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A non-commutative formula for the colored Jones function

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Abstract The colored Jones function of a knot is a sequence of Laurent polynomials that encodes the Jones polynomial of a knot and its parallels. It has been understood in terms of representations of quantum groups and Witten gave an intrinsic quantum field theory interpretation of the colored Jones function as the expectation value of Wilson loops of a 3-dimensional gauge theory, the Chern–Simons theory. We present the colored Jones function as an evaluation of the inverse of a non-commutative fermionic partition function. This result is in the form familiar in quantum field theory, namely the inverse of a generalized determinant. Our formula also reveals a direct relation between the Alexander polynomial and the colored Jones function of a knot and immediately implies the extensively studied Melvin–Morton–Rozansky conjecture, first proved by Bar–Natan and the first author about 10 years ago. Our results complement recent work of Huynh and Le, who also give a non-commutative formulae for the colored Jones function of a knot, starting from a non-commutative formula for the R matrix of the quantum group $U_q(\mathfrak{sl}_2)$; see Huynh and Le (in math.GT/0503296).

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1 Introduction

1.1 The Jones polynomial of a knot

In 1985, Jones discovered a celebrated invariant of knots, the Jones polynomial, [14]. Jones's original formulation of the Jones polynomial was given in terms of *representations of braid groups* and *Hecke algebras*, [14]. It soon became apparent that the Jones polynomial can be defined as a *state sum* of a *statistical mechanics* model that uses as input a planar projection of a knot, [15,29]. As soon as the Jones polynomial was discovered, it was compared with the better-understood Alexander polynomial of a knot. The latter can be defined using classical algebraic topology (such as the homology of the infinite cyclic cover of the knot complement), and its skein theory can be understood purely topologically. On the other hand, the Jones polynomial appears to be difficult to understand topologically, and there is a good reason for this, as was explained by Witten, [31]. Namely, the Jones polynomial can be thought of as the expectation value of Wilson loops of a 3-dimensional gauge theory, the Chern–Simons theory; in general, this is hard to understand. Witten's approach leads to a number of conjectures that relate limits of the Jones polynomial to geometric invariants of a knot, such as representations of the fundamental group of its complement into compact Lie groups. A recent approach to the Jones polynomial in terms of *D-modules* and *holonomic functions* seems to relate well to the hyperbolic geometry of knot complements, [8,10] and yet another approach to the Jones polynomial is via the *Kauffman bracket skein theory*, [16].

The goal of our paper is to present the colored Jones function as *an evaluation of the inverse of a non-commutative fermionic partition function*. This result is in the form very familiar in quantum field theory, namely the inverse of a generalized determinant. Hence there should be a quantum field theoretic derivation of it, which may teach us new things about how to compute path integrals in topological quantum field theory.

About 10 years ago, Melvin–Morton and Rozansky independently conjectured a relation among the limiting behavior of the colored Jones function of a knot and its Alexander polynomial (see Corollary 1.5), [25–27]. Bar-Natan and the first author reduced the conjecture about knot invariants to a statement about their *combinatorial weight systems*, and then proved it for all weight systems that come from *semisimple Lie algebras* using combinatorial Lie algebraic methods, [2]. A combinatorial description of the corresponding weight systems was obtained in [11]. Over the years, the MMR conjecture has received attention by many researchers who gave alternative proofs, [4,18,19,28,30].

A comparison of Theorem 1 and Theorem 2 reveals a direct relation between the Alexander polynomial and the colored Jones function. This should help us better understand the topological features of the colored Jones function.

We will introduce an auxiliary weighted directed graph, the arc-graph, that encodes transitions of walks along a planar projection of a knot. Our results are obtained by studying the non-negative integer flows on this arc-graph and applying the recently discovered q-MacMahon master theorem of [12].

1.2 Statement of the main result

Definition 1.1 *We consider* $5r$ *indeterminates* $r_i^-, r_i^+, u_i^-, u_i^+, z_i, 1 \leq i \leq r$. *Let* A *be a* r *by* r *matrix where each indeterminate appears at most once in an entry, and each entry is an indeterminate times a power of* q. *We assume* q *is an indeterminate which commutes with all other indeterminates. Moreover we assume that each column contains at most one* u *indeterminate, in its first or last entry different from* z. *Let* $L(A)$ *be the set of those columns of* A *where* u *appears in the last non-*z*-entry.*

We define a *noncommutative algebra* $\mathcal{A}(A)$ generated by the indeterminates which appear in A, modulo the commutation relations specified below. Consider any 2 by 2 minor of A consisting of rows i and i', and columns j and j' (where $1 \leq i < i' \leq r$, and $1 \leq j < j' \leq r$), writing $a = a_{ij}$, $b = a_{ij'}$, $c = a_{i'j}$, $d = a_{i'j'}$, we have the following commutation relations (we will use the symbol $=_q$ to denote 'equality up to a power of q'):

(1) The commutation in each row: $ba = q^{-2}ab$ if $b =_q u^-$ or $a =_q u^-$ and $ba = ab$ otherwise. The same rule is adapted for cd commutation.
(2) The bc commutation: $bc = q^{-1+s}cb$ if $c =_q u^s, b =_q r^{-+}$ or
$b =_q u^s, c =_q r^{-+}$ or
$c =_q u^s, b =_q u^{-+}, d =_q r^{-+}, a = z$ or
$b =_q u^s, c =_q u^{-+}, a =_q r^{-+}, d = z$.
$bc = q^{-1+s+s'}cb$ if $b =_q u^s, c =_q u^{s'}, a =_q r^{-+}, d =_q r^{-+}$, and
$bc = q^{-1}cb$ otherwise.
(3) Finally we require that A is right-quantum (see [12]), i.e.,

$$ca = qac, \qquad db = qbd, \qquad ad = da + q^{-1}cb - qbc.$$

Note that the commutation relations are such that each monomial in $\mathcal{A}(A)$ can be brought into a q-combination of *canonical monomials* $\prod_{i=1}^r a_{1i}^{m_{1i}} \cdots a_{ri}^{m_{ri}}$.

Definition 1.2 *We define* n-evaluation *of a canonical monomial* $\prod_{i=1}^r a_{1i}^{m_{1i}} \cdots a_{ri}^{m_{ri}}$ *to be zero if there is* ij *with* $m_{ij} > 0$ *and* $a_{ij} = z_k$ *and otherwise*

$$\mathrm{tr}_n \prod_{i=1}^r a_{1i}^{m_{1i}} \cdots a_{ri}^{m_{ri}} = \prod_{i \notin L(A)} \mathrm{tr}_n\, a_{1i}^{m_{1i}} \cdots a_{ri}^{m_{ri}} \prod_{i \in L(A)} \mathrm{tr}_n\, a_{ri}^{m_{ri}} \cdots a_{1i}^{m_{1i}},$$

and

$$\mathrm{tr}_n (u^{s_0})^{p_0} (r_{i_1}^{s_1})^{p_1} \cdots (r_{i_m}^{s_m})^{p_m} = q^{-s_0 p_0 n} \prod_{i=1}^m \prod_{j=0}^{p_i-1} (1 - t^{-s_i(n-j-p_0-\ldots p_{i-1})}).$$

We consider a generic planar projection $\mathcal{K}$ of an oriented zero framed knot with $r+1$ crossings and with no kinks, together with a special arc decorated with $\star$. Let K denote the corresponding *long knot* obtained by breaking the special arc. We will order the arcs of $\mathcal{K}$ so that they appear in increasing order as we walk in the direction of the knot, such that the special arc is last. We will also order the crossings of $\mathcal{K}$ such that arc a_i ends at the ith crossing, for $i = 1, \ldots, r+1$.

Note that $\mathcal{K}$ can be uniquely reconstructed from K, so that any invariant of knots gives rise to a corresponding invariant of long knots. We consider *transitions* of K: when we walk along K, we either go under a crossing (blue transition), or jump up at a crossing (red transition). Each transition from arc a_i to arc a_j is naturally equipped with a non-negative integer $\mathrm{rot}(a_i, a_j)$ which can be seen from K (see Definition B.6).

We define the r by r transition matrix $B_K = (b_{ij})$ as follows.

Definition 1.3

$$b_{ij} = \begin{cases} q^{-\mathrm{rot}(ij)} u_i^{\mathrm{sign}(i)} & \textit{if} \quad j = i+1 \\ q^{-\mathrm{rot}(ij)} r_i^{\mathrm{sign}(i)} & \textit{if} \quad a_i a_j \textit{is a red transition} \\ z_i & \textit{otherwise} \end{cases}$$

The next well-known theorem (see e.g. [2]) identifies the Alexander polynomial $\Delta(\mathcal{K})$ of a knot diagram $\mathcal{K}$ with the determinant of B_K.

Theorem 1 *For every knot diagram $\mathcal{K}$ we have:*

$$\Delta(\mathcal{K}, t) =_t \det(I - B_K)|_{q=1, z_i=0, u_i^{\mathrm{sign}(i)} = t^{-\mathrm{sign}(i)}, r_i^{\mathrm{sign}(i)} = (1 - t^{-\mathrm{sign}(i)})}.$$

Definition 1.4 *The quantum determinant of an r by r matrix $A = (a_{ij})$, introduced in [5], may be defined by*

$$\det_q(A) = \sum_{\pi \in S_r} (-q)^{-\mathrm{inv}(\pi)} a_{\pi(1)1} a_{\pi(2)2} \ldots a_{\pi(r)r},$$

where $\mathrm{inv}(\pi)$ *equals the number of pairs* $1 \leq i < j \leq r$ *for which* $\pi(i) > \pi(j)$. *Moreover we let*

$$\mathrm{Ferm}(A) = \sum_{J \subset \{1, \ldots, r\}} (-1)^{|J|} \det_q(A_J)$$

where A_J is the J by J submatrix of A.

If $q = 1$ then $\mathrm{Ferm}(A) = \det(I - A)$. Recall the MacMahon master theorem [24], known also as the *boson-fermion* correspondence

$$\frac{1}{\det(I - A)} = \sum_{n=0}^{\infty} \mathrm{tr}\, S^n(A),$$

where $S^n(A)$ is the n-th symmetric power of A.

The main result of this paper is as follows:

Theorem 2 *For every knot diagram $\mathcal{K}$ we can construct a matrix B'_K from B_K by a permutation of rows and columns so that*

$$J_n(\mathcal{K}, q) = q^{\delta(K,n)} 1/\mathrm{Ferm}(B'_K),$$

n-evaluated; $\delta(K, n)$ is an integer that can be computed easily from K (see Definition 3.3).

As an immediate consequence we obtain the seminal Melvin–Morton–Rozansky Conjecture (MMR in short), whose proof was first given by [2].

Corollary 1.5

$$\lim_{n\to\infty} J_n(\mathcal{K}, q^{1/n}) =_t \frac{1}{\Delta(\mathcal{K}, q)}.$$

Remark 1.6 The computational complexity of the Jones polynomial and its approximation is studied extensively and as far as we know, this cannot be said about non-commutative formulas. Hence, it may be enlightening to study our formula from a computation point of view.

Remark 1.7 Theorem 2 is inspired by work of Le, Zeilberger and the first author on a q-version of the *MacMahon Master Identity*, see [12]. Theorem 2 complements recent and elegant work of Le and Vu, who also give a noncommutative formulae for the colored Jones function of a knot, starting from a non-commutative formula for the R matrix of the quantum group $U_q(\mathfrak{sl}_2)$; see [13, Theorem. 1].

2 The zeta function of a graph and the quantum MacMahon Master theorem

One of main ingredients in our result is combinatorics of non-negative integer flows on digraphs. They appear in an expression of the zeta function.

Let us recall what is the zeta function of a digraph. We will consider digraphs (that is, graphs with oriented edges) with weights on their edges.

Let $G = (V, E)$ be a digraph with vertex set V and directed edges $E \subset V \times V$, and let $B = (\beta_e)_{e\in E}$ be a weight matrix for the edges of G. For edge e we denote by $s(e), t(e)$ the starting and terminal vertex of e. Bass–Ihara–Selberg defined a zeta function of a graph in analogy with number theory and dynamical systems, where the analogue of a prime number is a nonperiodic cycle. Let us define the latter.

A *pointed walk* on a digraph is a sequence $(e_1, \ldots, e_k)$ of edges such that the end of one coincides with the beginning of the next; we say that it is pointed at the beginning of e_1, which is also called a *base point*. A *pointed closed walk* is a path whose beginning and end vertex coincide. Two pointed closed walks are *equivalent* if they differ on the choice of base point only. By a *cycle* we will mean an equivalence class of pointed closed walks. A cycle c is *periodic* if $c = d^n$ for some closed walk d and some integer $n > 1$. Otherwise, it is called *nonperiodic*. Let $\mathcal{P}(G)$ denote the set of *nonperiodic cycles* of a digraph G. Using the weight function, we may define the weight $\beta(c)$ of a cycle c by $\beta(c) = \prod_{e \in c} \beta(e)$.

With the above preliminaries, Bass–Ihara–Selberg [3] define

Definition 2.1 *The* zeta function $\zeta(G, B)$ *of a weighted digraph* (G, B) *is defined by:*

$$\zeta(G, B) = \prod_{c \in \mathcal{P}(G)} \frac{1}{1 - \beta(c)}.$$

It follows by definition that

$$\zeta(G, B) = \sum_{c \text{ multisubsets of } \mathcal{P}(G)} \beta(c)$$

The actual definition of Bass–Ihara–Selberg uses more special weights for the edges (each edge is given the same weight), and is used to digraphs which are doubles (in the sense of replacing an unoriented edge by a pair of oppositely oriented edges) of undirected graphs.

Foata–Zeilberger proved that the zeta function is a rational function, and in fact given by the inverse of a determinant. Moreover, the zeta function is given by a sum over flows.

Definition 2.2 *A* flow f *on a digraph* G *is a function* $f : \mathrm{Edges}(G) \longrightarrow \mathbb{N}$ *of the edges of* G *that satisfies the (Kirkhoff)* conservation law

$$\sum_{e \text{ begins at } v} f(e) = \sum_{e \text{ ends at } v} f(e)$$

at all vertices v *of* G. *Let* $f(v)$ *denote this quantity and let* $\mathcal{F}(G)$ *denote the* set of flows *of a digraph* G.

If β is a weight function on the set of edges of G and f is a flow on G, then

- the *weight* $\beta(f)$ of f is given by $\beta(f) = \prod_e \beta(e)^{f(e)}$, where $\beta(e)$ is the weight of the edge e.
- The *multiplicity* at a vertex v with outgoing edges $e_1, e_2, \ldots$ is given by $\mathrm{mult}_v(f) = \binom{f(e_1)+f(e_2)+\ldots}{f(e_1), f(e_2), \ldots}$, and the *multiplicity* of f is given by $\mathrm{mult}(f) = \prod_v \mathrm{mult}_v(f)$.
- If A is a subset of edges then we let $f(A) = \sum_{e \in A} f(e)$.

Let us summarize Foata–Zeilberger's theorem [6, Theorem 1.1] here. For the sake of completeness we include its proof in Appendix 7.3.

Theorem 3 *If (G, B) is a weighted digraph, then*

$$\zeta(G,B) = \frac{1}{\det(I-B)} \tag{1}$$
$$= \sum_{f \in \mathcal{F}(G)} \beta(f)\,\mathrm{mult}(f). \tag{2}$$

Remark 2.3 For $r = 1$, the above Theorem states that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

where $x = b_{11}$. Thus, Theorem 3 is a version of the geometric series summation.

Another formula for the inverse of a determinant, the MacMahon master theorem, has been mentioned in the introduction. We will need its quantum version, proved in [12].

In r-dimensional *quantum algebra* we have r indeterminates x_i $(1 \le i \le r)$, satisfying the commutation relations $x_j x_i = q x_i x_j$ for all $1 \le i < j \le r$. Further we are given a right-quantum matrix A. We assume that the indeterminates of A commute with the x_i's. The following theorem has been proven recently in [12].

Theorem 4 *Let A be a right-quantum matrix of size r. For $1 \le i \le r$ let $X_i = \sum_{j=1}^{r} a_{ij} x_j$, and for any vector $(m_1, \ldots, m_r)$ of non-negative integers let $G_A(m_1, \ldots, m_r)$ be the coefficient of $x_1^{m_1} x_2^{m_2} \ldots x_r^{m_r}$ in $\prod_{i=1}^{r} X_i^{m_i}$. Then*

$$\sum_{m_1,\ldots,m_r=0}^{\infty} G_A(m_1, \ldots, m_r) = 1/\mathrm{Ferm}(A).$$

3 The arc-graph of a knot projection

Given a knot projection K, we define the arc-graph $G_{\mathcal{K}}$ as follows:

- The vertices of $G_{\mathcal{K}}$ are in 1-1 correspondence with the arcs of $\mathcal{K}$.
- The edges of $G_{\mathcal{K}}$ are in 1-1 correspondence with *transitions* of $\mathcal{K}$, when we walk along $\mathcal{K}$ and we either go under a crossing (blue edges), or jump up at a crossing (red edges) according to Fig. 1.

More formally, $G_{\mathcal{K}}$ is a weighted digraph defined as follows.

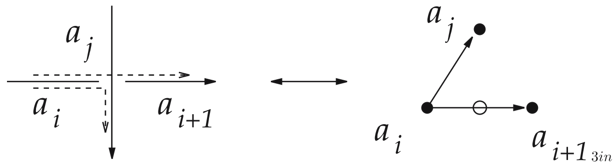


Fig. 1 From a planar projection to the arc-graph. Transitions in the planar projection are indicated by *dashed paths,* and the corresponding edges in the arc-graph are *blue* (depicted with a *small circle on them*) or *red*

Definition 3.1 *The* arc-graph $G_{\mathcal{K}}$ *has* $r+1$ *vertices* $1,\ldots,r+1$*,* $r+1$ blue directed edges $(v,v+1)$ *(v taken modulo* $r+1$*) and* $r+1$ red directed edges (u,v)*, where at the crossing u the arc that crosses over is labeled by* a_v*.*

The vertices of $G_{\mathcal{K}}$ are equipped with a sign, where $\mathrm{sign}(v)$ is the sign of the corresponding crossing v of $\mathcal{K}$, and the edges of $G_{\mathcal{K}}$ are equipped with a weight. The edge-weights are specified by matrix $W_{\mathcal{K}} = (\beta_{ij})$ where

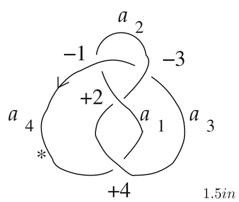
$$\beta_e = \begin{cases} t^{-\mathrm{sign}(v)} & \text{if} \quad e = (v, v+1), \\ 1 - t^{-\mathrm{sign}(v)} & \text{if} \quad e = (v, u). \end{cases}$$

Here t is a variable. Let W_K denote the matrix obtained from $W_{\mathcal{K}}$ by deleting the last row and column. Notice that $W_{\mathcal{K}}$ is formally stochastic (i.e., the sum of the rows of $I - W_{\mathcal{K}}$ is zero), but W_K is not.

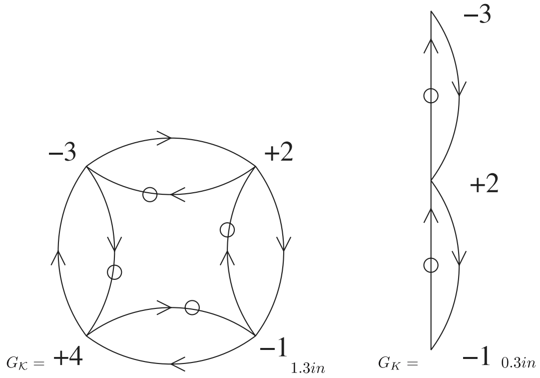
Let (G_K, W_K) denote the weighted digraph obtained by deleting the $r+1$ vertex from $G_{\mathcal{K}}$, together with all edges to and from it. We let V_K and E_K denote the set of vertices and edges of G_K.

It is clear from the definition that from every vertex of $G_{\mathcal{K}}$, the blue outdegree is 1, the red outdegree is 1, and the blue indegree is 1. It is also clear that $G_{\mathcal{K}}$ has a Hamiltonian cycle that consists of all the blue edges. We denote by e_i^b (e_i^r) the blue (red) edge *leaving vertex i* (Fig. 1).

Example 3.2 For the Fig. 8 knot we have:



Its arc-graph $G_{\mathcal{K}}$ with the ordering and signs of its vertices and $G_K = G_{\mathcal{K}} - \{4\}$ are given by



where the blue edges are the ones with circles on them. Moreover,

$$W_{\mathcal{K}} = \begin{bmatrix} 0 & t & 0 & 1-t \\ 1-\bar{t} & 0 & \bar{t} & 0 \\ 0 & 1-t & 0 & t \\ \bar{t} & 0 & 1-\bar{t} & 0 \end{bmatrix}.$$

Definition 3.3 *Let $\mathcal{K}$ be a knot projection. The* writhe *of $\mathcal{K}$, $\omega(\mathcal{K})$, is the sum of the signs of the crossings of $\mathcal{K}$, and* $\mathrm{rot}(K)$ *is the* rotation number *of K, defined as follows:* smoothen *all crossings of $\mathcal{K}$, and consider the oriented circles that appear; one of them is special, marked by $\star$. The number of circles different from the special one whose orientation agrees with the special one, minus the number of circles whose orientation is opposite to the special one is defined to be* $\mathrm{rot}(K)$. *We further let $\delta(K,n) = 1/2(n^2\omega(\mathcal{K}) + n\mathrm{rot}(K))$, and $\delta(K) = \delta(K,1)$.*

We remark that we define $\mathrm{rot}(e)$ for each edge e of $G_{\mathcal{K}}$ in Definition B.6.

4 The enhanced arc-graph and the Jones polynomial

In order to express the Jones polynomial as a function of the arc graph, we need to enhance the arc-graph as follows.

Definition 4.1 (a) *We introduce a* linear order $<_v$ *on the set of edges of G_K terminating at vertex v as follows. Recall that v corresponds to an arc a_v of K. If we travel on a_v along the orientation of K, we 'see' one by one the arcs corresponding to starting vertices of red arcs entering v: this gives the linear order of red arcs entering v. Finally there is at most one blue edge entering vertex v, and we make it less than all the red edges entering v.*

(b) *If f is a flow on G_K, we define the* rotation *and* excess *number of f by:*

$$\mathrm{rot}(f) = \sum_{e \in E_K} f(e)\mathrm{rot}(e), \quad \mathrm{exc}(f) = \sum_{v \in V_K} \mathrm{sign}(v) f(e_v^b) \sum_{e <_v e_v^r} f(e),$$
$$\delta(f) = \mathrm{exc}(f) - \mathrm{rot}(f), \tag{3}$$

where V_K and E_K are the set of vertices and edges of G_K, and rot(e) *is defined in Definition B.6.*

Let $\mathcal{S}(G)$ denote the set of all subgraphs C of G such that each component of C is a directed cycle. Note that $\mathcal{S}(G)$ may be identified with a finite subset of $\mathcal{F}(G)$ since the characteristic function of C is a flow.

The next theorem, due to Lin–Wang, expresses the Jones polynomial of a knot projection $\mathcal{K}$ in terms of the enhanced arc-graph of K. For the sake of completeness we include its proof in Appendix A.1.

Theorem 5 [21] *For every knot projection $\mathcal{K}$ we have:*

$$J(\mathcal{K}, t) = t^{\delta(K)} \sum_{c \in \mathcal{S}(G_K)} t^{\delta(c)} \beta(c).$$

We now give a similar formula for the *colored Jones function* J_n of a knot. We will normalize the colored Jones function so that it is the constant sequence $\{1\}$ for the unknot, and J_n is the quantum group invariant of knots that corresponds to the $(n+1)$-dimensional irreducible representation of $\mathfrak{sl}_2$.

Recall the operation of *cabling* $\mathcal{K}^{(n)}$ the knot projection $\mathcal{K}$ n times. Recall that $a_1, \ldots, a_{r+1}$ are the arcs of $\mathcal{K}$. Each a_l is in the cabling replaced by n^2 arcs $a^l_{i,j}$, $i, j = 1, \ldots, n$, with the agreement that the 'long arcs' obtained by cabling arc a_k will be a^k_{1j}, $j = 1, \ldots, n$ and the 'small arcs' obtained by cabling of crossing k will be denoted by a^k_{ij} for $i = 2, \ldots, n$ and $j = 1, \ldots, n$. Note that all crossings which replace the original crossing k have the same sign, equal to the sign of the crossing k. (see figure before Lemma 6.2).

We further let $K^{(n)}$ denote the link obtained from $\mathcal{K}^{(n)}$ by deletion of the n special long arcs a^{r+1}_{1j}, $j = 1, \ldots, n$.

Theorem 6 *For every knot $\mathcal{K}$ and every $n \in \mathbb{N}$, we have*

$$J_n(\mathcal{K}, t) = t^{\delta(K,n)} \sum_{c \in \mathcal{S}(G_{K^{(n)}})} t^{\delta(c)} \beta(c)$$

where $G_{K^{(n)}}$ is the arc graph of $K^{(n)}$.

Proof Let V_n denote the $(n+1)$-dimensional irreducible representation of the quantum group $U_q(\mathfrak{sl}_2)$, and let v_n denote a highest weight vector of V_n. Then, there is an inclusion $V_n \longrightarrow \otimes^n V_1$ that maps v_n to a nonzero multiple of $\otimes^n v_1$.

The result follows since cabling $\mathcal{K}$ corresponds to tensor product of representations and since $\omega(\mathcal{K}^{(n)}) = n^2\omega(\mathcal{K})$ and $\mathrm{rot}(K^{(n)}) = n\,\mathrm{rot}(K)$. □

For an integer m, we denote by

$$(m)_q = \frac{q^m - 1}{q - 1}$$

the *quantum integer m*. This defines the *quantum factorial* and the *quantum binomial coefficients* by

$$(m)_q! = (1)_q(2)_q \dots (m)_q, \qquad \binom{m}{n}_q = \frac{(m)_q!}{(n)_q!(m-n)_q!}$$

for natural numbers m, n with $n \leq m$. We also define

$$\mathrm{mult}_q(f) = \prod_v \binom{f(v)}{f(e_v^b)}_{q^{\mathrm{sign}(v)}}.$$

Theorem 7 *For every knot projection $\mathcal{K}$ we have:*

$$J_n(\mathcal{K}, t) = t^{\delta(K,n)} \sum_{f \in \mathcal{F}(G_K)} \mathrm{mult}_t(f) t^{\delta(f)} \prod_{v \in V_K} t^{-\mathrm{sign}(v) n f(e_v^b)} \prod_{ered; t(e)=v}$$
$$\times \prod_{j=0}^{f(e)-1} (1 - t^{-\mathrm{sign}(s(e))(n-j-\sum_{e' <_v e} f(e))}).$$

Remark 4.2 It simply follows that the contribution of a flow f to the sum in Theorem 7 is non-zero only if $f(v) \leq n$ for each vertex v. Thus, in the above sum, only finitely many terms contribute. As a result, when $n = 1$, Theorems 7 and 5 coincide.

5 Proof of Theorem 2

Theorem 7 is used in this section to prove the main Theorem 2. In the rest of the paper we then prove Theorem 7 from Theorem 6.

5.1 Row and column arcs order

Recall that we fix a generic planar projection $\mathcal{K}$ of an oriented knot with $r + 1$ crossings. We order the arcs of $\mathcal{K}$ so that they appear in increasing order as we walk in the direction of the knot, and we denote by K the *long knot* obtained by breaking the arc a_{r+1}. We also order the crossings of K so that arc a_i ends at the ith crossing, for $i = 1, \dots, r$.

Definition 5.1

(1) *We define two permutations S, T on the set of the arcs of K as follows. For arc a_i of K let $T(i) = T(i,1), \ldots, T(i,k_i)$ ($S(i) = S(i,1), \ldots, S(i,k_i)$ respectively) be the block of arcs of K terminating at (starting from) a_i and ordered along the orientation of a_i. Let T (S) be the permutation of the arcs of K defined by $T = T(1,1), \ldots, T(1,k_1), \ldots, T(r,k_r)$ ($S = S(r,k_r), \ldots, S(r,1), \ldots, S(1,1)$).*
(2) *We define permutation R of the arcs of K from T as follows: if a_i appears in S before the block $S(i)$ then replace $T(i,1), \ldots, T(i,k_i)$ by $T(i,k_i), \ldots, T(i,1)$.*
(3) *Similarly we define permutation C of the arcs of K from S as follows: if a_i appears in S after the block $S(i)$ then replace $S(i,k_i), \ldots, S(i,1)$ by $S(i,1), \ldots, S(i,k_i)$.*

Definition 5.2 *We define matrix $B'_K = (\gamma_{ij})$ to be obtained from B_K by taking the rows in the R order and the columns in the C order.*

We consider the commutation relations between the variables appearing in B'_K as in the Definition 1.1. In particular, B'_K is right-quantum.

5.2 Flows on G_K and monomials of $G_{B'_K}(m_1, \ldots m_r)$

We interpret each entry γ_{ij} with no z indeterminate as arc (ij) of the arc-graph G_K. Then each monomial in $G_{B'_K}(m_1, \ldots, m_r)$ corresponds to a flow on G_K with $indeg(i) = \text{outdeg}(i) = m_i, i = 1, \ldots, r$. If f is such a flow, we denote by $G(f)$ the sum of all monomials of $\sum G_{B'_K}(m_1, \ldots, m_r)$ corresponding to f. Summarizing we can write

Observation 1

$$\sum_{m_1,\ldots,m_r=0}^{\infty} G_{B'_K}(m_1, \ldots, m_r) = \sum_f G(f).$$

We denote by $C(f)$ the canonical monomial of a product (in arbitrary order) of the entries of B'_K corresponding to the edges of G_K, where the entry corresponding to each edge e appears $f(e)$ times.

Observation 2 Let C be a summand of $\prod_{i=1}^r (\sum_j \gamma_{ij} x_j)^{m_i}$, which contains m_i indeterminates $x_i, i = 1, \ldots, r$ and contributes to $G(f)$. For $1 \le v \le r$ and $1 \le j \le f(e_v^r)$ let $c(C, v, j)$ be the number of $\gamma_{e_v^b}$'s which need to be commuted through the j-th occurrence of $\gamma_{e_v^r}$ in order to get $C(f)$ from C. Then

$$C = x_1^{m_1} x_2^{m_2} \ldots x_r^{m_r} q^{-\text{rot}(f)} C(f) q^{\text{exc}(f)} \prod_{v=1}^{r} \prod_{j=1}^{f(e_v^r)} q^{\text{sign}(v) c(C,v,j)}.$$

Proof Let $X_{ij} = \gamma_{ij}x_j$. Hence C is a summand of the coefficient of $x_1^{m_1}x_2^{m_2}\dots x_r^{m_r}$ in $\prod_{i=1}^r(\sum_j X_{ij})^{m_i}$. For each j fixed the γ_{ij}'s appear ordered in C. In each canonical monomial, the γ_{ij}'s appear ordered by the second coordinate, and then by the first coordinate. Hence, in order to get a canonical monomial times $x_1^{m_1}x_2^{m_2}\dots x_r^{m_r}$ from a summand of $\prod_{i=1}^r(\sum_j X_{ij})^{m_i}$, we only need to commute X_{ij}'s so that they are ordered by the second coordinate. This means: if a_i appears in S *before* (*after* respectively) the block $S(i)$ then a_i appears in C *before* (*after* respectively)the block $S(i)$ BUT a_{i-1} appears in R *after* (*before* respectively) the block $T(i)$. Hence we need to commute

1. Each $X_{i-1,i}$ through each $X_{j-1,j}, j \in S(i)$ and each $X_{k-1,i}$ through each $X_{j-1,j}, j, k \in S(i), R(k) > R(j)$. The commutation in B'_K is such that we acquire each time $q^{\mathrm{sign}(j-1)}$. Hence we acquire in total $q^{\mathrm{exc}(f)}$ since we recall that $\mathrm{exc}(f) = \sum_v \mathrm{sign}(v) f(e_v^b) \sum_{e<e_v^r} f(e)$.
2. Each $X_{k-1,i}$ through each $X_{k-1,k}, k \in S(i)$. The commutation in B'_K is such that we acquire in total $q^{\mathrm{sign}(k-1)(c(C,k-1,1)+\dots c(C,k-1,f(e_{k-1}^r))}$.
3. The commutation in B'_K is such that if $i < i', j < j'$ and $X_{i,j'}, X_{i',j}$ do not appear in one of the previous two cases then $X_{i,j'}, X_{i',j} = X_{i',j}, X_{i,j'}$.

This finishes the proof. □

Corollary 5.3

$$G(f) = C(f)q^{\delta(f)}\prod_{v=1}^{r}\sum_{f(e_v^b)\ge c_1\ge\dots\ge c_{f(e_v^r)}\ge 0} q^{\mathrm{sign}(v)(c_1+\dots c_{f(e_v^r)})}.$$

Since

$$\sum_{f(e_v^b)\ge c_1\ge\dots\ge c_{f(e_v^r)}\ge 0} q^{\mathrm{sign}(v)(c_1+\dots c_{f(e_v^r)})} = \binom{f(v)}{f(e_v^b)}_{q^{\mathrm{sign}(v)}},$$

we have

Corollary 5.4

$$G(f) = q^{\delta(f)}\mathrm{mult}_q(f)C(f).$$

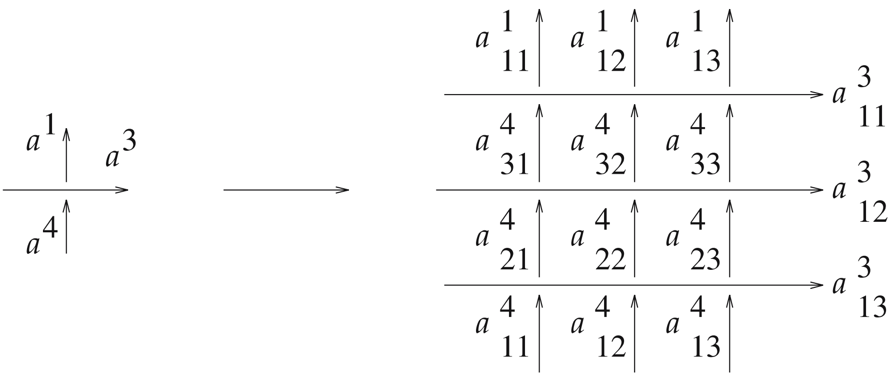
Proof (of Theorem 2) Theorem 7 tells us that

$$J_n(K,q) = q^{\delta(K,n)}\sum_{f\in\mathcal{F}(G_K)}\mathrm{mult}_t(f)t^{\delta(f)}\prod_{v\in V_K} t^{-\mathrm{sign}(v)nf(e_v^b)}\prod_{e\,\mathrm{red};t(e)=v}$$
$$\times\prod_{j=0}^{f(e)-1}(1-t^{-\mathrm{sign}(s(e))(n-(\sum_{e'<_v e}f(e))-j)}).$$

Comparing this with the Definition 1.2 of the n-evaluation and using Theorem 4, Observation 2 and Corollary 5.4, we can see that Theorem 2 follows. □

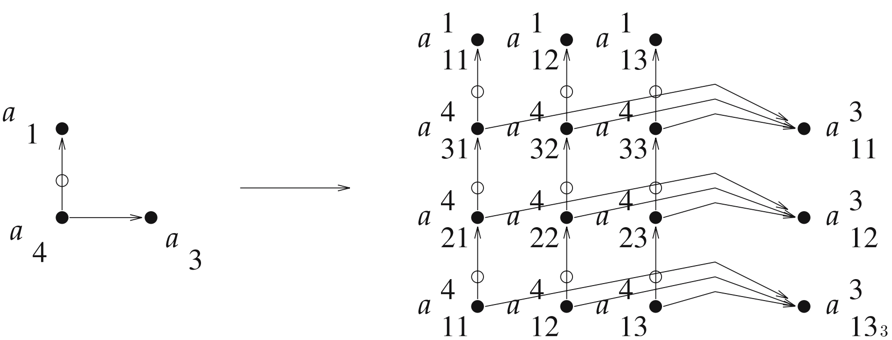
6 Cabling of the arc graph

Recall the operation of *cabling* $\mathcal{K}^{(n)}$ the knot projection $\mathcal{K}$ n times. Recall that $a_1, \ldots, a_{r+1}$ are the arcs of $\mathcal{K}$. Each a_l is in the cabling replaced by n^2 arcs $a^l_{i,j}$, $i,j = 1,\ldots,n$, with the agreement that the 'long arcs' obtained by cabling arc a_k will be a^k_{1j}, $j = 1,\ldots,n$ and the 'small arcs' obtained by cabling of crossing k will be denoted by a^k_{ij} for $i = 2,\ldots,n$ and $j = 1,\ldots,n$. Note that all crossings which replace the original crossing k have the same sign, equal to the sign of the crossing k. We make the following agreement: assume the parallel arcs $a^l_{i,1},\ldots,a^l_{i,n}$ go horizontally from left to right. Then $a^l_{i,1}$ is the upmost one. See figure below for part of Example 3.2 and of its 3-cabled version $\mathcal{K}^{(3)}$:



We further let $K^{(n)}$ denote the link obtained from $\mathcal{K}^{(n)}$ by deletion of the n special long arcs $a^{r+1}_{1j}, j = 1,\ldots,n$.

Next we consider the arc-graph of the cabling of $\mathcal{K}$. For example, part of the red-blue digraph $G_{\mathcal{K}}$ of Example 3.2 and of its 3-cabled version is depicted as follows:



where vertical edges are blue and horizontal edges are red.

We now define an n-cabling $G_K^{(n)}$ of the arc-graph G_K. Cabling of a planar projection is a local operation, and so is cabling of a digraph. In the language of combinatorics, we blow up the vertices of G using a suitable *gadget*. For a similar discussion, see also [11, Sect. 4].

Definition 6.1 *Fix a red–blue arc-graph G_K. Let $G_K^{(n)}$ denote the digraph with vertices a_j^v for v a vertex of G_K and $j = 1, \dots, n$. $G_K^{(n)}$ contains blue directed edges (a_j^l, a_j^{l+1}) with weight $t^{-\epsilon n}$ (where $\epsilon \in \{-1, +1\}$ is the sign of the crossing l) for each $l = 1, \dots, r-1$ and $j = 1, \dots, n$. Moreover, if (a_k, a_l) is a red directed edge of G_K, then $G_K^{(n)}$ contains red edges (a_i^k, a_j^l) for all $i, j = 1, \dots n$ with weight $t^{(j-1)}(1-t)$ resp. $t^{-(n-j)}(1-t^{-1})$, if the sign of the i crossing is -1 resp. $+1$. Notice that the weights of the red edges are independent of the index i.*

Lemma 6.2 *There is a 1–1 correspondence*

$$\{\textit{admissible subgraphs of } G_{K^{(n)}}\} \longleftrightarrow \{\textit{admissible subgraphs of } G_K^{(n)}\}.$$

We will denote the set of admissible even subgraphs of $G_K^{(n)}$ by $\mathcal{S}_n(G_K)$.

Proof Denote by p_j^k path $(a_{1j}^k, a_{2j}^k, \dots, a_{nj}^k)$ of $n-1$ blue edges in $G_{K^{(n)}}$, $k = 1, \dots, r$ and $j = 1, \dots, n$. There is a natural map $G_{K^{(n)}} \longrightarrow G_K^{(n)}$ which contracts each directed path p_j^k into its initial vertex, and deletes all vertices a_{ij}^{r+1}. Forgetting the weights, it is clear that the result of the contraction coincides with $G_K^{(n)}$.

$G_{K^{(n)}}$ has two types of vertices: a_{ij}^k for $k = 1, \dots, r+1$ and $i, j = 1, \dots, n$ and $i \neq 1$ (call these white) and a_{1j}^k, $k < r+1$ (call these black). The indegree of a white vertex is 1, but the indegree of a black vertex may be higher. The black vertices are the initial vertices of the paths p_j^k, hence the vertices of $G_K^{(n)}$. Let $E^{\{n\}}$ and $E^{(n)}$ denote the set of edges of $G_{K^{(n)}}$ and $G_K^{(n)}$ respectively. Then each edge e of $E^{(n)}$ replaces the unique directed path P_e of $G_{K^{(n)}}$ between the corresponding black end-vertices of e, which contains no other black vertices. If $E \subset E^{\{n\}}$ is an admissible even subgraph of $G_{K^{(n)}}$ then E is a vertex-disjoint union of directed cycles of $G_{K^{(n)}}$ and each directed cycle may be decomposed into directed paths between the black vertices. If each such directed path is replaced by a directed edge, we get an admissible even subgraph E' of $G_K^{(n)}$. This gives the 1–1 correspondence between the admissible even subgraphs without the weights. To realize that the weights are correct as well, we only need to compare the product of the weights in $G_{K^{(n)}}$ along P_e with the weights of e in $G_K^{(n)}$. □

Theorem 6 and Lemmas B.2, 6.2 imply that:

Lemma 6.3 *For every knot $\mathcal{K}$ and every $n \in \mathbb{N}$, we have*

$$J_n(\mathcal{K}, t) = t^{n\delta(K)} \sum_{c \in \mathcal{S}_n(G_K)} t^{\delta(c)} \beta(c).$$

Our next task is to figure out $\delta(c) = \mathrm{exc}(c) - \mathrm{rot}(c)$ for $c \in \mathcal{S}_n(G_K)$.
The following lemma is clear from the Definition 4.1:

Lemma 6.4 *If f is a flow on G_K and $\widetilde{f}$ is a lift of f to flow on $G_K^{(n)}$, for some n, then* $\mathrm{rot}(f) = \mathrm{rot}(\widetilde{f})$.

6.1 Comparison of excess numbers

Given an admissible subgraph c in $G_K^{(n)}$, let f be the corresponding flow in G_K, to which c projects, under the projection

$$\pi : G_K^{(n)} \to G_K.$$

In this section, we compare exc(f) (in Definition 4.1) with exc(c).

As we will see, the two excess numbers do not agree. In this section we will determine their difference.

We begin by introducing a partial ordering $\prec$ on the set of edges of $G_K^{(n)}$. We warn the reader that this ordering is different from the ordering $<_v$ of the edges of G_K entering vertex v, introduced in Definition 4.1.

Definition 6.5 *Consider two edges e and e' of $G_K^{(n)}$ which start at the vertices a_j^i and $a_{j'}^{i'}$ of $G_K^{(n)}$. We say that $e \prec e'$ if*

- *e, e' end at the same vertex v and $\pi(e) <_v \pi(e')$ in G_K, or*
- *$i = i'$ and* $\mathrm{sign}(i) = +$ *and $j < j'$, or*
- *$i = i'$ and* $\mathrm{sign}(i) = -$ *and $j' < j$*

Recall that $c \in \mathcal{S}_n(G_K)$ (c admissible) if and only if c is a collection of vertex disjoint directed cycles of $G_K^{(n)}$. Hence the ordering on the edges of c defined in Definition 6.5 induces a total ordering on each $\pi^{-1}(e)$, e edge of G_K.

This total ordering may be seen from the cabling of the knot in the same way as the ordering $<_v$ of Definition 4.1 may be seen from the knot: if we travel along an arc a_j^i, we see one by one the arcs corresponding to the starting vertices of edges of $\pi^{-1}(e)$, where e is an edge of G_K. This agrees with the total ordering on $\pi^{-1}(e)$ induced by $\prec$; see figure before 6.2.

Definition 6.6 *Consider two edges e and e' of $G_K^{(n)}$ which end at the vertices a_j^i and $a_{j'}^{i'}$ of $G_K^{(n)}$.*

$$X(e,e') = \begin{cases} 1 & \textit{if} \quad e,e' = \textit{red}, \quad e' \prec e, \quad \mathrm{sign}(s(e)) = +, \quad j < j' \\ 1 & \textit{if} \quad e = \textit{red},\ e' = \textit{blue}, \\ & \pi(e), \pi(e') \textit{ do not start at the same vertex}, \\ & e' \prec e, \quad \mathrm{sign}(s(e)) = +, \quad j < j' \\ 1 & \textit{if} \quad e,e' = \textit{red}, \quad e' \prec e, \quad \mathrm{sign}(s(e)) = -, \quad j' < j \\ 1 & \textit{if} \quad e = \textit{red},\ e' = \textit{blue}, \\ & \pi(e), \pi(e') \textit{ do not start at the same vertex}, \\ & e' \prec e, \quad \mathrm{sign}(s(e)) = -, \quad j' < j \\ 0 & \textit{otherwise} \end{cases}$$

$$Y(e,e') = \begin{cases} 1 & \textit{if} \quad e = \textit{red},\ e' = \textit{blue}, \\ & \pi(e), \pi(e') \textit{ start at the same vertex}, \quad e \prec e' \\ 0 & \textit{otherwise} \end{cases}$$

Lemma 6.7 *Let c be an admissible subgraph of $G_K^{(n)}$. Denote by f the flow on G_K which is the projection of c to G_K. Then*

$$\mathrm{exc}(c) = \mathrm{exc}(f) + \sum_e \mathrm{sign}(s(e)) \left(\sum_{e'} X(e,e') + Y(e,e') \right)$$

where the summations of e and e' are over the set of edges of c and $s(e)$ denotes the starting vertex of e.

Proof Consider a crossing v of K, and the corresponding n^2 crossings of K^n. We count the contribution to $\mathrm{exc}(c)$ of pairs (e,e') of edges of c such that

- e projects to e_v^b (the blue edge that starts at v), and e' does not project to e_v^r (the red edge that starts at v). This gives $\mathrm{exc}(f)$.
- e projects to e_v^r, and e' does not project to e_v^b. This gives the X-term in the formula.
- e projects to e_v^r, and e' projects to e_v^b. This gives the Y-term in the formula.

□

7 Sortings and multiplicities of flows

7.1 Sortings

In this section we introduce one of our key tools, which is a categorification of multiplicities of the flows on G_K. Let f be a flow on G_K. If e is an edge of G_K

then we let $F(e) \subset F$ be the set of $f(e)$ copies of e; we choose an *arbitrary* total order on each $F(e)$.

Let $F = \cup_{e \in E_K} F(e)$ and let $F_r \subset F$ consists of the union of $F(e)$, e red. Further let F_r^+ denote the subset of F_r consisting of the red edges which leave a vertex with $+$ sign, and we let $f_r^+ = |F_r^+|$. Analogously we define $F_r^-, \dots$.

Definition 7.1 *Fix a flow f on G_K. A* sorting *C of f is a function*

$$C : Vertices(G_K) \to 2^{F_r}$$

such that

- *C_1 is a collection of red edges that terminate in vertex 1, of cardinality $f(e_1^b)$.*
- *For each $2 \le i \le r$, $C_i \subseteq C_{i-1} \cup \{e \in F_r; e$ terminates in vertex $i\}$ of $f(e_i^b)$ elements.*

Let $\mathcal{C}(f)$ denote the set of all sortings of f.

Lemma 7.2 *Every flow f has* mult(f) *sortings.*

Proof Use that $\text{mult}(f, r) = 1$, $\{e \in F_r; e$ terminates in vertex $1\} = \{e \in F; e$ terminates in vertex $1\}$ and for each $2 \le i < r$, $\sum f(e) : e$ terminates at vertex i equals $|C_{i-1} \cup \{e \in F_r; e$ terminates in vertex $i\}|$. □

Definition 7.3 *We define $\mathcal{I}(f, n) = \{0, \dots, n-1\}^{F_r}$. If $v \in \mathcal{I}(f, n)$ then we define $f_r^-(v) = \sum_{e \in F_r^-} v_e$ and we define $f_r^+(v)$ analogously.*

Definition 7.4 (a) *Fix a flow f on G_K and a natural number n. An* n-sorting of *f is a pair $P = (C, v)$ where $C \in \mathcal{C}(f)$ and $v \in \mathcal{I}(f, n)$.*
(b) *If $P = (C, v)$ is an n-sorting of f then we define its weight $b(P)$ to be*

$$b(P) = t^{n(f_w^- - f_w^+)}(1-t)^{f_r^-} t^{f_r^-(v)} (1-t^{-1})^{f_r^+} t^{-(f_r^+)(n-1) + f_r^+(v)}.$$

(c) *Let $\mathcal{C}_n(f)$ denote the* set of all n-sortings of f.

The following lemma states that the n-sortings of f categorify multiplicities and weights of flows.

Lemma 7.5 *For every flow f on G_K and every n we have*

$$\beta(f)|_{t \to t^n} \text{mult}(f) = \sum_{P \in \mathcal{C}_n(f)} b(P).$$

Proof It follows by Lemma 7.2 that

$$\sum_{P \in \mathcal{C}(f,n)} b(P) = \text{mult}(f) t^{n(f_b^- - f_b^+)}(1-t)^{f_r^-}(1-t^{-1})^{f_r^+} \times \left(\sum_{v \in \mathcal{I}(f,n)} t^{f_r^-(v)} t^{-(f_r^+)(n-1) + f_r^+(v)} \right)$$

by a simple rearrangement

$$\mathrm{mult}(f)t^{n(f_b^- - f_b^+)}\left((1-t)^{f_r^-}\sum_{v\in\mathcal{I}(f,n)} t^{f_r^-(v)}\right)\left((1-t^{-1})^{f_r^+}\sum_{v\in\mathcal{I}(f,n)} t^{-(f_r^+)(n-1)+f_r^+(v)}\right)$$

by a *geometric series summation*

$$\beta(f)|_{t\to t^n}\mathrm{mult}(f).$$

□

7.2 Sortings and the Jones polynomial

Here we define admissible sortings and give a formula for the colored Jones function in terms of them.

Definition 7.6 *Fix a flow f of G_K and a natural number n. Let $P=(C,v)$ be an n-sorting of f. We say that P is* admissible *if*

- *For every two edges $e,e'\in F_r$ such that $v_e=v_{e'}$ and e ends in vertex i and e' ends in vertex j and $j\ge i$, there exists an l, $i\le l<j$ such that $e\notin C_l$.*

We denote by $\mathcal{AC}_n(f)$ the set of all admissible n-sortings of f, and by $\mathcal{S}_n(G_K,f)$ the set of all lifts of f to admissible subgraphs of $G_K^{(n)}$.

The next lemma explains the notion of admissible sortings.

Lemma 7.7 *There is a bijection Φ from $\mathcal{AC}_n(f)$ to $\mathcal{S}_n(G_K,f)$ such that $\beta(\Phi(P))=b(P)$. Moreover, if $P\in\mathcal{AC}_n(f)$ and π is the projection to G_K then for each $e\in E_K$, the fixed total order on $F(e)$ agrees with the total order $(\pi^{-1}(e),\prec)$ introduced in Definition 6.5.*

Proof Let $P=(C,v)$, $C=(C_1,\ldots,C_r)$, and $P\in\mathcal{AC}_n(f)$. In order to define $\Phi(P)$ we will define the image $\Phi(P,e)$ for each $e\in F$.

First we determine the ends of the lifts of the red edges as follows: if $e\in F_r$ then we let $t(\Phi(P,e))=a^{t(e)}_{v(e)+1}$.

Next we determine the blue edges of $\Phi(P)$ as follows: if $1\le i\le r$ then we let

$$\{s(\Phi(P,e))\mid e\in F(e_i^b)\}=\{a^i_{v_e+1}\mid e\in C_i\}.$$

This determines the beginnings of the blue edges, and hence also the ends of the blue edges.

It remains to specify the beginnings of the lifts of the red edges. Since P is admissible, observe that for each $1\le i\le r$, there are exactly $f(e_i^r)$ vertices a^i_j of indegree 1 in current $\Phi(P)$. Hence it remains to make each of them starting vertex of exactly one edge $\Phi(P,e)$, $e\in F(e_r^i)$. This is uniquely determined by the 'moreover' part of the Lemma. This finishes the definition of Φ. The equality for the weights follows easily, and the moreover part of the Lemma directly from the definition of Φ. To finish the proof we find the inverse to Φ.

Let $c \in \mathcal{S}_n(G_K, f)$. We construct $\Phi^{-1}(c) = (C(c), v(c))$ as follows: Let e be an edge of G_K. There is an order preserving bijection between the fixed total ordering $(F(e), <)$ and $(\pi^{-1}(e), \prec)$. If e' is an edge of c then we let e'_F be the corresponding edge of F.

First let e' be a red edge of c. We let $v(c)_{e'_F} = j$ where $t(e') = a^i_{j+1}$. Hence $v(c)$ encodes the ends of the red edges of c.

Next we define a predecessor $p(e)$ for each edge e of c. If e red then $p(e) = e$. If e blue then $p(e)$ is the red edge of c which terminates in the starting vertex of the longest blue path of c whose last edge is e. Note that $p(e)$ always exists and is unique since c is admissible.

Finally for $1 \leq i \leq r$ let $C_i = \{p(e)_F; e$ edge of c that terminates in some a^i_j that is a starting vertex of a blue edge of $c\}$. This finishes the construction of Φ^{-1}. □

Theorem 8 *We have:*

$$J_n(K)(t) = t^{\delta(K,n)} \sum_{f \in \mathcal{F}(G_K)} t^{\delta(f)} \sum_{P \in \mathcal{AC}_n(f)} t^{\mathrm{exc}(P)} b(P),$$

where $\mathrm{exc}(P) = \mathrm{exc}(\Phi(P)) - \mathrm{exc}(f)$.

Proof We have:

$$\begin{aligned} J_n(K)(t) &= t^{\delta(K,n)} \sum_{c \in \mathcal{S}_n(G_K)} t^{\delta(c)} \beta(c) \\ &= t^{\delta(K,n)} \sum_{f \in \mathcal{F}(G_K)} t^{\delta(f)} \sum_{c \in \mathcal{S}_n(G_K,f)} t^{\mathrm{exc}(c) - \mathrm{exc}(f)} \beta(c) \\ &= t^{\delta(K,n)} \sum_{f \in \mathcal{F}(G_K)} t^{\delta(f)} \sum_{P \in \mathcal{AC}_n(f)} t^{\mathrm{exc}(P)} b(P) \end{aligned}$$

□

7.3 Proof of Theorem 7

Definition 7.8 *Let $e \in F_r$. We define set $P(f, e)$ as follows: if $e \in F(e'), e_1 \in F(e'_1)$ then $e_1 \in P(f, e)$ if $t(e') = t(e'_1) = v$ and $e'_1 <_v e'$, or $e' = e'_1$ and $e_1 < e$ in our fixed total order of $F(e)$.*

Definition 7.9 *Let $e \in F_r$. We define*

- $\mathrm{def}_1(C, v, e) = |\{e' \in P(f, e) : v_{e'} < v_e\}|$,
- $\mathrm{def}_2(C, v, e) = |\{e' \in C_{d(e)} : v_{e'} < v_e\}|$, *where $d(e)$ is the biggest index such that $d(e) \geq t(e)$ and $e \notin C_{d(e)}$.*

Recall that $t(e)$ denotes the terminal vertex of e.

Proposition 7.10 *Let $P = (C, v)$ be an n-sorting of f. Then*

$$\mathrm{exc}(P) = \sum_{e \in F_r} \delta_1(e) + \delta_2(e),$$

where

$$\delta_1(e) = \begin{cases} |P(f,e)| - \mathrm{def}_1(C,v,e) & \textit{if } e \textit{ starts in a } + \textit{ vertex} \\ -\mathrm{def}_1(C,v,e) & \textit{if } e \textit{ starts in a } - \textit{ vertex}, \end{cases}$$

$$\delta_2(e) = \begin{cases} |C_{d(e)}| - \mathrm{def}_2(C,v,e) & \textit{if} \quad \mathrm{sign}(d(e)) = + \\ -\mathrm{def}_2(C,v,e) & \textit{if} \quad \mathrm{sign}(d(e)) = -. \end{cases}$$

Proof Let $e \in F_r$ and first assume $\mathrm{sign}(s(e)) = +$. Then

$$\begin{aligned} \delta_1(e) = |P(f,e)| - \mathrm{def}_1(C,v,e) = |\{e' : e' \in P(f,e) \cap F_r, v_e < v_{e'}\}| \\ +|\{e' \in C_{t(e)-1} : v_e < v_{e'}\}|. \end{aligned}$$

This equals, by definition 6.6 of function X and by definition of bijection Φ in lemma 7.7,

$$\mathrm{sign}(s(\Phi(e))) \sum_{e' \in F} X(\Phi(e), \Phi(e')).$$

We proceed analogously if $\mathrm{sign}(s(e)) = -$. Hence

$$\sum_{e \in F_r} \delta_1(e) = \sum_{e \in F_r} \mathrm{sign}(s(\Phi(e))) \sum_{e' \in F} X(\Phi(e), \Phi(e')).$$

Next we denote, for $e \in F_r$, by $D(e))$ the red edge of $\Phi(P)$ that starts at vertex $a^{d(e)}_{v(e)+1}$. Note that D is a bijection between F_r and the set of the red edges of $\Phi(P)$.

Now let $\mathrm{sign}(d(e)) = +$. Then $\delta_2(e)$ equals the number of $e' \in C_{d(e)}$ such that $v_e < v_{e'}$. This equals $\mathrm{sign}(d(e)) \sum_{e' \in F} Y(D(e), \Phi(e'))$. Again the case $\mathrm{sign}(d(e)) = -$ is analogous.

Hence we get

$$\sum_{e \in F_r} \delta_2(e) = \sum_{e \in F_r} \mathrm{sign}(d(e)) \sum_{e' \in F} Y(D(e), \Phi(e')).$$

This finishes the proof by lemma 6.7. □

Proof (of Theorem 7)
We use Theorem 8, Lemma 7.7 and Proposition 7.10:

$$J_n(K)(t) = t^{\delta(K,n)} \sum_{f\in\mathcal{F}(G_K)} t^{\delta(f)} \sum_{P\in\mathcal{AC}(f,n)} t^{\mathrm{exc}(P)} b(P)$$
$$= t^{\delta(K,n)} \sum_{f\in\mathcal{F}(G_K)} t^{\delta(f)} t^{n(f_b^- - f_b^+)} (1-t)^{f_r^-} (1-t^{-1})^{f_r^+}$$
$$\times \prod_{e\in F_r^+} t^{-(n-1-|P(f,e)|)} \prod_{e\in F_r, \mathrm{sign}(d(e))=+} t^{|C_{d(e)}|}$$
$$\times \sum_{(C,v)\in\mathcal{AC}(f,n)} \prod_{e\in F_r} t^{v_e - \mathrm{def}_1(C,v,e) - \mathrm{def}_2(C,v,e)}.$$

Let us recall that

$$\mathrm{mult}(f)_q = \prod_{v=1}^{r-2} \binom{f(v)}{f(e_v^b)}_{q^{\mathrm{sign}(v)}}.$$

At this point, we will use Appendix B.1. By Theorem 9 and Theorem 10 we get

$$J_n(K)(t) = t^{\delta(K,n)} \sum_{f\in\mathcal{F}(G_K)} t^{\delta(f)} t^{n(f_b^- - f_b^+)} (1-t)^{f_r^-} (1-t^{-1})^{f_r^+} t^{-(n-1)f_r^+}$$
$$\times \prod_{e\in F_r^+} t^{|P(f,e)|} \prod_{e\in F_r, \mathrm{sign}(d(e))=+} t^{|C_{d(e)}|} \prod_{v=1}^{r} \binom{f(v)}{f(e_v^b)}_{t^{-1}} \prod_{e\in F_r} (n-|P(f,e)|)_t$$
$$= t^{\delta(K,n)} \sum_{f\in\mathcal{F}(G_K)} t^{\delta(f)} \prod_{v=1}^{r} \binom{f(v)}{f(e_v^b)}_{t^{-1}} \prod_{v:\mathrm{sign}(v)=+} t^{f(e_v^r) f(e_v^b)}$$
$$\times t^{n(f_b^- - f_b^+)} (1-t)^{f_r^-} (1-t^{-1})^{f_r^+} t^{-(n-1)f_r^+} \prod_{e\in F_r^+} t^{|P(f,e)|} \prod_{e\in F_r} (n-|P(f,e)|)_t$$
$$= t^{\delta(K,n)} \sum_{f\in\mathcal{F}(G_K)} \mathrm{mult}_t(f) t^{\delta(f)} t^{n(f_b^- - f_b^+)} (1-t)^{f_r^-} (1-t^{-1})^{f_r^+}$$
$$\times \prod_{e\in F_r} (n-|P(f,e)|)_{t^{-\mathrm{sign}(s(e))}}$$
$$= t^{\delta(K,n)} \sum_{f\in\mathcal{F}(G_K)} \mathrm{mult}_t(f) t^{\delta(f)} \prod_{v\in V_K} t^{-\mathrm{sign}(v) n f(e_v^b)}$$
$$\times \prod_{e \,\mathrm{red}; t(e)=v} \prod_{j=0}^{f(e)-1} \left(1 - t^{-\mathrm{sign}(s(e))(n-j-\sum_{e'<_v e} f(e))}\right).$$

This finishes the proof. □

Appendix A. The zeta function of a graph and the Foata–Zeilberger formula

A.1 The Foata-Zeilberger formula

In this section we translate key combinatorial results of Foata and Zeilberger [6, Theorem 1.1] in the language of our paper, resulting in Theorem 3.

Consider the *complete graph* K_r with r vertices equipped with a weight martix $B = (b_{ij})$ of size r with independent commuting variables, and let $\mathcal{R} = \mathbb{Z}[[b_{ij}]]$. Let $X = \{x_1, \dots, x_r\}$ denote an alphabet on r letters and $X^\star$ denote the set of words on X.

Recall the notion of a *Lyndon word* $l \in X$, that is a word which is not a nontrivial power of another word, and is strictly smaller than any of its cyclic rearrangements. It follows by definition that

Lemma A.1 *There is a 1–1 correspondence between the set of nonperiodic cycles in K_r and the set of Lyndon words in X.*

Given a nonempty word $w = x_1 x_2 \dots x_m \in X$, Foata and Zeiberger define a function β_{circ} by

$$\beta_{\mathrm{circ}}(w) = b_{x_1,x_2} b_{x_2,x_3} \dots b_{x_{m-1},x_m} b_{x_m,x_1}$$

and $\beta_{circ}(w) = 1$ if w is the empty word. Every word $w \in X$ has a *unique factorization* as

$$w = l_1 l_2 \dots l_n$$

where l_i are Lyndon words in nonincreasing order $l_1 \geq l_2 \geq \dots \geq l_n$. Using this, Foata and Zeilberger define a map:

$$\beta_{\mathrm{dec}} : X^\star \longrightarrow \mathcal{R}$$

by $\beta_{\mathrm{dec}}(w) = \beta_{\mathrm{circ}}(l_1)\beta_{\mathrm{circ}}(l_2) \dots \beta_{\mathrm{circ}}(l_n)$ where $(l_1, \dots, l_n)$ is the unique factorization of w. For example, if $X = \{1,2,3,4,5\}$ and $w = 34512421231242$, then its factorization is given by $(l_1, l_2, l_3) = (345, 1242, 1231242)$ and $\beta_{\mathrm{dec}}(w) = b_{1,2}^3 b_{2,1}^2 b_{2,3} b_{2,4}^2 b_{3,1} b_{3,4} b_{4,2}^2 b_{4,5} b_{5,3}$.

Foata and Zeilberger define another map

$$\beta_{\mathrm{vert}} : X^\star \longrightarrow \mathcal{R}$$

as follows: if $w = x_1 x_2 \dots x_m$ is a word and $\widetilde{w} = \widetilde{x}_1 \widetilde{x}_2 \dots \widetilde{x}_m$ is the rearrangement of the letters of w in nondecreasing order, then they define

$$\beta_{\mathrm{vert}}(w) = b_{\widetilde{x}_1,x_1} b_{\widetilde{x}_2,x_2} \dots b_{\widetilde{x}_m,x_m}.$$

In [6, Theorem 1.1] they show that

$$\frac{1}{\det(I-B)} = \sum_{w\in X^\star} \beta_{\text{dec}}(w) \tag{4}$$

$$= \prod_{c\in\mathcal{P}(K_r)} \frac{1}{1-\beta(c)} \tag{5}$$

$$= \sum_{w\in X^\star} \beta_{\text{vert}}(w) \in \mathcal{R} \tag{6}$$

Let us now translate (6). Write a word w and its rearrangement $\widetilde{w}$ as an array

$$\Gamma(w) = \begin{bmatrix} \widetilde{w} \\ w \end{bmatrix}$$

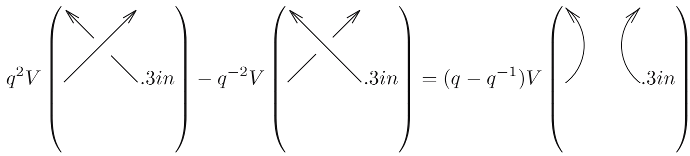
A rearrangement $\widetilde{w}$ of a word w is always of the form $\widetilde{w} = 1^{n_1}2^{n_2}\dots r^{n_r}$, and gives rise to a function $f_w : \text{Edges}(K_r) \longrightarrow \mathbb{N}$ on the edges of K_r defined by $f_w((i,j))$ is the number that the column vector $\begin{bmatrix} i \\ j \end{bmatrix}$ appears in $\Gamma(w)$. Since $\widetilde{w}$ is a rearrangement of w, it follows that f_w is a flow. It follows from 4 that this map $X^\star \longrightarrow \mathcal{F}(K_r)$ is onto, and it is easy to see that given an flow γ on K_r, the preimage under this map consists of mult(γ) words with the same β_{vert} weight, equal to the weight of γ. This together with Eq. (6) implies that

$$\frac{1}{\det(I-B)} = \sum_{f\in\mathcal{F}(K_r)} \beta(f)\,\text{mult}(f). \tag{7}$$

This, together with a specialization of the variables imply Theorem 3.

Appendix B. A state sum for the Jones polynomial

In this section we review the proof of Theorem 5. The Jones polynomial V of a link is determined by the skein theory:

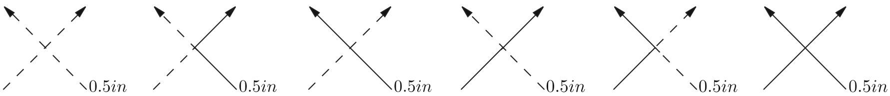


together with the initial condition $V(\text{unknot})(q) = q + q^{-1}$. We will be using a normalized version of the Jones polynomial defined by

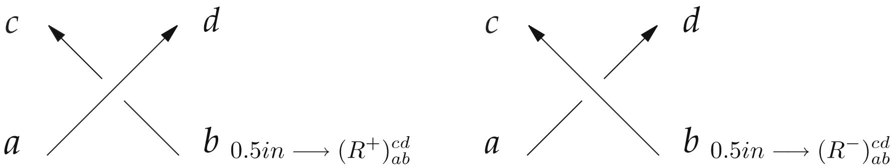
$$J(\mathcal{K})(t) = V(\mathcal{K})(t^{1/2})/V(\text{unknot})(t^{1/2}).$$

We review a state sum definition of the Jones polynomial V discussed by Turaev [29] (see also [14]) and further studied by Lin and Wang [21]. We recall the details of Turaev's general state sum construction, adapted to our special case.

Definition B.1 *Fix a planar projection $\mathcal{K}$ of a knot.*
(a) *Let $P_{\mathcal{K}}$ denote the planar digraph obtained from $\mathcal{K}$ by turning each crossing into a vertex. We call the edges of $P_{\mathcal{K}}$* partarcs *of $\mathcal{K}$.*
(b) *A* state *s of $\mathcal{K}$ is the assignment of* 0 *or* 1 *to each partarc of $\mathcal{K}$, such that at each crossing, the multiset of labels of the incoming edges equals to the multiset of labels of outgoing edges. In other words, at each crossing (positive or negative) a state looks like one of the following pictures,*



where edges colored by 0 *or* 1 *are depicted as dashed or solid respectively.*
(c) *The* local weight $\Pi_v(s)$ *of a vertex v of $P_{\mathcal{K}}$ of a state s is given by*



where R^+ and $R^- = (R^+)^{-1}$ is the R-matrix of the quantum group $U_q(\mathfrak{sl}_2)$ given by:

$$\begin{array}{lll} (R^+)^{0,0}_{0,0} = (R^+)^{1,1}_{1,1} = -q & (R^+)^{1,0}_{0,1} = (R^+)^{0,1}_{1,0} = 1 & (R^+)^{0,1}_{0,1} = \bar{q} - q \\ (R^-)^{0,0}_{0,0} = (R^-)^{1,1}_{1,1} = -\bar{q} & (R^-)^{1,0}_{0,1} = (R^-)^{0,1}_{1,0} = 1 & (R^-)^{1,0}_{1,0} = q - \bar{q} \end{array}$$

where $\bar{q} = q^{-1}$ and all other entries of the R matrix are zero.
(d) *The* weight $\Pi(s)$ *of a state s is defined by*

$$\Pi(s) = \prod_v \Pi_v(s).$$

Note that $(R^-)^{0,1}_{0,1} = (R^+)^{1,0}_{1,0} = 0$.
(e) *A state s is* admissible *iff* $\Pi(s) \neq 0$.

There is an involution $s \to s^c$ of states of $\mathcal{K}$, obtained by interchanging 0 by 1's.

Lemma B.2 (a) *There is a 1–1 correspondence*

$$\{\textit{states of } \mathcal{K}\} \longleftrightarrow \{\textit{even subgraphs of } P_{\mathcal{K}}\}.$$

(b) *There is a 1–1 correspondence*

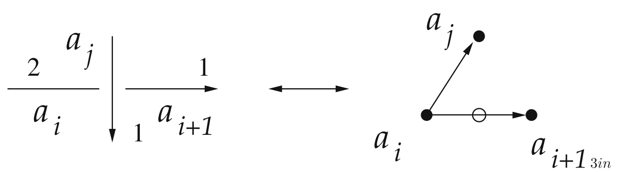
$$\{\textit{admissible states of } \mathcal{K}\} \longleftrightarrow \{\textit{collections of vertex-disjoint cycles of } G_{\mathcal{K}}\}.$$

Proof A state s gives rise to an *even subgraph* of the $P_{\mathcal{K}}$ (whose edges are the ones colored by 1 in s), also denoted by s. Part (a) follows.

Since every vertex of $P_{\mathcal{K}}$ has outdegree 2, it follows that the involution of states corresponds to the operation of taking the complement of an even subgraph in $P_{\mathcal{K}}$.

For part (b), observe that an admissible even subgraph s of $P_{\mathcal{K}}$ gives rise to an even subgraph of the arc-graph $G_{\mathcal{K}}$ with each indegree at most one: this follows since as mentioned above, $(R^+)_{0,1}^{0,1} = (R^-)_{1,0}^{1,0} = 0$, and so if we walk on s along the orientation of $\mathcal{K}$, we never 'jump down'; hence whenever we get to an arc of $\mathcal{K}$, we traverse it (along its orientation) until its end. Hence we can get to each arc at most once and s corresponds to an even subgraph of $G_{\mathcal{K}}$ where each indegree is at most one.

Conversely, an even subgraph of $G_{\mathcal{K}}$ gives rise to a flow on $P_{\mathcal{K}}$. This flow will be an admissible even subgraph of $P_{\mathcal{K}}$ if each indegree is at most one. The following figure illustrates the excluded possibilities, where the value of the flow is shown on the partarcs:

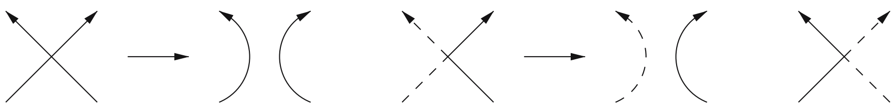


□

Definition B.3 *An even subgraph G of $G_{\mathcal{K}}$ is* admissible *if each indegree is at most one. In other words, G is a vertex-disjoint collection of directed cycles. Let $\mathcal{S}(G_{\mathcal{K}})$ denote the collection of admissible subgraphs of the arc-graph $G_{\mathcal{K}}$.*

Next we define rotation and excess numbers of states.

Definition B.4 (a) *The* rotation number $\mathrm{rot}(s)$ *of a state s is the number of counterclockwise circles colored by* 1 *minus the number of clockwise circles colored by* 1, *obtained from smoothening of s, i.e., by the replacement:*



at all crossings of s.

(b) *The* excess number $\mathrm{exc}(s)$ *of a state s is the sum of the signs of the crossings where* all *four edges are colored by* 1 *in s.*

With these preliminaries, the Jones polynomial is given by the state sum

$$V(\mathcal{K})(q) = (-q^2)^{-\omega(\mathcal{K})} \sum_{s \text{ admissible}} q^{\mathrm{rot}(s^c)-\mathrm{rot}(s)} \Pi(s).$$

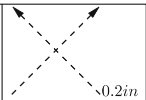
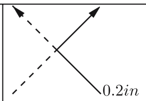
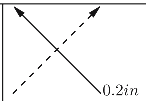
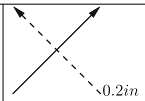
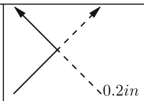
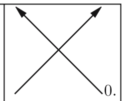
It was observed by Lin and Wang that the local weights of the R-matrix are proportional, up to a power of q to the weights of a random walk on $\mathcal{K}$. This is formalized in the following Lemma:

Lemma B.5 [21, Lemma 2.3] *For an admissible state s of* $\mathcal{K}$*, we have:*

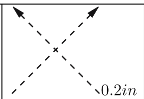
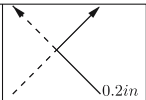
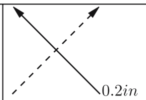
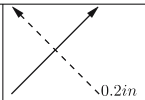
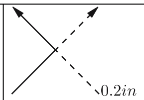
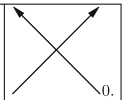
$$\Pi(s) = (-q)^{\omega(\mathcal{K})} q^{2\mathrm{exc}(s)} \beta(s)|_{t \to q^2}.$$

Proof First note that $\beta(s)$ is well defined since by Lemma B.2 there is a 1–1 correspondence between admissible states of K and even admissible subgraphs of $G_{\mathcal{K}}$, and each even subgraph is naturally a flow on $G_{\mathcal{K}}$.

Consider the following table of a state around a positive crossing:

						
R	$-q$	$\bar{q}-q$	1	1	0	$-q$
$-\bar{q}R$	1	$1-\bar{q}^2$	$-\bar{q}$	$-\bar{q}$	0	1
β	1	$1-\bar{q}^2$	$\bar{q}^2$	1	0	$\bar{q}^2$
q^{exc}	1	1	1	1	1	q^2
q^{err}	1	1	$-q$	$-\bar{q}$	1	1

and around a negative crossing:

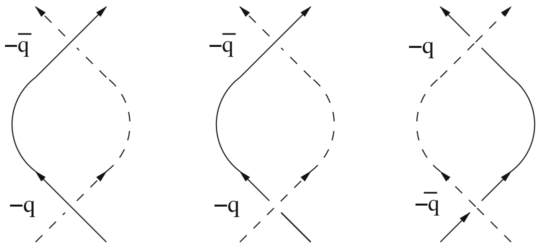
						
R	$-\bar{q}$	0	1	1	$q-\bar{q}$	$-\bar{q}$
$-qR$	1	0	$-q$	$-q$	$1-q^2$	1
β	1	0	1	q^2	$1-q^2$	q^2
q^{exc}	1	1	1	1	1	$\bar{q}^2$
q^{err}	1	1	$-q$	$-\bar{q}$	1	1

Here, $\beta(s)$ of a state s equals to the weight of the 1-part of s.

Inspection of these tables reveals that given a state s and a crossing of sign $\epsilon = \pm 1$, we have $R = (-q)^{\epsilon} q^{2\mathrm{exc}+\mathrm{err}} \beta$. Taking a product over all vertices, we obtain that

$$\Pi(s) = (-q)^{\omega(\mathcal{K})} q^{2\mathrm{exc}(s)+\mathrm{err}(s)} \beta(s)|_{t \to q^2}.$$

It remains to show that $q^{\mathrm{err}(s)} = 1$. $\mathrm{err}(s)$ is computed from a smoothening $\mathrm{smooth}(s)$ of s, which consists of a number of transversely intersecting circles colored by 0 or 1. Any two transverse planar circles intersect on an even number of points, which can be paired up by paths on each circle. A case by case argument shows that $\mathrm{err}(s) = 1$. Some cases of the local contributions to 'err' and their pairwise canceling is shown by:



This concludes the proof of the lemma. □

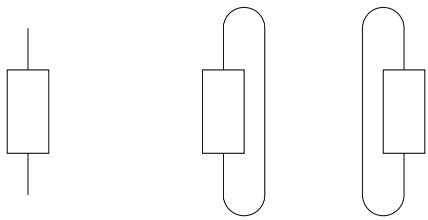
The involution on the set of states of $\mathcal{K}$ has further consequences discovered by Lin and Wang. Fix a partarc of $\mathcal{K}$ that borders the unbounded region of the planar projection and mark it by $\star$. Let $\mathcal{F}'(\mathcal{K}-\star)$ denote the set of all admissible states of $\mathcal{K}$ where $\star$ is colored by 0.

We will show first that

$$J(\mathcal{K})(t) = t^{\delta(K)} \sum_{s \in \mathcal{F}'(K-\star)} t^{\delta(s)} \beta(s). \tag{8}$$

We recall that $\delta(K) = 1/2(-\omega(K) + \mathrm{rot}(K))$ and $\delta(s) = \mathrm{exc}(s) - \mathrm{rot}(s)$.

Consider a long knot K^{long} depicted as a box and the two ways of closing it to obtain a knot $\mathcal{K}$ as follows:



Let a_i denote $V(K^{\mathrm{long}})$ with boundary conditions i, for $i = 0, 1$. Then, the two ways of closing K^{long} give:

$$V(K)(q) = qa_0 + q^{-1}a_1 = q^{-1}a_0 + qa_1$$

from which follows that $a_0 = a_1$ and thus $V(K)(q) = (q+q^{-1})a_0 = V(\mathrm{unknot})a_0$. Thus, (8) follows.

Next we introduce the rotation and excess numbers of a collection of vertex disjoint cycles of G_K, using Lemma B.2.

B.1 Rotation and excess numbers

We observe that there is an integer function rot on the set of the edges of $G_{\mathcal{K}}$ so that for each admissible state s and its corresponding (see Lemma B.2) admissible subgraph c of $G_{\mathcal{K}}$, $\mathrm{rot}(s) = \sum_{e \in c} \mathrm{rot}(e)$.

Definition B.6 *There is a Gauss map $d : \mathcal{K} \to S^1$ which together with the orientation of $\mathcal{K}$ and the counterclockwise orientation of S^1 induces a map*

$$H_1(P_{\mathcal{K}}, \mathbb{Z}) \longrightarrow H_1(S^1, \mathbb{Z}) \cong \mathbb{Z}.$$

The above composition is defined to be the rotation number rot. *We can think of the rotation number as an element of $H^1(P_{\mathcal{K}}, \mathbb{Z})$ represented by a 1-cocyle, that is a map*

$$\mathrm{rot} : \mathrm{Edges}(P_{\mathcal{K}}) \longrightarrow \mathbb{Z}.$$

Consider now the arc-graph $G_{\mathcal{K}}$ of $\mathcal{K}$. There is a canonical map

$$\mathrm{Edges}(G_{\mathcal{K}}) \longrightarrow 2^{\mathrm{Edges}(P_{\mathcal{K}})}$$

defined as follows: if (i, j) is an edge of $G_{\mathcal{K}}$, consider the ith crossing of $P_{\mathcal{K}}$, and start walking on the part of the arc a_j in a direction of the orientation of K, until the end of the arc a_j. This defines a collection of part-arcs that we associate to the edge (i, j) of $G_{\mathcal{K}}$. Taking the sum of the rotation numbers of these part-arcs, defines a map

$$\mathrm{rot} : \mathrm{Edges}(G_K) \longrightarrow \mathbb{Z}.$$

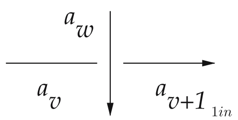
Next we show that exc' of next definition agrees with exc of Definition 4.1.

Definition B.7 *Let c be an admissible subgraph of G_K. We let* $\mathrm{exc}'(c)$ *equal to* $\mathrm{exc}(s)$*, where s is the corresponding admissible state (see Definition B.4 and Lemma B.2 for the correspondence).*

Lemma B.8 *For every admissible subgraph c of G_K, we have:*

$$\mathrm{exc}'(c) = \mathrm{exc}(c).$$

Proof $\mathrm{exc}'(c)$ is the sum of $\mathrm{sign}(v)$ where all 4 edges incident to a crossing v of K belong to c:



We will translate this in the language of the arc-graph G_K, using Figure 1. A crossing v as above determines a unique vertex of G_K (corresponding to the arc a_v ending at v) and a unique pair of edges (e, e') of G_K: e is the blue edge that starts at v, and e' is the unique edge of c that ends in w and signifies the transition on the arc a_w. The result follows. □

Proof (of Theorem 5)
Assume (after possibly changing the orientation of the knot, which does not change the Jones polynomial) that we mark by $\star$ the last partarc of an arc of $\mathcal{K}$. Lemma B.2, (8), and subsection B.1 conclude the proof of Theorem 5. □

Appendix C. A combinatorial counting of structures

In this section we consider structures on a set $[k] = \{1, \dots, k\}$, and their combinatorial countings.

Definition C.1 *Let k be a positive integer. A k-*structure *is a pair $S = (A, B)$ such that*

- *$A = (A_1, \dots, A_l)$, $B = (B_1, \dots, B_l)$ for some l, $A_i, B_i \subset [k]$, $A_i \neq \emptyset$ for all i,*
- *A is a partition of $\{1, \dots, k\}$ such that for every $i < j$, $x \in A_i, y \in A_j$ we have $x < y$.*
- *$B_i \subset \cup_{j<i} A_j$. In particular $B_1 = \emptyset$.*
- *B is monotonic. That is, if $x \in B_j \cap A_i$ then for each $j \geq j' > i$, $x \in B_{j'}$,*

Lemma C.2 *The number of k-structures S such that $|A_i| = a_i$ and $|B_i| = b_i$ for $i = 1, \dots, l$ is*

$$\prod_{i=2}^{l} \binom{a_{i-1} + b_{i-1}}{b_i}.$$

Proof B_i is an arbitrary subset of $A_{i-1} \cup B_{i-1}$ of b_i elements. □

Definition C.3 *Let S be a k-structure, $v \in \{0, \dots, n-1\}^{\{1,\dots,k\}}$ and $i \in A_x$ for some $x \leq l$.*

- *We let $|S| = (a, b)$, where $a = (|A_1|, \dots, |A_l|)$ and $b = (|B_1|, \dots, |B_l|)$.*
- *We let $m(|S|, i)$ be the number of $j \in A_x \cup B_x$ such that $j < i$. Note that $m(|S|, i)$ equals b_i plus the number of elements of A_x that are smaller than i and hence it depends only on $|S|$.*
- *We denote by $\mathrm{def}_1(S, v, i)$ the number of $j \in A_x \cup B_x$ such that $j < i$ and $v_j < v_i$,*
- *we denote by $\mathrm{def}_2(S, v, i)$ the number of $j \in B_{d(i)+1}$ such that $v_j < v_i$; $d(i)$ is the minimum index so that $d(i) \geq i$ and $i \notin B_{d(i)+1}$.*

Definition C.4 *Let S be a k-structure. We let $V(S, n) = \{v \in \{0, \dots, n-1\}^{\{1,\dots,k\}};$ if $\{i, j\} \subset A_m \cup B_m$ for some m then $v_i \neq v_j\}$.*

The following Theorem follows by comparing the definitions.

Theorem 9 *Let f be a flow on arc-graph G_K. Recall that $f_r(v) = \sum f(e)$ over all red edges of G_K terminating in vertex v of G_K, and $f_b(v)$ is defined analogously for the blue edges. We consider set F_r linearly ordered, first by the terminal vertices, and then by ordering $\prec$ which induces a linear ordering on the set $\cup F(e)$, over red edges e entering the same vertex (see Definition 6.5).*

There is a natural bijection between $\mathcal{AC}_n(f)$ and the set of all pairs (S, v) where S is an $|F_r|$-structure, $|S| = ((f_r(1), \ldots, f_r(r))(f_b(1), \ldots, f_b(r)))$ and $v \in V(S, n)$.

Theorem 10

$$\sum_{S:|S|=(a,b)} \sum_{v\in V(S,n)} \prod_{i=1}^{k} t^{v_i-\mathrm{def}_1(S,v,i)-\mathrm{def}_2(S,v,i)} = \prod_{i=1}^{k}(n-m(i))_t \prod_{i=1}^{l-1} \binom{a_i+b_i}{b_{i+1}}_{t^{-1}}.$$

In the proof we will use the following proposition.

Proposition C.5 *Let S be a k-structure. Then*

$$\sum_{v\in V(S,n)} \prod_{i=1}^{k} t^{v_i-\mathrm{def}_1(S,v,i)} = \prod_{i=1}^{k}(n-m(|S|,i))_t.$$

Proof Use induction on k. The inductive step follows from the following claim:
Claim Let $m(k) < n$ and fix different numbers $v_1, \ldots, v_{m(k)}$ between 0 and $n-1$. Then

- $$\sum_{v_k:v_k\neq v_i, i\leq m(k)} t^{v_k-\mathrm{def}_1(v,k)} = A - B + C,$$

  where $A = \sum_{v_k:v_k\neq v_i,i\leq m(k)} t^{v_k}$, $B = \sum_{i=1}^{m(k)} t^{n-i}$, and $C = \sum_{i=1}^{m(k)} t^{v_i}$.
- $A + C = \sum_{0\leq z\leq n-1} t^z$ and $A - B + C = \frac{1-t^{n-m(k)}}{1-t}$.

Note that the second part is simply true.

Let $v'_1 < \cdots < v'_{m(k)}$ be a reordering of $v_1, \ldots, v_{m(k)}$. We may write $v'_1 = n - i_1, \ldots, v'_{m(k)} = n - i_{m(k)}, 1 \leq i_{m(k)} < \cdots < i_1$. The LHS becomes

$$A - t^{n-i_1+1} - \cdots - t^{n-i_2-1} - t^{n-i_2+1} - \cdots - t^{n-i_{m(k)}-1} - t^{n-i_{m(k)}+1} - \cdots - t^{n-1} \\ + t^{n-i_1} + \cdots + t^{n-i_2-2} + t^{n-i_2-1} + \cdots + t^{n-m(k)-1}.$$

This equals to the RHS of the equality we wanted to show. The Proposition simply follows from the Claim. □

Proof (of Theorem 10)

We let $a'_i = \sum_{j\le i} a_i$.

$$\sum_{S:|S|=(a,b)} \sum_{v\in V(S,n)} \prod_{i=1}^{k} t^{v_i-\mathrm{def}_1(S,v,i)-\mathrm{def}_2(S,v,i)} = \sum_{B_2\subset A_1} \sum_{v_1,\dots,v_{a_1}} \prod_{i=1}^{a_1} t^{v_i-\mathrm{def}_1(v,i)-\mathrm{def}_2(B_2,v,i)}$$
$$\times \sum_{B_3,\dots,B_l} \sum_{v_{a_1+1},\dots,v_k} \prod_{i=a_1+1}^{k} t^{v_i-\mathrm{def}_1(v,i)-\mathrm{def}_2(B_3,\dots,B_l,v,i)}$$
$$= \sum_{v_1,\dots,v_{a_1}} \prod_{i=1}^{a_1} t^{v_i-\mathrm{def}_1(v,i)} \sum_{B_2\subset A_1} \prod_{i=1}^{a_1} t^{-\mathrm{def}_2(B_2,v,i)}$$
$$\times \sum_{B_3,\dots,B_l} \sum_{v_{a_1+1},\dots,v_k} \prod_{i=a_1+1}^{k} t^{v_i-\mathrm{def}_1(v,i)-\mathrm{def}_2(B_3,\dots,B_l,v,i)}$$
$$= \sum_{v_1,\dots,v_{a_1}} \prod_{i=1}^{a_1} t^{v_i-\mathrm{def}_1(v,i)} \sum_{B_2\subset A_1} \prod_{i=1}^{a_1} t^{-\mathrm{def}_2(B_2,v,i)}$$
$$\times \cdots \times \sum_{v_{a'_{l-2}+1},\dots,v_{a'_{l-1}}} \prod_{i=a'_{l-2}+1}^{a'_{l-1}} t^{v_i-\mathrm{def}_1(v,i)} \sum_{B_l} \prod_{i=a'_{l-2}+1}^{a'_{l-1}} t^{-\mathrm{def}_2(B_l,v,i)}$$
$$\times \sum_{v_{a'_{l-1}+1},\dots,v_k} \prod_{i=a'_{l-1}+1}^{k} t^{v_i-\mathrm{def}_1(v,i)}.$$

The last sum may be expressed using Proposition C.5, and we get

$$\sum_{v_1,\dots,v_{a_1}} \prod_{i=1}^{a_1} t^{v_i-\mathrm{def}_1(v,i)} \sum_{B_2\subset A_1} \prod_{i=1}^{a_1} t^{-\mathrm{def}_2(B_2,v,i)}$$
$$\times \cdots \times \sum_{v_{a'_{l-2}+1},\dots,v_{a'_{l-1}}} \prod_{i=a'_{l-2}+1}^{a'_{l-1}} t^{v_i-\mathrm{def}_1(v,i)} \sum_{B_l} \prod_{i=a'_{l-2}+1}^{a'_{l-1}} t^{-\mathrm{def}_2(B_l,v,i)} \times \prod_{i=a'_{l-1}+1}^{k} (n-m(i))_t$$
$$= \prod_{i=a'_{l-1}+1}^{k} (n-m(i))_t \sum_{v_1,\dots,v_{a_1}} \prod_{i=1}^{a_1} t^{v_i-\mathrm{def}_1(v,i)} \sum_{B_2\subset A_1} \prod_{i=1}^{a_1} t^{-\mathrm{def}_2(B_2,v,i)}$$
$$\times \cdots \times \prod_{i=a'_{l-2}+1}^{a'_{l-1}} (n-m(i))_t \binom{a_{l-1}+b_{l-1}}{b_l}_{t^{-1}}$$
$$= \prod_{i=1}^{k} (n-m(i))_t \prod_{i=1}^{l-1} \binom{a_i+b_i}{b_{i+1}}_{t^{-1}}.$$

□

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