PATTERNS OF THE V_2-POLYNOMIAL OF KNOTS

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ABSTRACT. Recently, Kashaev and the first author defined a sequence V_n of 2-variable knot polynomials with integer coefficients, coming from the R-matrix of a rank 2 Nichols algebra, the first polynomial been identified with the Links–Gould polynomial. In this note we present the results of the computation of the V_n polynomials for $n = 1, 2, 3, 4$ and discover applications and emerging patterns, including unexpected Conway mutations that seem undetected by the V_n-polynomials as well as by Heegaard Floer Homology and Khovanov Homology.

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1. INTRODUCTION

1.1. **A sequence of 2-variable knot polynomials.** Recently, Rinat Kashaev and the first author defined multivariable polynomials of knots using Nichols algebras [GK] with automorphisms. In our paper, we focus on the sequence of 2-variable polynomial invariants $V_n(t, q)$ of knots which come by applying a general construction of [GK] to one of the simplest Nichols f-algebras of rank 2 of diagonal type. This algebra depends on one variable q that determines the braiding and two variables (t_1, t_2) that determine the automorphism type, and when q is not a root of unity and (t_1, t_2, q) satisfy the relation $t_1 t_2 q^n = 1$ for some

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positive integer $n > 0$, then the Nichols algebra has a right Yetter–Drinfel'd f-module Y_n of dimension $4n$ [GK, Prop.7.4] and an explicit R-matrix T_n. Taking this as a black box, and parametrizing the three variables (t_2, t_2, q) satisfying $t_1 t_2 q^n = 1$ in terms of two variables (t, q) $(t_1, t_2) = (1/(q^{n/2}t, t/q^{n/2})$, it was shown in [GK] that Y_n comes equipped with an R-matrix T_n which leads to a matrix-valued knot invariant $K \mapsto J_{T_n}(K) \in \mathrm{End}(Y_n)$ as well as to a scalar-valued invariant $V_{K,n}(t, q) \in \mathbb{Z}[t^{\pm 1}, q^{\pm 1/2}]$ given by the $(1, 1)$-entry of $J_{T_n}(K)$.

It was advocated in [GK] that the sequence V_n of 2-variable knot polynomials has similarities and differences with the sequence of the Jones polynomial of a knot and its parallels, otherwise known as the colored Jones polynomial.

The polynomial invariant V_n satisfies

- the symmetry
$$V_{K,n}(t, q) = V_{K,n}(t^{-1}, q), \qquad V_{\overline{K},n}(t, q) = V_{K,n}(t, q^{-1}) \tag{1}$$
where the first equality comes from the involution exchanging t_1 and t_2, and in the second one $\overline{K}$ denotes the mirror image of K,
- conjecturally, the specialization
$$V_{K,n}(q^{n/2}, q) = 1, \qquad V_{K,n}(t, 1) = \Delta_K(t)^2 \tag{2}$$
where $\Delta_K(t) \in \mathbb{Z}[t^{\pm 1/2}]$ is the symmetrized (i.e., $\Delta_K(t) = \Delta_K(t^{-1})$), normalized (i.e., $\Delta_K(1) = 1$) Alexander polynomial,
- conjecturally, the relation $V_1 = \mathrm{LG}$ with the Links–Gould invariant [LG92],
- and conjecturally, the genus bound
$$\deg_t V_{K,n} \le 4g(K) \tag{3}$$
where the Seifert genus $g(K)$ is the smallest genus of a spanning surface of a knot. Here, by t-degree of a Laurent polynomial of t we mean the difference between the highest and the lowest power of t.

The paper [GK] stimulated a lot of subsequent work. The relation $V_1 = \mathrm{LG}$ is now known [HKST], and consequently the specialization (2) and the genus bounds (3) holds for $n = 1$ since they hold for the Links–Gould invariant [NvdV, KT]. On the other hand, it is known that the Links–Gould invariant does not detect Conway mutation, whereas the genus does, and hence the genus bounds (3) cannot be sharp for $n = 1$.

More recently, an expression of V_2 in terms of V_1 polynomial of a knot and its $(2, 1)$-parallel was obtained [GKK], and this proves both the specialization (2) and the genus bound (3) for $n = 2$.

But of all those properties of V_2, the last inequality is the most intriguing. Based on some experiments with few knots and 12 and 13 crossings, it was observed in [GK] that in all computed cases, the inequality (3) is in fact an equality. Is this an accident for knots with low numbers of crossings, or a new phenomenon? To decide one way or another, one needs an efficient way to compute the V_n-polynomials of knots, do so, and sieve the data. This is exactly what we did, and led to the results of our paper.

1.2. **Is the genus bound an equality for V_2?** Since we are talking about tables of knots and their invariants, we will be using the naming of the HTW table of knots up to 16 crossings [HTW98] imported in `SnapPy` [CDGW] and also in `KnotAtlas` [Kno].

crossings	11	12	13	14	15	16
knots	552	2176	9988	46972	253293	1388705
Loose knots	7	29	208	1220	6319	48174
Trivial Alex knots	2	2	15	36	118	499
V_1 genus bound <	7	20	173	974	5025	???
V_2 genus bound <	0	0	0	0	0	???

TABLE 1. Knot counts, up to mirror image.

The inequality (3) combined with the specialization (1) for $n = 2$ imply that

$$2 \deg_t A_K(t) \le \deg_t V_{K,2}(t,q) \le 4g(K)\,. \tag{4}$$

On the other hand,

$$\deg_t A_K(t) \le 2g(K) \tag{5}$$

with equality if and only if a knot is Alexander-tight, otherwise Alexander-loose. In our paper we abbreviate these two classes simply with tight/loose, similar to what people do in Heegaard Floer Homology and Khovanov Homology where they talk about HFK-thin/thick or Kh-thin/thick knots, but then they drop the HFK or Kh once the context is clear. Likewise, we use the term thin in our paper to mean HFK and Kh-thin. Note that alternating knots are tight [Mur58]. What's more, quasi-alternating knots (a class that includes all alternating knots) introduced by Ozsvath-Szabo in [OS05] are HFK and Khovanov-thin [MO08], and hence tight.

Combining the above two inequalities, it follows that the inequality (3) is in fact an equality for V_2 and all tight knots. Note next that there are no loose knots with ≤ 10 crossings. Moreover, the number of loose knots with ≤ 16 crossings is given in Table (1). Incidentally, the list of loose knots was compiled by computing in `SnapPy` the Alexander polynomial, and also the HFK (and in particular, the Seifert genus of a knot).

Among the loose knots, are the ones with trivial Alexander polynomial (also computed by `SnapPy`) which are in some sense extreme. The list of 7 loose knots (up to mirror) with 11 crossings is

$$11n34^*,\; 11n42^*,\; 11n45,\; 11n67,\; 11n73,\; 11n97,\; 11n152 \tag{6}$$

where the asterisque indicates that the knot has trivial Alexander polynomial, and the pair $(11n34, 11n42)$ is the famous Kinoshita–Terasaka and Conway pair of mutant knots. Their genus is given by $3, 2, 3, 2, 3, 2, 3$, the t-degree of the V_1-polynomial is $6, 6, 8, 6, 8, 6, 8$ and the t-degree of the V_2-polynomial is $12, 8, 12, 8, 12, 8, 12$, confirming the equality in (3) for $n = 2$.

polynomial	V_1	V_2	V_3	V_4
knots	≤ 14	≤ 14	≤ 11	≤ 9
Loose knots	≤ 16	≤ 15		
Trivial Alex knots	≤ 16	≤ 16		

TABLE 2. Table of computed values of V_n, the values indicate the number of crossings.

Table 2 summarizes the knots for which the V_n-polynomial was computed. The data is available in [GL24], with the convention that we replaced q by q^2 in V_1 and V_3 so that we obtain Laurent polynomials in t and q, as opposed to Laurent polynomials in t and $q^{1/2}$. For $n = 2$, we computed its values for all loose knots with at most 15 crossings and all trivial Alexander knots with at most 16 crossings. In all cases, we found that the inequality (3) is an equality for $n = 2$. Combined with the specialization and the genus bounds for V_2, this implies the following.

Proposition 1.1. Equality holds in (3) for $n = 2$ and for all knots with at most 15 crossings and all trivial Alexander polynomial knots with 16 crossings.

Remark 1.2. The relation between V_2 and V_1 discussed in Section 1.5 below, combined with the fact that $V_1 = \mathrm{LG}$ implies that the map

$$K \mapsto V_{K,2}(e^{\hbar N}, e^{\hbar}) \in \mathbb{Q}[N][\![\hbar]\!] \tag{7}$$

is a Vassiliev power series invariant of knots. Hence, if (3) is an equality for $n = 2$, it follows that Vassiliev invariants determine the Seifert genus of a knot. A celebrated method to detect the genus of the knot is Heegaard Floer Homology [OS06]. A second (conjectural) method to compute the genus of a knot uses hyperbolic geometry, and more specifically the degree of the twisted torsion polynomial $\tau_{K,3}(t)$ of a hyperbolic knot (twisted with the adjoint representation of the geometric representation of a hyperbolic knot); see [DGST10, Sec.1.6]. Curiously, the Conjecture 1.7 of [DGST10] was verified for all hyperbolic knots with at most 15 crossings.

1.3. **V_2-trivial Conway mutations.** A question that we discuss next is how strong is the new V_2 polynomial in separating knots. Given the values of the polynomial for knots up to 14 crossings, we searched for repetitions, taking into account mirror image, which changes $V_2(t, q)$ to $V_2(t, q^{-1})$. Here, we came across a new surprise. The V_2 polynomial separates knots with at most 11 crossings, but fails to separate 12 crossing knots, and the three pairs that we found are

$$(12n364,\ \overline{12n365}) \quad (12n421,\ \overline{12n422}) \quad (12n553,\ \overline{12n556})\,. \tag{8}$$

We tried the V_1 polynomial on them and it failed to separate them, and we tried the V_3-polynomial which also failed to separate. Yet, the genus inequality (3) was an equality for $n = 2$, which meant that these 3 pairs have equal genus (in each pair). We checked their HFK, computed by `SnapPy`, and their Khovanov Homology, computed by `KnotAtlas`, and a bit to our surprise, was equal in each pair. Looking at these 3 pairs more closely, we realized that they are in fact Conway mutants. A table of mutant knots with at most 15 crossings is given in Stoimenow [Sto]. As was pointed out to us by N. Dunfield, one can separate the knots in these pairs using the homology of their 5-fold covers, or the certified isometry signature of the complete hyperbolic structure.

Having found these unexpected pairs of Conway mutant knots, we tried knots with 13 crossings, where we now found 25 more pairs with exactly the same properties as above, given in Table (26). We then searched knots with 14 crossings, where we now found 189 pairs and 1 triple (up to mirror image) with the same properties as above. The V_2-equivalence classes of knots of size more than 1 with at most 12, 13 and 14 crossings are given Tables (25), (26) and (27a), respectively.

crossings	≤ 11	12	13	14
pairs	0	3	25	189
triples	0	0	0	1

TABLE 3. Number of V_2-equivalence classes of size more than 1 (up to mirror image).

Summarizing, we obtain the following.

Proposition 1.3. The V_2 polynomial separates knots with at most 14 crossings except for the pairs and triples in Tables (25), (26) and (27a)–(27d). The knots in those tuples with at most 14 crossings have

- equal V_1, V_2, V_3[1] and (for at most 13 crossings) V_4 polynomials,
- equal HFK and equal Khovanov Homology,
- and they are Conway mutant knots, in particular they have equal volumes, trace field, colored Jones polynomials, ADO polynomials and HOMFLY polynomial.

For lack of a better name, let us say that two knots are V_2-equivalent if they have equal V_2-polynomial. This notion is similar to the almost-mutant knots of [DGST10].

Question 1.4. Are V_2-equivalent knots always Conway mutant? Do they have equal equal HFK and equal Khovanov Homology? And why?

We can give a partial answer to this question as follows. The observed V_2-equivalence classes come in 3 flavors

$$1: \text{tight} + \text{thin}, \qquad 2: \text{tight} + \text{thick}, \qquad 3: \text{loose} + \text{thick} \tag{9}$$

and the counts of the V_2-equivalence classes of size more than one, according to their flavor is given by

$$1: 172, \qquad 2: 30, \qquad 3: 16. \tag{10}$$

Regarding the more numerous class 1, note that the HFK homology of an HFK-thin (resp., Kh-thin) knot is determined by its Alexander polynomial (resp., by the Jones polynomial and the signature). Since mutation does not change the Alexander polynomial, nor the Jones polynomial, nor the signature, it follows that mutant thin knots have equal HFK and equal Khovanov Homology. This gives an explanation of the above question for the class 1, which as was mentioned in the introduction, includes pairs of mutant quasi-alternating knots.

The tuples in the other two classes are not as well-understood. Our tables (given in the Appendix) give concrete examples of tight + thick or loose + thick knots. For instance, three pairs of tight + thick knots are

$$(13n1655,\ 13n1685)^2, \qquad (14n1370,\ 14n1395)^2, \qquad (14n1699,\ 14n1947)^2 \tag{11}$$

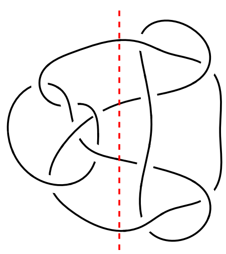
[1]with 2 exceptional pairs $(14n2423, 14n5868)^2$ and $(14n5822, 14n5852)^3$ that have different V_3-polynomials and they are of flavors 2 and 3, respectively.

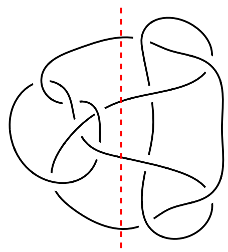
and three pairs of loose + thick knots are

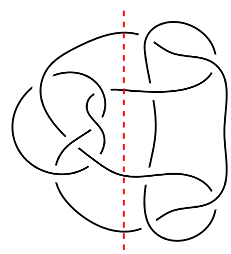
$$(13n372,\ 13n375)^3, \qquad (13n536,\ 13n551)^3, \qquad (13n1653,\ 13n1683)^3\,. \tag{12}$$

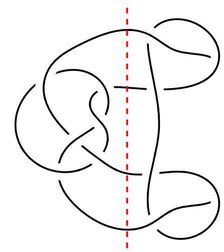
There are several methods of constructing knots with equal HFK and equal Khovanov Homology discussed for example in detail in Hedden–Watson, [HW18], but we do not know how to apply these constructions to generate our examples.

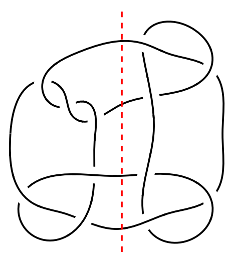
It is also worth noticing that all tight + thin knots listed in the appendix are $\overline{\mathrm{Kh}'}$-thin, as defined in [ORS13, Def.5.1].


$12n364$


$\overline{12n365}$


$12n421$


$\overline{12n422}$


$12n553$

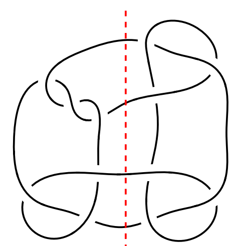

$\overline{12n556}$

FIGURE 1. The 3 pairs of knots from (8).

1.4. **Independence of the V_2 and the 2-loop polynomials.** The colored Jones polynomial can be decomposed into loop invariants, starting from 0-loop which is the inverse Alexander polynomial, and then going up the loops. In fact, the 2-loop invariant of the Kontsevich integral of a knot is essentially 2-variable polynomial invariant, and its image under the $\mathfrak{sl}_2(\mathbb{C})$-weight system is a 1-variable polynomial computed efficiently by Bar-Natan and van der Veen [BNvdVb, BNvdVa].

One may ask whether the V_2-polynomial is determined by the 2-loop invariant Z_2 of the Kontsevich integral. It is known that the map $K \mapsto Z_2(\mathrm{Wh}(K))$ is a degree 2 Vassiliev invariant of knots [Gar04, Thm.1], where $\mathrm{Wh}(K)$ denotes the Whitehead doubling of a 0-framed knot with a positive clasp. Since the vector space of degree 2 Vassiliev invariants is 1-dimensional generated by $a_2(K) = \Delta''(1)$, it follows that $Z_2(\mathrm{Wh}(K)) = c\, a_2(K)$ for a universal nonzero constant c. For the trefoil and its mirror image, we have $a_2(3_1) = a_2(\overline{3_1}) = 2$, which implies that

$$Z_2(\mathrm{Wh}(3_1)) = Z_2(\mathrm{Wh}(\overline{3_1}))\,. \tag{13}$$

On the other hand, $V_{\mathrm{Wh}(3_1),2}(t,q) \neq V_{\mathrm{Wh}(\overline{3_1}),2}(t,q)$, the exact values given in Equation (29) in the Appendix. This implies the following.

Proposition 1.5. The V_2-polynomial is not determined by the 2-loop part of the Kontsevich integral.

Based on limited computations available, it was observed in [GK] that Khovanov Homology alone, or HFK alone, or the colored Jones polynomial alone do not determine V_2.

1.5. **A relation between V_1 and V_2.** In a sense, the sequence of V_n-polynomials are similar to the sequence of the Jones polynomial of a knot [Jon87] and its $(n,1)$-parallels. In fact, it follows from the axioms of the TQFT that the Jones polynomial of a parallel of knot is a linear combination (with coefficients that are independent of the knot) of colored Jones polynomial, colored by the irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$; see [RT90, Tur94], and vice-versa.

It was recently conjectured in [GKK] that the V_n-polynomials of a knot are also linear combinations of the V_1-polynomial of a knot and its $(n',1)$-cables for $n' \leq n$, and a proof for $n=2$ was given there. We illustrate this relation here, and at the same time giving a consistency between the coefficients of the relation computed by the spectral decomposition of R-matrices and by representation theory in [GKK] with the computer-program that computes V_1 and V_2. The following relation holds for the unknot, 3_1, 4_1, 6_1, 6_2, 6_3, 7_7, 8_3, 8_4 and their mirrors (in total, 14 knots)

$$V_{K,2}(t^2,q^2) = c_{2,0}(t,q)V_{K(2,1),1}(t,q) + c_{2,-1}(t,q)V_{K,1}(t^2q^{-1},q) + c_{2,1}(t,q)V_{K,1}(t^2q,q)\,, \qquad (14)$$

where

$$c_{2,-1}(t,q) = \frac{t(t^2q^2-1)}{q(1+q^2)(t^2-1)}, \qquad c_{2,1}(t,q) = \frac{t^2-q^2}{qt(1+q^2)(t^2-1)}, \qquad c_{2,0}(t,q) = \frac{(q+t)(1+qt)}{(1+q^2)t} \qquad (15)$$

satisfy the symmetry $c_{2,1}(t,q) = c_{2,-1}(t^{-1},q)$, $c_{2,0}(t,q) = c_{2,0}(t^{-1},q)$.

Some values of (14) are given in the appendix.

1.6. **The V_2-polynomial of torus knots with two strings.** From the general setting, it follows that if β and γ are elements of a braid group of a fixed numbers of strands and K_n denotes the link obtained by the closure of $\beta^n\gamma$, then K_n is a knot if n lies in an arithmetic progression and the sequence $V_1(K_n)(t,q)$ is holonomic and satisfies a linear recursion relation with coefficients in $\mathbb{Z}[t^{\pm1},q^{\pm1}]$) coming from the minimal polynomial of the square of the R-matrix. This can be computed explicitly and leads to the answer. The above holds locally, if we replace tangle in a planar projection of a knot by $\beta^n\gamma$, and holds for any of the polynomial invariants that we discuss in this paper.

We illustrate this giving a recursion relation of the values of V_1 and V_2 for $T(2,2b+1)$-torus knots for an integer b. The minimal polynomial of the square of the R-matrix of V_1 is

$$(-1+x)(-t^2+q^2x)(-1+q^2t^2x) = -t^2+(q^2+t^2+q^2t^4)x-(q^2+q^4t^2+q^2t^4)x^2+q^4t^2x^3\,. \qquad (16)$$

It follows that $f_b(t,q) = V_{T(2,2b+1),1}(t,q)$ satisfies the recursion relation

$$-t^2f_b(t,q) + (q^2+t^2+q^2t^4)f_{b+1}(t,q) - (q^2+q^4t^2+q^2t^4)f_{b+2}(t,q) + q^4t^2f_{b+3}(t,q) = 0 \qquad (17)$$

for $b \in \mathbb{Z}$ with initial conditions

$$f_{-1}(t,q) = 1, \quad f_0(t,q) = 1, \quad f_1(t,q) = 1 + (q^{-1}+q^{-3})u + q^2u^2 \qquad (18)$$

where

$$u = t + t^{-1} - q - q^{-1} \,. \tag{19}$$

This and the $t \leftrightarrow t^{-1}$ symmetry of V_1 implies that $f_b(t,q) = q^{-2b}(t^{2b}+t^{-2b})+$(lower order terms) for $b \geq 0$, thus $\deg_t(f_b(t,q)) = 4b = 4 \cdot \mathrm{genus}(T(2,2b+1))$ for $b > 0$. It follows that inequality (3) for $n=2$ is an equality for $b \geq 0$. Since $\overline{T(2,2b+1)} = T(2,-2b-1)$, it follows that $f_b(t,q^{-1}) = f_{-b-1}(t,q)$ which then concludes that inequality (3) for $n=1$ is an equality for all 2-string torus knots.

Likewise, the minimal polynomial of the square of the R-matrix of V_2 is

$$(-1+x)(-t^2+q^2x)(-1+q^3x)(-t^2+q^4x)(-1+q^2t^2x)(-1+q^4t^2x) \tag{20}$$

which translates into a 6th order linear recursion relation for $g_b(t,q) = V_{T(2,2b+1),2}(t,q)$ with initial conditions

$$\begin{aligned}
g_{-1}(t,q) =& g_0(t,q^{-1}) = 1 \\
g_{-2}(t,q) =& g_1(t,q^{-1}) = 1 + (q+2q^3-q^4+q^5-q^6)u + (q^2+q^4-q^5)u^2 \\
g_{-3}(t,q) =& g_2(t,q^{-1}) = 1 + (2q+3q^3-q^4+3q^5-q^6+2q^7-q^8+q^9-2q^{10}+q^{11}-q^{12})u \\
&+ (4q^2+7q^4-3q^5+10q^6-6q^7+6q^8-7q^9+3q^{10}-3q^{11})u^2 \\
&+ (3q^3+6q^5-3q^6+6q^7-6q^8+3q^9-3q^{10})u^3 + (q^4+q^6-q^7+q^8-q^9)u^4
\end{aligned} \tag{21}$$

with u as in (18). As in the case of V_1, from the above recursion one deduces that inequality (3) for $n=2$ is in fact an equality for all 2-string torus knots.

We have performed the analogous calculation for the case of the V_3 and V_4 polynomials, and the conclusion is that inequality (3) for $n = 1, \dots, 4$ is in fact an equality for all 2-string torus knots.

1.7. **Positivity of the V_1 and V_2-polynomials of alternating knots?** The next topic that we discuss is a curious positivity observation for the coefficients of the V_1 and V_2 polynomials of alternating knots. Recall the number of alternating knots with at most 15 crossings (up to mirror image) given in Table 4 and taken from [HTW98].

crossings	3	4	5	6	7	8	9	10	11	12	13	14	15
# alt. knots	1	1	2	3	7	18	41	123	367	1288	4878	19536	85263

Table 4. Alternating knot counts, up to mirror image.

After computing the V_1 and V_2 polynomials in the following range of knots, we observed the following.

Proposition 1.6. For all alternating knots with ≤ 15 crossings, we have

$$V_1(t,-q) \in \mathbb{Z}_{\geq 0}[t^{\pm 1}, q^{\pm 1}] \,. \tag{22}$$

The same conclusion holds for V_2 for all alternating knots with ≤ 14 crossings.

The above positivity fails for $V_3(t,-q)$ and $V_4(t,-q)$ already both for the 3_1 and the 4_1 knots.

Question 1.7. Is this an accident of knots with low number of crossings or a hint of a relation of V_1 and V_2 with some categorification theory?

2. Computing the V_n-polynomials

A priori, the polynomial invariant of long knots based on an R-matrix on a d-dimensional vector space is a state sum of d^{2c} terms where c is the number of crossings of a planar projection of a knot knot, and in the case of the V_n polynomials, $d = 4n$. Even though the summand is sparse, a direct computation of the V_2 polynomial for knots with 8 crossings is unfeasible. A key observation is that every polynomial invariant of long knots of [GK] is given as a state sum and has a natural local tangle version due to the fact that oriented crossings are allowed to be oriented up/downwards but also sideways. The locality property of this polynomial is very important for its efficient computation, an idea that highlighted time and again in the work of Bar-Natan and van der Veen.

Given a planar diagram of a knot, the computation is assembled from the following parts:

(1) Convert the planar diagram into the planar diagram of a corresponding long knot, with up-pointing crossings, and record the rotation number for each arc;
(2) Correspond each crossing with the pre-computed R-matrices. Since the crossings are up-pointing, there are only two kinds of crossings (positive and negative);
(3) Tensor contract the crossings until the polynomial is obtained, with respect to a specific order that simplifies the computation. The contraction is done accordingly to the rotation numbers of arcs, which, together with the pre-computed curls, determine the signs of indeterminate terms.

Part (1) is done by directly calling the function `Rot[]` from [BNvdVa]. The part most critical to the speed of computation is to determine the order to which the contraction is performed. Before explaining our method of determining the order, we briefly review some standard terminologies of tensor contraction.

An *n-tensor* is a tuple indexed by a set of the form

$$\{1, \cdots, m_1\} \times \cdots \times \{1, \cdots m_n\},$$

where, for $k \in \{1, \cdots, n\}$, the integer m_k is the dimension of the *leg* k of the tensor.

For an n-tensor T, the entry corresponding to an element $\{i_1, \cdots, i_n\}$ in the index set is denoted as $T_{i_1,\cdots,i_n}$. Given two tensors T and T', if T has a leg k whose dimension m_k is the same as a leg k' of T', we can *contract* T and T' along the pair of legs k and k' to a new tensor T'', defined by

$$T''_{i_1,\cdots,\widehat{i_k},\cdots,i_n,j_1,\cdots,\widehat{j_{k'}},\cdots,j_{n'}} := \sum_{\substack{i_k, j_{k'} \in \{1,\cdots,m_k\} \\ i_k = j_{k'}}} T_{i_1,\cdots,i_k,\cdots,i_n} T'_{j_1,\cdots,j_{k'},\cdots,j_{n'}},$$

where the hats indicate that the indices under them are deleted. Similarly we can define contractions of contracting multiple pairs of legs and/or more than two tensors at once. By definition, assuming each multiplication and addition is of the same complexity (and ignoring the fact that we are adding and multiplying polynomials with integer coefficients in two variables, rather than integers), the time complexity of a contraction is described by

$$\frac{\text{Product of dimensions of all legs of tensors involved}}{\text{Product of dimensions of all pairs of legs contracted}}. \tag{23}$$

For example, the time complexity of contracting two 3-tensors along two pairs of legs into a 2-tensor, where all legs are of dimension d, is $\frac{d^3 \cdot d^3}{d \cdot d} = d^4$.

In our case, the tensors come from the R-matrices, which are indexed as 4-tensors with $4n$-dimensional legs for the V_n-polynomial. When associating the crossings with R-matrices in part (2), the four arcs of the crossings are also associated to the four legs of the corresponding 4-tensors according to a carefully arranged order. Tensor contraction of crossings in part (3) means to contract the associated tensors along their common legs (i.e. legs associated to same arcs) up to the sign provided by the curls and rotation numbers. After all arcs, except the entrance and exit arcs, have been contracted, we obtain a 2-tensor which is a $4n \times 4n$ diagonal matrix with identical diagonal entries which give the desired V_n-polynomial. This allows us to reduce the legs associated to the entrance and exit arcs to only one dimension before computing the contractions.

When all dimensions of legs are equal, taking the logarithm with the dimension as the base to (23) gives the following description of the time complexity

$$\log_{\mathrm{dim}}(\text{time complexity}) = \#\{\text{legs of tensors involved}\} - \#\{\text{pairs of legs contracted}\}, \tag{24}$$

where $\#S$ stands for the cardinality of a set S.

Therefore, to reduce the time complexity, we need to reduce the number of legs of tensors involved and increase the number of pairs of legs contracted per contraction. For our case, this means that we contract all contractible legs of two tensors at a time, and find an order of contractions so that the right hand side of (24) is as small as possible. The latter is the most difficult, and is well-known to be NP-hard for general graphs (instead of only the planar diagrams of knots) in the field of tensor network. Due to the less complex nature of planar diagrams of knots, we do not need to go as far as NP-hard, while being able to reduce the time complexity to a satisfactory level with some techniques. One approach to this is to prioritize contracting bigons as described in [MST], but we are taking a different approach here, described as the following:

(1) Input: a planar diagram of a long knot with rotation numbers obtained by `Rot[]` and R-matrices associated;
(2) While there are arcs other than the entrance and exit arcs remaining:
    (a) Find all pairs of crossings with common arcs (crossings may appear multiple times in different pairs);
    (b) Evaluate the time complexity of contracting each pair of crossings in (a) along all their common arcs according to (24) (the entrance and exit arcs do not count);
    (c) Contract the pair with the minimal time complexity (if there are multiple pairs with the minimal complexity, choose an arbitrary one).
(3) Output: Only the entrance and exit arcs remain and the desired V_n-polynomial is obtained.

In practice, the time complexity depends on which arc is split when the knot is converted into a long knot, so before actually performing the computations of R-matrices, we execute the above algorithm with each possible long knot as the input, contract formally without computing the R-matrices, record the time complexities in each step (c) and return the maximal time complexity recorded as the estimated complexity for the input long knot. After all these, we perform the actual computation on the long knot with the minimal estimated complexity.

For example, it takes a few minutes to compute the V_2-polynomial of the loose knots with 11 crossings from (6).

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Appendix A. V_2-equivalence classes of knots with at most 14 crossings

In this section we list the V_2-equivalence classes (up to mirror image) of knots with at most 14 crossings. As perhaps expected, the equivalence classes involve knots with the same number of crossings. Overline means mirror image. The counts are given in Table 3.

Below, we indicate the flavor of each equivalence class (defined in Equation (9)) by the corresponding number in the superscript. There are 218 tuples in total, 172 being tight + thin, 30 tight + thick, and 16 loose + thick.

First, we give the 3 pairs with 12 crossings.

$$(12n364,\ \overline{12n365})^1 \quad (12n421,\ \overline{12n422})^1 \quad (12n553,\ \overline{12n556})^1 \tag{25}$$

Next, we give the 25 pairs of knots with 13 crossings.

$$\begin{array}{cccc}
(13a141,\ 13a142)^1 & (13a199,\ 13a204)^1 & (13a906,\ 13a916)^1 & (13a1114,\ 13a1143)^1 \\
(13a1126,\ 13a1163)^1 & (13a1813,\ 13a1831)^1 & (13a1991,\ 13a2021)^1 & (13a1995,\ 13a2006)^1 \\
(13a2720,\ 13a2727)^1 & (13a2802,\ 13a2808)^1 & (13n370,\ 13n373)^1 & (13n372,\ 13n375)^3 \\
(13n404,\ 13n416)^1 & (13n406,\ 13n418)^1 & (13n534,\ 13n549)^1 & (13n536,\ 13n551)^3 \\
(13n875,\ 13n950)^1 & (13n1129,\ 13n1130)^1 & (13n1653,\ 13n1683)^3 & (13n1655,\ 13n1685)^2 \\
(13n1894,\ 13n2099)^1 & (13n2185,\ 13n2229)^1 & (13n2205,\ 13n2250)^3 & (13n2933,\ \overline{13n2956})^1 \\
(13n2937,\ 13n2955)^1 & & &
\end{array} \tag{26}$$

Next, we give the 189 pairs and one triple of knots with 14 crossings.

$$
\begin{array}{rrrr}
(14a34,\, 14a35)^1 & (14a96,\, 14a103)^1 & (14a454,\, 14a458)^1 & (14a518,\, 14a592)^1 \\
(14a533,\, 14a550)^1 & (14a608,\, 14a617)^1 & (14a675,\, 14a734)^1 & (14a718,\, 14a736)^1 \\
(14a989,\, 14a1017)^1 & (14a1047,\, 14a1170)^1 & (14a1268,\, 14a1362)^1 & (14a1445,\, 14a1449)^1 \\
(14a1522,\, 14a1532)^1 & (14a1767,\, 14a1860)^1 & (14a2205,\, 14a2215)^1 & (14a2253,\, 14a2256)^1 \\
(14a2609,\, 14a2618)^1 & (14a3400,\, 14a3433)^1 & (14a3403,\, 14a3432)^1 & (14a3407,\, 14a3436)^1 \\
(14a3409,\, 14a3438)^1 & (14a3419,\, 14a3439)^1 & (14a4041,\, 14a4998)^1 & (14a4147,\, 14a4939)^1 \\
(14a4901,\, 14a5698)^1 & (14a6467,\, \overline{14a6614})^1 & (14a6614,\, \overline{14a6467})^1 & (14a7193,\, 14a7216)^1 \\
(14a7196,\, 14a7269)^1 & (14a7200,\, 14a7249)^1 & (14a7207,\, 14a7272)^1 & (14a7210,\, 14a7275)^1 \\
(14a7215,\, 14a7698)^1 & (14a7219,\, 14a7260)^1 & (14a7258,\, 14a7263)^1 & (14a7264,\, 14a7274)^1 \\
(14a7446,\, 14a7477)^1 & (14a7527,\, 14a7598)^1 & (14a8017,\, 14a8107)^1 & (14a8096,\, 14a8115)^1 \\
(14a9707,\, 14a9711)^1 & (14a10116,\, \overline{14a10142})^1 & (14a10142,\, \overline{14a10116})^1 & (14a10405,\, 14a10410)^1 \\
(14a10407,\, 14a10434)^1 & (14a10411,\, 14a10453)^1 & (14a10414,\, 14a10417)^1 & (14a10415,\, 14a10435)^1 \\
(14a10439,\, 14a10456)^1 & (14a10853,\, 14a11544)^1 & (14a11793,\, 14a12325)^1 & (14a12335,\, 14a12344)^1 \\
(14a12816,\, 14a12833)^1 & (14a12868,\, 14a12876)^1 & (14a13431,\, 14a13433)^1 & (14a13473,\, 14a13475)^1 \\
(14n179,\, 14n182)^1 & (14n181,\, 14n184)^3 & (14n213,\, 14n226)^1 & (14n215,\, 14n228)^1 \\
(14n364,\, 14n386)^1 & (14n366,\, 14n388)^3 & (14n733,\, 14n810)^1 & (14n1366,\, 14n1393)^1
\end{array}
\tag{27a}
$$

$$
\begin{array}{rrrr}
(14n1370,\, 14n1395)^2 & (14n1374,\, 14n1397)^1 & (14n1378,\, 14n1399)^1 & (14n1380,\, 14n1401)^3 \\
(14n1692,\, 14n1704)^1 & (14n1697,\, 14n1945)^1 & (14n1699,\, 14n1947)^2 & (14n1701,\, 14n1949)^1 \\
(14n1752,\, 14n1753)^1 & (14n1762,\, 14n1839)^2 & (14n1764,\, 14n1841)^2 & (14n1766,\, 14n1843)^2 \\
(14n1768,\, 14n1775)^1 & (14n1770,\, 14n1835)^1 & (14n1772,\, 14n1837)^1 & (14n2007,\, 14n2032)^2 \\
(14n2148,\, 14n2372)^1 & (14n2150,\, 14n2374)^3 & (14n2295,\, 14n2376)^1 & (14n2297,\, 14n2378)^1 \\
(14n2299,\, 14n2380)^2 & (14n2423,\, 14n5868)^2 & (14n3418,\, 14n3823)^1 & (14n3422,\, 14n3825)^3 \\
(14n3444,\, 14n3836)^1 & (14n3448,\, 14n3829)^1 & (14n3450,\, 14n3831)^2 & (14n4577,\, 14n4583)^2 \\
(14n4579,\, 14n4585)^2 & (14n4657,\, 14n4665)^1 & (14n4925,\, 14n5085)^1 & (14n4930,\, 14n4931)^1 \\
(14n4933,\, 14n5087)^2 & (14n4938,\, 14n4939)^1 & (14n5822,\, 14n5852)^3 & (14n5854,\, 14n5862)^1 \\
(14n7506,\, 14n7559)^1 & (14n7566,\, 14n7673)^1 & (14n7575,\, 14n7675)^2 & (14n7577,\, 14n7677)^2 \\
(14n7586,\, 14n7678)^1 & (14n7593,\, 14n7680)^1 & (14n7597,\, 14n7682)^2 & (14n7599,\, 14n7684)^2 \\
(14n7603,\, 14n7686)^1 & (14n7617,\, 14n7628)^2 & (14n7636,\, 14n7688)^1 & (14n7638,\, 14n7690)^1 \\
(14n8225,\, 14n10806)^2 & (14n8291,\, \overline{14n8293})^1 & (14n8648,\, \overline{14n8649})^1 & (14n8650,\, \overline{14n8651})^1 \\
(14n8696,\, \overline{14n8697})^1 & (14n9075,\, \overline{14n9076})^1 & (14n9139,\, \overline{14n9140})^1 & (14n9142,\, \overline{14n9143})^1 \\
(14n9395,\, \overline{14n9396})^1 & (14n9398,\, \overline{14n9399})^1 & (14n9455,\, \overline{14n9456})^1 & (14n9458,\, \overline{14n9459})^1 \\
(14n9686,\, \overline{14n9687})^1 & (14n10002,\, 14n11740)^3 & (14n10503,\, 14n11641)^3 & (14n11679,\, 14n11981)^3
\end{array}
\tag{27b}
$$

$$\begin{array}{llll}
(14n14122,\, 14n14288)^1 & (14n14130,\, \overline{14n14216})^1 & (14n14134,\, 14n14215)^2 & (14n14136,\, \overline{14n14219})^3 \\
(14n14148,\, 14n14150)^1 & (14n14149,\, 14n14151)^1 & (14n14154,\, 14n14328)^1 & (14n14156,\, \overline{14n14157})^1 \\
(14n14158,\, \overline{14n14159})^1 & (14n14162,\, 14n14169)^1 & (14n14177,\, 14n14330)^3 & (14n14196,\, 14n14333)^1 \\
(14n14204,\, 14n14341)^1 & (14n14208,\, 14n15068)^2 & (14n14210,\, 14n14225)^1 & (14n14214,\, \overline{14n14131})^2 \\
(14n14223,\, \overline{14n14211})^2 & (14n14227,\, 14n14335)^1 & (14n14313,\, 14n14319)^3 & (14n14322,\, 14n14332)^2 \\
(14n14504,\, \overline{14n14506})^1 & (14n14508,\, \overline{14n14502})^1 & (14n14511,\, \overline{14n14513})^1 & (14n14516,\, \overline{14n14509})^1 \\
(14n14590,\, 14n14663)^1 & (14n14655,\, 14n14685)^1 & (14n14687,\, 14n15694)^1 & (14n14780,\, 14n14893)^2 \\
(14n14787,\, 14n14895)^1 & (14n14793,\, 14n14897)^1 & (14n14808,\, \overline{14n14804})^1 & (14n14924,\, \overline{14n14923})^1 \\
(14n14926,\, \overline{14n14925})^1 & (14n14931,\, \overline{14n14930})^1 & (14n15024,\, \overline{14n15022})^1 & (14n15059,\, \overline{14n15058})^1 \\
(14n15063,\, \overline{14n15062})^1 & (14n15066,\, \overline{14n15065})^1 & (14n15084,\, \overline{14n15083})^1 & (14n15103,\, \overline{14n15102})^1 \\
(14n15106,\, \overline{14n15105})^1 & (14n15173,\, \overline{14n15172})^1 & (14n15180,\, \overline{14n15179})^1 & (14n15202,\, \overline{14n15201})^1 \\
(14n15205,\, \overline{14n15204})^1 & (14n15208,\, \overline{14n15207})^1 & (14n15228,\, \overline{14n15227})^1 & (14n15232,\, \overline{14n15231})^1 \\
(14n15258,\, \overline{14n15257})^1 & (14n15727,\, 14n15756)^1 & (14n15729,\, 14n15758)^2 & (14n17934,\, \overline{14n17940})^1 \\
(14n17938,\, \overline{14n17936})^2 & (14n17942,\, 14n17946)^1 & (14n17949,\, \overline{14n17960})^2 & (14n17986,\, 14n18013)^1 \\
(14n17997,\, 14n18017)^1 & (14n18005,\, 14n18012)^2 & (14n18146,\, \overline{14n18144})^1 & (14n18208,\, \overline{14n18207})^1 \\
(14n19744,\, 14n19758)^2 & & &
\end{array} \tag{27c}$$

$$(14n14212,\, 14n14213,\, \overline{14n14222})^1 \tag{27d}$$

The knots in Tables (25), (26), (27a)–(27c) are Conway mutant, have equal HKF and Khovanov Homology and Equation (3) is an equality for $n = 2$. The knots in Tables (25), (26) have equal V_1, V_2, V_3 and V_4 polynomials and the ones in Tables (27a)–(27c) have equal V_1, V_2 and V_3 polynomials with the two exceptions as in the footnote of Proposition 1.3.

All but at most 12 of the tight + thin knots listed above are quasi-alternating knots. We examined them with a computer search, extending further the table of quasi-alternating knots with ≤ 12 crossings of Jablan [Jab]. All but the following 12 tight + thin knots were confirmed to be quasi-alternating:

$$\begin{gathered} 14n2378,\; 14n3448,\; 14n4925,\; 14n5085,\; 14n5862,\; 14n5854, \\ 14n7559,\; 14n7506,\; 14n14151,\; 14n14149,\; 14n14169,\; 14n14162\,. \end{gathered} \tag{28}$$

Appendix B. Values for Whitehead doubles and (2,1)-parallels

The values of $V_{\mathrm{Wh}(K),2}(t,q)$ for the first three nontrivial knots is given as follows, where $u = t + t^{-1} - q - q^{-1}$ is as in (19).

$$\begin{aligned}
V_{\mathrm{Wh}(3_1),2}(t,q) =& 1 + (-2 - 2q^{-2} + 2q^{-1} + 4q - 4q^2 + 4q^3 - 2q^4 - 2q^7 + 2q^8 - 2q^{10} + 2q^{11} + 2q^{15} - 2q^{16} + 2q^{17} \\
& - 2q^{18} - 2q^{20} + 4q^{22} - 2q^{23})u + (2 + 2q^{-2} - 2q^{-1} - 4q + 2q^2 - 4q^3 + 2q^4 + 4q^5 - 2q^6 + 4q^7 \\
& - 4q^8 + 4q^9 - 6q^{10} + 2q^{13} - 2q^{14} + 2q^{15} + 2q^{18} - 2q^{19} + 2q^{20} - 4q^{21} + 2q^{22})u^2, \\
V_{\mathrm{Wh}(\overline{3_1}),2}(t,q) =& 1 + (-2q^{-26} + 4q^{-25} - 2q^{-23} - 2q^{-21} + 2q^{-20} - 2q^{-19} + 2q^{-18} + 2q^{-14} - 2q^{-13} + 2q^{-11} \\
& - 2q^{-10} + 2q^{-9} - 4q^{-8} + 4q^{-7} - 6q^{-6} + 4q^{-5} - 4q^{-4} + 4q^{-3})u + (-2q^{-25} + 4q^{-24} - 2q^{-23} \\
& + 2q^{-22} - 2q^{-21} - 2q^{-18} + 2q^{-17} - 2q^{-16} + 6q^{-13} - 4q^{-12} + 4q^{-11} - 4q^{-10} + 2q^{-9} - 2q^{-8} \\
& - 2q^{-7} + 4q^{-6} - 4q^{-5} + 2q^{-4} - 2q^{-3} + 2q^{-2})u^2
\end{aligned} \tag{29}$$

and for fun,

$$
\begin{aligned}
V_{\mathrm{Wh}(4_1),2}(t,q) =&1 + (-14 - 2q^{-18} + 4q^{-17} + 2q^{-16} - 6q^{-15} - 4q^{-14} + 6q^{-13} + 6q^{-12} - 4q^{-11} - 8q^{-10} + 4q^{-9} \\
&+ 8q^{-8} - 4q^{-7} - 10q^{-6} + 16q^{-5} - 16q^{-4} + 20q^{-3} - 22q^{-2} + 18q^{-1} + 8q + 2q^2 - 4q^3 - 6q^4 + 8q^5 \\
&+ 4q^6 - 8q^7 - 4q^8 + 6q^9 + 6q^{10} - 4q^{11} - 6q^{12} + 2q^{13} + 4q^{14} - 2q^{15})u + (24 - 2q^{-17} + 4q^{-16} \\
&- 2q^{-14} - 4q^{-13} + 4q^{-12} + 2q^{-11} + 4q^{-10} - 16q^{-9} + 10q^{-8} + 8q^{-7} - 10q^{-6} - 4q^{-5} + 20q^{-4} \\
&- 28q^{-3} + 28q^{-2} - 28q^{-1} - 16q + 2q^2 + 12q^3 - 8q^4 - 10q^5 + 16q^6 - 4q^7 - 2q^8 - 4q^9 + 4q^{10} + 2q^{11} \\
&- 4q^{13} + 2q^{14})u^2
\end{aligned} \tag{30}
$$

with u as in (19).

Appendix C. Values for (2,1)-parallels of knots

We now give values of the $V_{K,1}$, $V_{K(2,1),1}$ and $V_{K,2}$ for some sample knots to explicitly confirm Equation (14).

$$
\begin{aligned}
V_{3_1,1}(t,q) =&1 + (q + q^3)u + q^2u^2, \\
V_{3_1,2}(t,q) =&1 + (q + 2q^3 - q^4 + q^5 - q^6)u + (q^2 + q^4 - q^5)u^2, \\
V_{3_1(2,1),1}(t,q) =&1 + (q^{-3} + 2q + 3q^3 + 2q^5 + q^7 - q^{13})u + (3 + 3q^{-2} + 6q^2 + 4q^4 + 4q^6 + 2q^8 - 2q^{10} - 2q^{12})u^2 \\
&+ (3q^{-1} + 3q + q^3 + q^5 + q^7 - q^{11})u^3 + u^4
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
V_{4_1,1}(t,q) =&1 + (-q^{-1} - q)u + u^2, \\
V_{4_1,2}(t,q) =&1 + (2 - q^{-3} + q^{-2} - 2q^{-1} - 2q + q^2 - q^3)u + (1 + q^{-2} - q^{-1} - q + q^2)u^2, \\
V_{4_1(2,1),1}(t,q) =&1 + (-q^{-7} + q^{-5} - 3q^{-3} - 3q^{-1} - q^3 - q^7)u + (2 + 2q^{-6} + 5q^{-4} + q^{-2} + 2q^4 + 2q^6)u^2 \\
&+ (q^{-5} + 3q^{-3} + 3q^{-1} + q^5)u^3 + q^{-2}u^4
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
V_{6_1,1}(t,q) =&1 + (-q^{-1} - 2q - q^3)u + (3 + q^2)u^2, \\
V_{6_1,2}(t,q) =&1 + (1 - q^{-3} - 2q^{-1} - 2q + 2q^2 - 2q^3 + 2q^4 - 2q^5 + q^6 - q^7)u \\
&+ (5 + 3q^{-2} - 2q^{-1} - 5q + 3q^2 - 3q^3 + 3q^4 - q^5 + q^6)u^2, \\
V_{6_1(2,1),1}(t,q) =&1 + (-q^{-7} + 2q^{-5} - 5q^{-3} - 8q^{-1} - 2q - q^{11} - q^{15})u + (15 + 6q^{-6} + 17q^{-4} + 16q^{-2} - 2q^2 - 4q^4 + 4q^8 \\
&+ 4q^{10} + 2q^{12} + 2q^{14})u^2 + (3q^{-5} + 10q^{-3} + 15q^{-1} + 3q - 2q^3 + 2q^9 + q^{13})u^3 + (1 + 3q^{-2})u^4
\end{aligned} \tag{33}
$$

$$
\begin{aligned}
V_{6_2,1}(t,q) =&1 + (-q^{-1} - q^5)u + (1 - q^2 - q^4)u^2 + (q + q^3)u^3 + q^2u^4, \\
V_{6_2,2}(t,q) =&1 + (1 - q^{-3} + q^{-2} - 2q^{-1} - q + q^2 - q^5 + q^6 - 2q^7 + 2q^8 - q^9)u + (-1 + q^{-2} - q^{-1} + 2q - 4q^2 + 5q^3 \\
&- 6q^4 + 5q^5 - 3q^6 + 2q^7 - q^8)u^2 + (-1 + q^{-1} + 3q - 3q^2 + 4q^3 - 4q^4 + 3q^5 - 2q^6 + q^7)u^3 \\
&+ (1 - q + 2q^2 - 2q^3 + 2q^4 - 2q^5 + q^6)u^4, \\
V_{6_2(2,1),1}(t,q) =&1 + (q^{-9} - 3q^{-7} + q^{-3} - 6q^{-1} + 4q^3 - 2q^5 - 2q^7 - q^{13} + q^{17} - q^{19})u + (7q^{-8} - q^{-6} - 5q^{-4} - q^{-2} - 8q^2 \\
&- 12q^4 - q^6 + q^8 + 2q^{10} + 2q^{12} - 2q^{18})u^2 + (21q^{-7} + 20q^{-5} + 6q^{-3} + 10q^{-1} + 18q + 14q^3 + 18q^5 + 9q^7 \\
&- 5q^9 + 2q^{11} - q^{13} + 5q^{15} + 3q^{17})u^3 + (64 + 35q^{-6} + 62q^{-4} + 64q^{-2} + 63q^2 + 41q^4 + 22q^6 - 12q^8 \\
&- 16q^{10} + 16q^{14} + 12q^{16})u^4 + (35q^{-5} + 73q^{-3} + 74q^{-1} + 51q + 29q^3 + 18q^5 - 13q^9 - 11q^{11} + 11q^{13} \\
&+ 13q^{15})u^5 + (25 + 21q^{-4} + 39q^{-2} + 6q^2 + 7q^4 - 6q^{10} + 6q^{14})u^6 + (7q^{-3} + 8q^{-1} + q^3 - q^{11} + q^{13})u^7 + q^{-2}u^8
\end{aligned} \tag{34}
$$

and finally,

$$
\begin{aligned}
V_{8_4,1}(t,q) =& 1 + (-q^{-3} - 2q^{-1} - q - q^3 - q^5)u + (3 + q^{-2} + q^4)u^2 + (q^{-1} + 6q + 5q^3)u^3 + (1 + 3q^2)u^4, \\
V_{8_4,2}(t,q) =& 1 + (2 - q^{-7} + q^{-6} - 2q^{-5} + q^{-4} - 2q^{-3} + q^{-2} - 3q^{-1} - 2q + 3q^2 - 2q^3 + q^4 - 2q^5 - q^7 + q^8 - q^9)u \\
&+ (8 + q^{-6} - q^{-5} + 3q^{-4} - 2q^{-3} + 3q^{-2} - 5q^{-1} - 7q + 8q^2 - 10q^3 + 8q^4 - 5q^5 + 6q^6 - 3q^7 + q^8)u^2 \\
&+ (-20 + q^{-5} - q^{-4} + 7q^{-3} - 7q^{-2} + 16q^{-1} + 24q - 27q^2 + 27q^3 - 22q^4 + 18q^5 - 9q^6 + 5q^7)u^3 \\
&+ (6 + q^{-4} - q^{-3} + 4q^{-2} - 4q^{-1} - 8q + 8q^2 - 7q^3 + 7q^4 - 5q^5 + 3q^6)u^4, \\
V_{8_4(2,1),1}(t,q) =& 1 + (-q^{-15} - 5q^{-7} - 3q^{-3} - 11q^{-1} + 2q + 6q^3 - 6q^5 - 4q^7 + 2q^9 - 2q^{11} - q^{13} - q^{19})u + (14 + 2q^{-14} \\
&+ 2q^{-12} + 11q^{-10} + 24q^{-8} - 2q^{-6} - 9q^{-4} + 14q^{-2} - 6q^2 - 11q^4 + 17q^8 + 8q^{10} + 8q^{12} + 2q^{14} - 2q^{16} \\
&+ 2q^{18})u^2 + (5q^{-13} + 8q^{-11} + 51q^{-9} + 127q^{-7} + 124q^{-5} + 89q^{-3} + 63q^{-1} + 56q + 46q^3 + 49q^5 + 52q^7 \\
&+ 27q^9 + 41q^{11} + 27q^{13} + 22q^{15} + 21q^{17})u^3 + (141 + 12q^{-12} + 28q^{-10} + 107q^{-8} + 259q^{-6} + 330q^{-4} \\
&+ 255q^{-2} + 84q^2 + 47q^4 + 90q^6 + 44q^8 + 44q^{10} + 60q^{12} + 60q^{14} + 44q^{16})u^4 + (13q^{-11} + 24q^{-9} + 89q^{-7} \\
&+ 237q^{-5} + 309q^{-3} + 215q^{-1} + 70q + 5q^3 + 39q^5 + 41q^7 + 20q^9 + 21q^{11} + 44q^{13} + 41q^{15})u^5 + (59 \\
&+ 6q^{-10} + 6q^{-8} + 39q^{-6} + 114q^{-4} + 132q^{-2} - 11q^2 + 9q^4 + 6q^6 + 12q^8 + 6q^{12} + 18q^{14})u^6 \\
&+ (q^{-9} + 10q^{-5} + 28q^{-3} + 23q^{-1} - 2q + q^5 + 2q^9 - 2q^{11} + 3q^{13})u^7 + (q^{-4} + 3q^{-2})u^8
\end{aligned}
\tag{35}
$$

Keep in mind that the genus of the $(2,1)$-parallel of K is twice the genus of K, and that the knots 3_1, 4_1, 6_1, 6_2 and 8_4 have genus $1,1,1,2,2$, hence we expect (and we find) that the V_1-polynomial of their $(2,1)$-parallel to have u-degree $2,2,2,4,4$ conforming the equality in (3) for the $(2,1)$-parallels of 3_1, 4_1, 6_1, 6_2 and 8_4.

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