A CONJECTURE ON KHOVANOV'S INVARIANTS

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Abstract. We formulate a conjectural formula for Khovanov's invariants of alternating knots in terms of the Jones polynomial and the signature of the knot.

Appendix A. A conjecture

This note is written as an appendix to the forthcoming paper of Bar-Natan, who explains the beautiful ideas of Khovanov's on a categorification of the Jones polynomial, [Kh1, Kh2]. Hence, we will assume familiarity with [B], the notation of which we will use below.

Given a knot L, we will denote by $Kh(L) \in BZ[t^{\pm 1}, q^{\pm}]$ the generating series of the ranks of the Khovanov homology of L, using rational coefficients.

It follows by the definition of Kh that

$$Kh(-1,q) = (q+q^{-1})J(q), \tag{1}$$

where J is the Jones polynomial, normalized by $J(\text{unknot}) = 1$. We formulate a conjecture regarding the opposite specialization $Kh(t,1)$ of $Kh(t,q)$:

Conjecture 1. *For all alternating knots L, we have that*

$$Kh(t,1) = 1 + J(it^{1/2})(it^{1/2})^{-\sigma} \tag{2}$$

where $i=\sqrt{-1}$ and σ is the signature of a knot.

Bar-Natan, Khovanov and the author have formulated a restriction on the support of the Kh invariants, as follows:

Conjecture 2. *For all alternating knots L, the support of Kh lies in the lines* $\deg(q) = 2\deg(t) + \sigma \pm 1$.

These restrictions are enough to determine the Khovanov invariants in terms of the Jones polynomial and the signature, as follows.

Proposition A.1. *If a knot satisfies Conjectures 1 and 2, then*

$$Kh(t,q) = \frac{q^\sigma}{qt^{1/2}+q^{-1}t^{-1/2}}\Big(J(iqt^{1/2})(iqt^{1/2})^{-\sigma}(q^2t^{1/2}+q^{-2}t^{-1/2}) + t^{1/2}+t^{-1/2}\Big).$$

Proof. Conjecture 2 implies that we can write

$$Kh(t,q) = \sum_n q^{2n+\sigma}t^n(qa_n + q^{-1}b_n) = q^\sigma(qA(q^2t)+q^{-1}B(q^2t))$$

for some Laurrent polynomials A and B. Equations (1) and (2) are linear equations on A and B, which can solve explicitly. The result follows. □

Corollary A.2. *If a knot satisfies Conjectures 1 and 2, then the Khovanov invariant determines the Jones polynomial and its signature, and vice-versa.*

Proof. Conjecture 1, together with Equation (1) imply that Kh determines J and σ. The converse follows from Proposition A.1. □

Here is a numerical example of the simplest nontrivial alternating knot, a trefoil 3_1.

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Example A.3. For the trefoil, we have $Kh(t,q) = q + q^3 + t^2q^5 + t^3q^9$, $J(q) = q^2 + q^6 - q^8$ and $\sigma = 2$.

Which knots satisfy Conjectures 1 and 2?

Remark A.4. The computer calculations of Bar-Natan [B] inform us that Conjectures 1 and 2 are true for all alternating knots with at most 10 crossings and for most (but not all) non-alternating knots with at most 10 crossings. Both conjectures fail on the same set

$$\{8_{19}, 9_{42}, 10_{124}, 10_{128}, 10_{132}, 10_{136}, 10_{139}, 10_{145}, 10_{152}, 10_{153}, 10_{154}, 10_{161}\}$$

of knots up to 10 crossings. It is remarkable that both conjectures (which are superficially independent) fail on the same small set of knots.

The next remark gives an alternative form of Conjecture 1 in terms of the special value of $Kh(t,-1)$. This special value is in some sense an Euler characteristic rather than a Poincare polynomial.

Remark A.5. Consider a knot that satisfies Conjecture 2. Then it satisfies Conjecture 1 iff:

$$Kh(t,-1) = -1 - J(it^{1/2})(it^{1/2})^{-\sigma} \tag{3}$$

The proof of this claim follows easily from the fact that $J(q)$ is an even function of q and that σ is an even integer.

References

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