $X \subset \mathbb{R}^d$, $d \geq 2$, *с параметрами* r *и* R *такое, что* $N(4R-\varepsilon)=1$*, но* X *не является правильной системой.*

Теорема доказывается предъявлением конструкции. Отметим, что имеющаяся конструкция дает множества Делоне с асимметричными $2R$-кластерами. Таким образом, в классе локально асимметричных множеств Делоне множество — правильное тогда и только тогда, когда все $4R$-кластеры эквивалентны, причем значение $4R$ нельзя уменьшить.

В этом контексте упомянем новый результат (см. теорему 8) о том, что если во множестве Делоне X все $2R$-кластеры центрально симметричны, то их эквивалентность, то есть $N(2R)=1$, гарантирует правильность множества X.

Однако наличие у $2R$-кластера нетривиальной группы симметрий, которая при этом не содержит центральной симметрии, является препятствием для получения хороших значений для радиуса $\bar{\rho}$, которые гарантируют правильность X, то есть такого $\bar{\rho}$, что $N(\bar{\rho})=1$ и, следовательно, X — правильная система.

Теорема 5. [М. И. Штогрин, Н. П. Долбилин]. *Пусть* $X \subset \mathbb{R}^d$ *— множество Делоне с параметрами* r *и* R*. Тогда при* $d=2$ *из равенства* $N(4R)=1$ *следует, что* X *— правильная система; а при* $d=3$ *из* $N(10R)=1$ *вытекает правильность системы* X*.*

В силу теоремы о $4R-\varepsilon$ последняя теорема для плоскости дает неулучшаемый результат. Что касается оценки $N(10R)=1$ для $d=3$, она представляется завышенной. Доказательство основано на лемме.

Лемма 1. [М. И. Штогрин [8]]. *Пусть* $X \subset \mathbb{R}^3$ *— множество Делоне с параметрами* r *и* R*. Если* $N(2R)=1$*, то любая ось поворота в группе* $S_x(2R)$ *— не выше 6-го порядка.*

По этой лемме порядок группы $S_x(2R)$ симметрий при условии $N(2R)=1$ ограничивается настолько, что применение критерия правильности непосредственно дает достаточность условия $N(14R)=1$. Благодаря дополнительным аргументам это требование удалось ослабить до $N(10R)=1$.

В заключение приведем три теоремы, которые будут доказаны в следующих параграфах. Две из них, теоремы 7 и 8, новые. Теорема 6 является обобщением критерия правильной системы.

Теорема 6. [Критерий кристалла; Н. П. Долбилин, М. И. Штогрин]. *Множество Делоне* $X \subset \mathbb{R}^d$ *с параметрами* r, R *является кристаллом, состоящим из* m *правильных систем, тогда и только тогда, когда при некотором* $\rho_0 > 0$ *выполняются два условия:*

(I) $N(\rho_0)=N(\rho_0+2R)=m$;

(II) $M_i(\rho_0)=M_i(\rho_0+2R)$ $\forall i \in [1,m]$.

Локальный критерий кристалла был сформулирован без приведения доказательства (хотя оно и имелось) в [9]. Ниже мы приводим доказательство, в котором часть, относящаяся к доказательству кристаллографичности группы симметрий множества, опирается на другую, более прозрачную идею.

Теорема 7. [Об антиподальности множества Делоне; Н. П. Долбилин]. *Пусть* X *— множество Делоне, в котором каждый* $2R$*-кластер* $C_x(2R)$ *антиподален относительно своего центра* x*. Тогда все множество* X *антиподально относительно каждой своей точки.*

Теорема 8. [Н. П. Долбилин]. *Пусть* X *— множество Делоне с* $N(2R)=1$ *и пусть* $2R$*-кластер* $C_x(2R)$ *симметричен относительно своего центра* x*. Тогда* X *является правильным множеством.*

2. Доказательство теоремы 6

Прежде всего прокомментируем условия (I) и (II) теоремы. Условие (I) означает, что при увеличении радиуса ρ_0 на $2R$ число классов кластеров не увеличивается. В рамках условия (II) ни в одном из m классов соответствующая группа «не падает» при $2R$-расширении ρ_0-кластеров. Смысл теоремы состоит в том, что обусловленная стабилизация двух параметров (число классов и порядок группы кластера) на отрезке $[\rho_0, \rho_0 + 2R]$ имплицирует на самом деле их стабилизацию на **всей** оставшейся полупрямой $[\rho_0 + 2R, \infty)$.

Лемма 2. [О $2R$-цепочке]. *Для каждой пары точек x и $x' \in X$, где X — множество Делоне с параметрами r и R, существует конечная последовательность точек $x_1 = x, x_2, \ldots, x_k = x'$ такая, что расстояние между двумя последовательными точками $|x_i x_{i+1}| < 2R$ для $i \in [1, k-1]$.*

Доказательство. Пусть $|xx'| \geq 2R$. Рассмотрим шар $B_z(R)$ с центром $z \in [xx']$ такой, что точка x лежит на его границе $\partial B_z(R)$. Так как $|xx'| \geq 2R$, то в $B_z(R)$ содержится точка $x_2 \in X$, где $x_2 \neq x'$, $x_2 \neq x$. Ясно, что $|xx_1| < 2R$. По неравенству треугольника $x'x_2 < x'x$.

Пусть $|x_2 x'| \geq 2R$. Рассмотрим шар $B_{z_2}(R)$ с центром $z_2 \in [x_2 x']$ такой, что точка x_2 лежит на границе $\partial B_z(R)$. Так как $|x_2 x'| \geq 2R$, то в $B_{z_2}(R)$ содержится точка $x_3 \in X$, где $x_3 \neq x'$, $x_3 \neq x_2$. Легко видеть, что $|x_2 x_3| < 2R$, и опять по неравенству треугольника получаем $x'x_3 < x'x_2$. Мы получаем последовательность попарно различных точек $x_1 (= x), x_2, x_3, \ldots \in X$ с условием $|x_1 x'| > |x_2 x'| > |x_3 x'| > \ldots$ Последовательность $x_1, x_2, \ldots,$ монотонно приближающаяся к x', содержится в шаре B радиуса $|xx'|$ в точке x'. Множество Делоне X удовлетворяет условию (r), поэтому пересечение $B \cap X$ есть конечное множество точек. Так как всякий раз, когда для точки x_i расстояние $x_i x' \geq 2R$, найдется точка $x_{i+1} \neq x_i$, $x_{i+1} \neq x'$. Но так как последовательность конечна, то в ней найдется точка x_{k-1}, такая что $x_{k-1} x' < 2R$. Итак, показано, что $2R$-цепочка от x к x' существует.

Лемма 3. [О $2R$-продолжении]. *Пусть X — множество Делоне, для которого выполняются оба условия теоремы 1, и пусть x и $x' \in Y_i$. Тогда если $f \in$ Iso — изометрия такая, что*

$$f : x \mapsto x' \text{ и } C_x(\rho_0) \mapsto C_{x'}(\rho_0), \quad (1)$$

то та же изометрия совмещает и концентрические кластеры на $2R$ большего радиуса:

$$f : C_x(\rho_0 + 2R) \mapsto C_{x'}(\rho_0 + 2R). \quad (2)$$

Доказательство. По условию (I) теоремы 6 для точек x и $x' \in Y_i$ их ρ_0-кластеры и $\rho_0 + 2R$-кластеры эквивалентны. Возьмем изометрию f (из условия (1) леммы) и какую-нибудь изометрию g такую, что $g : C_x(\rho_0 + 2R) \mapsto C_{x'}(\rho_0 + 2R)$. Изометрия g существует в силу эквивалентности указанных кластеров.

Рассмотрим суперпозицию изометрий $f^{-1} \circ g$ (порядок здесь: сначала g, затем f^{-1}). Тогда $f^{-1}(g(C_x(\rho_0))) = f^{-1}(C_{x'}(\rho_0)) = C_x(\rho_0)$. Итак, имеем

$$f^{-1} \circ g : x \mapsto x \text{ и } f^{-1} \circ g : C_x(\rho_0) \mapsto C_x(\rho_0). \quad (3)$$

Соотношение (3) показывает, что $f^{-1} \circ g$ является симметрией s ρ-кластера $C_x(\rho_0)$. В силу условия (II) теоремы 6 $s \in S_x(\rho_0 + 2R)$.

Из соотношения $f^{-1} \circ g = s$ следует $g \circ s^{-1} = f$. Тогда

$$f(C_x(\rho_0 + 2R)) = (g \circ s^{-1})(C_x(\rho_0 + 2R)) = g(s^{-1}(C_x(\rho_0 + 2R))) =$$
$$= g(C_x(\rho_0 + 2R)) = C_{x'}(\rho_0 + 2R).$$

Лемма доказана. □

Обозначим для удобства $G := \mathrm{Sym}(X)$.

Лемма 4. [Основная лемма]. *Пусть множество X удовлетворяет условиям (I) и (II) теоремы 6. Тогда группа G действует транзитивно на множестве Y_i при любом $i \in [1, m]$. Более того, для любых точек x, $x' \in Y_i$ и любой изометрии f такой, что $f(C_x(\rho_0 + 2R)) = C_{x'}(\rho_0 + 2R)$, верно $f \in \mathrm{Sym}(X)$*

Доказательство. Так как x, $x' \in Y_i$ имеют эквивалентные (ρ_0+2R)-кластеры, то существует изометрия, совмещающая эти кластеры. Количество таких изометрий равно порядку группы кластера, которая, вообще говоря, может быть нетривиальной. Пусть f — одна из таких изометрий. Докажем, что она является симметрией всего X.

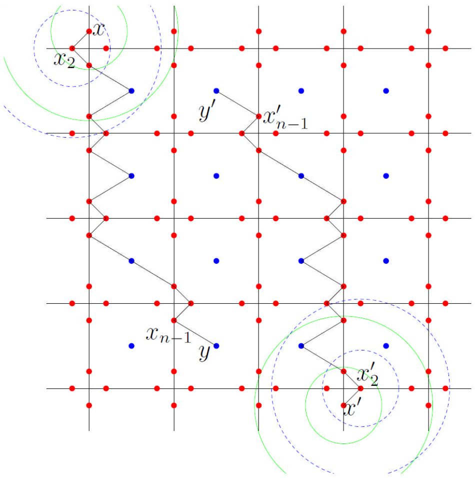

Рис. 2

Сначала докажем, что для произвольной точки $y \in X$ $f(y) \in X$. Соединим точки x и y $2R$-цепочкой $L = \{x_1 = x, x_2, \ldots, x_n = y : | x_i x_{i+1} | < 2R \,\forall i \in [0, n-1]\}$. По условию леммы 3

$$f(C_{x_1}(\rho_0 + 2R)) = C_{x'_1}(\rho_0 + 2R). \tag{4}$$

Так как $|x_1 x_2| < 2R$, отсюда следует, что $C_{x_2}(\rho_0) \subset C_{x_1}(\rho_0 + 2R)$. Поэтому в силу (4) имеем $f : C_{x_2}(\rho_0) \to C_{x'_2}(\rho_0)$. По лемме 3

$$f : C_{x_2}(\rho_0 + 2R) \to C_{x'_2}(\rho_0 + 2R). \tag{5}$$

Из того, что $|x_2 x_3| < 2R$, следует соотношение $C_{x_3}(\rho_0) \subset C_{x_2}(\rho_0 + 2R)$. Поэтому в силу (5) $f: C_{x_3}(\rho_0) \to C_{x'_3}(\rho_0)$. По лемме 3 имеем $f: C_{x_3}(\rho_0 + 2R) \to C_{x'_3}(\rho_0 + 2R)$. Продвигаясь вдоль цепочки L и повторяя это рассуждение конечное число раз, получим, что $2R$-цепочка $L \subset X$ при изометрии f переходит в некоторую $2R$-цепочку $L' \subset X$, а конечная точка y первой цепочки переходит в конечную точку y' второй. Итак, показано, что изометрия f отображает X **в** X. Покажем, что это отображение является отображением **на** все X. Рассмотрим произвольную точку $y'' \in X$ и покажем, что ее прообраз $f^{-1}(y'')$ также принадлежит X. Для этого рассмотрим обратное движение f^{-1}. Из соотношения (4) имеем $f^{-1} : C_{x_{1'}}(\rho_0 + 2R) \to C_{x_1}(\rho_0 + 2R)$. Соединим точки x'_1 с y'' $2R$-цепочкой. Двигаясь вдоль нее, получаем $f^{-1}(y'') = x'' \in X$. Таким образом, при отображении f в произвольную точку y'' переходит некоторая точка x'': $f(x'') = y''$. Лемма доказана. □

Лемма 5. *Если группа $G \subset \mathrm{Iso}(d)$ такова, что для некоторой точки $x \in \mathbb{R}^d$ ее орбита $G \cdot x$ — множество Делоне, то G — кристаллографическая группа.*

Доказательство. Обозначим $X := G \cdot x$, а через $V(X)$ — разбиение пространства на области Вороного относительно орбиты X. Область Вороного V_x для точки x есть выпуклый d-многогранник с конечным числом гиперграней. Это число можно ограничить сверху в терминах параметров r и R. Поэтому группа симметрий многогранника V_x конечна, более того, ее порядок может быть ограничен в зависимости от r и R.

Так как группа G действует на X транзитивно, то и на множестве многогранников (областей Вороного) действует транзитивно. Группа разбиения совпадает с группой $\mathrm{Sym}(X)$ множества X: $\mathrm{Sym}(X) \supset G$. Если точка x не является неподвижной ни для какого движения из G, то область Вороного является фундаментальной областью, которая компактна (условие 2 кристаллографической группы выполнено). Далее, для произвольно выбранной точки x' внутри или на границе замкнутой области Вороного V_x ее орбита будет дискретна (условие 1 в определении кристаллографической области).

Пусть стабилизатор $\mathrm{Stab}(x)$ точки x в группе G не тривиален. Так как $\mathrm{Stab}(x) \subset G$, то группа $\mathrm{Stab}(x)$ вместе с разбиением $V(X)$ оставляет инвариантной область V_x с центром x. Поэтому стабилизатор конечен. Фундаментальная область группы G — это подобласть многогранника. Следовательно, фундаментальная область группы G компактна. Орбита $G \cdot x'$ любой точки x' пересекается с каждой областью Вороного V_x лишь по конечному множеству, то есть $G \cdot x$ дискретна. Итак, G — кристаллографическая группа. Лемма доказана. □

По лемме 5 для любого $i \in [1, m]$ и $x \in Y_i$ множество $Y_i = G \cdot x$. Поэтому, чтобы по лемме 6 доказать кристаллографичность группы G, достаточно доказать, что Y_i есть множество Делоне.

Лемма 6. *В разбиении* $X = \biguplus_i Y_i$ *среди множеств* Y_i, $i \in [1, m]$, *хотя бы одно является множеством Делоне.*

Доказательство. Заметим, что так как X есть множество Делоне с параметрами r и R, любое его подмножество Y_i также удовлетворяет условию r.

Предположим, что множество Y_i не удовлетворяет второму условию ни при каком конечном R'. В этом случае существует бесконечная последовательность шаров $B_1, B_2, \dots, B_k, \dots$ с неограниченно растущими радиусами $R_1 < R_2 < \dots < R_k < \dots \to \infty$, пустых от точек из Y_i. Так как множество Y_i дискретно, то каждый из этих шаров B_k можно сдвинуть так, чтобы на его границе оказалась некоторая точка $x_k \in Y_i$. Так как все точки $x_k \in Y_i$ принадлежат G-орбите, то переведем каждую точку x_k вместе с шаром B_k изометрией $f_k \in \mathrm{Sym}(X)$. Таким образом, можно считать, что для каждого $k = 1, 2, \dots$ точка x лежит на границе пустого шара радиуса R_k. Будем по-прежнему обозначать эти шары через B_k. Обозначим через n_k единичный вектор, отложенный из точки x, направленный по радиусу шара B_k. Из последовательности $\{n_k\}$ выберем сходящуюся подпоследовательность $n_{k_j} \to n$.

Обозначим через Π гиперплоскость, проходящую через точку x перпендикулярно нормали n, а через Π^+ то из двух **открытых** полупространств, в которое смотрит нормаль n. Полупространство Π^+ не содержит ни одной точки из Y_1. Действительно, для любой фиксированной точки $z \in \Pi^+$ в подпоследовательности шаров с неограниченно увеличивающимися радиусами найдется шар B_{k_j}, который содержит точку z. А поскольку все шары пусты от точек из Y_i, то $z \notin Y_i$.

Итак, все точки из Y_i лежат в **замкнутом** полупространстве Π^-. Мы не исключаем, что все точки из Y_i могут лежать на самой гиперплоскости Π. Так как X есть множество Делоне, то полупространство Π^+ непусто от точек из X.

Для каждого $j \in [1, m]$, $j \neq i$, и точки $x \in Y_i$ выберем точку $z \in Y_j$, ближайшую к x. Заметим, что в силу того, что X — множество Делоне, ближайшие точки к x существуют, вообще говоря, их может быть несколько, но конечное число. Пусть минимум $|xz| = \delta_{ij}$. Очевидно, что в силу транзитивной группы G этот минимум не зависит от выбора точки $x \in Y_i$: для другой точки $x' \in Y_i$ найдется точка $z' \in Y_j$ с условием $|x'z'| = |xz| = \delta_{ij}$. Ясно, что $\delta_{ij} = \delta_{ji}$.

Обозначим $\delta_i := \max\limits_{j \in [1, m], j \neq i} \delta_{ij}$.

Рассмотрим плоскость $\Pi + \delta_i \mathbf{n}$. Так как для каждого j, $j \neq i$, а для любого $z \in X_j$ ближайшая к z точка из Y_i удалена не далее чем на δ_i, то **все** множество X лежит в полупространстве $(\Pi + \delta_i \mathbf{n})^-$, что противоречит условию (R) множества Делоне. Лемма доказана. □

Опираясь на доказанные леммы, делаем вывод: множество X с условиями (I) и (II) теоремы 1 можно представить как $X = G \cdot x_1 \cup G \cdot x_2 \cup \dots \cup G \cdot x_m$, где G — кристаллографическая группа и $G \cdot x_i = Y_i$, $i \in [1, m]$. Таким образом, X — это кристалл из m правильных систем = орбит. Теорема доказана.

3. Доказательство теоремы 7

Отметим, что мы не требуем здесь выполнения равенства $N(\rho) = 1$. Более того, не предполагаем даже, что X — множество конечного типа.

Рассмотрим точку $x \in X$ и определим для нее *спектр расстояний* как упорядоченное по возрастанию множество положительных чисел такое, что $\mathrm{R}e_x := \{\rho \in R_+ \mid \exists y \in X, |xy| = \rho\}$.

В силу условия (r) спектр расстояний (для любой данной точки из X) есть строго возрастающая последовательность $\{\rho_1, \rho_2, \dots, \rho_i, \dots\}$, $\rho_{i-1} < \rho_i$, сходящаяся к бесконечности.

Однако объединение таких спектров Re_x (по всем точкам $x \in X$) дискретно тогда и только тогда, когда X — множество конечного типа, что, напомним, не обусловлено в теореме 7.

Рассмотрим для данной точки x_0 спектр $\mathrm{Re}_{x_0} = \{\rho_i\}$ и докажем антиподальность кластеров $C_{x_0}(\rho_i)$ индукцией по индексу i. Пусть уже для всех $i \le k$ доказано, что ρ_i-кластеры $C_{x_0}(\rho_i)$ точек антиподальны. Заметим, что так как по условию теоремы $2R$-кластер любой точки антиподален, то можно считать, что $\rho_k \ge 2R$, а следующий элемент $\rho_{k+1} \in \mathrm{Re}_{x_0}$ уже строго больше $2R$. Установим антиподальность ρ_{k+1}-кластера $C_{x_0}(\rho_{k+1})$.

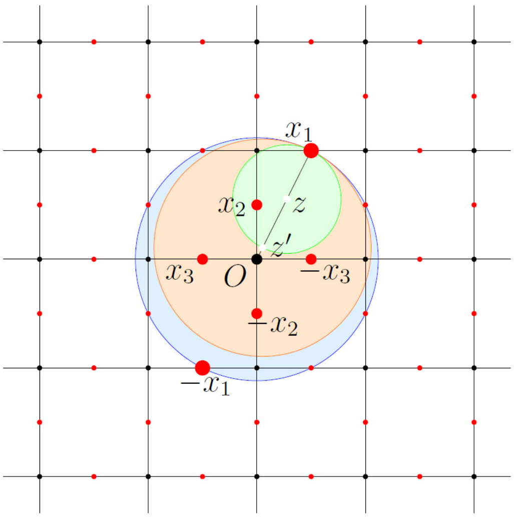

Рис. 3

Для простоты обозначений будем считать, что $x_0 = O$, то есть центр кластера совпадает с началом O координат. Итак, рассмотрим кластер $C_O(\rho_{k+1})$, в котором имеется точка (хотя бы одна) x_1 такая, что $|Ox_1| = \rho_{k+1} > \rho_k \ge 2R$.

Обозначим через $B_z(R)$ шар радиуса R такой, что он касается точки x_1 и центр z его лежит на отрезке $[0x_1]$, а через $B_{z'}(2R)$ — шар $h(B_z(R))$, где h — гомотетия с центром в x_1 и с коэффициентом 2 (рис. 3). Очевидно, что шары $B_z(R)$ и $B_{z'}(2R)$ касаются граничной сферы шара $B_O(\rho_{k+1})$. Так как $\rho_{k+1} > 2R$, центр z' большего шара также лежит на отрезке $[Ox_1]$. Поэтому имеем последовательные вложения: $B_z(R) \subset B_{z'}(2R) \subset B_O(\rho_{k+1})$. Более того, весь шар $B_{z'}(2R)$ за исключением точки x_1, лежит строго внутри шара $B_O(\rho)$.

По условию (R) шар $B_z(R)$ помимо x_1 содержит, по крайней мере, еще одну точку $x_2 \in X$: $x_2 \in B_z(R)$. Ясно, что

$$|x_2x_1| \le 2R. \tag{6}$$

Кластер $C_{x_2}(2R)$ по условию теоремы антиподален относительно его центра x_2, а в силу неравенства (6) $x_1 \in C_{x_2}(2R)$. Поэтому и антиподальная точке x_1 относительно x_2 точка x_3 ($x_2 = \dfrac{x_1 + x_3}{2}$) также принадлежит $C_{x_2}(2R)$. С другой стороны, легко видеть, что $x_3 \in B_{z'}(2R)$ и $x_3 \ne x_1$. Поэтому $|x_3O| < |x_1O| = \rho_{k+1}$. Аналогично имеем $|x_2O| < \rho_{k+1}$.

Итак, обе точки x_2 и x_3 принадлежат кластеру $C_O(\rho_k)$, который по предположению индукции антиподален относительно O. Поэтому антиподальные относительно центра этого кластера точки $-x_2$ и $-x_3$ также принадлежат кластеру $C_O(\rho_k)$. Ясно, что точка $-x_3$ принадлежит кластеру $C_{-x_2}(2R)$. Так как $C_{-x_2}(2R)$ также антиподален относительно $-x_2$, то точка $-x_3$ имеет в кластере $C_{-x_2}(2R)$ симметричную точку, которая, как легко видеть, симметрична точке x_1 относительно O. Таким образом, в кластере $C_O(\rho_{k+1})$ каждая точка x_1 такая, что $|x_1O| = \rho_{k+1}$, имеет антиподальную точку $-x_1$. Таким образом, доказано, что если $C_O(\rho_k)$ антиподален, то $C_O(\rho_{k+1})$ также антиподален. Теорема 7 доказана.

4. Доказательство теоремы 8

Итак, мы предполагаем, что все $2R$-кластеры в X центрально симметричны и, более того, все $2R$-кластеры эквивалентны. Зафиксируем $x \in X$, назовем точку $x' \in X$ *t-эквивалентной* точке x, если существует такая последовательность точек $x_1 = x, x_2, \dots, x_n = x' \in X$, что $y_i = \dfrac{x_i + x_{i+1}}{2} \in X$ и, более того,

$$X \cap [x_ix_{i+1}] = \{x_i, x_{i+1}, y_i\}, \quad |x_iy_i| \le 2R, \quad i \in [1, n-1]. \tag{7}$$

Заметим, что $x_i, x_{i+1} \in C_{y_i}(2R)$. Обозначим через X_x класс всех точек, t-эквивалентных x. Назовем описанную последовательность *t-цепочкой* с началом в x.

Лемма 7. *Класс X_x антиподален относительно любой точки $x' \in X_x$, а также любой точки y_i с условием* (7).

Доказательство. По теореме 7 центральная симметрия $\tau_{x'}$ в точке $x' \in X$ переводит X в себя. При этом, если $x' \in X_x$, то любая t-цепочка, начинающаяся в x', переходит в антиподальную цепочку с тем же началом. Симметрия в точке τ_{y_i}, где $y_i = \frac{x_i + x_{i+1}}{2}$, переводит любую t-цепочку с началом в x_i в t-цепочку с началом в x_{i+1}, так как $x_i, x_{i+1} \in X$, а t-цепочками, начинающимися в любой точке класса X_x, достигаются все точки из X_x и только из X_x. □

Лемма 8. *Класс X_x является множеством Делоне с параметром $\bar{R}$, причем $\bar{R} \le 2R$, где $\bar{R}$ и R — радиусы покрытия для множеств X_x и X, соответственно.*

Доказательство. Заметим, что значение $\bar{R}$ является радиусом покрытия множества X_x тогда и только тогда, когда для каждой точки $x' \in X_x$ любой шар радиуса $\bar{R}$, содержащий точку x' на своей границе, кроме нее содержит еще хотя бы одну точку y из X_x. Множество X имеет радиус R покрытия. Рассмотрим произвольный шар, касающийся точки x. Он содержит помимо нее по крайней мере еще одну точку $y \in X$. Так как $|xy| \le 2R$, то $y \in C_x(2R) \subset X$. Возьмем теперь шар радиуса $2R$, который по-прежнему касается точки x и центр которого лежит на том же луче, что и предыдущий шар. Он содержит точку $x_1 \in X$, симметричную точке x относительно точки y, то есть $y = \frac{x + x_1}{2}$. Таким образом, доказано, что любой шар радиуса $2R$, касающийся точки x из X_x, содержит другие точки из этого класса. Лемма доказана. □

Лемма 9. *Класс X_x есть решетка и любая трансляция этой решетки есть трансляция всего множества X.*

Доказательство. Множество X_x является множеством Делоне с параметрами $\bar{r}, \bar{R}$, так как $\bar{r} \ge r$ в силу $X_x \subseteq X$ и $R \le \bar{R} \le 2R$ по лемме 8.

Покажем, что для каждой пары точек $x, x' \in \bar{X}$ существует параллельный перенос t, являющийся симметрией класса X_x такой, что $x + t = x'$, $X_x + t = X_x$.

Рассмотрим цепочку $\{x = x_1, \ldots, x' = x_k\}$ и последовательность центральных симметрий τ_i в точках $y_i = \frac{x_{i-1} + x_i}{2}$ с условием (7), а также симметрию τ_x в точке x. В силу теоремы 7 каждая из этих симметрий есть симметрия всего множества X, а в силу леммы 7 это есть симметрия класса X_x. Посредством суперпозиции f симметрий τ_i можно отобразить x в x'. Если число симметрий, входящих в суперпозицию, нечетно, то можно добавить к суперпозиции еще одну симметрию $\tau_{x'}$. Тогда f является параллельным переносом. Мы получим параллельный перенос f такой, что f: $x + t = x'$; $\bar{X} + t = X_x$; $X + t = X$.

Так как класс $\bar{X}$ является дискретным множеством, на котором группа параллельных переносов действует транзитивно, то X является решеткой. □

Лемма 10. *Пусть $z \in X \setminus X_x$, тогда существует изометрия g такая, что $g : x \mapsto z$ и g: $X \to X$.*

Доказательство. Так как $N(2R) = 1$, то существует движение g, которое переводит $2R$-кластер $C_x(2R)$ в эквивалентный кластер $C_z(2R)$. Покажем, что $g : X_x \mapsto X_z$. Рассмотрим $x' \in X_x$, покажем, что $g(x') \in X_z$. Для этого построим t-цепочку $\{x_1 = x, x_2, \ldots, x_k = x'\} \subset X_x$, связывающую x с x'. Рассмотрим точку $y_1 = \frac{x_1 + x_2}{2}$ с условием (7). Ясно, что $y_1 \in C_z(2R)$. Изометрия g переводит точку y_1 в некоторую точку $y_1' \in C_z(2R)$. Отрезок $[zy_1']$, как и отрезок $[xy_1]$, пуст внутри от точек из X.

Рассмотрим кластер $C_{y_1'}(2R)$. Так как он центрально симметричен, то существует точка $z_2 \in C_{y_1'} \subset X$ такая, что $y_1' = \frac{z_1 + z_2}{2}$. Ясно, что по построению $z_2 \in X_z$.

Обозначим $t_1 = x_2 - x_1$ и $t_1' = z_2 - z_1$, здесь $z_1 = z$. Тогда по лемме 9

$$C_{x_2}(2R) = C_{x_1}(2R) + t_1 \text{ и } C_{z_2}(2R) = C_{z_1}(2R) + t_1'. \quad (8)$$

С другой стороны, имеем

$$g(C_{x_2}(2R)) = g(C_{x_1}(2R) + t_1) = C_{z_1}(2R) + g(t_1) = C_{z_1}(2R) + t_1'. \quad (9)$$

Из (8) и (9) вытекает, что $g(C_{x_2}(2R)) = C_{z_1}(2R) + t_1' = C_{z_2}(2R)$.

Эти рассуждения можно применить к кластеру $C_{x_2}(2R)$. Для этого обозначим $t_2 := x_3 - x_2$ и $t_2' = z_3 - z_2$. Соотношение (9) переписывается как

$$g(C_{x_3}(2R)) = g(C_{x_2}(2R) + t_2) = C_{z_2}(2R) + g(t_2) = C_{z_2}(2R) + t_{2'} = C_{z_3}(2R). \quad (10)$$

Повторяя эти рассуждения при продвижении вдоль t-цепочки $\{x, \ldots, x'\}$, получаем для $x' \in X_x$, что $g(x') = z' \in X_z$. Более того, g отображает каждую точку $x' \in X_x$ в $z' \in X_z$ вместе с ее ρ-кластером во всем множестве X: $g(C_{x'}(2R)) = C_{z'}(2R)$. Поэтому

$$g : \cup_{x' \in X_x} C_{x'}(2R) \to \cup_{z' \in X_x} C_{z'}(2R). \quad (11)$$

С другой стороны, в силу леммы 8 $2R$-окрестности точек из X_x образуют покрытие: $\cup_{x' \in X_x} B_{x'}(2R) = \mathbb{R}^d$. Следовательно, объединение $2R$-кластеров всех точек из X_x или из X_z совпадает с X: $\cup_{x' \in \bar{X}_x} C_{x'}(2R) = \cup_{z' \in X_z} C_{z'}(2R) = X$.

Поэтому из (11) следует, что $g : X \to X$. □

Список литературы

1. Локальный критерий правильности системы точек / Б. Н. Делоне, Н. П. Долбилин, М. И. Штогрин, Р. В. Галиулин // Докл. АН СССР. — 1976. — Т. 227, № 1. — С. 19–21.

2. Delaunay, B. Sur la sphere vide. A la memoire de Georges Voronoi / B. Delaunay // Изв. АН СССР. VII сер. Отд-е мат. и естеств. наук. — 1934. — № 6. — С. 793–800.

3. Делоне, Б. Н. Геометрия положительных квадратичных форм / Б. Н. Делоне // Успехи мат. наук. — 1937. — № 3. — С. 16–62.

4. Федоров, Е. С. Начала учения о фигурах / Е. С. Федоров. — М. ; Л. : Изд-во АН СССР, 1953. — 410 с.

5. Schoefliess, A. Kristallsysteme und Kristallstruktur / A. Schoefliess. — Leipzig, 1981.

6. Проблемы Гильберта : сборник / под ред. П. С. Александрова. — М. : Наука, 1969. — 240 с.

7. Bieberbach, L. Üeber die Bewegungsgruppen des *n*-dimensionalen Euklidischen Räumes / L. Bieberbach // Math. Ann. — 1911. — Vol. 70. — P. 207–336; 1912. — Vol. 72. — P. 400–412.

8. Штогрин, М. Об ограничении порядка оси паучка в локально правильной системе Делоне / М. Штогрин // Geometry, Topology, Algebra and Number Theory, Applications: Abstracts of the International Conference, dedicated to the 120-th anniversary of Boris Nikolaevich Delone (1890-1980) (Moscow, August 16–20, 2010). — Moscow : Steklov Mathematical Institute, 2010. — P. 168–169.

9. Dolbilin, N. P. A local criterion for a crystal structure / N. P. Dolbilin, M. I. Shtogrin // Abstracts of the 9th All-Union Geometrical Conference. — Kishinev, 1988. — P. 99.

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CRYSTAL CRITERION AND ANTIPODAL DELAUNAY SETS

N. P. Dolbilin

It is proved that a discrete set of points repeatability local configurations under certain conditions implies the so-called «global order», which includes the presence of a plurality of crystallographic symmetry group. It is also proved that the set of Delaunay, in which all $2R$-clusters are antipodal, that is centrally symmetric, is itself a centrally symmetric with respect to each of its points. Moreover, if in addition to this cluster are identical, then the set is correct, i. e. its symmetry group acts transitively.

This article based on a lecture delivered at the International Conference «Quantum topology» (5-17 July 2014), organized by the Laboratory of Quantum Topology of Chelyabinsk State University.

Keywords: *Delaunay set, cluster, the right system, crystallographic group.*

References

1. Delone B.N., Dolbilin N.P., Shtogrin M.I., Galiulin R.V. Lokal'nyy kriteriy pravil'nosti sistemy tochek [Local criterion of correctness of system of points]. *Doklady Akademii Nauk SSSR* [Reports of Academy of Sciences of the USSR], 1976, vol. 227, no. 1, pp. 19–21. (In Russ.).

2. Delaunay B. Sur la sphere vide. A la memoire de Georges Voronoi. *Izvestiya Akademii Nauk SSSR* [News of Academy of Sciences of the USSR], 1934, no 6, pp. 793–800.

3. Delone B.N. Geometriya polozhitel'nykh kvadratichnykh form [Geometry of positive square forms]. *Uspekhi matematicheskikh nauk* [Achievements of mathematical sciences], 1937, no 3, pp. 16–62. (In Russ.).

4. Fedorov E.S. *Nachala ucheniya o figurakh* [I began doctrines about figures]. Moscow, Leningrad, AN SSSR Publ., 1953, 410 p. (In Russ.).

5. Schoefliess A. *Kristallsysteme und Kristallstruktur*. Leipzig, 1981.

6. *Problemy Gil'berta: sbornik* [Gilbert's problems: collection]. Moskva, Nauka Publ., 1969, 240 p. (In Russ.).

7. Bieberbach L. Üeber die Bewegungsgruppen des n-dimensionalen Euklidischen Räumes. *Math. Ann.,* 1911, vol. 70, pp. 207–336; 1912, vol. 72, pp. 400–412.

8. Shtogrin M. Ob ogranichenii poryadka osi pauchka v lokal'no pravil'noy sisteme Delone [About restriction of an order of an axis of a spider in locally correct system Delon]. *Geometry, Topology, Algebra and Number Theory, Applications: Abstracts of the International Conference, dedicated to the 120-th anniversary of Boris Nikolaevich Delone (1890-1980) (Moscow, August 16–20, 2010)*. Moscow, 2010, pp. 168–169. (In Russ.).

9. Dolbilin N.P. A local criterion for a crystal structure. *Abstracts of the 9th All-Union Geometrical Conference.* Kishinev, 1988, pp. 99.

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ББК В151.5

ГЕОМЕТРИЧЕСКИЕ ПРЕДСТАВЛЕНИЯ ДЛЯ ЧЕТНЫХ КУБИЛЯЦИЙ*

Ф. Г. Кораблёв

Для любой четной кубиляции замкнутого трехмерного многообразия строится представление его фундаментальной группы в симметрическую группу степени 6. Вычисляются образы таких представлений для всех четных кубиляций сложности 1.

Ключевые слова: *кубиляция, многообразие, фундаментальная группа, представление, симметрическая группа.*

1. Предварительные сведения

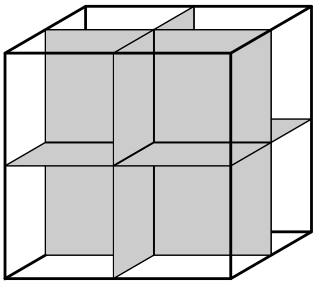

Рис. 1. *Разбиение куба*

Кубиляцией замкнутого ориентируемого трехмерного многообразия называется его представление в виде результата попарной склейки граней нескольких кубов. Под *сложностью* кубиляции будем понимать число кубов в ней. Каждый куб кубиляции C многообразия M разрезается тремя попарно перпендикулярными дисками на 8 кубов (рис. 1). В результате попарной склейки граней кубов эти диски склеиваются в полиэдр S_C, который является поверхностью с самопересечениями, погруженной в многообразие M. Будем говорить, что этот полиэдр S_C и кубиляция C *двойственны*.

Определение 1. Двумерный полиэдр P называется $\mathcal{C}$-специальным, если он удовлетворяет следующим условиям.

1. Каждая точка $x \in P$ имеет окрестность одного из следующих трех типов (рис. 2).

2. Объединение множества точек первого типа является набором открытых дисков, которые называются *2-компонентами.*

3. Объединение множества точек второго типа является набором открытых интервалов, которые называются *четверными линиями.*

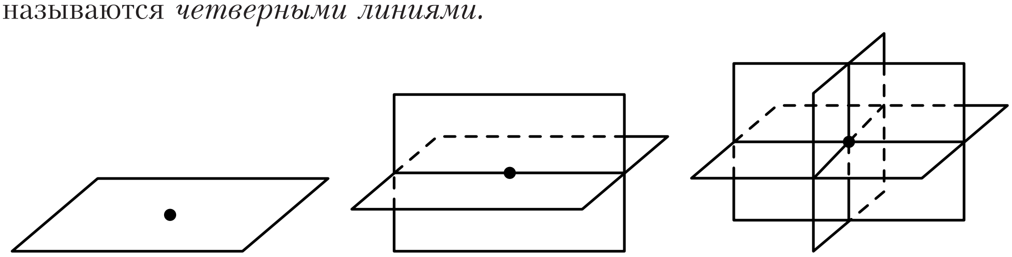

Рис. 2. *Три типа окрестностей точки в $\mathcal{C}$-специальном полиэдре*

Точки, имеющие окрестность третьего типа, называются *кубическими вершинами*, а объединение четверных линий и кубических вершин $\mathcal{C}$-специального полиэдра P — *особым графом* этого полиэдра.

Понятие $\mathcal{C}$-специального полиэдра аналогично понятию специального полиэдра (см. [1]). Отличие состоит в том, что специальные полиэдры задают триангуляции многообразий (в общем случае сингулярные), а $\mathcal{C}$-специальные полиэдры — кубиляции.

Определение 2. $\mathcal{C}$-специальный полиэдр P называется *утолщаемым*, если существует такая кубиляция C замкнутого многообразия M, что полиэдр P гомеоморфен полиэдру S_C.

* Работа выполнена при поддержке гранта РФФИ (проект № 14-01-00441) и гранта молодежных проектов (№ 14-1-НП-17)

Отметим, что каждый утолщаемый $\mathcal{C}$-специальный полиэдр однозначно задает замкнутое трехмерное многообразие, которое отвечает кубиляции, двойственной этому полиэдру.

Определение 3. $\mathcal{C}$-специальный полиэдр P называется *четным*, если все его 2-компоненты являются многоугольниками с четным числом сторон.

В работе [2] Рубинштейн предложил способ для каждой триангуляции многообразия, в которой каждое ребро примыкает к четному числу тетраэдров, построения представления его фундаментальной группы в группу подстановок S_4. Настоящая работа посвящена изучению аналогичных представлений фундаментальных групп четных утолщаемых $\mathcal{C}$-специальных полиэдров в симметрическую группу S_6. Такие представления мы будем называть *геометрическими*. Раздел 2 посвящен описанию геометрических представлений для случая четных кубиляций. Раздел 3 содержит результаты эксперимента по вычислению геометрических представлений для четных кубиляций сложности 1.

2. Геометрические представления

Пусть P — $\mathcal{C}$-специальный полиэдр, и пусть v — вершина его особого графа G_P. Окрестность вершины v в графе G_P состоит из шести дуг a_0, a_1, a_2, a_3, a_4 и a_5. Две дуги a_i и a_j будем называть *соседними*, если в окрестности вершины v в полиэдре P они инцидентны одному диску. В противном случае эти дуги будем называть *противоположными*. *Раскраской* вершины v называется биекция $\mathrm{col}_v\colon \{a_0, a_1, a_2, a_3, a_4, a_5\} \to \{0,1,2,3,4,5\}$, а числа $0,1,\ldots,5$ естественно называть *цветами*.

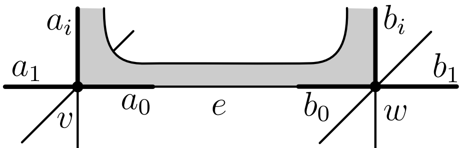

Рис. 3. *Перенос раскраски вдоль ребра*

Пусть v, w — две вершины графа G_P, инцидентные одному ребру e. Опишем процедуру *переноса* раскраски col_v вершины v вдоль ребра e. Раскраска col_w вершины w получается следующим образом: пусть окрестность вершины v в графе G_P состоит из дуг $a_0, \ldots, a_5$, а окрестность вершины w в графе G_P состоит из дуг $b_0, \ldots, b_5$. Для определенности будем считать, что $a_0, b_0 \subset e$, ребро a_1 противоположно ребру a_0, а дуга b_1 противоположна дуге b_0. Положим $\mathrm{col}_w(b_0) = \mathrm{col}_v(a_0)$ и $\mathrm{col}_w(b_1) = \mathrm{col}_v(a_1)$. Далее, если дуга b_i соседняя с дугой b_0, то положим $\mathrm{col}_v(b_i) = \mathrm{col}_v(a_i)$, где a_i — такая дуга, что в окрестности ребра e в полиэдре P дуги a_0, a_i, b_0 и b_i инцидентны одному диску (рис. 3).

Пусть v_0 — фиксированная вершина графа G_P. Построим отображение $\mathcal{R}_P : \pi_1(P, v_0) \to S_6$. Пусть γ — петля с началом и концом в вершине v_0. С точностью до гомотопии можно считать, что эта петля γ проходит по ребрам графа G_P. Выберем некоторую раскраску col_{v_0} и выполним последовательный перенос этой раскраски вдоль ребер, составляющих петлю γ. Получим новую раскраску col'_{v_0} вершины v_0. Образом $\mathcal{R}_P(\gamma)$ является биекция между образами раскрасок col_{v_0} и col'_{v_0}, которую можно отождествить с перестановкой из симметрической группы S_6.

Теорема 1. *Пусть P — четный утолщаемый $\mathcal{C}$-специальный полиэдр. Тогда отображение $\mathcal{R}_P$ является корректно определенным представлением группы $\pi_1(P, v_0)$ в группу S_6.*

Доказательство. Достаточно проверить, что если γ — путь по ребрам особого графа G_P полиэдра P, начинающийся и заканчивающийся в вершине v_0 и ограничивающий диск D, то образ $\mathcal{R}_P(\gamma)$ является тривиальным элементом в S_6.

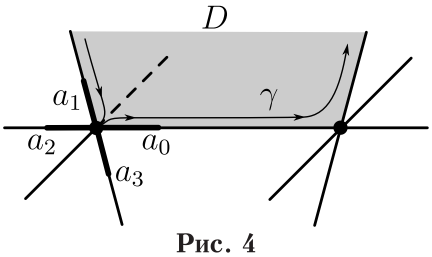

Рис. 4

Пусть col_{v_0} — некоторая раскраска вершины v_0, и пусть раскраска col'_{v_0} получается из нее переносом вдоль пути γ. Пусть окрестность вершины v_0 в графе G_P состоит из дуг $a_0, \ldots, a_5$. Для определенности будем считать, что $a_0, a_1 \subset \gamma$, дуга a_2 противоположна дуге a_0 и дуга a_3 противоположна дуге a_1 (рис. 4). Покажем, что $\mathrm{col}_{v_0}(a_i) = \mathrm{col}'_{v_0}(a_i)$, $i = 0,1,\ldots,5$.

Так как путь γ состоит из четного числа ребер, то $\mathrm{col}_{v_0}(a_0) = \mathrm{col}'_{v_0}(a_0)$ и $\mathrm{col}_{v_0}(a_1) = \mathrm{col}'_{v_0}(a_1)$. Также заметим, что при переносе раскраски вдоль любого ребра цвета противоположных дуг сохраняются. Следовательно, $\mathrm{col}_{v_0}(a_2) = \mathrm{col}'_{v_0}(a_2)$ и $\mathrm{col}_{v_0}(a_3) = \mathrm{col}'_{v_0}(a_3)$. Наконец, так как полиэдр P утолщаем, то при переносе раскраски цвета всех дуг, лежащих по одну сторону от диска D, совпадают. Следовательно, $\mathrm{col}_{v_0}(a_4) = \mathrm{col}'_{v_0}(a_4)$ и $\mathrm{col}_{v_0}(a_5) = \mathrm{col}'_{v_0}(a_5)$. □

Представление $\mathcal{R}_P : \pi_1(P, v_0) \to S_6$ называется *геометрическим*. Отметим, что если выбрать другую базисную вершину фундаментальной группы полиэдра P, либо выбрать другую раскраску этой базисной вершины, то в результате получится геометрическое представление, сопряженное с исходным.

3. Геометрические представления, заданные кубиляциями сложности 1

Был проведен эксперимент по вычислению геометрических представлений для четных кубиляций сложности 1, задающих замкнутые многообразия. Всего таких кубиляций оказалось 26. Для каждой из них был вычислен образ при геометрическом представлении фундаментальной группы соответствующего многообразия в группу S_6. Этот образ оказался изоморфен:

1) группе $\mathbb{Z}_2$ для 1 кубиляции;
2) группе $\mathbb{Z}_4$ для 2 кубиляций;
3) группе $\mathbb{Z}_6$ для 1 кубиляции;
4) группе Диэдра порядка 8 для 6 кубиляций;
5) прямому произведению $\mathbb{Z}_2 \times \mathbb{Z}_2$ для 3 кубиляций;
6) прямому произведению $\mathbb{Z}_2 \times \mathbb{Z}_4$ для 8 кубиляций;
7) прямому произведению $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ для 4 кубиляций;
8) прямому произведению $\mathbb{Z}_2 \times A_4$ для 1 кубиляции.

Список литературы

1. Matveev, S. V. Algorithmic Topology and Classification of 3-Manifolds / S. V. Matveev. — Berlin : Springer-Verlag Berlin Heidelberg, 2007. — 491 p.

2. Rubinstein, J. H. Triangulations of n-Manifolds / J. H Rubinstein // Mathematisches Forschungsinstitut Oberwolfach. — 2012. — Report No. 24. — P. 28–31.

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GEOMETRIC REPRESENTATIONS FOR EVEN CUBILATIONS

Ph. G. Korablev

For each even cubulation of closed 3-manifold we construct the representation of its fundamental group to the symmetric group of degree 6. Images of such representations for all even cubulations with complexity 1 are calculated.

Keywords: *cubulation, manifold, fundamental group, representation, symmetric group.*

References

1. Matveev S.V. *Algorithmic Topology and Classification of 3-Manifolds*. Berlin, Springer-Verlag Berlin Heidelberg, 2007. 491 p.

2. Rubinstein J.H. Triangulations of n-Manifolds. *Mathematisches Forschungsinstitut Oberwolfach*, 2012, no. 24, pp. 28–31.

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ББК В151.5

A NOTE ON THE UNITARITY PROPERTY OF THE GASSNER INVARIANT*

D. Bar-Natan

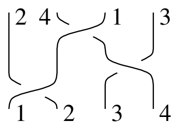
We give a 3-page description of the Gassner invariant (or representation) of braids (or pure braids), along with a description and a proof of its unitarity property.

Keywords: *braids, unitarity, Gassner, Burau.*

The unitarity of the Gassner representation [1] of the pure braid group was discussed by many authors (e. g. [2; 3; 4]) and from several points of view, yet without exposing how utterly simple the formulas turn out to be. Partially this is because the formulas are simplest when extended a "Gassner invariant" defined on the full braid group, but then it is not a representation and it is not unitary. Yet it has an easy "unitarity property"; see below. When the present author needed quick and easy formulas, he couldn't find them. This note is written in order to rectify this situation (but with no discussion of theory). I was heavily influenced by a similar discussion of the unitarity of the Burau representation in [5. Section 3.1.2].

Let n be a natural number. The braid group B_n on n strands is the group with generators σ_i, for $1 \le i \le n-1$, and with relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ when $|i-j| > 1$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ when $1 \le i \le n-2$. A standard way to depict braids, namely elements of B_n, is as follows:

$$b_0 = \sigma_1\sigma_3^{-1}\sigma_2 :$$



Braids are made of strands that are indexed 1 through n at the bottom. The generator σ_i denotes a positive crossing between the strand at position $\#i$ as counted just below the horizontal level of that crossing, and the strand just to its right. Note that with the strands indexed at the bottom, the two strands participating in a crossing corresponding to σ_i may have arbitrary indices, depending on the permutation induced by the braids below the level of that crossing.

Let t be a formal variable and let $U_i(t) = U_{n;i}(t)$ denote the $n \times n$ identity matrix with its 2×2 block at rows i and $i+1$ and columns i and $i+1$ replaced by $\begin{pmatrix} 1-t & 1 \\ t & 0 \end{pmatrix}$, as in the following example:

$$U_{5;3}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1-t & 1 & 0 \\ 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $U_i^{-1}(t)$ be the inverse of $U_i(t)$; it is the $n \times n$ identity matrix with the block at $\{i, i+1\} \times \{i, i+1\}$ replaced by $\begin{pmatrix} 0 & \bar{t} \\ 1 & 1-\bar{t} \end{pmatrix}$, where $\bar{t}$ denotes t^{-1}.

* This work was partially supported by NSERC grant RGPIN 262178. The full TeX sources are at http://drorbn.net/AcademicPensieve/2014-06/UnitarityOfGassner/. Updated less often: arXiv:1406.7632.

Let b be a braid $b = \prod_{\alpha=1}^{k} \sigma_{i_\alpha}^{s_\alpha}$ where the s_α are signs and where products are taken from left to right. Let j_α be the index of the "over" strand at crossing $\#\alpha$ in b. The Gassner invariant $\Gamma(b)$ of b b is given by

$$\Gamma(b) := \prod_{\alpha=1}^{k} U_{i_\alpha}^{s_\alpha}(t_{j_\alpha}).$$

It is a Laurent polynomial in n formal variables $t_1, \dots, t_n$, with coefficients in $\mathbb{Z}$.

As an example, for the braids in the Fig. 1, $\Gamma(\sigma_1\sigma_2\sigma_1) = U_1(t_1)U_2(t_1)U_1(t_2)$ and $\Gamma(\sigma_2\sigma_1\sigma_2) = U_2(t_2)U_1(t_1)U_2(t_1)$. The equality of these two matrix products constitutes the bulk of the proof of the well-definedness of Γ, and the rest is even easier. The verification of this equality is a routine exercise in 3×3 matrix multiplication. Impatient readers may find it in the *Mathematica* notebook [6] that accompanies this note.

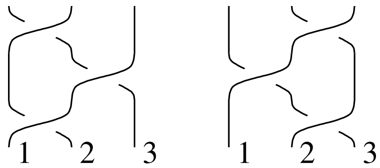


Fig. 1. *The braids* $\sigma_1\sigma_2\sigma_1$ *and* $\sigma_2\sigma_1\sigma_2$

A second example is the braid b_0 of the first figure. Here and in [6],

$$\Gamma(b_0) = U_1(t_1)U_3^{-1}(t_4)U_2(t_1) = \begin{pmatrix} 1-t_1 & 1-t_1 & 1 & 0 \\ t_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{t}_4 \\ 0 & t_1 & 0 & 1-\overline{t}_4 \end{pmatrix}.$$

Given a permutation $\tau = [\tau 1, \dots, \tau n]$ of $1, \dots, n$, let $\Omega(\tau)$ be the triangular $n \times n$ matrix

$$\Omega(\tau) := \begin{pmatrix} (1-t_{\tau 1})^{-1} & 0 & \cdots & 0 \\ 1 & (1-t_{\tau 2})^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & (1-t_{\tau n})^{-1} \end{pmatrix}$$

(diagonal entries $(1-t_{\tau i})^{-1}$, 1's below the diagonal, 0's above). Let ι denote the identity permutation $[1, 2, \dots, n]$.

Theorem 1. *Let b be a braid that induces a strand permutation $\tau = [\tau 1, \dots, \tau n]$ (meaning, the strand indices that appear at the top of b are $\tau 1, \tau 2, \dots, \tau n$). Let $\gamma = \Gamma(b)$ be the Gassner invariant of b. Then γ satisfies the "unitarity property"*

$$\Omega(\tau)\gamma^{-1} = \overline{\gamma}^T \Omega(\iota), \tag{1}$$

$$\textit{orequivalently,} \quad \gamma^{-1} = \Omega(\tau)^{-1}\overline{\gamma}^T\Omega(\iota),$$

where $\overline{\gamma}$ is γ subject to the substitution for all i $t_i \to \overline{t}_i := t_i^{-1}$, and $\overline{\gamma}^T$ is the transpose matrix of $\overline{\gamma}$.

Proof. A direct and simple-minded computation proves Equation (1) for $b = \sigma_i$ and for $b = \sigma_i^{-1}$, namely for $\gamma = U_i(t_i)$ and for $\gamma = U_i^{-1}(t_{i+1})$ (impatient readers see [6]), and then, clearly, using the second form of Equation (1), the statement generalizes to products with all the intermediate $\Omega(\tau)^{-1}\Omega(\tau)$ pairs cancelling out nicely. □

If the Gassner invariant Γ is restricted to pure braids, namely to braids that induce the identity permutation, it becomes multiplicative and then it can be called "the Gassner representation" (in general Γ can be recast as a homomorphism into $M_{n\times n}(\mathbb{Z}[t_i, \overline{t}_i]) \rtimes S_n$, where S_n acts on matrices by permuting the variables t_i appearing in their entries).

For pure braids $\Omega(\tau) = \Omega(\iota) =: \Omega$ and hence by conjugating (in the $t_i \to 1/t_i$ sense) and transposing Equation (1) and replacing γ by γ^{-1}, we find that the theorem also holds if Ω is replaced by $\bar{\Omega}^T$. Hence, extending the coefficients to $\mathbb{C}$, the theorem also holds if Ω is replaced by $\Psi := i\Omega - i\bar{\Omega}^T$, which is formally Hermitian ($\bar{\Psi}^T = \Psi$).

If the t_i's are specialized to complex numbers of unit norm then inversion is the same as complex conjugation. If also the t_i's are sufficiently close to 1 and have positive imaginary parts, then Ψ is dominated by its main diagonal entries, which are real, positive, and large, and hence Ψ is positive definite and genuinely Hermitian. Thus in that case, the Gassner representation is unitary in the standard sense of the word, relative to the inner product on $\mathbb{C}^n$ defined by Ψ.

We remark is that the Gassner representation easily extends to a representation of pure v/w-braids. See e. g. [7. Sections 2.1 and 2.2], where the generators σ_{ij} are described (they are *not* generators of the ordinary pure braid group). Simply set $\Gamma(\sigma_{ij})^{\pm 1} = U_{ij}^{\pm 1}$ where U_{ij} is the $n \times n$ identity matrix with its 2×2 block at rows i and j and columns i and j replaced by $\begin{pmatrix} 1 & 1-t_i \\ 0 & t_i \end{pmatrix}$. Yet on v/w-braids Γ does not satisfy the unitarity property of this note and I'd be very surprised if it is at all unitary.

We also remark that there is an alternative form Γ' for the Gassner representation of pure v/w-braids, defined by $\Gamma'(\sigma_{ij})^{\pm 1} = V_{ij}^{\pm 1}$ where V_{ij} is the $n \times n$ identity matrix with its 2×2 block at rows i and j and columns i and j replaced by $\begin{pmatrix} 1 & 1-t_j \\ 0 & t_i \end{pmatrix}$. Clearly, U_{ij} and V_{ij} are conjugate; $V_{ij} = D^{-1} U_{ij} D$, with D the diagonal matrix whose (i,i) entry is $1 - t_i$ for every i. Hence on ordinary pure braids and for appropriate values of the t_i's (as above), Γ' is also unitary, relative to the Hermitian inner product defined by the matrix $\Psi' := \bar{D}^T \Psi D = i\bar{D}^T(\Omega - \bar{\Omega}^T)D$ whose printed form is better avoided (yet it appears at the end of [6]).

References

1. Gassner B.J. On Braid Groups. *Ph. D. thesis*. New York, New York Univeristy Publ., 1959.

2. Long D.D. On the linear representation of braid groups. *Transactions of the American Mathematical Society*, 1989, vol. 311, no. 2, pp. 535–560.

3. Abdulrahim M.N. A faithfulness criterion for the Gassner representation of the pure braid group. *Proceedings of the American Mathematical Society,* 1997, vol. 125, no. 5, pp. 1249–1257.

4. Kirk P., Livingston C., Wang Z. The Gassner representation for string links. *Communications in Contemporary Mathematics*, 2001, vol. 3, no. 1, pp. 87–136.

5. Kassel C., Turaev C. *Braid Groups*. Springer GTM Publ., 2008. DOI: 10.1007/978-0-387-68548-9.

6. Bar-Natan D. Unitarity Of Gassner Demo.nb. *Mathematica noteboook*. Available at: http://drorbn.net/AcademicPensieve/2014-06/UnitarityOfGassner.

7. Bar-Natan D., Dancso Z. *Finite Type Invariants of w-Knotted Objects I: w-Knots and the Alexander Polynomial*. Available at: http://drorbn.net/AcademicPensieve/Projects/ WKO1/ and arXiv:1405.1956.

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О СВОЙСТВЕ УНИТАРНОСТИ ИНВАРИАНТА ГАССНЕРА

Д. Бар-Натанп

Дается трехстраничное описание кос (или чистых кос) инварианта Гасснера (его представления), а также определение и доказательство его свойства унитарности.

Ключевые слова: *коса, унитарность, Гасснер, Бурау.*

Список литературы

1. Gassner, B. J. On Braid Groups / B. J. Gassner // Ph. D. thesis. — New York : New York Univeristy, 1959.

2. Long, D. D. On the linear representation of braid groups / D. D. Long // Transactions of the American Mathematical Society. — 1989. — Vol. 311, № 2. — P. 535–560.

3. Abdulrahim, M. N. A faithfulness criterion for the Gassner representation of the pure braid group / M. N. Abdulrahim // Proceedings of the American Mathematical Society. — 1997. — Vol. 125, № 5. — P. 1249–1257.

4. Kirk, P. The Gassner representation for string links / P. Kirk, C. Livingston, Z. Wang // Communications in Contemporary Mathematics. — 2001. — Vol. 3, № 1. — P. 87–136.

5. Kassel, C. Braid Groups / C. Kassel, V. Turaev. — New York : Springer GTM, 2008.

6. Bar-Natan, D. Unitarity Of Gassner Demo.nb [Электронный ресурс] / D. Bar-Natan // Mathematica noteboook. URL: http://drorbn.net/AcademicPensieve/2014-06/UnitarityOf-Gassner.

7. Bar-Natan, D. Finite Type Invariants of w-Knotted Objects I: w-Knots and the Alexander Polynomial [Электронный ресурс] / D. Bar-Natan, Z. Dancso. URL: http://drorbn.net/AcademicPensieve/Projects/WKO1.

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QUANTUM INVARIANTS OF 3-MANIFOLDS ARISING FROM NON-SEMISIMPLE CATEGORIES*

M. De Renzi

This survey article covers some of the results contained in the papers by Costantino, Geer and Patureau and by Blanchet, Costantino, Geer and Patureau. In the first one the authors construct two families of Reshetikhin–Turaev-type invariants of 3-manifolds, N_r and N_r^0, using non-semisimple categories of representations of a quantum version of $\mathfrak{sl}_2$ at a $2r$-th root of unity with $r \geq 2$. The secondary invariants N_r^0 conjecturally extend the original Reshetikhin–Turaev quantum $\mathfrak{sl}_2$ invariants. The authors also provide a machinery to produce invariants out of more general ribbon categories which can lack the semisimplicity condition. In the second paper a renormalized version of N_r for $r \neq 0 \pmod 4$ is extended to a TQFT, and connections with classical invariants such as the Alexander polynomial and the Reidemeister torsion are found. In particular, it is shown that the use of richer categories pays off, as these non-semisimple invariants are strictly finer than the original semisimple ones: indeed they can be used to recover the classification of lens spaces, which Reshetikhin–Turaev invariants could not always distinguish.

Keywords: *q-binomial formula, dilogarithm identity.*

1. Modular categories

A (*strict*) *ribbon category* C is a (strict) monoidal category equipped with a braiding c, a twist ϑ and a compatible duality $(*, b, d)$. We will tacitly assume that all the ribbon categories we consider are strict. The *category* $\mathrm{Rib}_{\mathrm{C}}$ *of ribbon graphs over* C is the ribbon category whose objects are finite sequences $(V_1, \varepsilon_1), \ldots, (V_k, \varepsilon_k)$ where $V_i \in \mathrm{O}b(\mathrm{C})$ and $\varepsilon_i = \pm 1$ and whose morphisms are isotopy classes of C-colored ribbon graphs which are compatible with sources and targets.

Theorem 1. *If* C *is a ribbon category then there exists a unique (strict) monoidal functor* F: $\mathrm{Rib}_{\mathrm{C}} \to \mathrm{C}$ *such that (see Fig. 1):*

1) $F(V, +1) = V$ *and* $F(V, -1) = V^*$;
2) $F(X_{V,W}) = c_{V,W}$, $F(\varphi_V) = \vartheta_V$, $F(1[-1]_V) = b_V$ *and* $F(\frown_V) = d_V$;
3) $F(\Gamma_f) = f$.

The functor F is the *Reshetikhin–Turaev functor associated with* C.

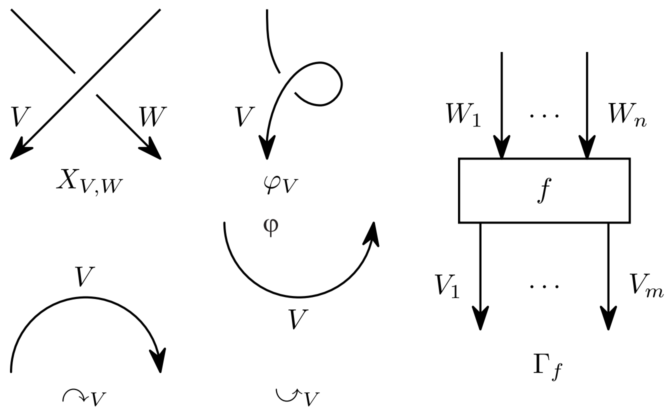

Fig. 1. *Elementary C-colored ribbon graphs*

Remark 1. Every i. e. Turaev functor F yields an invariant of framed oriented links colored with objects of C.

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A *ribbon Ab-category* is a ribbon category C whose sets of morphisms admit abelian group structures which make the composition and the tensor product of morphisms into $\mathbb{Z}$-bilinear maps. Then $K := \mathrm{End}_{\mathrm{C}}(\mathbb{1})$ becomes a commutative ring called the *ground ring of* C and all sets of morphisms are naturally endowed with K-module structures (the scalar multiplication being given by tensor products with elements of K on the left).

An object $V \in \mathrm{O}b(\mathrm{C})$ is *simple* if $\mathrm{End}(V) \simeq K$.

Remark 2. We will always suppose that all the ribbon Ab-categories we consider have a field $\mathbb{K}$ for ground ring.

A *semisimple category* is a ribbon Ab-category C together with a distinguished set of simple objects $\Gamma(\mathrm{C}) := \{V_i\}_{i\in I}$ such that:

1) there exists $0 \in I$ such that $V_0 = \mathbb{1}$;

2) there exists an involution $i \mapsto i^*$ of I such that $V_{i^*} \simeq V_i^*$;

3) for all $V \in \mathrm{O}b(\mathrm{C})$ there exist $i_1, \ldots, i_n \in I$ and maps $\alpha_j : V_{i_j} \to V$, $\beta_j : V \to V_{i_j}$ such that $\mathrm{id}_V = \sum_{j=1}^{n} \alpha_j \beta_j$ (we say that the set $\Gamma(\mathrm{C})$ *dominates* C);

4) for any distinct $i, j \in I$ we have $\mathrm{Hom}_{\mathrm{C}}(V_i, V_j) = 0$.

In a semisimple category we have the following results.

Lemma 1. *For all* $V, W \in \mathrm{O}b(\mathrm{C})$:

1) $\mathrm{Hom}_{\mathrm{C}}(V, W)$ *is a finite-dimensional* $\mathbb{K}$*-vector space*;

2) $\mathrm{Hom}_{\mathrm{C}}(V_i, V) = 0$ *for all but a finite number of* $i \in I$;

3) $\mathrm{Hom}_{\mathrm{C}}(V, W) \simeq \bigoplus_{i\in I} \mathrm{Hom}_{\mathrm{C}}(V, V_i) \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathrm{C}}(V_i, W)$ *where the inverse isomorphism is given by* $f \otimes g \mapsto g \circ f$ *on direct summands*;

4) *the* $\mathbb{K}$*-bilinear pairing* $\mathrm{Hom}_{\mathrm{C}}(V, W) \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathrm{C}}(W, V) \to \mathbb{K}$ *given by* $f \otimes g \mapsto \mathrm{tr}_{\mathrm{C}}(g \circ f)$ *is non-degenerate.*

Corollary 1. *The quantum dimension of simple objects is non-zero.*

Remark 3. Let $(f_i)_1, \ldots, (f_i)_{n_i}$ be a basis for the finite dimensional $\mathbb{K}$-vector space $\mathrm{Hom}_{\mathrm{C}}(V, V_i)$ and let $(g_i)^1, \ldots (g_i)^{n_i}$ denote the corresponding basis of $\mathrm{Hom}_{\mathrm{C}}(V_i, V)$ defined by $(f_i)_h \circ (g_i)^k = \delta_h^k \cdot \mathrm{id}_{V_i}$, which exists thanks to (*iv*) of the previous Proposition. Then we can write

$$\mathrm{id}_V = \sum_{i\in I} \sum_{h,k=1}^{n_i} (\lambda_i)_k^h \cdot \left[(g_i)^k \circ (f_i)_h\right].$$

But now we have $\delta_h^k \cdot \mathrm{id}_{V_i} = (f_i)_h \circ (g_i)^k = (f_i)_h \circ \mathrm{id}_V \circ (g_i)^k = (\lambda_i)_k^h \cdot \mathrm{id}_{V_i}$. Therefore

$$\mathrm{id}_V = \sum_{i\in I} \sum_{j=1}^{n_i} (g_i)^j \circ (f_i)_j. \tag{1}$$

Equation (1) is called the *fusion formula*.

A *premodular category* is a semisimple category $(\mathrm{C}, \Gamma(\mathrm{C}))$ such that $\Gamma(\mathrm{C})$ is finite. A *modular category* is a premodular category $(\mathrm{C}, \Gamma(\mathrm{C}) = \{V_i\}_{i\in I})$ such that the matrix $S = (F(S_{ij}))_{i,j\in I}$ with S_{ij} given by Fig. 2 is invertible.

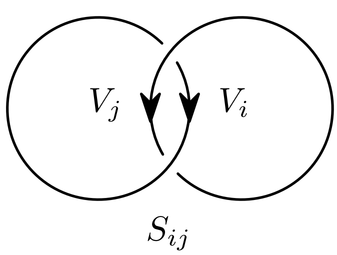


Fig. 2. *Positive Hopf link colored with* V_i *and* V_j

2. The Reshetikhin–Turaev invariants

The construction of Reshetikhin and Turaev associates with every premodular category C an invariant τ_C of 3-manifolds (which will always be assumed to be closed and oriented) provided C satisfies some non-degeneracy condition. Let us outline the general procedure in this context: let C be a premodular category, let $F\colon \mathrm{Rib}_C \to C$ be the associated i. e. Turaev functor and let Ω be the associated *Kirby color*

$$\Omega := \sum_{W \in \Gamma(C)} \dim_C(W) \cdot W.$$

It is known that every 3-manifold M^3 can be obtained by surgery along some framed link L inside S^3 (we write $S^3(L)$ for the result of this operation) and that two framed links yield the same 3-manifold if and only if they can be related by a finite sequence of Kirby moves. Therefore, in order to find an invariant of 3-manifolds, we can look for an invariant of framed links which remains unchanged under Kirby moves. For example let $L \subset S^3$ be a framed link giving a surgery presentation for M^3 and let $\vec{L}(\Omega)$ denote the C-colored ribbon graph obtained by assigning to each component an arbitrary orientation and the Kirby color Ω. Then by evaluating F on $\vec{L}(\Omega)$ we get a number in $\mathbb{K}$ and, thanks to the closure (up to isomorphism) of $\Gamma(C)$ under duality, we can prove that $F(\vec{L}(\Omega))$ is actually independent of the chosen orientation for L. Therefore we have a number $F(L(\Omega)) \in \mathbb{K}$ which depends only on the link L giving a surgery presentation for M. Let us see its behaviour under Kirby moves.

Proposition 1. [Slide]. *Let* $(C, \Gamma(C))$ *be a pre-modular category and let T be a* C*-colored ribbon graph. If T' is a* C*-colored ribbon graph obtained from T by performing a slide of an arc $e \subset T$ over a circle component $K \subset T$ colored by Ω, then $F(T') = F(T)$.*

This result crucially relies on the semisimplicity of C, which enables us to establish the fusion formula 1, and on the finiteness of $\Gamma(C)$, which enables us to define Kirby colors.

Now let us turn our attention towards blow-ups and blow-downs. Let $\Delta_\pm$ denote the image under F of a ±1-framed unknot colored by Ω. If $L' \subset S^3$ is a link obtained from $L \subset S^3$ by a ±1-framed blow-up then $F(L'(\Omega)) = \Delta_\pm \cdot F(L(\Omega))$. At the same time we have that the positive and negative signatures of the linking matrices of L' and L satisfy $\sigma_\pm(L') = \sigma_\pm(L) + 1$ *and* $\sigma_\mp(L') = \sigma_\mp(L)$. Therefore we are tempted to consider the ratio

$$\frac{F(L(\Omega))}{\Delta_+^{\sigma_+(L)} \cdot \Delta_-^{\sigma_-(L)}},$$

which is invariant under all Kirby moves. In order to be able to do so we must require from the premodular category C the following

Condition 1. $\Delta_+ \cdot \Delta_- \neq 0$.

Therefore, let C be a premodular category satisfying Condition 1 and let (M, T) be a pair consisting of a 3-manifold M and a closed C-colored ribbon graph $T \subset M$. If $L \subset S^3$ is any framed link yielding a surgery presentation for M and Γ_T is a C-colored ribbon graph in $S^3 \setminus L$ representing T then the *Reshetikhin–Turaev invariant associated with* C is

$$\tau_C(M, T) := \frac{F(L(\Omega) \cup \Gamma_T)}{\Delta_+^{\sigma_+(L)} \cdot \Delta_-^{\sigma_-(L)}}.$$

Remark 4. The actual Reshetikhin–Turaev invariant is given by the renormalization $D^{-b_1(M)-1}\tau_C(M)$ where $b_1(M)$ is the first Betti number of M and D is an element of $\mathbb{K}$ satisfying $D^2 = F(u(\Omega))$ with $u(\Omega)$ the Ω-colored 0-framed unknot. Note that such a D may not exists and we may have to manually adjoin it (compare with [3]).

Remark 5. The non-degeneracy condition is automatically satisfied by any modular category.

The most famous example of this construction, which yields the original invariants defined by Reshetikhin and Turaev, is obtained by considering a representation category of a quantum version of $\mathfrak{sl}_2$ at a root of unity. Let us recall the construction: fix an integer $r \geq 2$, set $q := e^{\pi i/r}$ and consider the quantum group $U_q(\mathfrak{sl}_2)$ generated (as a unital $\mathbb{C}$-algebra) by E, F, K, K^{-1} with relations

$$KK^{-1} = K^{-1}K = 1,$$
$$KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2}F,$$
$$[E,F] = \frac{K - K^{-1}}{q - q^{-1}}, \quad E^r = F^r = 0$$

and with comultiplication, counit and antipode given by

$$\Delta(E) = E \otimes K + 1 \otimes E, \quad \varepsilon(E) = 0, \quad S(E) = -EK^{-1},$$
$$\Delta(F) = F \otimes 1 + K^{-1} \otimes F, \quad \varepsilon(F) = 0, \quad S(F) = -KF,$$
$$\Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}, \quad \varepsilon(K^{\pm 1}) = 1, \quad S(K^{\pm 1}) = K^{\mp 1}.$$

A representation of $U_q(\mathfrak{sl}_2)$ is a *weight representation*, or a weight $U_q(\mathfrak{sl}_2)$-module, if it splits as a direct sum of eigenspaces for the action of K. The Hopf algebra structure on $U_q(\mathfrak{sl}_2)$ endows the category $U_q(\mathfrak{sl}_2)$-mod of finite-dimensional weight representations of $U_q(\mathfrak{sl}_2)$ with a natural monoidal structure and a compatible duality. Now let $\bar{U}_q(\mathfrak{sl}_2)$ denote the quantum group obtained from $U_q(\mathfrak{sl}_2)$ by adding the relation $K^{2r} = 1$. This condition forces all weights (eigenvalues for the action of K) to be integer powers of q for all representations of $\bar{U}_q(\mathfrak{sl}_2)$. Therefore we can consider the operator $q^{H \otimes H/2}$ defined on $V \otimes W$ for all weight $\bar{U}_q(\mathfrak{sl}_2)$-modules V and W by the following rule:

$$q^{H \otimes H/2}(v \otimes w) = q^{mn/2} v \otimes w$$

if $Kv = q^m v$ and $Kw = q^n w$, where $q^{mn/2}$ stands for $e^{\frac{mn\pi}{2r}i}$. Set $\{m\} := q^m - q^{-m}$ for all $m \in \mathbb{Z}$ and define

$$[n] := \frac{\{n\}}{\{1\}}, [n]! := [n][n-1]\cdots[1].$$

for all $n \in \mathbb{N}$. Consider the operator R defined on $V \otimes W$ for all weight $\bar{U}_q(\mathfrak{sl}_2)$-modules V and W as

$$q^{H \otimes H/2} \sum_{n=0}^{r-1} \frac{q^{n(n-1)/2}}{[n]!} \{1\}^n E^n \otimes F^n.$$

Finally consider the operator $q^{-H^2/2}$ determined on each weight $\bar{U}_q(\mathfrak{sl}_2)$-module V by the rule $q^{-H^2/2}(v) = q^{-n^2/2} v$ *if* $Kv = q^n v$, define the operator u as

$$q^{-H^2/2} \sum_{n=0}^{r-1} \frac{q^{3n(n-1)/2}}{[n]!} \{-1\}^n F^n K^{-n} E^n$$

and set $v := K^{r-1}u$. Then the category $\bar{U}_q(\mathfrak{sl}_2)$-mod of finite dimensional weight representations of $\bar{U}_q(\mathfrak{sl}_2)$ can be made into a ribbon Ab-category by considering the compatible braidings and twists given by

$$c_{V,W} = \tau \circ R : V \otimes W \to W \otimes V, \quad \vartheta_V = v^{-1} : V \to V,$$

where τ is the $\mathbb{K}$-linear map switching the two factors of the tensor product. Moreover $\bar{U}_q(\mathfrak{sl}_2)$-mod is quasi-dominated by a finite number of simple modules, and thus it can be made into a modular category by quotienting negligible morphisms. The invariant we obtain is denoted τ_r.

3. Relative G-premodular categories

To motivate the construction of non-semisimple invariants, let us consider the following different quantization of $\mathfrak{sl}_2$: let $U_q^H(\mathfrak{sl}_2)$ denote the quantum group obtained by adding to $U_q(\mathfrak{sl}_2)$ the additional generator H satisfying the following relations:

$$HK = KH, \quad [H,E] = 2E, \quad [H,F] = -2F,$$
$$\Delta(H) = H \otimes 1 + 1 \otimes H, \quad \varepsilon(H) = 0, \quad S(H) = -H.$$

Remark 6. The new generator H should be thought of as a logarithm of K and, even though we will not require the relation to hold true at the quantum group level, we will restrict ourselves to representations where it is satisfied.

The category $U_q^H(\mathfrak{sl}_2)$-mod of finite dimensional weight representations of $U_q^H(\mathfrak{sl}_2)$ where K acts like the operator q^H can be made into a ribbon Ab-category by means of the same R-matrix and ribbon element used for $\bar{U}_q(\mathfrak{sl}_2)$-mod.

Remark 7. The introduction of H is necessary in order to make sense of the formulas defining the operators R and u because the absence of the relation $K^{2r} = 1$ makes room for weights which are not integer powers of q. The operator $q^{H\otimes H/2}$ is then given by the rule

$$q^{H\otimes H/2}(v \otimes w) = q^{\lambda\mu/2} v \otimes w$$

if $Hv = \lambda v$ and $Hw = \mu w$, where q^{α} stands for $e^{\alpha\pi i/r}$ for all $\alpha \in \mathbb{C}$. The definition of $q^{-H^2/2}$ is analogous.

What is different in $U_q^H(\mathfrak{sl}_2)$-mod is that simple objects are not in a finite number: indeed for any $\alpha \in (\mathbb{C}\setminus\mathbb{Z}) \cup r\cdot\mathbb{Z}$ the r-dimensional module V_α generated by the highest weight vector v_0^α satisfying $Ev_0^\alpha = 0$ and $Hv_0^\alpha = (\alpha + r - 1)v_0^\alpha$ is simple and projective, and is called a *typical module* (see [4] for details). If we could put this richer category into the Reshetikhin–Turaev machinery we would perhaps find more refined 3-manifold invariants. It is indeed the case, but we need to face (among other things) the following obstructions:

1) every typical module has zero quantum dimension;

2) we cannot quotient negligible morphisms as this would kill all typical modules, and thus we are forced to work with a non-semisimple category;

3) typical modules are pairwise non-isomorphic, and therefore we have to deal with infinitely many isomorphism classes of simple objects.

Let us see how we can work around these obstacles. The idea is to generalize the Reshetikhin–Turaev construction to more general ribbon Ab-categories which have the previous set of obstructions.

Facing obstruction (*i*): Modified quantum dimension

To begin with let us take care of the vanishing quantum dimension problem. The strategy will be to use categories C which admit a modified dimension which does not vanish. In order to do so we need an *ambidextrous pair* (A, d), that is the given of a set of simple objects $\mathrm{A} \subset \mathrm{O}b(\mathrm{C})$ and a map $\mathrm{d} : \mathrm{A} \to \mathbb{K}^*$ with the following property: if T is an A-*graph*, i. e. a closed C-colored ribbon graph admitting at least one color in A, if $e \subset T$ is an arc colored by $V \in \mathrm{A}$ and if T_e denotes the element of $\mathrm{End}_{Rib_\mathrm{C}}((V,+))$ obtained by cutting open T at e, then $F'(T) := \mathrm{d}(V)\cdot\langle T_e\rangle$ is independent of the chosen A-colored arc e (here $\langle T_e\rangle$ denotes the unique element of $\mathbb{K}$ such that $F(T_e) = \langle T_e\rangle \cdot \mathrm{id}_V$).

Definition 1. A ribbon Ab-category C admitting an ambidextrous pair (A, d) is said to have *modified dimension* d. F' is the *modified* A-*graph invariant associated with* (A, d).

Example 1. In the category $U_q^H(\mathfrak{sl}_2)$-mod considered before we have indeed an ambidextrous pair. It is obtained by taking A to be the set of typical modules and by defining

$$\mathrm{d}(V_\alpha) = (-1)^{r-1}\prod_{j=1}^{r-1}\frac{q^j - q^{-j}}{q^{\alpha+r-j} - q^{-\alpha-r+j}} = (-1)^{r-1}\cdot r\cdot\frac{\sin(\alpha\pi / r)}{\sin(\alpha\pi)}.$$

Facing obstruction (*ii*): G-grading relative to X

Moving on to the subject of semisimplicity, we will ask our categories to have a distinguished family of semisimple full subcategories nicely arranged, i. e. indicized by an abelian group G in such a way that the tensor product respects the group operation. The aim of course is to work as much as possible in the semisimple part of the category and to leave aside the non-semisimple part.

Definition 2. Let C be a ribbon Ab-category. A full subcategory C′ of C is said to be *semisimple inside* C if it is dominated by a set $\Gamma(C')$ of simple objects of C′ such that for any distinct $V,\ W \in \Gamma(C')$ we have $\mathrm{Hom}_C(V,W)=0$.

Remark 8. We do not ask of $\Gamma(C')$ to contain $\mathbb{1}$ nor to be closed under duality up to isomorphism. In particular it may very well happen that the quantum dimension of a simple object of C' is zero.

Definition 3. We will say that a subset X of an abelian group G is *small* if G cannot be covered by any finite union of translated copies of X, i. e. if there exists no choice of $g_1,\ldots,g_k \in G$ such that $G \subset \cup_{i=1}^{k}(g_i + X)$. Let G be an abelian group and let $X \subset G$ be a small subset. A family of full subcategories $\{C_g\}_{g\in G}$ of a ribbon Ab-category category C gives a *G-grading relative to X* for C if:

1) C_g is semisimple inside C for all $g \in G \setminus X$;
2) $V \in Ob(C_g),\ V' \in Ob(C_{g'}) \Rightarrow V \otimes V' \in Ob(C_{g+g'})$;
3) $V \in Ob(C_g) \Rightarrow V^* \in Ob(C_{-g})$;
4) $V \in Ob(C_g),\ V' \in Ob(C_{g'}),\ g \neq g' \Rightarrow \mathrm{Hom}_C(V,V')=0$.

The elements of g which are not contained in X are called *generic* and a subcategory C_g indicized by a generic g is called a *generic subcategory*. A category C with a G-grading relative to X will be called a *G-graded category* for the sake of brevity.

Example 2. In the category $U_q^H(\mathfrak{sl}_2)$-mod considered before we have a relative G-grading too. Indeed we can take $G=\mathbb{C}/2\mathbb{Z}$, $X=\mathbb{Z}/2\mathbb{Z}$ and set $C_{\bar\alpha}$ equal to the full subcategory of modules whose weights are all congruent to α modulo 2. Then every $C_{\bar\alpha}$ with α not integer is semisimple inside $\bar{U}_q(\mathfrak{sl}_2)$-mod, being dominated by the typical modules it contains.

Facing obstruction (*iii*): Periodicity group

Finally, for the finiteness issue, we will proceed as follows: for a G-graded category C we will ask the sets of isomorphism classes of simple objects of all generic subcategories to be finitely partitioned in a way we can control.

Definition 4. A set $C \subset Ob(C)$ of objects of a ribbon Ab-category is a *commutative family* if the braiding and the twist are trivial on C, i. e. if we have $c_{W,V} \circ c_{V,W} = \mathrm{id}_{V\otimes W}$ and $\vartheta_V = \mathrm{id}_V$ for all $V,W \in C$.

Definition 5. Let Z be an abelian group and C be a ribbon Ab-category. A *realization* of Z in C is a commutative family $\{\varepsilon^t\}_{t\in Z}$ satisfying

$$\varepsilon^0=\mathbb{1},\quad \varepsilon^t \otimes \varepsilon^s = \varepsilon^{t+s},\quad \dim_C(\varepsilon^t)=1\ \ \forall t,s \in Z.$$

Any free realization of Z gives isomorphisms between the $\mathbb{K}$-vector spaces $\mathrm{Hom}_C(V,W)$ and $\mathrm{Hom}_C(V\otimes\varepsilon^t, W\otimes\varepsilon^t)$ for all choices of $V,W \in Ob(C)$ and $t \in Z$. Indeed the inverse of the map $f \mapsto f \otimes \mathrm{id}_{\varepsilon^t}$ is simply given by $g \mapsto g \otimes \mathrm{id}_{\varepsilon^{-t}}$. Therefore if V is simple then $V\otimes\varepsilon^t$ is simple too for all $t \in Z$. Thus any realization of Z induces an action of Z on (isomorphism classes of) objects of C given by the tensor product on the right with ε^t. Such a realization is *free* if this action is free.

Definition 6. An abelian group Z is the *periodicity group* of the G-graded category C if there exists a free realization of Z in C_0 whose action on $\Gamma(C_g)$ has a finite number of orbits for all $g \in G \setminus X$.

In this case there exists some finite set of representatives of Z-orbits $O(C_g) \subset \Gamma(C_g)$ for all generic g such that each simple module in $\Gamma(C_g)$ is isomorphic to some tensor product $W \otimes \varepsilon^t$ for $W \in O(C_g)$ and $t \in Z$.

Example 3. Once again the category $U_q^H(\mathfrak{sl}_2)$-mod considered before gives us an instance of this structure. Namely the periodicity group is $\mathrm{Z} = \mathbb{Z}$ and its free realization in $\mathrm{C}_{\bar{0}}$ is given by a 1-dimensional module ε^t for every $t \in \mathbb{Z}$ which is spanned by the non-zero vector v^t such that $Ev^t = Fv^t = 0$ and $Hv^t = 2rtv^t$.

The categories which will allow us to extend the Reshetikhin–Turaev construction to the non-semisimple case admit all of the structures we just introduced.

Definition 7. A *relative G-premodular category* is $(\mathrm{C}, G \supset X, (\mathrm{A}, \mathrm{d}), \mathrm{Z})$ where C is a G-graded category with modified dimension d and periodicity group Z satisfying the following compatibility conditions:

1) $\mathrm{A} \supset \Gamma(\mathrm{C}_g)$ for all $g \in G \backslash X$;

2) $c_{V,\varepsilon^t} = \psi(g,t) \cdot c^{-1}_{\varepsilon^t,V}$ for all $V \in \mathrm{O}b(\mathrm{C}_g), t \in \mathrm{Z}$ and for some $\mathbb{Z}$-bilinear pairing $\psi : G \times \mathrm{Z} \to \mathbb{K}^*$ (see Fig. 3).

Example 4. $U_q^H(\mathfrak{sl}_2)$-mod is a relative $\mathbb{C}/\mathbb{Z}$-premodular category as it can be shown that a skein relation like the one required in condition 2 of the previous definition holds.

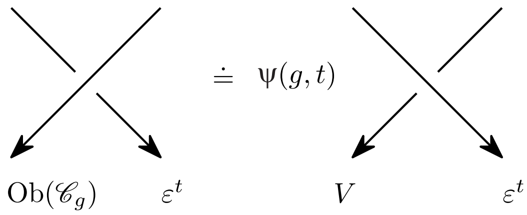


Fig. 3. *Skein-type relation for G and* Z
(*the ≐ sign stands for equality under F*)

4. Construction of non-semisimple invariants

We are ready to sketch a construction analogous to the one of Reshetikhin and Turaev which associates with each relative G-premodular category C an invariant of 3-manifolds provided C satisfies some non-degeneracy conditions. The idea will be to use the modified invariant F' as a basis for this construction exactly as the functor F was used as a basis for the standard case. Remember however that in order to compute F' on a C-colored ribbon graph T we will need to make sure that T is actually an A-graph.

Let us fix a relative G-premodular category C. The first thing we did in the construction of Reshetikhin–Turaev invariants was to color a framed link giving a surgery presentation for a 3-manifold M with the Kirby color Ω associated with some premodular category. Now, in C we do not have the concept of a Kirby color, but we can define an infinite family of modified Kirby colors.

Indeed if $g \in G$ is generic then the formal sum

$$\Omega_g := \sum_{W \in \mathrm{O}(\mathrm{C}_g)} \mathrm{d}(W) \cdot W$$

is a *modified Kirby color of degree g*.

Remark 9. It can be easily proved using the properties of the periodicity group Z that the modified dimension d factorizes through a map defined on Z-orbits on all generic subcategories, i.e. we have $\mathrm{d}(W \otimes \varepsilon^t) = \mathrm{d}(W)$ for all $W \in O(\mathrm{C}_g)$ and all $t \in \mathrm{Z}$. In particular the coefficients in the formal sum Ω_g are independent of the choice of the representatives of Z-orbits in $\Gamma(\mathrm{C}_g)$. Of course W and $W \otimes \varepsilon^t$ are not isomorphic if $t \neq 0$ but we will see that under certain circumstances this choice will not affect the value of F'.

Since we defined an infinite family of modified Kirby colors it is not clear which one should be used to color the components of a surgery link L for M^3. The right choice is to let the coloring be determined by a cohomology class $\omega \in H^1(M \setminus T; G) \simeq \mathrm{Hom}_{\mathbb{Z}}(H_1(M \setminus T, G)$ which is compatible with the C-coloring which is already present on T.

Definition 8. Let T be a C-colored ribbon graph inside M and ω be an element of $H^1(M \setminus T; G)$. For every arc $e \subset T$ let μ_e denote the homology class of a positive meridian around e. The triple (M, T, ω) is *compatible* if the color of e is an object of $C_{\langle \omega, \mu_e \rangle}$.

We will now look for an invariant of compatible triples (M, T, ω), where two triples (M_i, T_i, ω_i) for $i = 1, 2$ are considered to be equivalent if there exists an orientation preserving diffeomorphism $f : M_1 \to M_2$ such that $f(T_1) = T_2$ as C-colored ribbon graphs and $f^*(\omega_2) = \omega_1$.

Remark 10. We will have to be more careful and to keep track of the (isotopy class of the) diffeomorphism induced by each Kirby move.

The idea is to color each component L_i of a surgery link L with a modified Kirby color whose degree is determined by the evaluation $\langle \omega, \mu_i \rangle$, where μ_i denotes the homology class corresponding to a positive meridian of L_i. Thus, since modified Kirby colors are defined only for generic degrees, not all surgery presentations can be used to define the new invariant.

Definition 9. A compatible triple (M, T, ω) admits a *computable surgery presentation* $L = L_1 \cup \ldots \cup L_m \subset S^3$ if one of the following holds:

1) $L \neq \varnothing$ and $\langle \omega, \mu_i \rangle$ is generic for all $i = 1, \ldots, m$;

2) $L = \varnothing$ and T is an A-graph.

If L is a link yielding a computable surgery presentation for a compatible triple (M, T, ω) and we denote by $L(\omega)$ the C-colored link obtained by coloring each component L_i of L with $\Omega_{\langle \omega, \mu_i \rangle}$, then F' can be evaluated on $L(\omega) \cup \Gamma_T$, where Γ_T represents T inside $S^3 \setminus L$.

Remark 11. It can be shown that a sufficient condition for the existence of a computable surgery presentation for a compatible triple (M, T, ω) is that the image of ω is not entirely contained in the critical set X (when we regard ω as a map from $H_1(M \setminus T)$ to G).

Let us see what happens when we perform Kirby moves.

Remark 12. In order to be able to evaluate F' we can only consider Kirby moves between computable surgery presentations of (M, T, ω).

The slide of an arc $e \subset L \cup \Gamma_T$ over a component L_i of L corresponds to a change of basis in $H^1(M \setminus T)$ which amounts to susbstituting μ_i with $\mu_i \pm \mu_e$ (depending on orientations). This operation preserves $F'(L(\omega) \cup \Gamma_T)$.

Proposition 2. [Slide]. *Let T be an* A*-graph, let $e \subset T$ be an arc colored by $V \in \mathrm{Ob}(C_g)$, let $K \subset T$ be a knot component colored by Ω_h for some generic $h \in G$ and suppose that $g + h$ is generic too. If T' is an* A*-graph obtained from T by sliding e along K and switching the color of K to Ω_{g+h} (like in Fig. 4) then $F'(T') = F'(T)$.*

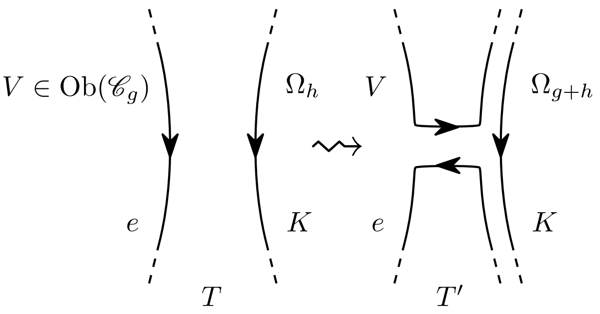


Fig. 4. *Subtraction*

To prove this proposition we need to establish a fusion formula (which can be done in the semisimple part of C exactly as before) and to use the skein-type relation in the definition of C in order to handle closed components colored with ε^t for $t \in \mathbb{Z}$. The color of K changes because $V \otimes W$ is an object of C_{g+h} for all $W \in O(C_h)$.

For what concerns blow-ups and blow-downs, we cannot compute F' directly on a detached ± 1-framed unknot as such a component should be colored with the modified Kirby color of degree 0 and it may very well happen that $0 \in X$ (which is the case in our previous example).

Proposition 3. [Blow-up and blow-down]. *Let T_+ be the* C*-colored ribbon graph given by Fig. 5 (a) with g generic in G. Then $\Delta_+ := \langle T_+ \rangle$ does not depend on the generic g nor on the object $U \in \mathrm{Ob}(C_g)$. The same holds for the analogous graph T_- (obtained by turning each overcrossing of T_+ into an undercrossing) and for $\Delta_- := \langle T_- \rangle$.*

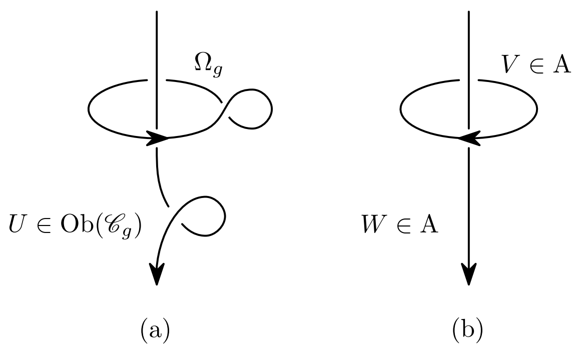


Fig. 5. *Blow-up of* +1*-framed meridian* (a) *and H-stabilization* (b)

Remark 13. The operation of blowing up a positive meridian of an arc in a ribbon graph T can replace the operation of blowing up an isolated unknotted componend provided T is non-empty. This is always the case for computable surgery presentations since there is always at least one arc colored in A.

Thus what we need in order to be able to define the invariant is once again to ask the condition $\Delta_+ \cdot \Delta_- \neq 0$. However, this time we need also another non-degeneracy condition which allows us to perform an operation called H-stabilization which is needed in the proof of the invariance of our construction. Namely, let $H(V,W)$ denote the C-colored ribbon graph given by Fig. 5 (b) for $V,W \in \mathrm{A}$. Then:

Condition 2.

1. $\Delta_+ \cdot \Delta_- \neq 0$.
2. $\langle H(V,W)\rangle \neq 0$ *for all* $V,W \in \mathrm{A}$.

Now we can state our result.

Theorem 2. *Let* C *be a relative G-premodular category satisfying the non-degeneracy Condition 2. Let L be a framed link giving a computable surgery presentation for a compatible triple* (M,T,ω) *and let* Γ_T *be a* C*-colored ribbon graph inside* $S^3 \setminus L$ *representing T. Then*

$$\mathrm{N}_{\mathrm{C}}(M,T,\omega) := \frac{F'(L(\omega) \cup \Gamma_T)}{\Delta_+^{\sigma_+(L)} \cdot \Delta_-^{\sigma_-(L)}}$$

is a well-defined invariant of (M,T,ω).

Remark 14. When $\mathrm{C} = U_q^H(\mathfrak{sl}_2)$-mod with $q = e^{\pi i/r}$ we write N_r instead of $\mathrm{N}_{U_q^H(\mathfrak{sl}_2)\text{-mod}}$.

The subtlety in the proof of this result is the following: if we have two different computable surgery presentations L and L' it may happen that the sequence of Kirby moves connecting them passes through some non-computable presentation. What we have to prove is that, up to passing to a different sequence of Kirby moves, we can make sure to get a computable presentation at each intermediate step.

This turns out to be true, but we have to allow an operation, called H-stabilization, which modifies the triple (M,T,ω) and which is defined as follows: let $e \subset T$ be an arc colored by $W \in \mathrm{A}$, let α be a positive 0-framed meridian of e disjoint from T and colored by $V \in \Gamma(\mathrm{C}_g)$ for some generic g and let $D^2 \subset S^3$ be a disc intersecting e once and satisfying $\partial D^2 = \alpha$. Now let T_H denote the A-graph $T \cup \alpha$ and let ω_H be the cohomology class coinciding with ω on homology classes contained in $M \setminus (T \cup D^2)$ and satisfying $\langle \omega_H, \mu_\alpha \rangle = g$ where μ_α is the homology class of a positive meridian of α. Then the compatible triple (M,T_H,ω_H) is said to be obtained by *H-stabilization of degree g* from (M,T,ω), and α is called the *stabilizing meridian*. Now (M,T_H,ω_H) is not equivalent to (M,T,ω) but we have the equality $\mathrm{N}_{\mathrm{C}}(M,T_H,\omega_H) = \langle H(V,W)\rangle \cdot \mathrm{N}_{\mathrm{C}}(M,T,\omega)$.

Returning to the proof of the Theorem, we split the argument into three steps: we begin by first proving the result in the case that T itself is an A-graph, that the initial and final surgery presentations are the same and that the sequence of Kirby moves involves only isotopies of Γ_T inside $S^3(L)$, i. e. slides of arcs of Γ_T over components of L (we call this sequence of moves an *isotopy inside* $S^3(L)$). This case can be easily treated by performing a single H-stabilization on (M,T,ω) whose degree is sufficiently generic. Indeed if we slide a stabilizing meridian on some component L_j of the computable link L which is colored by Ω_{h_j} we change the color of

L_j, provided the degree g of the H-stabilization satisfies $g + h_j \in G \setminus X$. This would impose a condition on the choice of the degree, but there surely exists a $g \in G$ which satisfies it because X is small. More generally, if $C \subset G$ denotes the finite set of (degrees of) colors appearing on L during the sequence of slides, we can choose the degree g of the H-stabilization in such a way that $(g + C) \cap X = \varnothing$. Thus, we can begin by sliding the stabilizing meridian α over all components of L, then we can follow the original sequence of Kirby moves and finally we can slide back α to its original position. What we will get is an equality of the form

$$\frac{F'(L(\omega) \cup \Gamma_T) \cdot \langle H(V,W) \rangle}{\Delta_+^{\sigma_+(L)} \cdot \Delta_-^{\sigma_-(L)}} = \frac{F'(L(\omega) \cup \Gamma'_T) \cdot \langle H(V,W) \rangle}{\Delta_+^{\sigma_+(L)} \cdot \Delta_-^{\sigma_-(L)}},$$

for some $V, W \in \mathrm{A}$, which proves the first step.

$$\begin{array}{ccc} L^0 \cup \Gamma_T^0 \to rs_1 & \cdots \to rs_k & L^k \cup \Gamma_T^k \\ \| & & \| \\ L \cup \Gamma_T & & L' \cup \Gamma'_T \end{array}$$

Fig. 6

The second step consists in proving the Theorem when T is an A-graph. If Fig. 6 is our sequence of Kirby moves, we perform an H-stabilization for each Kirby move s_h which makes some non-generic color appear. If s_h is a non-admissible slide over some component L_j^{h-1} we precede it by a slide of the corresponding stabilizing meridian α_h over L_j^{h-1}. If s_h is a non-admissible blow-up around some arc we perform it on the corresponding stabilizing meridian α_h instead and then we slide the arc over the newly created component. All degrees can be chosen so to adjust all colors, and the use of different stabilizations ensures the independence of the conditions. The tricky point is that a move s_ℓ which in the original sequence was a blow-down of a ±1-framed meridian of some arc or link component may now have become the blow-down of a component which is also linked to some of the stabilizing meridians we added. In this case, though, we can slide all these stabilizing meridians off, and this operation is an isotopy inside $S^3(L^{\ell-1})$. Remark that the configuration we get at this point is not necessarily admissible, but problems can arise only for blow-downs of meridians of arcs in $\Gamma_T^{\ell-1}$. Thus in this case we can perform a new H-stabilization, slide the arc off and slide the new stabilizing meridian over. This operation is yet another isotopy inside $S^3(L^{\ell-1})$ which yields a computable presentation. Therefore, thanks to the first step, the invariant does not change. In the end we get the original final presentation plus some stabilizing meridian linked to the rest of the graph. All these meridians can be slid back to their initial positions, and once again this operation is an isotopy inside $S^3(L')$.

The third step is the general case: now what we have to do is to blow-up two meridians of a component of L in such a way that its framing does not change. Then we can consider these new curves as part of T, falling back into the previous case, we can prove that we can undo our initial operation and that the result is not affected by our changes.

Extension to all compatible triples

There exist of course compatible triples which do not admit computable presentations. In order to include also this case in the construction we can build a second invariant $\mathrm{N}_{\mathrm{C}}^0$ which is defined for all compatible triples.

Remark 15. In categories where the quantum dimension of the objects of A is always zero (such as the categories in our example) this second invariant will vanish on all triples which admit computable presentations. Therefore in this case one should continue to use N_{C} to get topological informations.

For the definition of $\mathrm{N}_{\mathrm{C}}^0$ we will need the concept of connected sum of compatible triples. Let (M_1, T_1, ω_1) and (M_2, T_2, ω_2) be two compatible triples, let $M_3 = M_1 \# M_2$ be the connected sum along balls B_i inside $M_i \setminus T_i$ for $i = 1,2$ and set $T_3 = T_1 \sqcup T_2$. Then we have the chain of isomorphisms

$$H_1(M_3 \setminus T_3) \simeq H_1(M_1 \setminus (B_1 \cup T_1)) \oplus H_1(M_2 \setminus (B_2 \cup T_2)) \simeq H_1(M_1 \setminus T_1) \oplus H_1(M_2 \setminus T_2)$$

where the first one is induced by a Mayer–Vietoris sequence and the second one comes from excision. These maps induce an isomorphism

$$H^1(M_3 \setminus T_3; G) \simeq H^1(M_1 \setminus T_1; G) \oplus H^1(M_2 \setminus T_2; G).$$

Finally let ω_3 be the unique element of $H^1(M_3 \setminus T_3; G)$ which restricts to ω_i on $H^1(M_i \setminus T_i; G)$ for $i = 1,2$ via the previous isomorphism. The connected sum of (M_1, T_1, ω_1) and (M_2, T_2, ω_2) is defined as (M_3, T_3, ω_3). Now if the compatible triple (M, T, ω) does not admit any computable presentation consider the triple (S^3, u_V, ω_V) where u_V is a 0-framed unknot in S^3 colored by $V \in \mathrm{A}$ and ω_V is the unique cohomology class in $H^1(S^3 \setminus u_V; G)$ which makes the previous triple into a compatible one. Then we can define $\mathrm{N}^0_{\mathrm{C}}(M, T, \omega)$ to be

$$\frac{\mathrm{N}_{\mathrm{C}}((M,T,\omega)\#(S^3, u_V, \omega_V))}{\mathrm{d}(V)}.$$

Remark 16.

1. Just like before, when $\mathrm{C} = U_q^H(\mathfrak{sl}_2)$-mod with $q = e^{\pi i/r}$ we write N^0_r instead of $\mathrm{N}^0_{U_q^H(\mathfrak{sl}_2)\text{-mod}}$. As claimed earlier, N^0_r vanishes on computable presentations because in this category F' vanishes on split A-graphs, i. e. if T and T' are completely disjoint A-graphs then we have $F'(T \sqcup T') = F'(T)F(T') = 0$.

2. As it was mentioned in the abstract, N^0_r coincides with τ_r in a lot of cases, though in general their equality remains conjectural.

5. Case $r = 2$: Alexander polynomial, Reidemeister torsion and lens spaces

For the special case $r = 2$ we have that $q = i$ and the modified A-graph invariant F' associated with the category $U_i^H(\mathfrak{sl}_2)$-mod can be related to the multivariable Alexander polynomial. This fact, which was first observed by Murakami in [5], is exposed in detail in Viro's paper [6]. He defines an *Alexander invariant* $\underline{\Delta}^2$ for oriented trivalent graphs equipped with the following additional structure:

1) a half-integer framing (half-twists are allowed too);

2) a coloring with typical $U_i^H(\mathfrak{sl}_2)$-modules satisfying a condition like the ones shown in Fig. 7 around each vertex;

3) a cyclic ordering of the (germs of the) edges around each vertex.

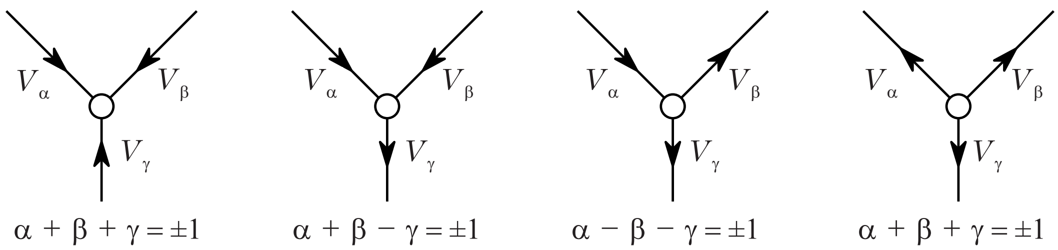


Fig. 7. *Admissible colorings*

Viro's construction uses a functor which is similar to the Reshetikhin–Turaev one, though the source category is not the category of colored ribbon graphs. It is indeed a category G^2 whose objects are the objects of $\mathrm{R}ib_{U_i^H(\mathfrak{sl}_2)\text{-mod}}$ which feature only typical colors and whose morphisms are (isotopy classes of) a non-closed version of the graphs mentioned above. In particular all vertices are either 3-valent (internal vertices) or 1-valent (boundary vertices). If such a graph Γ is closed, i.e. if it does not contain boundary vertices, and if its framing yields an orientable surface, then we can associate with it an A-graph T_Γ defined as follows:

consider the ordered basis

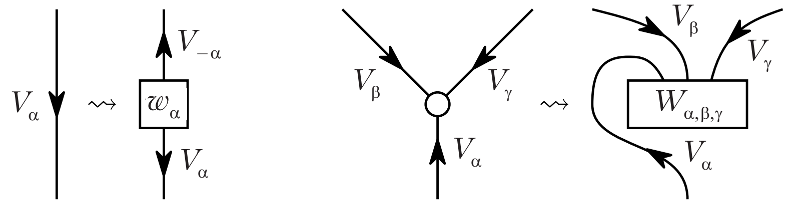
$$\left\{ v_0^\alpha, v_1^\alpha := \frac{i^{\frac{\alpha+1}{2}}}{[\alpha+1]} F v_0^\alpha \right\}$$

of the typical module V_α and let $\{\varphi_\alpha^0, \varphi_\alpha^1\}$ be its dual basis in V_α^*. Then we have an isomorphism $w_\alpha : V_\alpha \to V_{-\alpha}^*$ given by $v_j^\alpha \mapsto i^{-\left(\frac{\alpha+1}{2}-j\right)} \varphi_{-\alpha}^{1-j}$ for $j = 0,1$. Moreover, every time $\alpha, \beta, \gamma \in \mathbb{C} \setminus (2\mathbb{Z}+1)$ satisfy $\alpha + \beta + \gamma = \pm 1$, we can consider the morphism $W_{\alpha,\beta,\gamma} : \mathbb{C} \to V_\alpha \otimes V_\beta \otimes V_\gamma$ mapping 1 to $\sum_{2(j+k-h)=\alpha+\beta+\gamma+1} C_{j,k,h}^{\alpha,\beta,\gamma} v_j^\alpha \otimes v_k^\beta \otimes v_h^\gamma$ where the coefficients $C_{j,k,h}^{\alpha,\beta,\gamma}$ are derived from the Clebsch–Gordan quantum coefficients (compare with [7]) and are defined as

$$(-1)^{k-h} i^{\frac{\beta(k-1)-\alpha(j+1)+2(k+h-j-1)+j^2-k^2}{2}} \begin{bmatrix} 1-\gamma \\ 1-\gamma-h \end{bmatrix}^{-1} \begin{bmatrix} 1-\gamma \\ \alpha+\beta-\gamma+2 \end{bmatrix} \times$$

$$\times \sum_{t+s=h} (-1)^t i^{\frac{(2t-h)(2-\gamma-h)}{2}} \begin{bmatrix} \frac{\alpha+\beta+\gamma+1}{2} \\ j-t \end{bmatrix} \begin{bmatrix} \alpha-j+t+1 \\ \alpha-j+1 \end{bmatrix} \begin{bmatrix} \beta-k+s+1 \\ \beta-k+1 \end{bmatrix}.$$

Then we can construct T_Γ by replacing each edge of Γ which is not a connected component as shown in Fig. 8 (a) and each trivalent vertex of Γ as shown in Fig. 8 (b).



(*a*) (*b*)

Fig. 8. C-*colored ribbon graph* T_Γ *obtained from trivalent graph* Γ

Proposition 4. $F'(T_\Gamma) = (-2i)^{1-v/2} \underline{\Delta}^2(\overline{\Gamma})$ *where* v *is the number of vertices of* Γ *and* $\overline{\Gamma}$ *is obtained from* Γ *by inverting the orientation on each edge.*

This result is obtained by checking that the two expressions coincide for a set of elementary graphs (the trivial one, the Θ-graph and the tetrahedron graph) and by checking that they both satisfy the same set of relations which reduce the computation for an arbitrary graph to elementary ones (see [1] and [2] for details).

If $L = L_1 \sqcup \ldots \sqcup L_m$ is an oriented colored framed link whose j-th component L_j is colored with the typical module V_{α_j} then Viro shows that

$$\underline{\Delta}^2(L) = \nabla_L(i^{1+\alpha_1}, \ldots, i^{1+\alpha_m}) i^{\sum_{j,h=1}^{m} \frac{\alpha_j \alpha_h - 1}{2} \ell k(L_j, L_h)},$$

where ∇_L is the Alexander–Conway function of L. Therefore, if the framing is integral, Proposition 4 immediately gives

$$F'(L) = (-2i) \nabla_L(i^{1-\alpha_1}, \ldots, i^{1-\alpha_m}) i^{\sum_{j,h=1}^{m} \frac{\alpha_j \alpha_h - 1}{2} \ell k(L_j, L_h)}.$$

Now since $C_{\overline{1}}$ is semisimple inside $U_i^H(\mathfrak{sl}_2)$-mod we can take the critical set $X \subset \mathbb{C}/2\mathbb{Z}$ to be just $\{\overline{0}\}$. Therefore, thanks to Remark 11, every triple of the form $(M, \varnothing, \omega)$ with $\omega \neq 0$ is compatible and admits a computable presentation. In particular some computation (compare with [2]) yields

$$N_2(M,\varnothing,\omega) = 2\cdot 4^{m-\sigma_+(L)-\sigma_-(L)} i^{\sigma_-(L)-\sigma_+(L)-m-1}\cdot\left(\prod_{j=1}^{m}\frac{1}{i^{\alpha_j}-i^{-\alpha_j}}\right)\nabla_L(i^{\alpha_1},\ldots,i^{\alpha_m})\cdot i^{\sum_{j,h=1}^{m}\frac{\alpha_j(\alpha_h+2)}{2}\ell k(L_j,L_h)},$$

where $L = L_1 \sqcup \ldots \sqcup L_m$ is a surgery presentation for M and $\alpha_j := \langle\omega,\mu_j\rangle$. Thus N_2 recovers the Alexander − Conway function, which is known to be related to the Reidemeister torsion. Moreover N_2 yields a canonical normalization of the Reidemeister torsion which fixes the scalar indeterminacy. Indeed recall that the refined abelian Reidemeister torsion of M defined by Turaev (see [8] for example) is determined by the choice of a homomorphism $\varphi : H_1(M) \to \mathbb{C}^*$, of a homology orientation ω_M for M and of a Spin^c-structure $\sigma \in \mathrm{Spin}^c(M)$ (or equivalently of an Euler structure on M). We write $\tau^{\varphi}(M,\omega_M,\sigma)$ or, if M is oriented and we pick the canonical homology orientation associated with the orientation of M, simply $\tau^{\varphi}(M,\sigma)$. Now, if $(M,\varnothing,\omega)$ is a compatible triple as above, we can use the non-zero cohomology class ω to define the homomorphism $\varphi_\omega : H_1(M) \to \mathbb{C}^*$ given by $h \mapsto e^{i\pi\langle\omega,h\rangle} = i^{2\langle\omega,h\rangle}$.

Theorem 3. *Let M be a closed oriented 3-manifold endowed with a non-trivial cohomology class* $\omega \in H^1(M;\mathbb{C}/2\mathbb{Z})$. *Then for any complex spin structure* $\sigma \in \mathrm{Spin}^c(M)$ *we have*

$$\tau^{\varphi_\omega}(M,\sigma) = \frac{i^{b_1(M)+4\psi_{M,\sigma}(\omega)+1}}{2\cdot 4^{b_1(M)}}\, e N_2(M,\varnothing,\omega),$$

where $\psi_{M,\sigma}\colon H^1(M;\mathbb{C}/2\mathbb{Z}) \to \mathbb{C}/\mathbb{Z}$ *is the homomorphism obtained by first extending Deloup − Massuyeau's quadratic linking function* $\psi_{M,\sigma}\colon H_2(M;\mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ *(compare with* [9], *Definition* 2.2) *to a homomorphism* $\varphi^{\mathbb{C}}_{M,\sigma}\colon H_2(M;\mathbb{C}/\mathbb{Z}) \to \mathbb{C}/\mathbb{Z}$, *and then by considering the composition* $\varphi^{\mathbb{C}}_{M,\bar\sigma}\circ\frac{1}{2}\circ D$ *where* $D\colon H_2(M;\mathbb{C}/2\mathbb{Z}) \to H^1(M;\mathbb{C}/2\mathbb{Z})$ *is Poincaré duality,* $\frac{1}{2} : H^1(M;\mathbb{C}/2\mathbb{Z}) \to H^1(M;\mathbb{C}/\mathbb{Z})$ *is induced by the "division by 2" isomorphism between* $\mathbb{C}/2\mathbb{Z}$ *and* $\mathbb{C}/\mathbb{Z}$ *and* $\bar\sigma$ *is the image of* σ *under the standard involution of* $\mathrm{Spin}^c(M)$

This is proven by using the surgery formula for the Reidemeister torsion (see [8], section VIII.2, equation (2.b)): if L is a computable surgery presentation for $(M,\varnothing,\omega)$ then

$$\tau^{\varphi_\omega}(M,\sigma) = (-1)^{2m-\sigma_+(L)} e\prod_{j=1}^{m}\frac{i^{\alpha_j(k_j-1)}}{i^{\alpha_j}-i^{-\alpha_j}}\cdot\nabla_L(i^{\alpha_1},\ldots,i^{\alpha_m}),$$

where $\alpha_j := \langle\omega,\mu_j\rangle$ and $k_1,\ldots,k_m$ are the charges of the Spin^c-structure σ (see [8], section VII.2.2 for a definition).

We conclude with a Proposition which gives the value of N_2 for lens spaces and can be used to follow the path of the classical proof of their classification.

Proposition 5. *Let* $p > q > 0$ *be two coprime integers, let* $L(p,q)$ *be a lens space and consider a non zero cohomology class* $\omega \in H^1(L(p,q);\mathbb{C}/2\mathbb{Z})$. *Then*

$$N_2(L(p,q),\omega) = \frac{(-1)^{k(\omega)} e^{i\pi k(\omega)^2 p/q}}{2i\sin\frac{\pi k(\omega)q}{p}\sin\frac{\pi k(\omega)}{p}}$$

for some $k(\omega) \in \mathbb{Z}\setminus p\mathbb{Z}$.

Corollary 2. N_2 *classifies lens spaces.*

References

1. Costantino F., Geer N., Patureau-Mirand B. Quantum invariants of 3-manifolds via link surgery presentations and non-semi-simple categories. *J. of Topology,* 2014, vol. 7, is. 4, pp. 1005–1053.

2. Blanchet C., Costantino F., Geer N., Patureau-Mirand B. *Non-semisimple TQFTs, Reidemeister torsion and Kashaev's invariants*. Available at: http://arxiv.org/abs/1404.7289arXiv:1404.7289.

3. Turaev V. *Quantum Invariants of Knots and 3-Manifolds.* Berlin, Walter de Gruyter Publ., 2010. 592 p.

4. Costantino F., Geer N., Patureau-Mirand B. *Some remarks on the unrolled quantum group of* $\mathfrak{sl}(2)$. Available at: http://arxiv.org/abs/1406.0410arXiv:1406.0410.

5. Murakami J. The multi-variable Alexander polynomial and a one-parameter family of representations of $U_q(\mathfrak{sl}(2,\mathbb{C}))$ at $q^2=-1$. *Quantum Groups. Proceedings of Workshops held in the Euler International Mathematical Institute, Leningrad, Fall, 1990*. Berlin, Springer Publ., 1992, pp. 350–353.

6. Viro O. Quantum relatives of the Alexander polynomial. *St. Petersburg Math. J.,* 2007, vol. 18, no. 3, pp. 391–457.

7. Costantino F., Murakami J. On $\mathrm{SL}(2,\mathbb{C})$ quantum $6j$-symbols and its relation to thehyperbolic volume. *Quantum Topology,* 2013, vol. 4, is. 3, pp. 303–351.

8. Turaev V. *Torsions of 3-dimensional manifolds.* Berlin, Springer Publ., 2002. 196 p.

9. Deloup F., Massuyeau G. Quadratic functions and complex spin structures on three-manifolds. *Topology,* 2005, vol. 44, no. 3, pp. 509–555.

10. Geer N., Patureau-Mirand B., Turaev V. Modified Quantum dimensions and re-normalized link invariants. *Compositio Mathematica*, 2009, vol. 145, is. 1, pp. 196–212.

11. Ohtsuki T. *Quantum Invariants, a Study of Knots, 3-Manifolds and Their Sets*. Singapore, World Scientific, 2002. 508 p.

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КВАНТОВЫЕ ИНВАРИАНТЫ ТРЕХМЕРНЫХ МНОГООБРАЗИЙ, ВОЗНИКАЮЩИЕ ИЗ НЕПОЛУПРОСТЫХ КАТЕГОРИЙ

М. Де Рензи

Эта обзорная статья охватывает некоторые из результатов, содержащихся в работах Костантино, Гир, Патуреау и Бланше. В первой работе авторы строят два семейства инвариантов типа Решетихина — Тураева для трехмерных многообразий, N_r и N_r^0, используя для этого неполупростые категории представлений квантовой версии $\mathfrak{sl}_2$ в множество корней из единицы степени $2r$, $r \geq 2$. Второе семейство инвариантов N_r^0 предположительно обобщает оригинальные квантовые $\mathfrak{sl}_2$ инварианты Решетихина — Тураева. Авторы также развивают технику для построения инвариантов, возникающих из более общих ленточных категорий, которые могут и не обладать свойством полупростоты. Во второй работе перенормированная версия инварианта N_r при $r \neq 0 \pmod 4$ продолжается до TQFT, а также устанавливаются связи с классическими инвариантами, такими как полином Александера и кручение Рейдемейстера. В частности показано, что использование более богатых категорий имеет смысл, так как эти неполупростые инварианты более информативны, чем оригинальные полупростые инварианты: в самом деле, они могут быть использованы для классификации линзовых пространств, в то время как инварианты Решетихина — Тураев не всегда их различают.

Ключевые слова: *q-биномиальная формула, тождество дилогарифма.*

Список литературы

1. Costantino, F. Quantum invariants of 3-manifolds via link surgery presentations and non-semi-simple categories / F. Costantino, N. Geer, B. Patureau-Mirand // J. of Topology. — 2014. — Vol. 7, issue 4. — P. 1005–1053.

2. Non-semisimple TQFTs, Reidemeister torsion and Kashaev's invariants [Электронный ресурс] / C. Blanchet, F. Costantino, N. Geer, B. Patureau-Mirand. — URL: http://arxiv.org/abs/1404.7289arXiv:1404.7289.

3. Turaev, V. Quantum Invariants of Knots and 3-Manifolds / V. Turaev. — Berlin : Walter de Gruyter, 2010. — 592 p.

4. Costantino, F. Some remarks on the unrolled quantum group of $\mathfrak{sl}(2)$ [Электронный ресурс] / F. Costantino, N. Geer, B. Patureau-Mirand. — URL: http://arxiv.org/abs/1406.0410arXiv:1406.0410.

5. Murakami, J. The multi-variable Alexander polynomial and a one-parameter family of representations of $U_q(\mathfrak{sl}(2,\mathbb{C}))$ at $q^2 = -1$ / J. Murakami // Quantum Groups. Proceedings of Workshops held in the Euler International Mathematical Institute, Leningrad, Fall, 1990. — Berlin : Springer, 1992. — P. 350–353.

6. Viro, O. Quantum relatives of the Alexander polynomial / O. Viro // St. Petersburg Math. J. — 2007. Vol. 18, № 3. — P. 391–457.

7. Costantino, F. On $SL(2,\mathbb{C})$ quantum $6j$-symbols and its relation to thehyperbolic volume / F. Costantino, J. Murakami // Quantum Topology. — 2013. — Vol. 4, issue 3. — P. 303–351.

8. Turaev, V. Torsions of 3-dimensional manifolds / V. Turaev. Berlin : Springer, 2002. — 196 p.

9. Deloup, F. Quadratic functions and complex spin structures on three-manifolds / F. Deloup, G. Massuyeau // Topology. — 2005. — Vol. 44, № 3. — P. 509–555.

10. Geer, N. Modified Quantum dimensions and re-normalized link invariants / N. Geer, B. Patureau-Mirand, V. Turaev // Compositio Mathematica. — 2009. — Vol. 145, issue 1. — P. 196–212.

11. Ohtsuki, T. Quantum Invariants, a Study of Knots, 3-Manifolds and Their Sets / T. Ohtsuki. — Singapore : World Scientific, 2002. — 508 p.

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LINKS WITH TRIVIAL ALEXANDER MODULE AND NONTRIVIAL MILNOR INVARIANTS*

S. Garoufalidis

Cochran constructed many links with Alexander module that of the unlink and some nonvanishing Milnor invariants, using as input commutators in a free group and as an invariant the longitudes of the links. We present a different and conjecturally complete construction, that uses elementary properties of clasper surgery, and a different invariant, the tree-part of the LMO invariant. Our method also constructs links with trivial higher Alexander modules and nontrivial Milnor invariants.

Keywords: *Alexander module, Milnor invariants, claspers, Aarhus integral, LMO invariant.*

1. Introduction

1.1. History of the problem

Two of the best studied topological invariants a link L in S^3 are its *Alexander module* $A(L)$ which measures the homology of the universal abelian cover of $S^3 - L$, and its collection of *Milnor invariants* $\bar{\mu}(L)$, which are concordance (and sometimes link homotopy) invariants, defined modulo a recursive indeterminacy. Let us say that L has *trivial* Alexander module (resp. Milnor invariants) if $A(L) = A(\mathcal{O})$ (resp. $\bar{\mu}(L) = \bar{\mu}(\mathcal{O}) = 0$) for an unlink $\mathcal{O}$. Despite the indeterminacy of the Milnor invariants, note that the vanishing of all Milnor invariants is a well-defined statement.

Using the language of *longitudes* λ_i of components of L, Milnor showed that a link L has vanishing Milnor invariants iff $\lambda_i(L) \subset \pi_\omega$ for all i, where $\pi = \pi_1(S^3 - L)$ and $\pi_\omega = \cap_{n=1}^{\infty} \pi_n$ is the intersection of the *lower central series* π_n of π, defined by $\pi_1 = \pi$ and $\pi_{n+1} = [\pi_n, \pi]$, see [1]. L has trivial Alexander module iff there is a map $\pi \to F/[[F,F],[F,F]]$ which induces an isomorphism $\pi/[[\pi,\pi],[\pi,\pi]] \cong F/[[F,F],[F,F]]$.

It is natural to ask how independent are the conditions of trivial Alexander module and trivial Milnor invariants. In a sense, this question asks for a comparison between the lower central series and the commutator series of a link group.

In one direction, Levine showed that the vanishing of the Milnor invariants of a link L implies that a localization $A(L)_S$ of its Alexander module (although not the Alexander module itself) vanishes, where $S \subset \mathbb{Z}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$ is the multiplicative set of polynomials that evaluate to ± 1 at $t_1 = \ldots = t_r = 1$; see [2]. A boundary link has vanishing Milnor invariants, and its Alexander module splits as a direct sum of a trivial module and a torsion module. It was shown in [3] that all torsion modules with the appropriate symmetry can be realized.

In the opposite direction, if L has trivial Alexander module, then it is known that some low order Milnor invariants vanish [2; 4]. For example, all nonrepeated (link homotopy) invariants with at most 5 indices vanish. On the other hand, Cochran constructed a class of links with trivial Alexander module and nontrivial Milnor invariants; such links are not even be concordant to homology boundary links.

Cochran's construction used iteration, and used as a pattern certain elements in the lower central series of the free group. There is enough explicitness and control on the iteration that enabled Cochran to compute the longitudes directly and verify that these links have vanishing Alexander modules. Further, a geometric interpretation of Milnor invariants in terms of cycles

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on Seifert surfaces allowed Cochran to conclude that the constructed links have nontrivial Milnor invariants.

As an elementary application of the calculus of claspers, we will construct a plethora of links with vanishing Alexander module. For these links, we can compute the tree part of the LMO invariant (which can be identified with Milnor invariants, [5]), using formal Gaussian integration. As a result, we will construct many (and conjecturally all) links with trivial Alexander module and nontrivial Milnor invariants. The next definition explains the patterns that we will use in our construction.

Definition 1. Let $\mathcal{A}^{\text{tr}}(r)$ (or simply, $\mathcal{A}^{\text{tr}}$, in case r is clear) denote the vector space over $\mathbb{Q}$ generated by vertex-oriented unitrivalent trees, whose univalent vertices are labeled by r colors, modulo the AS and IHX relation. $\mathcal{A}^{\text{tr}}(r)$ is a graded vector space, where the degree of a graph is half the number of vertices. We will call a tree of degree 1 (with two univalent vertices and no trivalent ones) a *strut*.

A *pattern* β is an element of $\mathcal{A}^{\text{tr}}(r)$ which is represented by a tree which has a trivalent vertex v such that $\beta - v$ has no strut components.

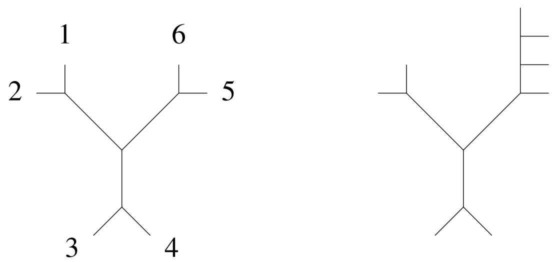


Fig. 1

Fig. 1 gives some examples of nonvanishing patterns.

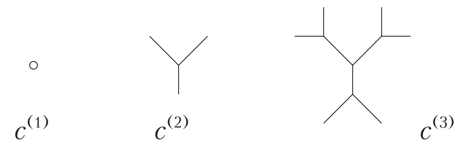
Theorem 1. *For every nonvanishing pattern* $\beta \in \mathcal{A}^{\text{tr}}_m(r)$ *there exists a link* $L(\beta)$ *with r components such that* $A(L(\beta)) = A(\mathcal{O})$, *all Milnor invariants of degree less than m vanish and some Milnor invariant of degree m do not.*

Our construction adapts without change to the case of links with trivial *higher Alexander modules*. Although classical, these modules appeared only recently in work of Cochran-Orr-Teichner [6] and subsequent work of Cochran, [7]. Given a group π, consider its *commutator series* defined by $\pi^{(0)} = \pi$ and $\pi^{(n+1)} = [\pi^{(n)}, \pi^{(n)}]$.

Definition 2. We will say that a link L in a homology sphere M has *trivial nth Alexander module* if it has a map $\pi \to F/F^{(n+1)}$ which induces an isomorphism $\pi/\pi^{(n+1)} \cong F/F^{(n+1)}$, where $\pi = \pi_1(M - L)$.

The next definition explains the n-patterns which we will use.

Definition 3. Let $c^{(n)}$ be a unitrivalent tree defined by



In other words, we are adding two univalent vertices in $c^{(n+1)}$ to each of the univalent vertices of $c^{(n)}$. An *n-pattern* $\beta^{(n)}$ is an element of $\mathcal{A}^{\text{tr}}(r)$ which is represented by a tree $\beta^{(n)}$ such that $c^{(n)} \subset \beta^{(n)}$ and $\beta^{(n)} - c^{(n)}$ has no strut components.

The proof of Theorem 1 generalizes without change to the following

Theorem 2. *For every nonvanishing n-pattern* $\beta^{(n)} \in \mathcal{A}^{\text{tr}}_m(r)$ *there exists a link* $L(\beta^{(n)})$ *with r components with trivial nth Alexander module, such that all Milnor invariants of degree less than m vanish and some Milnor invariant of degree m do not.*

2. Constucting links by surgery on claspers

2.1. What is surgery on a clasper?

As we mentioned in the introduction, we will construct links of Theorem 2 using *surgery on claspers*. Since claspers play a key role in geometric constructions, as well as in the theory of finite type invariant s, we include a brief discussion here. For a reference on claspers and their associated surgery, we refer the reader to [8; 9] and also to [10. Section 2] (where claspers were called clovers instead). It suffices to say that a clasper is a thickening of a trivalent graph, and it has a preferred set of loops, called the leaves. The degree of a clasper is the number of trivalent vertices (excluding those at the leaves). With our conventions, the smallest clasper is a Y-clasper (which has degree one and three leaves), so we explicitly exclude struts (which would be of degree zero with two leaves).

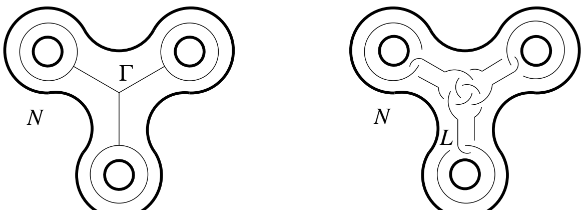
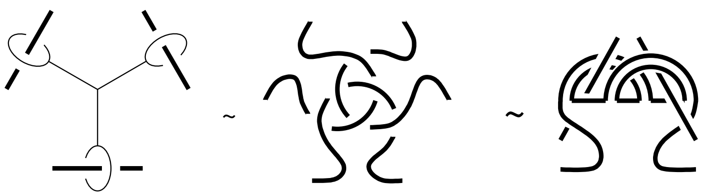


Fig. 2

A clasper G of degree 1 is an embedding $G : N \to M$ of a regular neighborhood of the graph Γ in a 3-manifold M. Surgery on G can be described by cutting $G(N)$ from M (which is a genus 3 handlebody), twisting by a fixed diffeomorphism of its boundary (which acts trivially on the homology of the boundary) and gluing back. We will denote the result of surgery by M_G. Alternatively, we can describe surgery on G by surgery on a framed six component link (the image of L) in M. The six component link consists of a 0-framed Borromean ring and an arbitrarily framed three component link, the so-called *leaves* of G. If one of the leaves bounds a 0-framed disk disjoint from the rest of G, then surgery on G does not change the ambient 3-manifold M, although it can change an embedded link in M. In particular, surgery on a clasper of degree 1 is shown as follows:



In general, surgery on a clasper G of degree n can be described in terms of simultaneous surgery on n claspers $G_1, \ldots, G_n$, which are obtained from G after breaking its edges and inserting Hopf links as follows:

2.2. A basic principle

Surgery on a clasper is described by twisting by a surface diffeomorphism that acts trivially on homology, thus we have the basic principle:

Clasper surgery preserves the homology

Surgery on claspers with leaves of a resticted type has already been studied and used successfully in [11] (where the leaves were assumed null homologous in a knot complement), [12] (and where the leaves where null homotopic) and [13] (where the leaves where in the kernel of a map to a free group). It is important to study not only 3-manifolds but rather pairs of 3-manifolds together with a representation of their fundamental group into a fixed group. Claspers adapt well to this point of view, as we explain next.

Consider a pair (N, ρ) of a 3-manifold N (possibly noncompact) and a representation $\rho : \pi_1(N) \to \Gamma$ for some group Γ. Consider a clasper $G \subset N$ whose leaves are mapped to 1 under ρ. We will call such claspers ρ-*null*, or simply *null*, if ρ is clear. Surgery on G gives rise to a 4-manifold W whose boundary consists of one copy of N and one copy of N_G. We may think that W is obtained by attaching $6n$ 2-handles on $N \times I$, where $n = \text{degree}(G)$. Since the cores of these handles lie in the kernel of ρ, it follows that ρ extends over W, and in particular restricts to a representation ρ_G on the end N_G of W.

Lemma 1. *We have* $H_*(N,\rho) \cong H_*(N_G,\rho_G)$.

Proof. Let $\widetilde{N}$ (resp. $\widetilde{N_G}$) denote the cover of N (resp. N_G) corresponding to ρ (resp. ρ_G). Surgery on G is equivalent to surgery on a collection $\{G_1,\dots,G_k\}$ of degree 1 claspers, constructed by inserting Hopf links in the edges of G. Each G_i lifts to a collection $\widetilde{G}_i$ of claspers in $\widetilde{N}$; let $\widetilde{G} = \widetilde{G}_1 \cup \dots \widetilde{G}_k$. Then, $\widetilde{N_G}$ can be identified with $(\widetilde{N})_{\widetilde{G}}$. Since clasper surgery preserves homology, the result follows. □

We will adapt the above lemma in the following situation. Suppose that G is a clasper in the complement of an unlink $X_0 = S^3 - \mathcal{O}$ of r components whose leaves are null homologous in X_0, and let (M,L) denote the result of surgery along G on the pair $(S^3,\mathcal{O})$. It follows that G lifts to a family $\widetilde{G}$ of claspers in $\widetilde{X}_0$ (the universal abelian cover of X) and that $\widetilde{X}$ is obtained from $\widetilde{X}_0$, by surgery on $\widetilde{G}$, where $X = M - L$. Since $A(L) = H_1(\widetilde{X},\widetilde{x})$, and clasper surgery preserves homology, it follows that $A(M,L) = A(\mathcal{O})$.

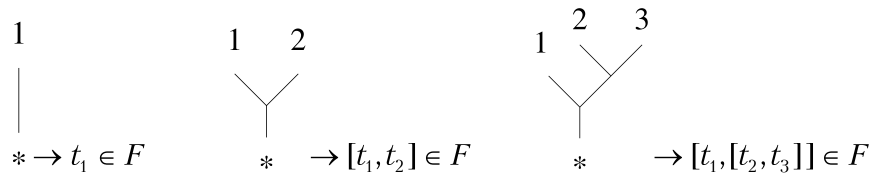
Remark 1. There are two known cases where surgery on a null clasper $G \subset X_0$ gives rise to a link (M,L) with vanishing Milnor invariants.

(a) If the leaves of G are null homotopic in X_0, then the constructed links would be boundary links, as was observed and used in [13]. Boundary links have vanishing Milnor invariants.

(b) If G is a connected clasper with at least one loop, then (M,L) is concordant to $(S^3,\mathcal{O})$, [14] and also [15]. Concordance preserves Milnor invariants.

With a bit more effort, we can arrange that $M = S^3$. For this, it suffices to assume that each connected component G_i of G has a 0-framed leaf l_i, such that the union of the leaves $\{l_i\}$ is an unlink in S^3.

To finalize the construction of Theorem 1, consider a pattern β, and a vertex v of β such that $\beta - v = T_1 \cup T_2 \cup T_3$ where T_i are rooted trees which are not struts. Each rooted tree T corresponds to an element $\phi(T) \in F$ via a map defined in pictures by:



If T is not a strut, then $\phi(T) \in [F,F]$. Given β as above, we will choose a clasper $G(\beta)$ of degree 1 such that its three leaves l_i satisfy $l_i = \phi(T_i) \in [F,F]$, for $i = 1,2,3$. Then, $L(\beta)$ is obtained from the unlink by clasper surgery on $G(\beta)$.

Finally, let us modify the above discussion for the construction of Theorem 2. Given an n-pattern $\beta^{(n)}$, let $G(\beta^{(n)})$ be a tree clasper of degree n in X_0, which consists of $c^{(n+1)}$ and 2^{n+1} leaves l_i (one in each univalent vertex of $c^{(n+1)}$). There is a 1-1 correspondence between the connected components T_i of $\beta^{(n)} - c^{(n)}$ and the leaves l_i of $G(\beta^{(n)})$. We will choose these leaves so that $l_i = \phi(T_i) \in F$, and we will let $L(b^{(n)})$ be obtained from the unlink by clasper surgery on $G(\beta^{(n)})$.

We need to show that $L(b^{(n)})$ has trivial nth Alexander module. Indeed, using the figures above that describe clasper surgery, it follows that clasper surgery on $G(\beta^{(n)})$ is equivalent to surgery on a clasper $G'(\beta^{(n)})$ of degree 1 whose leaves lie in $F^{(n)}$. This implies that the nth Alexander module of $L(b^{(n)})$ is trivial.

We end this section with a comment on pictures. To get *pictures* of the constructed links, one may use various descriptions of surgery on a clasper that were discussed at length by Goussarov and Habiro at [8; 9]. From our point of view though, these pictures are complicated and unnecessary, since not only claspers describe surgery adequately, but also the invariants which we will use behave well with respect to clasper surgery. This is the content of the next section.

3. Computing the tree part of the Aarhus integral

3.1. The Aarhus integral in brief

As was stated in the discussion of Theorem 1, we will not compute the Milnor invariants of the links $L(\beta)$ constructed via clasper surgery, but rather we will compute the tree-part of their Aarhus integral. The Aarhus integral is a graph version of stationary phase approximation that was introduced at [16–18]. Despite its intimidating name, it is a rather harmless combinatorial object which we now describe.

Consider a framed link $C \subset S^3 - \mathcal{O}$ and let $(M, L) = (S^3, \mathcal{O})_G$ denote the result of surgery on C. That is, M is the 3-manifold obtained from S^3 by surgery on C and L is the image of $\mathcal{O}$ after surgery. Assuming that M is a rational homology sphere (i.e., that the linking matrix of C has nonzero determinant) the Aarhus integral $Z(M, L)$ can be computed by the Kontsevich integral of the link $\mathcal{O} \cup C$ by integration as follows:

$$Z(M, L) = \int dX Z(S^3, \mathcal{O} \cup C)$$

(where X is a set of variables in 1-1 correspondence with the components of C). Let us briefly recall from [17] how this integration works. Consider an element

$$s = \exp\left(\frac{1}{2} \sum_{x,y \in X} \overset{x}{\underset{y}{|}}\, Q_{xy} \right) R,$$

with R a series of graphs that do not contain a strut whose legs are colored by X. Notice that Q and R, the *X-strutless part* of s, are uniquely determined by s. Then, the integration $\int dX(s)$ glues all the X-colored legs of R pairwise, using the negative inverse of the matrix Q. That is, when two legs x, y of R are glued, the resulting graph is multiplied by $-Q^{xy}$, the negative inverse of the matrix Q_{xy}.

It follows immediately that the *tree-part* $Z^{\mathrm{tr}}(M, L)$ of $Z(M, L)$ depends only on the tree-part $Z^{\mathrm{tr}}(S^3, \mathcal{O} \cup C)$ of $Z(S^3, \mathcal{O} \cup C)$.

3.2. Claspers and the Aarhus integral

Let us adapt the above discussion when the link C is one that describes clasper surgery. Consider a null clasper $G \subset S^3 - \mathcal{O}$ of degree 1 constructed from a pattern β and let $(M, L) = (S^3, \mathcal{O})_G$. Let $Z^{\min}(M, L)$ denotes the *lowest degree nonvanishing tree part* of $Z^{\mathrm{tr}}(M, L)$. Assuming that the pattern is nonvanishing, and after we choose string-link representatives of $L \cup G$, we will prove

Proposition 1. *We have* $Z^{\min}(M, L) = \beta \in \mathcal{A}^{\mathrm{tr}}$.

It is clear that this concludes Theorem 1.

Proof. (Of Proposition 1). Surgery on G is equivalent to surgery on a 6 component link $C = C^e \cup C^l$; see Subection 2.1. C^e is a borromean link and C^l consists of the leaves of G. In the obvious basis, the linking matrix of C is given by

$$\begin{pmatrix} 0 & I \\ I & \mathrm{lk}(C_i^l, C_j^l) \end{pmatrix},$$

and its negative inverse is given by

$$\begin{pmatrix} \mathrm{lk}(C_i^l, C_j^l) & -I \\ -I & 0 \end{pmatrix}.$$

In particular, a univalent vertex labeled by a leaf has to be glued to a univalent vertex labeled by the corresponding edge. Let $A_i = \{C_i^e, C_i^l\}$ denote the arms of G for $i = 1,2,3$. It is a key fact that surgery on any proper subcollection of the set $\{A_1, A_2, A_3\}$ of arms does not change the pair $(S^3, \mathcal{O})$. In other words, alternating with respect to the 8 subsets of the set of arms we have that $Z([(S^3,\mathcal{O}),G]) = Z([(S^3,\mathcal{O}),\{A_1,A_2,A_3\}])$. The nontrivial contributions to the left hand side come from the $(\mathcal{O} \cup C)$-strutless part of $Z(S^3, \mathcal{O} \cup C)$ that consists of graphs with legs on A_1 and on A_2 and on A_3.

What kind of diagrams in $Z^{\text{tr}}(S^3, \mathcal{O} \cup C)$ contribute to the above sum? Consider a disjoint union D of trees whose legs are labeled by $\mathcal{O} \cup C$. D must have a leg (i. e., univalent vertex) labeled by C_i^l or by C_i^e for each $i = 1,2,3$. If D has a leg labeled by C_i^l, then due to the shape of the gluing matrix, D must have a C_i^e-labeled leg. Thus, in all cases, D must have legs labeled by all three edges C_i^e of G.

Consider a tree T labeled by $\mathcal{O} \cup C$. If T has a C_i^e-labeled leg, then it must either have legs labeled by all three edges of G, or else it must have a leg labeled by C_i^l. Indeed, C_i^e is an unknot in a ball disjoint from $\mathcal{O} \cup C - \{C_i^l\}$, thus the rest of the trees have vanishing coefficient in $Z^{\text{tr}}(S^3, \mathcal{O} \cup C)$.

Consider further a *vortex* Y (that is, a unitrivalent graph of the shape Y with three univalent vertices and one trivalent one) whose legs are labeled by three leaves of G. Then, the coefficient of Y in $Z(S^3, \mathcal{O} \cup C)$ is 1.

Consider further a tree T with one univalent vertex labeled by a leaf C_i^l of G and all other vertices labeled by $\mathcal{O}$. Recall the corresponding rooted tree T_i which is a component of $\beta - v$. Then the coefficient of T in $Z^{\text{tr}}(S^3, \mathcal{O} \cup C)$ is zero if $\deg(T) < \deg(T_i)$ and equals to 1 if $T = T_i$. This, together with the above discussion and the gluing rules concludes the proof of Proposition 1. The argument is best illustrated by the Fig. 3. □

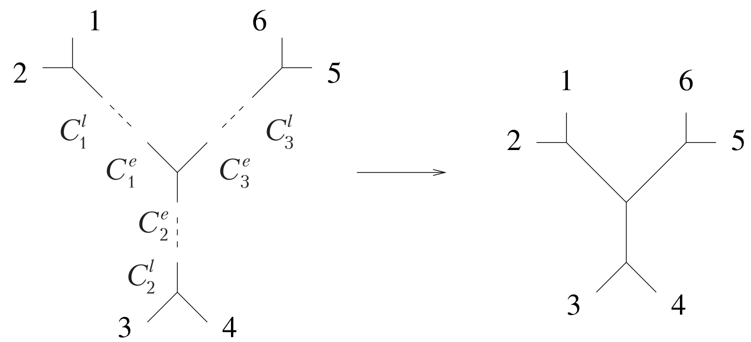


Fig. 3

The above proposition and its proof generalize easily to the case of claspers G corresponding to nonvanishing n-patterns $\beta^{(n)}$. In that case, if (M, L) denote the corresponding link, we still have that $Z^{\min}(M, L) = \beta^{(n)} \in \mathcal{A}^{\text{tr}}$ which implies Theorem 2.

Remark 2. In the above discussion we have silently chosen dotted Morse link representatives (or equivalently, string-link representatives) and we ought to have normalized the Aarhus integral. But this does not affect the lowest degree nonvanishing tree part.

The links constructed by clasper surgery in Theorem 1 include the links that Cochran constructed via Seifert surfaces.

Example 1. Does Section 2 construct every link with trivial Alexander module?

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The paper arose during conversations with the late J. Levine in Brandeis in the spring of 1999 and posted on the arXiv in its present form in 2002. After years of sitting idle on a desk, we were encouraged to publish it.

References

1. Milnor J. Link groups. *Ann. of Math.,* 1954, vol. 59, no. 2, pp. 177–195.

2. Levine J.P. Localization of link modules. *Amer. Math. Soc. Contemporary Math.,* 1983, vol. 20, pp. 213–229.

3. Garoufalidis S., Levine J.P. Analytic invariants of boundary links. *J. Knot Theory Ramifications*, 2002, vol. 11, no. 3, pp. 283–293.

4. Traldi L. Milnor's invariants and the completions of link modules. *Trans. Amer. Math. Soc.,* 1984, vol. 284, no. 1, pp. 401–424.

5. Habegger N., Masbaum G. The Kontsevich integral and Milnor's invariants. *Topology*, 2000, vol. 39, no. 6, pp. 1253–1289.

6. Cochran T.D., Orr K.E., Teichner P. Knot concordance, Whitney towers and L^2-signatures. *Ann. of Math.,* 2003, vol. 157, no. 2, pp. 433–519.

7. Cochran T.D. Noncommutative knot theory. *Algebr. Geom. Topol.,* 2004, vol. 4, pp. 347–398.

8. Gusarov M.N. Variations of knotted graphs. The geometric technique of n-equivalence. *St Petersburg Math J.*, 2000, vol. 12, no. 4, pp. 79–125.

9. Habiro K. Claspers and finite type invariants of links. *Geom. Topol.*, 2000, vol. 4, pp. 1–83.

10. Garoufalidis S., Gusarov M., Polyak M. Calculus of clovers and finite type invariants of 3-manifolds. *Geom. Topol.,* 2001, vol. 5, pp. 75–108.

11. Garoufalidis S., Rozansky L. The loop expansion of the Kontsevich integral, the null-move and S-equivalence. *Topology*, 2004, vol. 43, no. 5, pp. 1183–1210.

12. Garoufalidis S., Levine J.P. Homology surgery and invariants of 3-manifolds. *Geom. Topol.,* 2001, vol. 5, pp. 551–578.

13. Garoufalidis S., Kricker A. A rational noncommutative invariant of boundary links. *Geom. Topol.,* 2004, vol. 8, pp. 115–204.

14. Garoufalidis S., Levine J.P. Concordance and 1-loop clovers. *Algebr. Geom. Topol.,* 2001, vol. 1, pp. 687–697.

15. Conant J., Teichner P. Grope cobordism of classical knots. *Topology*, 2004, vol. 43, no. 1, pp. 119–156.

16. Bar-Natan D., Garoufalidis S., Rozansky L., Thurston D.P. The Aarhus integral of rational homology 3-spheres. I. A highly non trivial flat connection on S^3. *Selecta Math. (N.S.)*, 2002, vol. 8, no. 3, pp. 315–339.

17. Bar-Natan D., Garoufalidis S., Rozansky L., Thurston D.P. The Aarhus integral of rational homology 3-spheres. II. Invariance and universality. *Selecta Math. (N.S.).,* 2002, vol. 8, no. 3, pp. 341–371.

18. Bar-Natan D., Garoufalidis S., Rozansky L., Thurston D. P. The Aarhus integral of rational homology 3-spheres. III. Relation with the Le-Murakami-Ohtsuki invariant. *Selecta Math. (N.S.).,* 2004, vol. 10, no. 3, pp. 305–324.

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ЗАЦЕПЛЕНИЯ С ТРИВИАЛЬНЫМ МОДУЛЕМ АЛЕКСАНДЕРА И НЕТРИВИАЛЬНЫЕ ИНВАРИАНТЫ МИЛНОРА

С. Гаруфалидис

Кокран построил много зацеплений, для которых модуль Александера совпадет с модулем для тривиального зацепления, но некоторые инварианты Милнора нетривиальны. Для этого он использовал коммутаторы в свободной группе и параллели для зацеплений. Мы даем другую и гипотетически полную конструкцию, которая использует элементарные свойства класперных перестроек, а также строим новый инвариант, являющийся частью LMO-инварианта. Наш метод также позволяет построить зацепления с тривиальными модулями Александера высоких порядков и нетривиальными инвариантами Милнора.

Ключевые слова: *модель Александера, инварианты Милнора, класперы, интеграл Архуса, LMO-инвариант.*

Список литературы

1. Milnor, J. Link groups / J. Milnor // Ann. of Math. — 1954. — Vol. 59, № 2. — P. 177–195.

2. Levine, J. P. Localization of link modules / J. P. Levine // Amer. Math. Soc. Contemporary Math. — 1983. — Vol. 20. — P. 213–229.

3. Garoufalidis, S., Analytic invariants of boundary links / S. Garoufalidis, J. P. Levine // J. Knot Theory Ramifications. — 2002. — Vol. 11, № 3. — P. 283–293.

4. Traldi, L. Milnor's invariants and the completions of link modules / L. Traldi // Trans. Amer. Math. Soc. — 1984. — Vol. 284, № 1. — P. 401–424.

5. Habegger, N. The Kontsevich integral and Milnor's invariants / N. Habegger, G. Masbaum // Topology. — 2000. — Vol. 39, № 6. — P. 1253–1289.

6. Cochran, T. D. Knot concordance, Whitney towers and L^2-signatures / T. D. Cochran, K. E. Orr, P. Teichner // Ann. of Math. — 2003. — Vol. 157, № 2. — P. 433–519.

7. Cochran, T. D. Noncommutative knot theory / T. D. Cochran // Algebr. Geom. Topol. — 2004. — Vol. 4. —P. 347–398.

8. Gusarov, M. N. Variations of knotted graphs. The geometric technique of n-equivalence / M. N. Gusarov // St Petersburg Math J. — 2000. — Vol. 12, № 4. — P. 79–125.

9. Habiro, K. Claspers and finite type invariants of links / K. Habiro // Geom. Topol. — 2000. — Vol. 4. — P. 1–83.

10. Garoufalidis, S. Calculus of clovers and finite type invariants of 3-manifolds / S. Garoufalidis, M. Gusarov, M. Polyak // Geom. Topol. — 2001. — Vol. 5. — P. 75–108.

11. Garoufalidis, S. The loop expansion of the Kontsevich integral, the null-move and S-equivalence / S. Garoufalidis, L. Rozansky // Topology. — 2004. — Vol. 43, № 5. — P. 1183–1210.

12. Garoufalidis, S. Homology surgery and invariants of 3-manifolds / S. Garoufalidis, J. P. Levine // Geom. Topol. — 2001. — Vol. 5. — P. 551–578.

13. Garoufalidis, S. A rational noncommutative invariant of boundary links / S. Garoufalidis, A. Kricker // Geom. Topol. — 2004. — Vol. 8. — P. 115–204.

14. Garoufalidis, S. Concordance and 1-loop clovers / S. Garoufalidis, J. P. Levine // Algebr. Geom. Topol. — 2001. — Vol. 1. — P. 687–697.

15. Conant, J. Grope cobordism of classical knots / J. Conant, P. Teichner // Topology. — 2004. — Vol. 43, № 1. — P. 119–156.

16. Bar-Natan, D. The Aarhus integral of rational homology 3-spheres. I. A highly non trivial flat connection on S^3 / D. Bar-Natan, S. Garoufalidis, L. Rozansky, D. P. Thurston // Selecta Math. (N.S.). — 2002. — Vol. 8, № 3. — P. 315–339.
17. Bar-Natan, D. The Aarhus integral of rational homology 3-spheres. II. Invariance and universality / D. Bar-Natan, S. Garoufalidis, L. Rozansky, D. P. Thurston // Selecta Math. (N.S.). — 2002. — Vol. 8, № 3. — P. 341–371.
18. Bar-Natan, D. The Aarhus integral of rational homology 3-spheres. III. Relation with the Le-Murakami-Ohtsuki invariant / D. Bar-Natan, S. Garoufalidis, L. Rozansky, D. P. Thurston // Selecta Math. (N.S.). — 2004. — Vol. 10, № 3. — P. 305–324.

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ALGEBRAIC G-FUNCTIONS ASSOCIATED TO MATRICES OVER A GROUP-RING*

S. Garoufalidis, J. Bellissard

Given a square matrix with elements in the group-ring of a group, one can consider the sequence formed by the trace (in the sense of the group-ring) of its powers. We prove that the corresponding generating series is an algebraic G-function (in the sense of Siegel) when the group is free of finite rank. Consequently, it follows that the norm of such elements is an exactly computable algebraic number, and their Green function is algebraic. Our proof uses the notion of rational and algebraic power series in non-commuting variables and is an easy application of a theorem of Haiman. Haiman's theorem uses results of linguistics regarding regular and context-free language. On the other hand, when the group is free abelian of finite rank, then the corresponding generating series is a G-function. We ask whether the latter holds for general hyperbolic groups.

Keywords: *rational function, algebraic function, holonomic function, G-function, generating series, non-commuting variables, moment, hamiltonian, resolvant, regular language, context-free language, Hadamard product, group-ring, free probability, Schur complement method, free group, von Neumann algebra, polynomial Hamiltonian, spectral theory, norm.*

1. Introduction

1.1. Algebricity of the Green's function for the free group

Given a group G, consider the group-algebra $\mathbb{Q}[G]$, and define a *trace* map:

$$\mathrm{Tr} : \mathbb{Q}[G] \to \mathbb{C}, \quad \mathrm{Tr}(P) = \text{constant term of } P,$$

where the constant term is the coefficient of the identity element of G. Let $M_N(R)$ denote the set of N by N matrices with entries in a ring R. We can extend the trace to the algebra $M_N(\mathbb{Q}[G])$ by:

$$\mathrm{Tr} : M_N(\mathbb{Q}[G]) \to \mathbb{C}, \quad \mathrm{Tr}(P) = \sum_{j=1}^{N} \mathrm{Tr}(P_{jj}).$$

Definitiom 1. Given $P \in M_N(\mathbb{Q}[G])$, consider the sequence $(a_{P,n})$, $a_{P,n} = \mathrm{Tr}(P^n)$, and the *generating series* $R_P(z) = \sum_{n=0}^{\infty} a_{P,n} z^n$.

Let F_r denote the free group of rank r.

Theorem 1. *The Green's function $R_P(z)$ of every element P of $M_N(\mathbb{Q}[F_r])$ is algebraic.*

Theorem 1 appears in the cross-roads of several areas of research:

(a) operator algebras;

(b) free probability;

(c) linguistics and context-free languages;

(d) non-commutative combinatorics;

(e) mathematical physics.

In fact, Woess proves Theorem1 when $N = 1$ using linguistics and context-free languages; see [1; 2]. In [3] Sauer also gives a proof using linguistics, with emphasis the rationality of the Novikov–Shubin invariants. Voiculescu proves Theorem 1 using the R and S transforms of free probability; see [4; 5]. For additional results using free probability, see [6; 7] and also [8–10].

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It is well-known that Theorem 1 provides an exact calculation of the norm of $P \in M_N(\mathbb{Q}[F_r]) \subset M_N(L(F_r))$, where $L(F_r)$ denotes the *reduced* $\mathbb{C}^*$*-algebra* completion of the group-algebra $\mathbb{C}[F_r]$. For a detailed discussion, see the above references.

Our proof of Theorem 1 uses the notion of an algebraic function in non-commuting variables and a theorem of Haiman, which itself is based on a theorem of Chomsky − Schützenberger on context-free languages. A by-product of our proof is the fact that the moment generating series is a matrix of algebraic power series in non-commuting variables (see Proposition 1), which is a statement a priori stronger than Theorem 1.

An alternative proof of Theorem 1 uses methods from functional analysis, and most notably the *Schur complement method* (see below). We will discuss in detail the first proof and postpone the third proof to a later publication. Either proof explains the close relation between the differential properties of the generating function $R_P(z)$ and the word problem in G.

Our aim is to give a proof of algebraicity in the case of the free group, discuss holonomicity in the case of the free abelian group and formulate a question regarding holonomicity for hyperbolic groups. As it turns out, algebricity is well-studied in the above mentioned literature whereas holonomicity is largely absent.

1.2. Related work

Our paper was completed in the summer of 2007, and posted on the arxiv arXiv:0708.4234. In the fall of 2008, M. Kontsevich brought to the attention of the second author a related earlier paper of Sauer [3] from 2003 that gives a proof of Theorem 1 with emphasis on the Novikov−Shubin invariants. Sauer's and our work has been cited by M. Kontsevich in the Arbeitstagung talk Bonn 2011, and (from what we have heard) in other talks too. Theorem 1 keeps attracting attention in diverse areas of mathematics. In the summer 2013, C. Kassel informed the second author of related article of Kassel−Reutenauer [11] around the theme of Theorem 1. Kassel was unaware of Sauer's work and of our work. In view of the interest of Theorem 1 and its connections to several branches of mathematics, we were encouraged to submit our article for publication.

2. The case of the free abelian group

2.1. Holonomic, algebraic and G-functions

A priori, $R_P(z)$ is only a formal power series. However, it is easy to see that $(a_{P,n})$ is bounded exponentially by n, which implies that $R_P(z)$ defines an analytic function in a neighborhood of $z = 0$. The paper is concerned with differential / algebraic properties of the function $R_P(z)$. Algebraic and holonomic functions are well-studied objects. Let us recall their definition here.

Definition 2. (a) A *holonomic* function $f(z)$ is one that satisfies a linear differential equation with polynomial coefficients. In other words, we have:

$$c_d(z)f^{(d)}(z) + \ldots + c_0(z)f(z) = 0$$

where $c_j(z) \in \mathbb{Q}[z]$ for all $j = 0, \ldots, d$ and $f^{(j)}(z) = (d^j / dz^j)f(z)$.

(b) An *algebraic* function $f(z)$ is one that satisfies a polynomial equation $Q(f(z), z) = 0$ where $Q(y, z) \in \mathbb{Q}[y, z]$.

Lesser known to the combinatorics community are G-functions, which originated in the work Siegel on arithmetic problems in elliptic integrals, and transcendence problems in number theory; see [12]. G-functions originate naturally in:

(a) algebraic geometry, related to the regularity properties of the Gauss−Manin connection, see for example [13—15];

(b) arithmetic, see for example [16—18];
(c) enumerative combinatorics, as was recently shown in [19].

Definition 3. A *G-function* $f(z)=\sum_{n=0}^{\infty}a_n z^n$ is one which satisfies the following conditions:
(a) for every $n\in\mathbb{N}$, we have $a_n\in\overline{\mathbb{Q}}$;
(b) there exist a constant $C_f>0$ such that for every $n\in\mathbb{N}$ we have: $|a_n|\leqslant C_f^n$ (for every conjugate of a_n) and the common denominator of $a_0,\ldots,a_n$ is less than or equal to C_f^n;
(c) $f(z)$ is holonomic.

The next theorem summarizes the analytic continuation and the shape of the singularities of algebraic functions and *G*-functions. Part (a) follows from the general theory of differential equations (see eg. [20]), parts (b) and (d) follow from [21. Lem. 2.2] (see also [18] and [22]) and (c) follows from a combination of Katz's theorem, Chudnovsky's theorem and André's theorem; see [16. P. 706] and also [23].

Theorem 2. (a) *A holonomic function $f(z)$ can be analytically continued as a multivalued function in $\mathbb{C}\setminus\S_f$ where $\S_f\subset\overline{\mathbb{Q}}$ is the finite set of singular points of $f(z)$.*
(b) *Every algebraic function $f(z)$ is a G-function.*
(c) *In a neighborhood of a singular point $\lambda\in\S_f$, a G-function $f(z)$ can be written as a finite sum of germs of the form:*

$$(z-\lambda)^{\alpha_\lambda}(\log(z-\lambda))^{\beta_\lambda}h_\lambda(z-\lambda) \tag{2.1}$$

where $\alpha_\lambda\in\mathbb{Q}$, $\beta_\lambda\in\mathbb{N}$, and h_λ a holonomic G-function.
(d) *In addition, $\beta_\lambda=0$ if $f(z)$ is algebraic.*

Remark 1. Local expansions of the form (2.1) are known in the literature as Nilsson series (see [24]), and minimal order linear differential equations that they satisfy are known to be regular singular, with rational exponents $\{a_\lambda\}$ and quasi-unipotent monodromy. For a discussion, see [14; 15; 19] and references therein.

It is classical and easy to show that the existence of analytic continuation of a function implies the existence of asymptotic expansion of its Taylor series; see for example [12; 25] and also [26. Sec. 7] and [19].

Lemma 1. *If $f(z)=\sum_{n=0}^{\infty}a_n z^n$ is holonomic and analytic at $z=0$, then the nth Taylor coefficient a_n has an asymptotic expansion in the sense of Poincaré*

$$a_n\sim\sum_{\lambda\in\S}\lambda^{-n}n^{-\alpha_\lambda-1}(\log n)^{\beta_\lambda}\sum_{s=0}^{\infty}\frac{c_{\lambda,s}}{n^s}$$

where $\S_f$ is the set of singularities of f, $\alpha_\lambda,\beta_\lambda\in\mathbb{Q}$, and $c_{\lambda,s}\in\mathbb{C}$.

2.2. The case of the free abelian group

In this section we will summarize what is known about the generating functions $R_P(z)$ when $G=\mathbb{Z}^r$ is the free abelian group or rank *r*. The next theorem is shown in [19], using André main theorems from [16]. An alternative proof uses the regular holonomicity of the Gauss − Manin connection and the rationality of its exponents. This was kindly communicated to us by C. Sabbah (see also [27]). Holonomicity of $R_P(z)$ also follows from a fundamental result of Wilf–Zeilberger, explained in [19].

Theorem 3. [19]. *For every $P\in M_N(\mathbb{Q}[\mathbb{Z}^r])$, $R_P(z)$ is a G-function.*

2.3. A complexity remark

Given $P\in M_N(\mathbb{Q}[F_r])$ (resp. $P\in M_N(\mathbb{Q}[F_r])$), one may ask for the complexity of a minimal polynomial $Q(y,z)\in\mathbb{Q}[y,z]$ (resp. minimal degree differential operator $D(z,\partial_z)\in\mathbb{Q}\langle z,\partial z\rangle$) so that $Q(R_P(z),z)=0$ (resp. $D(z,\partial_z)R_P(z)=0$). One expects that the *y*-degree of $Q(y,z)$ and the ∂_z-degree of $D(z,\partial_z)$ is exponential in the *complexity* of P, where the latter can be defined to be the degree of P and the maximum of the absolute values of the coefficients of the enrties of P. This prohibits explicit calculations in general.

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3. A theorem of Haiman and a proof of Theorem 1

In [28] Haiman proves the following theorem.

Theorem 4. [28]. *Let K be a field with a rank 1 discrete valuation v; K_v its completion with respect to the metric induced by v. Let $f(x_1,\dots,x_r,y_1,\dots,y_r)$ be a rational power series over K in non-commuting indeterminants. Any coefficient of $f(x_1,\dots,x_r,x_1^{-1},\dots,x_r^{-1})$ converging over K_v is algebraic over K.*

Letting $K=\mathbb{Q}(z)$, and $K_v=\mathbb{Q}((z))$ the ring of formal Laurent series in z, and considering the element $(1-zP)^{-1}$, where $P\in\mathcal{M}_N(\mathbb{Q}[F_r])$, gives an immediate proof of Theorem 1.

In the next section we will give a detailed description of Haiman's argument which exhibits a close relation to linguistics, as well as an obstruction to generalizing Theorem 1 to groups other than the free group.

4. Algebraic and rational functions in noncommuting variables

4.1. Rational, algebraic and holonomic functions in one variable

In this section all functions will be analytic in a neighborhood of $z=0$. Let $\mathbb{Q}_0^{\mathrm{rat}}(z)$, $\mathbb{Q}_0^{\mathrm{alg}}(z)$ and $\mathbb{Q}_0^{\mathrm{hol}}(z)$ denote respectively the set of rational, algebraic and holonomic functions, analytic at $z=0$. Let $\mathbb{Q}[[z]]$ denote the set of formal power series in z. Using the injective Taylor series map around $z=0$, we will consider $\mathbb{Q}_0^{\mathrm{rat}}(z)$, $\mathbb{Q}_0^{\mathrm{alg}}(z)$ and $\mathbb{Q}_0^{\mathrm{hol}}(z)$ as subsets of $\mathbb{Q}[[z]]$: $\mathbb{Q}_0^{\mathrm{rat}}(z)\subset\mathbb{Q}_0^{\mathrm{alg}}(z)\subset\mathbb{Q}_0^{\mathrm{hol}}(z)\subset\mathbb{Q}[[z]]$. $\mathbb{Q}[[z]]$ has *two* multiplications:

- the usual multiplication of formal power series

$$\left(\sum_{n=0}^{\infty}a_nz^n\right)\cdot\left(\sum_{n=0}^{\infty}b_nz^n\right)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}a_kb_{n-k}\right)z^n,$$

- the *Hadamard product*

$$\left(\sum_{n=0}^{\infty}a_nz^n\right)\circledast\left(\sum_{n=0}^{\infty}b_nz^n\right)=\sum_{n=0}^{\infty}a_nb_nz^n.$$

With respect to the usual multiplication, $\mathbb{Q}[[z]]$ is an algebra and $\mathbb{Q}_0^{\mathrm{rat}}(z)$, $\mathbb{Q}_0^{\mathrm{alg}}(z)$ and $\mathbb{Q}_0^{\mathrm{hol}}(z)$ are subalgebras. In case two power series are convergent in a neighborhood of zero, so is their Hadamard product. Hadamard, Borel and Jungen studied the analytic continuation and the singularities of the Hadamard product of two functions; see [25; 29]. Their method used an integral representation of the Hadamard product, and a deformation of the contour of integration; see [25. Fig. 2, p. 303]. Let us summarize these classical results.

Theorem 5. (a) *If f and g are rational,so is $f\circledast g$.*

(b) *If f is rational and g is algebraic,then $f\circledast g$ is algebraic.*

(c) *If f and g are holonomic (resp. regular holonomic with rational exponents),so is $f\circledast g$.*

(d) *If f and g are algebraic,then $f\circledast g$ is not necessarily algebraic.*

For a proof, see Thm. 7, 8, E and the example of p. 298 from [25].

4.2. Rational and algebraic functions in noncommuting variables

In this section we discuss a generalization of the previous section to non-commuting variables. Let X be a finite set, and let X^* denote the free monoid on X. In other words, X consists of the set of all words in X, including the empty word e. Let $\mathbb{Q}\langle X\rangle$ (resp. $\mathbb{Q}\langle\langle X\rangle\rangle$) denote the algebra of polynomials (resp. formal power series) in non-commuting variables. In [30], Schützenberger defines the notion of a *rational* and an *algebraic* power series in non-commuting variables. Let $\mathbb{Q}^{\mathrm{rat}}\langle X\rangle$ and $\mathbb{Q}^{\mathrm{alg}}\langle X\rangle$ denote the sets of rational (resp. algebraic) power series. Then, we have an inclusion: $\mathbb{Q}^{\mathrm{rat}}\langle X\rangle \subset \mathbb{Q}^{\mathrm{alg}}\langle X\rangle \subset \mathbb{Q}\langle\langle X\rangle\rangle$. $\mathbb{Q}\langle\langle X\rangle\rangle$ has two multiplications:

- the usual multiplication of formal power series in non-commuting variables:

$$\left(\sum_{w\in X^*} a_w w\right)\cdot\left(\sum_{w\in X^*} b_w w\right) = \sum_{w\in X^*}\left(\sum_{w',w'':w'w''=w} a_{w'} b_{w''}\right) w;$$

- the Hadamard product:

$$\left(\sum_{w\in X^*} a_w w\right)\circledast\left(\sum_{w\in X^*} b_w w\right) = \sum_{w\in X^*} a_w b_w w.$$

With respect to the usual multiplication, $\mathbb{Q}\langle\langle X\rangle\rangle$ is a non-commutative algebra and $\mathbb{Q}^{\mathrm{rat}}\langle X\rangle$ and $\mathbb{Q}^{\mathrm{alg}}\langle X\rangle$ are subalgebras. We have the following analogue of Theorem 5.

Theorem 6. [30. Pro. 2.2]. (a) *If* $f\in\mathbb{Q}^{\mathrm{rat}}\langle X\rangle$ *and* $g\in\mathbb{Q}^{\mathrm{rat}}\langle X\rangle$, *then* $f\circledast g\in\mathbb{Q}^{\mathrm{rat}}\langle X\rangle$.

(b) *If* $f\in\mathbb{Q}^{\mathrm{rat}}\langle X\rangle$ *and* $g\in\mathbb{Q}^{\mathrm{alg}}\langle X\rangle$, *then* $f\circledast g\in\mathbb{Q}^{\mathrm{alg}}\langle X\rangle$.

Remark 2. The notion of rational and algebraic functions works for an arbitrary ring $\mathcal{R}$ of characteristic zero, instead of $\mathbb{Q}$. Theorem 6 is still valid.

4.3. Proof of Theorem 1

Let F_r denote the free group of rank r with generating set $\{u_1,\dots,u_r\}$, and

$$X=\{x_1,\dots,x_r,\overline{x}_1,\dots,\overline{x}_r\}.$$

Consider the monoid map:

$$\pi: X^*\to F_r,\quad \pi(x_i)=u_i,\quad \pi(\overline{x}_i)=u_i^{-1}.$$

The kernel $\mathrm{Ker}(\pi)$ of π is the set of those words in X which reduce to the identity under the relations $x_i\overline{x}_i=\overline{x}_i x_i=e$. Let $\Delta=\sum_{w\in\mathrm{Ker}(\pi)} w\in\mathbb{Q}\langle\langle X\rangle\rangle$. The next proposition is attributed to Chomsky–Schützenberger by Haiman. For a proof, see [28. Sec. 3].

Proposition 1. [31]. Δ *is algebraic.*

The map π has a right inverse (that satisfies $\pi\circ\iota=I_{F_r}$) $\iota: F_r\to X$, defined by mapping a reduced word in u_i to a corresponding word in X. For every $f\in\mathbb{Q}[F_r]$ we have a key relation between trace and Hadamard product: $\mathrm{Tr}(f)=\phi(\iota(f)\circledast\Delta)$, where ϕ is a $\mathbb{Q}$-linear map defined by:

$$\phi:\mathbb{Q}\langle X\rangle\to\mathbb{Q},\qquad \phi(w)=1\qquad \text{for } w\in X^*.$$

Now, fix $P\in M_N(\mathbb{Q}[F_r])$. Let Δ_N denote the N by N matrix with entries equal to Δ, and $\mathcal{R}=\mathbb{Q}(z)$. Let $P_z=z\iota(P)\in M_N(\mathcal{R}\langle X\rangle)$, $P_z^*=\sum_{n=0}^{\infty}P_z^n\in M_N(\mathcal{R}\langle\langle X\rangle\rangle)$. Notice that P_z^* is well-defined since P_z has no z-constant term.

Lemma 2. *We have* $P_z^*\in M_N(\mathcal{R}^{\mathrm{rat}}\langle X\rangle)$.

Proof. P_z^* satisfies the matrix equation $(1-P_z)P_z^*=I$ with entries in $\mathcal{R}\langle X\rangle$. □

Lemma 2, together with Propositions 1 and part (b) of 6 imply the following result, which we can think as a noncommutative analogue of Theorem 1.

Proposition 2. *For every* $P\in M_N(\mathbb{Q}[F_r])$, *we have* $\sum_{n=0}^{\infty} z^n(\iota(P))^n\circledast\Delta_N\in M_N(\mathcal{R}^{\mathrm{alg}}\langle X\rangle)$.

Consider the abelianization ring homomorphism $\psi:\mathcal{R}\langle\langle X\rangle\rangle\to\mathcal{R}[[X]]$, where $\mathcal{R}[[X]]$ is the formal power series ring in commuting variables. Haiman proves the following.

Proposition 3. [28. Prop. 3.3]. *If* $f\in\mathcal{R}^{\mathrm{alg}}\langle X\rangle$, *then* $\psi(f)$ *is algebraic over* $\mathcal{R}(X)$.

It follows that $\psi(P_z^* \circledast \Delta_N) \in M_N(\mathcal{R}^{\mathrm{alg}}(X))$. Consider now the subalgebra $\mathcal{R}^{\mathrm{conv}}[[X]]$ of $\mathcal{R}[[X]]$ that contains all elements of the form $\sum_{w\in X^*} a_w w$ where $a_w \in z^{l(w)}\mathbb{Q}[[z]]$, $l(w)$ denotes the length of w. Then, we can define an algebra map:

$$\phi_z : \mathcal{R}^{\mathrm{conv}}[[X]] \to \mathbb{Q}[[z]], \qquad \phi_z(w) = 1.$$

for $x \in X$.

Haiman shows that if $f \in \mathcal{R}^{\mathrm{alg}}(X) \cap \mathcal{R}^{\mathrm{conv}}[[X]]$, then $\phi_z(f) \in \mathbb{Q}^{\mathrm{alg}}$. To state our final conclusion, we define for $1 \le i, j \le N$, the sequence $(a_{P,n}^{ij})$ by $a_{P,n}^{ij} = \mathrm{Tr}((P^n)_{ij})$ and the matrix of generating series $A_P(z) \in M_N(\mathbb{Q}[[z]])$ by $(A_P(z))_{ij} = \sum_{n=0}^{\infty} a_{P,n}^{ij} z^n$.

Lemma 3. *We have:* $(\phi_z \circ \psi)(P_z^* \circledast \Delta_N) = A_P(z)$. *Thus,* $A_P(z) \in M_N(\mathbb{Q}_0^{\mathrm{alg}}(z))$.

Proof. The conclusion follows from the above discussion. □

Thus, the entries of $A_P(z)$ are algebraic functions, convergent at $z = 0$. Since by definition we have $R_P(z) = \sum_{i=1}^{N}(A_P(z))_{ii}$ it follows that $R_P(z) \in \mathbb{Q}_0^{\mathrm{alg}}(z)$. This completes the proof of Theorem 1. □

5. Some Linguistics

5.1. Regular and context-free languages

Haiman's proof uses the key Proposition 1 from linguistics. Let us recall some concepts from this field. See for example [32—34] and references therein. Given a finite set X (the alphabet), a language L is a collection of words in X. In other words, $\mathcal{L} \subset X^*$. The *generating series* F_L of a language is $F_{\mathcal{L}} = \sum_{w\in L} w \in \mathbb{Q}\langle\langle X\rangle\rangle$. It follows that for two languages $\mathcal{L}_1$ and $\mathcal{L}_2$ we have $F_{\mathcal{L}_1\cap\mathcal{L}_2} = F_{\mathcal{L}_1} \circledast F_{\mathcal{L}_2}$. A language L is called *rational* (resp. *context-free*) iff $F_L \in \mathbb{Q}^{\mathrm{rat}}(X)$ (resp. $F_L \in \mathbb{Q}^{\mathrm{alg}}(X)$). In this context, Theorem 6 takes the following form.

Theorem 7. [31]. (a) *If* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *are rational languages, so is* $\mathcal{L}_1 \cap \mathcal{L}_2$.

(b) *If* $\mathcal{L}_1$ *is rational and* $\mathcal{L}_2$ *is (unambiguous) context-free, then* $\mathcal{L}_1 \cap \mathcal{L}_2$ *is (unambiguous) context-free.*

It was pointed out to us independently by D. Zeilberger and F. Flajolet that the above theorem essentially proves Theorem 1.

5.2. Some questions

Let us end this short paper with some questions. Despite the similarity in their statements and the multitude of proofs, Theorems 1 and 3 have different assumptions, different proofs and different conclusions.

Consider a generating set X for a group G such that every element of G can be written as a word in X with nonnegative exponents. Given X and G, let $\mathcal{L}_X$ denote the set of all words in X that map to the identity in G. Deciding membership in $\mathcal{L}_X$ is the *word problem* in G.

Definition 4. *A group G has context-free word problem if it has a generating set X such that the language* $\mathcal{L}_X$ *is context-free.*

The proof of Theorem 1 applies to groups with a context-free word problem. Muller-Schupp classified those groups. In [35] Muller-Schupp prove that G has context-free word problem iff G has a free finite-index subgroup.

On the other hand, if G is the fundamental group of a hyperbolic manifold of dimension not equal to 2, then G does not have a free finite-index subgroup.

Thus, the linguistics proof of Theorem 1 does not apply to the case of hyperbolic groups in dimension three. Neither does it apply to the case of $\mathbb{Z}^r$ since the latter does not have context-free word problem.

Example 1. If P is a hyperbolic group and $P \in M_N(\mathbb{Q}[G])$, is it true that $R_P(z)$ is a G-function? The question may be relevant to low dimensional topology, when one tries to compute the ℓ^2-torsion of a hyperbolic manifold using Luecke's theorem; [36]. In that case, the matrix P comes from Fox (free differential) calculus of a presentation of the fundamental group G of the hyperbolic manifold. See also [37].

Example 2. Given $P \in M_N(\mathbb{Q}[F_r])$, consider the abelianization $P^{\mathrm{ab}} \in M_N(\mathbb{Q}[\mathbb{Z}^r])$, and the G-functions $R_P(z)$ and $R_{P^{\mathrm{ab}}}(z)$. How are the singularities of $R_P(z)$ and $R_{P^{\mathrm{ab}}}(z)$ related?

Example 3. What is a holonomic function in non-commuting variables?

6. A functional analysis interpretation of Theorem 1

The present paper is focusing on results and techniques inspired by algebra, non-commutative algebraic combinatorics. However it is worth mentioning that Theorem 1 has applications to problems coming from functional analysis, spectral theory, and the spectrum of Schrödinger operators. For instance, the Schrödinger equation describing the electron motion in a d-dimensional periodic crystal, can be well approximated by the difference equation on a lattice of same dimension. The corresponding operator can be seen as an element of the group ring of $\mathbb{Z}^d$. The function $R_P(z)$ defined previously is noting but the diagonal element of the resolvent and is used to compute the spectral measure, through the Charles de la Vallée Poussin theorem. There are instances for which, this operator is better approximated by the free group analog. For instance the *retracable path approximation* was used by Brinkman and Rice [38] in 1971 to treat the effect of spin-orbit coupling in the Hall effect, while it was used in [39] to compute the electronic Density of States when the electron is submitted to a random magnetic field. The same operator, seen as an element of the free group ring, is used to describe various infinite dimension approximations. The seminal work of Georges and Kotliar [40] used this free group approximation to give the first model known with a *Mott–Hubbard* transition.

Another domain in which the Theorem 1 may apply is the Voiculescu Theory of *Free Probability* [5; 41]. The so-called *R-transform* used to treat the convolution of free random variables, is also based upon the Schur complement formula. In particular the free central limit theorem asserts that a sum of identically distributed free random variable obey the semicircle law, is a special case of the present result.

Besides the two proofs of Theorem 1 discussed in this paper, the algebraic character of $R_P(z)$ can also be deduced from the used of the Schur complement method [42]. This is what makes the free group approximation so attractive to theoretical physicists. This method, also known under the name of *Feshbach method* [43—45] is used in many domains of Physics, Quantum Chemistry, Solid State Physics, Nuclear Physics, to reduce the Hilbert space to a finite dimensional one and make the problem amenable to numerical calculations. However, very few Mathematical Physicists have paid attention to the fact that algebraicity or holonomy can give rise to results concerning the explicit computation of the spectral radius, or more generally, to the band edges, of the Hamiltonian they consider. This later problem is known to be notably hard with other methods.

For the benefit of the reader, we include some history of that method. The Schur complement method [42] is widely used in numerical analysis under this name, while Mathematical Physicists prefer the reference to Feshbach [43]. In Quantum Chemistry, the common reference is Feshbach–Fano [46] or Feshbach–Löwdin [47]. This method is used in various algorithms in Quantum Chemistry (*ab initio* calculations), in Solid State Physics (the muffin tin approximation, LMTO) as well as in Nuclear Physics. The formula used above is found in the original paper of Schur [42. P. 217].

The formula has been proposed also by an astronomer Tadeusz Banachiewicz in 1937, even though closely related results were obtained in 1923 by Hans Boltz and in 1933 by Ralf Rohan [48]. Applied to the Green function of a selfadjoint operator with finite rank perturbation, it becomes the Kren formula [49].

Let us end this section with a small dictionary that compares our notions with those in physics:

$H \in M_N(\mathbb{Q}[F_r])$	Hamiltonian
$1/(z-H)$	resolvant
$1/zR_H(1/z)$	trace of the resolvant
$\mathrm{Tr}(H^n)$	nth moment of H

References

1. Woess W. Context-free languages and random walks on groups. *Discrete Math.,* 1987, vol. 67, no. 1, pp. 81–87.

2. Woess W. *Random walks on infinite graphs and groups*. Cambridge, Cambridge University Press, 2000.

3. Sauer R. Power series over the group ring of a free group and applications to Novikov–Shubin invariants. *High-dimensional manifold topology*. River Edge, World Sci. Publ., 2003, pp. 449–468.

4. Voiculescu D. Operations on certain non-commutative operator-valued random variables. *Astérisque*, 1995, no. 232, pp. 243–275.

5. Voiculescu D. Free probability and the von Neumann algebras of free groups. *Rep. Math. Phys.,* 2005, vol. 55, no. 1, pp. 127–133.

6. Kazuhiko A. Spectral theory on a free group and algebraic curves. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.,* 1984, vol. 31, no. 2, pp. 297–318.

7. Cherix P.-A., Valette A. On spectra of simple random walks on one-relator groups. *Pacific J. Math.,* 1996, vol. 175, no. 2, pp. 417–438.

8. Lalley S.P. Finite range random walk on free groups and homogeneous trees. *Ann. Probab.*, 1993, vol. 21, no. 4, pp. 2087–2130.

9. Lehner F. Computing norms of free operators with matrix coefficients. *Amer. J. Math.,* 1999, vol. 121, no. 3, pp. 453–486.

10. Lehner F. On the computation of spectra in free probability. *J. Funct. Anal.,* 2001, vol. 183, no. 2, pp. 451–471.

11. Kassel C., Reutenauer C. Algebraicity of the zeta function associated to a matrix over a free group algebra. *Algebra Number Theory,* 2014, vol. 8-2, pp. 497–511.

12. Siegel C.L. Über einige Anwendungen diophantischer Approximationen. *Gesammelte Abhandlungen, Band I.* 1966, pp. 209–266.

13. Deligne P. *Équations différentielles à points singuliers réguliers*. Berlin, Springer-Verlag Publ., 1970.

14. Katz N.M. Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin. *Inst. Hautes Études Sci. Publ. Math.*, 1970, no. 39, pp. 175–232.

15. Malgrange B. Intégrales asymptotiques et monodromie. *Ann. Sci. École Norm. Sup.,* 1974, vol. 7, no. 4, pp. 405–430.

16. Yves A. Séries Gevrey de type arithmétique. I. Théorèmes de pureté et de dualité. *Ann. of Math.,* 2000, vol. 151, no. 2, pp. 705–740.

17. Bombieri E. On G-functions. *Recent progress in analytic number theory. Vol. 2.* London, Academic Press, 1981, pp. 1–67.

18. Dwork B., Gerotto G., Sullivan F.-J. An introduction to G-functions. Princeton, Princeton University Press, 1994.

19. Garoufalidis S. G-functions and multisum versus holonomic sequences. *Adv. Math.,* 2009, vol. 220, no. 6, pp. 1945–1955.

20. Wasow W. *Asymptotic expansions for ordinary differential equations.* New York, Dover Publ., 1987.

21. Cormier O., Singer M.F., Trager B.M., Ulmer F. Linear differential operators for polynomial equations. *J. Symbolic Comput.,* 2002, vol. 34, no. 5, pp. 355–398.

22. Dwork M., Poorten A. J. van der. The Eisenstein constant. *Duke Math. J.,* 1992, vol. 65, no. 1, pp. 23–43.

23.Chambert-Loir A. Théorèmes d'algébricité en géométrie diophantienne (d'après J.-B. Bost, Y. André, D. & G. Chudnovsky). *Astérisque*, 2002, no. 282, exp. no. 886, viii, pp. 175–209.

24. Nilsson N. Some growth and ramification properties of certain integrals on algebraic manifolds. *Ark. Mat.,* 1965, vol. 5, pp. 463–476.

25. Jungen R. Sur les séries de Taylor n'ayant que des singularités algébrico-logarithmiques sur leur cercle de convergence. *Comment. Math. Helv.*, 1931, vol. 3, no. 1, — P. 266–306.

26. Costin O., Garoufalidis S. Resurgence of the Kontsevich–Zagier series. *Ann. Inst. Fourier (Grenoble)*, 2011, vol. 61, no. 3, pp. 1225–1258.

27. Duistermaat J.J., Kallen W. van der. Constant terms in powers of a Laurent polynomial. *Indag. Math. (N. S.)*, 1998, vol. 9, no. 2, pp. 221–231.

28. Haiman M. Noncommutative rational power series and algebraic generating functions. *European J. Combin.,* 1993, vol. 14, no. 4, pp. 335–339.

29. Borel E. Sur les singularités des séries de Taylor. *Bull. Soc. Math. France*, 1898, vol. 26, pp. 238–248.

30. Schützenberger M.P. On a theorem of R. Jungen. *Proc. Amer. Math. Soc.,* 1962, vol. 13, pp. 885–890.

31. Chomsky N., Schützenberger M.P. The algebraic theory of context-free languages. *Studies in Logic and the Foundations of Mathematics,* 1963, vol. 35, no. Computer Programming and Formal Systems, pp. 118–161.

32. Berstel J., Reutenauer C. *Rational series and their languages.* Berlin, Springer-Verlag Publ., 1988.

33. Linz P. *An introduction to formal languages and automata.* Burlington, Jones and Bartlett Publ., 1997.

34. Yan S.Y. *An introduction to formal languages and machine computation.* River Edge, World Sci. Publ., 1998.

35. Muller D.E., Paul E., Schupp P.E. Groups, the theory of ends, and context-free languages. *J. Comput. System Sci.,* 1983, vol. 26, no. 3, pp. 295–310.

36. Lück W. L^2-torsion and 3-manifolds. *Conf. Proc. Lecture Notes Geom. Topology, III*. Cambridge Int. Press, 1994. — P. 75–107.

37. Dasbach O.T., Lalin M.N. Mahler measure under variations of the base group. *Forum Math.,* 2009, vol. 21, no. 4, pp. 621–637.

38. Brinkman W.F., Rice T.M. Hall effect in the presence of strong spin-disorder scattering. *Physical Review B (Solid State),* 1971, vol. 4, no. 5, pp. 1566–1571.

39. Barelli A., Fleckinger R., Ziman T. Localization of two-dimensional electrons with uniform and random fluxes. *Europhysics Letters*, 1994, vol. 27, no. 7, pp. 531.

40. Georges, A.A., Kotliar G.B. Hubbard model in infinite dimensions. *Physical Review B.,* 1992, vol. 45, no. 12, pp. 6479–6483.

41. Voiculescu D. Aspects of free probability. *XIVth International Congress on Mathematical Physics.* Hackensack, World Sci. Publ., 2005, pp. 145–157.

42. Schur J. Der potenzreihen, die im innern des einheitskreises beschrnkt sind. *J. für die reine und angewandte Mathematik*, 1918, no. 148, pp. 122.

43. Feshbach H. Unified theory of nuclear reactions. *Annals of Physics (N. Y.)*, 1958, vol. 5, no. 4, pp. 357.

44. Herman F. A unified theory of nuclear reactions. II. *Annals of Physics*, 1962, vol. 19, pp. 287–313.

45. Feshbach H. The unified theory of nuclear reactions. III. Overlapping resonances. *Ann. Phys. (N. Y.)*, 1967, vol. 43, no. 3, pp. 410–420.

46. Fano U. Absorption spectrum of the noble gases near the limit of the arc spectrum. *Nuovo Cimento*, 1935, vol. 12, pp. 154–161.
47. Lowdin P.-O. Studies in perturbation theory. IV. Solution of eigenvalue problem by projection operator formalism. *J. of Mathematical Physics,* 1962, vol. 3, no. 5, pp. 969–982.
48. Puntanen S., Styan G. P. H. Historical introduction: Issai Schur and the early development of the Schur complementm. *Numerical Methods and Algorithms,* 2005, vol. 4, pp. 1–16.
49. Krein M. Concerning the resolvents of an Hermitian operator with the deficiency-index (m,m). *Doklady Akademii nauk SSSR* [Reports of academy of Sciences of the USSR], 1946, vol. 52, pp. 651–654.

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АЛГЕБРАИЧЕСКИЕ *G*-ФУКНЦИИ, АССОЦИИРОВАННЫЕ С МАТРИЦАМ НАД ГРУППОВЫМ КОЛЬЦОМ

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Для каждой матрицы с элементами из группового кольца некоторой группы можно построить последовательность следов (в смысле группового кольца) их степеней. Мы доказываем, что соответствующий производящий ряд является алгебраической *G*-функцией (в смысле Зигеля) в случае, когда группа является свободной конечного ранга. Следовательно, норма таких элементов является точно вычислимым алгебраическим числом, и их функция Грина является алгебраической. Наше доказательство использует понятия рациональных и алгебраических степенных рядов с некоммутирующими переменными и опирается на теорему Хаймана. В основе этой теоремы лежат результаты о регулярных и контекстно-свободных языках. С другой стороны, когда группа является свободной абелевой конечного ранга, то соответствующий производящий ряд представляет собой *G*-функцию. Вопрос состоит в том, выполняется ли это для любой гиперболической группы.

Ключевые слова: *рациональная функция, алгебраическая функция, голономная функция, G-функция, производящий ряд, некоммутирующие переменные, момент, гамильтониан, резольвенты, регулярный язык, контекстно-свободный язык, произведение Адамара, групповое кольцо, свободная вероятность, метод дополнений Шура, свободная группа, алгебра фон Неймана, полиномиальный Гамильтониан, спектральная теория, норма.*

Список литературы

1. Woess, W. Context-free languages and random walks on groups / W. Woess // Discrete Math. — 1987. — Vol. 67, № 1. — P. 81–87.
2. Woess, W. Random walks on infinite graphs and groups / W. Woess. — Cambridge : Cambridge University Press, 2000.
3. Sauer, R. Power series over the group ring of a free group and applications to Novikov–Shubin invariants / R. Sauer // High-dimensional manifold topology. — River Edge : World Sci. Publ., 2003. — P. 449–468.
4. Voiculescu, D. Operations on certain non-commutative operator-valued random variables / D. Voiculescu // Astérisque. — 1995. — № 232. — P. 243–275.

5. Voiculescu, D. Free probability and the von Neumann algebras of free groups / D. Voiculescu // Rep. Math. Phys. — 2005. — Vol. 55, № 1. — P. 127–133.
6. Kazuhiko, A. Spectral theory on a free group and algebraic curves / A. Kazuhiko // J. Fac. Sci. Univ. Tokyo Sect. IA Math. — 1984. — Vol. 31, № 2. — P. 297–318.
7. Cherix, P.-A. On spectra of simple random walks on one-relator groups / P.-A. Cherix, A. Valette // Pacific J. Math. — 1996. — Vol. 175, № 2. — P. 417–438.
8. Lalley, S. P. Finite range random walk on free groups and homogeneous trees / S. P. Lalley // Ann. Probab. — 1993. — Vol. 21, № 4. — P. 2087–2130.
9. Lehner, F. Computing norms of free operators with matrix coefficients / F. Lehner // Amer. J. Math. — 1999. — Vol. 121, № 3. — P. 453–486.
10. Lehner, F. On the computation of spectra in free probability / F. Lehner // J. Funct. Anal. — 2001. — Vol. 183, № 2. — P. 451–471.
11. Kassel, C. Algebraicity of the zeta function associated to a matrix over a free group algebra / C. Kassel, C. Reutenauer // Algebra Number Theory. — 2014. Vol. 8-2. — P. 497–511.
12. Siegel, C. L. Über einige Anwendungen diophantischer Approximationen / C. L. Siegel // Gesammelte Abhandlungen, Band I. — 1966. — P. 209–266.
13. Deligne, P. Équations différentielles à points singuliers réguliers / P. Deligne. — Berlin : Springer-Verlag, 1970.
14. Katz, N. M. Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin / N. M. Katz // Inst. Hautes Études Sci. Publ. Math. — 1970. — № 39. — P. 175–232.
15. Malgrange, B. Intégrales asymptotiques et monodromie / B. Malgrange // Ann. Sci. École Norm. Sup. — 1974. — Vol. 7, № 4. — P. 405–430.
16. Yves, A. Séries Gevrey de type arithmétique. I. Théorèmes de pureté et de dualité / A. Yves // Ann. of Math. — 2000. — Vol. 151, № 2. — P. 705–740.
17. Bombieri, E. On *G*-functions / E. Bombieri // Recent progress in analytic number theory. Vol. 2. — London : Academic Press, 1981. — P. 1–67.
18. Dwork, B. An introduction to *G*-functions / B. Dwork, G. Gerotto, F.-J. Sullivan. — Princeton, NJ : Princeton University Press, 1994.
19. Garoufalidis, S. *G*-functions and multisum versus holonomic sequences / S. Garoufalidis // Adv. Math. — 2009. — Vol. 220, № 6. — P. 1945–1955.
20. Wasow, W. Asymptotic expansions for ordinary differential equations / W. Wasow. — New York : Dover Publications Inc., 1987. — Reprint of the 1976 edition.
21. Linear differential operators for polynomial equations / O. Cormier, M. F. Singer, B. M. Trager, F. Ulmer // J. Symbolic Comput. — 2002. — Vol. 34, № 5. — P. 355–398.
22. Dwork, M. The Eisenstein constant / M. Dwork, A. J. van der Poorten // Duke Math. J. — 1992. — Vol. 65, № 1. — P. 23–43.
23.Chambert-Loir, A. Théorèmes d'algébricité en géométrie diophantienne (d'après J.-B. Bost, Y. André, D. & G. Chudnovsky) / A. Chamber-Loir // Astérisque. — 2002. — № 282, exp. no. 886, viii. — P. 175–209.
24. Nilsson, N. Some growth and ramification properties of certain integrals on algebraic manifolds / N. Nilsson // Ark. Mat. — 1965. — Vol. 5. — P. 463–476.
25. Jungen, R. Sur les séries de Taylor n'ayant que des singularités algébrico-logarithmiques sur leur cercle de convergence / R. Jungen // Comment. Math. Helv. — 1931. — Vol. 3, № 1. — P. 266–306.
26. Costin, O. Resurgence of the Kontsevich–Zagier series / O. Costin, S. Garoufalidis // Ann. Inst. Fourier (Grenoble). — 2011. — Vol. 61, № 3. — P. 1225–1258.
27. Duistermaat, J. J. Constant terms in powers of a Laurent polynomial / J. J. Duistermaat, W. van der Kallen // Indag. Math. (N. S.). — 1998. — Vol. 9, № 2. — P. 221–231.
28. Haiman, M. Noncommutative rational power series and algebraic generating functions / M. Haiman // European J. Combin. — 1993. — Vol. 14, № 4. — P. 335–339.
29. Borel, E. Sur les singularités des séries de Taylor / E. Borel // Bull. Soc. Math. France. — 1898. — Vol. 26. — P. 238–248.

30. Schützenberger, M. P. On a theorem of R. Jungen / M. P. Schützenberger // Proc. Amer. Math. Soc. — 1962. — Vol. 13. — P. 885–890.

31. Chomsky, N. The algebraic theory of context-free languages / N. Chomsky, M. P. Schützenberger // Studies in Logic and the Foundations of Mathematics. — 1963. — Vol. 35, № Computer Programming and Formal Systems. — P. 118–161.

32. Berstel, J. Rational series and their languages / J. Berstel, C. Reutenauer. — Berlin : Springer-Verlag, 1988.

33. Linz, P. An introduction to formal languages and automata / P. Linz. — Burlington : Jones and Bartlett Publishers, 1997. — 437 p.

34. Yan, S. Y. An introduction to formal languages and machine computation / S. Y. Yan. — River Edge : World Scientific Publishing, 1998.

35. Muller, D. E. Groups, the theory of ends, and context-free languages / D. E. Muller, E. Paul, P. E. Schupp // J. Comput. System Sci. — 1983. — Vol. 26, № 3. — P. 295–310.

36. Lück, W. L^2-torsion and 3-manifolds / W. Lück // Conf. Proc. Lecture Notes Geom. Topology, III. — Cambridge: Int. Press, MA, 1994. — P. 75–107.

37. Dasbach, O. T. Mahler measure under variations of the base group / O. T. Dasbach, M. N. Lalin // Forum Math. — 2009. — Vol. 21, № 4. — P. 621–637.

38. Brinkman, W. F. Hall effect in the presence of strong spin-disorder scattering / W. F. Brinkman, T. M. Rice // Physical Review B (Solid State). — 1971. — Vol. 4, № 5. — P. 1566–1571.

39. Barelli, A. Localization of two-dimensional electrons with uniform and random fluxes / A. Barelli, R. Fleckinger, T. Ziman // EPL (Europhysics Letters). — 1994. — Vol. 27, № 7. — P. 531.

40. Georges, A. A. Hubbard model in infinite dimensions / A. A. Georges, G. B. Kotliar // Physical Review B. — 1992. — Vol. 45, № 12. — P. 6479–6483.

41. Voiculescu, D. Aspects of free probability / D. Voiculescu // XIVth International Congress on Mathematical Physics. — Hackensack, NJ : World Sci. Publ., 2005. — P. 145–157.

42. Schur, J. Der potenzreihen, die im innern des einheitskreises beschrnkt sind / J. Schur // J. für die reine und angewandte Mathematik. — 1918. — № 148. — P. 122.

43. Feshbach, H. Unified theory of nuclear reactions / H. Feshbach // Annals of Physics (N. Y.). — 1958. — Vol. 5, № 4. — P. 357.

44. Herman, F. A unified theory of nuclear reactions. II / F. Herman // Annals of Physics. — 1962. — Vol. 19 (n. d.). — P. 287–313.

45. Feshbach, H. The unified theory of nuclear reactions. III. Overlapping resonances / H. Feshbach // Ann. Phys. (N. Y.). — 1967. — Vol. 43, № 3. — P. 410–420.

46. Fano, U. Absorption spectrum of the noble gases near the limit of the arc spectrum / U. Fano // Nuovo Cimento. — 1935. — Vol. 12. — P. 154–161.

47. Lowdin, P.-O. Studies in perturbation theory. IV. Solution of eigenvalue problem by projection operator formalism / P.-O. Lowdin // J. of Mathematical Physics. — 1962. — Vol. 3, № 5. — P. 969–982.

48. Puntanen, S. Historical introduction: Issai Schur and the early development of the Schur complement / S. Puntanen, G. P. H. Styan // Numerical Methods and Algorithms. — 2005. — Vol. 4. — P. 1–16.

49. Krein, M. Concerning the resolvents of an Hermitian operator with the deficiency-index (m,m) / M. Krein // Докл. акад. наук СССР. — 1946. — Vol. 52. — P. 651–654.

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THE Q-BINOMAL FORMULA AND THE ROGERS DILOGARITHM IDENTITY*

R. M. Kashaev

The q-binomial formula in the limit q → 1⁻ is shown to be equivalent to the Rogers five term dilogarithm identity.

Keywords: *q-binomial formula, dilogarithm identity.*

1. Introduction

For any $q, x \in]0,1[$ define a q-exponential function as an infinite product

$$\Phi(x) := 1/(x;q)_\infty, \quad (x;q)_\infty := \prod_{n\geq 0}(1-q^n x).$$

The finite product

$$(x;q)_k := \prod_{n=0}^{k-1}(1-q^n x), \quad \forall k \in \mathbb{Z}_{\geq 0}$$

can be expressed as a ratio of two q-exponentials:

$$(x;q)_k = \frac{(x;q)_\infty}{(xq^k;q)_\infty} = \frac{\Phi(xq^k)}{\Phi(x)}.$$

The q-binomial formula (see, for example, [1]) is given by the following identity

$$\sum_{n\geq 0}\frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z|<1, \tag{1}$$

which, by using the above notation, can also be written entirely in terms of the function $\Phi(x)$:

$$\sum_{n\geq 0}\frac{\Phi(aq^n)}{\Phi(q^{n+1})} z^n = \frac{\Phi(a)\Phi(z)}{\Phi(q)\Phi(az)} \tag{2}$$

The following expansion formulae

$$\Phi(x) = \sum_{n\geq 0}\frac{x^n}{(q;q)_n} \tag{3}$$

and

$$\frac{1}{\Phi(x)} = \sum_{n\geq 0}\frac{(-1)^n q^{n(n-1)/2} x^n}{(q;q)_n} \tag{4}$$

are both particular cases of the q-binomial formula.

The asymptotic formula

$$\Phi(x) \sim e^{-\mathrm{Li}_2(x)/\ln q}, \quad q \to 1^-$$

where $q \to 1^-$ means that q approaches 1 from inside of the unit disk,

$$\mathrm{Li}_2(x) := \sum_{n=1}^{\infty}\frac{x^n}{n^2}$$

* Author would like to thank Yu. Manin for posing this question.

is the Euler dilogarithm function, has been used in [2; 3] to give an interpretation to $\Phi(x)$ as a quantum version of the dilogarithm function. In particular, by using a formal reasoning coming from quantum mechanics, it has been shown that the quantum five term identity

$$\Phi(\mathrm{u})\Phi(\mathrm{v}) = \Phi(\mathrm{v})\Phi(-\mathrm{vu})\Phi(\mathrm{u}) \tag{5}$$

where $\Phi(\mathrm{u})$, $\Phi(\mathrm{v})$, and $\Phi(-\mathrm{vu})$ are elements in the algebra $\mathcal{A}_q = \mathbb{C}_q[[\mathrm{u},\mathrm{v}]]$ of formal power series in two elements u,v satisfying the commutation relation $\mathrm{uv} = q\mathrm{vu}$, in the limit $q \to 1^-$ reduces to the Rogers pentagonal identity for the dilogarithm

$$\mathrm{Li}_2(a) + \mathrm{Li}_2(z) = \mathrm{Li}_2(az) + \mathrm{Li}_2\left(\frac{a-az}{1-az}\right) + \mathrm{Li}_2\left(\frac{z-az}{1-az}\right) + \log\left(\frac{1-z}{1-az}\right)\log\left(\frac{1-a}{1-az}\right). \tag{6}$$

The purpose of this paper is to make the statement of the paper [2] mathematically rigorous. Namely, we first show that the identity (5) is related to the q-binomial formula (1) and then derive from the latter the Rogers identity (6) in the limit $q \to 1^-$. The main result follows.

Theorem 1. *Let $q, a, z \in]0,1[$. Then in the limit $q \to 1^-$ the q-binomial identity* (2) *reproduces the Rogers pentagonal identity* (6).

The rest of this paper is organized as follows. In Section 2 the equivalence between he q-binomial formula and the quantum pentagonal identity is explained, while Section 3 contains the proof of Theorem 1.

2. The q-binomial formula and the quantum pentagonal identity

The relation between the formulas (1) and (5) can be established by comparing the expansion coefficients of $a^m z^n$ in (1) and $\mathrm{v}^n\mathrm{u}^m$ in (5), respectively.

Proposition 1. *The q-binomial formula is equivalent to the following set of identities*

$$\frac{q^{mn}}{(q;q)_m (q;q)_n} = \sum_{k=0}^{\min(m,n)} \frac{(-1)^k q^{k(k-1)/2}}{(q;q)_{m-k}(q;q)_{n-k}(q;q)_k}, \quad \forall m,n \in \mathbb{Z}_{\geq 0}. \tag{7}$$

Proof. Let us write the q-binomial formula in the form

$$\sum_{n\geq 0} \frac{\Phi(aq^n)}{(q;q)_n} z^n = \frac{\Phi(a)\Phi(z)}{\Phi(az)}$$

or, using formula (3) in the left hand side, we have

$$\sum_{m,n\geq 0} \frac{q^{mn} a^m z^n}{(q;q)_m (q;q)_n} = \frac{\Phi(a)\Phi(z)}{\Phi(az)}.$$

Again, by using the expansion formulas (3), (4) in the right hand side, and equating the coefficients of the monomials $a^m z^n$ in both sides of the equality, we arrive at formula (7). □

Proposition 2. *The set of identities* (7) *is equivalent to the quantum five term identity* (5).

Proof. We multiply the both sides of (7) by $\mathrm{v}^n\mathrm{u}^m$ and sum over m and n. The result can be easily written in the form of equation (5) by using the commutation relation $\mathrm{uv} = q\mathrm{vu}$, and, in particular, the formula $\mathrm{v}^k\mathrm{u}^k q^{k(k-1)/2} = (\mathrm{vu})^k$.

3. Proof of Theorem 1

Lemma 1. *Let $k, l \in \mathbb{Z}$ be such that $k \leq l$ and $f_\pm : [k, l+1] \to \mathbb{R}_{\geq 0}$ be functions, where f_- is decreasing and f_+ is increasing. Then*

$$\sum_{n=k+1}^{l+1} f_-(n) \leq \int_k^{l+1} f_-(t)dt \leq \sum_{n=k}^{l} f_-(n), \tag{8}$$

$$\sum_{n=k}^{l} f_+(n) \leq \int_k^{l+1} f_+(t)dt \leq \sum_{n=k+1}^{l+1} f_+(n). \tag{9}$$

Proof. The inequality

$$f_-(n+1) \leq f_-(x) \leq f_-(n), \quad \forall n \in \mathbb{Z} \cap [k,l], \forall x \in [n,n+1],$$

implies that $f_-(n+1) \leq \int_n^{n+1} f_-(x)dx \leq f_-(n)$. Thus, summing over all possible n we arrive at formula (8). The proof of formula (9) is similar. □

Remark 1. The variables k and l in Lemma 1 can take infinite values $k = -\infty$ or $l = \infty$.

In what follows, for any function f: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ we shall use the notation

$$S(f) := \sum_{n\geq 0} f(n), \quad I(f) := \int_0^\infty f(t)dt.$$

If a decreasing function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is integrable on $\mathbb{R}_{\geq 0}$ then, as a particular case of Lemma 1, we have $S(f) - f(0) \leq I(f) \leq S(f)$ or equivalently

$$0 \leq S(f) - I(f) \leq f(0). \tag{10}$$

Example 1. The function $f(t) = -\ln(1 - q^t x)$ is decreasing and integrable on $\mathbb{R}_{\geq 0}$, and $S(f) = \ln \Phi(x)$,

$$I(f) = -\int_0^\infty \ln(1 - q^t x)dt = \frac{1}{\ln q}\int_0^x \ln(1-z)\frac{dz}{z} = -\frac{\mathrm{Li}_2(x)}{\ln q}.$$

Thus, for any $q, x \in]0,1[$, inequalities (10) imply that

$$1 \leq \Phi(x)e^{\mathrm{Li}_2(x)/\ln q} \leq \frac{1}{1-x}. \tag{11}$$

Lemma 2. *Let* $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *be an integrable function increasing in the segment* $[0, x_0]$ *and decreasing on the interval* $[x_0, \infty[$. *Let also* $n_0 \in \mathbb{Z}_{\geq 0}$ *be such that* $g(n) \leq g(n_0)$ *for all* $n \in \mathbb{Z}_{\geq 0}$ (n_0 *is equal either to* $[x_0]$ (*the integer part of* x_0) *or* $[x_0]+1$). *Then*

$$g(n_0) \leq \sum_{n\geq 0} g(n) \leq \int_0^\infty g(x)dx + g(n_0). \tag{12}$$

Proof. The inequality $g(n_0) \leq \sum_{n\geq 0} g(n)$ follows directly from the positivity of $g(x)$. To prove the second part of (12), note that we can apply Lemma 1 to functions $f_+ = g\,|_{[0,[x_0]]}$ and $f_- = g\,|_{[[x_0]+1,\infty[}$. Thus, the left hand sides of the inequalities in Lemma 1 take the forms

$$\sum_{n=0}^{[x_0]-1} g(n) \leq \int_0^{[x_0]} g(x)dx, \qquad \sum_{n=[x_0]+2}^{\infty} g(n) \leq \int_{[x_0]+1}^{\infty} g(x)dx.$$

Adding these to each other, we obtain

$$\sum_{n=0}^{\infty} g(n) - g([x_0]) - g([x_0]+1) \leq \int_0^\infty g(x)dx - \int_{[x_0]}^{[x_0]+1} g(x)dx$$

which, combined with the inequality

$$\int_{[x_0]}^{[x_0]+1} g(x)dx \geq g(n_0')$$

where $\{n_0, n_0'\}$ $\{n_0, n_{0'}\} = \{[x_0], [x_0]+1\}$, is equivalent to the second part of (12). □

Proposition 3. *There exists* $\varepsilon \in]0,1[$ *such that for any* $q \in]1-\varepsilon, 1[$ *the function*

$$g(x) = \frac{\Phi(aq^x)}{\Phi(q^{1+x})} z^x$$

where $a, z \in]0,1[$, *satisfies the conditions of Lemma 2*

Proof. The integrability of $g(x)$ is evident. We have the following formula for its derivative

$$\frac{g'(x)}{g(x)} = \ln z - \ln(q)(q-a)S(h_x)$$

where

$$h_x(t) = \frac{q^{x+t}}{(1-q^{1+x+t})(1-aq^{x+t})}$$

satisfies the conditions of Lemma 1 so that

$$S(h_x) \geq I(h_x) = -\frac{1}{\ln(q)(q-a)} \ln\left(\frac{z(1-aq^x)}{1-q^{1+x}}\right).$$

Evidently, the function $S(h_x)$ is decreasing in x. Assuming that $q > 1-z(1-a)$, we obtain

$$\frac{g'(0)}{g(0)} \geq \ln\left(\frac{z(1-a)}{1-q}\right) > 0.$$

Besides, it is easy to see that

$$\lim_{x\to\infty} \frac{g'(x)}{g(x)} = \ln z < 0.$$

Thus, we have shown that for $\varepsilon = z(1-a)$ and any $q \in]1-\varepsilon, 1[$ the continuous function $g'(x) / g(x)$ is decreasing, positive at $x = 0$ and negative for sufficiently large x, i. e. there exists unique $x_0 \in]0, \infty[$ such that $g'(x_0) = 0$ and all conditions of Lemma 2 are satisfied. □

Proposition 4.

$$\lim_{q\to 1^-} \ln(q) \ln S(g) = F(\xi_0), \quad \xi_0 = \frac{1-z}{1-az} \tag{13}$$

where $F(\xi) = \mathrm{Li}_2(\xi) - \mathrm{Li}_2(a\xi) + \ln(\xi)\ln(z)$.

Proof. For any $\xi \in]0,1[$ equation (11) implies that $\lim_{q\to 1^-} \ln(q)\ln(g(\ln\xi / \ln q)) = F(\xi)$. Thus, one has asymptotically $g((\ln\xi / \ln q) \sim e^{\frac{F(\xi)}{\ln q}}$, $q \to 1^-$, and, by using the steepest decent method, one has also $I(g) \sim e^{\frac{F(\xi_0)}{\ln q}}$, $q \to 1^-$, where $\xi_0 = (1-z)/(1-az) \in]0,1[$ is the unique solution of the equation $F'(\xi) = 0$. The asymptotic formula for $S(g)$ follows immediately from Lemma 2 after taking into account the fact that $x_0 \sim n_0 \sim \frac{\ln \xi_0}{\ln q}$, $q \to 1^-$, and, correspondingly, $g(n_0) \sim g(x_0) \sim I(g)$, $q \to 1^-$. □

Proof of Theorem 1. Using Lemma 1, we have immediately

$$\lim_{q\to 1^-} \ln(q) \ln\left(\frac{\Phi(a)\Phi(z)}{\Phi(q)\Phi(az)}\right) = \mathrm{Li}_2(1) + \mathrm{Li}_2(az) - \mathrm{Li}_2(a) - \mathrm{Li}_2(y).$$

Combining this formula with equation (13), we conclude that the q-binomial identity (2) leads to the following identity: $F(\xi_0) = \mathrm{Li}_2(1) + \mathrm{Li}_2(az) - \mathrm{Li}_2(a) - \mathrm{Li}_2(z)$ or explicitly,

$$\mathrm{Li}_2(\xi_0) - \mathrm{Li}_2(a\xi_0) + \ln(\xi_0)\ln(z) = \mathrm{Li}_2(1) + \mathrm{Li}_2(az) - \mathrm{Li}_2(a) - \mathrm{Li}_2(z)$$

which we rewrite in the form

$$\mathrm{Li}_2(a) + \mathrm{Li}_2(z) = \mathrm{Li}_2(az) + \mathrm{Li}_2(a\xi_0) + \mathrm{Li}_2(1) - \mathrm{Li}_2(\xi_0) - \ln(\xi_0)\ln(z).$$

Using the identity

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \mathrm{Li}_2(1) - \ln(x)\ln(1-x), \quad \forall x \in [0,1],$$

we rewrite it further

$$\mathrm{Li}_2(a) + \mathrm{Li}_2(z) = \mathrm{Li}_2(az) + \mathrm{Li}_2(a\xi_0) + \mathrm{Li}_2(1-\xi_0) + \ln(\xi_0)\ln\left((1-\xi_0) / z\right)$$

which is exactly the Rogers identity (6). □

References

1. Gasper G., Rahman M. Basic hypergeometric series. *Encyclopedia of Mathematics and its Applications*. Cambridge, Cambridge University Press, 2004.

2. Faddeev L.D., Kashaev R.M. Quantum dilogarithm. *Modern Phys. Lett. A.,* 1994, vol. 9, no. 5, pp. 427–434.

3. Kirillov A.N. Dilogarithm identities. *Progr. Theoret. Phys. Suppl.,* 1995, vol. 118, pp. 61–142.

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Q-БИНОМИАЛЬНАЯ ФОРМУЛА И ДИЛОГАРИФМИЧЕСКОЕ ТОЖДЕСТВО РОДЖЕРСА

Р. М. Кашаев

Показывается, что q-биномиальная формула в пределе при q → 1^- эквивалентна пятичленному дилогарифмическому тождеству Роджерса.

Ключевые слова: *q-биномиальная формула, дилогарифмическое тождество.*

Список литературы

1. Gasper, G. Basic hypergeometric series / G. Gasper, M. Rahman // Encyclopedia of Mathematics and its Applications. — Second edition. — Cambridge : Cambridge University Press, 2004.

2. Faddeev, L. D. Quantum dilogarithm / L. D. Faddeev, R. M. Kashaev // Modern Phys. Lett. A. — 1994. — Vol. 9, № 5. — P. 427–434.

3. Kirillov, A. N. Dilogarithm identities / A. N. Kirillov // Progr. Theoret. Phys. Suppl. — 1995. — Vol. 118. — P. 61–142.

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AN INTRODUCTION TO FINITE TYPE INVARIANTS OF KNOTS AND 3-MANIFOLDS DEFINED BY COUNTING GRAPH CONFIGURATIONS

C. Lescop

The finite type invariant concept for knots was introduced in the 90's in order to classify knot invariants, with the work of Vassiliev, Goussarov and Bar-Natan, shortly after the birth of numerous quantum knot invariants. This very useful concept was extended to 3-manifold invariants by Ohtsuki.

These introductory lectures show how to define finite type invariants of links and 3-manifolds by counting graph configurations in 3-manifolds, following ideas of Witten and Kontsevich.

The linking number is the simplest finite type invariant for 2-component links. It is defined in many equivalent ways in the first section. As an important example, we present it as the algebraic intersection of a torus and a 4-chain called a *propagator* in a configuration space.

In the second section, we introduce the simplest finite type 3-manifold invariant, which is the Casson invariant (or the Θ-invariant) of integer homology 3-spheres. It is defined as the algebraic intersection of three propagators in the same two-point configuration space.

In the third section, we explain the general notion of finite type invariants and introduce relevant spaces of Feynman Jacobi diagrams.

In Sections 4 and 5, we sketch an original construction based on configuration space integrals of universal finite type invariants for links in rational homology 3-spheres and we state open problems. Our construction generalizes the known constructions for links in $\mathbb{R}^3$ and for rational homology 3-spheres, and it makes them more flexible.

In Section 6, we present the needed properties of parallelizations of 3-manifolds and associated Pontrjagin classes, in details.

Keywords: *knots, 3-manifolds, finite type invariants, homology 3-spheres, linking number, Theta invariant, Casson-Walker invariant, Feynman Jacobi diagrams, perturbative expansion of Chern–Simons theory, configuration space integrals, parallelizations of 3-manifolds, first Pontrjagin class.*

Foreword

These notes contain some details about talks that were presented in the international conference "Quantum Topology" organized by Laboratory of Quantum Topology of Chelyabinsk State University in July 2014. They are based on the notes of five lectures presented in the ICPAM–ICTP research school of Meknès in May 2012. I thank the organizers of these two great events. I also thank Catherine Gille and Kévin Corbineau for useful comments on these notes.

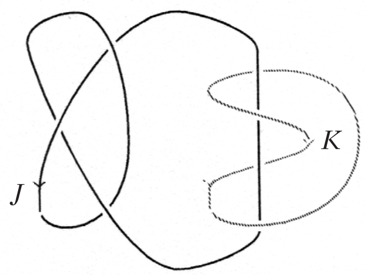
These notes have been written in an introductory way, in order to be understandable by graduate students. In particular, Sections 1, 2 and 6 provide an elementary self-contained presentation of the Θ-invariant. The notes also contain original statements (Theorems 5, 6, 7 and 8) together with sketches of proofs. Complete proofs of these statements, which generalize known statements, will be included in a monograph [1].

1. Various aspects of the linking number

1.1. The Gauss linking number of two disjoint knots in $\mathbb{R}^3$ the ambient space

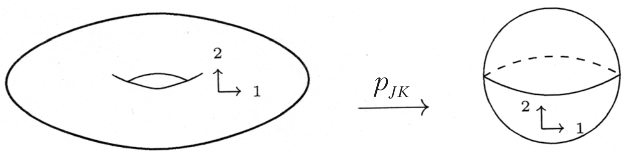
The modern powerful invariants of links and 3-manifolds that will be defined in Section 4 can be thought of as generalizations of the linking number. In this section, we warm up with several ways of defining this classical basic invariant. This allows us to introduce conventions and methods that will be useful througout the article.

Let S^1 denote the unit circle of $\mathbb{C}$: $S^1 = \{z; z \in \mathbb{C}, |z| = 1\}$. Consider two C^∞ embeddings $J: S^1 \hookrightarrow \mathbb{R}^3$ and $K: S^1 \hookrightarrow \mathbb{R}^3 \setminus J(S^1)$



and the associated *Gauss map*

$$\begin{array}{cccc} p_{JK}: & S^1\times S^1 & \hookrightarrow & S^2, \\ & (w,z) & \mapsto & \frac{1}{\| K(z)-J(w).\|}(K(z)-J(w)) \end{array}$$



Denote the standard area form of S^2 by $4\pi\omega_{S^2}$ so that ω_{S^2} is the homogeneous volume form of S^2 such that $\int_{S^2}\omega_{S^2}=1$. In 1833, Gauss defined the *linking number* of the disjoint *knots* $J(S^1)$ and $K(S^1)$, simply denoted by J and K, as an integral [2]. With modern notation, his definition reads

$$lk_G(J,K)=\int_{S^1\times S^1}p^*_{JK}(\omega_{S^2}).$$

It can be rephrased as $lk_G(J,K)$ *is the degree of the Gauss map* p_{JK}.

1.2. Some background material on manifolds without boundary, orientations, and degree

A *topological n-dimensional manifold M without boundary* is a Hausdorff topological space that is a countable union of open subsets U_i labeled in a set I $(i\in I)$, where every U_i is identified with an open subset V_i of $\mathbb{R}^n$ by a homeomorphism $\phi_i:U_i\to V_i$, called a *chart.* Manifolds are considered up to homeomorphism so that homeomorphic manifolds are considered identical.

For $r=0,\dots,\infty$, the topological manifold M *has a C^r-structure* or *is a C^r-manifold*, if, for each pair $\{i,j\}\subset I$, the map $\phi_j\circ\phi_i^{-1}$ defined on $\phi_i(U_i\cap U_j)$ is a C^r-diffeomorphism to its image. The notion of C^s-maps, $s\le r$, from such a manifold to another one can be naturally deduced from the known case where the manifolds are open subsets of some $\mathbb{R}^n$, thanks to the local identifications provided by the charts. C^r-manifolds are considered up to C^r-diffeomorphisms.

An *orientation* of a real vector space V of positive dimension is a basis of V up to a change of basis with positive determinant. When $V=\{0\}$, an orientation of V is an element of $\{-1,1\}$. For $n>0$, an orientation of $\mathbb{R}^n$ identifies $H_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{x\};\mathbb{R})$ with $\mathbb{R}$. (In these notes, we freely use basic algebraic topology, see [3] for example.) A homeomorphism h from an open subset U of $\mathbb{R}^n$ to another such V is *orientation-preserving* at a point x, if $h_*:H_n(U,U\setminus\{x\})\to H_n(V,V\setminus\{h(x)\})$ is orientation-preserving. If h is a diffeomorphism, h is orientation-preserving at x if and only if the determinant of the Jacobian T_xh is positive. If $\mathbb{R}^n$ is oriented and if the transition maps $\phi_j\circ\phi_i^{-1}$ are orientation-preserving (at every point) for $\{i,j\}\subset I$, the manifold M is *oriented*.

For $n=0$, 1, 2 or 3, any topological n-manifold may be equipped with a unique smooth structure (up to diffeomorphism) (See Theorem 10, below). Unless otherwise mentioned, our manifolds are *smooth* (i. e. C^∞), oriented and compact, and considered up oriented diffeomorphisms. Products are oriented by the order of the factors. More generally, unless otherwise mentioned, the order of appearance of coordinates or parameters orients manifolds.

A point y is *a regular value* of a smooth map p: $M \to N$ between two smooth manifolds M and N, if for any $x \in p^{-1}(y)$ the tangent map $T_x p$ at x is surjective. According to the Morse–Sard theorem [4. P. 69], the set of regular values of such a map is dense. If M is compact, it is furthermore open.

When M is oriented and compact, and when the dimension of M coincides with the dimension of N, the *differential degree* of p at a regular value y of N is the (finite) sum running over the $x \in p^{-1}(y)$ of the signs of the determinants of $T_x p$. In our case where M has no boundary, this differential degree is locally constant on the set of regular values, and it is the *degree* of p, if N is connected. See [5. Chapter 5].

Finally, recall a homological definition of the degree. Let $[M]$ denote the class of an oriented *closed* (i. e. compact, connected, without boundary) n-manifold in $H_n(M;\mathbb{Z})$. $H_n(M;\mathbb{Z}) = \mathbb{Z}[M]$. If M and N are two closed oriented n-manifolds and if f: $M \to N$ is a (continuous) map, then $H_n(f)([M]) = \deg(f)[N]$. In particular, for the Gauss map p_{JK} of Subsection 1.1, $H_2(p_{JK})([S^1 \times S^1]) = lk(J,K)[S^2]$.

1.3. The Gauss linking number as a degree

Since the differential degree of the Gauss map p_{JK} is locally constant, $lk_G(J,K) = \int_{S^1 \times S^1} p_{JK}^*(\omega)$ for any 2-form ω on S^2 such that $\int_{S^2} \omega = 1$.

Let us compute $lk_G(J,K)$ as the differential degree of p_{JK} at the vector Y that points towards us. The set $p_{JK}^{-1}(Y)$ is made of the pairs of points (w,z) where the projections of $J(w)$ and $K(z)$ coincide, and $J(w)$ is under $K(z)$. They correspond to the *crossings* ^{J}⤧^{K} and ^{K}⤧^{J} of the diagram.

In a diagram, a crossing is *positive* if we turn counterclockwise from the arrow at the end of the upper strand to the arrow of the end of the lower strand like ⤧. Otherwise, it is *negative* like ⤧.

For the positive crossing ^{J}⤧^{K}, moving $J(w)$ along J following the orientation of J, moves $p_{JK}(w,z)$ towards the South-East direction, while moving $K(z)$ along K following the orientation of K, moves $p_{JK}(w,z)$ towards the North-East direction, so that the local orientation induced by the image of p_{JK} around $Y \in S^2$ is $\left\langle Tp\frac{\partial}{\partial z}, Tp\frac{\partial}{\partial w}\right.$ which is $\left\langle 2, 1\right.$. Therefore, the contribution of a positive crossing to the degree is 1. Similarly, the contribution of a negative crossing is (-1).

We have just proved the following formula

$$\deg_Y(p_{JK}) = \#^J⤧^K - \#^K⤧^J$$

where # stands for the cardinality — here # ^{J}⤧^{K} is the number of occurences of ^{J}⤧^{K} in the diagram — so that

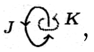
$$lk_G(J,K) = \#^J⤧^K - \#^K⤧^J.$$

Similarly, $\deg_{-Y}(p_{JK}) = \#^K⤧^J - \#^J⤧^K$ so that

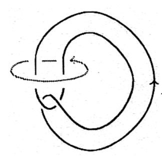
$$lk_G(J,K) = \#^K⤧^J - \#^J⤧^K = \frac{1}{2}(\#^J⤧^K + \#^K⤧^J) - \frac{1}{2}(\#^K⤧^J + \#^J⤧^K)$$

and $lk_G(J,K) = lk_G(K,J)$.

In our first example, $lk_G(J,K) = 2$. Let us draw some further examples.

For the *positive Hopf link* , $lk_G(J,K) = 1$.

For the *negative Hopf link* , $lk_G(J,K) = -1$.

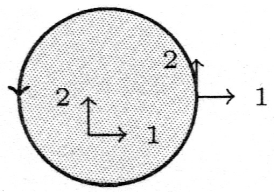
For the *Whitehead link* , $lk_G(J,K) = 0$.

1.4. Some background material on manifolds with boundary and algebraic intersections

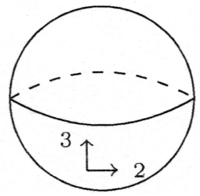
A *topological n-dimensional manifold M with possible boundary* is a Hausdorff topological space that is a union of open subsets U_i labeled in a set I, $(i \in I)$, where every U_i is identified with an open subset V_i of $]-\infty,0]^k \times \mathbb{R}^{n-k}$ by a chart $\phi_i : U_i \to V_i$. The *boundary* of $]-\infty,0]^k \times \mathbb{R}^{n-k}$ is made of the points $(x_1,\dots,x_n)$ of $]-\infty,0]^k \times \mathbb{R}^{n-k}$ such that there exists $i \leq k$ such that $x_i = 0$. The *boundary* of M is made of the points that are mapped to the boundary of $]-\infty,0]^k \times \mathbb{R}^{n-k}$.

For $r = 1,\dots,\infty$, the topological manifold M *is a C^r-manifold with ridges (or with corners) (resp. with boundary)*, if, for each pair $\{i,j\} \subset I$, the map $\phi_j \circ \phi_i^{-1}$ defined on $\phi_i(U_i \cap U_j)$ is a C^r-diffeomorphism to its image (resp. and if furthermore $k \leq 1$, for any i). Then the *ridges* of M are made of the points that are mapped to points $(x_1,\dots,x_n)$ of $]-\infty,0]^k \times \mathbb{R}^{n-k}$ so that there are at least two $i \leq k$ such that $x_i = 0$.

The tangent bundle to an oriented submanifold A in a manifold M at a point x is denoted by T_xA. The *normal bundle* T_xM / T_xA to A in M at x is denoted by $\mathfrak{V}_xA$. It is oriented so that (a lift of an oriented basis of) $\mathfrak{V}_xA$ followed by (an oriented basis of) T_xA induce the orientation of T_xM. The boundary ∂M of an oriented manifold M is oriented by the *outward normal first* convention. If $x \in \partial M$ is not in a ridge, the outward normal to M at x followed by an oriented basis of $T_x\partial M$ induce the orientation of M. For example, the standard orientation of the disk in the plane induces the standard orientation of the circle, counterclockwise, as the following picture shows.



As another example, the sphere S^2 is oriented as the boundary of the ball B^3, which has the standard orientation induced by (Thumb, index finger (2), middle finger (3)) of the right hand.



Two submanifolds A and B in a manifold M are transverse if at each intersection point x, $T_xM = T_xA + T_xB$. The transverse intersection of two submanifolds A and B in a manifold M is oriented so that the normal bundle to $A \cap B$ is $(\mathfrak{V}(A) \oplus \mathfrak{V}(B))$, fiberwise. If the two manifolds are of complementary dimensions, then the sign of an intersection point is +1 if the orientation of its normal bundle coincides with the orientation of the ambient space, that is if $T_xM = \mathfrak{V}_xA \oplus \mathfrak{V}_xB$ (as oriented vector spaces), this is equivalent to $T_xM = T_xA \oplus T_xB$ (as oriented vector spaces again, exercise). Otherwise, the sign is −1. If A and B are compact and if A and B are of complementary dimensions in M, their *algebraic intersection* is the sum of the signs of the intersection points, it is denoted by $\langle A,B \rangle_M$.

When M is an oriented manifold, $(-M)$ denotes the same manifold, equipped with the opposite orientation. In a manifold M, a k-dimensional *chain (resp. rational chain)* is a finite combination with coefficients in $\mathbb{Z}$ (resp. in $\mathbb{Q}$) of smooth k-dimensional oriented submanifolds C of M with boundary and ridges, up to the identification of $(-1)C$ with $(-C)$.

Again, unless otherwise mentioned, manifold are oriented. The boundary ∂ of chains is a linear map that maps a smooth submanifold to its oriented boundary. The canonical orientation of a point is the sign +1 so that $\partial[0,1] = \{1\} - \{0\}$.

Lemma 1. *Let A and B be two transverse submanifolds of a d-dimensional manifold M, of respective dimensions α and β, with disjoint boundaries. Then*

$$\partial(A \cap B) = (-1)^{d-\beta} \partial A \cap B + A \cap \partial B.$$

Proof. Note that $\partial(A \cap B) \subset \partial A \cup \partial B$. At a point $a \in \partial A$, $T_a M$ is oriented by $(\mathfrak{N}_a A, o, T_a \partial A)$, where o is the outward normal to A. If $a \in \partial A \cap B$, then o is also an outward normal to $A \cap B$, and $\partial(A \cap B)$ is cooriented by $(\mathfrak{N}_a A, \mathfrak{N}_a B, o)$ while $\partial A \cap B$ is cooriented by $(\mathfrak{N}_a A, o, \mathfrak{N}_a B)$. At a point $b \in A \cap \partial B$, $\partial(A \cap B)$ is cooriented by $(\mathfrak{N}_a A, \mathfrak{N}_a B, o)$ like $A \cap \partial B$. □

1.5. A general definition of the linking number

Lemma 2. *Let J and K be two rationally null-homologous disjoint cycles of respective dimensions j and k in a d-manifold M, where $d = j + k + 1$. There exists a rational $(j+1)$-chain Σ_J bounded by J transverse to K, and a rational $(k+1)$-chain Σ_K bounded by K transverse to J and for any two such rational chains Σ_J and Σ_K, $\langle J, \Sigma_K \rangle_M = (-1)^{j+1} \langle \Sigma_J, K \rangle_M$. In particular, $\langle J, \Sigma_K \rangle_M$ is a topological invariant of (J,K), which is denoted by $lk(J,K)$ and called the linking number of J and K.*

$$lk(J,K) = (-1)^{(j+1)(k+1)} lk(K,J).$$

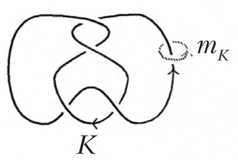
Proof. Since K is rationally null-homologous, K bounds a rational $(k+1)$-chain Σ_K. Without loss, Σ_K is assumed to be transverse to Σ_J so that $\Sigma_J \cap \Sigma_K$ is a rational 1-chain (which is a rational combination of circles and intervals). (As explained in [4. Chapter 3], generically, manifolds are transverse). According to Lemma 1,

$$\partial(\Sigma_J \cap \Sigma_K) = (-1)^{d+k+1} J \cap \Sigma_K + \Sigma_J \cap K.$$

Furthermore, the sum of the coefficients of the points in the left-hand side must be zero, since this sum vanishes for the boundary of an interval. This shows that $\langle J, \Sigma_K \rangle_M = (-1)^{d+k} \langle \Sigma_J, K \rangle_M$, and therefore that this rational number is independent of the chosen Σ_J and Σ_K. Since $(-1)^{d+k} \langle \Sigma_J, K \rangle_M = (-1)^{j+1} (-1)^{k(j+1)} \langle K, \Sigma_J \rangle_M$, $lk(J,K) = (-1)^{(j+1)(k+1)} lk(K,J)$. □

In particular, the *linking number* of two rationally null-homologous disjoint links J and K in a 3-manifold M is the algebraic intersection of a rational chain bounded by one of the knots and the other one.

For $\mathbb{K} = \mathbb{Z}$ or $\mathbb{Q}$, a $\mathbb{K}$-*sphere or (integer or rational) homology* 3-*sphere* (resp. a $\mathbb{K}$-*ball*) is a smooth, compact, oriented 3-manifold, without ridges, with the same $\mathbb{K}$-homology as the sphere S^3 (resp. as a point). In such a manifold, any knot is rationally null-homologous so that the linking number of two disjoint knots always makes sense.



A *meridian* of a knot K is the (oriented) boundary of a disk that intersects K once with a positive sign. Since a chain Σ_J bounded by a knot J disjoint from K in a 3-manifold M provides a rational cobordism between J and a combination of meridians of K, $[J] = lk(J,K)[m_K]$ in $H_1(M \setminus K; \mathbb{Q})$ where m_K is a meridian of K.

Lemma 3. *When K is a knot in a $\mathbb{Q}$-sphere or a $\mathbb{Q}$-ball M, $H_1(M \setminus K; \mathbb{Q}) = \mathbb{Q}[m_K]$, so that the equation $[J] = lk(J,K)[m_K]$ in $H_1(M \setminus K; \mathbb{Q})$ provides an alternative definition for the linking number.*

Proof. Exercise. □

The reader is also invited to check that $lk_G = lk$ as an exercise though it will be proved in the next subsection, see Proposition 1.

1.6. Generalizing the Gauss definition of the linking number and identifying the definitions

Lemma 4. *The map*

$$\begin{array}{cccc} p_{S^2}: & ((\mathbb{R}^3)^2 \setminus \mathrm{diag}) & \to & S^2 \\ & (x,y) & \mapsto & \frac{1}{\| y - x \|}(y - x) \end{array}$$

is a homotopy equivalence. In particular $H_i(p_{S^2})$: $H_i((\mathbb{R}^3)^2 \setminus \mathrm{diag};\mathbb{Z}) \to H_i(S^2;\mathbb{Z})$ *is an isomorphism for all i,* $(\mathbb{R}^3)^2 \setminus \mathrm{diag}$ *is a homology* S^2*, and* $[S] = \left(H_2(p_{S^2})\right)^{-1}[S^2]$ *is a canonical generator of* $H_2((\mathbb{R}^3)^2 \setminus \mathrm{diag};\mathbb{Z}) = \mathbb{Z}[S]$.

Proof. The map $(x,y) \mapsto (x, \| y-x \|, p_{S^2}(x,y))$ provides a homeomorphism from $(\mathbb{R}^3)^2 \setminus \mathrm{diag}$ to $\mathbb{R}^3 \times]0,\infty[\times S^2$. □

As in Subsection 1.1, consider a two-component link $J \sqcup K\colon S^1 \sqcup S^1 \hookrightarrow \mathbb{R}^3$. This embedding induces an embedding

$$\begin{array}{rccc} J\times K: & S^1\times S^1 & \hookrightarrow & ((\mathbb{R}^3)^2 \setminus \mathrm{diag}) \\ & (z_1,z_2) & \mapsto & (J(z_1),K(z_2)) \end{array}$$

the map p_{JK} of Subsection 1.1 reads $p_{S^2}\circ(J\times K)$, and since $H_2(p_{JK})[S^1\times S^1] = \deg(p_{JK})[S^2] = lk_G(J,K)[S^2]$ in $H_2(S^2;\mathbb{Z}) = \mathbb{Z}[S^2]$, $[J\times K] = H_2(J\times K)[S^1\times S^1] = lk_G(J,K)[S]$ in $H_2((\mathbb{R}^3)^2 \setminus \mathrm{diag};\mathbb{Z} = (\mathbb{Z}[S]$. We will see that this definition of lk_G generalizes to links in rational homology 3-spheres and then prove that our generalized definition coincides with the general definition of linking numbers in this case.

For a 3-manifold M, the normal bundle to the diagonal of M^2 in M^2 is identified with the tangent bundle to M, fiberwise, by the map

$$(u,v) \in \frac{(T_xM)^2}{\mathrm{diag}((T_xM)^2)} \mapsto (v-u) \in T_xM.$$

A *parallelization* τ of an oriented 3-manifold M is a bundle isomorphism $\tau: M\times\mathbb{R}^3 \to TM$ that restricts to $x\times\mathbb{R}^3$ as an orientation-preserving linear isomorphism from $x\times\mathbb{R}^3$ to T_xM, for any $x\in M$. It has long been known that any oriented 3-manifold is parallelizable (i. e. admits a parallelization). (It is proved in Subsection 6.2.) Therefore, a tubular neighborhood of the diagonal in M^2 is diffeomorphic to $M\times\mathbb{R}^3$.

Lemma 5. *Let M be a rational homology 3-sphere, let* ∞ *be a point of M. Let* $\breve{M} = (M \setminus \{\infty\})$*. Then* $\breve{M}^2 \setminus \mathrm{diag}$ *has the same rational homology as* S^2*. Let B be a ball in* $\breve{M}$ *and let x be a point inside B, then the class* $[S]$ *of* $x\times\partial B$ *is a canonical generator of* $H_2(\breve{M}^2 \setminus \mathrm{diag}\ \mathbb{Q}) = \mathbb{Q}[S]$.

Proof. In this proof, the homology coefficients are in $\mathbb{Q}$. Since $\breve{M}$ has the homology of a point, the Künneth Formula implies that $\breve{M}^2$ has the homology of a point. Now, by excision,

$$H_*(\breve{M}^2, \breve{M}^2 \setminus \mathrm{diag}) \cong H_*(\breve{M}\times\mathbb{R}^3, \breve{M}\times(\mathbb{R}^3\setminus 0)) \cong H_*(\mathbb{R}^3,S^2) \cong \begin{cases}\mathbb{Q} & \text{if } * = 3,\\ 0 & \text{otherwise.}\end{cases}$$

Using the long exact sequence of the pair $(\breve{M}^2, \breve{M}^2 \setminus \mathrm{diag}$, we get that $H_*(\breve{M}^2 \setminus diag;\mathbb{Q}) = H_*(S^2)$. □

Define the *Gauss linking number* of two disjoint links J and K in $\breve{M}$ so that

$$[(J\times K)(S^1\times S^1)] = lk_G(J,K)[S]$$

in $H_2(\breve{M}^2 \setminus \mathrm{diag};\mathbb{Q}$. Note that the two definitions of lk_G coincide when $\breve{M} = \mathbb{R}^3$.

Proposition 1. $lk_G = lk$.

Proof. First note that both definitions make sense when J and K are disjoint links: $[J\times K] = lk_G(J,K)[S]$ and $lk(J,K)$ is the algebraic intersection of K and a rational chain Σ_J bounded by J.

If K is a knot, then the chain Σ_J of $\breve{M}$ provides a rational cobordism C between J and a combination of meridians of K in $\breve{M}\setminus K$, and a rational cobordism $C\times K$ in $\breve{M}^2 \setminus \mathrm{diag}$, which allow us to see that $lk_G(\cdot,K)$ and $lk(\cdot,K)$ linearly depend on $[J]\in H_1(\breve{M}\setminus K)$. Thus we are left with the proof that $lk_G(m_K,K) = lk(m_K,K) = 1$. Since $lk_G(m_K,\cdot)$ linearly depends on $[K]\in H_1(\breve{M}\setminus m_K)$, we are left with the proof $lk_G(m_K,K) = 1$ when K is a meridian of m_K. Now, there is no loss in assuming that our link is a Hopf link in $\mathbb{R}^3$ so that the equality follows from the equality for the positive Hopf link in $\mathbb{R}^3$. □

For a 2-component link (J,K) in $\mathbb{R}^3$, the definition of $lk(J,K)$ can be rewritten as

$$lk(J,K) = \int_{J\times K} p^*_{S^2}(\omega) = \langle J \times K, p^{-1}_{S^2}(Y)\rangle_{(\mathbb{R}^3)^2\setminus \mathrm{diag}}$$

for any regular value Y of p_{JK}, and for any 2-form ω of S^2 such that $\int_{S^2}\omega = 1$. Thus, $lk(J,K)$ is the evaluation of a 2-form $p^*_{S^2}(\omega)$ of $(\mathbb{R}^3)^2 \setminus \mathrm{diag}$ at the 2-cycle $[J\times K]$, or it is the intersection of the 2-cycle $[J\times K]$ with a 4-manifold $p^{-1}_{S^2}(Y)$, which will later be seen as the interior of a prototypical propagator. We will adapt these definitions to rational homology 3-spheres in Subsection 2.3. The definition of the linking number that we will generalize in order to produce more powerful invariants is contained in Lemma 8.

2. Propagators and the Θ-invariant

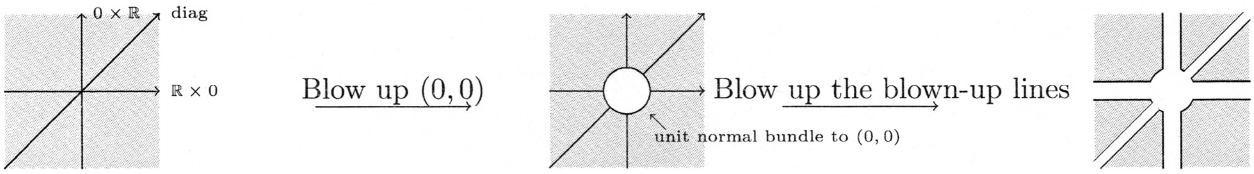
Propagators will be the key ingredient to define powerful invariants from graph configurations in Section 4. They are defined in Subsection 2.3 below after needed preliminaries. They allow us to define the Θ-invariant as an invariant of parallelized homology 3-balls in Subsection 2.4. The Θ-invariant is next turned to an invariant of rational homology 3-spheres in Subsection 2.6.

2.1. Blowing up in real differential topology

Let A be a submanifold of a smooth manifold B, and let $U\mathfrak{N}(A)$ denote its unit normal bundle. The fiber $U\mathfrak{N}_a(A) = (\mathfrak{N}_a(A)\setminus\{0\})/\mathbb{R}^{+*}$ of $U\mathfrak{N}(A)$ is oriented as the boundary of a unit ball of $\mathfrak{N}_a(A)$.

Here, *blowing up* such a submanifold A of codimension c of B means replacing A by $U\mathfrak{N}(A)$. For small open subspaces U_A of A, $\left((\mathbb{R}^c = \{0\}\cup]0,\infty[\times S^{c-1})\times U_A\right)$ is replaced by $([0,\infty[\times S^{c-1}\times U_A)$, so that the blown-up manifold $B\ell(B,A)$ is homeomorphic to the complement in B of an open tubular neighborhood (thought of as infinitely small) of A. In particular, $B\ell(B,A)$ is homotopy equivalent to $B\setminus A$. Furthermore, the blow up is canonical, so that the created boundary is $\pm U\mathfrak{N}(A)$ and there is a canonical smooth projection from $B\ell(B,A)$ to B such that the preimage of $a\in A$ is $U\mathfrak{N}_a(A)$. If A and B are compact, then $B\ell(B,A)$ is compact, it is a smooth compactification of $B\setminus A$.

In the following figure, we see the result of blowing up $(0,0)$ in $\mathbb{R}^2$, and the closures in $B\ell(\mathbb{R}^2,(0,0))$ of $\{0\}\times\mathbb{R}$, $\mathbb{R}\times\{0\}$ and the diagonal of $\mathbb{R}^2$, successively.



2.2. The configuration space $C_2(M)$

See S^3 as $\mathbb{R}^3\cup\infty$ or as two copies of $\mathbb{R}^3$ identified along $\mathbb{R}^3\setminus\{0\}$ by the (exceptionally orientation-reversing) diffeomorphism $x\mapsto x/\|x\|^2$. Then $B\ell(S^3,\infty) = \mathbb{R}^3\cup S^2_\infty$ where the unit normal bundle $(-S^2_\infty)$ to ∞ in S^3 is canonically diffeomorphic to S^2 via $p_\infty : S^2_\infty \to S^2$, where $x\in S^2_\infty$ is the limit of a sequence of points of $\mathbb{R}^3$ approaching ∞ along a line directed by $p_\infty(x)\in S^2$, $\partial B\ell(S^3,\infty) = S^2_\infty$.

Fix a rational homology 3-sphere M, a point ∞ of M, and $\check{M} = M\setminus\{\infty\}$. Identify a neighborhood of ∞ in M with the complement $\check{B}_{1,\infty}$ of the closed ball $B(1)$ of radius 1 in $\mathbb{R}^3$. Let $\check{B}_{2,\infty}$ be the complement of the closed ball $B(2)$ of radius 2 in $\mathbb{R}^3$, which is a smaller neighborhood of

∞ in M via the understood identification. Then BM $= M\setminus \breve{B}_{2,\infty}$ is a compact rational homology ball diffeomorphic to $B\ell(M,\infty)$.

Define the *configuration space* $C_2(M)$ as the compact 6-manifold with boundary and ridges obtained from M^2 by blowing up (∞,∞), the closures in $B\ell(M^2,(\infty,\infty))$ of $\{\infty\}\times\breve{M}$, $\breve{M}\times\{\infty\}$ and the diagonal of $\breve{M}^2$, successively. Then $\partial C_2(M)$ contains the unit normal bundle

$$\left(\frac{T\breve{M}^2}{\text{diag}}\setminus\{0\}\right)/\,\mathbb{R}^{+*}$$

to the diagonal of $\breve{M}^2$. This bundle is canonically isomorphic to the unit tangent bundle $U\breve{M}$ to $\breve{M}$ (again via the map $\left([(x,y)]\mapsto[y-x]\right)$).

Lemma 6. *Let $\breve{C}_2(M)=\breve{M}^2\setminus$ diag. The open manifold $C_2(M)\setminus\partial C_2(M)$ is $\breve{C}_2(M)$ and the inclusion $\breve{C}_2(M)\rightarrowtail C_2(M)$ is a homotopy equivalence. In particular, $C_2(M)$ is a compactification of $\breve{C}_2(M)$ homotopy equivalent to $\breve{C}_2(M)$. The manifold $C_2(M)$ is a smooth compact 6-dimensional manifold with boundary and ridges. There is a canonical smooth projection $p_{M^2}: C_2(M)\to M^2$, $\partial C_2(M)=(S^2_\infty\times\breve{M})\cup(-\breve{M}\times S^2_\infty)\cup U\breve{M}\pm p^{-1}_{M^2}(\infty,\infty)$.*

Proof. Let $B_{1,\infty}$ be the complement of the open ball of radius one of $\mathbb{R}^3$ in S^3. Blowing up (∞,∞) in $B^2_{1,\infty}$ transforms a neighborhood of (∞,∞) into the product $[0,1[\times S^5$. Explicitly, there is a map $\psi:[0,1[\times S^5\to B\ell(B^2_{1,\infty},(\infty,\infty))$, where $B\ell(B^2_{1,\infty},(\infty,\infty))\subset B\ell(M^2,(\infty,\infty))$, such that when $\lambda\in]0,1[$ and (x,y) is an element of the unit sphere S^5 of $(\mathbb{R}^3)^2$ such that $x\neq 0$ and $y\neq 0$,

$$\psi(\lambda,(x,y))=\left(\frac{1}{\lambda\parallel x\parallel^2}x,\frac{1}{\lambda\parallel y\parallel^2}y\right)$$

and such that ψ is a diffeomorphism onto its image, which is a neighborhood of the preimage of (∞,∞) under the blow-down map $B\ell(M^2,(\infty,\infty))\xrightarrow{p_1}M^2$. This neighborhood intersects $\infty\times\breve{M}$, $\breve{M}\times\infty$, and $\text{diag}(\breve{M}^2)$ as $\psi(]0,1[\times 0\times S^2)$, $\psi(]0,1[\times S^2\times 0)$ and $\psi(]0,1[\times(S^5\cap\text{diag}((\mathbb{R}^3)^2)))$, respectively. In particular, the closures of $\infty\times\breve{M}$, $\breve{M}\times\infty$, and $\text{diag}(\breve{M}^2)$ in $B\ell(M^2,(\infty,\infty))$ intersect the boundary $\psi(0\times S^5)$ of $B\ell(M^2,(\infty,\infty))$ as three disjoint spheres in S^5, and they read $\infty\times B\ell(M,\infty)$, $B\ell(M,\infty)\times\infty$ and $\text{diag}(B\ell(M,\infty)^2)$. Thus, the next steps will be three blow-ups along these three disjoint smooth manifolds.

These blow-ups will preserve the product structure $\psi([0,1[\times\cdot)$. Therefore, $C_2(M)$ is a smooth compact 6-dimensional manifold with boundary, with three *ridges* $S^2\times S^2$ in $p^{-1}_{M^2}(\infty,\infty)$. A neighborhood of these ridges in $C_2(M)$ is diffeomorphic to $[0,1[^2\times S^2\times S^2$. □

Lemma 7. *The map p_{S^2} of Lemma 4 smoothly extends to $C_2(S^3)$, and its extension p_{S^2} satisfies:*

$$p_{S^2}=\begin{cases}-p_\infty\circ p_1 & \text{on } S^2_\infty\times\mathbb{R}^3,\\ p_\infty\circ p_2 & \text{on } \mathbb{R}^3\times S^2_\infty,\\ p_2 & \text{on } U\mathbb{R}^3=\mathbb{R}^3\times S^2,\end{cases}$$

where p_1 and p_2 denote the projections on the first and second factor with respect to the above expressions.

Proof. Near the diagonal of $\mathbb{R}^3$, we have a chart of $C_2(S^3)$

$$\psi_d:\mathbb{R}^3\times[0,\infty[\times S^2\to C_2(S^3)$$

that maps $(x\in\mathbb{R}^3,\lambda\in]0,\infty[,y\in S^2)$ to $(x,x+\lambda y)\in(\mathbb{R}^3)^2$. Here, p_{S^2} extends as the projection onto the S^2 factor.

Consider the orientation-reversing embedding ϕ_∞

$$\begin{array}{rcl}\phi_\infty:\mathbb{R}^3 & \to & S^3,\\ \mu(x\in S^2) & \mapsto & \begin{cases}\infty & \text{if } \mu=0,\\ \dfrac{1}{\mu}x & \text{otherwise.}\end{cases}\end{array}$$

Note that this chart induces the already mentioned identification p_∞ of the ill-oriented unit normal bundle S^2_∞ to $\{\infty\}$ in S^3 with S^2. When $\mu \neq 0$,

$$p_{S^2}(\phi_\infty(\mu x), y \in \mathbb{R}^3) = \frac{\mu y - x}{\| \mu y - x \|}.$$

Then p_{S^2} can be smoothly extended on $S^2_\infty \times \mathbb{R}^3$ (where $\mu = 0$) by $p_{S^2}(x \in S^2_\infty, y \in \mathbb{R}^3) = -x$. Similarly, set $p_{S^2}(x \in \mathbb{R}^3, y \in S^2_\infty) = y$. Now, with the map ψ of the proof of Lemma 6, when x and y are not equal to zero and when they are distinct,

$$p_{S^2} \circ \psi((\lambda,(x,y))) = \frac{\frac{y}{\|y\|^2} - \frac{x}{\|x\|^2}}{\| \frac{y}{\|y\|^2} - \frac{x}{\|x\|^2} \|} = \frac{\|x\|^2 y - \|y\|^2 x}{\|\|x\|^2 y - \|y\|^2 x\|}$$

when $\lambda \neq 0$. This map naturally extends to $B\ell(M^2,(\infty,\infty))$ outside the boundaries of $\infty \times B\ell(M,\infty)$, $B\ell(M,\infty)\times\infty$ and $\mathrm{diag}(B\ell(M,\infty))$ by keeping the same formula when $\lambda = 0$.

Let us check that p_{S^2} smoothly extends over the boundary of the diagonal of $B\ell(M,\infty)$. There is a chart of $C_2(M)$ near the preimage of this boundary in $C_2(M)$

$$\psi_2 : [0,\infty[\times[0,\infty[\times S^2 \times S^2 \to C_2(S^3)$$

that maps $(\lambda \in]0,\infty[, \mu \in]0,\infty[, x \in S^2, y \in S^2)$ to $(\phi_\infty(\lambda x), \phi_\infty(\lambda(x+\mu y)))$ where p_{S^2} reads

$$(\lambda,\mu,x,y) \mapsto \frac{y - 2\langle x,y\rangle x - \mu x}{\| y - 2\langle x,y\rangle x - \mu x \|},$$

and therefore smoothly extends when $\mu = 0$. We similarly check that p_{S^2} smoothly extends over the boundaries of $(\infty \times B\ell(M,\infty))$ and $(B\ell(M,\infty)\times\infty)$. □

Let τ_s denote the standard parallelization of $\mathbb{R}^3$. Say that a parallelization

$$\tau : \breve{M} \times \mathbb{R}^3 \to T\breve{M}$$

of $\breve{M}$ that coincides with τ_s on $\breve{B}_{1,\infty}$ is *asymptotically standard*. According to Subsection 6.2, such a parallelization exists. Such a parallelization identifies $U\breve{M}$ with $\breve{M} \times S^2$.

Proposition 2. *For any asymptotically standard parallelization* τ *of* $\breve{M}$, *there exists a smooth map* $p_\tau : \partial C_2(M) \to S^2$ *such that*

$$p_\tau = \begin{cases} p_{S^2} & \text{on } p^{-1}_{M^2}(\infty,\infty), \\ -p_\infty \circ p_1 & \text{on } S^2_\infty \times \breve{M}, \\ p_\infty \circ p_2 & \text{on } \breve{M} \times S^2_\infty, \\ p_2 & \text{on } U\breve{M} \stackrel{\tau}{=} \breve{M} \times S^2, \end{cases}$$

where p_1 *and* p_2 *denote the projections on the first and second factor with respect to the above expressions.*

Proof. This is a consequence of Lemma 7. □

Since $C_2(M)$ is homotopy equivalent to $(\breve{M}^2 \setminus \mathrm{diag})$, according to Lemma 5, $H_2(C_2(M);\mathbb{Q}) = \mathbb{Q}[S]$ where the canonical generator $[S]$ is the homology class of a fiber of $U\breve{M} \subset \partial C_2(M)$. For a 2-component link (J,K) of $\breve{M}$, the homology class $[J \times K]$ of $J \times K$ in $H_2(C_2(M);\mathbb{Q})$ reads $lk(J,K)[S]$, according to Subsection 1.6 and to Proposition 1.

Define an *asymptotic rational homology* $\mathbb{R}^3$ as a pair $(\breve{M},\tau)$ where $\breve{M}$ is 3-manifold that reads as the union over $]1,2]\times S^2$ of a rational homology ball B_M and the complement $\breve{B}_{1,\infty}$ of the unit ball of $\mathbb{R}^3$, and τ is an asymptotically standard parallelization of $\breve{M}$. Since such a pair $(\breve{M},\tau)$ canonically defines the rational homology 3-sphere $M = \breve{M} \cup \{\infty\}$, "Let $(\breve{M},\tau)$ be an asymptotic rational homology $\mathbb{R}^3$" is a shortcut for "Let M be a rational homology 3-sphere equipped with an asymptotically standard parallelization τ of $\breve{M}$".

2.3. On propagators

Definition 1. Let $(\breve{M},\tau)$ be an asymptotic rational homology $\mathbb{R}^3$. A *propagating chain* of $(C_2(M),\tau)$ is a 4-chain $\mathcal{P}$ of $C_2(M)$ such that $\partial\mathcal{P}=p_\tau^{-1}(a)$ for some $a\in S^2$. A *propagating form* of $(C_2(M),\tau)$ is a closed 2-form ω_p on $C_2(M)$ whose restriction to $\partial C_2(M)$ reads $p_\tau^*(\omega)$ for some 2-form ω of S^2 such that $\int_{S^2}\omega=1$. Propagating chains and propagating forms are simply called *propagators* when their nature is clear from the context.

Example 1. Recall the map $p_{S^2}:C_2(S^3)\to S^2$ of Lemma 7. For any $a\in S^2$, $p_{S^2}^{-1}(a)$ is a propagating chain of $(C_2(S^3),\tau_s)$, and for any 2-form ω of S^2 such that $\int_{S^2}\omega=1$, $p_{S^2}^*(\omega)$ is a propagating form of $(C_2(S^3),\tau_s)$.

Propagating chains exist because the 3-cycle $p_\tau^{-1}(a)$ of $\partial C_2(M)$ bounds in $C_2(M)$ since $H_3(C_2(M);\mathbb{Q})=0$. Dually, propagating forms exist because the restriction induces a surjective map $H^2(C_2(M);\mathbb{R})\to H^2(\partial C_2(M);\mathbb{R})$ since $H^3(C_2(M),\partial C_2(M);\mathbb{R})=0$. Explicit constructions of propagating chains associated to Morse functions or Heegaard diagrams can be found in [6].

Lemma 8. *Let $(\breve{M},\tau)$ be an asymptotic rational homology $\mathbb{R}^3$. Let C be a two-cycle of $C_2(M)$. For any propagating chain $\mathcal{P}$ of $(C_2(M),\tau)$ transverse to C and for any propagating form ω_p of $(C_2(M),\tau)$, $[C]=\int_C\omega_p[S]=\langle C,\mathcal{P}\rangle_{C_2(M)}[S]$ in $H_2(C_2(M);\mathbb{Q})=\mathbb{Q}[S]$. In particular, for any two-component link (J,K) of $\breve{M}$ $lk(J,K)=\int_{J\times K}\omega_p=\langle J\times K,\mathcal{P}\rangle_{C_2(M)}$.*

Proof. Fix a propagating chain $\mathcal{P}$, the algebraic intersection $\langle C,\mathcal{P}\rangle_{C_2(M)}$ only depends on the homology class $[C]$ of C in $C_2(M)$. Similarly, since ω_p is closed, $\int_C\omega_p$ only depends on $[C]$. (Indeed, if C and C' cobound a chain D, $C\cap\mathcal{P}$ and $C'\cap\mathcal{P}$ cobound $\pm(D\cap\mathcal{P})$, and $\int_{\partial D=C'-C}\omega_p=\int_D d\omega_p$ according to the Stokes theorem.) Furthermore, the dependance on $[C]$ is linear. Therefore it suffices to check the lemma for a cycle that represents the canonical generator $[S]$ of $H_2(C_2(M);\mathbb{Q})$. Any fiber of $U\breve{M}$ is such a cycle. □

2.4. The Θ-invariant of (M,τ)

Note that the intersection of transverse (oriented) submanifolds is an associative operation, so that $A\cap B\cap C$ is well defined. Furthermore, for a connected manifold N, the class of a 0-cycle in $H_0(M;\mathbb{Q})=\mathbb{Q}[m]=\mathbb{Q}$ is a well-defined number, so that the *algebraic intersection* of several transverse submanifolds whose codimension sum is the dimension of the ambient manifold is well defined as the homology class of their (oriented) intersection. This extends to rational chains, multilinearly. Thus, for three such transverse submanifolds A, B, C in a manifold D, their algebraic intersection $\langle A,B,C\rangle_D$ is the sum over the intersection points a of the associated signs, where the sign of a is positive if and only if the orientation of D is induced by the orientation of $\mathfrak{N}_aA\oplus\mathfrak{N}_aB\oplus\mathfrak{N}_aC$.

Theorem 1. *Let $(\breve{M},\tau)$ be an asymptotic rational homology $\mathbb{R}^3$. Let $\mathcal{P}_a,\mathcal{P}_b$ and $\mathcal{P}_c$ be three pairwise transverse propagators of $(C_2(M),\tau)$ with respective boundaries $p_\tau^{-1}(a)$, $p_\tau^{-1}(b)$ and $p_\tau^{-1}(c)$ for three distinct points a,b and c of S^2, then $\Theta(M,\tau)=\langle\mathcal{P}_a,\mathcal{P}_b,\mathcal{P}_c\rangle_{C_2(M)}$ does not depend on the chosen propagators $\mathcal{P}_a,\mathcal{P}_b$ and $\mathcal{P}_c$. It is a topological invariant of (M,τ). For any three propagating chains ω_a,ω_b and ω_c of $(C_2(M),\tau)$,*

$$\Theta(M,\tau)=\int_{C_2(M)}\omega_a\wedge\omega_b\wedge\omega_c.$$

Proof. Since $H_4(C_2(M))=0$, if the propagator $\mathcal{P}_a$ is changed to a propagator $\mathcal{P}_a'$ with the same boundary, $(\mathcal{P}_a'-\mathcal{P}_a)$ bounds a 5-dimensional chain W transverse to $\mathcal{P}_b\cap\mathcal{P}_c$. The 1-dimensional chain $W\cap\mathcal{P}_b\cap\mathcal{P}_c$ does not meet $\partial C_2(M)$ since $\mathcal{P}_b\cap\mathcal{P}_c$ does not meet $\partial C_2(M)$. Therefore, up to a well-determined sign, the boundary of $W\cap\mathcal{P}_b\cap\mathcal{P}_c$ is $\mathcal{P}_a'\cap\mathcal{P}_b\cap\mathcal{P}_c-\mathcal{P}_a\cap\mathcal{P}_b\cap\mathcal{P}_c$. This shows that $\langle\mathcal{P}_a,\mathcal{P}_b,\mathcal{P}_c\rangle_{C_2(M)}$ is independent of $\mathcal{P}_a$ when a is fixed. Similarly, it is independent of $\mathcal{P}_b$ and $\mathcal{P}_c$ when b and c are fixed. Thus, $\langle\mathcal{P}_a,\mathcal{P}_b,\mathcal{P}_c\rangle_{C_2(M)}$ is a rational function on the connected set of triples (a,b,c) of distinct point of S^2. It is easy to see that this function is continuous. Thus, it is constant.

Let us similarly prove that $\int_{C_2(M)}\omega_a \wedge \omega_b \wedge \omega_c$ is independent of the propagating forms ω_a, ω_b and ω_c. Assume that the form ω_a, which restricts to $\partial C_2(M)$ as $p_\tau^*(\omega_A)$, is changed to ω'_a, which restricts to $\partial C_2(M)$ as $p_\tau^*(\omega'_A)$.

Lemma 9. *There exists a one-form η_A on S^2 such that $\omega'_A = \omega_A + d\eta_A$. For any such η_A, there exists a one-form η on $C_2(M)$ such that $\omega'_a - \omega_a = d\eta$, and the restriction of η to $\partial C_2(M)$ is $p_\tau^*(\eta_A)$.*

Proof. Since ω_a and ω'_a are cohomologous, there exists a one-form η on $C_2(M)$ such that $\omega'_a = \omega_a + d\eta$. Similarly, since $\int_{S^2}\omega'_A = \int_{S^2}\omega_A$, there exists a one-form η_A on S^2 such that $\omega'_A = \omega_A + d\eta_A$. On $\partial C_2(M)$, $d(\eta - p_\tau^*(\eta_A)) = 0$. Thanks to the exact sequence

$$0 = H^1(C_2(M)) \to H^1(\partial C_2(M)) \to H^2(C_2(M), \partial C_2(M)) \cong H_4(C_2(M)) = 0,$$

$H^1(\partial C_2(M)) = 0$. Therefore, there exists a function f from $\partial C_2(M)$ to $\mathbb{R}$ such that $df = \eta - p(\tau)^*(\eta_A)$ on $\partial C_2(M)$. Extend f to a C^∞ map on $C_2(M)$ and change η into $(\eta - df)$. □

Then

$$\int_{C_2(M)}\omega'_a \wedge \omega_b \wedge \omega_c - \int_{C_2(M)}\omega_a \wedge \omega_b \wedge \omega_c = \int_{C_2(M)} d(\eta \wedge \omega_b \wedge \omega_c) =$$
$$= \int_{\partial C_2(M)} \eta \wedge \omega_b \wedge \omega_c = \int_{\partial C_2(M)} p(\tau)^*(\eta_A \wedge \omega_B \wedge \omega_C) = 0$$

since any 5-form on S^2 vanishes. Thus, $\int_{C_2(M)}\omega_a \wedge \omega_b \wedge \omega_c$ is independent of the propagating forms ω_a, ω_b and ω_c. Now, we can choose the propagating forms ω_a, ω_b and ω_c, Poincaré dual to $\mathcal{P}_a$, $\mathcal{P}_b$ and $\mathcal{P}_c$, and supported in very small neighborhoods of $\mathcal{P}_a$, $\mathcal{P}_b$ and $\mathcal{P}_c$, respectively, so that the intersection of the three supports is a very small neighborhood of $\mathcal{P}_a \cap \mathcal{P}_b \cap \mathcal{P}_c$, where it can easily be seen that $\int_{C_2(M)}\omega_a \wedge \omega_b \wedge \omega_c = \langle \mathcal{P}_a, \mathcal{P}_b, \mathcal{P}_c \rangle_{C_2(M)}$. □

In particular, $\Theta(M,\tau)$ reads $\int_{C_2(M)}\omega^3$ for any propagating chain ω of $(C_2(M),\tau)$. Since such a propagating chain represents the linking number, $\Theta(M,\tau)$ can be thought of as the *cube of the linking number with respect to* τ.

When τ varies continuously, $\Theta(M,\tau)$ varies continuously in $\mathbb{Q}$ so that $\Theta(M,\tau)$ is an invariant of the homotopy class of τ.

2.5. Parallelisations of 3-manifolds and Pontrjagin classes

In this subsection, M denotes a smooth, compact oriented 3-manifold with possible boundary ∂M. Recall that such a 3-manifold is parallelizable.

Let $GL^+(\mathbb{R}^3)$ denote the group of orientation-preserving linear isomorphisms of $\mathbb{R}^3$. Let $C^0\left((M,\partial M),(GL^+(\mathbb{R}^3),1)\right)$ denote the set of maps $g:(M,\partial M) \to (GL^+(\mathbb{R}^3),1)$ from M to $GL^+(\mathbb{R}^3)$ that send ∂M to the unit 1 of $GL^+(\mathbb{R}^3)$. Let $[(M,\partial M),(GL^+(\mathbb{R}^3),1)]$ denote the group of homotopy classes of such maps, with the group structure induced by the multiplication of maps, using the multiplication in $GL^+(\mathbb{R}^3)$. For a map g in $C^0\left((M,\partial M),(GL^+(\mathbb{R}^3),1)\right)$, set

$$\begin{array}{rccc} \psi_{\mathbb{R}}(g): & M \times \mathbb{R}^3 & \to & M \times \mathbb{R}^3, \\ & (x,y) & \mapsto & (x, g(x)(y)). \end{array}$$

Let $\tau_M : M \times \mathbb{R}^3 \to TM$ be a parallelization of M. Then any parallelization τ of M that coincides with τ_M on ∂M reads $\tau = \tau_M \circ \psi_{\mathbb{R}}(g)$ for some $g \in C^0\left((M,\partial M),(GL^+(\mathbb{R}^3),1)\right)$.

Thus, fixing τ_M identifies the set of homotopy classes of parallelizations of M fixed on ∂M with the group $[(M,\partial M),(GL^+(\mathbb{R}^3),1)]$. Since $GL^+(\mathbb{R}^3)$ deformation retracts onto the group $SO(3)$ of orientation-preserving linear isometries of $\mathbb{R}^3$, $[(M,\partial M),(GL^+(\mathbb{R}^3),1)]$ is isomorphic to $[(M,\partial M),(SO(3),1)]$.

See S^3 as $B^3/\partial B^3$ where B^3 is the standard ball of radius 2π of $\mathbb{R}^3$ seen as $([0,2\pi]\times S^2)/(0\sim\{0\}\times S^2)$. Let $\rho: B^3 \to SO(3)$ map $(\theta \in [0,2\pi], v \in S^2)$ to the rotation $\rho(\theta, v)$ with axis directed by v and with angle θ. This map induces the double covering $\tilde{\rho}: S^3 \to SO(3)$, which orients $SO(3)$ and which allows one to deduce the first three homotopy groups of $SO(3)$ from the ones of S^3. They are $\pi_1(SO(3))=\mathbb{Z}/2\mathbb{Z}$, $\pi_2(SO(3))=0$ and $\pi_3(SO(3))=\mathbb{Z}[\tilde{\rho}]$. For $v\in S^2$, $\pi_1(SO(3))$ is generated by the class of the loop that maps $\exp(i\theta)\in S^1$ to the rotation $\rho(\theta,v)$.

Note that a map g from $(M,\partial M)$ to $(SO(3),1)$ has a degree $\deg(g)$, which may be defined as the differential degree at a regular value (different from 1) of g. It can also be defined homologically, by $H_3(g)[M,\partial M]=\deg(g)[SO(3),1]$.

The following theorem is proved in Section 6.

Theorem 2. *For any smooth compact connected oriented 3-manifold M, the group $[(M,\partial M),(SO(3),1)]$ is abelian, and the degree* $\deg\colon[(M,\partial M),(SO(3),1)]\to\mathbb{Z}$ *is a group homomorphism, which induces an isomorphism* $\deg\colon[(M,\partial M),(SO(3),1)]\otimes_{\mathbb{Z}}\mathbb{Q}\to\mathbb{Q}$. *When $\partial M=\emptyset$, (resp. when $\partial M=S^2$), there exists a canonical map p_1 from the set of homotopy classes of parallelizations of M (resp. that coincide with τ_s near S^2) such that for any map g in $C^0\left((M,\partial M),(SO(3),1)\right)$, for any trivialization τ of TM*

$$p_1(\tau\circ\psi_{\mathbb{R}}(g))-p_1(\tau)=2\deg(g).$$

The definition of the map p_1 is given in Subsection 6.5, it involves relative Pontrjagin classes. When $\partial M=\emptyset$, the map p_1 coincides with the map h that is studied by Kirby and Melvin in [7] under the name *Hirzebruch defect.* See also [8. § 3.1].

Since $[(M,\partial M),(SO(3),1)]$ is abelian, the set of parallelizations of M that are fixed on ∂M is an affine space with translation group $[(M,\partial M),(SO(3),1)]$.

Recall that $\rho\colon B^3\to SO(3)$ maps $(\theta\in[0,2\pi],v\in S^2)$ to the rotation with axis directed by v and with angle θ. Let M be an oriented connected 3-manifold with possible boundary. For a ball B^3 embedded in M, let $\rho_M(B^3)\in C^0\left((M,\partial M),(SO(3),1)\right)$ be a (continuous) map that coincides with ρ on B^3 and that maps the complement of B^3 to the unit of $SO(3)$. The homotopy class of $\rho_M(B^3)$ is well-defined.

Lemma 10. $\deg(\rho_M(B^3))=2$.

Proof. Exercise. □

2.6. Defining a $\mathbb{Q}$-sphere invariant from Θ

Recall that an asymptotic rational homology $\mathbb{R}^3$ is a pair $(\breve{M},\tau)$ where $\breve{M}$ is 3-manifold that reads as the union over $]1,2]\times S^2$ of a rational homology ball B_M and the complement $\breve{B}_{1,\infty}$ of the unit ball of $\mathbb{R}^3$, and that is equipped with an asymptotically standard parallelization τ.

In this subsection, we prove the following proposition.

Proposition 3. *Let $(\breve{M},\tau)$ be an asymptotic rational homology $\mathbb{R}^3$. For any map g in $C^0\left((B_M,B_M\cap\breve{B}_{1,\infty}),(SO(3),1)\right)$ trivially extended to $\breve{M}$,*

$$\Theta(M,\tau\circ\psi_{\mathbb{R}}(g))-\Theta(M,\tau)=\frac{1}{2}\deg(g).$$

Theorem 2 allows us to derive the following corollary from Proposition 3.

Corollary 1. $\Theta(M)=\Theta(M,\tau)-\frac{1}{4}p_1(\tau)$ *is an invariant of $\mathbb{Q}$-spheres.*

Lemma 11. $\Theta(M,\tau\circ\psi_{\mathbb{R}}(g))-\Theta(M,\tau)$ *is independent of τ. Set*

$$\Theta'(g)=\Theta(M,\tau\circ\psi_{\mathbb{R}}(g))-\Theta(M,\tau).$$

Then Θ' is a homomorphism from $[(B_M,B_M\cap\breve{B}_{1,\infty}),(SO(3),1)]$ to $\mathbb{Q}$.

Proof. For $d=a$, b or c, the propagator $\mathcal{P}_d$ of $(C_2(M),\tau)$ can be assumed to be a product $[-1,0]\times p^{-1}_{\tau|UB_M}(d)$ on a collar $[-1,0]\times UB_M$ of UB_M in $C_2(M)$. Since $H_3([-1,0]\times UB_M;\mathbb{Q})=0$, $\left(\partial([-1,0]\times p^{-1}_{\tau|UB_M}(d))\setminus(0\times p^{-1}_{\tau|UB_M}(d))\right)\cup(0\times p^{-1}_{\tau\circ\psi_{\mathbb{R}}(g)|UB_M}(d))$ bounds a chain G_d.

The chains G_a, G_b and G_c can be assumed to be transverse. Construct the propagator $\mathcal{P}_d(g)$ of $(C_2(M),\tau\circ\psi_{\mathbb{R}}(g))$ from $\mathcal{P}_d$ by replacing $[-1,0]\times p^{-1}_{\tau|UB_M}(d)$ by G_d on $[-1,0]\times UB_M$. Then

$\Theta(M,\tau\circ\psi_{\mathbb{R}}(g))-\Theta(M,\tau)$ is equal to $\langle G_a,G_b,G_c\rangle_{[-1,0]\times UB_M}$. Using τ to identify UB_M with $B_M\times S^2$ allows us to see that $\Theta(M,\tau\circ\psi_{\mathbb{R}}(g))-\Theta(M,\tau)$ is independent of τ. Then it is easy to observe that Θ' is a homomorphism from $[(B_M,\partial B_M),(SO(3),1)]$ to $\mathbb{Q}$. □

According to Theorem 2 and to Lemma 10, it suffices to prove that $\Theta'(\rho_M(B^3))=1$ in order to prove Proposition 3. It is easy to see that $\Theta'(\rho_M(B^3))=\Theta'(\rho)$. Thus, we are left with the proof of the following lemma.

Lemma 12. $\Theta'(\rho)=1$.

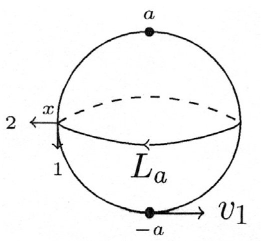
Again, see B^3 as $([0,2\pi]\times S^2)/(0\sim\{0\}\times S^2)$. We first prove the following lemma:

Lemma 13. *Let a be the North Pole. The point $(-a)$ is regular for the map*

$$\begin{array}{rccc}\rho_a: & B^3 & \to & S^2\\ & m & \mapsto & \rho(m)(a)\end{array}$$

and its preimage (cooriented by S^2 via ρ_a) is the knot $L_a=\{\pi\}\times E$, where E is the equator that bounds the Southern Hemisphere.

Proof. It is easy to see that $\rho_a^{-1}(-a)=\pm\{\pi\}\times E$.



Let $x\in\{\pi\}\times E$. When m moves along the great circle that contains a and x from x towards $(-a)$ in $\{\pi\}\times S^2$, $\rho(m)(a)$ moves from $(-a)$ in the same direction, which will be the direction of the tangent vector v_1 of S^2 at $(-a)$, counterclockwise in our picture, where x is on the left. Then in our picture, S^2 is oriented at $(-a)$ by ν_1 and by the tangent vector v_2 at $(-a)$ towards us. In order to move $\rho(\theta,v)(a)$ in the ν_2 direction, one increases θ so that L_a is cooriented and oriented like in the figure. □

Proof of Lemma 12. We use the notation of the proof of Lemma 11 and we construct an explicit G_a in $[-1,0]\times UB^3\overset{\tau_s}{=}[-1,0]\times B^3\times S^2$.

When $\rho(m)(a)\neq -a$, there is a unique geodesic arc $[a,\rho(m)(a)]$ with length ($\ell\in[0,\pi[$) from a to $\rho(m)(a)=\rho_a(m)$. For $t\in[0,1]$, let $X_t(m)\in[a,\rho_a(m)]$ be such that the length of $[X_0(m)=a,X_t(m)]$ is $t\ell$. This defines X_t on $(M\setminus L_a)$, $X_1(m)=\rho_a(m)$. Let us show how the definition of X_t smoothly extends on the manifold $B\ell(B^3,L_a)$ obtained from B^3 by blowing up L_a.

The map ρ_a maps the normal bundle to L_a to a disk of S^2 around $(-a)$, by an orientation-preserving diffeomorphism on every fiber (near the origin). In particular, ρ_a induces a map $\tilde\rho_a$ from the unit normal bundle to L_a to the unit normal bundle to $(-a)$ in S^2, which preserves the orientation of the fibers. Then for an element y of the unit normal bundle to L_a in M, define $X_t(y)$ as before on the half great circle $[a,-a]_{\tilde\rho_a(-y)}$ from a to $(-a)$ that is tangent to $\tilde\rho_a(-y)$ at $(-a)$ (so that $\tilde\rho_a(-y)$ is an outward normal to $[a,-a]_{\tilde\rho_a(-y)}$ at $(-a)$). This extends the definition of X_t, continuously.

The whole sphere is covered with degree (-1) by the image of $([0,1]\times U\mathfrak{V}_x(L_a))$, where the fiber $U\mathfrak{V}_x(L_a)$ of the unit normal bundle to L_a is oriented as the boundary of a disk in the fiber of the normal bundle. Let $G_h(a)$ be the closure of $\left(\cup_{t\in[0,1],m\in(B^3\setminus L_a)}(m,X_t(m))\right)$ in UB^3. $G_h(a)=\cup_{t\in[0,1],m\in B\ell(B^3,L_a)}\left(p_{B^3}(m),X_t(m)\right)$. Then

$$\partial G_h=-(B^3\times a)+\cup_{m\in B^3}(m,\rho_a(m))+\cup_{t\in[0,1]}X_t(-\partial B\ell(S^3,L_a))$$

where $(-\partial B\ell(S^3,L_a))$ is oriented like $\partial N(L_a)$ so that the last summand reads $(-L_a\times S^2)$ because the sphere is covered with degree (-1) by the image of $([0,1]\times U\mathfrak{V}_x(L_a))$.

Let D_a be a disk bounded by L_a in B^3. Set $G(a)=G_h(a)+D_a\times S^2$ so that

$$\partial G(a)=-(B^3\times a)+\cup_{m\in B^3}(m,\rho_a(m)).$$

Now let ι be the endomorphism of UB^3 over B^3 that maps a unit vector to the opposite one. Set $G_a = [-1,-2/3]\times B^3 \times a + \{-2/3\}\times G(a) + [-2/3,0]\times \cup_{m\in B^3}(m,\rho_a(m))$ and $G_{-a} = [-1,-1/3]\times B^3\times(-a) + \{-1/3\}\times \iota(G(a)) + [-1/3,0]\times \cup_{m\in B^3}(m,\rho(m)(-a))$. Then $G_a \cap G_{-a} = [-2/3,-1/3]\times L_a \times (-a) + \{-2/3\}\times D_a \times(-a) - \{-1/3\}\times \cup_{m\in D_a}(m,\rho_a(m))$. Finally, $\Theta'(\rho)$ is the algebraic intersection of $G_a \cap G_{-a}$ with $\mathcal{P}_c(\rho)$ in $C_2(M)$. This intersection coincides with the algebraic intersection of $G_a \cap G_{-a}$ with any propagator of $(C_2(M),\tau)$ according to Lemma 8. Therefore $\Theta'(\rho) = \langle \mathcal{P}_a, G_a \cap G_{-a}\rangle_{[-1,0]\times S^2\times B^3} = -\deg_a(\rho_a : D_a \to S^2)$. The orientation of L_a allows us to choose $(-D_a)$ as the Northern Hemisphere, the image of this hemisphere under ρ_a covers the sphere with degree 1 so that $\Theta'(\rho)=1$. □

3. An introduction to finite type invariants

This section contains the needed background from the theory of finite type invariants. It allows us to introduce the target space generated by Feynman–Jacobi diagrams, for the general invariants presented in Section 4, in a progressive way.

Theories of finite type invariants are useful to characterize invariants. Such a theory allowed Greg Kuperberg and Dylan Thurston to identify $\Theta/6$ with the Casson invariant λ for integer homology 3-spheres, in [9]. The invariant λ was defined by Casson in 1984 as an algebraic count of conjugacy classes of irreducible representations from $\pi_1(M)$ to $SU(2)$. See [10–12]. The Kuperberg—Thurston result above was generalized to the case of rational homology 3-spheres in [13; Theorem 2.6 and Corollary 6.14]. Thus, for any rational homology 3-sphere M, $\Theta(M) = 6\lambda(M)$, where λ is the Walker generalization of the Casson invariant to rational homology 3-spheres, which is normalized like in [10–12] for integer homology 3-spheres, and like $\frac{1}{2}\lambda_W$ for rational homology 3-spheres with respect to the Walker normalisation λ_W of [14].

For invariants of knots and links in $\mathbb{R}^3$, the base of the theory of finite type invariants was mainly established by Bar-Natan in [15]. A more complete review of this theory has been written by Chmutov, Duzhin and Mostovoy in [16]. For integer homology 3-spheres, the theory was started by Ohtsuki in [17] and further developed by Goussarov, Habiro, Le and others. See [18–20]. Delphine Moussard developed a theory of finite type invariants for rational homology 3-spheres in [21]. Her suitable theory is based on the Lagrangian-preserving surgeries defined below.

3.1. Lagrangian-preserving surgeries

Definition 2. An *integer (resp. rational) homology handlebody* of genus g is a compact oriented 3-manifold A that has the same integral (resp. rational) homology as the usual solid handlebody $\mathcal{H}_g$ of Fig. 1.

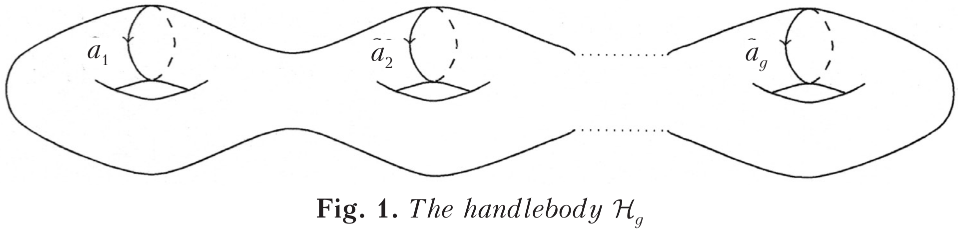


Fig. 1. *The handlebody* $\mathcal{H}_g$

Exercise 1. Show that if A is a rational homology handlebody of genus g, then ∂A is a genus g surface.

The *Lagrangian* $\mathcal{L}_A$ of a compact 3-manifold A is the kernel of the map induced by the inclusion from $H_1(\partial A;\mathbb{Q})$ to $H_1(\partial A;\mathbb{Q})$.

In Fig. 1, the Lagrangian of $\mathcal{H}_g$ is freely generated by the classes of the curves a_i.

Definition 3. An *integral (resp. rational) Lagrangian-Preserving (or LP) surgery (A'/A)* is the replacement of an integral (resp. rational) homology handlebody A embedded in the interior of a 3-manifold M by another such A' whose boundary is identified with ∂A by an orientation-preserving diffeomorphism that sends $\mathcal{L}_A$ to $\mathcal{L}_{A'}$. The manifold $M(A'/A)$ obtained by such an LP-surgery reads $M(A'/A) = (M \setminus \mathrm{Int}(A)) \cup_{\partial A} A'$. (This only defines the topological structure of $M(A'/A)$, but we equip $M(A'/A)$ with its unique smooth structure.)

Lemma 14. *If (A'/A) is an integral (resp. rational) LP-surgery, then the homology of $M(A'/A)$ with $\mathbb{Z}$-coefficients (resp. with $\mathbb{Q}$-coefficients) is canonically isomorphic to $H_*(M;\mathbb{Z})$ (resp. to $H_*(M;\mathbb{Q})$). If M is a $\mathbb{Q}$-sphere, if (A'/A) is a rational LP-surgery, and if (J,K) is a two-component link of $M \setminus A$, then the linking number of J and K in M and the linking number of J and K in $M(A'/A)$ coincide.*

Proof. Exercise. □

3.2. Definition of finite type invariants

Let $\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$. A $\mathbb{K}$-valued *invariant* of oriented 3-manifolds is a function from the set of 3-manifolds, considered up to orientation-preserving diffeomorphisms to $\mathbb{K}$. Let $\coprod_{i=1}^n S_i^1$ denote a disjoint union of n circles, where each S_i^1 is a copy of S^1. Here, an *n-component link* in a 3-manifold M is an equivalence class of smooth embeddings $L: \coprod_{i=1}^n S_i^1 \hookrightarrow M$ under the equivalence relation that identifies two embeddings L and L' if and only if there is an orientation-preserving diffeomorphism h of M such that $h(L) = L'$. A *knot* is a one-component link. A *link invariant* (resp. a *knot invariant*) is a function of links (resp. knots). For example, Θ is an invariant of $\mathbb{Q}$-spheres and the linking number is a rational invariant of two-component links in rational homology 3-spheres.

In order to study a function, it is usual to study its derivative, and the derivative of its derivative The derivative of a function is defined from its variations. For a function f from $\mathbb{Z}^d = \oplus_{i=1}^d \mathbb{Z}e_i$ to $\mathbb{K}$, one can define its first order derivatives $\frac{\partial f}{\partial e_i}: \mathbb{Z}^d \to \mathbb{K}$ by

$$\frac{\partial f}{\partial e_i}(z) = f(z+e_i) - f(z)$$

and check that all the first order derivatives of f vanish if and only if f is constant. Inductively define an n-order derivative as a first order derivative of an $(n-1)$-order derivative for a positive integer n. Then it can be checked that all the $(n+1)$-order derivatives of a function vanish if and only if f is a polynomial of degree not greater than n. In order to study topological invariants, we can similarly study their variations under *simple operations.*

Below, X denotes one of the following sets:

- $\mathbb{Z}^d$;
- the set $\mathcal{K}$ of knots in $\mathbb{R}^3$, the set $\mathcal{K}_n$ of n-component links in $\mathbb{R}^3$;
- the set $\mathcal{M}$ of $\mathbb{Z}$-spheres, the set $\mathcal{M}_Q$ of $\mathbb{Q}$-spheres.

And $\mathcal{O}(X)$ denotes a set of *simple operations* acting on some elements of X.

For $X = \mathbb{Z}^d$, $\mathcal{O}(X)$ will be made of the operations $(z \to z \pm e_i)$.

For knots or links in $\mathbb{R}^3$, the *simple operations* will be *crossing changes.* A *crossing change ball* of a link L is a ball B of the ambient space, where $L \cap B$ is a disjoint union of two arcs α_1 and α_2 properly embedded in B, and there exist two disjoint topological disks D_1 and D_2 embedded in B, such that, for $i \in \{1,2\}$, $\alpha_i \subset \partial D_i$ and $(\partial D_i \setminus \alpha_i) \subset \partial B$. After an isotopy, the projection of (B,α_1,α_2) reads  or  (the corresponding pairs (ball,arcs) are isomorphic, but they are regarded in different ways), a *crossing change* is a change that does not change L outside B and that modifies L inside B by a local move ($\rightarrow$) or ($\rightarrow$). For the move ($\rightarrow$), the crossing change is *positive*, it is ($\rightarrow$) *negative* for the move ($\rightarrow$).

For integer (resp. rational) homology 3-spheres, the simple operations will be integral (resp. rational) *LP-surgeries of genus* 3.

Say that crossing changes are *disjoint* if they sit inside disjoint 3-balls. Say that LP-surgeries (A'/A) and (B'/B) in a manifold M are *disjoint* if A and B are disjoint in M. Two operations on $\mathbb{Z}^d$ are always *disjoint* (even if they look identical). In particular, disjoint operations commute, (their result does not depend on which one is performed first). Let $\underline{n} = \{1,2,\ldots,n\}$. Consider the vector space $\mathcal{F}_0(X)$ freely generated by X over $\mathbb{K}$. For an element x of X and n pairwise disjoint operations $o_1,\ldots,o_n$ acting on x, define

$$[x;o_1,\ldots,o_n] = \sum_{I\subseteq \underline{n}} (-1)^{\# I} x((o_i)_{i\in I}) \in \mathcal{F}_0(X)$$

where $x((o_i)_{i\in I})$ denotes the element of X obtained by performing the operations o_i for $i \in I$ on x. Then define $\mathcal{F}_n(X)$ as the $\mathbb{K}$-subspace of $\mathcal{F}_0(X)$ generated by the $[x;o_1,\ldots,o_n]$, for all $x\in X$ equipped with n pairwise disjoint simple operations. Since

$$[x;o_1,\ldots,o_n,o_{n+1}] = [x;o_1,\ldots,o_n] - [x(o_{n+1});o_1,\ldots,o_n],$$

$\mathcal{F}_{n+1}(X) \subseteq \mathcal{F}_n(X)$, for all $n\in\mathbb{N}$.

Definition 4. A $\mathbb{K}$-valued function f on X, uniquely extends as a $\mathbb{K}$-linear map of

$$\mathcal{F}_0(X)^* = Hom(\mathcal{F}_0(X);\mathbb{K}),$$

which is still denoted by f. For an integer $n\in\mathbb{N}$, the invariant (or function) f is of *degree* $\leq n$ if and only if $f(\mathcal{F}_{n+1}(X))=0$. The *degree* of such an invariant is the smallest integer $n\in\mathbb{N}$ such that $f(\mathcal{F}_{n+1}(X))=0$. An invariant is of *finite type* if it is of degree n for some $n\in\mathbb{N}$. This definition depends on the chosen set of operations $\mathcal{O}(X)$. We fixed our choices for our sets X, but other choices could lead to different notions. See [18].

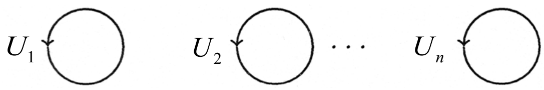
Let $\mathcal{I}_n(X) = (\mathcal{F}_0(X)/\mathcal{F}_{n+1}(X))^*$ be the space of invariants of degree at most n. Of course, for all $n\in\mathbb{N}$, $\mathcal{I}_n(X)\subseteq \mathcal{I}_{n+1}(X)$.

Example 2. $\mathcal{I}_n(\mathbb{Z}^d)$ is the space of polynomials of degree at most n on $\mathbb{Z}^d$. (Exercise.)

Lemma 15. *Any n-component link in* $\mathbb{R}^3$ *can be transformed to the trivial n-component link below by a finite number of disjoint crossing changes.*

Proof. Let L be an n-component link in $\mathbb{R}^3$. Since $\mathbb{R}^3$ is simply connected, there is a homotopy that carries L to the trivial link. Such a homotopy $h:[0,1]\times\coprod_{i=1}^{n} S^1_i \to \mathbb{R}^3$ can be chosen, so that $h(t,\cdot)$ is an embedding except for finitely many times t_i, $0<t_1<\ldots<t_i<t_{i+1}<t_k<1$ where $h(t_i,\cdot)$ is an immersion with one double point and no other multiple points, and the link $h(t,\cdot)$ changes exactly by a crossing change when t crosses a t_i. (For an alternative elementary proof of this lemma, see [22. Subsection 7.1].) □

In particular, a degree 0 invariant of n-component links of $\mathbb{R}^3$ must be constant, since it is not allowed to vary under a crossing change.



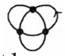
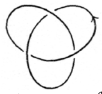
Exercise 2. 1. Check that $\mathcal{I}_1(\mathcal{K}) = \mathbb{K}c_0$, where c_0 is the constant map that maps any knot to 1.

2. Check that the linking number is a degree 1 invariant of 2-component links of $\mathbb{R}^3$.

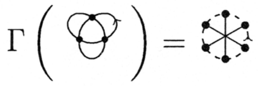
3. Check that $\mathcal{I}_1(\mathcal{K}_2) = \mathbb{K}c_0 \oplus \mathbb{K}lk$, where c_0 is the constant map that maps any two-component link to 1.

3.3. Introduction to chord diagrams

Let f be a knot invariant of degree at most n. We want to evaluate $f([K;o_1,\ldots,o_n])$ where the o_i are disjoint negative crossing changes

to be performed on a knot K. Such a $[K;o_1,\ldots,o_n]$ is usually represented as a *singular knot with n double points* that is an immersion of a circle with n transverse double points , where each double point  can be desingularized in two ways, the positive one  and the negative one  and K is obtained from the singular knot by desingularizing all the crossings in the positive way, which is  in our example. Note that the sign of the desingularization is defined from the orientation of the ambient space.

Define the *chord diagram* $\Gamma([K;o_1,\ldots,o_n])$ associated to $[K;o_1,\ldots,o_n]$ as follows. Draw the preimage of the associated singular knot with n double points as an oriented dashed circle equipped with the $2n$ preimages of the double points and join the pairs of preimages of a double point by a plain segment called a *chord*.

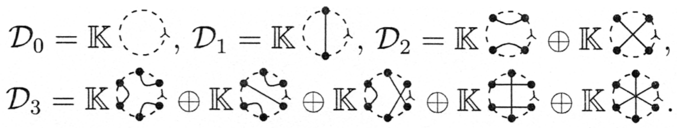


Formally, a *chord diagram* with n chords is a cyclic order of the $2n$ ends of the n chords, up to a permutation of the chords and up to exchanging the two ends of a chord.

Lemma 16. *When f is a knot invariant of degree at most n,* $f([K;o_1,\ldots,o_n])$ *only depends on* $\Gamma([K;o_1,\ldots,o_n])$.

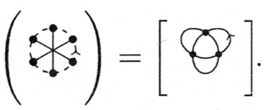
Proof. Since f is of degree n, $f([K;o_1,\ldots,o_n])$ is invariant under a crossing change outside the balls of the o_i, that is outside the double points of the associated singular knot. Therefore, $f([K;o_1,\ldots,o_n])$ only depends on the cyclic order of the $2n$ arcs involved in the o_i on K. □

Let $\mathcal{D}_n$ be the $\mathbb{K}$-vector space freely generated by the n chord diagrams on S^1. Then



Lemma 17. *The map* ϕ_n *from* $\mathcal{D}_n$ *to* $\dfrac{\mathcal{F}_n(\mathcal{K})}{\mathcal{F}_{n+1}(\mathcal{K})}$ *that maps* Γ *to some* $[K;o_1,\ldots,o_n]$ *whose diagram is* Γ *is well-defined and surjective.*

Proof. Use the arguments of the proof of Lemma 16. □

For example, ϕ_3 .

The kernel of the composition of ϕ_n^* and the restriction below

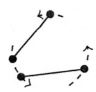
$$\mathcal{I}_n(\mathcal{K}) = \left(\frac{\mathcal{F}_0(\mathcal{K})}{\mathcal{F}_{n+1}(\mathcal{K})}\right)^* \to \left(\frac{\mathcal{F}_n(\mathcal{K})}{\mathcal{F}_{n+1}(\mathcal{K})}\right)^* \overset{\phi_n^*}{\to} \mathcal{D}_n^*$$

is $\mathcal{I}_{n-1}(\mathcal{K})$. Thus, $\dfrac{\mathcal{I}_n(\mathcal{K})}{\mathcal{I}_{n-1}(\mathcal{K})}$ injects into $\mathcal{D}_n^*$ and $\mathcal{I}_n(\mathcal{K})$ is finite dimensional for all n. Furthermore,

$$\frac{\mathcal{I}_n(\mathcal{K})}{\mathcal{I}_{n-1}(\mathcal{K})} = \mathrm{Hom}\left(\frac{\mathcal{F}_n(\mathcal{K})}{\mathcal{F}_{n+1}(\mathcal{K})};\mathbb{K}\right).$$

An *isolated chord* in a chord diagram is a chord between two points of S^1 that are consecutive on the circle.

Lemma 18. *Let D be a diagram on* S^1 *that contains an isolated chord. Then* $\phi_n(D) = 0$. *Let* D^1, D^2, D^3, D^4 *be four n-chord diagrams that are identical outside three portions of circles where they look like:*

$D^1 =$  , $D^2 =$  , $D^3 =$  *and* $D^4 =$  ,

then $\phi_n(-D^1 + D^2 + D^3 - D^4) = 0$.

Proof. For the first assertion, observe that $\phi_n($  $) = [$  $] - [$  $]$. For the second one, see [22. Lemma 2.21], for example. □

Let $\mathcal{D}_n$ denote the quotient of $\mathcal{D}_n$ by the *four-term relation,* which is the quotient of $\mathcal{D}_n$ by the vector space generated by the $(-D^1 + D^2 + D^3 - D^4)$ for all the 4-tuples (D^1, D^2, D^3, D^4) as above. Call $(1T)$ the relation that identifies a diagram with an isolated chord with 0 so that $\mathcal{D}_n/(1T)$ is the quotient of $\mathcal{D}_n$ by the vector space generated by diagrams with an isolated chord.

According to Lemma 18 above, the map ϕ_n induces a map

$$\bar{\phi}_n : \mathcal{D}_n/(1T) \to \frac{\mathcal{F}_n(\mathcal{K})}{\mathcal{F}_{n+1}(\mathcal{K})}.$$

The fundamental theorem of *Vassiliev invariants* (which are finite type knot invariants) can now be stated.

Theorem 3. *There exists a family of linear maps* $\left(Z_n^K : \mathcal{F}_0(\mathcal{K}) \to \mathcal{D}_n\right)_{n\in\mathbb{N}}$ *such that:*

- $Z_n^K(\mathcal{F}_{n+1}(\mathcal{K})) = 0$;
- Z_n^K *induces the inverse of* $\bar{\phi}_n$ *from* $\frac{\mathcal{F}_n(\mathcal{K})}{\mathcal{F}_{n+1}(\mathcal{K})}$ *to* $\mathcal{D}_n/(1T)$;

In particular $\frac{\mathcal{F}_n(\mathcal{K})}{\mathcal{F}_{n+1}(\mathcal{K})} \cong \mathcal{D}_n/(1T)$ *and* $\frac{\mathcal{I}_n(\mathcal{K})}{\mathcal{I}_{n-1}(\mathcal{K})} \cong (\mathcal{D}_n/(1T))^*$.

This theorem has been proved by Kontsevich and Bar-Natan in [15] using the *Kontsevich integral* $Z^K = (Z_n^K)_{n\in\mathbb{N}}$ described in [23] and in [16. Chapter 8], for $\mathbb{K} = \mathbb{R}$. It is also true when $\mathbb{K} = \mathbb{Q}$.

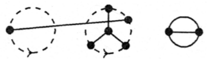
Remark 1. The Kontsevich integral has been generalized to a functor from the category of framed tangles to a category of Jacobi diagrams by Le and Murakami in [24]. Le and Murakami showed how to derive the i. e.Turaev quantum invariants of framed links in $\mathbb{R}^3$ defined in [25; 26] from their functor, in [24. Theorem 10].

3.4. More spaces of diagrams

Definition 5. A *uni-trivalent graph* Γ is a 6-tuple $(H(\Gamma), E(\Gamma), U(\Gamma), T(\Gamma), p_E, p_V)$ where $H(\Gamma)$, $E(\Gamma)$, $U(\Gamma)$ and $T(\Gamma)$ are finite sets, which are called the set of half-edges of Γ, the set of edges of Γ, the set of univalent vertices of Γ and the set of trivalent vertices of Γ, respectively, $p_E : H(\Gamma) \to E(\Gamma)$ is a two-to-one map (every element of $E(\Gamma)$ has two preimages under p_E) and $p_V : H(\Gamma) \to U(\Gamma) \coprod T(\Gamma)$ is a map such that every element of $U(\Gamma)$ has one preimage under p_V and every element of $T(\Gamma)$ has three preimages under p_V, up to isomorphism. In other words, Γ is a set $H(\Gamma)$ equipped with two partitions, a partition into pairs (induced by p_E), and a partition into singletons and triples (induced by p_V), up to the bijections that preserve the partitions. These bijections are the *automorphisms* of Γ.

Definition 6. Let C be an oriented one-manifold. A *Jacobi diagram* Γ *with support* C, also called Jacobi diagram on C, is a finite uni-trivalent graph Γ equipped with an isotopy class of injections i_Γ of the set $U(\Gamma)$ of univalent vertices of Γ into the interior of C. A *vertex-orientation* of a Jacobi diagram Γ is an *orientation* of every trivalent vertex of Γ, which is a cyclic order on the set of the three half-edges which meet at this vertex. A Jacobi diagram is *oriented* if it is equipped with a vertex-orientation.

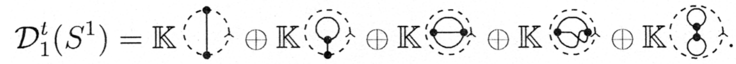
Such an oriented Jacobi diagram Γ is represented by a planar immersion of $\Gamma \cup C$ where the univalent vertices of $U(\Gamma)$ are located at their images under i_Γ, the one-manifold C is represented by dashed lines, whereas the diagram Γ is plain. The vertices are represented by big points. The local orientation of a vertex is represented by the counterclockwise order of the three half-edges that meet at it.

Here is an example of a picture of a Jacobi diagram Γ on the disjoint union $M = S^1 \coprod S^1$ of two circles: .

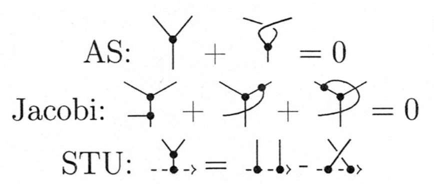
The *degree* of such a diagram is half the number of all the vertices of Γ.

Of course, a chord diagram of $\mathcal{D}_n$ is a degree n Jacobi diagram on S^1 without trivalent vertices.

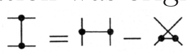
Let $\mathcal{D}_n^t(C)$ denote the $\mathbb{K}$-vector space generated by the degree n oriented Jacobi diagrams on C.



Let $\mathcal{D}_n^t(C)$ denote the quotient of $\mathcal{D}_n^t(C)$ by the following relations AS, Jacobi and STU:



As before, each of these relations relate oriented Jacobi diagrams which are identical outside the pictures where they are like in the pictures.

Remark 2. Lie algebras provide nontrivial linear maps, called *weight systems* from $\mathcal{D}_n^t(C)$ to $\mathbb{K}$, see [15] and [22. Section 6]. In the weight system constructions, the Jacobi relation for the Lie bracket ensures that the maps defined for oriented Jacobi diagrams factor through the Jacobi relation. In [27], Pierre Vogel proved that the maps associated to Lie (super)algebras are sufficient to detect nontrivial elements of $\mathcal{D}_n^t(C)$ until degree 15, and he exhibited a non trivial element of $\mathcal{D}_{16}^t(\varnothing)$ that cannot be detected by such maps. The Jacobi relation was originally called IHX by Bar-Natan in [15] because, up to AS, it can be written as .

Set $\mathcal{D}_n(\varnothing) = \mathcal{D}_n(\varnothing;\mathbb{K}) = \mathcal{D}_n^t(\varnothing)$.

When $C \neq \varnothing$, let $\mathcal{D}_n(C) = \mathcal{D}_n(C;\mathbb{K})$ denote the quotient of $\mathcal{D}_n^t(C) = \mathcal{D}_n^t(C;\mathbb{K})$ by the vector space generated by the diagrams that have at least one connected component without univalent vertices. Then $\mathcal{D}_n(C)$ is generated by the oriented Jacobi diagrams whose (plain) connected components contain at least one univalent vertex.

Proposition 4. *The natural map from $\mathcal{D}_n$ to $\mathcal{D}_n(S^1)$ induces an isomorphism from $\mathcal{D}_n$ to $\mathcal{D}_n(S^1)$.*

Sketch of proof. The natural map from $\mathcal{D}_n$ to $\mathcal{D}_n(S^1)$ factors though $4T$ since, according to STU,

in $\mathcal{D}_n^t(S^1)$. Since STU allows us to inductively write any oriented Jacobi diagram whose connected components contain at least a univalent vertex as a combination of chord diagrams, the induced map from $\mathcal{D}_n$ to $\mathcal{D}_n(S^1)$ is surjective. In order to prove injectivity, one constructs an inverse map. See [22. Subsection 3.4]. □

The Le fundamental theorem on *finite type invariants of $\mathbb{Z}$-spheres* is the following one.

Theorem 4. *There exists a family* $\left(Z_n^{LMO} : \mathcal{F}_0(\mathcal{M}) \to \mathcal{D}_n(\varnothing)\right)_{n\in\mathbb{N}}$ *of linear maps such that:*

- $Z_n^{LMO}(\mathcal{F}_{2n+1}(\mathcal{M})) = 0$;
- Z_n^{LMO} *induces an isomorphism from* $\dfrac{\mathcal{F}_{2n}(\mathcal{M})}{\mathcal{F}_{2n+1}(\mathcal{M})}$ *to* $\mathcal{D}_n(\varnothing)$;

- $\dfrac{\mathcal{F}_{2n-1}(\mathcal{M})}{\mathcal{F}_{2n}(\mathcal{M})} = \{0\}.$

In particular $\dfrac{\mathcal{F}_{2n}(\mathcal{M})}{\mathcal{F}_{2n+1}(\mathcal{M})} \cong \mathcal{D}_n(\varnothing)$ *and* $\dfrac{\mathcal{I}_{2n}(\mathcal{M})}{\mathcal{I}_{2n-1}(\mathcal{M})} \cong \mathcal{D}_n^*(\varnothing)$.

This theorem has been proved by Le [20] using the Le–Murakami–Ohtsuki invariant $Z^{LMO} = (Z_n^{LMO})_{n\in\mathbb{N}}$ of [28]. As explained in [29], this LMO invariant contains the quantum Witten–Reshetikhin-invariants of rational homology 3-spheres defined in [30].

In [21], Delphine Moussard obtained a similar fundamental theorem for *finite type invariants of* $\mathbb{Q}$*-spheres* using the configuration space integral Z_{KKT} described in [9; 31] and in Theorem 5 below.

As in the knot case, the hardest part of these theorems is the construction of an invariant $Z = (Z_n)_{n\in\mathbb{N}}$ that has the required properties. We will define such an invariant by "counting Jacobi diagram configurations" in Subsection 4.3 and explain why it satisfies the required so-called universality properties in Subsection 4.4.

3.5. Multiplying diagrams

Set $\mathcal{D}^t(C) = \prod_{n\in\mathbb{N}} \mathcal{D}_n^t(C)$ and $\mathcal{D}(C) = \prod_{n\in\mathbb{N}} \mathcal{D}_n(C)$.

Assume that a one-manifold C is decomposed as a union of two one-manifolds $C = C_1 \cup C_2$ whose interiors in C do not intersect. Define the *product associated to this decomposition*:

$$\mathcal{D}^t(C_1) \times \mathcal{D}^t(C_2) \to \mathcal{D}^t(C)$$

as the continuous bilinear map which maps $([\Gamma_1],[\Gamma_2])$ to $[\Gamma_1 \coprod \Gamma_2]$, if Γ_1 is a diagram with support C_1 and if Γ_2 is a diagram with support C_2, where $\Gamma_1 \coprod \Gamma_2$ denotes their disjoint union.

In particular, the disjoint union of diagrams turns $\mathcal{D}(\varnothing)$ into a commutative algebra graded by the degree, and it turns $\mathcal{D}^t(C)$ into a $\mathcal{D}(\varnothing)$-module, for any 1-dimensional manifold C.

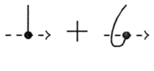
An orientation-preserving diffeomorphism from a manifold C to another one C' induces an isomorphism from $\mathcal{D}_n(C)$ to $\mathcal{D}_n(C')$, for all n.

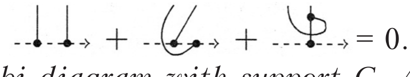
Let $I = [0,1]$ be the compact oriented interval. If $I = C$, and if we identify I with $C_1 = [0,1/2]$ and with $C_2 = [1/2,1]$ with respect to the orientation, then the above process turns $\mathcal{D}(I)$ into an algebra where the elements with non-zero degree zero part admit an inverse.

Proposition 5. *The algebra* $\mathcal{D}([0,1])$ *is commutative. The projection from* $[0,1]$ *to* $S^1 = [0,1]/(0 \sim 1)$ *induces an isomorphism from* $\mathcal{D}_n([0,1])$ *to* $\mathcal{D}_n(S^1)$ *for all n, so that* $\mathcal{D}(S^1)$ *inherits a commutative algebra structure from this isomorphism. The choice of a connected component* C_j *of C equips* $\mathcal{D}(C)$ *with an* $\mathcal{D}([0,1])$*-module structure* $\#_j$*, induced by the inclusion from* $[0,1]$ *to a little part of* C_j *outside the vertices, and the insertion of diagrams with support* $[0,1]$ *there.*

In order to prove this proposition, we present a useful trick in diagram spaces.

First adopt a convention. So far, in a diagram picture, or in a chord diagram picture, the plain edge of a univalent vertex, has always been attached on the left-hand side of the oriented one-manifold. Now, if k plain edges are attached on the other side on a diagram picture, then we agree that the corresponding represented element of $\mathcal{D}_n^t(M)$ is $(-1)^k$ times the underlying diagram. With this convention, we have the new antisymmetry relation in $\mathcal{D}_n^t(M)$:

 = 0, and we can draw the STU relation like the Jacobi relation:



Lemma 19. *Let* Γ_1 *be a Jacobi diagram with support C. Assume that* $\Gamma_1 \cup C$ *is immersed in the plane so that* $\Gamma_1 \cup C$ *meets an open annulus A embedded in the plane exactly along* $n+1$ *embedded arcs* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *and* β*, and one vertex* ν *so that:*

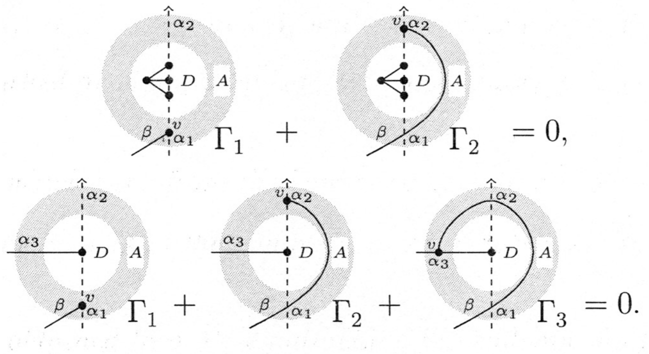
1) the α_i may be dashed or plain, they run from a boundary component of A to the other one;

2) β is a plain arc which runs from the boundary of A to $v \in \alpha_1$;

3) the bounded component D of the complement of A does not contain a boundary point of C.

Let Γ_1 be the diagram obtained from Γ_1 by attaching the endpoint v of β to α_i instead of α_1 on the same side, where the side of an arc is its side when going from the outside boundary component of A to the inside one ∂D. Then $\sum_{i=1}^{n} \Gamma_i = 0$ in $\mathcal{D}^t(C)$.

Examples 3.

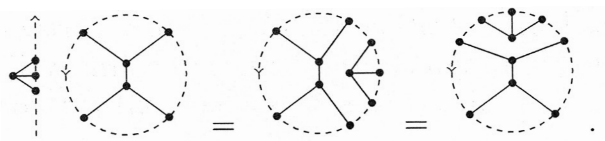


Proof. The second example shows that the STU relation is equivalent to this relation when the bounded component D of $\mathbb{R}^2 \setminus A$ intersects Γ_1 in the neighborhood of a univalent vertex on C. Similarly, the Jacobi relation is easily seen as given by this relation when D intersects Γ_1 in the neighborhood of a trivalent vertex. Also note that AS corresponds to the case when D intersects Γ_1 along a dashed or plain arc. Now for the Bar-Natan [15. Lemma 3.1] proof. See also [27. Lemma 3.3]. Assume without loss that v is always attached on the right-hand-side of the α's. Add to the sum the trivial (by Jacobi and STU) contribution of the sum of the diagrams obtained from Γ_1 by attaching v to each of the three (dashed or plain) half-edges of each vertex w of $\Gamma_1 \cup C$ in D on the left-hand side when the half-edges are oriented towards w. Now, group the terms of the obtained sum by edges of $\Gamma_1 \cup C$ where v is attached, and observe that the sum is zero edge by edge by AS.

Proof of Proposition 5. To each choice of a connected component C_j of C, we associate an *$\mathcal{D}(I)$-module structure $\#_j$ on $\mathcal{D}(C)$*, which is given by the continuous bilinear map:

$$\mathcal{D}(I) \times \mathcal{D}(C) \to \mathcal{D}(C)$$

such that: if Γ' is a diagram with support C and if Γ is a diagram with support I, then $([\Gamma],[\Gamma'])$ is mapped to the class of the diagram obtained by inserting Γ along C_j outside the vertices of Γ, according to the given orientation. For example,



As shown in the first example that illustrates Lemma 19, the independence of the choice of the insertion locus is a consequence of Lemma 19, where Γ_1 is the disjoint union $\Gamma \coprod \Gamma'$ and Γ_1 intersects D along $\Gamma \cup I$. This also proves that $\mathcal{D}(I)$ is a commutative algebra. Since the morphism from $\mathcal{D}(I)$ to $\mathcal{D}(S^1)$ induced by the identification of the two endpoints of I amounts to quotient out $\mathcal{D}(I)$ by the relation that identifies two diagrams that are obtained from one another by moving the nearest univalent vertex to an endpoint of I near the other endpoint, a similar application of Lemma 19 also proves that this morphism is an isomorphism from $\mathcal{D}(I)$ to $\mathcal{D}(S^1)$. (In this application, β comes from the inside boundary of the annulus.) □

4. Configuration space construction of universal finite type invariants

In this section, we finally describe the promised invariants, which generalize both the linking number and Θ. These invariants count configurations of Jacobi diagrams with support some link, in an asymptotic rational homology $\mathbb{R}^3$. In Subsection 4.1, we introduce the relevant configuration spaces. In Subsection 4.2, we define integrals over these spaces from propagating forms. The wanted invariants are obtained by combining these integrals in Subsection 4.3. These integrals will be expressed in terms of algebraic intersections, which involve propagating chains, in Subsection 5.3. Important universality properties of the constructed invariants are presented in Subsection 4.4.

4.1. Configuration spaces of links in 3-manifolds

Let $(\breve{M},\tau)$ be an asymptotic rational homology $\mathbb{R}^3$. Let C be a disjoint union of k circles S^1_i, $i\in\underline{k}$, and let $L:C\to\breve{M}$ denote a C^∞ embedding from C to $\breve{M}$. Let Γ be a Jacobi diagram with support C. Let $U=U(\Gamma)$ denote the set of univalent vertices of Γ, and let $T=T(\Gamma)$ denote the set of trivalent vertices of Γ. A *configuration* of Γ is an embedding c: $U\cup T\hookrightarrow\breve{M}$ whose restriction $c_{|U}$ to U may be written as $L\circ j$ for some injection $j:U\hookrightarrow C$ in the given isotopy class $[i_\Gamma]$ of embeddings of U into the interior of C. Let $\breve{C}(L;\Gamma)=\left\{c:U\cup T\rightarrowtail\breve{M};\exists j\in[i_\Gamma],c_{|U}=L\circ j\right\}$ denote the set of these configurations. In $\breve{C}(L;\Gamma)$, the univalent vertices move along $L(C)$ while the trivalent vertices move in the ambient space, and $\breve{C}(L;\Gamma)$ is naturally an open submanifold of $C^U\times\breve{M}^T$.

An *orientation* of a set of cardinality at least 2 is a total order of its elements up to an even permutation.

Cut each edge of Γ into two half-edges. When an edge is oriented, define its *first* half-edge and its *second* one, so that following the orientation of the edge, the first half-edge is met first. Recall that $H(\Gamma)$ denotes the set of half-edges of Γ.

Lemma 20. *When Γ is equipped with a vertex-orientation, orientations of the manifold $\breve{C}(L;\Gamma)$ are in canonical one-to-one correspondence with orientations of the set $H(\Gamma)$.*

Proof. Since $\breve{C}(L;\Gamma)$ is naturally an open submanifold of $C^U\times\breve{M}^T$, it inherits $\mathbb{R}^{\#U+3\#T}$-valued charts from $\mathbb{R}$-valued orientation-preserving charts of C and $\mathbb{R}^3$-valued orientation-preserving charts of $\breve{M}$. In order to define the orientation of $\mathbb{R}^{\#U+3\#T}$, one must identify its factors and order them (up to even permutation). Each of the factors may be labeled by an element of $H(\Gamma)$: the $\mathbb{R}$-valued local coordinate of an element of C corresponding to the image under j of an element of U sits in the factor labeled by the half-edge of U; the 3 cyclically ordered (by the orientation of $\breve{M}$) $\mathbb{R}$-valued local coordinates of the image under a configuration c of an element of T live in the factors labeled by the three half-edges that are cyclically ordered by the vertex-orientation of Γ, so that the cyclic orders match. □

The dimension of $\breve{C}(L;\Gamma)$ is $\#U(\Gamma)+3\#T(\Gamma)=2\#E(\Gamma)$ where $E=E(\Gamma)$ denotes the set of edges of Γ. Since $n=n(\Gamma)=\frac{1}{2}(\#U(\Gamma)+\#T(\Gamma))$, $\#E(\Gamma)=3n-\#U(\Gamma)$.

4.2. Configuration space integrals

A *numbered* degree n Jacobi diagram is a degree n Jacobi diagram Γ whose edges are oriented, equipped with an injection $j_E:E(\Gamma)\hookrightarrow\underline{3n}$. Such an injection numbers the edges. Note that this injection is a bijection when $U(\Gamma)$ is empty. Let $\mathcal{D}^e_n(C)$ denote the set of numbered degree n Jacobi diagrams with support C without *looped edges* like $-\!\!\bigcirc$.

Let Γ be a numbered degree n Jacobi diagram. The orientations of the edges of Γ induce the following orientation of the set $H(\Gamma)$ of half-edges of Γ: order $E(\Gamma)$ arbitrarily, and order the half-edges as (first half-edge of the first edge, second half-edge of the first edge,…, second half-edge of the last edge). The induced orientation is called the *edge-orientation* of $H(\Gamma)$. Note that it does not depend on the order of $E(\Gamma)$. Thus, as soon as Γ is equipped with a vertex-orientation $o(\Gamma)$, the edge-orientation of Γ orients $\breve{C}(L;\Gamma)$.

An edge e oriented from a vertex v_1 to a vertex v_2 of Γ induces the following canonical map

$$\begin{array}{rccl} p_e: & \breve{C}(L;\Gamma) & \to & C_2(M), \\ & c & \mapsto & (c(v_1),c(v_2)). \end{array}$$

For any $i\in\underline{3n}$, let $\omega(i)$ be a propagating form of $(C_2(M),\tau)$. Define $I(\Gamma,o(\Gamma),(\omega(i))_{i\in\underline{3n}})$ as

$$I(\Gamma,o(\Gamma),(\omega(i))) = \int_{\breve{C}(L;\Gamma)} \bigwedge_{e\in E(\Gamma)} p_e^*(\omega(j_E(e)))$$

where $\breve{C}(L;\Gamma)$ is equipped with the orientation induced by the vertex-orientation $o(\Gamma)$ and by the edge-orientation of Γ.

The convergence of this integral is a consequence of the following proposition, which will be proved in Subsection 5.1.

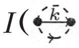
Proposition 6. *There exists a smooth compactification $C(L;\Gamma)$ of $\breve{C}(L;\Gamma)$ where the maps p_e smoothly extend.*

According to this proposition, $\bigwedge_{e\in E(\Gamma)}p_e^*(\omega(j_E(e)))$ smoothly extends to $C(L;\Gamma)$, and $\int_{(\breve{C}(L;\Gamma),o(\Gamma))}\bigwedge_{e\in E(\Gamma)}p_e^*(\omega(j_E(e)))$ is equal to $\int_{(C(L;\Gamma),o(\Gamma))}\bigwedge_{e\in E(\Gamma)}p_e^*(\omega(j_E(e)))$.

Examples 1. For any three propagating forms $\omega(1)$, $\omega(2)$ and $\omega(3)$ of $(C_2(M),\tau)$,

$I(S_i^1$  $S_j^1,\ (\omega(i))_{i\in\underline{3}}) = lk(K_i,K_j)$, $I($  $,(\omega(i))_{i\in\underline{3}}) = \Theta(M,\tau)$

for any numbering of the (plain) diagrams (exercise).

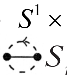
Let us now study the case of $I($  $S_j^1,(\omega(i))_{i\in\underline{3}})$, which depends on the chosen propagating forms, and on the diagram numbering.

A *dilation* is a homothety with positive ratio.

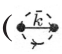
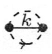
Let U^+K_j denote the fiber space over K_j made of the tangent vectors to the knot K_j of $\breve{M}$ that orient K_j, up to dilation. The fiber of U^+K_j is made of one point, so that the total space of this *unit positive tangent bundle to* K_j is K_j. Let U^-K_j denote the fiber space over K_j made of the opposite tangent vectors to K_j, up to dilation.

For a knot K_j in $\breve{M}$, define the two-point configuration space $\breve{C}(K_j;$  $S_i^1)$ as

$$\{(K_j(z),K_j(z\exp(i\theta)));(z,\theta)\in S^1\times]0,2\pi[\}.$$

Let $C_j = C(K_j;$  $S_j^1)$ be the closure of $\breve{C}(K_j;$  $S_j^1)$ in $C_2(M)$. This closure is diffeomorphic to $S^1\times[0,2\pi]$ where $S^1\times 0$ is identified with U^+K_j, $S^1\times\{2\pi\}$ is identified with U^-K_j and $\partial C(K_j;$  $S_j^1)=U+K_j-U^-K_j$.

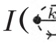
Lemma 21. *For any $i\in\underline{3}$, let $\omega(i)$ and $\omega'(i)=\omega(i)+d\eta(i)$ be propagating forms of $(C_2(M),\tau)$, where $\eta(i)$ is a one-form on $C_2(M)$. Then*

$I($  $S_j^1,(\omega'(i))_{i\in\underline{3}}) - I($  $S_j^1,(\omega(i))_{i\in\underline{3}}) = \int_{U+K_j}\eta(k)-\int_{U-K_j}\eta(k).$

Proof. Apply the Stokes theorem to $\int_{C_j}(\omega'(k)-\omega(k))=\int_{C_j}d\eta(k)$. □

Exercise 2. Find a knot K_j of $\mathbb{R}^3$ and a form $\eta(k)$ of $C_2(\mathbb{R}^3)$ such that the right-hand side of Lemma 21 does not vanish. (Use Lemma 9, hints can be found in Subsection 5.2.)

Say that a propagating form ω of $(C_2(M),\tau)$ is *homogeneous* if its restriction to $\partial C_2(M)$ is $p_\tau^*(\omega_{S^2})$ for the homogeneous volume form ω_{S^2} of S^2 of total volume 1.

Lemma 22. *For any $i \in \underline{3}$, let $\omega(i)$ be a homogeneous propagating form of $(C_2(M),\tau)$. Then* $I($$S^1_j,(\omega(i))_{i\in\underline{3}})$ *does not depend on the choices of the $\omega(i)$, it is denoted by $I_\theta(K_j,\tau)$.*

Proof. Apply Lemma 21 with $\eta_A = 0$, so that $\eta(k) = 0$ in Lemma21. □

4.3. An invariant for links in $\mathbb{Q}$-spheres from configuration spaces

Let $\mathbb{K} = \mathbb{R}$. Let $[\Gamma, o(\Gamma)]$ denote the class in $\mathcal{D}^t_n(C)$ of a numbered Jacobi diagram Γ of $\mathcal{D}^e_n(C)$ equipped with a vertex-orientation $o(\Gamma)$, then $I(\Gamma, o(\Gamma), (\omega(i))_{i\in\underline{3n}})[\Gamma, o(\Gamma)] \in \mathcal{D}^t_n(C)$ is independent of the orientation of $o(\Gamma)$, it will be simply denoted by $I(\Gamma,(\omega(i))_{i\in\underline{3n}})[\Gamma]$.

Theorem 5. *Let $(\breve{M},\tau)$ be an asymptotic rational homology $\mathbb{R}^3$. Let $L : \coprod_{j=1}^k S^1_j \hookrightarrow \breve{M}$ be an embedding. For any $i \in \underline{3n}$, let $\omega(i)$ be a homogeneous propagating form of $(C_2(M),\tau)$. The sum*

$$\sum_{\Gamma\in\mathcal{D}^e_n(C)} \frac{(3n-\#E(\Gamma))!}{(3n)!2^{\#E(\Gamma)}} I(\Gamma,(\omega(i))_{i\in\underline{3n}})[\Gamma]$$

in $\mathcal{D}^t_n(\coprod_{j=1}^k S^1_j)$ is independent of the chosen $\omega(i)$. It only depends on the diffeomorphism class of (M,L), on $p_1(\tau)$ and on the $I_\theta(K_j,\tau)$, for the components K_j of L. It is denoted by $Z_n(L,\breve{M},\tau)$.

More precisely, set $Z(L,\breve{M},\tau) = (Z_n(L,\breve{M},\tau))_{n\in\mathbb{N}} \in \mathcal{D}^t(\coprod_{j=1}^k S^1_j)$. There exist two constants $\alpha \in \mathcal{D}(S^1;\mathbb{Q})$ and $\beta \in \mathcal{D}(\varnothing;\mathbb{Q})$ such that the product of $\exp(-\frac{1}{4}p_1(\tau)\beta)$ by

$$\prod_{j=1}^k \left(\exp(-I_\theta(K_j,\tau)\alpha)\,\#_j\right) Z(L,\breve{M},\tau),$$

where $\exp(-I_\theta(K_j)\alpha)$ acts on $Z(L,\breve{M},\tau)$, on the copy S^1_j of S^1 as indicated by the subscript j, only depends on the diffeomorphism class of (M,L). It is denoted by $Z(L,M)$,

$$Z(L,M) \in \mathcal{D}^t(\coprod_{j=1}^k S^1_j;\mathbb{Q}).$$

Furthermore, if $\breve{M} = \mathbb{R}^3$, then the projection $Z^u(L,S^3)$ of $Z(L,S^3)$ on $\mathcal{D}(\coprod_{j=1}^k S^1_j)$ is a universal finite type invariant of links in $\mathbb{R}^3$, i. e. Z^u_n satisfies the properties stated for Z^K_n in Theorem 3. It is the configuration space invariant studied by Altschьler, Freidel [32], Dylan Thurston [33], Sylvain Poirier [34] and others. If $k = 0$, then $Z(\varnothing,M)$ is the Kontsevich configuration space invariant $Z_{KKT}(M)$, which is a universal invariant for $\mathbb{Z}$-spheres according to a theorem of Kuperberg and Thurston [9; 13], and which was completed to a universal finite type invariant for $\mathbb{Q}$-spheres by Delphine Moussard [21].*

The proof of this theorem is sketched in Section 5.

Under its assumptions, let ω_0 be a homogeneous propagating form of $(C_2(M),\tau)$, let ι be the involution of $C_2(M)$ that permutes two elements in $\breve{M}^2 \setminus \text{diagonal}$, set $\omega = \frac{1}{2}(\omega_0 - \iota_*(\omega_0))$, and set $\omega(i) = \omega$ for any i.

Let $\mathrm{Aut}(\Gamma)$ be the set of automorphisms of Γ, which is the set of permutations of the half-edges that map a pair of half-edges of an edge to another such and a triple of half-edges that contain a vertex to another such, and that map half-edges of univalent vertices on a component K_j to half-edges of univalent vertices on K_j so that the cyclic order among such vertices is preserved. Set

$$\beta_\Gamma = \frac{(3n-\#E(\Gamma))!}{(3n)!2^{\#E(\Gamma)}}.$$

* After work of many people including Witten [35], Guadagnini, Martellini and Mintchev [36], Kontsevich [37; 38], Bott and Taubes [39], Bar-Natan [40], Axelrod and Singer [41; 42].

Then $\sum_{\Gamma\in\mathcal{D}_n^e(C)}\beta_\Gamma I(\Gamma,(\omega(i))_{i\in\underline{3n}})[\Gamma]$ reads

$$\sum_{\Gamma \text{ unnumbered, unoriented}} \frac{1}{\#\mathrm{Aut}(\Gamma)} I(\Gamma,(\omega)_{i\in\underline{3n}})[\Gamma]$$

where the latter sum runs over the degree n Jacobi diagrams on C without looped edges.

Indeed, for a numbered graph Γ, there are $\frac{1}{\beta_\Gamma}$ ways of renumbering it, and $\#\mathrm{Aut}(\Gamma)$ of them will produce the same numbered graph.

4.4. On the universality proofs

Theorem 6. *Let $y, z \in \mathbb{N}$. Recall $\underline{y} = \{1,2,\dots,y\}$. Set $(\underline{z+y}) = \{y+1, y+2,\dots, y+z\}$. Let $\breve{M}$ be an asymptotically standard $\mathbb{Q}$-homology $\mathbb{R}^3$. Let L be a link in $\breve{M}$. Let $(B_b)_{\in\underline{y}}$ be a collection of pairwise disjoint balls in $\breve{M}$ such that every B_b intersects L as a ball of a crossing change that contains a positive crossing c_b, and let $L((B_b)_{b\in\underline{y}})$ be the link obtained by changing the positive crossings c_b to negative crossings. Let $(A_a)_{a\in(\underline{z+y})}$ be a collection of pairwise disjoint rational homology handlebodies in $\breve{M}\setminus(L\cup_{b=1}^{y} B_b)$. Let (A'_a/A_a) be rational LP surgeries in $\breve{M}$. Set $X = [M, L; (A'_a / A_a)_{a\in(\underline{z+y})}, (B_b, c_b)_{b\in\underline{y}}]$ and define $Z_n(X)$ as the sum over all subsets I of $\underline{y+z}$ of the terms $(-1)^{\#I} Z_n\left(L((B_b)_{b\in I\cap\underline{y}}), M((A'_a / A_a)_{a\in I\cap(\underline{z+y})})\right)$. If $2n < 2y+z$, then $Z_n(X)$ vanishes.*

Sketch of proof. As in [13], one can use (generalized) propagators for the $M((A'_a / A_a)_{a\in I\cap(\underline{z+y})})$ that coincide for different I wherever it makes sense (for example, for configurations that do not involve points in surgered pieces A_a). See also [9]. Then contributions to the alternate sum of the integrals over parts that do not involve at least one point in an A_a or in an A'_a, for all a cancel. Assume that every crossing change is performed by moving only one strand. Again, contributions to the alternate sum of the integrals that do not involve at least one point on a moving strand cancel. Furthermore, if the moving strand of c_b is moved very slightly, and if no other vertex is constrained to lie on the other strand in the ball of the crossing change, then the alternate sum is close to zero. Thus in order to produce a contribution to the alternate sum, a graph must have at least $(2y+z)$ vertices. See [32] or [22. Section 5.4], and [13. Section 3] for more details. □

This implies that Z_n^u is of degree at most n for links in $\mathbb{R}^3$, and that Z_n is of degree at most $2n$ for $\mathbb{Z}$-spheres or $\mathbb{Q}$-spheres.

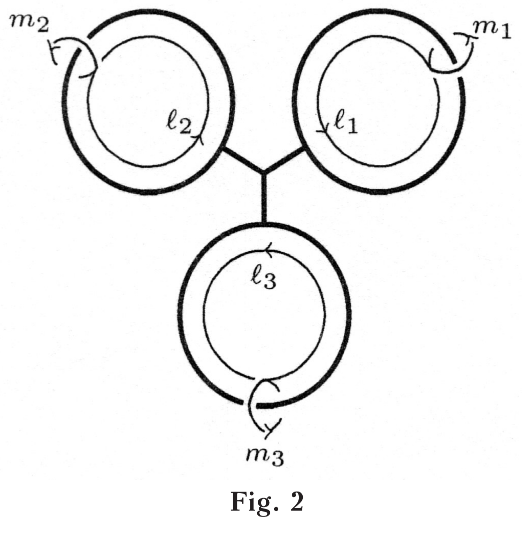


Fig. 2

Now, under the hypotheses of Theorem 6, assume that A_a is the standard genus 3 handlebody with three handles with meridians $m_j^{(a)}$ and longitudes $l_j^{(a)}$ such that $\langle m_i^{(a)}, \ell_j^{(a)}\rangle_{\partial A_a} = \delta_{ij}$. See A_a as a thickening of the trivalent graph on Fig. 2.

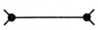
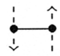
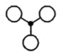
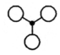
Also assume that A'_a is an integer homology handlebody. In $A_a \cup_{\partial A_a} (-A'_a)$, there is a surface S_j such that $\partial(S_j \cap A_a) = m_j^{(a)}$. Assume that $\langle S_1, S_2, S_3\rangle_{A_a\cup_{\partial A_a}(-A'_a)} = 1$. (For example, choose A'_a such that $A_a \cup_{\partial A_a} (-A'_a) = (S^1)^3$, like in the case of the Matveev Borromean surgery of [43].) Assume that the $l_j^{(a)}$ bound surfaces $D_j^{(a)}$ in $\breve{M}$.

Assume that the collection of surfaces $\{D_j^{(a)}\}_{a\in(\underline{z+y}), j\in\underline{3}}$ reads $\{D_{p,1}\}_{p\in\underline{P}} \sqcup \{D_{p,2}\}_{p\in\underline{P}}$ so that for any $q \in \underline{P}$, for $\delta \in \underline{2}$, if $D_{q,\delta} = D^{(a(q,\delta))}_{j(q,\delta)}$, the interior of $D_{q,\delta}$ intersects

$$L \cup \bigcup_{a \in (\underline{z}+y)} \left(A_a \cup \cup_{j \in \underline{3}, D^{(a)}_j \neq D_{q,\delta}} D^{(a)}_j \right) \cup \cup_{b \in \underline{y}} (B_b)$$

only in $A_{a(q,3-\delta)} \cup D^{(a(q,3-\delta))}_{j(q,3-\delta)}$.

Note that $\langle D_{q,\delta}, \ell^{(a(q,3-\delta))}_{j(q,3-\delta)} \rangle_M = lk(\partial D_{q,1}, \partial D_{q,2})$.

Example 3. Note that these assumptions are realised in the following case. Start with an embedding of a Jacobi diagram Γ whose univalent vertices belong to chords (plain edges between two univalent vertices) on $\cup_{i=1}^k S^1_i$ in $\check{M}$. Assume that the trivalent vertices of Γ are labeled in $(\underline{z}+y)$, and assume that its chords are labeled in $\underline{y}$. Apply the following operations replace edges  without univalent vertices by , replace a chord  labeled by b by a crossing change c_b  $\to$  in a ball B_b that is a neighborhood of the plain edge. Thicken the trivalent graph  associated to the trivalent vertex labeled by a, and call it A_a. Then the surfaces $D^{(a)}_j$ are the disks bounded by the small loops of .

Conversely, under the assumptions before the example, define the following vertex-oriented Jacobi diagram $\Gamma([M, L; (A'_a/A_a)_{a \in (\underline{z}+y)}, (B_b, c_b)_{b \in \underline{y}}])$ on $\cup_{i=1}^k S^1_i$, with:

- two univalent vertices joined by a chord for each crossing change ball B_b at the corresponding places on $\cup_{i=1}^k S^1_i$ (in $L^{-1}(B_b)$),
- one trivalent vertex for each A_a, where the three adjacent half-edges of the vertex correspond to the three $D^{(a)}_j$, with the fixed cyclic order,

such that any pair of half-edges corresponding to some $D_{p,1}$ and its friend $D_{p,2}$ forms an edge between two trivalent vertices.

Theorem 7. *Under the assumptions above, let* $X = [M, L; (A'_a / A_a)_{a \in (\underline{z}+y)}, (B_b, c_b)_{b \in \underline{y}}]$. *When* $2n = 2y + z$,

$$Z_n(X) = \left(\prod_{p \in \underline{P}} lk(\partial D_{p,1}, \partial D_{p,2}) \right) [\Gamma(X)] \bmod 1T \left(\text{or in } \frac{A^t_n(\coprod_{j=1}^k S^1_j)}{(1T)} \right).$$

Sketch of proof. When $z = 0$, the proof of Theorem 6 can be pushed further in order to prove the result like in [32] or [22. Section 5.4]. In general, when $y = 0$, it is a consequence of the main theorem in [13] (Theorem 2.4). The general result can be obtained by mixing the arguments of [13. Section 3] with the arguments of the link case. □

This theorem is the key to proving the universality of Z^u among Vassiliev invariants for links in $\mathbb{R}^3$ and to proving the universality of Z among finite type invariants of $\mathbb{Z}$-spheres. This universality implies that all finite type invariants factor through Z.

Remark 3. Theorem 7 with Z^{LMO} instead of Z is proved in [20], when $y = 0$, when the (A'_a / A_a) are Matveev's Borromean surgeries and when the $D^{(a)}_j$ are disks such that $lk(\partial D_{p,1}, \partial D_{p,2}) = 1$. Then the main theorem of [44] implies Theorem 7 with Z^{LMO} instead of Z, when $y = 0$ and when the A_a and the A'_a are integral homology handlebodies.

5. Compactifications, anomalies, proofs and questions

In this section, we state Theorem 8. This is another version of Theorem 5, which leads to a definition of Z involving algebraic intersections rather than integrals in Subsection 5.3. It is based on the concept of *straight links* introduced in Subsection 5.2.

This section also contains sketches of proofs of Theorems 5 and 8. We begin with the introduction of appropriate compactifications of configuration spaces to justify the convergence of our integrals stated in Proposition 6.

5.1. Compactifications of configuration spaces

Let N be a finite set. See the elements of M^N as maps $m\colon N \to M$.

For a non-empty $I \subseteq N$, let E_I be the set of maps that map I to ∞. For $I \subseteq N$ such that $\# I \geq 2$, let Δ_I be the set of maps that map I to a single element of M. When I is a finite set, and when V is a vector space of positive dimension, $\breve{S}_I(V)$ denotes the space of injective maps from I to V up to translation and dilation. When $\# I \geq 2$, $\breve{S}_I(V)$ embeds in the compact space $S_I(V)$ of non-constant maps from I to V up to translation and dilation.

Lemma 23. *The fiber of the unit normal bundle to Δ_I in M^N over a configuration m is $S_I(T_{m(I)}M)$.*

Proof. Exercise. □

Let $\breve{C}_N(M)$ denote the space of injective maps from N to $\breve{M}$. Define a compactification $C_N(M)$ of $\breve{C}_N(M)$ by generalizing the previous construction of $C_2(M) = C_{\underline{2}}(M)$ as follows.

Start with M^N. Blow up E_N, which is the point $m = \infty^N$ such that $m^{-1}(\infty) = N$. Then for $k = \# N, \# N - 1, \dots, 3, 2$, in this decreasing order, successively blow up the (closures of the preimages under the composition of the previous blow-down maps of the) Δ_I such that $\# I = k$ (choosing an arbitrary order among them) and, next, the (closures of the preimages under the composition of the previous blow-down maps of the) E_J such that $\# J = k-1$ (again, choosing an arbitrary order among them).

Lemma 24. *The successive manifolds that are blown-up in the above process are smooth and transverse to the boundaries. The manifold $C_N(M)$ is a smooth compact $(3\# N)$-manifold independent of the possible order choices in the process. For $i, j \in N$, $i \neq j$, the map*

$$\begin{array}{cccc} p_{i,j}: & \breve{C}_N(M) & \to & C_2(M) \\ & m & \mapsto & (m(i), m(j)) \end{array}$$

smoothly extends to $C_N(M)$.

Sketch of proof. A configuration m_0 of M^N induces the following partition $\mathbb{C}(m_0)$ of

$$N = m_0^{-1}(\infty) \coprod \coprod_{x \in \breve{M} \cap m_0(N)} m_0^{-1}(x).$$

Pick disjoint neighborhoods V_x in M of the points x of $m_0(N)$ that are furthermore in $\breve{M}$ for x in $\breve{M}$ and that are identified with balls of $\mathbb{R}^3$ by C^∞-charts. Consider the neighborhood $\prod_{x \in m_0(N)} V_x^{m_0^{-1}(x)}$ of m_0 in M^N. The first blow-ups that transformed this neighborhood are:

- the blow-up of $E_{m_0^{-1}(\infty)}$ if $m_0^{-1}(\infty) \neq \emptyset$, which changed (a smaller neighborhood of $\infty^{m_0^{-1}(\infty)}$ in) $V_\infty^{m_0^{-1}(\infty)}$ to $[0, \varepsilon_\infty[\times S^{3\# m_0^{-1}(\infty) - 1}$;
- and the blow-ups of the $\Delta_{m_0^{-1}(x)}$, for the $x \in \breve{M}$ such that $\# m_0^{-1}(x) \geq 2$, which changed (a smaller neighborhood of $x^{m_0^{-1}(x)}$ in) $V_x^{m_0^{-1}(x)}$ to $[0, \varepsilon_x[\times F(U_x^{m_0^{-1}(x)})$, where $U_x \subset V_x$ and $F(U_x^{m_0^{-1}(x)})$ fibers over U_x, and the fiber over $y \in U_x$ is $S_{m_0^{-1}(x)}(T_y M)$.

When considering how the next blow-ups affect the preimage of a neighborhood of m_0, we can restrict to our new factors.

First consider a factor $[0, \varepsilon_x[\times F(U_x^{m_0^{-1}(x)})$. Picking $i \in m_0^{-1}(x)$ and fixing a Riemannian structure on TU_x identifies $S_{m_0^{-1}(x)}(T_y M)$ with the space of maps $c : m_0^{-1}(x) \to T_y M$ such that

$c(i)=0$ and $\sum_{j\in m_0^{-1}(x)}\|c(j)\|^2=1$. Then (λ,c) is identified with $y|\lambda c$ in $V_x^{m_0^{-1}(x)}$ (where V_x is identified with an open subset of $\mathbb{R}^3$), for $\lambda\neq 0$. Now, $[0,\varepsilon_x[\times F(U_x^{m_0^{-1}(x)})$ must be blown-up along its intersections with the preimage closures of the Δ_I such that $\#I\geq 2$, $I\subset m_0^{-1}(x)$ and I is maximal. These intersections respect the product structure by $[0,\varepsilon_x[$ and the fibration over U_x so that we only need to understand the blow-ups of the intersections of the Δ_I with a fiber of $F(U_x^{m_0^{-1}(x)})$. These are nothing but configurations in a ball of $\mathbb{R}^3$, and we can iterate our process.

Now consider the possible factor $[0,\varepsilon_\infty[\times S^{3\#m_0^{-1}(\infty)-1}$ and blow up its intersections with the preimage closures of the E_J for $J\subset m_0^{-1}(\infty)$ maximal and with the preimage closures of the Δ_I with $I\subset m_0^{-1}(\infty)$ in an order compatible with the algorithm. Here, $S^{3\#m_0^{-1}(\infty)-1}$ is the unit sphere of $(\mathbb{R}^3_\infty)^{m_0^{-1}(\infty)}$. A point $d\in(\mathbb{R}^3_\infty)^{m_0^{-1}(\infty)}$ is in the preimage closure of E_J under the previous blow-up if $d(J)=0$. In particular, the E_J and the Δ_I again read as products by $[0,\varepsilon_\infty[$, and we study what happens near a given d of $S^{3\#m_0^{-1}(\infty)-1}$. For such a d, we proceed as before if $d^{-1}(0)=\varnothing$. Otherwise the factor of $d^{-1}(0)$ must be treated differently, namely by blowing up $0^{d^{-1}(0)}$ in $S^{3\#m_0^{-1}(\infty)-1}$. Then iterate.

This produces a compact manifold $C_N(M)$ with boundary and ridges, which is finally independent of the order of the blow-ups (when this order is compatible with the algorithm), since it is locally independent. The interior of $C_N(M)$ is $\breve{C}_N(\breve{M})$. Since the blow-ups separate all the pairs of points at some scale, p_e naturally extends there. The introduced local coordinates show that the extension is smooth. See [31. Section 3] for more details.

Lemma 25. *The closure of $\breve{C}(L;\Gamma)$ in $C_{V(\Gamma)}(M)$ is a smooth compact submanifold of $C_{V(\Gamma)}(M)$, which is denoted by $C(L;\Gamma)$.*

Proof. Exercise. □

Proposition 6 is a consequence of Lemmas 24 and 25.

5.2. Straight links

A one-chain c of S^2 is *algebraically trivial* if for any two points x and y outside its support, the algebraic intersection of an arc from x to y transverse to c with c is zero, or equivalently if the integral of any one form of S^2 along c is zero.

Let $(\breve{M},\tau)$ be an asymptotic rational homology $\mathbb{R}^3$. Say that K_j is *straight* with respect to τ if the curve $p_\tau(U^+K_j)$ of S^2 is algebraically trivial (recall the notation from Proposition 2 and Subsection 4.2). A link is *straight* with respect to τ if all its components are. If K_j is straight, then $p_\tau(\partial C(K_j;\overset{\leftrightarrow}{\bullet\bullet}S^1_j))$ is algebraically trivial.

Lemma 26. *Recall $C_j=C(K_j;\overset{\leftrightarrow}{\bullet\bullet}S^1_j)$, $C_j\subset C_2(M)$. If $p_\tau(\partial C_j)$ is algebraically trivial, then for any propagating chain $\mathcal{P}$ of $(C_2(M),\tau)$ transverse to C_j and for any propagating form ω_p of $(C_2(M),\tau)$,*

$$\int_{C_j}\omega_p=\langle C_j,\mathcal{P}\rangle_{C_2(M)}=I_\theta(K_j,\tau)$$

where $I_\theta(K_j,\tau)$ is defined in Lemma 22. In particular, $I_\theta(K_j,\tau)\in\mathbb{Q}$ and $I_\theta(K_j,\tau)\in\mathbb{Z}$ when M is an integer homology 3-sphere.

Proof. Exercise. Recall Lemmas 9 and 21. □

Proposition 7. *Let $\breve{M}$ be an asymptotically standard $\mathbb{Q}$-homology $\mathbb{R}^3$. For any parallel $K_\parallel$ of a knot K in $\breve{M}$, there exists an asymptotically standard parallelization $\tilde{\tau}$ homotopic to τ, such that K is straight with respect to $\tilde{\tau}$, and $I_\theta(K_j,\tilde{\tau})=lk(K,K_\parallel)$ or $I_\theta(K_j,\tilde{\tau})=lk(K,K_\parallel)+1$.*

For any embedding $K:S^1\to\breve{M}$ that is straight with respect to τ, $I_\theta(K,\tau)$ is the linking number of K and a parallel of K.

Sketch of proof. For any knot embedding K, there is an asymptotically standard parallelization $\tilde{\tau}$ homotopic to τ such that $p_{\tilde{\tau}}(U^+K)$ is one point. Thus K is straight with respect to $(M,\tilde{\tau})$. Then $\tilde{\tau}$ induces a parallelization of K, and $I_\theta(K,\tilde{\tau})$ is the linking number of K with the parallel induced by $\tilde{\tau}$. (Exercise).

In general, for two homotopic asymptotically standard parallelizations τ and $\tilde{\tau}$ such that K is straight with respect to τ and $\tilde{\tau}$, $I_\theta(K,\tau)-I_\theta(K,\tilde{\tau})$ is an even integer (exercise) so that $I_\theta(K,\tau)$ is always the linking number of K with a parallel of K.

In $\mathbb{R}^3$ equipped with τ_s, any link is represented by an embedding L that sits in a horizontal plane except when it crosses under, so that the non-horizontal arcs crossing under are in vertical planes. Then the non-horizontal arcs have an algebraically trivial contribution to $p_\tau(U^+K_j)$, while the horizontal contribution can be changed by adding kinks  or  so that L is straight with respect to τ_s. In this case $I_\theta(K_j,\tau_s)$ is the *writhe* of K_j, which is the number of positive self-crossings of K_j minus the number of negative self-crossings of K_j. In particular, up to isotopy of L, $I_\theta(K_j,\tau_s)$ can be assumed to be ± 1. (Exercise).

Similarly, for any number ι that is congruent mod $2\mathbb{Z}$ to $I_\theta(K,\tau)$ there exists an embedding K' isotopic to K and straight such that $I_\theta(K',\tau)=\iota$. (Exercise).

5.3. Rationality of Z

Let us state another version of Theorem 5 using straight links instead of homogeneous propagating forms. Recall $\beta_\Gamma = \dfrac{(3n-\#E(\Gamma))!}{(3n)!2^{\#E(\Gamma)}}$.

Theorem 8. *Let* $(\check{M},\tau)$ *be an asymptotic rational homology* $\mathbb{R}^3$. *Let* $L:\coprod_{j=1}^k S^1_j \hookrightarrow \check{M}$ *be a straight embedding with respect to* τ. *For any* $i\in\underline{3n}$, *let* $\omega(i)$ *be a propagating form of* $(C_2(M),\tau)$. *Then*

$$\sum_{\Gamma\in\mathcal{D}^e_n(C)} \beta_\Gamma I(\Gamma,(\omega(i))_{i\in\underline{3n}})[\Gamma]\in\mathcal{D}^t_n\left(\coprod_{j=1}^k S^1_j\right)$$

is independent of the chosen $\omega(i)$. *It is denoted by* $Z^s_n(L,\check{M},\tau)$. *In particular, with the notation of Theorem 5,* $Z^s_n(L,\check{M},\tau)=Z_n(L,\check{M},\tau)$.

This version of Theorem 5 allows us to replace the configuration space integrals by algebraic intersections in configuration spaces, and thus to prove the rationality of Z for straight links as follows.

For any $i\in\underline{3n}$, let $\mathcal{P}(i)$ be a propagating chain of $(C_2(M),\tau)$. Say that a family $(\mathcal{P}(i))_{i\in\underline{3n}}$ is *in general* $3n$ *position* with respect to L if for any $\Gamma\in\mathcal{D}^e_n(C)$, the $p_e^{-1}(\mathcal{P}(j_E(e)))$ are pairwise transverse chains in $C(L;\Gamma)$. In this case, define $I(\Gamma,o(\Gamma),(\mathcal{P}(i))_{i\in\underline{3n}})$ as the algebraic intersection in $(C(L;\Gamma),o(\Gamma))$ of the codimension 2 rational chains $p_e^{-1}(\mathcal{P}(j_E(e)))$. If the $\omega(i)$ are propagating forms of $(C_2(M),\tau)$ Poincaré dual to the $\mathcal{P}(i)$ and supported in sufficiently small neighborhoods of the $\mathcal{P}(i)$, then $I(\Gamma,o(\Gamma),(\mathcal{P}(i))_{i\in\underline{3n}})=I(\Gamma,o(\Gamma),(\omega(i))_{i\in\underline{3n}})$ for any $\Gamma\in\mathcal{D}^e_n(C)$, and $I(\Gamma,o(\Gamma),(\omega(i))_{i\in\underline{3n}})$ is rational, in this case.

5.4. On the anomalies

The constants $\alpha=(\alpha_n)_{n\in\mathbb{N}}$ and $\beta=(\beta_n)_{n\in\mathbb{N}}$ of Theorem 5 are called *anomalies.* The anomaly β is the opposite of the constant ξ defined in [31. Section 1.6], $\beta_{2n}=0$ for any integer n, and $\beta_1=\frac{1}{12}[$$]$ according to [31. Proposition 2.45]. The computation of β_1 can also be deduced from Corollary 1.

We define α below. Let $v\in S^2$. Let $\mathcal{L}_v$ denote the linear map

$$\begin{array}{rccc}\mathcal{L}_v: & \mathbb{R} & \to & \mathbb{R}^3\\ & 1 & \mapsto & v.\end{array}$$

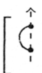
Let Γ be a numbered Jacobi diagram on $\mathbb{R}$. Define $\breve{C}(\mathcal{L}_v;\Gamma)$ like in Subsection 4.1 where the line $\mathcal{L}_v$ of $\mathbb{R}^3$ replaces the link L of $\breve{M}$. Let $\breve{Q}(v;\Gamma)$ be the quotient of $\breve{C}(\mathcal{L}_v;\Gamma)$ by the translations parallel to $\mathcal{L}_v$ and by the dilations. Then the map p_{e,S^2} associated to an edge e of Γ maps a configuration to the direction of the vector from its origin to its end in S^2. It factors through $\breve{Q}(v;\Gamma)$, which has two dimensions less. Now, define $\breve{Q}(\Gamma)$ as the total space of the fibration over S^2 whose fiber over v is $\breve{Q}(v;\Gamma)$. The configuration space $\breve{Q}(\Gamma)$ carries a natural smooth structure, it can be compactified as before, and it can be oriented as follows, when a vertex-orientation $o(\Gamma)$ is given. Orient $\breve{C}(L_v;\Gamma)$ as before, orient $\breve{Q}(v;\Gamma)$ so that $\breve{C}(\mathcal{L}_v;\Gamma)$ is locally homeomorphic to the oriented product (translation vector z in $\mathbb{R}v$, ratio of homothety $\lambda \in]0,\infty[$) $\times \breve{Q}(v;\Gamma)$ and orient $\breve{Q}(\Gamma)$ with the (base($=S^2$)$\oplus$fiber) convention. (This can be summarized by saying that the S^2-coordinates replace (z,λ).)

Proposition 8. *For* $i \in \underline{3n}$, *let* $\omega(i,S^2)$ *be a two-form of* S^2 *such that* $\int_{S^2}\omega(i,S^2) = 1$. *Define* $I(\Gamma,o(\Gamma),\omega(i,S^2))$ *as*

$$\int_{\breve{Q}(\Gamma)} \bigwedge_{e\in E(\Gamma)} p^*_{e,S^2}(\omega(j_E(e),S^2)).$$

Let $D^c_n(\mathbb{R})$ *denote the set of connected numbered diagrams on* $\mathbb{R}$ *with at least one univalent vertex, without looped edges. Define the element* $2\alpha_n$ *of* $A(\mathbb{R})$ *as*

$$\sum_{\Gamma\in D^c_n(\mathbb{R})} \frac{(3n - \# E(\Gamma))!}{(3n)!2^{\# E(\Gamma)}} I(\Gamma,o(\Gamma),\omega(i,S^2))[\Gamma,o(\Gamma)].$$

Then α_n *does not depend on the chosen* $\omega(i,S^2)$, $\alpha_1 = \frac{1}{2}\Big[$  $\Big]$ *and* $\alpha_{2k} = 0$ *for all* $k \in \mathbb{N}$.

The series $\alpha = \sum_{n\in\mathbb{N}}\alpha_n$ *is called the Bott and Taubes anomaly.*

Proof. The independence of the choices of the $\omega(i,S^2)$ will be a consequence of Lemma 27 below. Let us prove that $\alpha_{2k} = 0$ for all $k \in \mathbb{N}$. Let Γ be a numbered graph and let $\overline{\Gamma}$ be obtained from Γ by reversing the orientations of the $(\# E)$ edges of Γ. Consider the map r from $\breve{Q}(\overline{\Gamma})$ to $\breve{Q}(\Gamma)$ that composes a configuration by the multiplication by (-1) in $\mathbb{R}^3$. It sends a configuration over $v \in S^2$ to a configuration over $(-v)$, and it is therefore a fibered map over the orientation-reversing antipode of S^2. Equip Γ and $\overline{\Gamma}$ with the same vertex-orientation. Then our map r is orientation-preserving if and only if $\# T(\Gamma) + 1 + \# E(\Gamma)$ is even. Furthermore for all the edges e of $\overline{\Gamma}$, $p_{e,S^2} \circ r = p_{e,S^2}$, then since $\# E = n + \# T$,

$$I(\overline{\Gamma},o(\Gamma),\omega(i,S^2)) = (-1)^{n+1} I(\Gamma,o(\Gamma),\omega(i,S^2)).$$ □

It is known that $\alpha_3 = 0$ and $\alpha_5 = 0$ [34]. Furthermore, according to [45], α_{2n+1} is a combination of diagrams with two univalent vertices, and $Z^u(S^3,L)$ is obtained from the Kontsevich integral by inserting d times the plain part of 2α on each degree d connected component of a diagram.

5.5. The dependence on the forms in the invariance proofs

The variation of $I(\Gamma,o(\Gamma),(\omega(j))_{j\in\underline{3n}})$ when some $\omega(i = j_E(f \in E(\Gamma)))$ is changed to $\omega(i) + d\eta$ for a one-form η on $C_2(M)$ reads

$$\int_{C(L;\Gamma)} \left(p^*_f(d\eta) \wedge \bigwedge_{e\in(E(\Gamma)\setminus\{f\})} p^*_e(\omega(j_E(e))) \right),$$

where $C(L;\Gamma)$ is equipped with the orientation induced by $o(\Gamma)$. According to the Stokes theorem, it reads

$$\int_{\partial C(L;\Gamma)} \left(p^*_f(\eta) \wedge \bigwedge_{e\in(E(\Gamma)\setminus\{f\})} p^*_e(\omega(j_E(e))) \right)$$

where the integral along $\partial(C(L;\Gamma),o(\Gamma))$ is actually the integral along the codimension one faces of $C(L;\Gamma)$, which are considered as open. Such a codimension one face only involves one blow-up.

For any non-empty subset B of $V(\Gamma)$, the codimension one face associated to the blow-up of E_B in $M^{V(\Gamma)}$ is denoted by $F(\Gamma,\infty,B)$, it lies in the preimage of $\infty^B \times \breve{M}^{V(\Gamma)\setminus B}$, in $C(L;\Gamma)$.

The other codimension one faces are associated to the blow-ups of the Δ_B in $M^{V(\Gamma)}$, for subsets B of $V(\Gamma)$ of cardinality at least 2. The face of $C(L;\Gamma)$ associated to Δ_B is denoted by $F(\Gamma;B)$. Let $b \in B$. Assume that $b \in U(\Gamma)$ if $U(\Gamma) \cap B \neq \emptyset$. The image of $F(\Gamma;B)$ in $M^{V(\Gamma)}$ is in the set of maps m of Δ_B that define an injection from $(V(\Gamma)\setminus B) \cup \{b \in B\}$ to $\breve{M}$, which factors through an injection isotopic to the restriction of i_Γ on $U(\Gamma) \cap \left((V(\Gamma)\setminus B) \cup \{b\}\right)$. This set of maps $\breve{C}_{(V(\Gamma)\setminus B)\cup\{b\}}(\breve{M}, i_\Gamma)$ is a submanifold of $\breve{C}_{(V(\Gamma)\setminus B)\cup\{b\}}(\breve{M})$. Thus, $F(\Gamma;B)$ is a bundle over $\breve{C}_{(V(\Gamma)\setminus B)\cup\{b\}}(\breve{M}, i_\Gamma)$.

When B has no univalent vertices, the fiber over a map m is the space $\breve{S}_B(T_{m(b)})$ of injective maps from B to $T_{m(b)}$ up to translations and dilations.

When B contains univalent vertices of a component K_j, the fiber over m is the submanifold $\breve{S}_B(T_{m(b)}M,\Gamma)$ of $\breve{S}_B(T_{m(b)}M)$, made of the configurations that map the univalent vertices of B to a line of $T_{m(b)}M$ directed by U^+K_j at $m(b)$, in an order prescribed by Γ. If B does not contain all the univalent vertices of Γ on S^1_j, this order is unique. Otherwise, $F(\Gamma,B)$ has $\#(B \cap U(\Gamma))$ connected components corresponding to the total orders that induce the cyclic order of $B \cap U(\Gamma)$.

When B is a subset of the set of vertices $V(\Gamma)$ of a numbered graph Γ, $E(\Gamma_B)$ denotes the set of edges of Γ between two elements of B (edges of Γ are plain), and Γ_B is the subgraph of Γ made of the vertices of B and the edges of $E(\Gamma_B)$.

Lemma 27. *Let $(\breve{M},\tau)$ be an asymptotic rational homology $\mathbb{R}^3$. Let $C = \coprod_{j=1}^k S^1_j$.*

For $i \in \underline{3n}$, let $\omega(i)$ be a closed 2-form on $[0,1]\times C_2(M)$ whose restriction to $\{t\}\times C_2(M)$ is denoted by $\omega(i,t)$, for any $t \in [0,1]$. Assume that for $t \in [0,1]$, $\omega(i,t)$ restricts to $(\partial C_2(M)\setminus UB_M)$ as $p^*_\tau(\omega(i,t)(S^2))$, for some two-form $\omega(i,t)(S^2)$ of S^2 such that $\int_{S^2}\omega(i,t)(S^2) = 1$. Set $Z_n(t) = \sum_{\Gamma\in\mathcal{D}^e_n(C)} \beta_\Gamma I(\Gamma,(\omega(i,t))_{i\in\underline{3n}})[\Gamma]$ in $\mathcal{D}^t_n(\coprod_{j=1}^k S^1_j)$. Then $Z_n(1) - Z_n(0) = \sum_{(\Gamma,B)} I(\Gamma,B)$ where the sum runs over the set

$$\left\{(\Gamma,B);\Gamma \in \mathcal{D}^e_n(C), B \subset V(\Gamma), \# B \geq 2; \Gamma_B \text{ is a connected component of } \Gamma\right\}$$

and

$$I(\Gamma,B) = \beta_\Gamma \int_{[0,1]\times F(\Gamma,B)} \bigwedge_{e\in E(\Gamma)} p^*_e(\omega(j_E(e)))[\Gamma].$$

Under the assumptions of Theorem 5 (where the $\omega(i)$ are homogeneous) or Theorem 8 (where L is straight with respect to τ), when (M,L,τ) is fixed, $Z_n(L,\breve{M},\tau)$ is independent of the chosen $\omega(i)$.

In particular, when $k=0$, $Z(\breve{M},\tau)$ coincides with the Kontsevich configuration space integral invariant described in [31].

Furthermore, the α_n of Proposition 8 are also independent of the forms $\omega(i,S^2)$.

Sketch of proof. According to the Stokes theorem, for any $\Gamma \in \mathcal{D}^e_n(C)$,

$$I(\Gamma,(\omega(i,1))_{i\in\underline{3n}}) - I(\Gamma,(\omega(i,0))_{i\in\underline{3n}}) = \sum_F \int_{[0,1]\times F} \bigwedge_{e\in E(\Gamma)} p^*_e(\omega(j_E(e)))$$

where the sum runs over the codimension one faces F of $C(L;\Gamma)$. Below, we sketch the proof that the only contributing faces are the faces $F(\Gamma,B)$ such that $\# B \geq 2$ and Γ_B is a connected component of Γ, or equivalently, that the other faces do not contribute.

Like in [31. Lemma 2.17] faces $F(\Gamma,\infty,B)$ do not contribute. When the product of all the p_e factors through a quotient of $[0,1]\times F(\Gamma,B)$ of smaller dimension, the face $F(\Gamma,B)$ does not contribute. This allows us to get rid of

- the faces $F(\Gamma,B)$ such that B is not a pair of univalent vertices of Γ, and Γ_B is not connected (see [31. Lemma 2.18]);
- the faces $F(\Gamma,B)$ such that $\# B \geq 3$ where Γ_B has a univalent vertex that was trivalent in Γ (see [31. Lemma 2.19]).

We also have faces that cancel each other, for graphs that are identical outside their Γ_B part:

- the faces $F(\Gamma,B)$ (that are not already listed) such that Γ_B has at least a bivalent vertex cancel (mostly by pairs) by the parallelogram identification (see [31. Lemma 2.20]);
- the faces $F(\Gamma,B)$ where Γ_B is an edge between two trivalent vertices cancel by triples, thanks to the Jacobi (or IHX) relation (see [31. Lemma 2.21]);
- similarly, two faces where B is made of two (necessarily consecutive in C) univalent vertices of Γ cancel $(3n - \# E(\Gamma))$ faces $F(\Gamma',B')$ where $\Gamma'_{B'}$ is an edge between a univalent vertex of Γ and a trivalent vertex of Γ, thanks to the STU relation.

Thus, we are left with the faces $F(\Gamma,B)$ such that Γ_B is a (plain) connected component of Γ, and we get the wanted formula for $(Z_n(1)-Z_n(0))$.

In the anomaly case, the same analysis of faces leaves no contributing faces, so that the α_n are independent of the forms $\omega(i,S^2)$ in Proposition 8.

Back to the behaviour of $Z(L,\breve{M},\tau)$ under the assumptions of Theorem 5 or Theorem 8, assume that (M,L,τ) is fixed and apply the formula of the lemma to compute the variation of $Z_n(L,\breve{M},\tau)$ when some propagating chain $\omega(i,0)$ of $(C_2(M),\tau)$ is changed to some other propagating chain $\omega(i,1)=\omega(i,0)+d\eta$. According to Lemma 9, under our assumptions, η can be chosen so that $\eta = p_\tau^*(\eta_{S^2})$ on $\partial C_2(M)$ and $\eta_{S^2}=0$ if $\omega(i,0)$ and $\omega(i,1)$ are homogeneous. Define $\omega(i)=\omega(i,0)+d(t\eta)$ on $[0,1]\times C_2(M)$ ($t\in[0,1]$), and extend the other $\omega(j)$ trivially.

Then $(Z_n(1)-Z_n(0))$ vanishes if $\omega(i,0)$ and $\omega(i,1)$ are homogeneous, as all the involved $I(\Gamma,B)$ do, so that $Z_n(L,\breve{M},\tau)$ is independent from the chosen homogeneous propagating forms $\omega(i)$ of $C_2(M,\tau)$ in Theorem 5. Now, assume that L is straight.

When $i \notin j_E(E(\Gamma))$, the integrand of $I(\Gamma,B)$ factors through the natural projection of $[0,1]\times F(\Gamma,B)$ onto $F(\Gamma,B)$, so that $I(\Gamma,B)=0$. Assume $i = j_E(e_i \in E(\Gamma))$, then $I(\Gamma,B)$ equals

$$\beta_\Gamma \int_{[0,1]\times F(\Gamma,B)} p_{e_i}^*(d(t\eta)) \wedge \bigwedge_{e\in E(\Gamma)\setminus e_i} p_e^*(\omega(j_E(e)).$$

The form $\bigwedge_{e\in E(\Gamma_B)} p_e^*(\omega(j_E(e))$ pulls back through $[0,1]\times F(\Gamma_B,B)$, and through $F(\Gamma_B,B)$ when $e_i \notin E(\Gamma_B)$, so that, for dimension reasons, $I(\Gamma,B)$ vanishes unless $e_i \in E(\Gamma_B)$. Therefore, we assume $e_i \in E(\Gamma_B)$.

When B contains no univalent vertices, $I(\Gamma,B)$ factors through the integral along $[0,1]\times \cup_{m(b)\in\breve{M}} \breve{S}_B(T_{m(b)}M)$ of

$$p_{e_i}^*(d(t\eta)) \wedge \bigwedge_{e\in E(\Gamma_B)\setminus e_i} p_e^*(\omega(j_E(e))).$$

Here the parallelization τ identifies the bundle $\cup_{m(b)\in\breve{M}} \breve{S}_B(T_{m(b)}M)$ with $\breve{M}\times\breve{S}_B(\mathbb{R}^3)$, and the integrand factors through the projection of $[0,1]\times\breve{M}\times\breve{S}_B(\mathbb{R}^3)$ onto $[0,1]\times\breve{S}_B(\mathbb{R}^3)$ whose dimension is smaller (by 3). In particular, $I(\Gamma,B)=0$ in this case, the independence of the choice of the $\omega(i)$ is proved when $k=0$ (when the link is empty), and $Z(\breve{M},\tau)$ coincides with the Kontsevich configuration space integral invariant described in [31].

Let us now study the sum of the $I(\Gamma,B)$, where $(\Gamma\setminus\Gamma_B)$ is a fixed labeled graph and Γ_B is a fixed numbered connected diagram with at least one univalent vertex on S_j^1.

This sum factors through the integral along $[0,1]\times\cup_{m(b)\in K_j}\breve{S}_B(T_{m(b)}M,\Gamma)$ of

$$p_{e_i}^*(d(t\eta))\wedge\bigwedge_{e\in E(\Gamma_B)\setminus e_i}p_e^*(\omega(j_E(e))).$$

At a collapse, the univalent vertices of Γ_B are equipped with a linear order, which makes Γ_B a numbered graph $\tilde{\Gamma}_B$ on $\mathbb{R}$. The corresponding connected component of $[0,1]\times\cup_{m(b)\in K_j}\breve{S}_B(T_{m(b)}M,\Gamma)$ reads $[0,1]\times\cup_{x\in U^+K_j}\breve{Q}(p_\tau(x);\tilde{\Gamma}_B)$ ($\breve{Q}(v;\tilde{\Gamma}_B)$ was defined in Subsection 5.4). This allows us to see the contribution of such a connected component as the integral of a one-form (defined by partial integrations) over $p_\tau(U^+K_j)$. Such an integral is zero when K_j is straight. □

Now, Theorem 8 is a corollary of Theorem 5 (which is not yet completely proved).

5.6. The dependence on the parallelizations in the invariance proofs

Recall that $\mathcal{D}_n^t(C)$ splits according to the number of connected components without univalent vertices of the graphs. Then it is easy to observe that

$$Z(L,\breve{M},\tau)=\sum_{n\in\mathbb{N}}Z_n(L,\breve{M},\tau)=Z^u(L,\breve{M},\tau)Z(M;\tau)$$

where Z^u is obtained from Z by sending the graphs with components that have no univalent vertices to 0, and $Z(M;\tau)=Z(\emptyset,\breve{M},\tau)$. According to [31, Theorem 1.9], $Z(M)=Z(M;\tau)\exp\left(-\frac{1}{4}p_1(\tau)\beta\right)$ is a topological invariant of M. Here, we will now focus on $Z^u(L,\breve{M},\tau)$, and define it with a given homogeneous propagating form, $\omega=\omega(i)$ for all i, so that $Z^u(L,\breve{M},\tau)$ is an invariant of the diffeomorphism class of $(L,\breve{M},\tau)$. We study its variation under a continuous deformation of τ and we prove the following lemma.

Lemma 28. *Let $(\tau(t))_{t\in[0,1]}$ define a smooth homotopy of asymptotically standard parallelizations of $\breve{M}$. Then $\frac{\partial}{\partial t}Z^u(L,\breve{M},\tau(t))$ is equal to*

$$\left(\sum_{j=1}^k\frac{\partial}{\partial t}I_\theta(K_j,\tau(t))\alpha\,\#_j\right)Z^u(L,\breve{M},\tau(t)).$$

Proof. Set $Z_n(t)=Z_n^u(L,\breve{M},\tau(t))$, observe that Z_n (which is valued in a finite-dimensional vector space) is differentiable thanks to the expression of $Z_n(t)-Z_n(0)$ in Lemma 27 (any function $\int_{[0,t]\times C}\omega$ for a smooth compact manifold C and a smooth form ω on $[0,1]\times C$ is differentiable with respect to t). Now, the forms associated to edges of Γ_B do not depend on the configuration of $(V(\Gamma)\setminus B)$. They will be integrated along $[0,1]\times(\cup_{m(b)\in K_j}\breve{S}_B(T_{m(b)}M,\Gamma_B))$, while the other ones will be integrated along $\breve{C}(L;\Gamma\setminus\Gamma_B)$ at $u\in[0,1]$.

Therefore, the global variation $(Z(t)-Z(0))$ reads

$$\sum_{j=1}^k\int_0^t\left(\sum_{\Gamma_B\in\mathcal{D}^c(\mathbb{R})}\beta_{\Gamma_B}I_B(u)[\Gamma_B]\#_j\right)Z(u)du$$

where $\mathcal{D}^c(\mathbb{R})=\cup_{n\in\mathbb{N}}\mathcal{D}_n^c(\mathbb{R})$ and $I_B(u)$ is the integral along $\{c\in\cup_{m(b)\in K_j}\breve{S}_B(T_{m(b)}M,\Gamma_B)\}$ of $\left(\bigwedge_{e\in E(\Gamma_B)}p_e^*(\omega_{S^2})\right)(u,c)$.

Define $I(\Gamma_B,K_j)(t)$ as the integral along

$$\{(u,c);u\in[0,t],c\in\cup_{m(b)\in K_j}\breve{S}_B(T_{m(b)}M,\Gamma_B)\}$$

of $\bigwedge_{e\in E(\Gamma_B)}p_e^*(\omega_{S^2})(u,c)$, so that $I_B(u)=\frac{\partial}{\partial u}I(\Gamma_B,K_j)(u)du$. Therefore, $\frac{\partial}{\partial t}Z(t)$ reads

$$\sum_{j=1}^{k}\left(\sum_{\Gamma_B\in\mathcal{D}^c(\mathbb{R})}\beta_{\Gamma_B}\frac{\partial}{\partial t}I(\Gamma_B,K_j)(t)[\Gamma_B]\#_j\right)Z(t)$$

and we are left with the computation of $\frac{\partial}{\partial t}I(\Gamma_B,K_j)(t)$.

The restriction of $p_{\tau(.)}$ from $[0,1]\times U^+K_j$ to S^2 induces a map

$$p_{a,\tau,\Gamma_B}:[0,1]\times\cup_{m(b)\in K_j}\breve{S}_B(T_{m(b)}M,\Gamma_B)\to\breve{Q}(\Gamma_B)$$

for any Γ_B,

$$I(\Gamma_B,K_j)(t)=\int_{Im(p_{a,\tau,\Gamma_B})}\bigwedge_{e\in E(\Gamma_B)}p_e^*(\omega_{S^2}).$$

Integrating $\bigwedge_{e\in E(\Gamma_B)}p_e^*(\omega_{S^2})[\Gamma_B]$ along the fiber in $\breve{Q}(\Gamma_B)$ yields a two-form on S^2, which is homogeneous, because everything is. Thus this form reads $2\alpha(\Gamma_B)\omega_{S^2}[\Gamma_B]$ where $\alpha(\Gamma_B)\in\mathbb{R}$, and where $\sum_{\Gamma_B\in\mathcal{D}^c(\mathbb{R})}\beta_{\Gamma_B}\alpha(\Gamma_B)[\Gamma_B]=\alpha$. Therefore

$$I(\Gamma_B,K_j)(t)=2\beta_{\Gamma_B}\alpha(\Gamma_B)\int_{[0,t]\times U^+K_j}p_{\tau(.)}^*(\omega_{S^2}).$$

Since $\frac{\partial}{\partial t}\int_{[0,t]\times U^+K_j}p_{\tau(.)}^*(\omega_{S^2})=\frac{1}{2}\frac{\partial}{\partial t}I_\theta(K_j,\tau(t))$, we conclude easily. □

Then the derivative of

$$\prod_{j=1}^{k}\exp(-I_\theta(K_j,\tau(t))\alpha)\#_j\,Z^u(L,\breve{M},\tau(t))$$

vanishes so that this expression does not change when τ smoothly varies.

5.7. End of the proof of Theorem 5

Thanks to [31. Theorem 1.9], in order to conclude the (sketch of) proof of Theorem 5, we are left with the proof that

$$\prod_{j=1}^{k}\left(\exp(-I_\theta(K_j,\tau)\alpha)\#_j\right)Z^u(L,\breve{M},\tau)$$

does not depend on the homotopy class of τ.

When τ changes in a ball that does not meet the link, the forms can be changed only in the neighborhoods of the unit tangent bundle to this ball. Using Lemma 27 again, the variation will be seen on faces $F(\Gamma,B)$, where Γ_B has at least one univalent vertex, and where the forms associated to the edges of Γ_B do not depend on the parameter in [0,1] so that their product vanishes. In particular,

$$\prod_{j=1}^{k}\left(\exp(-I_\theta(K_j,\tau)\alpha)\#_j\right)Z^u(L,\breve{M},\tau)$$

is invariant under the natural action of $\pi_3(SO(3))$ on the homotopy classes of parallelizations.

We now examine the effect of the twist of the parallelization by a map $g:(B_M,1)\to(SO(3),1)$. Without loss, assume that $p_\tau(U^+K_j)=v$ for some v of S^2 and that g maps K_j to rotations with axis v. We want to compute $Z^u(L,\breve{M},\tau\circ\psi_{\mathbb{R}}(g))-Z^u(L,\breve{M},\tau)$. Identify UB_M with $B_M\times S^2$ via τ. There exists a form ω on $[0,1]\times B_M\times S^2$ that reads $p_\tau^*(\omega_{S^2})$ on $\partial([0,1]\times B_M\times S^2)\setminus(1\times B_M\times S^2)$ and that reads $p^*_{\tau\circ\psi_{\mathbb{R}}(g)}(\omega_{S^2})$ on $1\times B_M\times S^2$. Extend this form to a form Ω on $[0,1]\times C_2(M)$, that restricts to $0\times\partial C_2(M)$ as $p_\tau^*(\omega_{S^2})$, and to $1\times\partial C_2(M)$ as $p^*_{\tau\circ\psi_{\mathbb{R}}(g)}(\omega_{S^2})$, where $p_{\tau\circ\psi_{\mathbb{R}}(g)}=p_\tau\circ\psi_{\mathbb{R}}(g^{-1})$ on $B_M\times S^2$ so that $p^*_{\tau\circ\psi_{\mathbb{R}}(g)}(\omega_{S^2})=\psi_{\mathbb{R}}(g^{-1})^*\left(p_\tau^*(\omega_{S^2})\right)$, there. Let $\mathcal{D}_n^{e,u}(C)$ denote the set of diagrams of $\mathcal{D}_n^e(C)$ without components without univalent vertices. Define $Z_n(t)\in\mathcal{D}_n(\coprod_{j=1}^{k}S_j^1)$ by

$$Z_n(t)=\sum_{\Gamma\in\mathcal{D}_n^{e,u}(C)}\beta_\Gamma I(\Gamma,(\Omega_{|t\times C_2(M)})_{i\in\underline{3n}})[\Gamma].$$

For $\Gamma_B \in \mathcal{D}^c(\mathbb{R})$, define $I(\Gamma_B, K_j, \Omega)(t)$ as the integral along

$$\{(u,c); u \in [0,t], c \in \cup_{m(b)\in K_j} \breve{S}_B(T_{m(b)}M, \Gamma_B)\}$$

of $\bigwedge_{e\in E(\Gamma_B)} p_e^*(\Omega)[\Gamma_B]$.

Set $\beta_j(t) = \sum_{\Gamma_B \in \mathcal{D}^c(\mathbb{R})} \beta_{\Gamma_B} I(\Gamma_B, K_j, \Omega)(t)$ and $\gamma_j(t) = \frac{\partial}{\partial t}\beta_j(t)$. Thanks to Lemma 27, like in the proof of Lemma 28, $Z(t)$ is differentiable, and $Z'(t) = \left(\sum_{j=1}^k \gamma_j(t) \#_j\right) Z(t)$.

By induction on the degree, it is easy to see that this equation determines $Z(t)$ as a function of the $\beta_j(t)$ and $Z(0)$ whose degree 0 part is 1, and that $Z(t) = \prod_{j=1}^k \exp(\beta_j(t)) \#_j Z(0)$.

Extend Ω over $[0,2]\times C_2(M)$ so that its restriction to $[1,2]\times B_M \times S^2$ is obtained by applying $\left(\psi_{\mathbb{R}}(g^{-1})\right)^*$ to the Ω translated, and extend all the introduced maps, then $\gamma_j(t+1) = \gamma_j(t)$ because everything is carried by $\left(\psi_{\mathbb{R}}(g^{-1})\right)^*$. In particular $\beta_j(2) = 2\beta_j(1)$.

Now, $Z(2) = Z^u(L, M, \tau\circ\psi_{\mathbb{R}}(g)^2)$ is equal to

$$\prod_{j=1}^k \exp((I_\theta(K_j, \tau\circ\psi_{\mathbb{R}}(g)^2) - I_\theta(K_j,\tau))\alpha) \#_j Z(0),$$

where $Z(0) = Z^u(L, \breve{M}, \tau)$, since g^2 is homotopic to the trivial map outside a ball (see Lemma 29, 2). By induction on the degree of diagrams, this shows

$$\beta_j(2) = (I_\theta(K_j, \tau\circ\psi_{\mathbb{R}}(g)^2) - I_\theta(K_j,\tau))\alpha.$$

Conclude by observing that under our assumptions, where $I_\theta(K_j, \tau\circ\psi_{\mathbb{R}}(g)^i)$ is the linking number of K_j and its parallel induced by $\tau\circ\psi_{\mathbb{R}}(g)^i$, $I_\theta(K_j, \tau\circ\psi_{\mathbb{R}}(g)^2) - I_\theta(K_j,\tau)$ is equal to $2(I_\theta(K_j, \tau\circ\psi_{\mathbb{R}}(g)) - I_\theta(K_j,\tau))$. This finishes the (sketch of) proof of Theorem 5 in general.

5.8. Some open questions

1. A Vassiliev invariant is *odd* if it distinguishes some knot from the same knot with the opposite orientation. Are there odd Vassiliev invariants?
2. More generally, do Vassiliev invariants distinguish knots in S^3?
3. According to a theorem of Bar-Natan and Lawrence [46], the LMO invariant fails to distinguish rational homology 3-spheres with isomorphic H_1, so that, according to a Moussard theorem [21], rational finite type invariants fail to distinguish $\mathbb{Q}$-spheres. Do finite type invariants distinguish $\mathbb{Z}$-spheres?
4. Find relationships between Z or other finite type invariants and Heegaard Floer homologies. See [6] to get propagators associated to Heegaard diagrams. Also see related work by Shimizu and Watanabe [47; 48].
5. Compare Z with the LMO invariant Z_{LMO}.
6. Compute the anomalies α and β.
7. Find surgery formulae for Z.
8. Kricker defined a lift $\tilde{Z}^K$ of the Kontsevich integral Z^K (or the LMO invariant) for null-homologous knots in $\mathbb{Q}$-spheres [49; 50]. The Kricker lift is valued in a space $\tilde{A}$ that is mapped to $\mathcal{D}_n(S^1)$ by a map H, which allows one to recover Z^K from $\tilde{Z}^K$. The space $\tilde{A}$ is a space of trivalent diagrams whose edges are decorated by rational functions whose denominators divide the Alexander polynomial. Compare the Kricker lift $\tilde{Z}^K$ with the equivariant configuration space invariant $\tilde{Z}^c$ of [51] valued in the same diagram space $\tilde{A}$. See [52] for alternative definitions and further properties of $\tilde{Z}^c$.
9. Is Z obtained from $\tilde{Z}^c$ in the same way as Z^K is obtained from $\tilde{Z}^K$?

6. More on parallelizations of 3-manifolds and Pontrjagin classes

In order to make the definition of Θ complete, we give a detailed self-contained presentation of $p_1(\tau)$. In this section, M is a smooth oriented connected 3-manifold with possible boundary.

6.1. $[(M,\partial M),(SO(3),1)]$ is an abelian group.

Again, see S^3 as $B^3/\partial B^3$ and see B^3 as $([0,2\pi]\times S^2)/(0\sim\{0\}\times S^2)$. Recall that $\rho : B^3 \to SO(3)$ maps $(\theta\in[0,2\pi], v\in S^2)$ to the rotation $\rho(\theta,v)$ with axis directed by v and with angle θ.

Also recall that the group structure of $[(M,\partial M),(SO(3),1)]$ is induced by the multiplication of maps, using the multiplication of $SO(3)$.

Any $g\in C^0\left((M,\partial M),(SO(3),1)\right)$ induces a map

$$H_1(g) : H_1(M,\partial M)\to\left(H_1(SO(3),1)=\frac{\mathbb{Z}}{2\mathbb{Z}}\right),$$

where coefficients are in $\mathbb{Z}$ unless otherwise mentioned, so that $H_1(g)=H_1(g;\mathbb{Z})$ and $H_1(M,\partial M)=H_1(M,\partial M;\mathbb{Z})$. Since

$$H_1\left(M,\partial M;\frac{\mathbb{Z}}{2\mathbb{Z}}\right)=H_1(M,\partial M)/2H_1(M,\partial M)=H_1(M,\partial M)\otimes_{\mathbb{Z}}\frac{\mathbb{Z}}{2\mathbb{Z}},$$

$\mathrm{Hom}\left(H_1(M,\partial M),\frac{\mathbb{Z}}{2\mathbb{Z}}\right)$ is isomorphic to

$$\mathrm{Hom}\left(H_1\left(M,\partial M;\frac{\mathbb{Z}}{2\mathbb{Z}}\right),\frac{\mathbb{Z}}{2\mathbb{Z}}\right)=H^1\left(M,\partial M;\frac{\mathbb{Z}}{2\mathbb{Z}}\right)$$

and the image of $H_1(g)$ under the above isomorphisms is denoted by $H^1(g;\mathbb{Z}/2\mathbb{Z})$. (Formally, this $H^1(g;\mathbb{Z}/2\mathbb{Z})$ denotes the image of the generator of $H^1(SO(3),1;\mathbb{Z}/2\mathbb{Z})=\mathbb{Z}/2\mathbb{Z}$ under $H^1(g;\mathbb{Z}/2\mathbb{Z})$ in $H^1(M,\partial M;\mathbb{Z}/2\mathbb{Z})$.)

Lemma 29. *Let M be an oriented connected 3-manifold with possible boundary. Recall that $\rho_M(B^3)\in C^0\left((M,\partial M),(SO(3),1)\right)$ is a map that coincides with ρ on a ball B^3 embedded in M and that maps the complement of B^3 to the unit of $SO(3)$.*

1. Any homotopy class of a map g from $(M,\partial M)$ to $(SO(3),1)$, such that $H^1(g;\mathbb{Z}/2\mathbb{Z})$ is trivial, belongs to the subgroup $\langle[\rho_M(B^3)]\rangle$ of $[(M,\partial M),(SO(3),1)]$ generated by $[\rho_M(B^3)]$.

2. For any $[g]\in[(M,\partial M),(SO(3),1)]$, $[g]^2\in\langle[\rho_M(B^3)]\rangle$.

3. The group $[(M,\partial M),(SO(3),1)]$ is abelian.

Proof. Let $g\in C^0\left((M,\partial M),(SO(3),1)\right)$. Assume that $H^1(g;\mathbb{Z}/2\mathbb{Z})$ is trivial. Choose a cell decomposition of M with respect to its boundary, with only one three-cell, no zero-cell if $\partial M=\varnothing$, one zero-cell if $\partial M=\varnothing$, one-cells, and two-cells. Then after a homotopy relative to ∂M, we may assume that g maps the one-skeleton of M to 1. Next, since $\pi_2(SO(3))=0$, we may assume that g maps the two-skeleton of M to 1, and therefore that g maps the exterior of some 3-ball to 1. Now g becomes a map from $B^3/\partial B^3=S^3$ to $SO(3)$, and its homotopy class is $k[\tilde\rho]$ in $\pi_3(SO(3))=\mathbb{Z}[\tilde\rho]$. Therefore g is homotopic to $\rho_M(B^3)^k$. This proves the first assertion.

Since $H^1(g^2;\mathbb{Z}/2\mathbb{Z})=2H^1(g;\mathbb{Z}/2\mathbb{Z})$ is trivial, the second assertion follows.

For the third assertion, first note that $[\rho_M(B^3)]$ belongs to the center of $[(M,\partial M),(SO(3),1)]$ because it can be supported in a small ball disjoint from the support (preimage of $SO(3)\setminus\{1\}$) of a representative of any other element. Therefore, according to the second assertion any square will be in the center. Furthermore, since any commutator induces the trivial map on $\pi_1(M)$, any commutator is in $\langle[\rho_M(B^3)]\rangle$. In particular, if f and g are elements of $[(M,\partial M),(SO(3),1)]$, $(gf)^2=(fg)^2=(f^{-1}f^2g^2f)(f^{-1}g^{-1}fg)$ where the first factor equals $f^2g^2=g^2f^2$. Exchanging f and g yields $f^{-1}g^{-1}fg=g^{-1}f^{-1}gf$. Then the commutator, which is a power of $[\rho_M(B^3)]$, has a vanishing square, and thus a vanishing degree. Then it must be trivial. □

6.2. Any oriented 3-manifold is parallelizable

In this subsection, we prove the following standard theorem. The spirit of our proof is the same as the Kirby proof in [53. P. 46]. But instead of assuming familiarity with the obstruction theory described by Steenrod in [54. Part III], we use this proof as an introduction to this theory.

Theorem 9. [Stiefel]. *Any oriented* 3*-manifold is parallelizable.*

Lemma 30. *The restriction of the tangent bundle TM to an oriented* 3*-manifold M to any closed (non-necessarily orientable) surface S immersed in M is trivializable.*

Proof. Let us first prove that this bundle is independent of the immersion. It is the direct sum of the tangent bundle to the surface and of its normal one-dimensional bundle. This normal bundle is trivial when S is orientable, and its unit bundle is the 2-fold orientation cover of the surface, otherwise. (The orientation cover of S is its 2-fold orientable cover, which is trivial over annuli embedded in the surface). Then since any surface S can be immersed in $\mathbb{R}^3$, the restriction $TM_{|S}$ is the pull-back of the trivial bundle of $\mathbb{R}^3$ by such an immersion, and it is trivial. □

Then using Stiefel—Whitney classes, the proof of Theorem 9 quickly goes as follows. Let M be an orientable smooth 3-manifold, equipped with a smooth triangulation. (A theorem of Whitehead proved in the Munkres book [55] ensures the existence of such a triangulation.) By definition, the *first Stiefel—Whitney class* $w_1(TM) \in H^1(M;\mathbb{Z}/2\mathbb{Z} = \pi_0(GL(\mathbb{R}^3)))$ seen as a map from $\pi_1(M)$ to $\mathbb{Z}/2\mathbb{Z}$ maps the class of a loop c embedded in M to 0 if $TM_{|c}$ is orientable and to 1 otherwise. It is the obstruction to the existence of a trivialization of TM over the one-skeleton of M. Since M is orientable, the first Stiefel—Whitney class $w_1(TM)$ vanishes and TM can be trivialized over the one-skeleton of M. The *second Stiefel—Whitney class* $w_2(TM) \in H^2(M;\mathbb{Z}/2\mathbb{Z} = \pi_1(GL^+(\mathbb{R}^3)))$ seen as a map from $H_2(M;\mathbb{Z}/2\mathbb{Z})$ to $\mathbb{Z}/2\mathbb{Z}$ maps the class of a connected closed surface S to 0 if $TM_{|S}$ is trivializable and to 1 otherwise. The second Stiefel—Whitney class $w_2(TM)$ is the obstruction to the existence of a trivialization of TM over the two-skeleton of M, when $w_1(TM) = 0$. According to the above lemma, $w_2(TM) = 0$, and TM can be trivialized over the two-skeleton of M. Then since $\pi_2(GL^+(\mathbb{R}^3)) = 0$, any parallelization over the two-skeleton of M can be extended as a parallelization of M. □

We detail the involved arguments below without mentioning Stiefel—Whitney classes, (actually by almost defining $w_2(TM)$). The elementary proof below can be thought of as an introduction to the obstruction theory used above.

Elementary proof of Theorem 9. Let M be an oriented 3-manifold. Choose a triangulation of M. For any cell c of the triangulation, define an arbitrary trivialization $\tau_c : c \times \mathbb{R}^3 \to TM_{|c}$ such that τ_c induces the orientation of M. This defines a trivialization $\tau^{(0)} : M^{(0)} \times \mathbb{R}^3 \to TM_{|M^{(0)}}$ of M over the 0-skeleton $M^{(0)}$ of M. Let $C_k(M)$ be the set of k-cells of the triangulation. Every cell is equipped with an arbitrary orientation. For an edge $e \in C_1(M)$ of the triangulation, on ∂e, $\tau^{(0)}$ reads $\tau^{(0)} = \tau_e \circ \psi_{\mathbb{R}}(g_e)$ for a map $g_e : \partial e \to GL^+(\mathbb{R}^3)$. Since $GL^+(\mathbb{R}^3)$ is connected, g_e extends to e, and $\tau^{(1)} = \tau_e \circ \psi_{\mathbb{R}}(g_e)$ extends $\tau^{(0)}$ to e. Doing so for all the edges extends $\tau^{(0)}$ to a trivialization $\tau^{(1)}$ of the one-skeleton $M^{(1)}$ of M.

For an oriented triangle t of the triangulation, on ∂t, $\tau^{(1)}$ reads $\tau^{(1)} = \tau_t \circ \psi_{\mathbb{R}}(g_t)$ for a map $g_t : \partial t \to GL^+(\mathbb{R}^3)$. Let $E(t,\tau^{(1)})$ be the homotopy class of g_t in $(\pi_1(GL^+(\mathbb{R}^3)) = \pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}$, $E(t,\tau^{(1)})$ is independent of τ_t. Then $E(.,\tau^{(1)}) : C_2(M) \to \mathbb{Z}/2\mathbb{Z}$ is a cochain. When $E(.,\tau^{(1)}) = 0$, $\tau^{(1)}$ may be extended to a trivialization $\tau^{(2)}$ over the two-skeleton of M, as before.

Since $\pi_2(GL^+(\mathbb{R}^3)) = 0$, $\tau^{(2)}$ can next be extended over the three-skeleton of M, that is over M.

Let us now study the obstruction cochain $E(.,\tau^{(1)})$ whose vanishing guarantees the existence of a parallelization of M.

If the map g_e associated to e is changed to $d(e)g_e$ for some $d(e)$: $(e,\partial e) \to (GL^+(\mathbb{R}^3),1)$ for every edge e, define the associated trivialization $\tau^{(1)\prime}$, and the cochain $D(\tau^{(1)},\tau^{(1)\prime}) \to \mathbb{Z}/2\mathbb{Z}$

that maps e to the homotopy class of $d(e)$. Then $(E(\cdot,\tau^{(1)\prime})-E(\cdot,\tau^{(1)}))$ is the coboundary of $D(\tau^{(1)},\tau^{(1)\prime})$.

Let us show that $E(\cdot,\tau^{(1)})$ is a cocycle. Consider a 3-simplex T, then $\tau^{(0)}$ extends to T. Without loss of generality, assume that τ_T coincides with this extension, that for any face t of T, τ_t is the restriction of τ_T to t, and that the above $\tau^{(1)\prime}$ coincides with τ_T on the edges of ∂T. Then $E(\cdot,\tau^{(1)\prime})(\partial T)=0$. Since a coboundary also maps ∂T to 0, $E(\cdot,\tau^{(1)\prime})(\partial T)=0$.

Now, it suffices to prove that the cohomology class of $E(\cdot,\tau^{(1)})$ (which is actually $w_2(TM)$) vanishes in order to prove that there is an extension $\tau^{(1)\prime}$ of $\tau^{(0)}$ on $M^{(1)}$ that extends on M.

Since $H^2(M;\mathbb{Z}/2\mathbb{Z}) = \mathrm{Hom}(H_2(M;\mathbb{Z}/2\mathbb{Z});\mathbb{Z}/2\mathbb{Z})$, it suffices to prove that $E(\cdot,\tau^{(1)})$ maps any 2-dimensional $\mathbb{Z}/2\mathbb{Z}$-cycle C to 0.

We represent the class of such a cycle C by a non-necessarily orientable closed surface S as follows. Let $N(M^{(0)})$ and $N(M^{(1)})$ be small regular neighborhoods of $M^{(0)}$ and $M^{(1)}$ in M, respectively, such that $N(M^{(1)})\cap(M\setminus N(M^{(0)}))$ is a disjoint union, running over the edges e, of solid cylinders B_e identified with $]0,1[\times D^2$. The core $]0,1[\times\{0\}$ of $B_e=]0,1[\times D^2$ is a connected part of the interior of the edge e. ($N(M^{(1)})$ is thinner than $N(M^{(0)})$.)

Construct S in the complement of $N(M^{(0)})\cup N(M^{(1)})$ as the intersection of the support of C with this complement. Then the closure of S meets the part $[0,1]\times S^1$ of every $\overline{B_e}$ as an even number of parallel intervals from $\{0\}\times S^1$ to $\{1\}\times S^1$. Complete S in $M\setminus N(M^{(0)})$ by connecting the intervals pairwise in $\overline{B_e}$ by disjoint bands. After this operation, the boundary of the closure of S is a disjoint union of circles in the boundary of $N(M^{(0)})$, where $N(M^{(0)})$ is a disjoint union of balls around the vertices. Glue disjoint disks of $N(M^{(0)})$ along these circles to finish the construction of S.

Extend $\tau^{(0)}$ to $N(M^{(0)})$, assume that $\tau^{(1)}$ coincides with this extension over $M^{(1)}\cap N(M^{(0)})$, and extend $\tau^{(1)}$ to $N(M^{(1)})$. Then $TM_{|S}$ is trivial, and we may choose a trivialization τ_S of TM over S that coincides with our extension of $\tau^{(0)}$ over $N(M^{(0)})$, over $S\cap N(M^{(0)})$. We have a cell decomposition of $(S,S\cap N(M^{(0)}))$ with only 1-cells and 2-cells, where the 2-cells of S are in one-to-one canonical correspondence with the 2-cells of C, and one-cells bijectively correspond to bands connecting two-cells in the cylinders B_e. These one-cells are equipped with the trivialization of TM induced by $\tau^{(1)}$. Then we can define 2-dimensional cochains $E_S(\cdot,\tau^{(1)})$ and $E_S(\cdot,\tau_S)$ from $C_2(S)$ to $\mathbb{Z}/2\mathbb{Z}$ as before, with respect to this cellular decomposition of S, where $(E_S(\cdot,\tau^{(1)}) - E_S(\cdot,\tau_S))$ is again a coboundary and $E_S(\cdot,\tau_S)=0$ so that $E_S(C,\tau^{(1)})=0$, and since $E(C,\tau^{(1)})=E_S(C,\tau^{(1)})$, $E(C,\tau^{(1)})=0$ and we are done. □

6.3. The homomorphism induced by the degree on $[(M,\partial M),(SO(3),1)]$

Let S be a non-necessarily orientable closed surface embedded in the interior of M, and let τ be a parallelization of M. We define a twist $g(S,\tau)\in C^0\left((M,\partial M),(SO(3),1)\right)$ below.

The surface S has a tubular neighborhood $N(S)$, which is a $[-1,1]$-bundle over S that admits (orientation-preserving) bundle charts with domains $[-1,1]\times D$ for disks D of S so that the changes of coordinates restrict to the fibers as $\pm$Identity. Then

$$g(S,\tau):(M,\partial M)\to(GL^+(\mathbb{R}^3),1)$$

is the continuous map that maps $M\setminus N(S)$ to 1 such that $g(S,\tau)((t,s)\in[-1,1]\times D)$ is the rotation with angle $\pi(t+1)$ and with axis $p_2(\tau^{-1}(\nu_s)=(s,p_2(\tau^{-1}(\nu_s))))$ where $\nu_s=T_{(0,s)}([-1,1]\times s)$ is the tangent vector to the fiber $[-1,1]\times s$ at $(0,s)$. Since this rotation coincides with the rotation with opposite axis and with opposite angle $\pi(1-t)$, our map $g(S,\tau)$ is a well-defined continuous map.

Clearly, the homotopy class of $g(S,\tau)$ only depends on the homotopy class of τ and on the isotopy class of S. When $M=B^3$, when τ is the standard parallelization of $\mathbb{R}^3$, and when $\frac{1}{2}S^2$ denotes the sphere $\frac{1}{2}\partial B^3$ inside B^3, the homotopy class of $g\left(\frac{1}{2}S^2,\tau\right)$ coincides with the homotopy class of ρ.

Lemma 31. *$H^1(g(S,\tau);\mathbb{Z}/2\mathbb{Z})$ is the mod 2 intersection with S.*

The map $H^1(.;\mathbb{Z}/2\mathbb{Z})$ from $[(M,\partial M),(SO(3),1)]$ to $H^1(M,\partial M;\mathbb{Z}/2\mathbb{Z})$ is onto.

Proof. The first assertion is obvious, and the second one follows since $H^1(M,\partial M;\mathbb{Z}/2\mathbb{Z})$ is the Poincaré dual of $H_2(M;\mathbb{Z}/2\mathbb{Z})$ and since any element of $H^2(M;\mathbb{Z}/2\mathbb{Z})$ is the class of a closed surface. □

Lemma 32. *The degree is a group homomorphism* $\deg\colon[(M,\partial M),(SO(3),1)]\to\mathbb{Z}$ *and* $\deg(\rho_M(B^3)^k)=2k$.

Proof. It is easy to see that $\deg(fg)=\deg(f)+\deg(g)$ when f or g is a power of $[\rho_M(B^3)]$. Let us prove that $\deg(f^2)=2\deg(f)$ for any f. According to Lemma 31, there is an unoriented embedded surface S_f of the interior of C such that $H^1(f;\mathbb{Z}/2\mathbb{Z})=H^1(g(S_f,\tau);\mathbb{Z}/2\mathbb{Z})$ for some trivialization τ of TM. Then, according to Lemma 29, $fg(S_f,\tau)^{-1}$ is homotopic to some power of $\rho_M(B^3)$, and we are left with the proof that the degree of g^2 is $2\deg(g)$ for $g=g(S_f,\tau)$. This can easily be done by noticing that g^2 is homotopic to $g(S_f^{(2)},\tau)$ where $S_f^{(2)}$ is the boundary of the tubular neighborhood of S_f. In general, $\deg(fg)=\frac{1}{2}\deg((fg)^2)=\frac{1}{2}\deg(f^2g^2)=\frac{1}{2}\left(\deg(f^2)+\deg(g^2)\right)$, and the lemma is proved. □

Lemmas 29 and 32 imply the following lemma.

Lemma 33. *The degree induces an isomorphism* $\deg\colon[(M,\partial M),(SO(3),1)]\otimes_{\mathbb{Z}}\mathbb{Q}\to\mathbb{Q}$. *Any group homomorphism* $\phi:[(M,\partial M),(SO(3),1)]\to\mathbb{Q}$ *reads* $\frac{1}{2}\phi(\rho_M(B^3))\deg$.

6.4. First homotopy groups of the groups $SU(n)$

Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. Let $n\in\mathbb{N}$. The stabilization maps induced by the inclusions

$$\begin{array}{llll} i: & GL(\mathbb{K}^n) & \hookrightarrow & GL(\mathbb{K}\oplus\mathbb{K}^n) \\ & g & \mapsto & \left(i(g):(x,y)\mapsto(x,g(y))\right) \end{array}$$

will be denoted by i. Elements of $GL(\mathbb{K}^n)$ are represented by matrices whose columns contain the coordinates of the images of the basis elements, with respect to the standard basis of $\mathbb{K}^n$.

See S^3 as the unit sphere of $\mathbb{C}^2$ so that its elements are the pairs (z_1,z_2) of complex numbers such that $|z_1|^2+|z_2|^2=1$.

The group $SU(2)$ is identified with S^3 by the homeomorphism

$$\begin{array}{llll} m_r^{\mathbb{C}}: & S^3 & \to & SU(2) \\ & (z_1,z_2) & \mapsto & \begin{bmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{bmatrix} \end{array}$$

so that the first non trivial homotopy group of $SU(2)$ is $\pi_3(SU(2))=\mathbb{Z}[m_r^{\mathbb{C}}]$.

The long exact sequence associated to the fibration

$$SU(n-1)\overset{i}{\rightarrowtail}SU(n)\to S^{2n-1}$$

shows that $i_*^n:\pi_j(SU(2))\to\pi_j(SU(n+2))$ is an isomorphism for $j\le 3$ and $n\ge 0$, and in particular, that $\pi_j(SU(4))=\{1\}$ for $j\le 2$ and $\pi_3(SU(4))=\mathbb{Z}[i^2(m_r^{\mathbb{C}})]$ where $i^2(m_r^{\mathbb{C}})$ is the following map

$$\begin{array}{llll} i^2(m_r^{\mathbb{C}}): & (S^3\subset\mathbb{C}^2) & \to & SU(4) \\ & (z_1,z_2) & \mapsto & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z_1 & -\bar{z}_2 \\ 0 & 0 & z_2 & \bar{z}_1 \end{bmatrix}. \end{array}$$

6.5. Definition of relative Pontrjagin numbers

Let M_0 and M_1 be two compact connected oriented 3-manifolds whose boundaries have collars that are identified by a diffeomorphism. Let $\tau_0 : M_0 \times \mathbb{R}^3 \to TM_0$ and $\tau_1 : M_1 \times \mathbb{R}^3 \to TM_1$ be two parallelizations (which respect the orientations) that agree on the collar neighborhoods of $\partial M_0 = \partial M_1$. Then the *relative Pontrjagin number* $p_1(\tau_0, \tau_1)$ is the Pontrjagin obstruction to extending the trivialization of $TW \otimes \mathbb{C}$ induced by τ_0 and τ_1 across the interior of a signature 0 cobordism W from M_0 to M_1. Details follow.

Let M be a compact connected oriented 3-manifold. A *special complex trivialization* of TM is a trivialization of $TM \otimes \mathbb{C}$ that is obtained from a trivialization $\tau_M : M \times \mathbb{R}^3 \to TM$ that induces the orientation of M by composing $(\tau_M^{\mathbb{C}} = \tau_M \otimes_{\mathbb{R}} \mathbb{C}) : M \times \mathbb{C}^3 \to TM \otimes \mathbb{C}$ by

$$\begin{array}{rccc} \psi(G): & M \times \mathbb{C}^3 & \to & M \times \mathbb{C}^3 \\ & (x,y) & \mapsto & (x, G(x)(y)) \end{array}$$

for a map $G : M \to SL(3,\mathbb{C})$. The definition and properties of relative Pontrjagin numbers, which are given with more details below, are valid for pairs of special complex trivializations.

The *signature* of a 4-manifold is the signature of the intersection form on its $H_2(\cdot;\mathbb{R})$ (number of positive entries minus number of negative entries in a diagonalised version of this form). Also recall that any closed oriented three-manifold bounds a compact oriented 4-dimensional manifold whose signature may be arbitrarily changed by connected sums with copies of $\mathbb{C}P^2$ or $-\mathbb{C}P^2$. A *cobordism from M_0 to M_1* is a compact oriented 4-dimensional manifold W with corners whose boundary ∂W is equal to $-M_0 \cup_{\partial M_0 \sim 0 \times \partial M_0} (-[0,1] \times \partial M_0) \cup_{\partial M_1 \sim 1 \times \partial M_0} M_1$, and is identified with an open subspace of one of the products $[0,1[\times M_0$ or $]0,1] \times M_1$ near ∂W, as Fig. 3 suggests.

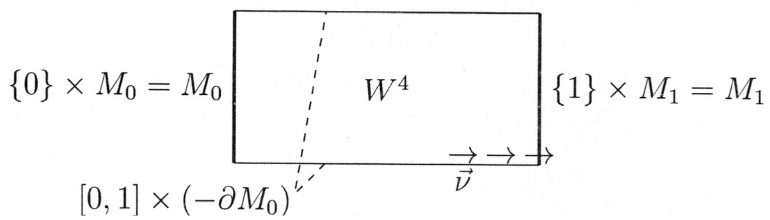


Fig. 3

Let $W = W^4$ be such a cobordism from M_0 to M_1, with signature 0. Consider the complex 4-bundle $TW \otimes \mathbb{C}$ over W. Let $\vec{\nu}$ be the tangent vector to $[0,1] \times \{pt\}$ over ∂W (under the identifications above), and let $\tau(\tau_0, \tau_1)$ denote the trivialization of $TW \otimes \mathbb{C}$ over ∂W that is obtained by stabilizing either τ_0 or τ_1 into $\vec{\nu} \oplus \tau_0$ or $\vec{\nu} \oplus \tau_1$. Then the obstruction to extending this trivialization to W is the relative first *Pontrjagin class* $p_1(W; \tau(\tau_0, \tau_1))[W, \partial W]$ of the trivialization, which belongs to $H^4(W, \partial W; \mathbb{Z} = \pi_3(SU(4))) = \mathbb{Z}[W, \partial W]$.

Now, we specify our sign conventions for this Pontrjagin class. They are the same as in [56]. In particular, p_1 is the opposite of the second Chern class c_2 of the complexified tangent bundle. See [56. P. 174]. More precisely, equip M_0 and M_1 with Riemannian metrics that coincide near ∂M_0, and equip W with a Riemannian metric that coincides with the orthogonal product metric of one of the products $[0,1] \times M_0$ or $[0,1] \times M_1$ near ∂W. Equip $TW \otimes \mathbb{C}$ with the associated hermitian structure. The determinant bundle of TW is trivial because W is oriented, and $\det(TW \otimes \mathbb{C})$ is also trivial. Our parallelization $\tau(\tau_0, \tau_1)$ over ∂W is special with respect to the trivialization of $\det(TW \otimes \mathbb{C})$. Up to homotopy, assume that $\tau(\tau_0, \tau_1)$ is unitary with respect to the hermitian structure of $TW \otimes \mathbb{C}$ and the standard hermitian form of $\mathbb{C}^4$. Since $\pi_i(SU(4)) = \{0\}$ when $i < 3$, the trivialization $\tau(\tau_0, \tau_1)$ extends to a special unitary trivialization τ outside the interior of a 4-ball B^4 and defines

$$\tau : S^3 \times \mathbb{C}^4 \to (TW \otimes \mathbb{C})_{|S^3}$$

over the boundary $S^3=\partial B^4$ of this 4-ball B^4. Over this 4-ball B^4, the bundle $TW\otimes C$ admits a trivialization $\tau_B : B^4\times\mathbb{C}^4 \to (TW\otimes\mathbb{C})_{|B^4}$. Then $\tau_B^{-1}\circ\tau(v\in S^3, w\in\mathbb{C}^4) = (v,\phi(v)(w))$, for a map $\phi : S^3 \to SU(4)$ whose homotopy class reads

$$[\phi] = -p_1(W;\tau(\tau_0,\tau_1))[i^2(m_r^{\mathbb{C}})] \in \pi_3(SU(4)).$$

Define $p_1(\tau_0,\tau_1) = p_1(W;\tau(\tau_0,\tau_1))$.

Proposition 9. *The first Pontrjagin number $p_1(\tau_0,\tau_1)$ is well-defined by the above conditions.*

Proof. According to the Nokivov additivity theorem, if a closed (compact, without boundary) 4-manifold Y reads $Y = Y^+\cup_X Y^-$ where Y^+ and Y^- are two 4-manifolds with boundary, embedded in Y that intersect along a closed 3-manifold X (their common boundary, up to orientation) then

$$\text{signature}(Y) = \text{signature}(Y^+) + \text{signature}(Y^-).$$

According to a Rohlin theorem (see [57] or [58. P. 18]), when Y is a compact oriented 4-manifold without boundary, $p_1(Y) = 3\,\text{signature}(Y)$.

We only need to prove that $p_1(\tau_0,\tau_1)$ is independent of the signature 0 cobordism W. Let W_E be a 4-manifold of signature 0 bounded by $(-\partial W)$. Then $W\cup_{\partial W} W_E$ is a 4-dimensional manifold without boundary whose signature is (signature(W_E) + signature(W) = 0) by the Novikov additivity theorem. According to the Rohlin theorem, the first Pontrjagin class of $W\cup_{\partial W} W_E$ is also zero. On the other hand, this first Pontrjagin class is the sum of the relative first Pontrjagin classes of W and W_E with respect to $\tau(\tau_0,\tau_1)$. These two relative Pontrjagin classes are opposite and therefore the relative first Pontrjagin class of W with respect to $\tau(\tau_0,\tau_1)$ does not depend on W.

Similarly, it is easy to prove the following proposition.

Proposition 10. *Under the above assumptions except for the assumption on the signature of the cobordism W, $p_1(\tau_0,\tau_1) = p_1(W;\tau(\tau_0,\tau_1)) - 3\,\text{signature}(W_E)$.*

6.6. On the groups $SO(3)$ and $SO(4)$

In this subsection, we describe $\pi_3(SO(4))$ and the natural maps from $\pi_3(SO(3))$ to $\pi_3(SO(4))$ and to $\pi_3(SU(4))$.

The *quaternion field* $\mathbb{H}$ is the vector space $\mathbb{C}\oplus\mathbb{C}j$ equipped with the multiplication that maps $(z_1+z_2j, z_1'+z_2'j)$ to $(z_1z'_1 - z_2\overline{z'_2}) + (z_2\overline{z'_1} + z_1z'_2)j$, and with the conjugation that maps (z_1+z_2j) to $\overline{z_1+z_2j} = \overline{z_1} - z_2j$. The norm of (z_1+z_2j) is the square root of $|z_1|^2+|z_2|^2 = (z_1+z_2j)\overline{z_1+z_2j}$, it is multiplicative. Setting $k=ij$, $(1,i,j,k)$ is an orthogonal basis of $\mathbb{H}$ with respect to the scalar product associated to the norm. The unit sphere of $\mathbb{H}$ is the sphere S^3, which is equipped with the corresponding group structure. There are two group morphisms from S^3 to $SO(4)$ induced by the multiplication in $\mathbb{H}$,

$$\begin{array}{rccl} m_\ell : & S^3 & \to & (SO(\mathbb{H}) = SO(4)) \\ & x & \mapsto & m_\ell(x) : v \mapsto x.v \\ \overline{m_r} : & S^3 & \to & SO(\mathbb{H}) \\ & y & \mapsto & (\overline{m_r}(y) : v \mapsto v.\overline{y}). \end{array}$$

Together, they induce the group morphism

$$\begin{array}{ccc} S^3\times S^3 & \to & SO(4) \\ (x,y) & \mapsto & (v\mapsto x.v.\overline{y}). \end{array}$$

The kernel of this group morphism is $\mathbb{Z}/2\mathbb{Z}(-1,-1)$ so that this morphism is a two-fold covering. In particular, $\pi_3(SO(4)) = \mathbb{Z}[m_\ell]\oplus\mathbb{Z}[\overline{m_r}]$.

For $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$ and $n\in\mathbb{N}$, the $\mathbb{K}$ (euclidean or hermitian) oriented vector space with the direct orthonormal basis $(v_1,\dots,v_n)$ is denoted by $\mathbb{K}\langle v_1,\dots,v_n\rangle$. There is also the following group morphism

$$\begin{array}{rcl}\tilde{\rho}: \ S^3 & \to & SO(\mathbb{R}<i,j,k>) = SO(3)\\ x & \mapsto & \left(v \mapsto (v \mapsto x.v.\overline{x})\right)\end{array}$$

whose kernel is $\mathbb{Z}/2\mathbb{Z}(-1)$. This morphism $\tilde{\rho}$ is also a two-fold covering.

Lemma 34. *This definition of $\tilde{\rho}$ coincides with the previous one, up to homotopy.*

Proof. It is clear that the two maps coincide up to homotopy, up to orientation since both classes generate $\pi_3(SO(3)) = \mathbb{Z}$. We take care of the orientation using the outward normal first convention to orient boundaries, as usual. An element of S^3 reads $\cos(\theta)+\sin(\theta)v$ for a unique $\theta\in[0,\pi]$ and a unit quaternion v with real part zero, which is unique when $\theta\notin\{0,\pi\}$. In particular, this defines a diffeomorphism ϕ from $]0,\pi[\times S^2$ to $S^3\setminus\{-1,1\}$. We compute the degree of ϕ at $\phi(\pi/2,i)$. The space $\mathbb{H}$ is oriented as $\mathbb{R}\oplus\mathbb{R}\langle i,j,k\rangle$, where $\mathbb{R}\langle i,j,k\rangle$ is oriented by the outward normal to S^2, which coincides with the outward normal to S^3 in $\mathbb{R}^4$, followed by the orientation of S^2. In particular since cos is an orientation-reversing diffeomorphism at $\pi/2$, the degree of ϕ is 1 and ϕ preserves the orientation. Now $(\cos(\theta)+\sin(\theta)v)w\overline{(\cos(\theta)+\sin(\theta)v)} = R(\theta,v)(w)$ where $R(\theta,v)$ is a rotation with axis v for any v. Since $R(\theta,i)(j) = \cos(2\theta)j+\sin(2\theta)k$, the two maps $\tilde{\rho}$ are homotopic. One can check that they are actually conjugate. □

Define

$$\begin{array}{rcl}m_r: \ S^3 & \to & (SO(\mathbb{H}) = SO(4))\\ y & \mapsto & (m_r(y): v\mapsto v.y).\end{array}$$

Lemma 35. *In* $\pi_3(SO(4)) = \mathbb{Z}[m_\ell]\oplus\mathbb{Z}[\overline{m_r}]$, $i_*([\tilde{\rho}]) = [m_\ell]+[\overline{m_r}] = [m_\ell]-[m_r]$.

Proof. The π_3-product in $\pi_3(SO(4))$ coincides with the product induced by the group structure of $SO(4)$. □

Lemma 36. *Recall that m_r denotes the map from the unit sphere S^3 of $\mathbb{H}$ to $SO(\mathbb{H})$ induced by the right-multiplication. Denote the inclusions $SO(n)\subset SU(n)$ by c. Then in $\pi_3(SU(4))$, $c_*([m_r]) = 2[i^2(m_r^{\mathbb{C}})]$.*

Proof. Let $\mathbb{H}+I\mathbb{H}$ denote the complexification of $\mathbb{R}^4=\mathbb{H}=\mathbb{R}\langle 1,i,j,k\rangle$. Here, $\mathbb{C}=\mathbb{R}\oplus I\mathbb{R}$. When $x\in\mathbb{H}$ and $v\in S^3$, $c(m_r)(v)(Ix) = Ix.v$, and $I^2=-1$. Let $\varepsilon=\pm1$, define

$$\mathbb{C}^2(\varepsilon) = \mathbb{C}\left\langle \frac{\sqrt{2}}{2}(1+\varepsilon Ii), \frac{\sqrt{2}}{2}(j+\varepsilon Ik)\right\rangle.$$

Consider the quotient $\mathbb{C}^4/\mathbb{C}^2(\varepsilon)$. In this quotient, $Ii=-\varepsilon 1$, $Ik=-\varepsilon j$, and since $I^2=-1$, $I1=\varepsilon i$ and $Ij=\varepsilon k$. Therefore this quotient is isomorphic to $\mathbb{H}$ as a real vector space with its complex structure $I=\varepsilon i$. Then it is easy to see that $c(m_r)$ maps $\mathbb{C}^2(\varepsilon)$ to 0 in this quotient. Thus $c(m_r)(\mathbb{C}^2(\varepsilon)) = \mathbb{C}^2(\varepsilon)$. Now, observe that $\mathbb{H}+I\mathbb{H}$ is the orthogonal sum of $\mathbb{C}^2(-1)$ and $\mathbb{C}^2(1)$. In particular, $\mathbb{C}^2(\varepsilon)$ is isomorphic to the quotient $\mathbb{C}^4/\mathbb{C}^2(-\varepsilon)$, which is isomorphic to $(\mathbb{H}; I=-\varepsilon i)$ and $c(m_r)$ acts on it by the right multiplication. Therefore, with respect to the orthonormal basis $\frac{\sqrt{2}}{2}(1-Ii, j-Ik, 1+Ii, j+Ik)$, $c(m_r)(z_1+z_2j)$ reads

$$\begin{bmatrix} z_1 & -\overline{z}_2 & 0 & 0\\ z_2 & \overline{z}_1 & 0 & 0\\ 0 & 0 & \overline{z}_1 = x_1 - Iy_1 & -z_2\\ 0 & 0 & \overline{z}_2 & z_1 = x_1+Iy_1 \end{bmatrix}.$$

Therefore, the homotopy class of $c(m_r)$ is the sum of the homotopy classes of

$$(z_1 + z_2 j) \mapsto \begin{bmatrix} m_r^{\mathbb{C}}(z_1, z_2) & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$(z_1 + z_2 j) \mapsto \begin{bmatrix} 1 & 0 \\ 0 & m_r^{\mathbb{C}} \circ \iota(z_1, z_2) \end{bmatrix}$$

where $\iota(z_1, z_2) = (\overline{z_1}, \overline{z_2})$. Since the first map is conjugate by a fixed element of $SU(4)$ to $i_*^2(m_r^{\mathbb{C}})$, it is homotopic to $i_*^2(m_r^{\mathbb{C}})$, and since ι induces the identity on $\pi_3(S^3)$, the second map is homotopic to $i_*^2(m_r^{\mathbb{C}})$, too. □

The following lemma finishes to determine the maps $c_* : \pi_3(SO(4)) \to \pi_3(SU(4))$ and $c_* i_* : \pi_3(SO(3)) \to \pi_3(SU(4))$.

Lemma 37. $c_*([\overline{m_r}]) = c_*([m_\ell]) = -2[i^2(m_r^{\mathbb{C}})]$, $c_*(i_*([\tilde{\rho}])) = -4[i^2(m_r^{\mathbb{C}})]$.

Proof. According to Lemma 35, $i_*([\tilde{\rho}]) = [m_\ell] + [\overline{m_r}] = [m_\ell] - [m_r]$. Using the conjugacy of quaternions, $m_\ell(v)(x) = v.x = \overline{\overline{x}.\overline{v}} = \overline{\overline{m_r}(v)(\overline{x})}$. Therefore m_ℓ is conjugated to $\overline{m_r}$ via the conjugacy of quaternions, which lies in $(O(4) \subset U(4))$.

Since $U(4)$ is connected, the conjugacy by an element of $U(4)$ induces the identity on $\pi_3(SU(4))$. Thus, $c_*([m_\ell]) = c_*([\overline{m_r}]) = -c_*([m_r])$, and $c_*(i_*([\tilde{\rho}])) = -2c_*([m_r])$. □

6.7. Relating the relative Pontrjagin number to the degree

We finish proving Theorem 2 by proving the following proposition.

Proposition 11. *Let M_0 and M be two compact connected oriented 3-manifolds whose boundaries have collars that are identified by a diffeomorphism. Let $\tau_0 : M_0 \times \mathbb{C}^3 \to TM_0 \otimes \mathbb{C}$ and $\tau : M \times \mathbb{C}^3 \to TM \otimes \mathbb{C}$ be two special complex trivializations (which respect the orientations) that coincide on the collar neighborhoods of $\partial M_0 = \partial M$. Let $[(M, \partial M), (SU(3), 1)]$ denote the group of homotopy classes of maps from M to $SU(3)$ that map ∂M to 1. For any $g : (M, \partial M) \to (SU(3), 1)$, define*

$$\begin{array}{rccc} \psi(g): & M \times \mathbb{C}^3 & \to & M \times \mathbb{C}^3 \\ & (x, y) & \mapsto & (x, g(x)(y)) \end{array}$$

then $p_1(\tau_0, \tau \circ \psi(g)) - p_1(\tau_0, \tau) = p_1(\tau, \tau \circ \psi(g)) = -p_1(\tau \circ \psi(g), \tau) = p'_1(g)$ is independent from τ_0 and τ, p'_1 induces an isomorphism from the group $[(M, \partial M), (SU(3), 1)]$ to $\mathbb{Z}$, and, if g is valued in $SO(3)$, then $p'_1(g) = 2\deg(g)$.

In order to prove this proposition, we first prove the following lemma.

Lemma 38. *Under the hypotheses of Proposition 11, $\left(p_1(\tau_0, \tau \circ \psi(g)) - p_1(\tau_0, \tau)\right)$ is independent from τ_0 and τ.*

Proof. Indeed, $\left(p_1(\tau_0, \tau \circ \psi(g)) - p_1(\tau_0, \tau)\right)$ can be defined as the obstruction to extending the following trivialization of the complexified tangent bundle to $[0,1] \times M$ restricted to the boundary. This trivialization is $T[0,1] \oplus \tau$ on $(\{0\} \times M) \cup ([0,1] \times \partial M)$ and $T[0,1] \oplus \tau \circ \psi(g)$ on $\{1\} \times M$. But this obstruction is the obstruction to extending the map $\tilde{g}$ from $\partial([0,1] \times M)$ to $SU(4)$ that maps $(\{0\} \times M) \cup ([0,1] \times \partial M)$ to 1 and that coincides with $i(g)$ on $\{1\} \times M$, regarded as a map from $\partial([0,1] \times M)$ to $SU(4)$, over $([0,1] \times M)$. This obstruction, which lies in $\pi_3(SU(4))$ since $\pi_i(SU(4)) = 0$, for $i < 3$, is independent of τ_0 and τ.

Proof of Proposition 11. Lemma 38 guarantees that p'_1 defines two group homomorphisms to $\mathbb{Z}$ from $[(M, \partial M), (SU(3), 1)]$ and from $[(M, \partial M), (S0(3), 1)]$. Since $\pi_i(SU(3))$ is trivial for $i < 3$ and since $\pi_3(SU(3)) = \mathbb{Z}$, the group of homotopy classes $[(M, \partial M), (SU(3), 1)]$ is generated by the class of a map that maps the complement of a 3-ball B to 1 and that factors through a map that generates $\pi_3(SU(3))$. By definition of the Pontrjagin classes, p_1' sends such a generator to ± 1 and it induces an isomorphism from $[(M,\partial M),(SU(3),1)]$ to $\mathbb{Z}$.

According to Lemma 29 and to Lemma 23, the restriction of p_1' to $[(M,\partial M),(SO(3),1)]$ must read $p'_1(\rho_M(B^3))\frac{\deg}{2}$, and we are left with the proof of the following lemma.

Lemma 39. $p_1'(\rho_M(B^3)) = 4$.

Let $g = \rho_M(B^3)$, we can extend $\tilde{g}$ (defined in the proof of Lemma 38) by the constant map with value 1 outside $[\varepsilon,1]\times B^3 \cong B^4$ and, in $\pi_3(SU(4))$ $[c(\tilde{g}_{|\partial B^4})] = -p_1(\tau,\tau\circ\psi(g))[i^2(m_r^{\mathbb{C}})]$. Since $\tilde{g}_{|\partial B^4}$ is homotopic to $c\circ i(\tilde{\rho})$, Lemma 37 allows us to conclude. □

7. Other complements

7.1. More on low-dimensional manifolds

Piecewise linear (or PL) n-manifolds can be defined as the C^i-manifolds of Subsection 1.2 by replacing C^i with piecewise linear (or PL).

When $n \le 3$, the above notion of PL-manifold coincides with the notions of smooth and topological manifold, according to the following theorem. This is not true anymore when $n > 3$. See [59].

Theorem 10. *When $n \le 3$, the category of topological n-manifolds is isomorphic to the category of PL n-manifolds and to the category of C^r n-manifolds, for $r = 1,\dots,\infty$.*

For example, according to this statement, which contains several theorems (see [31]), any topological 3-manifold has a unique C^∞-structure. Below $n = 3$.

The equivalence between the categories $C^i, i = 1,2,\dots,\infty$ follows from work of Whitney in 1936 [60]. In 1934, Cairns [61] provided a map from the C^1-category to the PL category, which shows the existence of a triangulation for C^1-manifolds, and he proved that this map is onto [62. Theorem III] in 1940. Moise [63] proved the equivalence between the topological category and the PL category in 1952. This diagram was completed by Munkres [64. Theorem 6.3] and Whitehead [65] in 1960 by their independent proofs of the injectivity of the natural map from the C^1-category to the topological category.

References

1. Lescop C. *Invariants of links and 3-manifolds from graph configurations.* (In preparation).

2. Gauss K. Zur mathematischen Theorie der electrodynamischen Wirkungen. *Werke*. Göttingen, Königl. Ges. Wiss., 1877, pp. 601–630.

3. Greenberg M.J. *Lectures on algebraic topology.* New York, Amsterdam, W. A. Benjamin Inc., 1967.

4. Hirsch M.W. *Differential topology*. New York, Springer-Verlag Publ., 1994.

5. Milnor J.W. *Topology from the differentiable viewpoint*. Princeton, Princeton University Press, 1997.

6. Lescop C.A *formula for the Θ-invariant from Heegaard diagrams*. Available at: http://arxiv.org/abs/1209.3219.

7. Kirby R., Melvin P. Canonical framings for 3-manifolds. *Turkish J. Math.*, 1999, vol. 23, no. 1, pp. 89–115.

8. Hirzebruch F.E.P. Hilbert modular surfaces. *Enseignement Math.,* 1973, vol. 19, no. 2, pp. 183–281.

9. Kuperberg G., Thurston D. *Perturbative 3-manifold invariants by cut-and-paste topology.* Available at: http://arxiv.org/pdf/math/9912167.pdf

10. Akbulut S., McCarthy J.D. *Casson's invariant for oriented homology 3-spheres*. Princeton, Princeton University Press, 1990.

11. Guillou L., Marin A. Notes sur l'invariant de Casson des sphères d'homologie de dimension trois. *Enseign. Math.*, 1992, vol. 38, no. 3-4, pp. 233–290.

12. Marin A. Un nouvel invariant pour les sphères d'homologie de dimension trois (d'après Casson). *Astérisque*, 1988, no. 161–162, pp. 151–164.

13. Lescop C. *Splitting formulae for the Kontsevich-Kuperberg-Thurston invariant of rational homology 3-spheres.* Available at: http://arxiv.org/abs/math/0411431.

14. Walker K. *An extension of Casson's invariant*. Princeton, Princeton University Press, 1992.

15. Bar-Natan D. On the Vassiliev knot invariants. *Topology*, 1995, vol. 34, no. 2, pp. 423–472.

16. Chmutov S., Duzhin S., Mostovoy J. *Introduction to Vassiliev knot invariants*. Cambridge, Cambridge University Press, 2012.

17. Ohtsuki T. Finite type invariants of integral homology 3-spheres. *J. Knot Theory Ramifications*, 1996, vol. 5, no. 1, pp. 101–115.

18. Garoufalidis S., Goussarov M., Polyak M. Calculus of clovers and finite type invariants of 3-manifolds. *Geom. Topol.*, 2001, vol. 5, pp. 75–108.

19. Habiro K. Claspers and finite type invariants of links. *Geom. Topol.,* 2000, vol. 4, pp. 1–83.

20. Le T. T. Q. *An invariant of integral homology 3-spheres which is universal for all finite type invariants*. Available at: http://arxiv.org/abs/q-alg/9601002.

21. Moussard D. Finite type invariants of rational homology 3-spheres. *Algebr. Geom. Topol.,* 2012, vol. 12, pp. 2389–2428.

22. Lescop C. Knot invariants and configuration space integrals. *Geometric and topological methods for quantum field theory.* Berlin, Springer Publ., 2005, pp. 1–57.

23. Chmutov S., Duzhin S. The Kontsevich integral. *Acta Appl. Math.,* 2001. vol. 66, no. 2, pp. 155–190.

24. Le T.T.Q., Murakami J. The universal Vassiliev-Kontsevich invariant for framed oriented links. *Compositio Math.,* 1996, vol. 102, no. 1, pp. 41–64.

25. Turaev V.G. Operator invariants of tangles, and 3-matrices. *Izv. Akad. Nauk SSSR. Ser. Mat.*, 1989, vol. 53, no. 5, pp. 1073–1107.

26. Reshetikhin N.Y., Turaev V.G. Ribbon graphs and their invariants derived from quantum groups. *Comm. Math. Phys.*, 1990, vol. 127, no. 1, pp. 1–26.

27. Vogel P. Algebraic structures on modules of diagrams. *J. Pure Appl. Algebra,* 2011, vol. 215, no. 6, pp. 1292–1339.

28. Le T.T.Q., Murakami J., Ohtsuki T. On a universal perturbative invariant of 3-manifolds. *Topology*, 1998, vol. 37, no. 3, pp. 539–574.

29. Kuriya T., Le T.T.Q., Ohtsuki T. The perturbative invariants of rational homology 3-spheres can be recovered from the LMO invariant. *J. Topol.,* 2012, vol. 5, no. 2, pp. 458–484.

30. Reshetikhin N.Y., Turaev V.G. Invariants of 3-manifolds via link polynomials and quantum groups. *Invent. Math.,* 1991, vol. 103, no. 3, pp. 547–597.

31. Lescop C. *On the Kontsevich-Kuperberg-Thurston construction of a configuration-space invariant for rational homology 3-spheres*.Available at: http://arxiv.org/abs/math/0411088.

32. Altschüler D. Freidel L. Vassiliev knot invariants and Chern-Simons perturbation theory to all orders. *Comm. Math. Phys.,* 1997, vol. 187, no. 2, pp. 261–287.

33. Thurston D. *Integral expressions for the Vassiliev knot invariants*. Available at: http://arxiv.org/abs/math/9901110.

34. Poirier S. The configuration space integral for links in $\mathbb{R}^3$. *Algebr. Geom. Topol.*, 2002, vol. 2, pp. 1001–1050.

35. Witten E. Quantum field theory and the Jones polynomial. *Comm. Math. Phys.,* 1989, vol. 121, no. 3, pp. 351–399.

36. Guadagnini E., Martellini M., Mintchev M. Wilson lines in Chern-Simons theory and link invariants. *Nuclear Phys. B.,* 1990, vol. 330, no. 2–3, pp. 575–607.

37. Kontsevich M. *Feynman diagrams and low-dimensional topology.* Available at: http://www.ihes.fr/~maxim/TEXTS/Feynman%20%20diagrams%20and%20low-dimensional%20topology.pdf.

38. Kontsevich M. *Vassiliev's knot invariants.* Available at: http://www.ihes.fr/~maxim/TEXTS/VassilievKnot.pdf.

39. Bott R., Taubes C. On the self-linking of knots. *J. Math. Phys.,* 1994, vol. 35, no. 10, pp. 5247–5287.

40. Bar-Natan D. Perturbative Chern-Simons theory. *J. Knot Theory Ramifications,* 1995, vol. 4, no. 4, pp. 503–547.

41. Axelrod S., Singer I.M. *Chern-Simons perturbation theory.* Available at: http://arxiv.org/abs/hep-th/9110056.

42. Axelrod S., Singer I.M. Chern-Simons perturbation theory. II. *J. Differential Geom.,* 1994, vol. 39, no. 1, pp. 173–213.

43. Matveev S.V. Generalized surgeries of three-dimensional manifolds and representations of homology spheres. *Mat. Zametki,* 1987, vol. 42, no. 2, pp. 268–278.

44. Auclair E., Lescop C. Clover calculus for homology 3-spheres via basic algebraic topology. *Algebr. Geom. Topol.,* 2005, vol. 5, pp. 71–106.

45. Lescop C. About the uniqueness of the Kontsevich integral. *J. Knot Theory Ramifications*, 2002, vol. 11, no. 5, pp. 759–780.

46. Bar-Natan D., Lawrence R. A rational surgery formula for the LMO invariant. *Israel J. Math.,* 2004, vol. 140, pp. 29–60.

47. Shimizu T. *An invariant of rational homology 3-spheres via vector fields.* Available at: http://arxiv.org/abs/1311.1863.

48. Watanabe T. *Higher order generalization of Fukaya's Morse homotopy invariant of 3-manifolds I. Invariants of homology 3-spheres.* Available at: http://arxiv.org/abs/1202.5754v2.

49. Kricker A. *The lines of the Kontsevich integral and Rozansky's rationality conjecture*. Available at: http://arxiv.org/abs/math/0005284.

50. Garoufalidis S., Kricker A. A rational noncommutative invariant of boundary links. *Geom. Topol.,* 2004, vol. 8, pp. 115–204.

51. Lescop C. *Invariants of knots and 3-manifolds derived from the equivariant linking pairing*. Available at: http://en.youscribe.com/catalogue/educational-resources/education/others/invariants-of-knots-and-3-manifolds-derived-from-the-equivariant-1711545.

52. Lescop C. *A universal equivariant finite type knot invariant defined from configuration space integrals.* Available at : http://arxiv.org/abs/1306.1705.

53. Kirby R.C. *The topology of 3-manifolds*. Berlin, Springer-Verlag, 1989.

54. Steenrod N. *The Topology of Fibre Bundles*. Princeton, Princeton University Press, 1951.

55. Munkres J.R. *Elementary differential topology*. Princeton, Princeton University Press, 1966.

56. Milnor J.W. *Characteristic classes*. Princeton, Princeton University Press, 1974.

57. Rohlin V.A. New results in the theory of four-dimensional manifolds. *Doklady Akad. Nauk SSSR,* 1952, vol. 84, pp. 221–224.

58. *À la recherche de la topologie perdue*. L. Guillou, A. Marin (eds.). Boston, Birkhäuser Boston Inc., 1986.

59. Kuiper N.H. A short history of triangulation and related matters. *History of topology*. Amsterdam, North-Holland Publ., 1999, pp. 491–502.

60. Whitney H. Differentiable manifolds. *Ann. of Math.,* 1936, vol. 37, no. 3, pp. 645–680.

61. Cairns S.S. Triangulation of the manifold of class one. *Bull. Amer. Math. Soc.,* 1935, vol. 41, no. 8, pp. 549–552.

62. Cairns S.S. Homeomorphisms between topological manifolds and analytic manifolds. *Ann. of Math.*, 1940, vol. 41, no. 2, pp. 796–808.

63. Moise E.E. Affine structures in 3-manifolds. V. The triangulation theorem and Hauptvermutung. *Ann. of Math.,* 1952, vol. 56, no. 2, pp. 96–114.

64. Munkres J. Obstructions to the smoothing of piecewise-differentiable homeomorphisms. *Ann. of Math.,* 1960, vol. 72, pp. 521–554.

65. Whitehead J.H.C. Manifolds with transverse fields in euclidean space. *Ann. of Math.,* 1961, vol. 73, no. 2, pp. 154–212.

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ВВЕДЕНИЕ В ТЕОРИЮ ИНВАРИАНТОВ КОНЕЧНОГО ТИПА УЗЛОВ И ТРЕХМЕРНЫХ МНОГООБРАЗИЙ, ОПРЕДЕЛЯЕМЫХ КАК ЧИСЛО КОНФИГУРАЦИЙ В ГРАФЕ

К. Лескоп

Концепция инвариантов конечного типа для узлов была предложена в 90-х гг. в работах Васильева, Гусарова и Бар-Натана с целью классификации инвариантов узлов вскоре после появления многочисленных квантовых инвариантов узлов. Эта очень полезная концепция была расширена Отсуки до случая инвариантов трехмерных многообразий.

В статье показывается, как определить инварианты конечного типа для узлов и трехмерных многообразий путем подсчета конфигураций графа в трехмерных многообразиях. Мы следуем идеям Виттена и Концевича.

Число зацеплений является простейшим инвариантом конечного типа для двухкомпонентных зацеплений. Он определяется несколькими эквивалентными способами в первом разделе. В качестве важного примера приводится его определение как алгебраическое пересечение тора и 4-цепи, называемое пропагатором в конфигурационном пространстве.

Во втором разделе мы вводим простейший инвариант конечного типа для трехмерных многообразий — инвариант Кассона (или Θ-инвариант) целочисленных гомологических 3-сфер. Он определяется как алгебраическое пересечение трех пропагаторов в одном и том же двухточечном конфигурационном пространстве.

В третьем разделе описано общее понятие инварианта конечного типа и введены соответствующие пространства диаграмм Фейнмана — Якоби.

В разделах 4 и 5 мы даем набросок оригинальной конструкции, основанной на интегралах конфигурационного пространства универсальных инвариантов конечного типа для зацеплений в рациональных гомологических сферах, а также формулируем несколько нерешенных проблем. Наша конструкция обобщает известные конструкции для зацеплений в $\mathbb{R}^3$ и для рациональных гомологических 3-сфер, что делает ее более гибкой.

В разделе 6 детально описываны необходимые свойства параллелизаций трехмерных многообразий и соответствующих классов Понтрягина.

Ключевые слова: *узлы, трехмерные многообразия, инварианты конечных типа, гомологические 3-сферы, число зацеплений, тета-инвариант, инвариант Кассона-Уолкера, диаграммы Фейнмана-Якоби, расширение теории Черна-Саймонса, интегралы конфигурационного пространства,параллелизация трехмерных многообразий,первый класс Понтрягина.*

Список литературы

1. Lescop, C. Invariants of links and 3-manifolds from graph configurations / C. Lescop. — (In preparation).

2. Gauss, K. Zur mathematischen Theorie der electrodynamischen Wirkungen manuscript, first published in his / K. Gauss // Werke. — Göttingen : Königl. Ges. Wiss., 1877. — P. 601–630.

3. Greenberg, M. J. Lectures on algebraic topology / M. J. Greenberg. — New York ; Amsterdam : W. A. Benjamin, Inc., 1967.

4. Hirsch, M. W. Differential topology / M. W. Hirsch. — New York : Springer-Verlag, 1994.

5. Milnor, J. W. Topology from the differentiable viewpoint / J. W. Milnor. — Princeton, NJ : Princeton University Press, 1997.

6. Lescop, C. A formula for the Θ-invariant from Heegaard diagrams [Электронный ресурс] / C. Lescop. — URL : http://arxiv.org/abs/1209.3219.

7. Kirby, R. Canonical framings for 3-manifolds / R. Kirby, P. Melvin // Turkish J. Math. — 1999. — Vol. 23, № 1. — P. 89–115.

8. Hirzebruch, F. E. P. Hilbert modular surfaces / F. E. P. Hirzebruch // Enseignement Math. — 1973. — Vol. 19, № 2. — P. 183–281.

9. Kuperberg, G. Perturbative 3-manifold invariants by cut-and-paste topology [Электронный ресурс] / G. Kuperberg, D. Thurston. — URL : http://arxiv.org/pdf/math/9912167.pdf

10. Akbulut, S. Casson's invariant for oriented homology 3-spheres / S. Akbulut, J. D. McCarthy. — Princeton, NJ : Princeton University Press, 1990.

11. Guillou, L. Notes sur l'invariant de Casson des sphères d'homologie de dimension trois / L. Guillou, A. Marin // Enseign. Math. — 1992. — Vol. 38, № 3–4. — P. 233–290.

12. Marin, A. Un nouvel invariant pour les sphères d'homologie de dimension trois (d'après Casson) / A. Marin // Astérisque. — 1988. — № 161–162. — P. 151–164.

13. Lescop, C. Splitting formulae for the Kontsevich-Kuperberg-Thurston invariant of rational homology 3-spheres [Электронный ресурс] / C. Lescop. — URL : http://arxiv.org/abs/math/0411431.

14. Walker, K. An extension of Casson's invariant / K. Walker. — Princeton, NJ : Princeton University Press, 1992.

15. Bar-Natan, D. On the Vassiliev knot invariants / D. Bar-Natan // Topology. — 1995. — Vol. 34, № 2. — P. 423–472.

16. Chmutov, S. Introduction to Vassiliev knot invariants / S. Chmutov, S. Duzhin, J. Mostovoy. — Cambridge : Cambridge University Press, 2012.

17. Ohtsuki, T. Finite type invariants of integral homology 3-spheres / T. Ohtsuki // J. Knot Theory Ramifications. — 1996. — Vol. 5, № 1. — P. 101–115.

18. Garoufalidis, S. Calculus of clovers and finite type invariants of 3-manifolds / S. Garoufalidis, M. Goussarov, M. Polyak // Geom. Topol. — 2001. — Vol. 5. — P. 75–108.

19. Habiro, K. Claspers and finite type invariants of links / K. Habiro // Geom. Topol. — 2000. — Vol. 4. — P. 1–83.

20. Le, T. T. Q. An invariant of integral homology 3-spheres which is universal for all finite type invariants [Электронный ресурс] / T. T. Q. Le. — URL: http://arxiv.org/abs/q-alg/9601002.

21. Moussard, D. Finite type invariants of rational homology 3-spheres / D. Moussard // Algebr. Geom. Topol. — 2012. — Vol. 12. — P. 2389–2428.

22. Lescop, C. Knot invariants and configuration space integrals / C. Lescop // Geometric and topological methods for quantum field theory. — Berlin : Springer, 2005. — P. 1–57.

23. Chmutov, S. The Kontsevich integral / S. Chmutov, S. Duzhin // Acta Appl. Math. — 2001. — Vol. 66, № 2. — P. 155–190.

24. Le, T. T. Q. The universal Vassiliev-Kontsevich invariant for framed oriented links / T. T. Q. Le, J. Murakami // Compositio Math. — 1996. — Vol. 102, № 1. — P. 41–64.

25. Turaev, V. G. Operator invariants of tangles, and 3-matrices / V. G. Turaev // Izv. Akad. Nauk SSSR. Ser. Mat. — 1989. — Vol. 53, № 5. — P. 1073–1107.

26. Reshetikhin, N. Y. Ribbon graphs and their invariants derived from quantum groups / N. Y. Reshetikhin, V. G. Turaev // Comm. Math. Phys. — 1990. — Vol. 127, № 1. — P. 1–26.

27. Vogel, P. Algebraic structures on modules of diagrams / P. Vogel // J. Pure Appl. Algebra. — 2011. — Vol. 215, № 6. — P. 1292–1339.

28. Le, T. T. Q. On a universal perturbative invariant of 3-manifolds / T. T. Q. Le, J. Murakami, T. Ohtsuki // Topology. — 1998. —Vol. 37, № 3. — P. 539–574.

29. Kuriya, T. The perturbative invariants of rational homology 3-spheres can be recovered from the LMO invariant / T. Kuriya, T. T. Q. Le, T. Ohtsuki // J. Topol. — 2012. — Vol. 5, № 2. — P. 458–484.

30. Reshetikhin, N. Y. Invariants of 3-manifolds via link polynomials and quantum groups / N. Y. Reshetikhin, V. G. Turaev // Invent. Math. — 1991. —Vol. 103, № 3. — P. 547–597.

31. Lescop, C. On the Kontsevich-Kuperberg-Thurston construction of a configuration-space invariant for rational homology 3-spheres [Электронный ресурс] / C. Lescop. — URL : http://arxiv.org/abs/math/0411088.

32. Altschüler, D. Vassiliev knot invariants and Chern-Simons perturbation theory to all orders / D. Altschüler, L. Freidel // Comm. Math. Phys. — 1997. — Vol. 187, № 2. — P. 261–287.

33. Thurston, D. Integral expressions for the Vassiliev knot invariants [Электронный ресурс] / D. Thurston. — URL : http://arxiv.org/abs/math/9901110.

34. Poirier, S. The configuration space integral for links in $\mathbb{R}^3$ / S. Poirier // Algebr. Geom. Topol. — 2002. — Vol. 2. — P. 1001–1050.

35. Witten, E. Quantum field theory and the Jones polynomial / E. Witten // Comm. Math. Phys. — 1989. — Vol. 121, № 3. — P. 351–399.

36. Guadagnini, E. Wilson lines in Chern-Simons theory and link invariants / E. Guadagnini, M. Martellini, M. Mintchev // Nuclear Phys. B. — 1990. — Vol. 330, № 2-3. — P. 575–607.

37. Kontsevich, M. Feynman diagrams and low-dimensional topology [Электронный ресурс] / M. Kontsevich. — URL : http://www.ihes.fr/~maxim/TEXTS/Feynman%20%20diagrams%20and%20low-dimensional%20topology.pdf.

38. Kontsevich, M. Vassiliev's knot invariants / M. Kontsevich. — URL : http://www.ihes.fr/~maxim/TEXTS/VassilievKnot.pdf.

39. Bott, R. On the self-linking of knots / R. Bott, C. Taubes // J. Math. Phys. — 1994. — Vol. 35, № 10. — P. 5247–5287.

40. Bar-Natan, D. Perturbative Chern-Simons theory / D. Bar-Natan // J. Knot Theory Ramifications. — 1995. — Vol. 4, № 4. — P. 503–547.

41. Axelrod, S. Chern-Simons perturbation theory / S. Axelrod, I. M. Singer. — URL: http://arxiv.org/abs/hep-th/9110056.

42. Axelrod, S. Chern-Simons perturbation theory. II / S. Axelrod, I. M. Singer // J. Differential Geom. — 1994. — Vol. 39, № 1. — P. 173–213.

43. Matveev, S. V. Generalized surgeries of three-dimensional manifolds and representations of homology spheres / S. V. Matveev // Mat. Zametki. — 1987. — Vol. 42, № 2. — P. 268–278.

44. Auclair, E. Clover calculus for homology 3-spheres via basic algebraic topology / E. Auclair, C. Lescop // Algebr. Geom. Topol. — 2005. — Vol. 5. — P. 71–106.

45. Lescop, C. About the uniqueness of the Kontsevich integral / C. Lescop // J. Knot Theory Ramifications. —2002. — Vol. 11, № 5. — P. 759–780.

46. Bar-Natan, D. A rational surgery formula for the LMO invariant / D. Bar-Natan, R. Lawrence // Israel J. Math. — 2004. — Vol. 140. — P. 29–60.

47. Shimizu, T. An invariant of rational homology 3-spheres via vector fields / T. Shimizu. — URL: http://arxiv.org/abs/1311.1863.

48. Watanabe, T. Higher order generalization of Fukaya's Morse homotopy invariant of 3-manifolds I. Invariants of homology 3-spheres [Электронный ресурс] / T. Watanabe. — URL : http://arxiv.org/abs/1202.5754v2.

49. Kricker, A. The lines of the Kontsevich integral and Rozansky's rationality conjecture [Электронный ресурс] / A. Kricker. — URL: http://arxiv.org/abs/math/0005284.

50. Garoufalidis, S. A rational noncommutative invariant of boundary links / S. Garoufalidis, A. Kricker // Geom. Topol. — 2004. — Vol. 8. — P. 115–204.

51. Lescop, C. Invariants of knots and 3-manifolds derived from the equivariant linking pairing / C. Lescop. — URL : http://en.youscribe.com/catalogue/educational-resources/education/others/invariants-of-knots-and-3-manifolds-derived-from-the-equivariant-1711545.

52. Lescop, C. A universal equivariant finite type knot invariant defined from configuration space integrals [Электронный ресурс] / C. Lescop. — URL : http://arxiv.org/abs/1306.1705

53. Kirby, R. C. The topology of 3-manifolds / R. C. Kirby // Lecture Notes in Mathematics. Vol. 1374. — Berlin : Springer-Verlag, 1989.

54. Steenrod, N. The Topology of Fibre Bundles / N. Steenrod. — Princeton, NJ : Princeton University Press, 1951.

55. Munkres, J. R. Elementary differential topology / J. R. Munkres. — Princeton, NJ : Princeton University Press, 1966.

56. Milnor, J. W. Characteristic classes / J. W. Milnor, J. D. Stasheff. Princeton, NJ : Princeton University Press, 1974.

57. Rohlin, V. A. New results in the theory of four-dimensional manifolds / V. A. Rohlin // Doklady Akad. Nauk SSSR. — 1952. — Vol. 84. — P. 221–224.

58. À la recherche de la topologie perdue / L. Guillou, A. Marin (eds.). — Boston, MA : Birkhäuser Boston Inc., 1986. — 244 p.

59. Kuiper, N. H. A short history of triangulation and related matters / N. H. Kuiper // History of topology. — Amsterdam : North-Holland, 1999. — P. 491–502.

60. Whitney, H. Differentiable manifolds / H. Whitney // Ann. of Math. — 1936. — Vol. 37, № 3. — P. 645–680.

61. Cairns, S. S. Triangulation of the manifold of class one / S. S. Cairns // Bull. Amer. Math. Soc. — 1935. — Vol. 41, № 8. — P. 549–552.

62. Cairns, S. S. Homeomorphisms between topological manifolds and analytic manifolds / S. S. Cairns // Ann. of Math. — 1940. — Vol. 41, № 2. — P. 796–808.

63. Moise, E. E. Affine structures in 3-manifolds. V. The triangulation theorem and Hauptvermutung / E. E. Moise // Ann. of Math. — 1952. — Vol. 56, № 2. — P. 96–114.

64. Munkres, J. Obstructions to the smoothing of piecewise-differentiable homeomorphisms / J. Munkres // Ann. of Math. — 1960. — Vol. 72. — P. 521–554.

65. Whitehead, J. H. C. Manifolds with transverse fields in euclidean space / J. H. C. Whitehead // Ann. of Math. — 1961. — Vol. 73, № 2. — P. 154–212.

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ON KNOTS AND LINKS IN LENS SPACES

M. Manfredi, M. Mulazzani

We shortly review some recent results about knots and links in lens spaces. A disk diagram is described together with a Reidemeister-type theorem concerning equivalence. The lift of knots/links in the 3-sphere is discussed, showing examples of different knots and links having equivalent lift. The essentiality respect to the lift of classical invariants on knots/links in lens spaces is discussed.

Keywords: *knot, link, lens space, lift, fundamental quandle, group of the link, twisted Alexander polynomial.*

1. Introduction

The interest on knots and links in lens space arises from several topological reasons. One of the most important is the Berge conjecture about knots in $\mathbf{S}^3$ admitting lens space surgeries, that can be translated into a conjecture about knots in lens spaces admitting $\mathbf{S}^3$ surgeries [1; 2]. Moreover the interest does not come only from geometric topology, but also from physics [3] and biology [4].

The first step on the study of knots and links in lens spaces is to find a suitable representation: there are several possible diagrams for links in lens spaces, as mixed link diagrams [5], band diagrams [6] and grid diagrams [1] among the others. Using band diagrams Gabrovsek obtained in [7] a tabulation of prime knots up to 4 crossings. For a detailed introduction about knots and links in lens spaces, together with a vaste bibliography, see [8].

Let L be a link in the lens space $L(p,q)$ and let P: $\mathbf{S}^3 \to L(p,q)$ be the (universal) cyclic covering, the lift of L is the link $\widetilde{L} = P^{-1}(L) \subset \mathbf{S}^3$. In [9] is described an algorithm producing a diagram of $\widetilde{L}$, starting from a disk diagram of L. This paper aims to investigating the behavior of the lift with respect to other invariants that have already been defined in [10; 11], namely: the fundamental quandle, the group of the link, the first homology group and the twisted Alexander polynomials. To be more precise, exploiting the different knots and links with equivalent lift described in [9], we show whether the considered invariants for L are or not essential, that is to say, whether they cannot or can be defined directly on the lift $\widetilde{L}$. A draft of this work can be found in [8].

The work about essential invariants extends also to the HOMFLY-PT invariant of Cornwell [12], the Link Floer Homology [1] (both of these results can be found in [13]) and the Kauffman Bracket Skein Module of [6] (the result can be found in [8]).

The setting of this paper is the *Diff* category (of smooth manifolds and smooth maps). Every result also holds in the *PL* category, and in the *Top* category if we consider only tame links, that is to say, we exclude wild knots.

2. Diagrams and equivalence of links in lens spaces

In this section, we describe two equivalent definitions of lens spaces that we are going to use through the paper. Then we introduce links in lens spaces and their equivalence. At last we describe a representation of them by disk diagrams and in this context we prove a Reidemeister-type theorem.

2.1. Lens spaces

Apart from $\mathbf{S}^3$, lens spaces are the simplest class of closed connected 3-manifolds. Usually they are defined as a quotient of the 3-sphere as follows. Let p, q be two coprime integers such that $0 \le q < p$. Regard $\mathbf{S}^3$ as the unit sphere in $\mathbb{C}^2$. Consider the diffeomorphism that sends (z_1, z_2) in $(e^{\frac{2\pi i}{p}} z_1, e^{\frac{2\pi q i}{p}} z_2)$, and the cyclic group $G_{p,q}$ generated by this diffeomorphism. Clearly $G_{p,q}$ is isomorphic to $\mathbb{Z}_p$ and it acts freely and in a properly discontinuous way on $\mathbf{S}^3$. Therefore the quotient space is a 3-manifold, the lens space $L(p,q)$.

Another possible definition of lens spaces is the following one. Consider $B^3 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, x_1^2 + x_2^2 + x_3^2 \le 1\}$ and let E_+ and E_- be respectively the upper and the lower closed hemisphere of ∂B^3. Call B_0^2 the equatorial disk, defined by the intersection of the plane $x_3 = 0$ with B^3. Let $g_{p,q}$: $E_+ \to E_+$ be the rotation of $2\pi q/p$ around the x_3 axis as in Fig. 2.1, and let f_3: $E_+ \to E_-$ be the reflection with respect to the plane $x_3 = 0$. The lens space $L(p,q)$ is the quotient of B^3 by the equivalence relation on ∂B^3 which identifies $x \in E_+$ with $f_3 \circ g_{p,q}(x) \in E_-$. We denote by F: $B^3 \to B^3/\sim$ the quotient map. Usually on ∂B^3 there are two points in each equivalence class, with the exception of the equator $\partial B_0^2 = E_+ \cap E_-$ where each class contains p points.

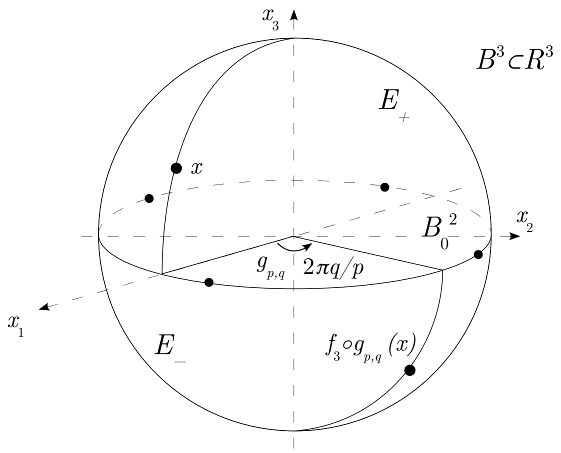


Fig. 2.1. *Lens model of* $L(p,q)$

It is easy to see that $L(1,0) \cong \mathbf{S}^3$ since $g_{1,0} = \mathrm{Id}_{E_+}$. Furthermore, $L(2,1)$ is $\mathbb{RP}^3$, since we obtain the usual model of the projective space where opposite points of ∂B^3 are identified.

Proposition 1. [14]. *The lens spaces* $L(p,q)$ *and* $L(p',q')$ *are diffeomorphic (as well as homeomorphic) if and only if* $p' = p$ *and* $q' \equiv \pm q^{\pm 1} \bmod p$.

2.2. Links in lens spaces and their equivalence

A link L in a lens space $L(p,q)$ is a pair $(L(p,q), L)$, where L is a submanifold of $L(p,q)$ diffeomorphic to the disjoint union of ν copies of $\mathbf{S}^1$, with $\nu \ge 1$. We call *component* of L each connected component of L. When $\nu = 1$ the link is called a *knot*. We usually refer to $L \subset L(p,q)$ meaning the pair $(L(p,q), L)$. A link $L \subset L(p,q)$ is *trivial* if its components bound embedded pairwise disjoint 2-disks in $L(p,q)$.

We consider on the set of links in $L(p,q)$ two different definitions of equivalence. The stronger one is the equivalence up to ambient isotopy: two links $L, L' \subset L(p,q)$ are called *isotopy-equivalent* if there exists a smooth map H: $L(p,q) \times [0,1] \to L(p,q)$ where, if we define $h_t(x) := H(x,t)$, then $h_0 = \mathrm{id}_{L(p,q)}$, $h_1(L) = L'$ and h_t is a diffeomorphism of $L(p,q)$ for each $t \in [0,1]$.

The weaker equivalence is up to diffeomorphism of pairs: two links L and L' in $L(p,q)$ are *diffeo-equivalent* if there exists a diffeomorphism of pairs h: $(L(p,q), L) \to (L(p,q), L')$, that is to say a diffeomorphism h: $L(p,q) \to L(p,q)$ such that $h(L) = L'$. This diffeomorphism is not necessarily orientation-preserving. Two isotopy-equivalent links L and L' in $L(p,q)$ are necessarily diffeo-equivalent, since from the ambient isotopy $H: L(p,q)\times[0,1] \to L(p,q)$, the map $h_1: (L(p,q),L) \to (L(p,q),L')$ is a diffeomorphism of pairs.

The two definitions coincide for links in $\mathbf{S}^3$ if only orientation preserving diffeomorphisms are considered. For the lens spaces, this fact is no longer true, as we can see from the structure of the groups of the isotopy classes of diffeomorphisms of lens spaces obtained in [15] and [16].

When necessary we will specify whether an orientation on the links is considered.

2.3. Disk diagrams

For the case $L(1,0) = \mathbf{S}^3$ links may be represented by the usual diagram coming from the regular projection onto a plane. This idea has been generalized in [11] for every $p>1$: if the lens model of $L(p,q)$ is considered and the link is regularly projected onto the equatorial disk, then a disk diagram for links in lens spaces is this projection together with overpasses and underpasses specifications for each crossing. The case $\mathbb{RP}^3=L(2,1)$ is described also in [17]. In order to have a more comprehensible diagram, we index the boundary points of the projection, so that $+i$ and $-i$ represent identified endpoints respectively in E_+ and in E_-. An example is shown in Fig. 2.2.

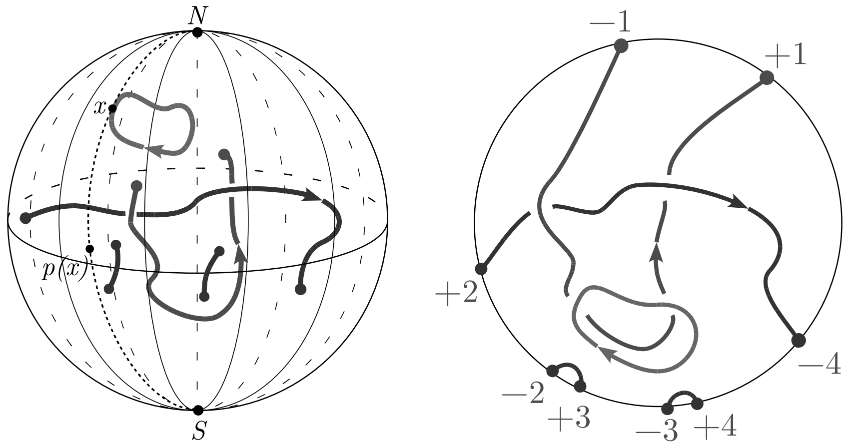


Fig. 2.2. *A link in $L(9,1)$ and its corresponding disk diagram*

2.4. Generalized Reidemeister moves

The equivalence of links in $\mathbf{S}^3$ can be studied through the Reidemeister theorem. We generalize this theorem for unoriented links in lens spaces up to isotopy equivalence. The oriented case is analogous. The *generalized Reidemeister moves* on a diagram of a link $L\subset L(p,q)$, are the moves R_1, R_2, R_3, R_4, R_5, R_6 and R_7 of Fig. 2.3. Observe that, when $p = 2$ the moves R_5 and R_6 are equal, and R_7 is trivial, thus we re-obtain the result of [17].

Theorem 1. [11]. *Two links L_0 and L_1 in $L(p,q)$ are isotopy-equivalent if and only if their diagrams can be joined by a finite sequence of generalized Reidemeister moves $R_1, \dots, R_7$ and diagram isotopies, when $p>2$. If $p=2$, moves $R_1, \dots, R_5$ are sufficient.*

2.5. Standard form of the disk diagram

A disk diagram is defined *standard* if the labels on its boundary points, read according to the orientation on ∂B_0^2, are $(+1,\dots,+t,-1,\dots,-t)$.

Proposition 2. [9]. *Every disk diagram can be reduced to a standard disk diagram.*

Indeed, if $p=2$, the signs of the boundary points of the disk diagram can be exchanged by performing an isotopy on the link (that preserves the projection); if $p>2$, a finite sequence of R_6 moves can be applied to the disk diagram in order to bring all the plus-type boundary points close to each other. An example is shown in Fig. 2.4.

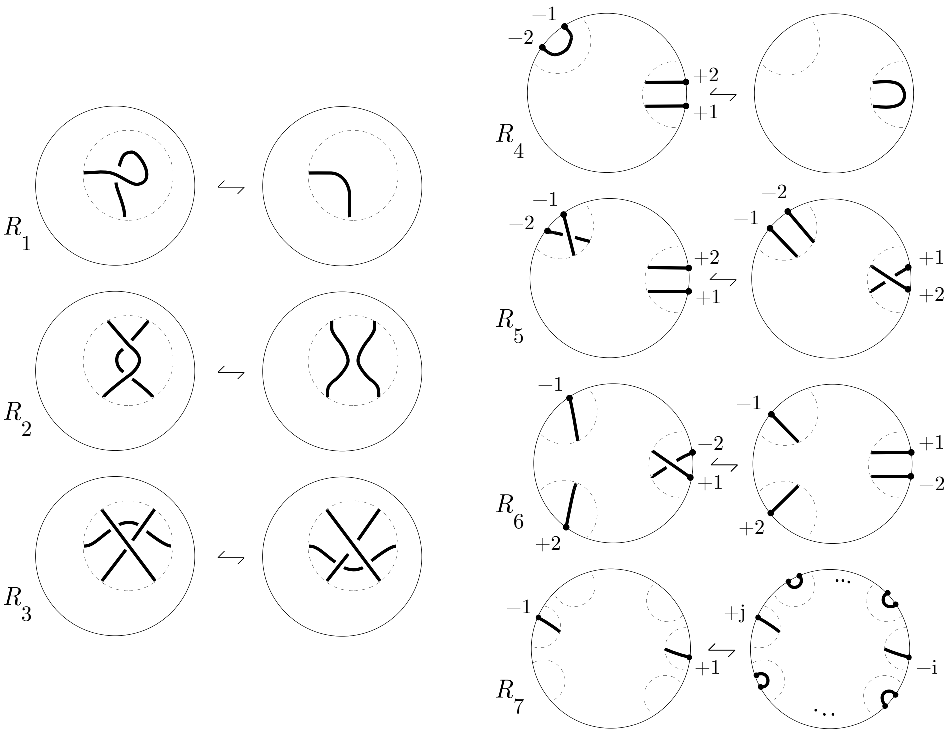


Fig. 2.3. *Generalized Reidemeister moves for links in lens spaces*

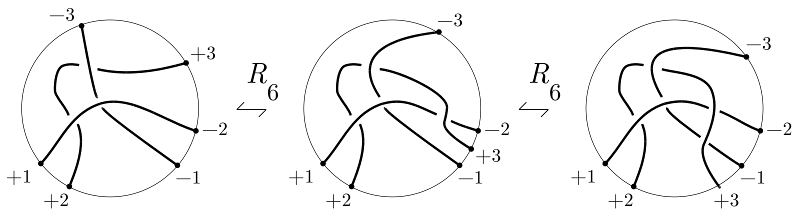


Fig. 2.4. *Example of R_6-reduction to a standard disk diagram*

3. Lift and essential invariants

In this section we deal with the following powerful invariant: let L be a link in the lens space $L(p,q)$, the *lift* $\tilde{L}$ is the counterimage $P^{-1}(L)$ in $\mathbf{S}^3$ under the universal cyclic covering $P\colon \mathbf{S}^3 \to L(p,q)$. Clearly the lift is an isotopy-equivalence invariant for the homotopy lifting property. The main result of this section is the construction of a diagram for the lift $\tilde{L}$ from a standard disk diagram of L. This result is the key to find different links with equivalent lift. Thus the lift is not a complete invariant for links, but it becomes complete with some further assumption. We conclude by defining precisely what is an essential invariant of links in lens spaces.

3.1. Lift component number

Let L be a link in $L(p,q)$, denote with ν its number of components, and with $\delta_1, \ldots, \delta_\nu$ the homology class in $H_1(L(p,q)) \cong \mathbb{Z}_p$ of the i-th component L_i of L. In Lemma 1 it will be described how to compute the homology classes directly from a disk diagram.

Proposition 3. [9] *Given a link $L \subset L(p,q)$, the number of components of $\tilde{L}$ is* $\sum_{i=1}^{\nu} \mathrm{gsd}(\delta_i, p)$.

A knot $K \subset L(p,q)$ is *primitive-homologous* if its homology class δ is coprime with p; clearly its lift is still a knot.

3.2. Diagram for the lift via disk diagrams

The construction of a diagram for $\tilde{L}\subset\mathbf{S}^3$ starting from a disk diagram of $L\subset L(p,q)$ is explained by the following two theorems. The case of $L(2,1)\cong\mathbb{RP}^3$ is outlined in [17]. As usual, the generators of the braid group on t strands are $\sigma_1,\dots,\sigma_{t-1}$. The Garside braid Δ_t on t strands is defined by $(\sigma_{t-1}\sigma_{t-2}\cdots\sigma_1)(\sigma_{t-1}\sigma_{t-2}\cdots\sigma_2)\cdots(\sigma_{t-1})$ and it is illustrated in Fig. 3.1.

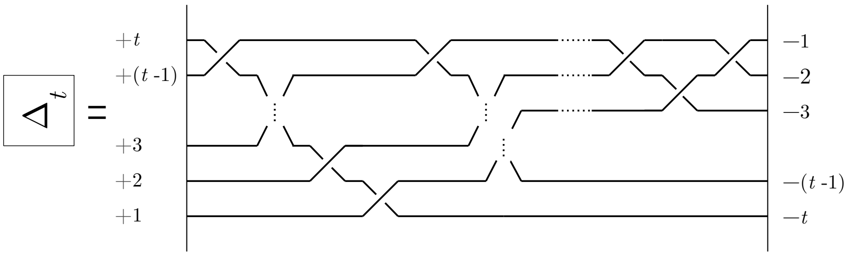


Fig. 3.1. *The braid* Δ_t

Theorem 1. [9]. *Let L be a link in the lens space L(p,q) and let D be a standard disk diagram for L; then a diagram for the lift* $\tilde{L}\subset\mathbf{S}^3$ *can be found as follows (refer to Fig. 3.2):*

- *consider p copies* $D_1,\dots,D_P$ *of the standard disk diagram D;*
- *for each* $i = 1,\dots, p-1$, *using the braid* Δ_t^{-1}, *connect the diagram* D_{i+1} *with the diagram* D_i, *joining the boundary point* $-j$ *of* D_{i+1} *with the boundary point* $+j$ *of* D_i*;*
- *connect* D_1 *with* D_P *via the braid* Δ_t^{2q-1}, *where the boundary points are connected as in the previous case.*

The proof can be found in [9] and the example of Fig. 3.4 gives the main idea of the construction.

The planar diagram of the lift of Theorem 2 has not the least possible number of crossings. Indeed if we reverse upside down D_2, reverse twice D_3, and so on, the braid Δ_t^{-1} between the disks becomes the trivial one, moving the crossings close to Δ_t^{2q-1} so that a simplification to Δ_t^{2q-p} is possible. In order to describe this construction, we define the reverse disk diagram $\overline{D}$ of D: it is the diagram that can be obtained by considering the image of D under a symmetry with respect to an external line and then exchanging all overpasses / underpasses.

Theorem 3. [9]. *Let L be a link in the lens space L(p,q) and let D be a standard disk diagram for L; then a diagram for the lift* $\tilde{L}\subset\mathbf{S}^3$ *can be found as follows (refer to Fig. 3.3):*

- *consider p copies* $D_1,\dots D_p$ *of the standard disk diagram D, then denote* $F_i = D_i$ *if i is odd, and* $F_i = \overline{D}_i$ *if i is even;*
- *for each* $i = 1,\dots, p-1$, *using a trivial braid, connect the diagram* F_{i+1} *with the diagram* F_i *joining the boundary point* $-j$ *of* F_{i+1} *with the boundary point* $+j$ *of* F_i*;*
- *connect* F_1 *with* F_p *via the braid* Δ_t^{2q-p}, *where the boundary points are connected as in the previous case.*

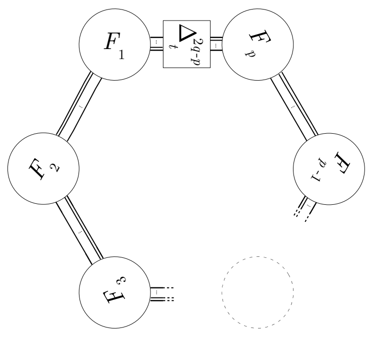


Fig. 3.2. *A diagram of the lift in* $\mathbf{S}^3$ *of a link in* $L(p,q)$

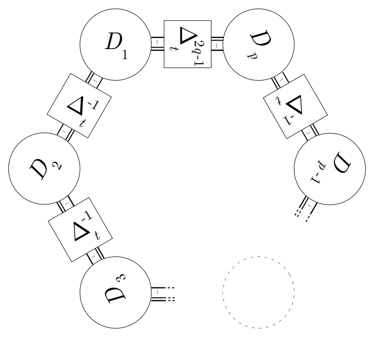


Fig. 3.3. *Another diagram of the lift in* $\mathbf{S}^3$ *of a link in* $L(p,q)$

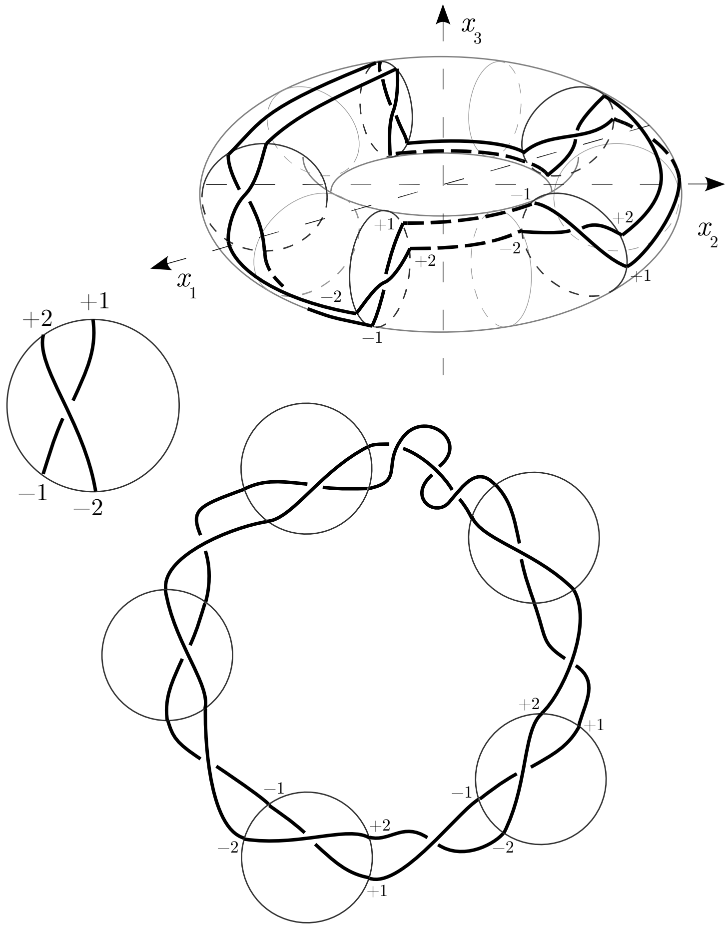


Fig. 3.4. *Lift in* $\mathbf{S}^3$ *of a link in* $L(5{,}2)$

3.3. Different links with equivalent lift

In [9] a short tabulation of the lifts of a particular class of links in lens spaces — that can be easily described by a braid — is performed in order to investigate the existence of different links with equivalent lift. Two pairs of links with such a property are shown in Example 1 and 2. In order to distinguish the links of each pair we will use invariants such as the group of the link and the Alexander polynomial that will be respectively defined in Sections 5 and 6.

Example 1. Different knots in $L\left(p,\frac{p\pm1}{2}\right)$ **with trivial knot lift**

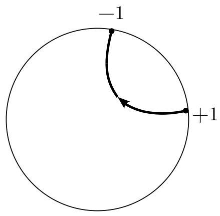
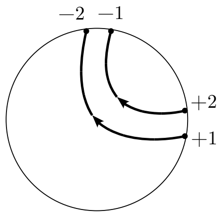
The knots K_1 and K_2 in $L\left(p,\frac{p\pm1}{2}\right)$ with p odd, depicted in Table 1, both lift to the unknot.

The homology class $[K]=\delta\subset H_1(L(p,q))\cong\mathbb{Z}_p$ of a knot in $L(p,q)$ can be $0,1,\dots,p-1$, but since we do not consider the orientation of the knots, we have to identify $\pm\delta$, so that the knots are partitioned into $\lfloor p/2\rfloor+1$ classes: $\delta=0,1,\dots,\lfloor p/2\rfloor$, where $\lfloor x\rfloor$ denotes the integer part of x. If two knots have different homology classes, they are necessarily not isotopy-equivalent.

Since $[K_1]=1$ and $[K_2]=2$, the knots are not isotopy-equivalent when $p>3$. An interesting fact is that K_1 and K_2 turn out to be diffeo-equivalent for $L(5{,}2)$. The diffeomorphism realizing this equivalence is the generator σ_- of the group of isotopy classes of diffeomorphisms of the lens space $L\left(p,\frac{p\pm1}{2}\right)$, described in [15]. It is possible to show that for $p>5$, the knots K_1 and K_2 are also not diffeo-equivalent.

Table 1

Geometric invariants of K_1 and K_2 in $L\left(p, \frac{p \pm 1}{2}\right)$

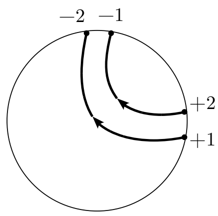
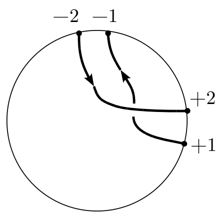
	K_1 	K_2 
ν	1	1
$[K] \subset H_1(L(p,q))$	1	2
lift	unknot	unknot
$\pi_1(L(p,q) \setminus K)$	$\mathbb{Z}$	$\mathbb{Z}$
$H_1(L(p,q) \setminus K)$	$\mathbb{Z}$	$\mathbb{Z}$
$\overline{\Delta}^1(t)$	1	1

Example 2. Different links in $L(4,1)$ lifting to the Hopf link

The knot L_A and the link L_B in $L(4,1)$ described in Table 2 have a different number of components, hence they are not diffeo-equivalent (and consequently also not isotopy-equivalent); beside this, they both lift to the Hopf link.

Table 2

Geometric invariants of L_A and L_B in $L(4,1)$

	L_A 	L_B 
ν	1	2
$[K] \subset H_1(L(p,q))$	2	1,1
lift	Hopf link	Hopf link
$\pi_1(L(p,q) \setminus L)$	$\langle a, f \mid af^{-1}af^{-3} = 1 \rangle$	$\langle a, f \mid af = fa \rangle$
$H_1(L(p,q) \setminus L)$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}$
$\overline{\Delta}^1(t)$	$t + 1$	$t - 1$
$\overline{\Delta}^{-1}(t)$	1	

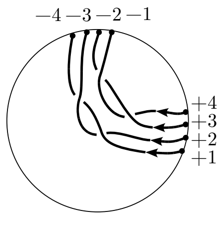
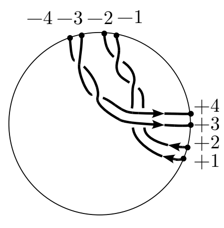
The previous examples consist of pairs of links that are easy to distinguish, because they have different numbers of components or different homology classes. In [9] a wide family of links that have got equivalent lift is shown. This family can be found by cabling Example 2 with particular braids. The simplest example that we can extract from this family, with the same number of components and the same homology class for each component, is the following one.

Example 3. Different links in $L(4,1)$ with cables of Hopf link as lift

The links $A_{2,2}$ and $B_{2,2}$ of Table 3 have the same number of components $n = 2$ and each of these components has the same homology class $\delta = 2$, anyway the computation of the Alexander polynomial of $A_{2,2}$ and $B_{2,2}$ (see Section 6 for details) shows that the links are not diffeo-equivalent. Their lift is equivalent because from the construction, we insert a braid into each arc of L_A and L_B so that each component of the Hopf link resulting from the lift has the same cabling.

Table 3

Geometric invariants of $A_{2,2}$ and $B_{2,2}$ in $L(4,1)$

	$A_{2,2}$	$B_{2,2}$
		
ν	2	2
$[K]\subset H_1(L(p,q))$	2,2	2,2
$H_1(L(p,q)\setminus L)$	$\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}_2$
$\overline{\Delta}^1(t)$	t^7+t^6-t-1	$t^7-t^6+t^5-t^4+t^3-t^2+t-1$
$\overline{\Delta}^{-1}(t)$	t^6+1	$t^6+t^4+t^2+1$

3.4. When the lift is a complete invariant

Since the lift of links in lens spaces comes from a cyclic covering, it is a natural question to ask if it is a complete invariant, at least for some family of knots.

As a consequence of Examples 1, 2 and 3, the lift is not a complete invariant for unoriented knots and links in $L(p,q)$, both up to diffeomorphism and up to isotopy. The problem of understanding whether the lift is a complete invariant can be referred also to oriented links. The answer is slightly different.

First of all, an orientation on the previous counter-examples allows us to find new examples, consisting of different oriented links in lens spaces having equivalent oriented lift.

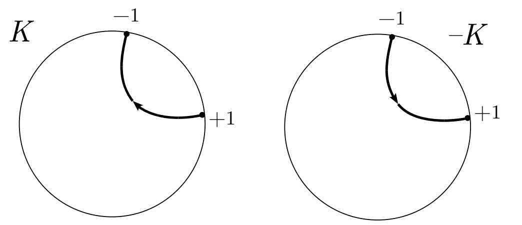


Fig. 3.5. *Two not isotopy-equivalent oriented knots with equivalent trivial lift in* $L(3,1)$

Moreover another family of counter-examples arises in the case of isotopy-equivalence. If we take an oriented knot $K\subset L(p,q)$ such that $\tilde{K}$ is invertible (i. e., it is equivalent to the knot with reversed orientation), then also the knot $-K\subset L(p,q)$ with reversed orientation has the same oriented lift. Usually $-K$ is not isotopy-equivalent to K because the homology class changes. A really simple example is provided by the two knots in $L(3,1)$ illustrated in Fig. 3.5: they both lift to the trivial knot, nevertheless they have different homology classes ($[K] = 1$ and $[-K] = 2$). For links something similar happens, but you have to be careful to the orientation of each component.

We can say something more in the case of oriented primitive-homologous knots when they are considered up to diffeo-equivalence, by the following theorem of Sakuma — also proved by Boileau and Flapan — about freely periodic knots.

A knot K in $\mathbf{S}^3$ is said to be freely periodic if there is a free cyclic action on $\mathbf{S}^3$ that fix K. Clearly this action produces a lens space and a knot inside it, that lifts to K. In this case, if $\mathrm{Diff}^*(\mathbf{S}^3,K)$ is the group of diffeomorphisms of the pair $(\mathbf{S}^3,K)$, which preserves the orientation of both $\mathbf{S}^3$ and K, up to isotopy, then a symmetry G of a knot K in $\mathbf{S}^3$ is a finite subgroup of $\mathrm{Diff}^*(\mathbf{S}^3,K)$, up to conjugation.

Theorem 4. [18; 19]. *Suppose that a knot $K \subset \mathbf{S}^3$ has free period p. Then there is a unique symmetry G of K realizing it, provided that (i) K is prime, or (ii) K is composite and the slope is specified.*

If we translate this theorem into the language of knots in lens spaces, the specification of the slope is equivalent to fixing the parameter q of the lens space. As a consequence, two primitive-homologous knots K and K' in $L(p,q)$ with equivalent non-trivial lift are necessarily diffeo-equivalent in $L(p,q)$.

From [15] and [16], we know that the group of the isotopy classes of diffeomorphisms of $L(p,q)$ is not trivial, thus Theorem 4 does not provide a complete answer about the equivalence of K and K' up to ambient isotopy.

3.5. Essential invariants

An invariant of links in $\mathbf{S}^3$ turns out to be an invariant of links in lens spaces when it is computed on their lifts. This operation produces a lot of invariants. On the contrary, an invariant for links in lens spaces which can not merely be computed in the lift is called an *essential* invariant.

Different links with equivalent lift are an useful tool to check whether an invariant I of links in lens spaces is essential: just find two different links L and L' with equivalent lift such that $I(L) \neq I(L')$. From now on, the paper will focus on checking whether several geometric invariants of links in lens spaces are essential or not: the fundamental quandle, the group of the link, the first homology group of the complement and the twisted Alexander polynomials.

4. Fundamental quandle

The fundamental quandle is a very strong invariant of links in the 3-sphere: in fact it is a complete diffeo-invariant. The fundamental quandle can be defined also for links in lens spaces [10; 20]: is it still a complete invariant? This question is strictly related also to the essentiality of the invariant.

Given an oriented link $L \subset L(p,q)$, let $N(L)$ denote an open tubular neighborhood of L, consider the manifold $Q = L(p,q)\setminus N(L)$ and fix a base point $*$ in it. Let Γ_L be the set of based homotopy classes of paths from $*$ to $\partial N(L)$ (the homotopy endpoint can move freely on $\partial N(L)$). We can define an operation $\circ$ on this set: for every a and b in Γ_L, consider the toric component of $\partial N(L)$ containing the starting point of b and let m be a meridian of this torus, the operation $a \circ b$ gives the class of the path $bmb^{-1}a$. The set Γ_L with the operation $\circ$ is a distributive groupoid or equivalently, a quandle (see [10]). The algebraic structure $(\Gamma_L, \circ)$ is the *fundamental quandle* of an oriented link L in $L(p,q)$.

Proposition 4. *The fundamental quandle of a link in a lens space is isomorphic to the fundamental quandle of its lift in $\mathbf{S}^3$.*

Proof. According to [20. Lemma 5.4], the fundamental quandle is invariant under cyclic coverings, and if we consider the cyclic covering P: $(\mathbf{S}^3 \setminus \tilde{L}) \to (L(p,q)\setminus L)$, we get the assertion. □

A consequence of this result is the following corollary.

Corollary 1. *The fundamental quandle of links in lens spaces is an inessential diffeo-invariant.*

The fundamental quandle of a link in a 3-manifold is a geometric invariant that can be explicitly computed on a diagram only for links in $\mathbf{S}^3$ [10] and in $\mathbb{RP}^3$ [21]. Proposition 4 allows us to compute the fundamental quandle of a link L in lens spaces by computing the fundamental quandle of the lift $\widetilde{L}$.

Theorem 4 can be combined to Proposition 4 to get the following statement.

Corollary 2. *The fundamental quandle of oriented primitive-homologous knots in lens spaces classifies them up to diffeo-equivalence, unless the fundamental quandle is trivial.*

For the case $\mathbb{RP}^3 = L(2,1)$, this result is directly stated in [22], where it is generalized also for non primitive-homologous knots.

Theorem 5. [22. Theorem 2]. *Two knots in $\mathbb{RP}^3$ are diffeo-equivalent if and only if the corresponding fundamental quandles are isomorphic.*

As a direct consequence of Proposition 4 and Theorem 5 we get the following theorem.

Theorem 6. *Two knots in $\mathbb{RP}^3$ are diffeo-equivalent if and only if the corresponding lifts are equivalent.*

We cannot generalize Corollary 2 to knots in all lens spaces up to diffeomorphism (and hence also up to isotopy) because of Example 1.

Remark 1. The fundamental quandle of knots in lens spaces is not a complete diffeo-invariant for $L\left(p,\frac{p\pm 1}{2}\right)$ with odd $p > 5$. Moreover it is not a complete isotopy-invariant also in the case of $L(5,2)$.

A similar statement for $L(4,1)$ follows from Examples 2 and 3.

By [23], we can compute other invariants of links in lens spaces derived from the quandle theory, such as quandle co-cycles invariants. If they are an invariant of the quandle, then they are inessential. If we consider bi-quandles instead, there is an example in [21] for links in the projective space where the co-cycle invariant seems more significant.

If we want a quandle-like structure that results essential we should turn to the oriented augmented fundamental rack [20], which is a complete invariant of framed links in 3-manifolds, and can be computed using mixed link diagrams.

5. Group of the link and homology

In this section we focus on the properties of the group of the link L in lens spaces, that is to say, the fundamental group of the complement $L(p,q) \setminus L$. After giving a presentation on disk diagram of the group, we compute it on several examples, in order to show that the group is an essential diffeo-invariant and that Norwood theorem about knots in $\mathbf{S}^3$ holds no longer in $L(p,q)$.

5.1. Group of the link

We follow the presentation given in [11], that is a generalization of the Wirtinger theorem as the presentation of [24] for the case of the projective space $L(2,1)$.

Let L be a link in $L(p,q)$ described by a disk diagram. Assume $p > 1$. Fix an orientation for L, which induces an orientation on the projection of the link. We can prove that if we reverse the orientation, the corresponding group is isomorphic to the former one. In order to find a presentation, perform an R_1 move on each overpass of the diagram having both endpoints on the boundary of the disk; in this way every overpass has at most one boundary point. Then label the overpasses as follows: $A_1, \dots, A_t$ are the ones ending in the upper hemisphere, namely in $+1, \dots, +t$, while $A_{t+1}, \dots, A_{2t}$ are the overpasses ending in $-1, \dots, -t$. The overpasses with no boundary points are labelled by $A_{2t+1}, \dots, A_r$. For each $i = 1, \dots, t$, let $\varepsilon_i = +1$ if, according to the link orientation, the overpass A_i starts from the point $+i$; otherwise, if A_i ends in the point $+i$, let $\varepsilon_i = -1$.

Associate to each overpass A_i a generator a_i, which is a loop around the overpass as in the classical Wirtinger theorem, oriented following the left hand rule. Moreover let f be the generator of the fundamental group of the lens space illustrated in Fig. 5.1. The relations are the following:

W: $w_1, \ldots, w_s$ are the classical Wirtinger relations for each crossing, that is to say $a_i a_j a_i^{-1} a_k^{-1} = 1$ or $a_i a_j^{-1} a_i^{-1} a_k = 1$, according to Fig. 5.2;

L: l is the lens relation $a_1^{\varepsilon_1} \ldots a_t^{\varepsilon_t} = f^p$;

M: $m_1, \ldots, m_t$ are relations (of conjugation) between loops corresponding to overpasses with identified endpoints on the boundary. If $t=1$ the relation is $a_2^{\varepsilon_1} = a_1^{-\varepsilon_1} f^q a_1^{\varepsilon_1} f^{-q} a_1^{\varepsilon_1}$. Otherwise, consider the point $-i$ and, according to equator orientation, let $+j$ and $+j+1$ (mod$t$) be the plus-type points adjacent to it. We distinguish two cases:

- if $-i$ lies on the diagram between -1 and $+1$, then the relation m_i is

$$a_{t+i}^{e_i} = \left(\prod_{k=1}^{j} a_k^{e_k} \right)^{-1} f^q \left(\prod_{k=1}^{i-1} a_k^{e_k} \right) a_i^{e_i} \left(\prod_{k=1}^{i-1} a_k^{e_k} \right)^{-1} f^{-q} \left(\prod_{k=1}^{j} a_k^{e_k} \right);$$

- otherwise, the relation m_i is

$$a_{t+i}^{e_i} = \left(\prod_{k=1}^{j} a_k^{e_k} \right)^{-1} f^{q-p} \left(\prod_{k=1}^{i-1} a_k^{e_k} \right) a_i^{e_i} \left(\prod_{k=1}^{i-1} a_k^{e_k} \right)^{-1} f^{p-q} \left(\prod_{k=1}^{j} a_k^{e_k} \right).$$

Consider the lens space model depicted in Fig. 2.1, let N be the point $(0,0,1)$ of B^3 and $F\colon B^3 \to L(p,q)$ be the quotient map.

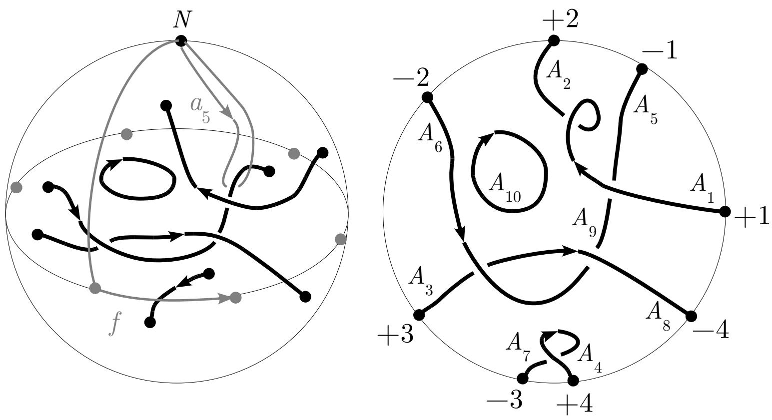


Fig. 5.1. *Example of overpasses labelling for a link in $L(6,1)$*

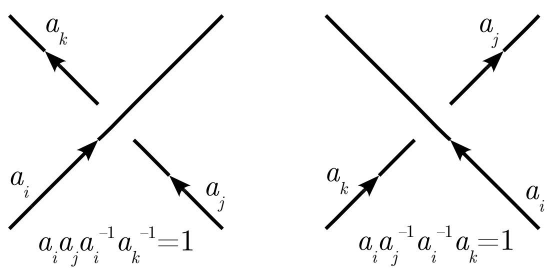


Fig. 5.2. *Wirtinger relations*

Theorem 7. [11]. *Let $*=F(N)$, then the group of the link $L \subset L(p,q)$ is:*

$$\pi_1(L(p,q) \setminus L, *) = \langle a_1, \ldots, a_r, f \mid w_1, \ldots, w_s, l, m_1, \ldots, m_t \rangle.$$

In the special case of $L(2,1) = \mathbb{RP}^3$, the presentation is equivalent (via Tietze transformations) to the one given in [24].

5.2. First homology group

At first, a way to compute the homology class of a knot directly from a disk diagram is shown and it is really useful because it is the easiest isotopy invariant. Then the method to determine, directly from the diagram, the first homology group of links in $L(p,q)$ is found. Differently from the $\mathbf{S}^3$ case, a non-trivial torsion part may appears and it is useful for the computation of twisted Alexander polynomials.

Consider a diagram of an oriented knot $K \subset L(p,q)$ and let ε_i be as defined in the previous section. Define $\delta_K = \sum_{i=1}^{t} \varepsilon_i$.

Lemma 1. *If $K \subset L(p,q)$ is an oriented knot and $[K]$ is the homology class of K in $H_1(L(p,q))$, then $[K] = \delta_K$.*

By abelianizing the presentation of the group of the link of Theorem 7, we get the first homology group of the complement of a link in lens spaces.

Corollary 3. [11]. *Let L be a link in $L(p,q)$, with components $L_1, \dots L_\nu$. For each $j = 1,\dots,\nu$, let $\delta_j = [L_j] \in \mathbb{Z}_p = H_1(L(p,q))$. Then $H_1(L(p,q) \setminus L) \cong \mathbb{Z}^\nu \oplus \mathbb{Z}_d$, where $d = \mathrm{gsd}(\delta_1, \dots \delta_\nu, p)$.*

5.3. Norwood theorem

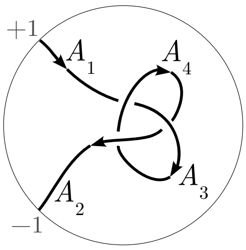


Fig. 5.3. *The knot $K_1 \# T$ in $L(p,q)$*

A theorem of Norwood [25] states that every knot in the 3-sphere admitting a presentation for its group with only two generators is prime. For every lens space $L(p,q)$ with $p > 1$, we now show a knot that has a minimal presentation of the group with two generators, but it is not prime; as a consequence, the Norwood theorem cannot be generalized to lens spaces.

Example 4. Let T be the trefoil knot in $\mathbf{S}^3$. Let K_1 be the knot of the previous example and consider the connected sum $K_1 \# T$ in $L(p,q)$, as Fig. 5.3 shows.

$$\pi_1(L(p,q) \setminus (K_1 \# T), *) = \langle a_1, a_2, a_3, a_4, f \mid a_1 a_4 a_3^{-1} a_4^{-1} = 1,\ a_4 a_3 a_2^{-1} a_3^{-1} = 1,$$

$$a_3 a_2 a_4^{-1} a_2^{-1} = 1, a_1 = f^p,\ a_2 = a_1^{-1} f^q a_1 f^{-q} a_1 \rangle = \langle a_3, f \mid f^{-p} a_3 f^p a_3 f^{-p} a_3^{-1} = 1 \rangle.$$

5.4. Essentiality of the group and the homology

Theorem 8. *The group of the link is an essential diffeo-invariant, as a consequence of Example 2.*

Theorem 9. *The homology group $H_1(L(p,q) \setminus L)$ is an essential diffeo-invariant too, as we can see from Table 2. Moreover the homology class of a knot is an essential isotopy-invariant (see Table 1).*

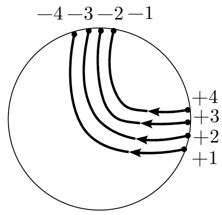
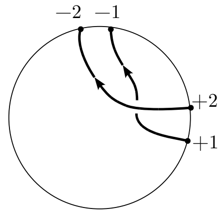
Example 5. The last example consists of the two links M_1 and M_2 in $L(5,2)$ on Table 4. They have not diffeo-equivalent lift, more precisely, one lift is the link $L4a1$ of the Knot Atlas, while the other one is its mirror image. In this case the groups are isomorphic, hence sometimes the lift may be stronger than the group of the link.

6. Twisted Alexander polynomials

In this section we describe a class of twisted Alexander polynomials of links in lens spaces. This class consists of those polynomials with 1-dimensional representation over particular Noetherian unique factorization domains that take into account the torsion part of the group of the link. The goal is to investigate whether they are an essential invariant.

Table 4

Geometric invariants of M_1 and M_2 in $L(5,2)$

	M_1	M_2
		
ν	2	2
$[K] \subset H_1(L(p,q))$	2,2	1,1
$\pi_1(L(p,q)\setminus L)$	$\langle a, f \mid af^2 = f^2a\rangle$	$\langle a, f \mid af^2 = f^2a\rangle$
$H_1(L(p,q)\setminus L)$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}$
$\overline{\Delta}^1(t)$	t^2-1	t^2-1

6.1. The computation of the twisted Alexander polynomials

The twisted Alexander polynomials are defined in the following way (for further references see [11; 26; 27]). Given a finitely generated group π, denote with $H = \pi / \pi'$ its abelianization and let $G = H / \mathrm{Tors}(H)$. Take a presentation $\pi = \langle x_1, \dots, x_m | r_1, \dots, r_n\rangle$ and consider the Alexander–Fox matrix A associated to the presentation, that is $A_{ij} = \mathcal{P}\left(\frac{\partial r_i}{\partial x_j}\right)$, where $\mathcal{P}$ is the natural projection $\mathbb{Z}[F(x_1, \dots, x_m)] \to \mathbb{Z}[\pi] \to \mathbb{Z}[H]$ and $\frac{\partial r_i}{\partial x_j}$ is the Fox derivative of r_i. Moreover let $E(\pi)$ be the first elementary ideal of π, which is the ideal of $\mathbb{Z}[H]$ generated by the $(m-1)$-minors of A. For each homomorphism σ: $\mathrm{Tors}(H) \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ we can define a twisted Alexander polynomial $\Delta^\sigma(\pi)$ of π as follows: fix a splitting $H = \mathrm{Tors}(H) \times G$ and consider the ring homomorphism that we still denote with σ: $\mathbb{Z}[H] \to \mathbb{C}[G]$ sending (f, g), with $f \in \mathrm{Tors}(H)$ and $g \in G$, to $\sigma(f)g$, where $\sigma(f) \in \mathbb{C}^*$. The ring $\mathbb{C}[G]$ is a unique factorization domain and we set $\Delta^\sigma(\pi) = \gcd(\sigma(E(\pi)))$. This is an element of $\mathbb{C}[G]$ defined up to multiplication by elements of G and non-zero complex numbers. If $\Delta(\pi)$ denotes the classical Alexander polynomial we have $\Delta^1(\pi) = \alpha\Delta(\pi)$, with $\alpha \in \mathbb{C}^*$.

6.2. Twisted Alexander polynomials are essential invariants

If $L \subset L(p,q)$ is a link in a lens space then the σ-*twisted Alexander polynomial* of L is $\Delta^\sigma_L = \Delta^\sigma(\pi_1(L(p,q)\setminus L))$. Since in this case $\mathrm{Tors}(H) = \mathbb{Z}_d$ then $\sigma(\mathrm{Tors}(H))$ is contained in the cyclic group generated by ζ, where ζ is a d-th primitive root of the unity. Note that $\Delta^\sigma_L \in \mathbb{C}[G]$ is defined up to multiplication by $\zeta^h g$, with $g \in G$.

If L has at least two components we can consider the projection φ: $\mathbb{C}[G] = \mathbb{C}[t_1, \dots, t_m, t_1^{-1}, \dots, t_m^{-1}] \to \mathbb{C}[t, t^{-1}]$, sending each variable t_i to t. The *one-variable* twisted Alexander polynomial of L is $\overline{\Delta}^\sigma_L = \varphi(\Delta^\sigma_L)$. The computation of $\overline{\Delta}^\sigma_L$ for knots in arbitrary lens spaces has been implemented in a program using Mathematica code: the input is a knot diagram in $L(p,q)$ given through a generalization of the Dowker–Thistlewaithe code (see [28; 29]). Thanks to this program we obtained the results listed in Tables 1, 2, 3 and 4.

Theorem 10. *The twisted Alexander polynomials are essential diffeo-invariants (see Tables 2 and 3 even the lift may be stronger than the Alexander polynomial (see Table 4). Moreover they are not complete invariants (see Table 1).*

6.3. Relationship between Alexander invariants of L and $\tilde{L}$

The lift of links in the lens space $L(p,q)$ can be seen as a freely p-periodic link (also said a (p,q)-lens links in $\mathbf{S}^3$ [30]). Exploiting known results about freely periodic links, we can relate the invariants of the link to the corresponding invariants of its lift. The first question that deserves our interest is the following one: does the Alexander polynomial of the lift depend on the twisted Alexander polynomials of the link in lens spaces? Hartley provided the answer for the Alexander polynomial of freely periodic knots: in [31] there is a formula connecting the twisted Alexander polynomials in the case that both $K \subset L(p,q)$ and $\tilde{K} \subset \mathbf{S}^3$ are knots. Furthermore, Chbili has shown in [32] an interesting characterization for the multi-variable Alexander polynomial of the lift of braid links in lens spaces.

Can we find pieces of information about the twisted Alexander polynomials of a link $L \subset L(p,q)$ from the Alexander polynomial of its lift? From Tables 2 and 3 we see that this is not possible, neither for knots nor for links. Another interesting counter-example for this question is the next one: considering the unknot and the local trefoil in $L(2,1)$, their lifts are the unlink with two components and two split trefoils respectively. The twisted Alexander polynomials of these links in $L(2,1)$ are different, their lifts in $\mathbf{S}^3$ are different, but their lifts have the same Alexander polynomial (equal to zero).

References

1. Baker K., Grigsby J.E., Hedden M. Grid diagrams for lens spaces and combinatorial knot Floer homology. *Int. Math. Res. Not. (IMRN)*, 2008, vol. 10, art. ID rnm024.

2. Berge J. *Some knots with surgeries yielding lens spaces.* (Unpublished).

3. Stevan S. *Torus Knots in Lens Spaces & Topological Strings.* Avialable at: http://arxiv.org/abs/1308.5509.

4. Buck D., Mauricio M. Connect sum of lens spaces surgeries: application to Hin recombination. *Math. Proc. Cambridge Philos. Soc.,* 2011, vol. 150, pp. 505–525.

5. Lambropoulou S., Rourke C.P. Markov's theorem in 3-manifolds. *Topology Appl.,* 1997, vol. 78, pp. 95–122.

6. Hoste J., Przytycki J.H. The $(2,\infty)$-skein module of lens spaces; a generalization of the Jones polynomial. *J. Knot Theory Ramifications*, 1993, vol. 2, pp. 321–333.

7. Gabrov ek B. Classification of knots in lens spaces. *Ph.D. Thesis.* Slovenia, University of Ljubljana, 2013.

8. Manfredi E. Knots and links in lens spaces. *Ph D. Thesis*. Italy, University of Bologna, 2014.

9. Manfredi E. Lift in the 3-sphere of knots and links in lens spaces. *J. Knot Theory Ramifications*, 2014, vol. 23, pp. 1450022.

10. Matveev S.V. Distributive groupoids in knot theory. *Math. USSR Sb.,* 1984, vol. 47, pp. 73–83.

11. Cattabriga A., Manfredi E., Mulazzani M. On knots and links in lens spaces. *Topology Appl.,* 2013, vol. 160, pp. 430–442.

12. Cornwell C. A polynomial invariant for links in lens spaces. *J. Knot Theory Ramifications*, 2012, vol. 21, pp. 1250060.

13. Cattabriga A., Manfredi E., Rigolli L. *Equivalence of two diagram representations of links in lens spaces and essential invariants.* Avialable at: http://arxiv.org/abs/1312.2230.

14. Brody E.J. The topological classification of the lens spaces. *Ann. of Math.,* 1960, vol. 71, pp. 163–184.

15. Bonahon F. Diffótopies des espaces lenticulaires. *Topology*, 1983, vol. 22, pp. 305–314.

16. Hodgson C., Rubinstein J.H. *Involutions and isotopies of lens spaces*. Berlin : Springer Publ., 1985.

17. Drobotukhina Y.V. An analogue of the Jones polynomial for links in $\mathbb{R}P^3$ and a generalization of the Kauffman–Murasugi theorem. *Leningrad Math. J.,* 1991, vol. 2, pp. 613–630.

18. Sakuma M. Uniqueness of symmetries of knots. *Math. Z.,* 1986, vol. 192, pp. 225–242.

19. Boileau M., Flapan E. Uniqueness of free actions on $\mathbf{S}^3$ respecting a knot. *Canad. J. Math.,* 1987, vol. 39, pp. 969–982.

20. Fenn R., Rourke C. Racks and links in codimension two. *J. Knot Theory Ramifications*, 1992, vol. 1, pp. 343–406.

21. Gorkovets D.V. Cocycle invariants for links in projective space. *Vestnik Chelyabinskogo gosudarstvennogo universiteta. Matematika. Mekhanika. Informatica* [Bulletin of Chelyabinsk State University. Mathematics. Mechanics. Informatics], 2010, vol. 12, pp. 88–97.

22. Gorkovets D.V. Distributive groupoids for knots in projective space. *Vestnik Chelyabinskogo gosudarstvennogo universiteta. Matematika. Mekhanika. Informatica* [Bulletin of Chelyabinsk State University. Mathematics. Mechanics. Informatics], 2008, vol. 10, pp. 89–93.

23. Carter J. Scott. A survey of quandle ideas. *Introductory lectures on knot theory. Ser. Knots Everything,46.* Hackensack, World Sci. Publ., 2012.

24. Huynh V.Q., T.T.Q. Le. Twisted Alexander polinomial of links in the projective space. *J. Knot Theory Ramifications,* 2008, vol. 17, pp. 411–438.

25. Norwood F.H. Every two-generator knot is prime. *Proc. Amer. Math. Soc.,* 1982, vol. 86, pp. 143–147.

26. Wada M. Twisted Alexander polynomial for finitely presentable groups. *Topology*, 1994, vol. 33, pp. 241–256.

27. Friedl S., Vidussi S. A survey of twisted Alexander polynomials. *The mathematics of knots. Contrib. Math. Comput. Sci. 1.* Heidelberg, Springer Publ., 2011.

28. Dowker, C. H., Thistlethwaite M.B. Classifications of knot projections. *Topology Appl.,* 1983, vol. 16, pp. 19–31.

29. Doll H., Hoste J. A tabulation of oriented link. *Math. Comp.,* 1991, vol. 57, pp. 747–761.

30. Chbili N. A new criterion for knots with free periods. *Ann. Fac. Sci. Toulouse Math.,* 2003, vol. 12, pp. 465–477.

31. Hillman J.A., Livingston C., Naik S. Twisted Alexander polynomials of periodic knots. *Algebr. Geom. Topol.,* 2006, vol. 6, pp. 145–169.

32. Chbili N. The multi-variable Alexander polynomial of lens braids. *J. Knot Theory Ramifications,* 2002, vol. 11, pp. 1323–1330.

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ОБ УЗЛАХ И ЗАЦЕПЛЕНИЯХ В ЛИНЗОВЫХ ПРОСТРАНСТВАХ

Э. Манфреди, М. Мулаццани

Дается короткий обзор некоторых недавних результатов об узлах и зацеплениях в линзовых пространствах. Описываются дисковые диаграммы вместе с касающимся эквивалентности аналогом теоремы Райдемайстера. Рассмотрено поднятие узлов и зацеплений в трехмерную сферу, приводится несколько примеров различных узлов и зацеплений, обладающих эквивалентными поднятиями. Обсуждается существенность относительно поднятия классических инвариантов узлов и зацеплений в линзовых пространствах.

Ключевые слова: *узел, зацепление, линзовое пространство, поднятие, фундаментальный квандл, группа зацепления, скрученный полином Александра.*

Список литературы

1. Baker, K. Grid diagrams for lens spaces and combinatorial knot Floer homology / K. Baker, J. E. Grigsby, M. Hedden // Int. Math. Res. Not. (IMRN). — 2008. — Vol. 10. — Art. ID rnm024.

2. Berge, J. Some knots with surgeries yielding lens spaces / J. Berge. — (Unpublished manuscript).

3. Stevan, S. Torus Knots in Lens Spaces & Topological Strings [Электронный ресурс] / S. Stevan. — URL : http://arxiv.org/abs/1308.5509.

4. Buck, D. Connect sum of lens spaces surgeries: application to Hin recombination / D. Buck, M. Mauricio // Math. Proc. Cambridge Philos. Soc. — 2011 — Vol. 150. — P. 505–525.

5. Lambropoulou, S. Markov's theorem in 3-manifolds / S. Lambropoulou, C. P. Rourke // Topology Appl. — 1997. — Vol. 78. — P. 95–122.

6. Hoste, J. The $(2,\infty)$-skein module of lens spaces; a generalization of the Jones polynomial / J. Hoste, J. H. Przytycki // J. Knot Theory Ramifications. — 1993. — Vol. 2. — P. 321–333.

7. Gabrov ek, B. Classification of knots in lens spaces / B. Gabrov ek. // Ph.D. Thesis. — Slovenia : University of Ljubljana, 2013.

8. Manfredi, E. Knots and links in lens spaces / E. Manfredi // Ph D. Thesis. — Italy : University of Bologna, 2014.

9. Manfredi, E. Lift in the 3-sphere of knots and links in lens spaces / E. Manfredi // J. Knot Theory Ramifications. — 2014. — Vol. 23. — P. 1450022.

10. Matveev, S. V. Distributive groupoids in knot theory / S. V. Matveev // Math. USSR Sb. — 1984. — Vol. 47. — P. 73–83.

11. Cattabriga, A. On knots and links in lens spaces / A. Cattabriga, E. Manfredi, M. Mulazzani // Topology Appl. — 2013. — Vol. 160. — P. 430–442.

12. Cornwell, C. A polynomial invariant for links in lens spaces / C. Cornwell // J. Knot Theory Ramifications. — 2012. — Vol. 21. —P. 1250060.

13. Cattabriga, A. Equivalence of two diagram representations of links in lens spaces and essential invariants [Электронный ресурс] / A. Cattabriga, E. Manfredi, L. Rigolli. — URL : http://arxiv.org/abs/1312.2230.

14. Brody, E. J. The topological classification of the lens spaces / E. J. Brody // Ann. of Math. — 1960. — Vol. 71. — P. 163–184.

15. Bonahon, F. Diffótopies des espaces lenticulaires / F. Bonahon // Topology. — 1983. — Vol. 22. — P. 305–314.

16. Hodgson, C. Involutions and isotopies of lens spaces / C. Hodgson, J. H. Rubinstein. — Berlin : Springer, 1985.

17. Drobotukhina, Y. V. An analogue of the Jones polynomial for links in $\mathbb{R}P^3$ and a generalization of the Kauffman–Murasugi theorem / Y. V. Drobotukhina // Leningrad Math. J. — 1991. — Vol. 2. — P. 613–630.

18. Sakuma, M. Uniqueness of symmetries of knots / M. Sakuma // Math. Z. — 1986. — Vol. 192. — P. 225–242.

19. Boileau, M. Uniqueness of free actions on $\mathbf{S}^3$ respecting a knot / M. Boileau, E. Flapan // Canad. J. Math. — 1987. — Vol. 39. — P. 969–982.

20. Fenn, R. Racks and links in codimension two / R. Fenn, C. Rourke // J. Knot Theory

Ramifications. — 1992. — Vol. 1. — P. 343–406.
21. Gorkovets, D. V. Cocycle invariants for links in projective space / D. V. Gorkovets // Bulletin of Chelyabinsk State University. Mathematics. Mechanics. Informatics. — 2010. — Vol. 12. — P. 88–97.
22. Gorkovets, D. V. Distributive groupoids for knots in projective space / D. V. Gorkovets // Bulletin of Chelyabinsk State University. Mathematics. Mechanics. Informatics. — 2008. — Vol. 10. — P. 89–93.
23. Scott Carter, J. A survey of quandle ideas / J. Scott Carter // Introductory lectures on knot theory. Ser. Knots Everything, 46. — Hackensack, NJ : World Sci. Publ., 2012.
24. Huynh, V. Q. Twisted Alexander polinomial of links in the projective space / V. Q. Huynh, T. T. Q. Le // J. Knot Theory Ramifications. — 2008. — Vol. 17. — P. 411–438.
25. Norwood, F. H. Every two-generator knot is prime / F. H. Norwood // Proc. Amer. Math. Soc. — 1982. — Vol. 86. — P. 143–147.
26. Wada, M. Twisted Alexander polynomial for finitely presentable groups / M. Wada // Topology. — 1994. — Vol. 33. — P. 241–256.
27. Friedl, S. A survey of twisted Alexander polynomials / S. Friedl, S. Vidussi // The mathematics of knots. Contrib. Math. Comput. Sci. 1. — Heidelberg : Springer, 2011. — P. 45–94.
28. Dowker, C. H. Classifications of knot projections / C. H. Dowker, M. B. Thistlethwaite // Topology Appl. — 1983. — Vol. 16. — P. 19–31.
29. Doll, H. A tabulation of oriented link / H. Doll, J. Hoste // Math. Comp. — 1991. — Vol. 57. — P. 747–761.
30. Chbili, N. A new criterion for knots with free periods / N. Chbili // Ann. Fac. Sci. Toulouse Math. — 2003. — Vol. 12. — P. 465–477.
31. Hillman, J. A. Twisted Alexander polynomials of periodic knots / J. A. Hillman, C. Livingston, S. Naik // Algebr. Geom. Topol. — 2006. — Vol. 6. — P. 145–169.
32. Chbili, N. The multi-variable Alexander polynomial of lens braids / N. Chbili // J. Knot Theory Ramifications. — 2002. — Vol. 11. — P. 1323–1330.

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A NOTE ON THE GROTHENDIECK GROUP OF AN ADDITIVE CATEGORY*

D. E. V. Rose

There are two abelian groups which can naturally be associated to an additive category $\mathcal{A}$: the split Grothendieck group of $\mathcal{A}$ and the triangulated Grothendieck group of the homotopy category of (bounded) complexes in $\mathcal{A}$. We prove that these groups are isomorphic. Along the way, we deduce that the 'Euler characteristic' of a complex in $\mathcal{A}$ is invariant under homotopy equivalence, a result which has implications for (de)categorification.

Keywords: *Grothendieck group, additive category, categorification.*

1. Introduction

A categorification of an algebraic structure is typically given by an additive category (often possessing additional structure) from which the original structure can be recovered by taking the Grothendieck group; see for instance [1] for the abelian case. In certain categorifications of quantum invariants of tangles, the categorification is accomplished by first finding an additive category which categorifies an algebraic structure and then passing to the homotopy category of complexes to give the categorification of the tangle invariant (see [2] and [3]). The categorified tangle invariant decategorifies to give the original tangle invariant by taking the 'Euler characteristic' of the complex, the alternating sum of the terms in the complex, viewed as an element of the split Grothendieck group of the additive category. Since the homotopy category is triangulated, the natural decategorification of this category is its triangulated Grothendieck group. This posits the question: are these two Grothendieck groups isomorphic? This question can equivalently be stated: is the Euler characteristic of a complex in an additive (but not necessarily abelian) category invariant under homotopy equivalence?

We answer both these questions in the affirmative:

Theorem 1. *Let $\mathcal{A}$ be an additive category and $\mathcal{K}^b(\mathcal{A})$ denote the homotopy category of bounded complexes in $\mathcal{A}$. The split Grothendieck group of $\mathcal{A}$ is isomorphic to the triangulated Grothendieck group of $\mathcal{K}^b(\mathcal{A})$.*

Theorem 2. *Let $A^\bullet \simeq B^\bullet$ be homotopy equivalent complexes in $\mathcal{K}^b(\mathcal{A})$, then*

$$\sum_{i=-\infty}^{\infty}(-1)^i\left\langle A^i\right\rangle=\sum_{i=-\infty}^{\infty}(-1)^i\left\langle B^i\right\rangle,$$

where $\langle\cdot\rangle$ denotes the corresponding element in the split Grothendieck group of $\mathcal{A}$.

Of course, this result is not surprising; indeed, in the case that $\mathcal{A}$ is abelian, the analog of Theorem 2 is an easy exercise in homological algebra. Nevertheless, the proof presented here is unexpectedly non-trivial and the general result is of interest to the categorification community as non-abelian additive categories arise naturally in this field. Furthermore, this result seems to have been implicitly assumed in the categorification literature, while a proof has up to now not appeared.

We present the relevant background on additive categories and Grothendieck groups in Section 2. In Section 3 we prove Theorems 1 and 2 and in Section 4 we mention a slight generalization of Theorem 2 which is used in [4].

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2. Background

Let $\mathcal{A}$ be an additive category. Recall that this means that $\mathcal{A}$ has a zero object, finite biproducts, and that $\mathrm{Hom}_{\mathcal{A}}(A_1, A_2)$ is an abelian group for any objects A_1, A_2 in $\mathcal{A}$ with addition distributing over composition.

Definition 1. The *split Grothendieck group* of $\mathcal{A}$, denoted $K_{\oplus}(\mathcal{A})$, is the abelian group generated by isomorphism classes $\langle A\rangle$ of objects in $\mathcal{A}$ modulo the relations $\langle A_1 \oplus A_2\rangle = \langle A_1\rangle + \langle A_2\rangle$ for all objects A_1, A_2 in $\mathcal{A}$.

Recall that the Grothendieck group of an abelian category is the abelian group generated by isomorphism classes $\langle A\rangle$ of objects modulo the relations $\langle A_2\rangle = \langle A_1\rangle + \langle A_3\rangle$ for every short exact sequence $0\to A_1\to A_2\to A_3\to 0$ in $\mathcal{A}$. We can think of Definition 1 as the analog of this notion in an additive category where we impose relations corresponding to the only notion of exact sequence that makes sense, the split exact sequences $0\to A_1\to A_1\oplus A_2\to A_2\to 0$.

Suppose now that $\mathcal{C}$ is not only additive, but triangulated.

Definition 2. The *triangulated Grothendieck group*, denoted $K_{\Delta}(\mathcal{C})$, is the abelian group generated by isomorphism classes $\langle C\rangle$ of objects in $\mathcal{C}$ quotiented by the relation $\langle C_2\rangle = \langle C_1\rangle + \langle C_3\rangle$ for all distinguished triangles $C_1\to C_2\to C_3$.

Again, we think of distinguished triangles as the analogs of short exact sequences in $\mathcal{C}$.

3. Grothendieck Groups of Additive Categories

Now fix an additive category $\mathcal{A}$. Let $\mathcal{K}^b(\mathcal{A})$ denote the homotopy category of bounded (cochain) complexes in $\mathcal{A}$. Let $A^\bullet = (A^k \xrightarrow{d^k} \cdots \xrightarrow{d^{l-1}} A^l)$ be a bounded complex and let $A[m]^\bullet$ denote the complex shifted up by m in homological degree. We will underline the term in homological degree zero when it is not clear from the context. The distinguished triangle $A^\bullet \to 0 \to A[-1]^\bullet$ gives that

$$\langle A[-1]^\bullet\rangle = -\langle A^\bullet\rangle \tag{1}$$

and the triangle $A^k = (\underline{A}^{k+1} \xrightarrow{d^{k+1}} \cdots \xrightarrow{d^{l-1}} A^l) \to A[-k-1]^\bullet$ shows (via induction) that

$$\langle A^\bullet\rangle = \chi\langle A^\bullet\rangle \tag{2}$$

in $K_{\Delta}(\mathcal{K}^b(\mathcal{A}))$. Here $\chi(A^\bullet) := \sum_{i=-\infty}^{\infty} (-1)^i \left\langle A^i\right\rangle$ and A^i is shorthand for the complex with the object A^i in degree zero and all other terms zero. From this, we see that $K_{\Delta}(\mathcal{K}^b(\mathcal{A}))$ and $K_{\oplus}(\mathcal{A})$ are generated by the same elements.

Given complexes $A_1^\bullet$ and $A_2^\bullet$, the distinguished triangle $A_1^\bullet \to A_1^\bullet \oplus A_2^\bullet \to A_2^\bullet$ shows that

$$\langle (A_1 \oplus A_2)^\bullet\rangle = \langle A_1^\bullet\rangle + \langle A_2^\bullet\rangle. \tag{3}$$

It follows that there is a surjective map $K_{\oplus}(\mathcal{A}) \to K_{\Delta}(\mathcal{K}^b(\mathcal{A}))$.

To prove Theorem 1, it suffices to show that this map is injective or equivalently that there are no additional relations imposed on $K_{\Delta}(\mathcal{K}^b(\mathcal{A}))$ other than those given in equations (1), (2) and (3). Given a map $A_1 \xrightarrow{f} A_2$, these equations show that

$$\left\langle \mathrm{cone}(f)^\bullet \right\rangle = \sum_{j=\infty}^{\infty}\left((-1)^j \left\langle A_2^j\right\rangle + (-1)^{j+1}\left\langle A_1^j\right\rangle\right) = \left\langle A_2^\bullet\right\rangle - \left\langle A_1^\bullet\right\rangle$$

so distinguished triangles of the form

$$A_1 \xrightarrow{f} A_2 \to \mathrm{cone}(f)^\bullet \tag{5}$$

contribute no new relations. Since all distinguished triangles are isomorphic to those of the form (5) and isomorphism in $\mathcal{K}^b(\mathcal{A})$ is homotopy equivalence, it suffices to prove Theorem 2.

To this end, suppose that $\varphi\colon A_1^\bullet \to A_2^\bullet$ is a homotopy equivalence. The following result from [5] is given in the context of the category of abelian groups, but the proof sketched there carries over to arbitrary additive categories.

Lemma 1. *A chain map* $\varphi\colon A_1^\bullet \to A_2^\bullet$ *is a homotopy equivalence iff* $\mathrm{cone}(\varphi)^\bullet$ *is null-homotopic.*

The distinguished triangle $A_1^\bullet \xrightarrow{\varphi} A_2^\bullet \to \mathrm{cone}(\varphi)^\bullet$ together with (4) then show that Theorem 2 (and hence Theorem 1) follows from the next result.

Proposition 1. *Let* $A^\bullet$ *be a null-homotopic complex in* $\mathcal{K}^b(\mathcal{A})$, *then* $\chi(A^\bullet) = 0$ *when viewed as an element of* $K_\oplus(\mathcal{A})$.

Proof. We may assume that $A^\bullet = A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{2k}} A^{2k+1}$ contains all of the non-zero terms of $A^\bullet$. It suffices to show that

$$\bigoplus_{i=0}^{k} A^{2i} \cong \bigoplus_{i=0}^{k} A^{2i+1}$$

in $\mathcal{A}$, which we shall do by explicitly writing down the matrices giving the isomorphism.

Since $A^\bullet$ is null-homotopic there exist maps $A^j \xrightarrow{h^j} A^{j-1}$ so that $id_j = d^{j-1}h^j + h^{j+1}d^j$ which imply the relations $h^j h^{j+1} \dots h^{j+2l+1} = d^{j-2}h^{j-1}h^j \dots h^{j+2l+1} + h^j \dots h^{j+2l+1} h^{j+2l+2} d^{j+2l+1}$.

Consider now the maps

$$R : \bigoplus_{i=0}^{k} A^{2i} \to \bigoplus_{i=0}^{k} A^{2i+1}, \quad L : \bigoplus_{i=0}^{k} A^{2i} \to \bigoplus_{i=0}^{k} A^{2i}$$

given by the matrices in (6) and (7), where $\{\alpha_i\}$ are integers defined by the recursion $\alpha_0 = 1$, $\alpha_1 = -1$, and $\alpha_i = -\sum_{j=0}^{i-1} \alpha_j \alpha_{i-1-j}$. It is easy to see that in fact $\alpha_i = (-1)^i c_i$ where c_i is the i^{th} Catalan number.

$$R = \begin{pmatrix} d^0 & \alpha_0 h^2 & \alpha_1 h^2h^3h^4 & \alpha_2 h^2 \dots h^6 & \dots & \alpha_{k-1} h^2 \dots h^{2k} \\ 0 & d^2 & \alpha_0 h^4 & \alpha_1 h^4h^5h^6 & \dots & \alpha_{k-2} h^4 \dots h^{2k} \\ 0 & 0 & d^4 & \alpha_0 h^6 & \dots & \alpha_{k-3} h^6 \dots h^{2k} \\ 0 & 0 & 0 & d^6 & \dots & \alpha_{k-4} h^8 \dots h^{2k} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & d^{2k} \end{pmatrix} \tag{6}$$

$$L = \begin{pmatrix} \alpha_0 h^1 & \alpha_1 h^1h^2h^3 & \alpha_2 h^1 \dots h^7 & \alpha_3 h^1 \dots h^7 & \dots & \alpha_k h^1 \dots h^{2k+1} \\ d^1 & \alpha_0 h^3 & \alpha_1 h^3h^4h^5 & \alpha_2 h^1 \dots h^7 & \dots & \alpha_{k-1} h^3 \dots h^{2k+1} \\ 0 & d^3 & \alpha_0 h^5 & \alpha_1 h^5h^6h^7 & \dots & \alpha_{k-2} h^5 \dots h^{2k+1} \\ 0 & 0 & d^5 & \alpha_0 h^7 & \dots & \alpha_{k-3} h^7 \dots h^{2k+1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \alpha_0 h^{2k+1} \end{pmatrix} \tag{7}$$

We now compute the entries of the matrices RL and LR. For $i < j$ we have

$$(RL)_{ij} = \alpha_{j-i} d^{2i-2}h^{2i-1} \dots h^{2j-1} + \alpha_0\alpha_{j-i-1}h^{2i} \dots h^{2j-1} + \dots + \alpha_{j-i-1}\alpha_0 h^{2i} \dots h^{2j-1} +$$
$$+ \alpha_{j-i}h^{2i} \dots h^{2j}d^{2j-1} = \alpha_{j-i}(d^{2i-2}h^{2i-1} \dots h^{2j-1} + h^{2i} \dots h^{2j}d^{2j-1} - h^{2i} \dots h^{2j-1}) = 0$$

and $(RL)_{ij} = 0$ for $i > j$. We also compute $(RL)_{jj} = \alpha_0(d^{2j-2}h^{2j-1} + h^{2j}d^{2j-1}) = id_{2j-1}$ which shows that $RL = id$. Similarly, for $i < j$ we have

$$(LR)_{ij} = \alpha_{j-i}d^{2i-3}h^{2i-2} \dots h^{2j-2} + \alpha_0\alpha_{j-i-1}h^{2i-1} \dots h^{2j-2} + \dots +$$
$$+ \alpha_{j-i-1}\alpha_0 h^{2i-1} \dots h^{2j-2} + \alpha_{j-i}h^{2i-1} \dots h^{2j-1}d^{2j-2} =$$
$$= \alpha_{j-i}(d^{2i-3}h^{2i-2} \dots h^{2j-2} + h^{2i-1} \dots h^{2j-1}d^{2j-2} + h^{2i-1} \dots h^{2j-2}) = 0$$

and $(LR)_{ij} = 0$ for $i > j$. We also see that $(LR)_{jj} = \alpha_0(d^{2j-3}h^{2j-2} + h^{2j-1}d^{2j-2}) = id_{2j-2}$ so $LR = id$. □

4. An extension to $\mathcal{K}^+(\mathcal{A})$

In the categorification of colored link invariants, one has to additionally consider the homotopy category of semi-infinite complexes in an additive category, see [4; 6; 7] (or [8–10] for the abelian/derived case). We can extend Proposition 1 to the category $\mathcal{K}^+(\mathcal{A})$ of bounded below complexes in $\mathcal{A}$. If $A^\bullet$ is such a complex and is null-homotopic, the infinite stable limit as $k \to \infty$ of the matrices R and L gives an isomorphism

$$\coprod_{i=-\infty}^{\infty} A^{2i} \cong \coprod_{i=-\infty}^{\infty} A^{2i+1}$$

where $\coprod$ denotes the categorical coproduct (a similar result holds for the category of bounded above complexes). If the category $\mathcal{A}$ is such that we can define a notion of Euler characteristic, this can be used to show that null-homotopic complexes have zero Euler characteristic; for example, see Section 5.2 of [11] for complete details and Section 3.3.1 of [10] for the analogous result in the setting of the derived category of an abelian category.

References

1. Khovanov M., Mazorchuk V., Stroppel C. A brief review of abelian categorifications. *Theory Appl. Categ.,* 2009, vol. 22, no. 19, pp. 479–508.

2. Bar-Natan D. Khovanov's homology for tangles and cobordisms. *Geom. Topol.*, 2005, vol. 9, pp. 1443–1499.

3. Morrison S., Nieh A. On Khovanov's cobordism theory for $\mathfrak{su}$ knot homology. *J. Knot Theory Ramifications*, 2008, vol. 17, no. 9, pp. 1121–1173.

4. Rose D. E. V. A categorification of quantum $\mathfrak{sl}_3$ projectors and the $\mathfrak{sl}_3$ Reshetikhin–Turaev invariant of tangles. *Quantum Topol.,* 2014, vol. 5, no. 1, pp. 1–59.

5. Spanier E. H. *Algebraic topology*. New York, Springer-Verlag Publ., 1966.

6. Cooper B., Krushkal V. Categorification of the Jones–Wenzl projectors. *Quantum Topol.,* 2012, vol. 3, no. 2, pp. 139–180. (arXiv:1005.5117).

7. Rozansky L. *An infinite torus braid yields a categorified Jones–Wenzl projector.* Available at: http://arxiv.org/abs/1005.3266.

8. Frenkel I., Stroppel C., Sussan J. Categorifying fractional Euler characteristics, Jones–Wenzl projectors and 3*j*-symbols. *Quantum Topol.,* 2012, vol. 3, no. 2, pp. 181–253. (arXiv:1007.4680).

9. Achar P. N., Stroppel C. *Completions of Grothendieck groups.* Available at: http://arxiv.org/abs/1105.2715.

10. Cautis S. *Clasp technology to knot homology via the affine Grassmannian.* Available at: http://arxiv.org/abs/1207.2074.

11. Rose D. E. V. Categorification of quantum $\mathfrak{sl}_3$ projectors and the $\mathfrak{sl}_3$ Reshetikhin–Turaev invariant of framed tangles. *Ph.D. thesis*. New York, Duke University Publ., 2012.

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О ГРУППЕ ГРОТЕНДИКА АДДИТИВНЫХ КАТЕГОРИЙ

Д. Е. В. Роуз

Есть две абелевых группы, которые могут быть естественным образом ассоциированы с аддитивной категорией $\mathcal{A}$: расщепленная группа Гротендика категории $\mathcal{A}$ и триангулированная группа Гротендика гомотопической категории (ограниченных) комплексов в $\mathcal{A}$. Доказывается, что эти группы изоморфны. Попутно получается, что «Эйлерова характеристика» комплекса в $\mathcal{A}$ является инвариантом относительно гомотопической эквивалентности. Этот результат имеет значение для (де)категорификации.

Ключевые слова: *группа Гротендика, аддитивная категория, категорификация.*

Список литературы

1. Khovanov, M. A brief review of abelian categorifications / M. Khovanov, V. Mazorchuk, C. Stroppel // Theory Appl. Categ. — 2009. — Vol. 22, № 19. — P. 479–508.

2. Bar-Natan, D. Khovanov's homology for tangles and cobordisms / D. Bar-Natan // Geom. Topol. — 2005. — Vol. 9. — P. 1443–1499.

3. Morrison, S. On Khovanov's cobordism theory for $\mathfrak{su}$ knot homology / S. Morrison, A. Nieh // J. Knot Theory Ramifications. — 2008. — Vol. 17, № 9. — P. 1121–1173.

4. Rose, D. E. V. A categorification of quantum $\mathfrak{sl}_3$ projectors and the $\mathfrak{sl}_3$ Reshetikhin–Turaev invariant of tangles / D. E. V. Rose // Quantum Topol. — 2014. — Vol. 5, № 1. — P. 1–59.

5. Spanier, E. H. Algebraic topology / E. H. Spanier. — 2nd ed. — New York : Springer-Verlag, 1966.

6. Cooper, B. Categorification of the Jones–Wenzl projectors / B. Cooper, V. Krushkal // Quantum Topol. — 2012. — Vol. 3, № 2. — P. 139–180.

7. Rozansky, L. An infinite torus braid yields a categorified Jones–Wenzl projector [Электронный ресурс] / L. Rozansky. — URL : http://arxiv.org/abs/1005.3266.

8. Frenkel, I. Categorifying fractional Euler characteristics, Jones–Wenzl projectors and $3j$-symbols / I. Frenkel, C. Stroppel, J. Sussan // Quantum Topol. — 2012. — Vol. 3, № 2. — P. 181–253. (arXiv:1007.4680).

9. Achar, P. N. Completions of Grothendieck groups [Электронный ресурс] / P. N. Achar, C. Stroppel. — URL : http://arxiv.org/abs/1105.2715.

10. Cautis, S. Clasp technology to knot homology via the affine Grassmannian [Электронный ресурс] / S. Cautis. — URL : http://arxiv.org/abs/1207.2074.

11. Rose, D. E. V. Categorification of quantum $\mathfrak{sl}_3$ projectors and the $\mathfrak{sl}_3$ Reshetikhin–Turaev invariant of framed tangles / D. E. V. Rose // Ph.D. thesis. — USA : Duke University, 2012. (arXiv:1109.1745v2 [math.GT]).

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